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Métodos mixtos con mallas híbridas para problemas elípticos en dominios poliedrales.

Tesis presentada para optar al título de Doctor de la Universidad de Buenos Aires en el área Ciencias Matemáticas

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Resumen

En esta Tesis introducimos un Método combinado de Elementos Finitos y Virtuales en dimensión tres para la aproximación mixta de un problema elíptico modelo para el operador de Laplace en un poliedro arbitrario. El método es analizado por completo cuando las mallas constan de prismas rectos de base triangular, pirámdes y tetraedros. Los espacios locales discretizantes coinciden con los espacios de orden mínimo de Raviart-Thomas sobre tetraedros y prismas, y constituyen una extensión de estos a elementos piramidales. Probamos que el esquema discreto es bien planteado y probamos estimaciones óptimas de error sobre mallas que admiten elementos anisótropos. En particular, nuestras estimaciones de error de interpolación local para el espacio discreto son óptimas y anisótropas en prismas rectos anisótropos. La motivación para trabajar con elementos anisótropos es que en distintas situaciones en aproximaciones por elementos finitos mixtos es necesario el uso de mallas con elementos elongados. Este es el caso, por ejemplo, con la ecuación de Poisson en un poliedro Ω con aristas cóncavas y vértices entrantes, que en forma mixta puede escribirse como

$$\begin{cases} \boldsymbol{u} &= -\nabla p \quad \text{en } \Omega \\ \operatorname{div} \boldsymbol{u} &= f \quad \text{en } \Omega \\ p &= 0 \quad \text{en } \partial\Omega. \end{cases}$$

En este caso la variable vectorial de la solución, $\boldsymbol{u}$, no está en $H^1(\Omega)$ en el caso general debido a las singularidades de aristas y vértices. En particular, cerca de las aristas cóncavas, $\boldsymbol{u}$ es más regular en la dirección a lo largo de éstas que transversalmente, y consecuentemente las mallas tienen que ser adecuadamente refinadas para recuperar el orden óptimo de convergencia con respecto al número de grados de libertad. Tales mallas contienen elementos arbitrariamente alargados en la dirección de las aristas singulares.

Asimismo, proponemos un proceso de mallado para construir una familia de mallas que nos permite obtener estimaciones de error de aproximacion global óptimas cuando la solución del problema modelo presenta singularidades de arista o vértice puesto que las mallas resultan, por construcción, adecuadamente graduadas y adaptadas a las singularidades, como mencionamos en el párrafo anterior.

Además en la presente Tesis obtuvimos cotas de estabilidad y de error de interpolación local para Elementos Finitos prismáticos anisótropos de orden arbitrario, tanto para la clase de elementos conformes en $H(\mathbf{curl})$ como para la clase de elementos conformes en $H(\operatorname{div})$, que constituyen resultados adicionales que extienden algunos hechos teóricos que probamos para el problema principal de la tesis.

También presentamos cotas de estabilidad y de error de interpolación local para Elementos Finitos piramidales anisótropos de orden bajo, tanto para la clase de elementos conformes en $H(\mathbf{curl})$ como para la clase de elementos conformes en $H(\operatorname{div})$. Este resultado está incluido para mostrar una variante a nuestro

método principal, esto es, un método solamente con elementos finitos. Con respecto a esta variante, como mostramos explícitamente en la Tesis, las funciones de forma, generadoras de estos últimos espacios de Elementos Finitos, son racionales y son singulares, aunque acotadas, en la pirámide de referencia. Esta es una razón por la cual consideramos que nuestra aproximación FE–VE combinada presenta una ventaja que es evitar la evaluación de funciones con dichas propiedades en implementaciones en computadoras.

Abstract

In this Thesis we introduce a combined Finite and Virtual Element Method in dimension three for the mixed approximation of a model elliptic problem for the Laplace operator on an arbitrary polyhedron. The method is fully analysed when the meshes are made up of triangularly right prisms, pyramids and tetrahedra. The local discrete spaces coincide with the lowest order Raviart-Thomas spaces on tetrahedra and prisms, and extend them to pyramidal elements. The discrete scheme is well posed and optimal error estimates are proved on meshes which allow for anisotropic elements. In particular, local interpolation error estimates for the discrete element space are optimal and anisotropic on anisotropic right prisms. The motivation to work with anisotropic elements is that in several situations in mixed finite element approximations the use of meshes with narrow elements is needed. This is the case for instance when dealing with the Poisson equation in a polyhedron Ω with concave edges or vertices, which in mixed form can be written as

$$\begin{cases} \boldsymbol{u} &= -\nabla p \quad \text{in } \Omega \\ \operatorname{div} \boldsymbol{u} &= f \qquad \text{in } \Omega \\ p &= 0 \qquad \text{on } \partial\Omega. \end{cases}$$

In this case the vectorial variable of the solution, $\boldsymbol{u}$, is in general not in $H^1(\Omega)$ due to vertex and edges singularities. In particular, close to concave edges, $\boldsymbol{u}$ is expected to be more regular in its direction than transversally to it, and consequently the mesh has to be accordingly refined in order to recover optimal order of convergence with respect to the number of degrees of freedom. Those meshes contain elements that are arbitrarily elongated in the direction of concave edges.

Likewise, we propose a meshing process to construct a mesh that allows us to obtain optimal global approximation error estimates when the solution has edge or vertex singularities as the mesh results, by construction, suitably graded and adapted to the singularities.

Furthermore, in the present Thesis we obtained local anisotropic stability and interpolation error estimates for arbitrary order Prismatic Finite Elements in both the **curl**–conforming and div–conforming classes of elements, which are additional results that extend some theoretical facts we proved for the main problem of the Thesis.

Moreover, we present local anisotropic stability and interpolation error estimates for lowest order Pyramidal Finite Elements constructed in the literature for both the **curl**–conforming and div–conforming classes of elements. This result is included to show a variant to our main method, that is one with only Finite Elements. Regarding this variant, as we show explicitly in the Thesis, the shape functions spanning the pyramidal Finite Element spaces are rational functions and are singular, yet bounded, in the reference pyramid. This is a reason why we considered that our combined FE–VE approach presents the advantage of avoiding the evaluation of functions with those properties in computer implementations.

Contents

Introducción

Un objeto físico se dice anisótropo si exhibe propiedades físicas vectoriales con magnitudes distintas cuando se las mide en direcciones distintas (cfr. [28]). En el caso de un método numérico diremos que la familia de mallas usada es anisótropa si contiene sucesiones de elementos tales que el orden de decrecimiento de sus tamaños a lo largo de una dirección es superior al correspondiente a otras direcciones independientes. Formalizaremos e ilustraremos esto en el Capítulo 1.

En varias situaciones en aproximaciones por elementos finitos mixtos es necesario el uso de mallas anisótropas. Este es el caso, por ejemplo, cuando se trata con la ecuación de Poisson en un poliedro $\Omega \subseteq \mathbb{R}^3$ con aristas y vértices cóncavos, la cual, introduciendo la variable vectorial $\boldsymbol{u} = \nabla p$ puede escribirse en forma mixta como

$$\begin{cases} \boldsymbol{u} = \nabla p & \text{in } \Omega \\ -\text{div}\, \boldsymbol{u} = f & \text{in } \Omega \qquad (f \in L^2(\Omega)) \\ p = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

La formulación variacional mixta estándar es hallar $\boldsymbol{u} \in H(\text{div}, \Omega)$ and $p \in L^2(\Omega)$ tales que

$$\begin{array}{lrcl} \forall \boldsymbol{v} \in H(\text{div}, \Omega) & a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) & = & 0 \\ \forall q \in L^2(\Omega) & b(\boldsymbol{u}, q) & = & (-f, q) \end{array} \tag{2}$$

en donde

$$a(\boldsymbol{v}, \boldsymbol{w}) = \int_\Omega \boldsymbol{v} \cdot \boldsymbol{w}\, d\boldsymbol{x}, \qquad b(\boldsymbol{v}, q) = \int_\Omega q\, \text{div}\, \boldsymbol{v}\, d\boldsymbol{x}.$$

Dado que Ω no es convexo, la solución $\boldsymbol{u}$ de (2) no pertenece a $H^1(\Omega)$ en el caso general por tener singularidades de arista y vértice. En particular, cerca de aristas cóncavas, se espera que la solución sea más regular a lo largo de la dirección de esa arista que transversalmente a ella (ver [5]), y es por eso que las mallas tienen que ser adecuadamente graduadas y refinadas para recuperar el orden óptimo de convergencia de la sucesión de soluciones numéricas con respecto al número de grados de libertad (ver [4, 5]). Ese tipo de mallas contiene elementos que son tan estrechos como se quiera en dirección ortogonal a las aristas cóncavas, con tal de tomarlos lo suficientemente cerca de ellas. Si tomáramos una sucesión de mallas como esta hecha únicamente con tetraedros, incurriríamos en el uso de subfamilias de estos que no verifican ciertas condiciones que son necesarias para el análisis anisótropo, y es aquí donde nuestro método propuesto con elementos de geometrías diferentes cobra sentido.

Para aclarar lo último y el principal resultado de esta Tesis (cfr. [30]) ponemos las siguientes definiciones y resultados previos relacionados.

Primero, T satisface la *propiedad del vértice regular* con una constante $\bar{c} > 0$ (escrito $T \in \mathcal{RVP}(\bar{c})$) si T tiene un vértice $\boldsymbol{x}_T$ tal que, tomando M_T como la matriz cuyas columnas son los vectores unitarios en las direcciones de las aristas que comparten a $\boldsymbol{x}_T$, entonces $|\det M_T| > \bar{c}$.

Una propiedad geométrica menos restrictiva es la siguiente. Diremos que un tetraedro T satisface la *propiedad del ángulo máximo* con parámetro $\bar{\alpha}$ (escrito $T \in \mathcal{MAC}(\bar{\alpha})$) si los ángulos de las caras de T y entre caras son menores a $\bar{\alpha}$.

Con estas dos nociones en mente, citamos el siguiente resultado de [1]. Si T es un tetraedro en $\mathcal{RVP}(\bar{c})$ y $\boldsymbol{r}_{k,T}$ es el interpolador de Raviart-Thomas (ver [34, 39]), entonces existe una $C > 0$ que depende solamente de $\bar{c}$ tal que para toda $\boldsymbol{u} \in H^1(T)^3$

$$\|\boldsymbol{u} - \boldsymbol{r}_{k,T}\boldsymbol{u}\|_{L^2(T)^3} \leqslant C \left\{ \sum_{1 \leqslant i \leqslant 3} h_i \|\partial_{\xi_i} \boldsymbol{u}\|_{L^2(T)^3} + h_T \|\operatorname{div} \boldsymbol{u}\|_{L^2(T)} \right\} \tag{3}$$

donde escribimos ξ_i para las coordenadas locales con origen en el vértice regular y h_i para las longitudes de las aristas incidentes a él y h_T para el diámetro de T.

Por otro lado, si $T \in \mathcal{MAC}(\bar{\alpha})$ entonces existe una $C > 0$ que depende solamente de $\bar{\alpha}$ tal que para toda $\boldsymbol{u} \in H^1(T)^3$ vale

$$\|\boldsymbol{u} - \boldsymbol{r}_{k,T}\boldsymbol{u}\|_{L^2(T)^3} \leqslant C h_T \sum_{1 \leqslant i \leqslant 3} \|\partial_{\xi_i} \boldsymbol{u}\|_{L^2(T)^3} \tag{4}$$

(ξ_i son coordenadas locales a partir de un vértice de T, pero no necesariamente tenemos uno regular).

Observamos que la desigualdad (4) es estrictamente más débil que (3), dado que hay elementos que satisfacen la condición $\mathcal{MAC}(\bar{\alpha})$ para $\bar{\alpha}$ fijo, pero con parámetro $\mathcal{RVP}$ arbitrariamente pequeño, haciendo que se degenere la cantidad $C(\bar{c})$ en (3). Ponemos un ejemplo en la Figura 1. Además, como está dicho en [1] mediante un contraejemplo, la desigualdad (3) no puede ser probada bajo condición de ángulo máximo solamente.

Volviendo al segundo párrafo, lo que decíamos es que es posible construir mallas graduadas anisótropas para problemas elípticos en dominios con singularidades que consistan solamente de tetraedros que satisfacen condicón $\mathcal{MAC}(\bar{\alpha})$ para $\alpha < \pi$, pero estos elementos no satisfacen la condición de $\mathcal{RVP}(\bar{c})$ para ningún parámetro uniforme positivo $\bar{c}$. En consecuencia, la estimación del error de interpolación (3) no puede ser usada de manera global y en cambio (4) debe ser tomada, y por esto las propiedades de anisotropía de las mallas pueden no traer ninguna ventaja. En otras palabras, en la segunda desigualdad una *derivada grande* en la dirección ξ_i no estaría necesariamente compensada por un h_i pequeño, así que tendríamos que hacer pequeño al diámetro h_T y refinar las mallas en todas las direcciones, que es lo que queremos evitar, y perderíamos orden $\mathcal{O}(h)$ en la convergencia del error numérico con la relación asintótica $h \sim N_{\mathcal{T}_h}^{-1/3}$

(aquí $\mathcal{T}_h$ es una malla y $N_{\mathcal{T}_h}$ es su cardinal; el significado concreto meaning del parámetro de mallado h, que no es el diámetro de sus elementos, como sí lo es en el caso de las mallas uniformes, se hará claro en la Subsección 5.3.1 cuando propongamos nuestro proceso de mallado).

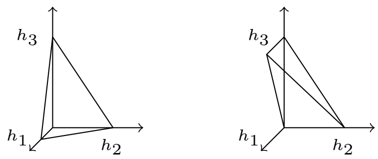


Figure 1 – Tetrahedra satisfying $\mathcal{MAC}(\pi/2)$. The tetrahedron at the right satisfies $\mathcal{RVP}(\bar{c})$ with a poor constant $\bar{c}$ close to 0.

Una idea para sobrellevar la mencionada dificultad, para el caso en que Ω es un dominio cilíndrico poliedral, fue propuesta en [23]. En este caso, cuando f está en $L^2(\Omega)$, la solución puede exhibir solamente singularidades a lo largo de aristas cóncavas. Los autores proponen un método mixto de Raviart–Thomas en mallas graduadas de prismas triangulares y probaron estimaciones óptimas de error por medio de resultados de interpolación con anisotropía y de este modo los tetraedros que no satisfacen una propiedad $\mathcal{RVP}$ uniforme son evitados. También proponen un método similar con mallas de tetraedros anisótropos graduados obtenido subdividiendo cada prisma entre tres tetraedros. Por supuesto, estas mallas contienen a los tetraedros malosm, y para obtener estimaciones de error óptimas el precio pagado es el de requerir más regularidad al dato f, más precisamente, debe pertenecer a un espacio de Sobolev con pesos.

Uno de los resultados presentados en esta Tesis extiende los resultados de [23] puesto que nuestro método fue buscado con el propósito de poder lidiar con la aproximación mixta de (1) para $f \in L^2(\Omega)$, y también para un poliedro cualquiera Ω. Como estos dominios no necesariamente admiten una partición en términos de prismas rectos y como también nosotros quisiéramos evitar pedir más regularidad al dato f, proponemos una discretización basada en mallas híbridas consistentes en prismas combinados con tetraedros, con brechas interelementales rellenadas con pirámides. Obtenemos la estimación del error de aproximación

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega)^3} + \|p - p_h\|_{L^2(\Omega)} \leqslant C\,h\|f\|_{L^2(\Omega)}, \tag{5}$$

en donde $\boldsymbol{u}_h$ y p_h son las aproximaciones de las soluciones $\boldsymbol{u}$ y p, por medio de estimaciones con normas de Sobolev con pesos (para $\boldsymbol{u}$) tratando las singularidades de manera localizada. Primero introducimos y analizamos por completo un nuevo método combinado de Elementos Finitos y Virtuales en un poliedro cuando las mallas son construidas con los mencionados poliedros elementales. Las estimaciones de interpolación locales obtenidas para los espacios discretos son óptimas y anisótropas en prismas anisótropos, lo que nos permite obtener estimaciones óptimas de error de aproximación cuando la solución tiene tanto singularidades de aristas como de vértices, usando mallas graduadas adecuadamente que incluyen elementos anisótropos, y que solamente incluyen tetraedros con un parámetro de condición $\mathcal{RVP}$ uniforme positivo.

No solamente probamos el resultado para una familia abstracta de tales mallas con las mencionadas propiedades, sino que también mostramos un proceso general de mallado que comienza aislando y clasificando las singularidades del dominio (cfr. Teorema 1.2.7), y con eso probamos la existencia de una familia de mallas como la requerimos, mediante su construcción.

Los espacios discretos V_h correspondientes a las mallas propuestas fueron obtenidos satisfaciendo las siguientes condiciones, adecuadas para una discretización en $\mathbb{R}^3$.

1. Conformidad.

2. Propiedades de aproximación anisótropas y óptimas.

3. Generalidad de dominio.

Tal como es sugerido en [15] para el caso $2D$, presentamos V_h como un espacio de elementos virtuales que coincide localmente con el espacio original $3D$ de Raviart–Thomas en tetraedros y prismas, y lo extiende de manera natural a elementos piramidales. En particular, las componentes normales de las funciones discretas son constantes sobre las caras de los elementos, coincidiendo de manera conforme a través de caras comunes a elementos de distinta geometría. De este modo el requerimiento 1 fue satisfecho. Una ventaja de esta presentación es que la definición de los espacios locales es independiente de la geometría del elemento, ver Sección 3.1. Como ya comentamos, el ítem 3 es verificado mediante la incorporación de pirámides de base paralelogramo. Con respecto al punto 2, presentamos estimaciones óptimas de estabilidad y de interpolación anisótropas en varias partes aunque no todas ellas son usadas en la aplicación final, en algunos casos porque los espacios funcionales para los cuales son usadas no se ven involucrados en el problema modelo (2) y en otros casos porque nuestro esquema de mallado funciona sin requerir que los elementos de todos los tipos presenten anisotropía.

También podrían ser consideradas mallas con elementos de geometría más general. De hecho esta es una de las principales propiedades de los métodos de elementos virtuales. Nosotros decidimos restringirnos a unas pocas geometrías elementales pues nuestro principal objetivo fue admitir mallas anisótropas, y con estimaciones anisótropas uniformemente válidas. Una dificultad que aparece cuando se consideran otras geometrías elementales (como prismas oblícuos) es que los espacios virtuales locales no necesariamente se ve preservado por transformaciones *push–forward* basadas en aplicaciones afines (nosotros usamos una propiedad de **curl** nulo que no es invariante en esta situación), con la consecuencia de que los argumentos de reescalado estándar son difíciles de usar (ver Sección 3.1). Sin embargo, esto no deviene en una limitación en los dominios poliedrales que podemos tratar con nuestro método, pues podemos restringirnos a usar prismas rectos duplicando, en el peor de los casos, el número de elementos de la primera malla, y así aumentando el número de elementos en una malla cualquiera posterior en un factor constante.

Nuestro método presenta distinta cantidad de grados de libertad a través de la malla, pues prismas, tetraedros y pirámides tienen distinta cantidad de caras, y también admite distintas geometrías para los grados de libertad pues estamos trabajando con caras triangulares y cuadriláteras al mismo tiempo. Como consecuencia de esto, nuestro método es en cierto sentido una generalización de los métodos de elementos virtuales clásicos, en los cuales todos los elementos tienen la misma geometría arbitraria, y de cada elemento de toma el mismo número de grados de libertad, todos del mismo tipo (por ejemplo, evaluación en los mismos puntos en el borde de cada elemento).

Las mallas híbridas que incluyen tetraedros y prismas (y hasta hexaedros) pueden ser necesarias para satisfaces las demandas de la geometría específica de un problema (regiones no triviales) o para alcanzar cálculos eficientes. Si se requiere que estas mallas eviten nodos colgantes entonces a menudo incluirán pirámides; ver por ejemplo [37]. Varios artículos contienen la construcción de elementos finitos conformes en pirámides, algunos de los cuales son [12] para elementos H^1 y [25, 36] para elementos $H(\mathrm{div})$ y $H(\mathbf{curl})$, el primero para orden bajo y el segundo para orden arbitrario. En [36] los autores prueban que no es posible construir elementos finitos H^1 útiles en pirámides usando solamente funciones polinomiales y muestran que, en el caso $H(\mathrm{div})$, todos los espacios construidos en la literatura contienen funciones no polinomiales.

Queremos finalizar esta introducción mencionando otros resultados que obtuvimos, presentados como resultados adicionales porque no fueron usados para el problema principal de la Tesis, y que pueden ser vistos como extensiones de ciertos resultados teóricos.

Probamos estimaciones locales anisótropas de estabilidad y de error de interpolación para elementos finitos prismáticos tanto para las clases de elementos **curl**–conformes como para las div–conformes. Nuestro método usa solo el caso de orden bajo de las estimaciones en $H(\mathrm{div})$ y dejamos una versión de orden alto del método para investigaciones futuras.

Finalmente, obtuvimos y presentamos estimaciones locales anisótropas de estabilidad y de error de interpolación para los elementos finitos piramidales introducidos en [25, 36] tanto para las clases de elementos **curl**–conformes como para las div–conformes. Como mostramos en el Capítulo 6, las funciones de forma que generan estos espacios son racionales y singulares, aunque acotadas, en la pirámide de referencia. Por esta razón consideramos que nuestro método combinado FE–VE presenta una ventaja, la de evitar las evaluaciones de funciones con esas propiedades en implementaciones en computadoras.

Introduction

A physical object is anisotropic if it exhibits vectorial physical properties with different magnitudes when measured in different directions (cfr. [28]). In the case of a numerical method we say that the family of meshes used is anisotropic if it contains sequences of elements such that the decrease order of their sizes along one coordinate direction is higher than along other independent directions. We will formalize and illustrate this in Chapter 1.

In several situations in mixed finite element approximations the use of anisotropic meshes is needed. This is the case for instance when dealing with the Poisson equation in a polyhedron $\Omega \subseteq \mathbb{R}^3$ with concave edges and vertices, which, introducing the vectorial variable $\boldsymbol{u} = \nabla p$ can be written in mixed form as

$$\begin{cases} \boldsymbol{u} = \nabla p & \text{in } \Omega \\ -\operatorname{div} \boldsymbol{u} = f & \text{in } \Omega \qquad (f \in L^2(\Omega)) \\ p = 0 & \text{on } \partial\Omega. \end{cases} \tag{6}$$

The standard mixed variational formulation is to find $\boldsymbol{u} \in H(\operatorname{div}, \Omega)$ and $p \in L^2(\Omega)$ such that

$$\begin{array}{lrcl} \forall \boldsymbol{v} \in H(\operatorname{div}, \Omega) & a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) & = & 0 \\ \forall q \in L^2(\Omega) & b(\boldsymbol{u}, q) & = & (-f, q) \end{array} \tag{7}$$

with

$$a(\boldsymbol{v}, \boldsymbol{w}) = \int_\Omega \boldsymbol{v} \cdot \boldsymbol{w} \, d\boldsymbol{x}, \qquad b(\boldsymbol{v}, q) = \int_\Omega q \operatorname{div} \boldsymbol{v} \, d\boldsymbol{x}.$$

With Ω being non–convex, the solution $\boldsymbol{u}$ of (7) is not in $H^1(\Omega)$ in the general case due to vertex and edges singularities. In particular, close to concave edges, the solution is expected to be more regular along the direction of that edge than transversally to it (see [5]), and that is why the mesh has to be accordingly graded and refined in order to recover the optimal order of convergence of the sequence of numerical solutions with respect to the number of degrees of freedom (see [4, 5]). Those meshes contain elements which are arbitrarily narrow in direction orthogonal to the concave edges. If we tried to construct such a sequence of meshes using only tetrahedra we would incur in using subfamilies of them which do not fullfill certain conditions needed for the anisotropic analysis, and there is where our proposed method with elements of different geometries becomes meaningful.

To explain the latter further and the main result of the present Thesis (cfr. [30]) we put the following definitions and related previous results.

First, we say that a tetrahedron T satisfies the *regular vertex property* with a constant $\bar{c} > 0$ (written $T \in \mathcal{RVP}(\bar{c})$) if T has a vertex $\boldsymbol{x}_T$ such that, if M_T is the matrix made up with the unitary vectors in the directions of the edges sharing $\boldsymbol{x}_T$ as columns, then $|\det M_T| > \bar{c}$.

A less restrictive geometrical property is the following. We say that a tetrahedron T satisfies the *maximum angle condition* with paramenter $\bar{\alpha}$ (writen $T \in \mathcal{MAC}(\bar{\alpha})$) if the angles of the faces of T and between faces are less than $\bar{\alpha}$.

With these two notions in mind we need to start with the following result from [1]. If T is a tetrahedron in $\mathcal{RVP}(\bar{c})$ and $\boldsymbol{r}_{k,T}$ is the Raviart-Thomas interpolation operator (see [34, 39, 41]), then there is a positive $C(\bar{c})$ such that for all $\boldsymbol{u} \in H^1(T)^3$

$$\|\boldsymbol{u} - \boldsymbol{r}_{k,T}\boldsymbol{u}\|_{L^2(T)^3} \leqslant C \left\{ \sum_{1 \leqslant i \leqslant 3} h_i \|\partial_{\xi_i} \boldsymbol{u}\|_{L^2(T)^3} + h_T \|\operatorname{div} \boldsymbol{u}\|_{L^2(T)} \right\} \quad (8)$$

where we wrote ξ_i for the local coordinates with origin at the regular vertex and h_i for the lengths of the edges incident to it and h_T for the diameter of T.

On the other hand, if $T \in \mathcal{MAC}(\bar{\alpha})$ then there exists a positive $C(\bar{\alpha})$ depending only on $\bar{\alpha}$ such that for all $\boldsymbol{u} \in H^1(T)^3$ it holds

$$\|\boldsymbol{u} - \boldsymbol{r}_{k,T}\boldsymbol{u}\|_{L^2(T)^3} \leqslant C h_T \sum_{1 \leqslant i \leqslant 3} \|\partial_{\xi_i} \boldsymbol{u}\|_{L^2(T)^3} \quad (9)$$

(ξ_i are local coordinates at a vertex of T, but we don't necessarily have a regular one).

As an observation we point out that inequality (9) is strictly weaker than (8), since there are elements satisfying MAC($\bar{\alpha}$) for a fixed $\bar{\alpha}$ with arbitrarily small $\mathcal{RVP}$ parameter $\bar{c}$, making the constant $C(\bar{c})$ in (8) to degenerate. An example is depicted in Figure 2. Furthermore, as stated in [1] by means of a counterexample, inequality (8) can't be proved under the maximum angle condition only.

Now returning to clarify what was said in the second paragraph, it is possible to construct anisotropic graded meshes to tacle numerically elliptic problems in domains with singularities, consisting those meshes exclusively of tetrahedral elements all satisfying a $\mathcal{MAC}(\bar{\alpha})$ condition for a fixed $\bar{\alpha} < \pi$, but unfortunately those elements do not satisfy an $\mathcal{RVP}(\bar{c})$ condition for any uniform positive parameter $\bar{c}$. That is due to the existence of tetrahedra with bounded maximal angle but poor regular vertex constant. Consequently, interpolation error estimate (8) can not be globally used to estimate the error approximation, but (9) has to be taken, and therefore the anisotropic properties of the meshes may give no profit. In other words, if we look at both inequalities, in the second case a *big derivative* in the direction ξ_i would not necessarily be compensated with a small h_i, so we would have to make the diameter h_T smaller and refine the meshes in all the directions, which is what we want to avoid, and we would lose $\mathcal{O}(h)$ order of convergence of the numerical error with the asymptotic relation $h \sim N_{\mathcal{T}_h}^{-1/3}$ (here $\mathcal{T}_h$

is a mesh and $N_{\mathcal{T}_h}$ is its cardinal; the concrete meaning of the meshing parameter h, which isn't the diameter of the elements, as it is in the case of uniform meshes, will be made clear in Subsection 5.3.1 when we propose our meshing procedure).

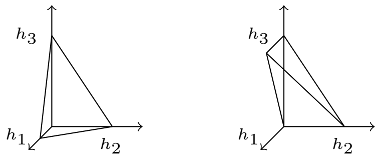


Figure 2 – Tetrahedra satisfying $\mathcal{MAC}(\pi/2)$. The tetrahedron at the right satisfies $\mathcal{RVP}(\overline{c})$ with a poor constant $\overline{c}$ close to 0.

An idea to overcome the mentioned difficulty, for the case of Ω being a cylindrical polyhedral domain, was proposed in [23]. In this case, when f is in $L^2(\Omega)$, the solution may exhibit only singularities along concave edges. Then the authors proposed a lowest order mixed Raviart–Thomas method on graded anisotropic meshes made up of triangular right prisms and they proved optimal error estimates by means of adequate anisotropic interpolation results. In this way, tetrahedra which do not satisfy a uniform regular vertex property are avoided.

Also in [23] a mixed Raviart–Thomas method is proposed on the tetrahedral anisotropic graded mesh which is obtained by splitting the prismatic elements into three tetrahedra. Of course, these kind of meshes contain the bad elements which are avoided with the prismatic ones, and in order to obtain optimal approximation error estimates, the price payed is to require additional regularity on the right hand side f, precisely, it has to belong to a weighted Sobolev space.

One of the results presented in this thesis extends the result in [23] since our method was seeked in order to be able to deal with the mixed approximation of (6) for $f \in L^2(\Omega)$, and also for a general polyhedral domain Ω. As such domains not always admit a partition by means of right prisms and since we also would like to avoid to require more regularity to f as mentioned in the previous paragraph, we propose a discretization based on hybrid meshes that consists of prisms combined with tetrahedra, with interelemental gaps filled with pyramids. We obtain the approximation error estimate

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega)^3} + \|p - p_h\|_{L^2(\Omega)} \leqslant C\, h \|f\|_{L^2(\Omega)} \tag{10}$$

with $\boldsymbol{u}_h$ and p_h being the approximations of the solutions $\boldsymbol{u}$ and p of (6) by means of weighted Sobolev norms treating the singularities in a localized manner. Firstly we introduced and analyzed a new combined Finite and Virtual Element Method on a polyhedron which is fully analyzed when the meshes are made up of the mentioned elementary polyhedra. Local interpolation error estimates for the discrete element spaces are optimal and anisotropic on anisotropic right prisms, which can be used to obtain optimal approximation error estimates when the solution has edge singularities as well as vertex singularities, as pointed out at the beginning of this introduction, by using suitably graded and adapted meshes which allow for anisotropic elements, and which only includes tetrahedra with a uniform positive regular vertex constant, precisely avoiding the use of tetrahedra with arbitrary small $\mathcal{RVP}$ constant.

Not only we proved the result for an abstract family of meshes with the previous properties, but also we show a general meshing procedure which starts by isolating and classifying the singularities (cfr. Theorem 1.2.7), and so we prove the existence of such a family of anisotropic graded hybrid meshes by constructing it.

The discrete spaces V_h corresponding to the proposed meshes were obtained fulfilling the following conditions, suitable for discretizations in $\mathbb{R}^3$.

1. Conformity: The space V_h is $H(\text{div})$-conforming.
2. Optimal and anisotropic approximation properties: Optimal interpolation error estimates are valid even on this family of meshes which do not satisfy the standard shape–regularity condition (cfr. Definition 1.1.2 and see [14, 19]).
3. Domain generality: The space is well defined on conforming meshes without restricting the considered domains to few special polyhedra.

As suggested in [15] for the 2D case, we present V_h as a virtual element space which locally coincides with the original lowest order 3D Finite Element Raviart-Thomas space on tetrahedra and right prisms, and naturally extends it to pyramidal elements. In particular, normal components of the discrete functions are constant on the faces of the elements, fitting well across different shape's elements. In this way requirement 1 is verified. One advantage of this presentation, is that the definition of the local spaces is independent of the geometry of the element, see Section 3.1. Also, in [11], an analogous space, but of arbitrary order, was introduced to discretize an acoustic flow free vibration problem in a bounded rigid cavity in $\mathbb{R}^2$. As we already commented, item 3 in the previous list is fulfilled by incorporating the pyramids of parallelogram basis to a grid with triangular right prisms and tetrahedra. With respect to point 2, we present optimal anisotropic local stability and interpolation error estimates in several parts although not all of them are used in the application, in some cases because the functional spaces they are used for are not involved in the mixed model elliptic problem (7) and in other cases because our meshing strategy requires not all the element types to present anisotropy.

Meshes with more general polyhedral–shaped elements can be considered. That is one of the Virtual Elements Method's main features indeed. But we decided to restrict ourselves to few shapes since our main objective is to allow for meshes with anisotropic elements, and with uniformly valid anisotropic estimates. One difficulty when other shapes are considered (like oblique prisms, for instance) is that the local virtual element space is not preserved by *push–forward* transformations based on affine mappings (we require a vanishing **curl** property that is not preserved in this situation), and so the standard rescaling arguments are hard to use (see Section 3.1). However, this is not a limitation on the polyhedra that we can treat with our procedure, since we can restrict ourselves to right prisms by duplicating the elements of the first mesh in the worst case, and so augmenting the number of elements by a constant factor.

Our method presents different number of degrees of freedom through the mesh, as prisms, tetrahedra and pyramids have different number of faces, and also allows for different planar geometry for the degrees of freedom, since we are dealing with triangular and rectangular faces at the same time. As a consequence, our method is in some sense a generalization of the classical Virtual Element Methods, in which all the elements have the same arbitrary geometry, and each element yields the same number of degrees of freedom, all of the same type (for example, evaluation at the same points on the boundary of each element).

Hybrid meshes that include tetrahedral and prismatic (and even hexahedral) elements may be needed to satisfy the demands of a specific problem geometry (complex regions) or to reach efficient calculations. If these meshes are to avoid hanging nodes then they will in general contain pyramids, see for instance [37]. Several articles contain introduction and analysis of conforming finite elements on pyramids; some of them are [12] for H^1–elements and [25, 36] for $H(\mathrm{div})$– and $H(\mathbf{curl})$–elements, the first one for lowest order and the second one for higher order. In [36] the authors prove that it is not possible to construct useful H^1–finite elements on pyramids using polynomial functions only, and they survey that, in the $H(\mathrm{div})$ case, all the spaces constructed in the literature contain non–polynomial functions.

We finish this introduction mentioning other results obtained which we regard as additional results because they weren't used for the main problem of the Thesis and can be seen as extensions of some theoretical results.

We proved local anisotropic stability and interpolation error estimates for arbitrary order Prismatic Finite Elements in both the **curl**–conforming and div–conforming classes of elements. Our method uses only the least order case of this estimates, an we leave a *high order* version of the method for further research.

Lastly, we obtained and present also local anisotropic stability and interpolation error estimates for the lowest order Pyramidal Finite Elements constructed in [25, 36] for both the **curl**–conforming and div–conforming classes of elements. As we show explicitly in Chapter 6, the shape functions spanning these Finite Element spaces are not polynomial and are singular, yet bounded, in the reference pyramid. This is a reason why we considered that our combined FE–VE approach presents an advantage, which is avoiding the evaluation of functions with those properties in computer implementations.

Chapter 1

Preliminaries

Introducción al capítulo

En este capítulo recolectamos definiciones y propiedades dentro de la teoría que desarrollamos y fijamos nuestra elección de notación, estándar en la mayoría de los casos en la literatura.

Introduction to the chapter

In this chapter we gather definitions and properties within the theory we develop and fix our chosen notation, which is standard in most cases in the literature.

1.1 Preliminaries

1.1.1 Definitions and Notations

For the first definition we refer the reader to [14, 19].

1.1.1 Definition. *A Finite Element in $\mathbb{R}^n$ is defined by the following triple (E, P_E, Σ):*

1. *E is an open non empty polytope of $\mathbb{R}^n$ with Lipschitz–continuous boundary.*
2. *P_E is a finite–dimensional space of real–valued scalar or vectorial functions defined over E.*
3. *Σ is a finite set of linearly independent linear functionals acting on a functional space containing P_E. The elements of Σ are often referred to as* degrees of freedom *or* moments *of the Finite Element.*

1.1.2 Definition. *Given a Finite Element E set*

$$\begin{aligned} h_E &= \text{ diameter of the element } E. \\ \rho_E &= \text{ supremum of the diameters of the spheres contained in } E. \end{aligned}$$

Consider a family of meshes $\{\mathcal{T}_n\}_{n\in\mathbb{N}}$ such that $\max_{E\in\mathcal{T}_n} h_E$ tends to zero as n tends to infinity.

i) The family of meshes is isotropic if there is a positive constant σ such that for each n and each $E \in \mathcal{T}_n$

$$h_E \leqslant \sigma \rho_E.$$

ii) The family of meshes is anisotropic if it is not isotropic (cfr. Figure 1.1).

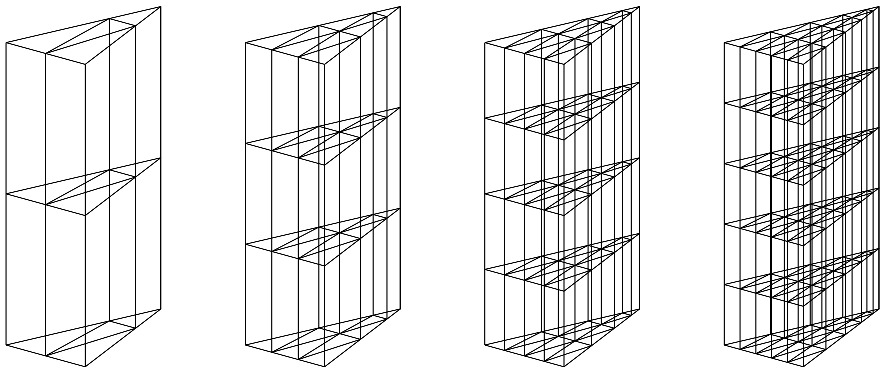


Figure 1.1 – Elements of an anisotropic family of meshes.

1.1.3 Definition. *If $V(E)$ denotes a functional space over an element E from a mesh $\mathcal{T}_n$ for a domain Ω, the symbol $V(\mathcal{T}_n)$ denotes the space of functions in Ω whose restrictions to each $E \in \mathcal{T}_n$ belongs to $V(E)$.*

1.1.2 Polynomials

The finite elements involved in this Thesis are built with piecewise polynomial functions or rational functions (that is, quotients of polynomials). As we will see, the virtual elements also involve functions of these two kinds amongst others. For that reason we state some handy notations here.

1.1.4 Definition. *For an element $E \subseteq \mathbb{R}^3$*

$$P_k(E) = \{\, \textit{polynomials of degree less than or equal to } k \textit{ over } E \,\}.$$

$$\tilde{P}_k(E) = \{\, \textit{homogeneous polynomials of degree } k \textit{ over } E \,\}.$$

In the case of a line segment $\boldsymbol{e}$ or a subdomain f of a plane (typically an edge or a face of a polyhedron) we will write interchangeably

$$P_k(\boldsymbol{e}) = \quad P_k(t) \quad = \{\, \textit{polynomials of degree less than or equal to } k \textit{ over } \boldsymbol{e} \,\},$$

$$P_k(f) = P_k(t_1, t_2) = \{\, \textit{polynomials of degree less than or equal to } k \textit{ in } \xi_1, \xi_2 \textit{ on } f \},$$

where we have used an orthogonal coordinate system (ξ_1, ξ_2) *in the plane containing* f *and the variable* ξ *along the edge* $\boldsymbol{e}$*. Moreover, we will observe that*

$$P_k(\boldsymbol{e}) = \{p|_{\boldsymbol{e}} \,:\, p \in P_k\} \quad \textit{and} \quad P_k(f) = \{p|_f \,:\, p \in P_k\}.$$

The tensor product of polynomials.

1.1.5 Definition. *Given two polynomials*

$$p(x) = \sum_i a_i x^i \quad \textit{and} \quad q(y) = \sum_j b_j y^j$$

the tensor product of p *and* q *is the following polynomial of* separate variables

$$p \otimes q(x, y) = \sum_{i,j} a_i b_j x^i y^j. \tag{1.1}$$

Smilarly if

$$p(x, y) = \sum_{i,j} a_{i,j} x^i y^j \quad \textit{and} \quad q(z) = \sum_k b_k z^k$$

then

$$p \otimes q(x, y, z) = \sum_{i,j,k} a_{i,j} b_k x^i y^j z^k. \tag{1.2}$$

If $\mathbb{Q}_1$ *and* $\mathbb{Q}_2$ *are two spaces of polynomials as above, then*

$$\mathbb{Q}_1 \otimes \mathbb{Q}_2 = \{p \otimes q \,|\, p \in \mathbb{Q}_1, q \in \mathbb{Q}_2\}. \tag{1.3}$$

1.1.6 Remark. *An observation that we will use several times is that, from the definition,* $\dim \mathbb{Q}_1 \otimes \mathbb{Q}_2 = \dim \mathbb{Q}_1 \dim \mathbb{Q}_2$.

1.1.3 Functional Spaces and Trace Spaces

Let $\boldsymbol{x}$, $\boldsymbol{y}$, … denote points in $\mathbb{R}^n$ and let $d\boldsymbol{x}$, $d\boldsymbol{y}$, …denote Lebesgue measure. If Ω is a measurable set, $1 \leqslant p \leqslant \infty$ and f is a real or complex valued measurable function, we say $f \in L^p(\Omega)$ if

$$\|f\|_{L^p(\Omega)} \;=\; \|f\|_{0,p,\Omega} := \left\{ \int_\Omega |f(\boldsymbol{x})|^p \, d\boldsymbol{x} \right\}^{1/p} < \infty$$

with the usual modification when $p = \infty$.

Let $\mathbb{Z}_{\geqslant 0}$ denote the set of nonnegative integers. A *multi–index* $\boldsymbol{\alpha}$ is an n–tuple of nonnegative integers: $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in \mathbb{Z}_{\geqslant 0}$, $1 \leqslant i \leqslant n$. With multi–indices we will establish the following notations:

$$
\begin{aligned}
|\boldsymbol{\alpha}| &= \sum_i \alpha_i, \\
\boldsymbol{\alpha} \leqslant \boldsymbol{\beta} &\quad \text{iff} \quad \alpha_i \leqslant \beta_i, \quad 1 \leqslant i \leqslant n, \\
\boldsymbol{\alpha} + \boldsymbol{\beta} &= (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n), \\
\boldsymbol{\alpha} - \boldsymbol{\beta} &= (\max\{\alpha_1 - \beta_1, 0\}, \ldots, \max\{\alpha_n - \beta_n, 0\}), \\
\boldsymbol{\alpha}! &= \Pi_{i=1}^n \alpha_i!, \\
\boldsymbol{x}^{\boldsymbol{\alpha}} &= \Pi_{i=1}^n x_i^{\alpha_i} \quad \text{and} \\
\partial^{\boldsymbol{\alpha}} = \left(\frac{\partial}{\partial \boldsymbol{x}}\right)^{\boldsymbol{\alpha}} &= \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} \\
&= \Pi_{i=1}^n \left(\frac{\partial}{\partial x_i}\right)^{\alpha_i}.
\end{aligned}
$$

If $|\alpha| = 0$, $\partial^\alpha f = f$.

Let Ω be an open set with Lipschitz boundary, let $\mathcal{C}^\infty(\Omega)$ denote the space of infinitely differentiable functions in Ω. Let $\mathscr{D}(\Omega)$ denote the vectorial subspace of $\mathcal{C}^\infty(\Omega)$ functions that have compact support in Ω, called the *space of test functions* with the usual topology (cfr. [40], page 136) for which we say a sequence $\phi_n \to 0$ in the topology of $\mathscr{D}(\Omega)$ if and only if there is a compact $K \subseteq \Omega$ which contains the support of every ϕ_n, and $\partial^\alpha \phi_n \to 0$ uniformly on K, as $n \to \infty$, for every multi–index $\boldsymbol{\alpha}$. The dual $\mathscr{D}'(\Omega)$ of $\mathscr{D}(\Omega)$ is called the space of *distributions* on Ω. If $\phi \in \mathscr{D}'(\Omega)$ and α is a multi–index, $\partial^\alpha \phi$ is called a distributional or weak derivative of ϕ, where $\partial^\alpha \phi$ is defined by

$$(\partial^\alpha \phi)(f) = (-1)^{|\alpha|} \phi(\partial^\alpha f), \qquad f \in \mathscr{D}(\Omega).$$

A distribution $\phi \in \mathscr{D}'(\Omega)$ will be identified with a function ψ defined on Ω if for each $f \in \mathscr{D}(\Omega)$, $\psi f \in L^1(\Omega)$ and $\phi(f) = \int_\Omega \psi f \, d\boldsymbol{x}$. In this case we shall let ϕ denote the identified function, ψ, as well.

If $m \in \mathbb{Z}_{\geqslant 0}$ and if for each multi–index α with $|\alpha| \leqslant m$, $\partial^\alpha \phi$ is given by a function such that

$$\|\phi\|_{W^{m,p}(\Omega)} = \|\phi\|_{m,p,\Omega} := \left\{ \sum_{|\alpha| \leqslant m} \|\partial^\alpha \phi\|^p_{L^p(\Omega)} \right\}^{1/p} < \infty,$$

then we will say $\phi \in W^{m,p}(\Omega)$. For $\phi \in W^{m,p}(\Omega)$ let

$$|\phi|_{W^{m,p}(\Omega)} = |\phi|_{m,p,\Omega} := \left\{ \sum_{|\alpha| = m} \|\partial \phi\|_{L^p(\Omega)} \right\}^{1/p}.$$

In the previous notation we will skip the number 2 for the special case $p = 2$ and write $\|\phi\|_{W^{m,2}(\Omega)} = \|\phi\|_{m,\Omega}$, $|\phi|_{W^{m,2}(\Omega)} = |\phi|_{m,\Omega}$. Sometimes we will write, in the vectorial case for $\boldsymbol{u} = (u_1, u_2, u_3)$, $|\boldsymbol{u}|^p = |u_1|^p + |u_2|^p + |u_3|^p$.

In the definitions of the following two functional spaces, the differential operators must be understood in the distributional sense.

As in page 26 of [24] we make the present definition.

1.1.7 Definition. *Let $\Omega \subseteq \mathbb{R}^3$ be a Lipschitz domain.*

$$H(div, \Omega) := \{\boldsymbol{u} \in L^2(\Omega)^3 \ : \ div\, \boldsymbol{u} \in L^2(\Omega)\},$$
$$\|\boldsymbol{u}\|_{H(div,\Omega)} := \left(\|\boldsymbol{u}\|^2_{L^2(\Omega)^3} + \|div\, \boldsymbol{u}\|^2_{L^2(\Omega)}\right)^{1/2}.$$

As in page 32 of [24] we make the present definition.

1.1.8 Definition. *Let $\Omega \subseteq \mathbb{R}^3$ be a Lipschitz domain.*

$$H(\boldsymbol{curl}, \Omega) := \{\boldsymbol{u} \in L^2(\Omega)^3 \ : \ \boldsymbol{curl}\, \boldsymbol{u} \in L^2(\Omega)^3\},$$
$$\|\boldsymbol{u}\|_{H(\boldsymbol{curl},\Omega)} := \left(\|\boldsymbol{u}\|^2_{L^2(\Omega)^3} + \|\boldsymbol{curl}\, \boldsymbol{u}\|^2_{L^2(\Omega)^3}\right)^{1/2}.$$

As in page 4 of [32] we make the present definition.

1.1.9 Definition.

$$W^p(\boldsymbol{curl}, \Omega) = \{\boldsymbol{u} \in W^{1,p}(\Omega)^3 \ : \ \boldsymbol{curl}\, \boldsymbol{u} \in W^{1,1}(\Omega)^3\},$$
$$\|\boldsymbol{u}\|_{W^p(\boldsymbol{curl},\Omega)} = \|\boldsymbol{u}\|_{W^{1,p}(\Omega)} + \|\boldsymbol{curl}\, \boldsymbol{u}\|_{W^{1,1}(\Omega)}. \tag{1.4}$$

The concept of a component that is normal to the boundary can be extended to $H(\mathrm{div}, \Omega)$ using a density argument. This is what we call *the normal trace*. Let $\boldsymbol{\nu}$ be, in the following, the unit outward normal to $\partial\Omega$. For a function $\boldsymbol{v} \in \mathcal{C}^\infty(\overline{\Omega})^3$ the normal trace $\gamma_{\boldsymbol{\nu}}$ is defined simply by

$$\gamma_{\boldsymbol{\nu}}(\boldsymbol{v}) := \boldsymbol{v}|_{\partial\Omega} \cdot \boldsymbol{\nu}. \tag{1.5}$$

The proof of next Theorem can be found in [33], page 53 and uses merely the density of $\mathcal{C}^\infty(\overline{\Omega})^3$ in $H(\mathrm{div}, \Omega)$.

1.1.10 Theorem. *The operator $\gamma_{\boldsymbol{\nu}}$ defined in (1.5) can be extended by continuity to a continuous linear map $\gamma_{\boldsymbol{\nu}}$ from $H(div, \Omega)$ onto $(H^{1/2}(\partial\Omega))'$.*

With regard to trace properties in $H(\mathbf{curl})$, we must verify that functions in this space have a well–defined tangential trace (cfr. page 34 of [24] and page 59 of [33]). The next Definition and proof of the Theorem after it can be found in [18] and gives the form that the surface degrees of freedom in $H(\mathbf{curl})$–Conforming Elements will take.

For a smooth vector function $\boldsymbol{v} \in \mathcal{C}^\infty(\overline{\Omega})^3$ we define the two traces

$$\gamma_t(\boldsymbol{v}) = \boldsymbol{\nu} \times \boldsymbol{v}|_{\partial\Omega} \tag{1.6}$$

$$\gamma_T(\boldsymbol{v}) = (\boldsymbol{\nu} \times \boldsymbol{v}|_{\partial\Omega}) \times \boldsymbol{\nu} \tag{1.7}$$

Theorem 3.29 in [33] states that (1.6) can be extended by continuity to a continuous linear map from $H(\mathbf{curl}, \Omega)$ into $(H^{1/2}(\partial\Omega)')^3$

1.1.11 Definition.

$$Y(\partial\Omega) = \{\boldsymbol{f} \in (H^{1/2}(\partial\Omega)')^3 \ : \ \textit{there exists } \boldsymbol{u} \in H(\boldsymbol{curl}, \Omega) \ \textit{with } \gamma_t(\boldsymbol{u}) = \boldsymbol{f}\}, \tag{1.8}$$

$$\|\boldsymbol{f}\|_{Y(\partial\Omega)} = \inf_{\boldsymbol{u} \in H(\boldsymbol{curl},\Omega), \gamma_t(\boldsymbol{u}) = \boldsymbol{f}} \|\boldsymbol{u}\|_{H(\boldsymbol{curl},\Omega)}.$$

So we can consider the epimorphism $\gamma_t : H(\mathbf{curl}, \Omega) \to Y(\partial\Omega)$ with which we have the following Theorem (cfr. Theorem 3.31 in [33]).

1.1.12 Theorem. *The map $\gamma_T : H(\boldsymbol{curl}, \Omega) \to Y(\partial\Omega)'$ is well–defined. For any $\boldsymbol{v}$, $\boldsymbol{\phi}$ in $H(\boldsymbol{curl}, \Omega)$*

$$\int_\Omega \boldsymbol{curl}\, \boldsymbol{v} \cdot \boldsymbol{\phi}\, d\boldsymbol{x} - \int_\Omega \boldsymbol{v} \cdot \boldsymbol{curl}\, \boldsymbol{\phi}\, d\boldsymbol{x} = \langle \gamma_t(\boldsymbol{v}), \gamma_T(\boldsymbol{\phi}) \rangle_{\partial\Omega} \tag{1.9}$$

($\langle \cdot, \cdot \rangle$ denotes a duality pairing).

1.1.13 Definition. *We shall say that a domain Ω has the* segment property *if for every x in the boundary of Ω there exists an open set U_x and a nonzero vector y_x such that $x \in U_x$ and if $z \in \overline{\Omega} \cap U_x$, then $z + ty_x \in \Omega$ for $0 < t < 1$.*

A domain having this property must have $(n-1)$–dimensional boundary and cannot simultaneously lie on both sides of any given part of its boundary. Since the polyhedra we will consider to build finite elements are convex and obviously satisfy the segment property, with techniques that can be learned from Chapter 3 in the book [2] we can prove the following Proposition for our elements.

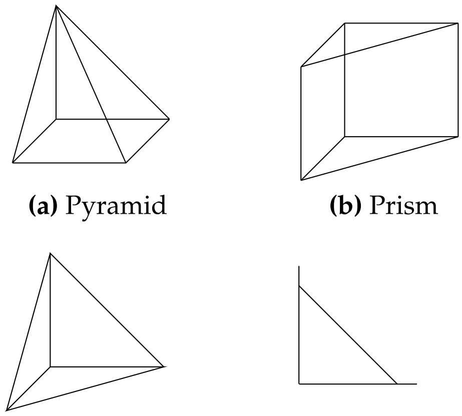
1.1.14 Proposition. *The space $\mathcal{C}^\infty(\bar{E})^3$ is dense in $W^p(\boldsymbol{curl}, E)$ with its norm defined in (1.4).*

With elementary applications of integration by parts for distributional derivatives we get the following two Lemata.

1.1.15 Lemma. *Let two disjoint Lipschitz domains Ω_1 and Ω_2 be given in $\mathbb{R}^3$, such that $\overline{\Omega_1} \cap \overline{\Omega_2}$ is an 2–dimensional surface f of positive measure. Take $\Omega = \Omega_1 \cup \Omega_2 \cup f$. Let $\boldsymbol{u}_1 \in H(div, \Omega_1)$ and $\boldsymbol{u}_2 \in H(div, \Omega_2)$. Consider*

$$\boldsymbol{u} = \begin{cases} \boldsymbol{u}_1, & \textit{on } \Omega_1 \\ \boldsymbol{u}_2, & \textit{on } \Omega_2 \end{cases}.$$

Then $\boldsymbol{u}$ is in $H(div, \Omega)$ if and only if the normal traces of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ coincide on f.



(a) Pyramid **(b)** Prism **(c)** Tetrahedron **(d)** Triangle

Figure 1.2 – Reference Elements

1.1.16 Lemma. *Let two disjoint Lipschitz domains* Ω_1 *and* Ω_2 *be given in* $\mathbb{R}^3$*, such that* $\overline{\Omega_1} \cap \overline{\Omega_2}$ *is an* 2*–dimensional surface* f *of positive measure. Take* $\Omega = \Omega_1 \cup \Omega_2 \cup f$*. Let* $\boldsymbol{u}_1 \in H(\boldsymbol{curl}, \Omega_1)$ *and* $\boldsymbol{u}_2 \in H(\boldsymbol{curl}, \Omega_2)$*. Consider*

$$\boldsymbol{u} = \begin{cases} \boldsymbol{u}_1, & on\ \Omega_1 \\ \boldsymbol{u}_2, & on\ \Omega_2 \end{cases}.$$

Then $\boldsymbol{u}$ *is in* $H(\boldsymbol{curl}, \Omega)$ *if and only if the tangential traces* $\gamma_T(\cdot)$ *of* $\boldsymbol{u}_1$ *and* $\boldsymbol{u}_2$*, as defined in (1.9), coincide on* f*.*

1.1.17 Definition. *Let* $\hat{T}$ *be the triangle* $\{0 < x_1 + x_2 < 1, x_1 > 0, x_2 > 0\}$*. The reference prism is* $\hat{T} \times \{0 < x_3 < 1\}$ *(cfr. Figure 1.2b).*

1.1.18 Definition. *The reference pyramid* $\hat{E}$ *is the polyhedron* $\{\boldsymbol{x} \in \mathbb{R}^3 : 0 < x_3 < 1, 0 < x_1 < 1 - x_3, 0 < x_2 < 1 - x_3\}$ *with vertices at* $(0,0,0)'$*,* $(1,0,0)'$*,* $(0,1,0)'$*,* $(1,1,0)'$ *and* $(0,0,1)'$ *(cfr. Figure 1.2a).*

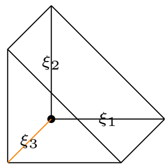
1.1.19 Definition. *The reference tetrahedron is (cfr. Figure 1.2c)* $\{\boldsymbol{x} \in \mathbb{R}^3 : x_1 > 0, x_2 > 0, x_3 > 0, x_1 + x_2 + x_3 < 1\}$*.*

1.2 Regularity of the solution for a model Elliptic Problem

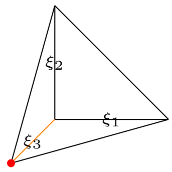
We are concerned in solving the following problem.

1.2.1 Problem. *Suppose we have a simply connected Lipschitz polyhedron* $\Omega \subseteq \mathbb{R}^3$*. Given* $f \in L^2(\Omega)$*, solve the mixed formulation for the Poisson Problem written as*

$$\begin{aligned} \boldsymbol{u} &= \nabla p \\ -div\, \boldsymbol{u} &= f \\ p|_{\partial\Omega} &= 0. \end{aligned}$$



(a) Prismatic Macroelement.



(b) Tetrahedral Macroelement.

Figure 1.3 – Macroelements.

1.2.2 Problem (Weak formulation.). *Suppose we have a simply connected Lipschitz polyhedron $\Omega \subseteq \mathbb{R}^3$. Given $f \in L^2(\Omega)$, find $(\boldsymbol{u}, p) \in H(div, \Omega) \times L^2(\Omega)$ such that for all $(\boldsymbol{v}, q) \in H(div, \Omega) \times L^2(\Omega)$*

$$\int_\Omega \boldsymbol{u} \cdot \boldsymbol{v} \, d\boldsymbol{x} + \int_\Omega p \, div \, \boldsymbol{v} \, d\boldsymbol{x} = 0$$
$$- \int_\Omega q \, div \, \mathbf{u} \, d\boldsymbol{x} = \int_\Omega f q \, d\boldsymbol{x}.$$

In the case in which the polyhedron is not convex there will be re–entrant vertices and edges whose interior angles are obtuse. In the following we are going to recall how to formalize the singularities in a polyhedron and how to classify the regularity of the solution of Problem 1.2.1.

As it is been said in the introduction of [5], regularity investigations go back to the pioneering work of Kondrat'ev [31]. One of the ways the theory has been developed in is the characterization of the solution by weighted Sobolev spaces it belongs to. For this topic we refer to [20].

Recall the descriptions in pages 1119 of [4] and 522 of [5] for the following definitions.

1.2.3 Definition. *Suppose that a vertex $S \in \partial\Omega$ is the origin of our Cartesian system of coordinates. Let C_S be the infinite polyhedral cone of $\mathbb{R}^3$ which coincides with Ω in a neighbourhood of S. Set $G_S = C_S \cap S^2(S)$, the intersection of C_S with the unit sphere centered at S. The vertex singular exponent related to S is defined by $\lambda_{\boldsymbol{v},S} := -1/2 + \sqrt{\lambda_{\boldsymbol{v},1} + 1/4}$ where $\lambda_{\boldsymbol{v},n} > 0$, $n \in \mathbb{N}$, are the eigenvalues, in increasing order, of a positive Laplace–Beltrami operator Δ' on $G_{\boldsymbol{v}}$ with Dirichlet boundary conditions. For any edge $A_{S,j}$, $1 \leqslant j \leqslant J_S$ of $\partial\Omega$ incident to a vertex S, the edge singular exponent is $\lambda_{\boldsymbol{e},S,j} := \pi/\omega_{S,j}$ where $\omega_{S,j}$ is the interior angle between the faces sharing $A_{S,j}$. We say that*

1. *$A_{S,j}$ is singular if $\lambda_{\boldsymbol{e},S,j} < 1$.*
2. *S is singular if $\lambda_{\boldsymbol{v},S} < \dfrac{1}{2}$.*

The reason why we call *singular* an edge or a vertex is that the solutions of elliptic problems in non–convex domains have singularities at those vertices or approaching those edges.

1.2.4 Definition. *Let $\Omega = \cup_{\ell=1}^{N} \Lambda_\ell$ be a decomposition of Ω in* **prismatic** *or* **tetrahedral** *macro–elements having, each one of them, at most a singular edge and a singular vertex. For any $\ell = 1, \ldots, L$, we set $\lambda_{\boldsymbol{v}}^{(\ell)} = \lambda_{\boldsymbol{v},S}$ if Λ_ℓ contains a singular vertex S of Ω, otherwise we take $\lambda_{\boldsymbol{v}}^{(\ell)} = \infty$, and $\lambda_{\boldsymbol{e}}^{(\ell)} = \lambda_{\boldsymbol{e},S,j}$ if Λ_ℓ contains one singular edge $A_{S,j}$ of Ω, otherwise we take $\lambda_{\boldsymbol{e}}^{(\ell)} = \infty$.*

In each macro–element Λ_ℓ let $\boldsymbol{\xi}^{(\ell)} = (\xi_1^{(\ell)}, \xi_2^{(\ell)}, \xi_3^{(\ell)})$ be a cartesian coordinate system such that the singular vertex, if existing, is located at the origin, and the singular edge, if existing, is contained in the $\xi_3^{(\ell)}$ axis (we will drop the ℓ's when they are not needed). In the case Λ_ℓ contains both singularities, the edge whithin $\xi_3^{(\ell)}$ is incident to the vertex; in case we had a singular edge and no singular vertex, the origin is at one end of the edge.

Let

$$\begin{aligned} R^{(\ell)}(\boldsymbol{\xi}) &= \left((\xi_1^{(\ell)})^2 + (\xi_2^{(\ell)})^2 + (\xi_3^{(\ell)})^2\right)^{1/2} \\ r^{(\ell)}(\boldsymbol{\xi}) &= \left((\xi_1^{(\ell)})^2 + (\xi_2^{(\ell)})^2\right)^{1/2} \\ \theta^{(\ell)}(\boldsymbol{\xi}) &= \frac{r(\boldsymbol{\xi})}{R(\boldsymbol{\xi})} \end{aligned}$$

be the distance to the origin, the radial *distance to the $\xi_3^{(\ell)}$–axis and the* angular distance *from the $\xi_3^{(\ell)}$–axis, respectively.*

The macro–element decomposition corresponds to the fact that the sequence of meshes we will introduce at the end of this thesis has a first coarse term consisting only of prisms and tetrahedra.

To talk about the regularity of a solution in these *non–convex* domains we will make use of the following weighted Sobolev spaces.

1.2.5 Definition. *Suppose Ω is a non–convex polihedral domain where $\Lambda \subseteq \Omega$ is a subdomain such that $\bar{\Lambda}$ contains at most one singular vertex S or at most one singular edge $A_{S,j}$ of Ω. Given two positive parameters β and δ we define the norm*

$$\|v\|_{\beta,\delta}^{1,2} := \left\{ \sum_{|\boldsymbol{\alpha}| \leqslant 1} \|R^{\beta-1+|\boldsymbol{\alpha}|} \theta^{\delta-1+|\boldsymbol{\alpha}|} \, D^{\boldsymbol{\alpha}} v\|_{L^2(\Lambda)}^2 \right\}^{1/2} . \qquad (1.10)$$

The symbol $V_{\beta,\delta}^{1,2}(\Lambda)$ will denote the space

$$V_{\beta,\delta}^{1,2}(\Lambda) = \left\{ v \in \mathscr{D}'(\Lambda) : \|v\|_{\beta,\delta}^{1,2} < \infty \right\} . \qquad (1.11)$$

1.2.6 Remark. *If in the last definitions we had $\beta = \delta$, which will hold true in the case of just an edge singularity, then we would write $V^{1,2}_{\delta,\delta}(\Lambda) = V^{1,2}_{\delta}(\Lambda)$ and, consistently,*

$$\|v\|^{1,2}_{\delta} := \left\{ \sum_{|\boldsymbol{\alpha}|\leqslant 1} \|r^{\delta-1+|\boldsymbol{\alpha}|} D^{\boldsymbol{\alpha}} v\|^2_{L^2(\Lambda)} \right\}^{1/2}.$$

We have the following regularity result, for which we refer again to [5].

1.2.7 Theorem. *The solutions $\boldsymbol{u}$ and p of Problem 1.2.2 satisfy*

$$p \in H^1(\Omega)$$

and for each ℓ

$$\boldsymbol{u} = \boldsymbol{u}_r + \boldsymbol{u}_s \tag{1.12}$$

with $\boldsymbol{u}_r \in H^1(\Omega)$ and, for any $\beta_\ell > \frac{1}{2} - \lambda^{(\ell)}_{\boldsymbol{v}}$, $\delta_\ell > 1 - \lambda^{(\ell)}_{\boldsymbol{e}}$,

$$\boldsymbol{u}_s \cdot \xi_i \in V^{1,2}_{\beta_\ell,\delta_\ell}(\Lambda_\ell), \quad i = 1,2, \qquad \boldsymbol{u}_s \cdot \xi_3 \in V^{1,2}_{\beta_\ell,0}(\Lambda_\ell)$$

Furthermore, the following estimates hold:

$$\|\boldsymbol{u}_r\|_{H^1(\Omega)} \leqslant C\,\|f\|_{L^2(\Omega)} \tag{1.13}$$

$$\|\boldsymbol{u}_s \cdot \xi_i\|_{V^{1,2}_{\beta_\ell,\delta_\ell}(\Lambda_\ell)} \leqslant C\,\|f\|_{L^2(\Omega)} \tag{1.14}$$

$$\|\boldsymbol{u}_s \cdot \xi_3\|_{V^{1,2}_{\beta_\ell,0}(\Lambda_\ell)} \leqslant C\,\|f\|_{L^2(\Omega)}. \tag{1.15}$$

1.2.8 Remark. *Note that it is always possible to take $0 < \beta_\ell = \delta_\ell < 1$ in the previous Theorem and note the dependence of β and δ on the macro–element, which we made explicit by putting the subindex ℓ, meaning the result is about local regularity.*

1.2.9 Definition. *For a subdomain Λ the vectorial weighted space we are using will be denoted*

$$\mathcal{V}_{\beta,\delta}(\Lambda) = V^{1,2}_{\beta,\delta}(\Lambda)^2 \times V^{1,2}_{\beta,0}(\Lambda)$$
$$\mathcal{V}_{\delta}(\Lambda) = V^{1,2}_{\delta}(\Lambda)^2 \times V^{1,2}_{0}(\Lambda)$$

with the usual product norm.

1.3 Polynomial Approximation

1.3.1 Definition. *Given $f \in \mathcal{C}^k(\Omega)$ the k–th degree Taylor polynomial of f centered at $\boldsymbol{y} \in \Omega$, $T^k_{\boldsymbol{y}} f$, is defined by*

$$(T^k_{\boldsymbol{y}} f)(\boldsymbol{x}) = \sum_{|\boldsymbol{\alpha}|\leqslant k} \frac{f^{(\boldsymbol{\alpha})}(\boldsymbol{y})}{\boldsymbol{\alpha}!}(\boldsymbol{x}-\boldsymbol{y})^{\boldsymbol{\alpha}}. \tag{1.16}$$

1.3.2 Definition. *Ω is star–shaped with respect to B if, for all $x \in \Omega$, the closed convex hull of $\{x\} \cup B$ is a subset of Ω.*

1.3.3 Definition. *Assume that Ω is star–shaped with respect to a set $B \subseteq \Omega$ of positive measure. Given an integer $k \geqslant 0$ and $f \in W^{k+1,p}(\Omega)$ we introduce the averaged Taylor polynomial of f, $\mathcal{Q}_{k,B} f \in P_k$ defined by*

$$(\mathcal{Q}_{k,B} f)(\boldsymbol{x}) = \frac{1}{|B|} \int_B (T^k_{\boldsymbol{y}} f)(\boldsymbol{x})\, dy \tag{1.17}$$

with $T^k_{\boldsymbol{y}}$ as in (1.16) but with distributional derivatives.

Given a field $\boldsymbol{u} = (u_1, u_2, u_3)' \in W^{m,p}(\Omega)^3$, the vectorial averaged Taylor polynomial of $\boldsymbol{u}$ is defined componentwise as

$$\mathbf{Q}_{m,B}\, \boldsymbol{u} = (\mathcal{Q}_{m,B}\, u_1, \mathcal{Q}_{m,B}\, u_2, \mathcal{Q}_{m,B}\, u_3). \tag{1.18}$$

The Taylor polynomials defined in the present section were taken from [22].

1.3.4 Lemma. *Let $\boldsymbol{\beta}$ be a multi–index such that $|\boldsymbol{\beta}| \leqslant m$, then*

$$\partial^{\boldsymbol{\beta}} \mathcal{Q}_{m,B} f = \mathcal{Q}_{m-|\boldsymbol{\beta}|,B} \partial^{\boldsymbol{\beta}} f. \tag{1.19}$$

1.3.5 Lemma. *Let $\Omega \subseteq \mathbb{R}^n$ be an open connected set with diameter d which is star–shaped with respect to a set $B \subseteq \Omega$ of positive measure. Given $p \geqslant 1$ and an integer $k \geqslant 0$ and $f \in W^{k+1,p}(\Omega)$ there exists a positive $C = C(k,n)$ such that, for $|\boldsymbol{\beta}| \geqslant k+1$,*

$$\|\partial^{\boldsymbol{\beta}}(f - \mathcal{Q}_{k,B} f)\|_{L^p(\Omega)} \leqslant C \frac{|\Omega|^{1/p}}{|B|^{1/p}} d^{k-|\boldsymbol{\beta}|+1} |f|_{k+1,p,\Omega}.$$

In particular, if Ω is convex,

$$\|\partial^{\boldsymbol{\beta}}(f - \mathcal{Q}_{k,\Omega} f)\|_{L^p(\Omega)} \leqslant C d^{k-|\boldsymbol{\beta}|+1} |f|_{k+1,p,\Omega}.$$

Proof. We will extend the proof in [22] for general p. In view of Lemma 1.3.4 we may only prove the estimate for the case $|\boldsymbol{\beta}| = 0$ and, for $|\boldsymbol{\beta}| > 0$, apply it to $\partial^{\boldsymbol{\beta}} f - \mathcal{Q}_{k-|\boldsymbol{\beta}|,B} \partial^{\boldsymbol{\beta}} f$.

Consider $|\boldsymbol{\beta}| = 0$ and take q as the Hölder conjugate of $p \geqslant 1$. By density we may assume $f \in \mathcal{C}^\infty(\Omega)$. Firt use Taylor's Theorem

$$f(\boldsymbol{x}) - (T^k_{\boldsymbol{y}} f)(\boldsymbol{x}) = (k+1) \sum_{|\boldsymbol{\alpha}|=k+1} \frac{(\boldsymbol{x}-\boldsymbol{y})^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!} \int_0^1 \partial^{\boldsymbol{\alpha}} f(t\boldsymbol{y} + (1-t)\boldsymbol{x})\, t^k\, dt$$

which implies

$$f(\boldsymbol{x}) - \mathcal{Q}_{k,B} f(\boldsymbol{x}) = \frac{k+1}{B} \sum_{|\boldsymbol{\alpha}|=k+1} \int_B \int_0^1 \frac{(\boldsymbol{x}-\boldsymbol{y})^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!} \partial^{\boldsymbol{\alpha}} f(t\boldsymbol{y} + (1-t)\boldsymbol{x})\, t^k\, dt\, d\boldsymbol{y}.$$

By Hölder's inequality twice (once in a finite dimensional version) we have

$$\int_\Omega |f(\boldsymbol{x}) - \mathcal{Q}_{k,B} f(\boldsymbol{x})|^p \, d\boldsymbol{x} \leqslant$$

$$\leqslant C \frac{d^{p(k+1)}}{|B|^p} \sum_{|\boldsymbol{\alpha}|=k+1} \int_\Omega \left(\int_B \int_0^1 |\partial^{\boldsymbol{\alpha}} f(t\boldsymbol{y} + (1-t)\boldsymbol{x})|^p \, dt \, d\boldsymbol{y} \right) \left(\frac{|B|}{qk+1} \right)^{p/q}$$

$$= \frac{C}{(qk+1)^{p/q}} \frac{d^{p(k+1)}}{|B|} \sum_{|\boldsymbol{\alpha}|=k+1} \int_\Omega \int_B \int_0^1 |\partial^{\boldsymbol{\alpha}} f(t\boldsymbol{y} + (1-t)\boldsymbol{x})|^p \, dt \, d\boldsymbol{y} d\boldsymbol{x}.$$

For any multi–index $\boldsymbol{\alpha}$ we split

$$\int_\Omega \int_B \int_0^1 |\partial^{\boldsymbol{\alpha}} f(t\boldsymbol{y} + (1-t)\boldsymbol{x})|^p \, dt \, d\boldsymbol{y} d\boldsymbol{x} =$$

$$= \int_\Omega \int_B \int_0^{1/2} |\partial^{\boldsymbol{\alpha}} f(t\boldsymbol{y} + (1-t)\boldsymbol{x})|^p \, dt \, d\boldsymbol{y} d\boldsymbol{x} + \int_\Omega \int_B \int_{1/2}^1 |\partial^{\boldsymbol{\alpha}} f(t\boldsymbol{y} + (1-t)\boldsymbol{x})|^p \, dt \, d\boldsymbol{y} d\boldsymbol{x}. \tag{1.20}$$

Let $\phi_{\boldsymbol{\alpha}}$ be the extension by zero of $\partial^{\boldsymbol{\alpha}} f$ to $\mathbb{R}^n$. By Fubini's Theorem and change of variables, the first term in (1.20) is less than or equal to

$$\int_B \int_0^{1/2} \int_{\mathbb{R}^n} |\phi_{\boldsymbol{\alpha}}(\boldsymbol{\zeta})|^p (1-t)^{-n} \, d\boldsymbol{\zeta} \, dt \, d\boldsymbol{y} \leqslant 2^{n-1} |B| \|\partial^{\boldsymbol{\alpha}} f\|_{p,\Omega}^p.$$

The second term in (1.20) is less than or equal to

$$\int_\Omega \int_{1/2}^1 \int_{\mathbb{R}^n} |\phi_{\boldsymbol{\alpha}}(t\boldsymbol{y})|^p \, d\boldsymbol{y} \, dt \, d\boldsymbol{x} = \int_\Omega \int_{1/2}^1 \int_{\mathbb{R}^n} |\phi_{\boldsymbol{\alpha}}(\boldsymbol{\zeta})|^p t^{-n} \, d\boldsymbol{\zeta} \, dt \, d\boldsymbol{x} \leqslant 2^{n-1} |\Omega| \|\partial^{\boldsymbol{\alpha}} f\|_{p,\Omega}^p.$$

Summing theese up for any $\boldsymbol{\alpha}$ of order $k+1$ we obtain

$$\|f - \mathcal{Q}_{k,B} f\|_p^p \leqslant \frac{C}{(qk+1)^{p/q}} d^{p(k+1)} \frac{|\Omega|}{|B|} |f|_{p,k+1,\Omega}^p.$$

□

For the following paragraphs we refer to the exposition in Theorem 3.2 of [21].

1.3.6 Theorem. *Let $m \geqslant 0$ and $p, \bar{p} \in [1, \infty]$. Suppose*

$$\frac{1}{\bar{p}} - \frac{1}{p} + \frac{m+1}{3} \geqslant 0$$

and that there exists σ with

$$0 < \sigma \leqslant \max \left\{ \left\lfloor \frac{m+1}{3} \right\rfloor, \frac{1}{\bar{p}} - \frac{1}{p} + \frac{m+1}{3}, \min \left\{ 1 - \frac{1}{\bar{p}}, \frac{1}{\bar{p}} \right\} \right\},$$

then there is a positive C depending only on m,σ and $\Omega \subseteq \mathbb{R}^3$ such that, for all $g \in W^{m+1,p}(\Omega)$

$$\|\partial^{\boldsymbol{\beta}}(g - \mathcal{Q}_m g)\|_{L^{\bar{p}}(\Omega)} \leqslant C|g|_{W^{m+1,p}(\Omega)} \tag{1.21}$$

whenever $0 \geqslant |\boldsymbol{\beta}| \geqslant m + 1$.

Chapter 2

Some Vectorial Finite Elements

Introducción al capítulo

Los elementos $H(\text{div})$–conformes fueron definidos para construir un operador de interpolación natural para campos vectoriales con componentes normales continuas. Análogamente, los otros elementos considerados son conformes en $H(\mathbf{curl})$, que son usados para interpolar campos vectoriales con componentes tangenciales continuas. Estos son los casos de las soluciones de las ecuaciones de Maxwell armónicas en el tiempo [33], y de la formulación mixta del problema de Poisson [13].

A pesar de poner la construcción y explicitar algunas propiedades para ambas clases de elementos aquí, dejamos las estimaciones que demostramos para el caso $H(\mathbf{curl})$ para el Capítulo 6, pues no serán usadas en la aplicación final (Capítulo 5).

Adoptaremos la convención de que los objetos como funciones, variables, vectores normales en el Prisma de referencia $\hat{E}$ tendrán un sombrero como $\hat{\boldsymbol{u}}, \hat{\boldsymbol{x}}, \hat{\boldsymbol{n}}, \ldots$ al mismo tiempo que el sombrero denotará una transformación de *pullback* correspondiente a una aplicación de $\hat{E}$ hacia un elemento físico E.

Para los espacios de funciones polinomiales de tipo producto tensorial involucrados en la construcción de los elementos $H(\mathbf{curl})$–conformes o $H(\text{div})$–conformes requeriremos de la siguiente notación.

2.0.1 Notación. *Polinomios definidos en un prisma:*

$$P_{k,l} := P_k(\hat{x}_1, \hat{x}_2) \otimes P_l(\hat{x}_3),$$
$$Q_{k,l,m} := P_k(\hat{x}_1) \otimes P_l(\hat{x}_2) \otimes P_m(\hat{x}_3).$$

Dada una cara f de un elemento poliedral, en la cual tenemos variables locales (ξ_1, ξ_2), el espacio $Q_{k,l}$ será definido como

$$Q_{k,l} = P_k(\xi_1) \otimes P_l(\xi_2).$$

Introduction to the chapter

$H(\text{div})$–conforming elements were defined to determine a natural interpolation operator for fields with continuous normal components. Similarly, the other elements considered are $H(\mathbf{curl})$–conforming, which are used to interpolate fields with continuous tangential components. These two are the cases of the solutions of time harmonic Maxwell's equations, and the solution of the mixed formulation of the Poisson problem. Although we put the construction and state some properties for both kinds of elements here, we leave the estimates proved for the $H(\mathbf{curl})$ case for Chapter 6, as they are not used in the main application in Chapter 5.

We will adopt the convention that objects like functions, variables, normals in the reference prism $\hat{E}$ will have a hat as in $\hat{\boldsymbol{u}}, \hat{\boldsymbol{x}}, \hat{\boldsymbol{n}}, \ldots$ and then also the hat will denote a pullback transformation corresponding to a mapping from $\hat{E}$ onto a physical element E.

For tensor product polynomial spaces involved in the construction of $H(\mathbf{curl})$ or $H(\text{div})$ elements we will need the following notation.

2.0.2 Notation. *For polynomials defined over polyhedra we add the following symbols:*

$$\begin{aligned} P_{k,l} &:= P_k(\hat{x}_1, \hat{x}_2) \otimes P_l(\hat{x}_3), \\ Q_{k,l,m} &:= P_k(\hat{x}_1) \otimes P_l(\hat{x}_2) \otimes P_m(\hat{x}_3), \end{aligned}$$

and for a given face f on a polyhedron, in which we have local variables (ξ_1, ξ_2), the space $Q_{k,l}$ will be defined as

$$Q_{k,l} = P_k(\xi_1) \otimes P_l(\xi_2).$$

2.1 Prismatic Finite Elements

2.1.1 $H(\mathbf{div})$–Conforming Element on Prisms

With the notation introduced in Subsection 1.1.2 we consider the polynomial space (see [34])

$$\begin{aligned} D_k &= P^2_{k-1}(\hat{x}_1, \hat{x}_2) \oplus \widetilde{P}_{k-1}(\hat{x}_1, \hat{x}_2)\hat{\boldsymbol{x}}, \\ \hat{\boldsymbol{x}} &= (\hat{x}_1, \hat{x}_2). \end{aligned} \tag{2.1}$$

2.1.1 Definition. *Given a non–negative integer k, we define a prismatic $H(div)$–Conforming Finite Element of degree k by the following triple.*

1. *$\hat{E}$ is the reference prism in Figure 1.2b.*

2. *The polynomial space $P_{\hat{E}}$ is*

$$\begin{aligned} P_{\hat{E}} = \{\hat{\boldsymbol{v}} = (\hat{v}_1, \hat{v}_2, \hat{v}_3)' :\ & (\hat{v}_1, \hat{v}_2)' \in D_k \otimes P_{k-1}(\hat{x}_3), \\ & \hat{v}_3 \in P_{k-1,k}\}. \end{aligned} \tag{2.2}$$

3. *The degrees of freedom are of two types, surface and volumen integrals.*

$$\hat{\rho}_{\hat{f},\hat{q}}(\hat{\boldsymbol{v}}) = \iint_{\hat{f}} (\hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{n}})\hat{q}\, d\hat{S} \quad \text{for } \hat{q} \in P_{k-1}(\hat{f}), \text{ if } \hat{f} = \hat{f}_3 \text{ or } \hat{f}_4; \tag{2.3}$$

$$\hat{\rho}_{\hat{f},\hat{q}}(\hat{\boldsymbol{v}}) = \iint_{\hat{f}} (\hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{n}})\hat{q}\, d\hat{S} \quad \text{for } \hat{q} \in Q_{k-1,k-1}(\hat{f}), \text{ if } \hat{f} = \hat{f}_1, \hat{f}_2 \text{ or } \hat{f}_5; \tag{2.4}$$

$$\hat{\rho}_{\hat{\boldsymbol{r}}}(\hat{\boldsymbol{v}}) = \int_{\hat{E}} (\hat{v}_1\, \hat{r}_1 + \hat{v}_2\, \hat{r}_2)\, d\hat{\boldsymbol{x}} \quad \text{for } \hat{r}_1, \hat{r}_2 \in P_{k-2,k-1}; \tag{2.5}$$

$$\hat{\rho}_{\hat{\boldsymbol{r}}}(\hat{\boldsymbol{v}}) = \int_{\hat{E}} \hat{v}_3\, \hat{r}_3\, d\hat{\boldsymbol{x}} \quad \text{for } \hat{r}_3 \in P_{k-1,k-2}. \tag{2.6}$$

The following result can be found in page 66 of [34].

2.1.2 Lemma. *The degrees of freedom of the Finite Element in Definition 2.1.1 are unisolvent in $\hat{E}$.*

2.1.3 Definition. *Let $P_{\hat{E}}$ be as in (2.2). The interpolator $\boldsymbol{r}_{\hat{E}} : W^{1,1}(\hat{E}) \to P_{\hat{E}}$ is defined as the operator such that, for each $\hat{\boldsymbol{u}} \in W^{1,1}(\hat{E})$, $\boldsymbol{r}_{\hat{E}}\hat{\boldsymbol{u}}$ is defined as the unique element in $P_{\hat{E}}$ satisfying*

$$\hat{\rho}_{\hat{f},\hat{q}}\,(\hat{\boldsymbol{u}} - \boldsymbol{r}_{\hat{E}}\hat{\boldsymbol{u}}) = 0 \quad \text{for } \hat{\rho}_{\hat{f},\hat{q}} \text{ as in (2.3) and (2.4)} \tag{2.7}$$

$$\hat{\rho}_{\hat{\boldsymbol{r}}}\,(\hat{\boldsymbol{u}} - \boldsymbol{r}_{\hat{E}}\hat{\boldsymbol{u}}) = 0 \quad \text{for } \hat{\rho}_{\hat{\boldsymbol{r}}} \text{ as in (2.5) and (2.6).} \tag{2.8}$$

Because of the degrees of freedom (2.3) and (2.4) we had to restrict the condition $\hat{\boldsymbol{u}} \in H(\mathrm{div}, \hat{E})$ to the one of $\hat{\boldsymbol{u}} \in W^{1,1}(\hat{E})$ in order to define the interpolation operator. The reason is that condition $\hat{\boldsymbol{u}} \in H(\mathrm{div}, \hat{E})$ does not suffice to have normal traces over individual faces of a polyhedron. The proof of the following Lemma follows from Trace Theorems in Sobolev spaces as in [33].

2.1.4 Lemma. *The operator of Definition 2.1.3 is well defined and bounded.*

2.1.5 Proposition. *On the Finite Element of definition 2.1.1.* $\dim(P_{\hat{E}}) = k^2\,(k+2) + k\,(k+1)^2/2$ *which is, at the same time, equal to the number of its independent degrees of freedom.*

Proof. From Remark 1.1.6 and (2.1) it is really straightforward to do

$$\dim(P^2_{k-1}(\hat{x}_1, \hat{x}_2) \oplus \widetilde{P}_{k-1}(\hat{x}_1, \hat{x}_2)) \otimes P_{k-1}(\hat{x}_3) + \dim P_{k-1,k} = $$
$$= (k(k+1) + k)k + \frac{k(k+1)}{2}(k+1)$$

and the same quantity is obtained by summing up the dimensions of all the polynomial spaces on the right–hand sides of (2.3)–(2.6). □

2.1.6 Remark. *The interpolation operator can be written*

$$\hat{\boldsymbol{r}}_k \hat{\boldsymbol{u}} = \sum_{\hat{f},\hat{q}} \hat{\rho}_{\hat{f},\hat{q}}(\hat{\boldsymbol{u}})\, \hat{\boldsymbol{v}}_{\hat{f},\hat{q}} + \sum_{\hat{\boldsymbol{r}}} \hat{\rho}_{\hat{\boldsymbol{r}}}(\hat{\boldsymbol{u}})\, \hat{\boldsymbol{v}}_{\hat{\boldsymbol{r}}} \tag{2.9}$$

where, in the first sum, $\hat{f}$ are the faces of the reference element and, in both sums, $\{\hat{q}\}$ and $\{\hat{\boldsymbol{r}}\}$ are fixed bases of the test spaces defining the degrees of freedom (2.3)–(2.6), and finally $\{\hat{\boldsymbol{v}}_{\hat{f},\hat{q}}\} \cup \{\hat{\boldsymbol{v}}_{\hat{\boldsymbol{r}}}\}$ is a basis of $P_{\hat{E}}$ which is dual to the basis of degrees of freedom $\{\hat{\rho}_{\hat{f},\hat{q}} \,|\, \hat{f}, \hat{q}\} \cup \{\hat{\rho}_{\hat{\boldsymbol{r}}} \,|\, \hat{\boldsymbol{r}}\}$.

2.1.2 $H(\mathbf{curl})$–Conforming Element on Prisms

First we introduce two more polynomial spaces on the reference prism of Figure 1.2b. $\hat{T}$ denotes the triangle in Definition 1.1.17 and $\hat{I} = [0,1]$.

2.1.7 Definition. *For an integer $k \geqslant 1$, let $R_k(\hat{T})$ denote the space of polynomials, defined over the triangle $\hat{T}$, given by*

$$R_k(\hat{T}) := P_{k-1}(\hat{T})^2 \oplus S_k(\hat{T}) \tag{2.10}$$

where

$$S_k(\hat{T}) := \{\boldsymbol{p} \in \widetilde{P}_k^2 \;:\; \boldsymbol{p} \cdot \hat{\boldsymbol{x}} = 0\}, \quad \hat{\boldsymbol{x}} = (\hat{x}_1, \hat{x}_2)'. \tag{2.11}$$

Table 2.1 – Notation for the faces and positive normals of the reference prism.

$\hat{f}_1$	$\subseteq$	$\{\hat{x}_1 = 0\}$	$\hat{\boldsymbol{n}}_1$	$=$	$(-1,0,0)'$
$\hat{f}_2$	$\subseteq$	$\{\hat{x}_2 = 0\}$	$\hat{\boldsymbol{n}}_2$	$=$	$(0,-1,0)'$
$\hat{f}_3$	$\subseteq$	$\{\hat{x}_3 = 0\}$	$\hat{\boldsymbol{n}}_3$	$=$	$(0,0,-1)'$
$\hat{f}_4$	$\subseteq$	$\{\hat{x}_3 = 1\}$	$\hat{\boldsymbol{n}}_4$	$=$	$(0,0,1)'$
$\hat{f}_5$	$\subseteq$	$\{\hat{x}_1 + \hat{x}_2 = 1\}$	$\hat{\boldsymbol{n}}_5$	$=$	$2^{-1/2}(1,1,0)'$

2.1.8 Definition. *Given a natural k, the $H(\mathbf{curl})$–Conforming Finite Element of degree k is defined by the following triple.*

1. *$\hat{E}$ is the reference prism in Definition 1.1.17.*
2. *The polynomial space $P_{\hat{E}}$ is*

$$P_{\hat{E}} = R_k(\hat{T}) \otimes P_k(\hat{I}) \times P_k(\hat{T}) \otimes P_{k-1}(\hat{I}). \tag{2.12}$$

Table 2.2 – Notation for the edges and positive tangents of the reference prism.

$\hat{\boldsymbol{e}}_1$	$=$	$\{(\hat{x}_1, 0, 0)^t \,:\, 0 \leqslant \hat{x}_1 \leqslant 1\}$	$\hat{\boldsymbol{\tau}}_1$	$=$	$(1, 0, 0)'$
$\hat{\boldsymbol{e}}_2$	$=$	$\{(0, \hat{x}_2, 0)^t \,:\, 0 \leqslant \hat{x}_2 \leqslant 1\}$	$\hat{\boldsymbol{\tau}}_2$	$=$	$(0, 1, 0)'$
$\hat{\boldsymbol{e}}_3$	$=$	$\{(0, 0, \hat{x}_3)^t \,:\, 0 \leqslant \hat{x}_3 \leqslant 1\}$	$\hat{\boldsymbol{\tau}}_3$	$=$	$(0, 0, 1)'$
$\hat{\boldsymbol{e}}_4$	$=$	$\{(\hat{x}_1, 0, 1)^t \,:\, 0 \leqslant \hat{x}_1 \leqslant 1\}$	$\hat{\boldsymbol{\tau}}_4$	$=$	$(0, 0, 1)'$
$\hat{\boldsymbol{e}}_5$	$=$	$\{(0, \hat{x}_2, 1)^t \,:\, 0 \leqslant \hat{x}_2 \leqslant 1\}$	$\hat{\boldsymbol{\tau}}_5$	$=$	$(0, 0, 1)'$
$\hat{\boldsymbol{e}}_6$	$=$	$\{(1, 0, \hat{x}_3)^t \,:\, 0 \leqslant \hat{x}_3 \leqslant 1\}$	$\hat{\boldsymbol{\tau}}_6$	$=$	$(0, 0, 1)'$
$\hat{\boldsymbol{e}}_7$	$=$	$\{(0, 1, \hat{x}_3)^t \,:\, 0 \leqslant \hat{x}_3 \leqslant 1\}$	$\hat{\boldsymbol{\tau}}_7$	$=$	$(0, 0, 1)'$
$\hat{\boldsymbol{e}}_8$	$=$	$\{(\hat{x}_1, 1 - \hat{x}_1, 1)^t \,:\, 0 \leqslant \hat{x}_1 \leqslant 1\}$	$\hat{\boldsymbol{\tau}}_8$	$=$	$2^{-1/2}(1, -1, 0)'$
$\hat{\boldsymbol{e}}_9$	$=$	$\{(\hat{x}_1, 1 - \hat{x}_1, 0)^t \,:\, 0 \leqslant \hat{x}_1 \leqslant 1\}$	$\hat{\boldsymbol{\tau}}_9$	$=$	$2^{-1/2}(1, -1, 0)'$

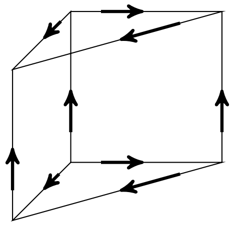


Figure 2.1 – Directions of positive unit tangents (cfr. Table 2.2).

3. *The degrees of freedom are (cfr. Tables 2.1 and 2.2):*

$$\hat{\varphi}_{\hat{e},\hat{q}}(\hat{\boldsymbol{u}}) = \int_{\hat{\boldsymbol{e}}} \hat{q}\,\hat{\boldsymbol{u}} \cdot d\hat{\boldsymbol{\alpha}}, \qquad \hat{q} \in P_{k-1}(\hat{\boldsymbol{e}}), \textit{for each edge } \hat{\boldsymbol{e}}; \tag{2.13}$$

$$\hat{\varphi}_{\hat{f},\hat{\boldsymbol{q}}}(\hat{\boldsymbol{u}}) = \iint_{\hat{f}} \hat{\boldsymbol{u}} \times \hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{q}}\,d\hat{S}, \qquad \hat{\boldsymbol{q}} = (\hat{q}_1, \hat{q}_2, 0) \in P^2_{k-2} \times \{0\},$$
$$\textit{for each face } f = \hat{f}_3 \textit{ or } \hat{f}_4; \tag{2.14}$$

$$\hat{\varphi}_{\hat{f}_1,\hat{\boldsymbol{q}}}(\hat{\boldsymbol{u}}) = \iint_{\hat{f}_1} \hat{\boldsymbol{u}} \times \hat{\boldsymbol{n}}_1 \cdot \hat{\boldsymbol{q}}\,d\hat{S}, \qquad \hat{\boldsymbol{q}} = (0, \hat{q}_3, \hat{q}_2),\ \hat{q}_3 \in Q_{k-2,k-1},$$
$$\hat{q}_2 \in Q_{k-1,k-2}; \tag{2.15}$$

$$\hat{\varphi}_{\hat{f}_2,\hat{\boldsymbol{q}}}(\hat{\boldsymbol{u}}) = \iint_{\hat{f}_2} \hat{\boldsymbol{u}} \times \hat{\boldsymbol{n}}_2 \cdot \hat{\boldsymbol{q}}\,d\hat{S}, \qquad \hat{\boldsymbol{q}} = (\hat{q}_3, 0, \hat{q}_1),\ \hat{q}_3 \in Q_{k-2,k-1},$$
$$\hat{q}_1 \in Q_{k-1,k-2}; \tag{2.16}$$

$$\hat{\varphi}_{\hat{f}_5,\hat{\boldsymbol{q}}}(\hat{\boldsymbol{u}}) = \iint_{\hat{f}_5} \hat{\boldsymbol{u}} \times \hat{\boldsymbol{n}}_5 \cdot \hat{\boldsymbol{q}}\,d\hat{S}, \qquad \hat{\boldsymbol{q}} = (0, \hat{q}_3, \hat{q}_1),\ \hat{q}_3 \in Q_{k-2,k-1},$$
$$\hat{q}_1 \in Q_{k-1,k-2}; \tag{2.17}$$

$$\hat{\varphi}_{\hat{\boldsymbol{r}}}(\hat{\boldsymbol{u}}) = \int_{\hat{E}} \hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{r}}\,d\hat{\boldsymbol{x}}, \qquad \hat{r}_1, \hat{r}_2 \in P_{k-2,k-2},$$
$$\hat{r}_3 \in P_{k-3,k-1}. \tag{2.18}$$

In the following Remark we make an explicitation of the elements of $P_{\hat{E}}$.

2.1.9 Remark. *Take* $\hat{\mathbf{s}} = (\hat{s}_1, \hat{s}_2) \in S_k$ *as defined in (2.11). Set* $\hat{s}_1 = \sum_{i+j=k} a_{i,j}\,\hat{x}_1^i \hat{x}_2^j$, $\hat{s}_2 = \sum_{i+j=k} b_{i,j}\,\hat{x}_1^i \hat{x}_2^j$. *By definition is*

$$\begin{aligned} 0 &= \hat{x}_1 \hat{s}_1 + \hat{x}_2 \hat{s}_2 \\ &= a_{k,0}\,\hat{x}_1^{k+1} + b_{0,k}\,\hat{x}_2^{k+1} + \sum_{i+j=k} (a_{i-1,j+1} + b_{i,j})\,\hat{x}_1^i \hat{x}_2^{j+1} \end{aligned}$$

so $a_{k,0} = b_{0,k} = 0$ *and for all pair* (i,j) *with* $i + j = k$ *and* $i \geqslant 1$ *it holds the relation* $a_{i-1,j+1} = -b_{i,j}$. *Then*

$$\begin{aligned} \hat{s}_1 &= \sum_{i+j=k,\, j\geqslant 1} a_{i,j}\,\hat{x}_1^i \hat{x}_2^j &&= \hat{x}_2 \sum_{i+j=k,\, j\geqslant 1} a_{i,j}\,\hat{x}_1^i \hat{x}_2^{j-1} \\ \hat{s}_2 &= -\sum_{i+j=k,\, j\geqslant 1} a_{i,j}\,\hat{x}_1^{i+1} \hat{x}_2^{j-1} &&= -\hat{x}_1 \sum_{i+j=k,\, j\geqslant 1} a_{i,j}\,\hat{x}_1^i \hat{x}_2^{j-1}. \end{aligned}$$

So any $\hat{\boldsymbol{p}} \in P_{\hat{E}}$ *may be written as*

$$\begin{aligned} \hat{\boldsymbol{p}} &= (\hat{p}_1, \hat{p}_2, \hat{p}_3)' \\ &= (\hat{\xi}_1 + \hat{x}_2\,\hat{h}, \hat{\xi}_2 - \hat{x}_1\,\hat{h}, \hat{\xi}_3), \end{aligned} \tag{2.19}$$

$$\begin{aligned} \hat{\xi}_1, \hat{\xi}_2 &\in P_{k-1,k}, \\ \hat{\xi}_3 &\in P_{k,k-1}, \\ \hat{h} &\in \widetilde{P}_{k-1}(\hat{x}_1, \hat{x}_2) \otimes P_k(\hat{x}_3). \end{aligned}$$

An illustrative example.

2.1.10 Example (edge elements of lowest order.)**.**

$$\hat{\boldsymbol{u}}\,(\hat{x}_1, \hat{x}_2, \hat{x}_3) = \begin{pmatrix} a_1 + a_3\hat{x}_2 + a_4\hat{x}_3 + a_6\hat{x}_2\hat{x}_3 \\ a_2 - a_3\hat{x}_1 + a_5\hat{x}_3 - a_6\hat{x}_1\hat{x}_3 \\ a_7 + a_8\hat{x}_1 + a_9\hat{x}_2 \end{pmatrix},$$

$$(\hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{\tau}})|_{\hat{\boldsymbol{e}}} \in P_0(\hat{\boldsymbol{e}}).$$

The following can be proved exactly as in [33], Lemma 5.38.

2.1.11 Remark. *Given* $p > 2$*, the degrees of freedom* (2.13)–(2.18) *are well defined and bounded as linear functionals from* $W^{1,p}(\hat{E})$ *to* $\mathbb{R}$*.*

2.1.12 Remark (dimension of the space (2.12))**.** *Recall the space* S_k *as*

$$S_k(\hat{T}) = \{\boldsymbol{p} \in \widetilde{P}_k^2 \;:\; \boldsymbol{p} \cdot \boldsymbol{x} = 0\}.$$

To know its dimension consider

$$\begin{aligned} \phi \;:\; \widetilde{P}_k^2 &\longrightarrow \widetilde{P}_{k+1} \\ \phi(\boldsymbol{p}) \quad &:= \; \boldsymbol{p} \cdot \boldsymbol{x} \\ &:= \; \hat{x}_1\hat{p}_1 + \hat{x}_2\hat{p}_2. \end{aligned}$$

It results

$$\begin{aligned} S_k(\hat{T}) &= \ker(\phi) \\ \dim(S_k(\hat{T})) &= \dim(\widetilde{P}_k^2) - \dim(\mathrm{Im}(\phi)). \end{aligned}$$

Now, any $\hat{p} \in \widetilde{P}_{k+1}$ *is*

$$\hat{x}_1(a_{k+1,0}\hat{x}_1^k + a_{k,1}\hat{x}_1^{k-1}\hat{x}_2 + \ldots + a_{1,k}\hat{x}_2^k) + \hat{x}_2(a_{0,k+1}\hat{x}_2^k) = \hat{x}_1\hat{p}_1 + \hat{x}_2\hat{p}_2$$

where precisely $\hat{p}_1$ *y* $\hat{p}_2$ *belong to* $\widetilde{P}_k$*, so* ϕ *is onto. So it holds*

$$\begin{aligned} \dim S_k(\hat{T}) &= \dim(\widetilde{P}_k^2) - \dim(\widetilde{P}_{k+1}) \\ &= 2\dim(\widetilde{P}_k) - \dim(\widetilde{P}_{k+1}) \\ &= 2(k+1) - (k+2) \\ &= k, \end{aligned}$$

so

$$\dim R_k(\hat{T}) = k(k+1) + k$$
$$= k(k+2)$$

and finally

$$\dim R_k(\hat{T}) \otimes P_k(\hat{I}) = k(k+1)(k+2).$$

Immediately we arrive at $\dim P_{\hat{E}} = 3\frac{k(k+1)(k+2)}{2}$.

The following result, only for the reference element, can be found in [34, page 75].

2.1.13 Lemma. *The degrees of freedom of the finite element in Definition 2.1.8 are unisolvent in* $\hat{E}$.

Later we will establish the unisolvence of a general finite element of this family on an arbitrary physical prism. The most important consequence of unisolvence is the existence of an interpolation operator defined in terms of the set of degrees of freedom.

2.1.14 Definition. *Given* $p > 2$, *we define the* $H(\mathbf{curl})$*–conforming interpolation operator on the reference prism* $\boldsymbol{w}_{\hat{E}} : W^{1,p}(\hat{E}) \to P_{\hat{E}}$ *on each* $\hat{\boldsymbol{u}}$ *as the unique element* $\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}}$ *such that*

$$\hat{\varphi}_{\hat{\mathbf{e}},\hat{p}}\left(\hat{\boldsymbol{u}} - \boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}}\right) = 0 \qquad \text{for all } \hat{\varphi}_{\hat{\mathbf{e}},\hat{p}} \text{ as in (2.13).} \tag{2.20}$$
$$\hat{\varphi}_{\hat{f},\hat{\boldsymbol{q}}}\left(\hat{\boldsymbol{u}} - \boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}}\right) = 0 \qquad \text{for all } \hat{\varphi}_{\hat{f},\hat{\boldsymbol{q}}} \text{ as in (2.14)–(2.17)} \tag{2.21}$$
$$\hat{\varphi}_{\boldsymbol{r}}\left(\hat{\boldsymbol{u}} - \boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}}\right) = 0 \qquad \text{for all } \hat{\varphi}_{\boldsymbol{r}} \text{ as in (2.18).} \tag{2.22}$$

2.1.15 Remark. *The interpolator in Definition 2.1.14 is well defined and bounded. Because of the presence of degrees of freedom (2.13) we can't define the operator for an arbitrary field in* $H(\mathbf{curl}, \hat{E})$, *which is why we put the hypothesis* $p > 2$ *(cfr. [33], page 134, and [2], Theorem 5.4). The reason is that the condition* $\hat{\boldsymbol{u}} \in H(\mathbf{curl}, \hat{E})$ *is not sufficient to have tangential traces along individual edges of a polyhedron.*

2.1.16 Remark. *The interpolation operator can be written*

$$\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}} = \sum_{\hat{\mathbf{e}},\hat{p}} \hat{\varphi}_{\hat{\mathbf{e}},\hat{p}}(\hat{\boldsymbol{u}})\, \hat{\boldsymbol{v}}_{\hat{\mathbf{e}},\hat{p}} + \sum_{\hat{f},\hat{\boldsymbol{q}}} \hat{\varphi}_{\hat{f},\hat{\boldsymbol{q}}}(\hat{\boldsymbol{u}})\, \hat{\boldsymbol{v}}_{\hat{f},\hat{\boldsymbol{q}}} + \sum_{\hat{\boldsymbol{r}}} \hat{\varphi}_{\hat{\boldsymbol{r}}}(\hat{\boldsymbol{u}})\, \hat{\boldsymbol{v}}_{\hat{\boldsymbol{r}}}, \tag{2.23}$$

where, in the first sum, $\hat{\mathbf{e}}$ *are the edges of the reference element, in the second sum,* $\hat{f}$ *are the faces of the reference element and, in all of the sums,* $\{\hat{p}\}$, $\{\hat{\boldsymbol{q}}\}$ *and* $\{\hat{\boldsymbol{r}}\}$ *are fixed bases of the test spaces defining the degrees of freedom (2.13)–(2.18), and finally* $\{\hat{\boldsymbol{v}}_{\hat{\mathbf{e}},\hat{p}}\} \cup \{\hat{\boldsymbol{v}}_{\hat{f},\hat{\boldsymbol{q}}}\} \cup \{\hat{\boldsymbol{v}}_{\hat{\boldsymbol{r}}}\}$ *is a basis of* $P_{\hat{E}}$ *which is dual to the basis of degrees of freedom* $\{\hat{\varphi}_{\hat{\mathbf{e}},\hat{p}} \,|\, \hat{\boldsymbol{e}}, \hat{p}\} \cup \{\hat{\varphi}_{\hat{f},\hat{\boldsymbol{q}}} \,|\, \hat{f}, \hat{\boldsymbol{q}}\} \cup \{\hat{\varphi}_{\hat{\boldsymbol{r}}} \,|\, \hat{\boldsymbol{r}}\}$.

2.2 Tetrahedral Finite Elements

2.2.1 $H(\mathbf{div})$–Conforming Element on Tetrahedra

For $k > 0$, let

$$P_{\hat{E}} = (P_{k-1}(\hat{E}))^3 + P_{k-1}(\hat{E})\,\hat{\boldsymbol{x}} \tag{2.24}$$
$$= (P_{k-1}(\hat{E}))^3 \oplus \widetilde{P}_{k-1}(\hat{E})\,\hat{\boldsymbol{x}}. \tag{2.25}$$

In Chapter 5 of [33] we find the following two results.

2.2.1 Lemma. *The dimension of $P_{\hat{E}}$ is $1/2(k+3)(k+1)k$.*

2.2.2 Lemma. *div $P_{\hat{E}} = P_{k-1}(\hat{E})$.*

2.2.3 Definition. *Given a non–negative integer k, the $H(div)$–Conforming Finite Element of degree k is defined by the following triple.*

1. *$\hat{E}$ is the reference tetrahedron of Definition 1.1.19.*
2. *$P_{\hat{E}}$ is the polynomial space in (2.24).*
3. *The degrees of freedom are*

$$\iint_{\hat{f}} \hat{q}\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}\,d\hat{S} \quad \text{for all } \hat{q}\in P_{k-1}(\hat{f}) \qquad \text{for all face } \hat{f} \text{ of } \hat{E}$$
$$\int_{\hat{E}} \hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{q}}\,d\hat{\boldsymbol{x}} \qquad \text{for all } \hat{\boldsymbol{q}}\in (P_{k-2}(\hat{E}))^3.$$

2.3 Differentials, coordinate transformations, unisolvence and conformity of Prismatic Finite Elements

The motivation of the following exposition is to extend and complete what is said about coordinate transformations in [19], [25] and Section 3.9 of [33].

We want the same setting to build Finite Elements on arbitrary contiguous prisms belonging to a fixed mesh, all of them being an affine image of the reference prism. In order to do so we transform the set $\hat{E}$ and show how to pull–back and forward the scalars and fields in the discrete local spaces and the corresponding degrees of freedom. Consider the application $\hat{\boldsymbol{x}} \longmapsto \boldsymbol{x} = F_E(\hat{\boldsymbol{x}})$, where

$$F_E\hat{\boldsymbol{x}} = M_E\hat{\boldsymbol{x}} + \boldsymbol{x}_0 \tag{2.26}$$

for an invertible matrix M_E, that transforms $\hat{E}$ onto an element E of the mesh.

A scalar function $\hat{p} \in H^1(\hat{E})$ is transformed into a scalar function p on E with

$$p \circ F_E = \hat{p}. \tag{2.27}$$

As it holds (cfr. [19])

$$\nabla p = M^{-t}\hat{\nabla}\hat{p} \circ F_E^{-1}, \tag{2.28}$$

then it results $p \in H^1(E)$. Gradients are taken with respect to the local coordinates of E and $\hat{E}$ in each case. The same will apply for every finite element and for div and **curl** .

Now let $\hat{\boldsymbol{u}} \in H(\mathbf{curl}, \hat{E})$. We want to assign to it a function $\boldsymbol{u}$ on E. As for $\hat{p}$ and p as before it holds $\nabla p \in H(\mathbf{curl}, E)$ and $\hat{\nabla}\hat{p} \in H(\mathbf{curl}, \hat{E})$, equality (2.28) suggests the following transformation.

$$\boldsymbol{u} \circ F_E = M_E^{-t}\hat{\boldsymbol{u}}. \tag{2.29}$$

With this definition we have $\boldsymbol{u} \in H(\mathbf{curl}, E)$ and also

$$(\mathbf{curl}\, \boldsymbol{u}) \circ F_E = \frac{1}{\det M_E} M_E(\mathbf{curl}\, \hat{\boldsymbol{u}}), \tag{2.30}$$

(cfr. Lemma 3.57, page 77 and Corollary 3.58 in [33]). For the important example, needed in anisotropic analysis, with

$$M_E = \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix}$$

it holds

$$\boldsymbol{h}^{\boldsymbol{\alpha}}(\partial^{\boldsymbol{\alpha}}\boldsymbol{u}) \circ F_E = M_E^{-t}\hat{\partial}^{\boldsymbol{\alpha}}\hat{\boldsymbol{u}} \tag{2.31}$$

that is,

$$\boldsymbol{h}^{\boldsymbol{\alpha}}(\partial^{\boldsymbol{\alpha}}\boldsymbol{u})\hat{} = \hat{\partial}^{\boldsymbol{\alpha}}\hat{\boldsymbol{u}}. \tag{2.32}$$

Now we proceed in the same way for $H(\mathrm{div})$. The relation $\hat{\boldsymbol{u}} \in H(\mathbf{curl}, \hat{E})$ implies $\mathbf{curl}\, \hat{\boldsymbol{u}} \in H(\mathrm{div}, \hat{E})$ so (2.30) shows that to transform $\hat{\boldsymbol{u}} \in H(\mathrm{div}, \hat{E})$ into $\boldsymbol{u} \in H(\mathrm{div}, E)$ we must do it via

$$\boldsymbol{u} \circ F_E = \frac{1}{\det M_E} M_E\hat{\boldsymbol{u}}. \tag{2.33}$$

If $\boldsymbol{u}$ and $\hat{\boldsymbol{u}}$ are related by (2.33), then with a $\mathscr{D}(E)$ density argument we get

$$(\mathrm{div}\ \boldsymbol{u}) \circ F_E = (\det M_E)^{-1}\mathrm{div}\ \hat{\boldsymbol{u}}. \tag{2.34}$$

and hence $\boldsymbol{u} \in H(\mathrm{div}, E)$ if and only if $\hat{\boldsymbol{u}} \in H(\mathrm{div}, \hat{E})$.

Normals and tangents are transformed as follows (cfr. page 265 of [24]). Let $\hat{\boldsymbol{n}}$ be the unit outward normal to $\hat{E}$. If $\hat{\boldsymbol{x}} \in \partial\hat{E}$ and $\boldsymbol{n}$ is defined by

$$\boldsymbol{n}(F_E\hat{\boldsymbol{x}}) = \frac{M_E^{-t}\hat{\boldsymbol{n}}(\hat{\boldsymbol{x}})}{\|M_E^{-t}\hat{\boldsymbol{n}}(\hat{\boldsymbol{x}})\|}, \tag{2.35}$$

then $\boldsymbol{n}$ is a unit normal to E. Second, let $\hat{\boldsymbol{\tau}}$ be any unit vector tangent to $\partial\hat{E}$ at $\hat{\boldsymbol{x}}$. If $\boldsymbol{\tau}$ is given by

$$\boldsymbol{\tau}(F_E\hat{\boldsymbol{x}}) = \frac{M_E\hat{\boldsymbol{\tau}}(\hat{\boldsymbol{x}})}{\|M_E\hat{\boldsymbol{\tau}}(\hat{\boldsymbol{x}})\|}, \tag{2.36}$$

then $\boldsymbol{\tau}$ is a unit vector tangent to ∂E at $F_E\hat{\boldsymbol{x}}$. Surface differentials are changed in the following way.

$$dS = \|M_E^{-t}\hat{\boldsymbol{n}}\|\,|\det M_E|\,d\hat{S}. \tag{2.37}$$

First a key result that establishes a relation between the interpolation operators. It will used as an important step in the proof of the stability of the edge element as well as in the proof of the stability of the face elements.

2.3.1 Remark. *In the reference [33], Lemma 5.40 in page 135 and the first Paragraph of Section 5.7 in page 149 state the following facts.*

Take the interpolation operator $\boldsymbol{w}_{\hat{E}}$ *of Definition 2.1.14, the operator* $\boldsymbol{r}_{\hat{E}}$ *of Definition 2.1.3 and set* $\pi_{\hat{E}}^{\perp}$ *as the* L^2*–orthogonal projection onto* $P_k(\hat{E})$*, then*

1. *For all sufficiently smooth* $\hat{\mathbf{u}}$ *such that the interpolants* $\boldsymbol{w}_{\hat{E}}\hat{\mathbf{u}}$ *and* $\boldsymbol{r}_{\hat{E}}\,\textbf{\textit{curl}}\,\hat{\mathbf{u}}$ *are both defined, then*

$$\textbf{\textit{curl}}\,\boldsymbol{w}_{\hat{E}}\hat{\mathbf{u}} = \boldsymbol{r}_{\hat{E}}\,\textbf{\textit{curl}}\,\hat{\mathbf{u}}. \tag{2.38}$$

2. *For all sufficiently smooth* $\hat{\mathbf{u}}$ *such that the interpolants* $\boldsymbol{r}_{\hat{E}}\hat{\mathbf{u}}$ *and* $\pi_{\hat{E}}^{\perp}\,div\,\hat{\mathbf{u}}$ *are both defined, then*

$$div\ \boldsymbol{r}_{\hat{E}}\hat{\mathbf{u}} = \pi_{\hat{E}}^{\perp}\,div\ \hat{\mathbf{u}}. \tag{2.39}$$

The last result can be expressed saying that the following diagram commutes:

$$\begin{array}{ccccc}
\mathrm{H}(\mathbf{curl},\hat{E}) & \xrightarrow{\mathbf{curl}} & \mathrm{H}(\mathrm{div},\hat{E}) & \xrightarrow{\mathrm{div}} & L^2(\hat{E}) \\
\cup & & \cup & & \cup \\
V & \xrightarrow{\mathbf{curl}} & W & \xrightarrow{\mathrm{div}} & Q \\
\boldsymbol{w}_{\hat{E}}\Big\downarrow & & \boldsymbol{r}_{\hat{E}}\Big\downarrow & & \pi_{\hat{E}}^{\perp}\Big\downarrow \\
V_{\hat{E}} & \xrightarrow{\mathbf{curl}} & W_{\hat{E}} & \xrightarrow{\mathrm{div}} & Q_{\hat{E}}
\end{array}$$

where $V_{\hat{E}}$, $W_{\hat{E}}$ and $Q_{\hat{E}}$ are the spaces (2.12), (2.2) and $P_0(\hat{E})$ respectively.

2.3.2 Definition. *Suppose Ω is the union of two elements E_1 and E_2 with a common face f, we say that the family of unisolvent Finite Elements (E, P_E, Σ) is conforming in some functional space $\mathbb{W}$ if, writing π_1, π_2 the interpolation operators determined by the degrees of freedom Σ_1 and Σ_2, on E_1 and E_2, the field defined as*

$$\boldsymbol{w} = \begin{cases} \pi_1(\boldsymbol{u}|_{E_1}), & \text{on } E_1 \\ \pi_2(\boldsymbol{u}|_{E_2}), & \text{on } E_2 \end{cases}$$

results in $\mathbb{W}(E_1 \cup E_2)$.

2.3.3 Lemma. *Let $\hat{E}$ be the reference prism (1.1.17) and let $P_{\hat{E}}$ be the space in (2.12), that is,*

$$P_{\hat{E}} = R_k(\hat{x}_1, \hat{x}_2) \otimes P_k(\hat{x}_3) \times P_k(\hat{x}_1, \hat{x}_2) \otimes P_{k-1}(\hat{x}_3).$$

Provided the matrix M_E is block–diagonal in the following manner

$$M_E = \begin{pmatrix} m_{1,1} & m_{1,2} & 0 \\ m_{2,1} & m_{2,2} & 0 \\ 0 & 0 & m_{3,3} \end{pmatrix},$$

then $P_{\hat{E}}$ is invariant by transformation (2.29).

Proof. Starting with transformation (2.29) take $\hat{\boldsymbol{u}} = (\hat{u}_1, \hat{u}_2, \hat{u}_3)' \in P_{\hat{E}}$ and $\boldsymbol{u} := (M_E^{-t}\hat{\boldsymbol{u}}) \circ F_E^{-1}$. Let

$$M_2 = \begin{pmatrix} m_{1,1} & m_{1,2} \\ m_{2,1} & m_{2,2} \end{pmatrix}$$

and let $m_{i,j}^{(-1)}$ denote the coefficients of the inverse. By definition, $(\hat{u}_1, \hat{u}_2)' = \hat{q}(\hat{\boldsymbol{p}} + \hat{\boldsymbol{s}})$ for some $\hat{\boldsymbol{p}}$, $\hat{\boldsymbol{s}}$ as in the spaces (2.11) and (2.10) and $\hat{q} \in P_k(\hat{x}_3)$. Using the blocks of M_E and the tensor nature of $P_{\hat{E}}$,

$$\begin{aligned}(u_1(\boldsymbol{x}), u_2(\boldsymbol{x}))' &= (\hat{q} M_2^{-t}(\hat{\boldsymbol{p}} + \hat{\boldsymbol{s}}))(M_E^{-1}\boldsymbol{x} - M_E^{-1}\boldsymbol{x}_E) \\ &= \hat{q}(m_{3,3}^{(-1)}(x_3 - x_{E,3}))(M_2^{-t}(\hat{\boldsymbol{p}} + \hat{\boldsymbol{s}}))(M_2^{-1}\boldsymbol{x} - M_2^{-1}\boldsymbol{x}_E).\end{aligned}$$

Now, $\hat{q}(m_{3,3}^{(-1)}(x_3 - x_{E,3}))$ is a polynomial $q(x_3)$ with the same degree as $\hat{q}$ is with respect to $\hat{x}_3$. Furthermore, as by Remark 2.1.9 $\hat{\boldsymbol{s}}$ has degree $\leqslant k-1$, expression $(M_2^{-t}(\hat{\boldsymbol{p}} + \hat{\boldsymbol{s}}))(M_2^{-1}\boldsymbol{x} - M_2^{-1}\boldsymbol{x}_E)$ can be seen as $M_2^{-t}(\boldsymbol{p}(x_1, x_2) + \hat{\boldsymbol{s}}(M_2^{-1}\boldsymbol{x}))$ with $\boldsymbol{p}$ having the expected degrees. And now

$$M_2^{-t}\hat{\boldsymbol{s}}(M_2^{-1}\boldsymbol{x}) \cdot (x_1, x_2)' = \hat{\boldsymbol{s}}(M_2^{-1}(x_1, x_2)') \cdot M_2^{-1}(x_1, x_2)' = 0.$$

The invariance of the third component of elements in $P_{\hat{E}}$ is an immediate consequence of the affine nature of the coordinate change. □

Now we are stating definitions and properties for the finite element space in the case in which the prisms are not necessarily right, that is, the matrix of the affine transformation from the reference element needs no longer to be as in Lemma 2.3.3. We are calling these prisms *oblique*.

2.3.4 Definition. *Let an oblique prism $E = F_E(\hat{E})$ be given as the affine image under transformation* (2.26) *from $\hat{E}$. Recall the reference space $P_{\hat{E}}$ from* (2.12). *The local $H(\mathbf{curl})$ finite element space is defined as*

$$P_E = \{M_E^{-t}\hat{\boldsymbol{u}} \circ F_E^{-1} \,|\, \hat{\boldsymbol{u}} \in P_{\hat{E}}\}.$$

2.3.5 Lemma. *Let a physical oblique prism $E = F_E(\hat{E})$ for F_E as in (2.26). Given $\hat{\boldsymbol{u}}$ defined in $\hat{E}$ let $\boldsymbol{u}$ in E be determined by (2.29), and let $\hat{\boldsymbol{\tau}}$, $\boldsymbol{\tau}$, $\hat{\boldsymbol{n}}$ and $\boldsymbol{n}$ be tangents and normals to edges and faces satisfying $F_E(\hat{\boldsymbol{e}}) = \boldsymbol{e}$ and $F_E(\hat{f}) = f$ in $\hat{E}$ and E, and related to each other by (2.36) and (2.35). Suppose we define degrees of freedom of $\boldsymbol{u}$ on E as*

$$\varphi_{\boldsymbol{e}_i,p}(\boldsymbol{u}) = \int_{\boldsymbol{e}} q\,\boldsymbol{u} \cdot d\boldsymbol{\alpha} \qquad q \in P_{k-1}, \text{ for each edge } \boldsymbol{e}; \tag{2.40}$$

$$\varphi_{f,\boldsymbol{q}}(\boldsymbol{u}) = \int_f \boldsymbol{u} \cdot \boldsymbol{q}_0 \, dS, \qquad \boldsymbol{q}_0 = (\det M_E \|M_E^{-t}\hat{\boldsymbol{n}}\|)^{-1} M_E\, \hat{\boldsymbol{q}}_T, \quad \hat{\boldsymbol{q}}_T := (\hat{\boldsymbol{n}} \times \hat{\boldsymbol{q}}) \times \hat{\boldsymbol{n}}, \quad \hat{\boldsymbol{q}} \text{ as in (2.14)–(2.17)}. \tag{2.41}$$

$$\varphi_{\boldsymbol{r}}(\boldsymbol{u}) = \int_E \boldsymbol{u} \cdot \boldsymbol{r}\, d\boldsymbol{x}, \qquad \boldsymbol{r} = (\det M_E)^{-1} M_E\, \hat{\boldsymbol{r}} \circ F_E^{-1}, \quad \hat{\boldsymbol{r}} \in P_{k-2,k-2} \times P_{k-3,k-1}. \tag{2.42}$$

Provided $\det M_E > 0$, the degrees of freedom (2.40)–(2.42) are identical to the referential degrees of freedom (2.13)–(2.18).

2.3.6 Remark. *For this Lemma we don't need the matrix M_E to be as in Lemma 2.3.3, so it is more general. If the matrix M_E is not block diagonal as we stated before, the local space in a physical prism element does not necessarily have the same structure as in the reference prism. This is not a restriccion, since in our main Theorem and in our applications we needed only right prisms.*

Proof of Lemma 2.3.5. For the degrees of freedom over the edges take a parameterization $\hat{\boldsymbol{\alpha}}(t) : I = [0,1] \to \hat{\mathrm{e}}$ such that

$$\hat{\boldsymbol{\tau}} = \frac{\dot{\hat{\boldsymbol{\alpha}}}(t)}{\|\dot{\hat{\boldsymbol{\alpha}}}(t)\|}.$$

As a jacobian maps tangents into tangents, $\boldsymbol{\alpha}(t) = M_E\,\hat{\boldsymbol{\alpha}}(t) + \boldsymbol{x}_E$ parameterizes $\boldsymbol{e}$. Then

$$\begin{aligned}\int_{\boldsymbol{e}} (q\boldsymbol{u}) \cdot d\boldsymbol{\alpha} &= \int_I q\boldsymbol{u}(F\hat{\boldsymbol{\alpha}}(t))) \cdot M_E\,\dot{\hat{\boldsymbol{\alpha}}}(t)\,dt \\ &= \int_I \hat{q}(\hat{\boldsymbol{\alpha}}(t)) M_E^{-t}\hat{\boldsymbol{u}}(\hat{\boldsymbol{\alpha}}(t)) \cdot M_E\,\dot{\hat{\boldsymbol{\alpha}}}(t)\,dt \\ &= \int_{\hat{\mathrm{e}}} (\hat{q}\hat{\boldsymbol{u}}) \cdot d\hat{\boldsymbol{\alpha}}.\end{aligned}$$

Continuing with the surface degrees of freedom,

$$\int_{\hat{f}} \hat{\boldsymbol{v}} \times \hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{q}} \times \hat{\boldsymbol{n}}\, d\hat{S} = \int_{\hat{f}} (\hat{\boldsymbol{n}} \times \hat{\boldsymbol{q}}) \times \hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{u}}\, d\hat{S}$$
$$= \int_{\hat{f}} \det M_E \| M_E^{-t} \hat{\boldsymbol{n}} \| \boldsymbol{q}_0(F\hat{\boldsymbol{x}}) \cdot \boldsymbol{u}(F\hat{\boldsymbol{x}}) d\hat{S}$$
$$= \int_f \boldsymbol{q}_0(\boldsymbol{x}) \cdot \boldsymbol{u}(\boldsymbol{x}) dS$$

Finally, for the volume degrees of freedom,

$$\int_E \boldsymbol{v} \cdot \boldsymbol{r}\, d\boldsymbol{x} = \int_{\hat{E}} M_E^{-t} \hat{\boldsymbol{v}}(F_E^{-1}\boldsymbol{x}) \cdot (\det M_E)^{-1} M_E \hat{\boldsymbol{r}}(F_E^{-1}\boldsymbol{x})\, d\hat{\boldsymbol{x}}. \tag{2.43}$$
$$= \int_{\hat{E}} \hat{\boldsymbol{v}}(\hat{\boldsymbol{x}}) \cdot \hat{\boldsymbol{r}}(\hat{\boldsymbol{x}})\, d\hat{\boldsymbol{x}}. \tag{2.44}$$

□

In the following corollary we complete Theorem 8 in page 75 of [34] which has only the case of the reference element.

2.3.7 Corollary. *The finite element in Definition 2.1.8 determines a family of $H(\textbf{curl})$–conforming and unisolvent finite elements provided we use the degrees of freedom transformations (2.40), (2.41) and (2.42).*

Proof. Transform the degrees of freedom from Lemma 2.3.5 back to $\hat{E}$ and use unisolvence over the reference element. □

2.3.8 Corollary. *For any $\boldsymbol{u} \in W^{1,p}(E)$ expressions (2.40)–(2.42) determine a well defined k–th order local interpolate $\boldsymbol{w}_E\, \boldsymbol{u}$ defined as the unique finite element function in P_E such that*

$$\varphi_{\boldsymbol{e},q}\,(\boldsymbol{u} - \boldsymbol{w}_E\, \boldsymbol{u}) = 0 \quad \textit{for } \varphi_{\boldsymbol{e},q} \textit{ as in (2.40)} \tag{2.45}$$
$$\varphi_{f,\boldsymbol{q}}\,(\boldsymbol{u} - \boldsymbol{w}_E\, \boldsymbol{u}) = 0 \quad \textit{for } \varphi_{f,\boldsymbol{q}} \textit{ as in (2.41)} \tag{2.46}$$
$$\varphi_{\boldsymbol{r}}\,(\boldsymbol{u} - \boldsymbol{w}_E\, \boldsymbol{u}) = 0 \quad \textit{for } \varphi_{\boldsymbol{r}} \textit{ as in (2.42).} \tag{2.47}$$

This will be used in the proof of local interpolation estimates.

2.3.9 Lemma. *Provided $\det M^t > 0$, the edge element interpolators satisfy*

$$\boldsymbol{w}_{\hat{E}} \hat{\boldsymbol{u}}(\hat{\boldsymbol{x}}) = M^t \boldsymbol{w}_E \boldsymbol{u}(F_E(\hat{\boldsymbol{x}})) \tag{2.48}$$

That is, the interpolator commutes with the coordinate change (2.29).

Proof. By definition, for all degree of freeom φ in (2.40)–(2.42) it holds

$$\varphi(\boldsymbol{u} - \boldsymbol{w}_E \boldsymbol{u}) = 0.$$

By the invariance of degrees of freedom proved in Lemma 2.3.5, it follows

$$\hat{\varphi}\left((\boldsymbol{u} - \boldsymbol{w}_E \boldsymbol{u})\hat{}\right) = \hat{\varphi}\left(\hat{\boldsymbol{u}} - (\boldsymbol{w}_E \boldsymbol{u})\hat{}\right) = 0$$

which implies $\boldsymbol{w}_{\hat{E}} \hat{\boldsymbol{u}} = \hat{\boldsymbol{w}}_E (\boldsymbol{w}_E \boldsymbol{u})\hat{}$. The space $P_{\hat{E}}$ was proven to be invariant under (2.29), so $\boldsymbol{w}_{\hat{E}} \hat{\boldsymbol{u}} = (\boldsymbol{w}_E \boldsymbol{u})\hat{}$. □

In the following paragraph we make all the previous exposition for the divergence finite elements.

2.3.10 Lemma. *Let $\hat{E}$ be the reference prism (1.1.17) and let $P_{\hat{E}}$ be the space in (2.2). Provided the matrix M_E is block–diagonal as in Lemma 2.3.3, then $P_{\hat{E}}$ is invariant by transformation (2.33).*

Proof. The property follows with the same direct approach as Lemma 2.3.3 and in a much simpler case. □

2.3.11 Definition. *Let an oblique prism $E = F_E(\hat{E})$ be given as the affine image under transformation (2.26) from $\hat{E}$. Recall the reference space $P_{\hat{E}}$ from (2.2). The local $H(div)$ finite element space is defined as*

$$P_E = \left\{ \det M_E^{-1} M_E \hat{\boldsymbol{u}} \circ F_E^{-1} \,|\, \hat{\boldsymbol{u}} \in P_{\hat{E}} \right\}.$$

2.3.12 Lemma. *Let an oblique prism $E = F_E(\hat{E})$ for F_E as in (2.26). Given $\hat{\boldsymbol{u}} \in W^{1,1}(\hat{E})$ let $\boldsymbol{u}$ in E be determined by (2.33), and let $\hat{\boldsymbol{n}}$ and $\boldsymbol{n}$ be normals to faces satisfying $F_E(\hat{f}) = f$ in $\hat{E}$ and E, and related to each other by (2.35). Suppose we define degrees of freedom of $\boldsymbol{u}$ on E as*

$$\rho_{f,q}(\boldsymbol{v}) = \int_f (\boldsymbol{v}\cdot\boldsymbol{n}) q \, dS \quad \text{for } q = \hat{q}\circ F_E^{-1}, \hat{q} \in P_{k-1}(\hat{f}), \quad \text{if } \hat{f} = \hat{f}_3 \text{ or } \hat{f}_4; \tag{2.49}$$

$$\rho_{f,q}(\boldsymbol{v}) = \int_f (\boldsymbol{v}\cdot\boldsymbol{n}) q \, dS \quad \text{for } q = \hat{q}\circ F_E^{-1}, \hat{q} \in Q_{k-1,k-1}(\hat{f}), \quad \text{if } \hat{f} = \hat{f}_1, \hat{f}_2 \text{ or } \hat{f}_5; \tag{2.50}$$

$$\rho_{\boldsymbol{r}}(\boldsymbol{v}) = \int_E \boldsymbol{v}\cdot\boldsymbol{r}\, d\boldsymbol{x} \quad \text{for } \boldsymbol{r} = M_E^{-t}\hat{\boldsymbol{r}}\circ F_E^{-1}, \quad \hat{\boldsymbol{r}} \in (P_{k-2,k-1})^2 \times P_{k-1,k-2}. \tag{2.51}$$

Provided $\det M_E > 0$, the degrees of freedom (2.49)–(2.51) are identical to the referential degrees of freedom (2.3)–(2.6).

Proof. On one hand, by (2.33) and (2.37),

$$\begin{aligned}\int_f \boldsymbol{v}\cdot\boldsymbol{n}\boldsymbol{q}\, dS &= \int_f (\det M_E)^{-1}\hat{\boldsymbol{v}}(F_E^{-1}\boldsymbol{x})\cdot \|M_E^{-t}\hat{\boldsymbol{n}}\|^{-1}\hat{\boldsymbol{n}}\hat{\boldsymbol{q}}(F_E^{-1}\boldsymbol{x})\, dS \\ &= \int_{\hat{f}} \hat{\boldsymbol{v}}(\hat{\boldsymbol{x}})\cdot\hat{\boldsymbol{n}}\hat{\boldsymbol{q}}(\hat{\boldsymbol{x}})\, d\hat{S}.\end{aligned}$$

On the other hand,

$$\int_E \boldsymbol{v}\cdot\boldsymbol{r}\, d\boldsymbol{x} = \int_{\hat{E}} M_E\hat{\boldsymbol{v}}\cdot M_E^{-t}\hat{\boldsymbol{r}}\, d\hat{\boldsymbol{x}} = \int_{\hat{E}} \hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{r}}\, d\hat{\boldsymbol{x}}. \tag{2.52}$$

□

2.3.13 Lemma. *For a right prism E let P_E be the image of the space $P_{\hat{E}}$ of (2.2) under transformation (2.33) for a block diagonal matrix M_E as in Lemma 2.3.3. If $\boldsymbol{u} \in P_E$ is such that all the degrees of freedom (2.49) or (2.50) vanish on the respective face f, then the normal component of $\boldsymbol{u}$ vanishes identically on that face f.*

Proof. This fact is stated in the proof of Theorem 4, page 66 of [34] only for the reference prism. By transforming the degrees of freedom to the faces of $\hat{E}$ and back to the faces of E using Lemma 2.3.12 we get the result. □

2.3.14 Lemma. *Let P_E be the space of Lemma 2.3.13. If $\boldsymbol{u} \in P_E$ is such that all the degrees of freedom (2.49)–(2.51) vanish, then $\boldsymbol{u}$ vanishes identically on E.*

Proof. In the case of the reference prism $\hat{E}$, we refer to the proof of Theorem 4 in page 66 of [34]. For the general case, by the invariance of the finite element space P_E under the transformation (2.33) (see Lemma 2.3.10) we have our result. □

2.3.15 Corollary. *The finite element in Definition 2.1.1 determines a family of $H(div)$–conforming and unisolvent finite elements provided we use the degrees of freedom transformations (2.49), (2.50) and (2.51).*

2.3.16 Corollary. *For any $\boldsymbol{v} \in W^{1,1}(E)$ expressions (2.49)–(2.51) determine a well defined local interpolate $\boldsymbol{r}_E\,\boldsymbol{u}$ defined as the unique finite element function in P_E such that*

$$\rho_{f,\boldsymbol{q}}\,(\boldsymbol{v} - \boldsymbol{r}_E\,\boldsymbol{u}) = 0 \quad \text{for } \rho_{f,\boldsymbol{q}} \text{ as in (2.49) and (2.50)} \tag{2.53}$$

$$\rho_{\boldsymbol{r}}\,(\boldsymbol{v} - \boldsymbol{r}_E\,\boldsymbol{u}) = 0 \quad \text{for } \rho_{\boldsymbol{r}} \text{ as in (2.51).} \tag{2.54}$$

The proof of the following important corollary is like the one of (2.48).

2.3.17 Corollary. *Given $\boldsymbol{u} \in W^{1,1}(E)$, then*

$$\boldsymbol{r}_{\hat{E}}\hat{\boldsymbol{u}} = \det M_E\,(M_E^{-1}\,\boldsymbol{r}_E\,\boldsymbol{u}) \circ F_E, \tag{2.55}$$

that is, the $H(div)$ interpolator commutes with the coordinate change (2.33).

2.3.18 Remark. *Unisolvence and conformity of the finite elements on tetrahedra mentioned in Section 2.2 are proved in Sections 5.4 and 5.5 of [33].*

Chapter 3

Virtual Elements

Introducción al capítulo

Los Elementos Virtuales son definidos en términos de un dado problema. Es nestra intención proponer un esquema para un método combinado de elementos Finitos/Virtuales como una generalización de los Elementos Finitos $H(\text{div})$–conformes en mallas consistentes en poliedros de geometría arbitraria. En este Capítulo presentamos un desarrollo de los espacios, las formas bilineales y las propiedades concernientes a los elementos virtuales. Trabajamos con tetraedros, prismas triangulares y pirámides y, en presencia de estas últimas, nuestro esquema FEM/VEM se ubica en el marco de los Elementos Finitos no–polinomiales.

El método de elementos virtuales (VEM) fue introducido recientemente [9] como una generalización de elementos H^1–conformes para elementos de geometría arbitraria y como una generalización de las Diferencias Finitas Miméticas a grado arbitrario de precisión. Una extensión de la discretización de campos vectoriales $H(\text{div})$–conformes y aproximaciones por elementos finitos mixtos fue propuesta en [15] en dimensión 2. Además, en [10] un VEM mixto fue analizado para la aproximación de problemas lineales elípticos con coeficientes variables.

Un espacio de elementos virtuales puede contener funciones no *polinomiales a trozos* y principalmente contiene funciones que son desconocidas y no pueden ser evaluadas en ningún punto. En el VEM los espacios y los grados de libertad son tomados de tal manera que la matriz de rigidez elemental puede ser computada sin conocer estas funciones no polinomiales, en nuestro caso gracias al Teorema de la Divergencia.

Introduction to the chapter

The Virtual Elements are customarly defined in terms of a given problem. Our intention is to propose a combined Finite/Virtual Elements Method scheme as a generalization of $H(\text{div})$–conforming Finite Elements in meshes consisting of polyhedra of arbitrary kind. In this Chapter we develop the spaces, bilinear forms and properties concerning virtual elements. We deal with tetrahedra, triangular prisms and pyramids and, in the presence of the latter, our FEM/VEM scheme is put in the framework of non–polynomial Finite Elements.

The virtual element method (VEM) has been recently introduced [9] as a generalization of H^1-conforming finite elements to arbitrary element–geometry and as a generalization of Mimetic Finite Differences to arbitrary degree of accuracy and arbitrary continuity properties. An extension to the discretization of $H(\text{div})$–conforming vector fields and mixed finite element approximations has been proposed in [15] in the two dimensional case. Furthermore, in [10] a mixed VEM has been analysed for the approximation of general linear elliptic problems with variable coefficients. The virtual element space can contain non piecewise polynomial functions and mainly functions which are a priori unknown, in the sense that they can't be evaluated in any point. In the VEM approach, the space and the degrees of freedom are taken in such a way that the elementary stiffness matrix can be computed without actually computing these non–polynomial functions, but just using the degrees of freedom. In this respect, a key point in this approach and particularly in our case is that, given an element E, if $\boldsymbol{u} = \nabla q_2$ for a known polynomial q_2, then for a field $\boldsymbol{v}$ the quantity

$$\int_E \boldsymbol{u} \cdot \boldsymbol{v} \, d\boldsymbol{x}$$

can be computed if $\operatorname{div} \boldsymbol{v}$ on E and the outer normal component $\boldsymbol{v} \cdot \mathbf{n}$ of $\boldsymbol{v}$ on ∂E are known polynomials, by using the divergence Theorem.

3.1 Definition and Construction

First we recall Problem 1.2.2 and add some notation for the bilinear forms. Let $V := H(\text{div}, \Omega)$, $Q := L^2(\Omega)$ and

$$a(\boldsymbol{v}, \boldsymbol{w}) = \int_\Omega \boldsymbol{v} \cdot \boldsymbol{w} \, d\boldsymbol{x} \quad \text{and} \quad b(\boldsymbol{v}, q) = \int_\Omega q \operatorname{div} \boldsymbol{v} \, d\boldsymbol{x}.$$

Then Problem 1.2.2 reads to find $\boldsymbol{u} \in V$ and $p \in Q$ such that for every $\boldsymbol{v} \in V$ and every $q \in Q$

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) &= 0 \\ b(\boldsymbol{u}, q) &= (-f, q). \end{aligned}$$

We start supposing we are given with a conforming polihedral triangulation $\mathcal{T}_h$ of Ω to define the virtual spaces V_h and Q_h as discretizations of V and Q respectively. For now we will suppose the aspect ratios of tetrahedra and pyramids are bounded in terms of a constant independent of h. However, in Chapter 5 we will construct a mesh using the three types of polihedra mentioned before, and we will see that the last assumtion is not a restriction. We will assume a strictly decreasing parameter h tending to zero to mitigate the abuse of notation present in expressions like "$\boldsymbol{u}_h$" and "$\mathcal{T}_h$". Then we are able to write, for example, $\boldsymbol{u}_h$ or $\boldsymbol{u}_{\mathcal{T}_h}$ indistinctly.

For $E \in \mathcal{T}_h$ the local space of virtual vector fields will be

$$V_h(E) = \Big\{ \boldsymbol{v} \in H(\mathrm{div}, E) \cap H(\mathbf{curl}, E) \ : \\ \boldsymbol{v} \cdot \boldsymbol{n}|_f \in P_0(f) \text{ for all face } f \text{ of } E, \\ \mathrm{div}\, \boldsymbol{v} \in P_0(E) \text{ and } \mathbf{curl}\, \boldsymbol{v} = 0 \Big\} \tag{3.1}$$

and the global space V_h will consist of functions defined piecewise with the former local spaces:

$$V_h{=}V_h(\mathcal{T}_h) := \Big\{ \boldsymbol{v} \in H(\mathrm{div}, \Omega) : \boldsymbol{v}|_E \in V_h(E), \text{for all element } E \in \mathcal{T}_h \Big\}.$$

The condition on the **curl** is put because we are considering gradients, as we will see later. The scalar discrete space we will consider is

$$Q_h = P_0(\mathcal{T}_h) \tag{3.2}$$

meaning the functions that are constant on each element of $\mathcal{T}_h$. As expected, with V_h we consider the $H(\mathrm{div}, \Omega)$ norm, $\|\boldsymbol{v}\|^2_{V_h} = \|\boldsymbol{v}\|^2_{L^2(\Omega)} + \|\mathrm{div}\, \boldsymbol{v}\|^2_{L^2(\Omega)}$, and with Q_h we consider the $L^2(\Omega)$ norm. As Problem 1.2.2 has solution in $H(\mathrm{div}, \Omega)$ it suggests to use a face–like operator as virtual interpolator. For that reason, let the degrees of freedom be

$$\iint_f \boldsymbol{v} \cdot \boldsymbol{n} \, dS \qquad \text{for all face } f \text{ of } \mathcal{T}_h. \tag{3.3}$$

With *faces of* $\mathcal{T}_h$ we mean the family of all faces forming the boundary of the elements, with $\boldsymbol{n}$ being a unit normal vector, chosen for each face among the two possibilities in the case of neighboring elements.

Something that has been already established for Finite Elements in Chapter 2 must be proved for our Virtual Elements.

3.1.1 Lemma. *Given a polyhedron $E \in \mathcal{T}_h$, the degrees of freedom (3.3) corresponding to the faces of E are unisolvent in $V_h(E)$.*

Proof. Existence. Let $n_{f,E}$ be the number of faces of E and take real numbers $\{\alpha_i\}_{i=1}^{n_{f,E}}$. Let g be the piecewise constant function on ∂E satisfying, for all $1 \leqslant i \leqslant n_{f,E}$,

$$\iint_{f_i} g \, dS = \alpha_i$$

and let d be the constant function in E such that

$$\int_E d \, d\boldsymbol{x} = \sum_i \alpha_i.$$

As this proof is performed locally on a fixed element E, we can fix the degrees of freedom in (3.3) so that the normal vector points outward. Then we consider the auxiliary problem of seeking a solution of

$$\Delta\phi = d \quad \text{in } E, \qquad \frac{\partial\phi}{\partial\boldsymbol{n}} = g \quad \text{on } \partial E. \tag{3.4}$$

By definition we obtain the compatibility condition

$$\int_E d\,d\boldsymbol{x} = \iint_{\partial E} g\,dS$$

so the solution ϕ to the problem (3.4) exists. Now we take $\boldsymbol{u} := \nabla\phi$ and it holds immediately that $\operatorname{div}\boldsymbol{u}$ is constant in E, $\boldsymbol{u}\cdot\boldsymbol{n}$ is constant on each face of ∂E and $\mathbf{curl}\,\boldsymbol{u} = 0$. So $\boldsymbol{u}$ lies in $V_h(E)$ and also for all $1 \leqslant i \leqslant n_{f,E}$ $\iint_{f_i} \boldsymbol{u}\cdot\boldsymbol{n}\,dS = \alpha_i$.

Uniqueness. Suppose that $\boldsymbol{v} \in V_h(E)$ has vanishing degrees of freedom. Condition $\mathbf{curl}\,\boldsymbol{v} = 0$ implies $\boldsymbol{v} = \nabla\phi$ for certain ϕ. Now, since $\operatorname{div}\boldsymbol{v}$ is constant on E, the relation

$$0 = \int_{\partial P} \boldsymbol{v}\cdot\boldsymbol{n}\,dS$$

implies $\operatorname{div}\boldsymbol{v} = 0$ by Green Theorem. Then, the potential ϕ satisfies

$$\Delta\phi = 0 \quad \text{in } E, \qquad \frac{\partial\phi}{\partial\boldsymbol{n}} = 0 \quad \text{on } \partial E$$

which means it is a constant, and it follows $\boldsymbol{v} = 0$. □

With this Lemma already demonstrated we are able to consider an $H(\mathrm{div})$–like local interpolation operator well defined.

3.1.2 Corollary. *For every $\boldsymbol{v} \in H^1(E)^3$ there exists a $V_h(E)$–interpolant $I\boldsymbol{v}$ defined as the unique element in $V_h(E)$ such that for every face f of E*

$$\iint_f I\boldsymbol{v}\cdot\boldsymbol{n}\,dS = \iint_f \boldsymbol{v}\cdot\boldsymbol{n}\,dS.$$

3.1.3 Lemma. *Consider the projection P_0 onto the constants on E. It holds*

$$\mathit{div}\,I\boldsymbol{v} = P_0\,\mathit{div}\,\boldsymbol{v}.$$

Proof.

$$\begin{aligned}\int_E (\operatorname{div} I\boldsymbol{v} - \operatorname{div}\boldsymbol{v})\,d\boldsymbol{x} &= \int_E \operatorname{div}(I\boldsymbol{v} - \boldsymbol{v})\,d\boldsymbol{x}\\ &= \iint_{\partial E} (I\boldsymbol{v} - \boldsymbol{v})\cdot\boldsymbol{n}\,dS\\ &= \sum_{f\subseteq\partial E} \iint_f (I\boldsymbol{v} - \boldsymbol{v})\cdot\boldsymbol{n}\,dS = 0\end{aligned}$$

□

3.1.4 Proposition. *Recall the space D_k introduced in (2.1). Then*

1. *if E is a right prism, we have*

$$V_h(E) = D_1(x, y) \times P_1(z), \tag{3.5}$$

with x, y, z being the variables in a cartesian system of coordinates in which the z–axis is orthogonal to the planes containing the triangular faces of E.

2. *If E is a tetrahedron, we have*

$$V_h(E) = P_0^3(\boldsymbol{x}) + P_0\boldsymbol{x}, \tag{3.6}$$

with $\boldsymbol{x} = (x, y, z)'$ being the vector of variables in a cartesian system of coordinates.

Proof. In the case of a prism E, the space $D_1(x, y) \otimes P_1(z)$ can be written as

$$\{\boldsymbol{v} \in P_1(E) \, : \, \boldsymbol{v} = (a + \gamma x, b + \gamma y, c + dz), a, b, c, d, \gamma \in \mathbb{R})\}$$

and then we see immediately that it has the same dimension as $V_h(E)$ which has dimension five. Finally, recalling that the div and **curl** operators can be computed in the chosen local variables, given $\boldsymbol{v} \in D_1(x, y) \otimes P_1(z)$ we can compute $\boldsymbol{v}|_f \cdot \boldsymbol{n}_f$ (on each face f of E), div $\boldsymbol{v}$ and **curl** $\boldsymbol{v}$ to verify explicitly that the space on the right side of (3.5) fulfills the definition of $V_h(E)$ given in (3.1). Exactlty the same argument works for the case of a tetrahedron E, in which we deal with two vectorial spaces of dimension four that are such that every element in

$$P_0^3(\boldsymbol{x}) + P_0\boldsymbol{x} = \{\boldsymbol{v} \in P_1(E) \, : \, \boldsymbol{v} = (a + \gamma x, b + \gamma y, c + \gamma z)', a, b, c, \gamma \in \mathbb{R}\}$$

fulfills the conditions defining $V_h(E)$. □

3.1.5 Remark. *By Proposition 3.1.4 the spaces $V_h(E)$ coincide with the lowest order $H(div)$–conforming local Finite Element spaces introduced in (2.2) for the prismatic case and in (2.24) for the tetrahedral case.*

3.1.6 Lemma. *The definition of the lowest order $H(div)$ Finite Elements on right prisms or tetrahedra is independent of the choice of the cartesian axes (as long as the z-axis is perpendicular to the trianguar faces in case of prisms).*

Proof. This can be proved by hand. Let us prove the case of a prism E. Let $(x, y, z)'$ and $(x', y', z')'$ be two cartesian coordinate systems satisfying the required properties. For them there is a change of coordinates such as

$$\begin{aligned} x &= p + \alpha x' - \beta y' \\ y &= q + \beta x' - \alpha y' \\ z &= r + z' \end{aligned}$$

for α and β satisfying $\alpha^2 + \beta^2 = 1$. Let $\boldsymbol{v}$ be an element in $V_h(E)$. So

$$\begin{aligned}\boldsymbol{v}(x, y, z) &= (a + \gamma x, b + \gamma y, c + dz)' \\ &= ((a + \gamma p) + \gamma(\alpha x' - \beta y'), \\ &\quad\ (b + \gamma q) + \gamma(\beta x' + \alpha y'), \\ &\quad\ (c + dr) + dz')'.\end{aligned}$$

Then, the components of $\boldsymbol{v}$ in the new coordinate versors are

$$\begin{cases} \boldsymbol{v} \cdot (\alpha, \beta, 0)' = (\alpha a + \beta b + \gamma(\alpha p + \beta q)) + \gamma x' =: a' + \gamma x' \\ \boldsymbol{v} \cdot (-\beta, \alpha, 0)' = (-\beta a + \alpha b + \gamma(-\beta p + \alpha q)) + \gamma y' =: b' + \gamma y' \\ \boldsymbol{v} \cdot (0, 0, 1)' = (c + dr) + dz' =: c' + dz'. \end{cases}$$

It follows that, in the $x'y'z'$ system,

$$\boldsymbol{v}(x', y', z') = (a' + \gamma x', b' + \gamma y', c' + dz')' \in D_1(x', y') \otimes P_1(z').$$

□

Next, together with the finite dimensional spaces already defined, we present discretized version of the bilinear forms. The evaluation of the form $b(\cdot, \cdot)$ at a pair $(\boldsymbol{v}, q) \in V_h \times Q_h$ can be computed using the degrees of freedom (3.3) applied to $\boldsymbol{v}$. For if $q \in Q_h$, then

$$b(\boldsymbol{v}, q) = \int_\Omega q \operatorname{div} \boldsymbol{v} \, d\boldsymbol{x} = \sum_{E \in \mathcal{T}_h} \int_E q \operatorname{div} \boldsymbol{v} \, d\boldsymbol{x} = \sum_{E \in \mathcal{T}_h} \iint_{\partial E} q \boldsymbol{v} \cdot \boldsymbol{n} \, dS.$$

So in the case of the form $b_h(\cdot, \cdot)$ we put simply

$$b_h(\boldsymbol{v}, q) = b(\boldsymbol{v}, q) = \sum_{E \in \mathcal{T}_h} b^E(\boldsymbol{v}, q), \tag{3.7}$$

where $b^E(\boldsymbol{v}, q) = \int_E q \operatorname{div} \boldsymbol{v} \, d\boldsymbol{x}$. For the bilinear form $a(\cdot, \cdot)$, we can decompose it as

$$a(\boldsymbol{v}, \boldsymbol{w}) = \sum_{E \in \mathcal{T}_h} \int_E \boldsymbol{v} \cdot \boldsymbol{w} \, d\boldsymbol{x}$$

but this is not computable in terms of the degrees of freedom. The following construction is done in order to get a calculable discrete form and also we will introduce a term for the stability.

For each element $E \in \mathcal{T}_h$, let the space $W(E)$ be defined by

$$\begin{aligned}W(E) &= \{\boldsymbol{w} \in V_h(E) : \boldsymbol{w} = \nabla q_2, \text{ for some } q_2 \in P_2(E)\} \\ &= V_h(E) \cap \nabla P_2(E),\end{aligned}$$

and with these we consider $W(\mathcal{T}_h) = \{\boldsymbol{w} : \boldsymbol{w}|_E \in W(E) \text{ for each } E \in \mathcal{T}_h\}$. Inspecting arbitrary elements in (3.5) and (3.6) yields the following result.

3.1.7 Lemma. *When $E \in \mathcal{T}_h$ is a tetrahedron or a prism, then $W(E) = V_h(E)$.*

Again, observe that if $\boldsymbol{v} \in V_h(E)$ and $\boldsymbol{w} \in W(E)$, $a^E(\boldsymbol{w}, \boldsymbol{v})$ can be computed using the degrees of freedom of $\boldsymbol{v}$, because it holds

$$\begin{aligned} a^E(\boldsymbol{w}, \boldsymbol{v}) := \quad & \int_E \boldsymbol{w} \cdot \boldsymbol{v}\, d\boldsymbol{x} \quad = \int_E \nabla q_2 \cdot \boldsymbol{v}\, d\boldsymbol{x} \\ = \quad & - \int_E q_2 \mathrm{div}\, \boldsymbol{v}\, d\boldsymbol{x} + \iint_{\partial E} q_2 \boldsymbol{v} \cdot \boldsymbol{n}\, dS. \end{aligned}$$

This means we can consider an auxiliary projection operator π^E from $H(\mathrm{div}, E)$ onto $W(E)$ defined by

$$a^E(\boldsymbol{v} - \pi^E \boldsymbol{v}, \boldsymbol{w}) = 0 \qquad \text{for all } \boldsymbol{w} \in W(E) \tag{3.8}$$

fully computable in terms of the degrees of freedom of $\boldsymbol{v}$.

We need the following last object to complete the discretization of $a(\cdot,\cdot)$. Thanks to Lemma 3.1.1 we can consider a basis $B = \{\boldsymbol{v}_{i,E}\}$ of $V_h(E)$ dual to the functionals (3.3), that is, for every $1 \leqslant i, j \leqslant n_{f,E}$

$$\iint_{f_j} \boldsymbol{v}_{i,E} \cdot \boldsymbol{n}_j\, dS = \delta_{i,j}. \tag{3.9}$$

What we are considering is the inner product $\langle(\boldsymbol{v})_B, (\boldsymbol{w})_B\rangle$ of the coordinates of the fields $\boldsymbol{v}, \boldsymbol{w} \in V_h(E)$ with respect to this local dual basis B.

Finally, the local discrete form a_h^E is stated in the following definition.

3.1.8 Definition. *Given an element $E \in \mathcal{T}_h$, for $\boldsymbol{v}$ and $\boldsymbol{w}$ in $V_h(E)$*

$$a_h^E(\boldsymbol{v}, \boldsymbol{w}) := a^E(\pi^E \boldsymbol{v}, \pi^E \boldsymbol{w}) + h_E^{-1} \langle(\boldsymbol{v} - \pi^E \boldsymbol{v})_B, (\boldsymbol{w} - \pi^E \boldsymbol{w})_B\rangle \tag{3.10}$$

where h_E is the diameter of E. The discrete global form a_h is defined by the following equation.

$$a_h(\boldsymbol{v}, \boldsymbol{w}) = \sum_{E \in \mathcal{T}_h} a_h^E(\boldsymbol{v}|_E, \boldsymbol{w}|_E), \qquad \textit{for } \boldsymbol{v} \textit{ and } \boldsymbol{w} \in V_h. \tag{3.11}$$

One fact to note about the projection π^E in (3.8) is that, by Proposition 3.1.4, it coincides with the identity $I : W(E) \to W(E)$ whenever the element $E \in \mathcal{T}_h$ is a Prism or a Tetrahedron so that, immediately, we have the following property.

3.1.9 Remark. *If E is a tetrahedron or a prism, then $a_h^E(\boldsymbol{v}, \boldsymbol{w}) = a^E(\boldsymbol{v}, \boldsymbol{w})$ for all $\boldsymbol{v}, \boldsymbol{w} \in V_h(E)$.*

Now we state a key result concerning the stability of the discrete form a_h^E and which explains the form of the term $h_E^{-1}\langle\,\cdot\,,\cdot\,\rangle$ in (3.10). In what follows we analyse this term in expression (3.10) when E is a pyramid. Let $B_E = \{\boldsymbol{v}_i \,:\, 1 \leqslant i \leqslant 5\}$

denote the dual basis of the degrees of freedom (3.3) and let f_j, $1 \leqslant j \leqslant 5$, denote the faces of E. Then $\boldsymbol{v} \in V_h(E)$ has a unique expression as

$$\boldsymbol{v} = \sum_{i=1}^{5} a_i \boldsymbol{v}_i = \sum_{i=1}^{5} \left\{ \iint_{f_i} \boldsymbol{v} \cdot \boldsymbol{n} \, dS \right\} \boldsymbol{v}_i. \tag{3.12}$$

Let $|||\cdot|||_E$ denote the following norm on $V_h(E)$

$$|||\boldsymbol{v}|||_E^2 := \frac{1}{h_E} \sum_{i=1}^{5} a_i^2. \tag{3.13}$$

Since the space is finite dimensional, for certain constants c_E and C_E we have

$$c_E \|\boldsymbol{v}\|_{L^2(E)} \leqslant |||\boldsymbol{v}|||_E \leqslant C_E \|\boldsymbol{v}\|_{L^2(E)}, \tag{3.14}$$

for all $\boldsymbol{v} \in V_h(E)$. Now the purpose of the next Proposition is to prove that C_E and c_E can be taken depending only on the aspect ratio of E.

3.1.10 Proposition. *Let E be a pyramid, and consider the basis B_E of $V_h(E)$ used in (3.12), and the associated discrete norm $|||\cdot|||_E$ introduced in (3.13). Then there exist constants C_E and c_E depending only on the aspect ratio of E such that (3.14) holds true for all $\boldsymbol{v} \in V_h$.*

Proof. First we note that if $\boldsymbol{v} \in V_h(E)$ is given by $\boldsymbol{v} = \sum_{i=1}^{5} a_i \boldsymbol{v}_i$ then there is a function ϕ of zero mean over E for which $\boldsymbol{v} = \nabla\phi$ and also $\Delta\phi = d$ on E, $\frac{\partial \phi}{\partial \boldsymbol{n}} = g$ on ∂E for

$$g|_{f_i} = \frac{a_i}{|f_i|}, \quad 1 \leqslant i \leqslant 5, \qquad |E| \, d = \sum_{i=1}^{5} a_i. \tag{3.15}$$

Given $q \in H^1(E)$, integrating by parts $\Delta\phi$ times q gives

$$\int_E \nabla\phi \cdot \nabla q \, d\boldsymbol{x} = -\int_E dq \, d\boldsymbol{x} + \int_{\partial E} gq \, dS.$$

Let $\overline{q}$ denote the mean of q over E. By the last expression, as d is constant, we have

$$\begin{aligned} \|\boldsymbol{v}\|_{L^2(E)} = \|\nabla\phi\|_{L^2(E)} &= \sup_{\substack{q \in H^1(E), \|\nabla q\|_{0,E}=1 \\ \overline{q}=0}} \int_E \nabla\phi \cdot \nabla q \, d\boldsymbol{x} \\ &= \sup_{\substack{q \in H^1(E), \|\nabla q\|_{0,E}=1 \\ \overline{q}=0}} \int_{\partial E} gq \, dS. \end{aligned} \tag{3.16}$$

As a consequence of the definition of d and g in (3.15), we have obtained an expression of the norm $\|\boldsymbol{v}\|_{L^2(E)}$ in terms of the coefficients of $\boldsymbol{v}$ with respecto to basis B_E. Now let $\hat{E}$ be a fixed reference pyramid, and let $F : \hat{E} \to E$ be an affine mapping from $\hat{E}$ onto E (remember that only pyramids with parallelogram basis are considered), which can be written as

$$\boldsymbol{x} = F(\hat{\boldsymbol{x}}) = M_E\, \hat{\boldsymbol{x}} + \mathbf{c}.$$

Given $q \in H^1(E)$ let $\hat{q} := q \circ F$. There exists constants c_0 and c_1 depending only on the aspect ratio of E such that

$$\frac{c_0}{h_E}\|\nabla q\|^2_{L^2(E)} \leqslant \|\nabla \hat{q}\|^2_{L^2(\hat{E})} \leqslant \frac{c_1}{h_E}\|\nabla q\|^2_{L^2(E)}, \tag{3.17}$$

and, on the other hand, it holds

$$\int_{\hat{E}} \hat{q}\, d\hat{\boldsymbol{x}} = 0 \qquad \Longleftrightarrow \qquad \int_E q\, d\boldsymbol{x} = 0.$$

Now setting $|J(\hat{\boldsymbol{x}})|_{f_i}| = |f_i|/|\hat{f}_i| \sim h_E^2$ we have

$$\int_{\partial E} gq\, dS = \|\nabla \hat{q}\|_{0,E} \left(\int_{\partial \hat{E}} \frac{\hat{g}\, \hat{q}\, |J|}{\|\nabla \hat{q}\|_{0,\hat{E}}}\, d\hat{S} \right).$$

It follows that

$$\|\boldsymbol{v}\|_{L^2(E)} = \sup \left\{ \|\nabla \hat{q}\|_{L^2(\hat{E})} \left(\int_{\partial \hat{E}} \frac{g|J|\hat{q}}{\|\nabla \hat{q}\|_{L^2(\hat{E})}} \right) : \right.$$
$$\left. q \in H^1(E),\ \|\nabla q\|_{L^2(E)} = 1,\ \overline{q} = 0 \right\},$$

and taking (3.17) into account we obtain

$$\|\boldsymbol{v}\|_{L^2(E)} \leqslant c_1^{1/2} h_E^{-1/2} \sup \left\{ \int_{\partial \hat{E}} g|J|\hat{q}\, d\hat{S} : \right.$$
$$\left. \hat{q} \in H^1(\hat{E}), \|\nabla \hat{q}\|_{L^2(\hat{E})} = 1, \overline{q} = 0 \right\} \tag{3.18}$$

and

$$\|\boldsymbol{v}\|_{L^2(E)} \geqslant c_0^{1/2} h_E^{-1/2} \sup \left\{ \int_{\partial \hat{E}} g|J|\hat{q}\, d\hat{S} : \right.$$
$$\left. \hat{q} \in H^1(\hat{E}), \|\nabla \hat{q}\|_{L^2(\hat{E})} = 1, \int_{\hat{E}} \hat{q}\, d\hat{\boldsymbol{x}} = 0 \right\}. \tag{3.19}$$

We remark that

$$\int_{\hat{E}} d|\det M_E|\, d\hat{\boldsymbol{x}} = \int_{\partial \hat{E}} g|J|\, d\hat{S}.$$

Now, let $\{\hat{\boldsymbol{v}}_i\}$ be the basis of $V_h(\hat{E})$ which is dual to the degrees of freedom over the faces of $\hat{E}$, and let

$$\hat{a}_i = g|_{f_i}\,|J|_{f_i}|\,|\hat{f}_i|, \qquad 1 \leqslant i \leqslant 5. \tag{3.20}$$

Take $\hat{\boldsymbol{v}} = \sum_{i=1}^{5} \hat{a}_i \hat{\boldsymbol{v}}_i$. From (3.14) applied to the pyramid $\hat{E}$ we know that

$$c_{\hat{E}}^2 \|\hat{\boldsymbol{v}}\|_{L^2(\hat{E})}^2 \leqslant \frac{1}{h_{\hat{E}}} \sum_{i=1}^{5} \hat{a}_i^2 \leqslant C_{\hat{E}}^2 \|\hat{\boldsymbol{v}}\|_{L^2(\hat{E})}^2, \tag{3.21}$$

and using (3.16) for $\hat{E}$ instead of E we have

$$\|\hat{\boldsymbol{v}}\|_{L^2(\hat{E})} = \sup_{\substack{\hat{q}\in H^1(\hat{E}), \|\nabla \hat{q}\|_{L^2(\hat{E})}=1 \\ \int_{\hat{E}} \hat{q}=0}} \int_{\partial \hat{E}} g|J|\hat{q}\, d\hat{S}.$$

It follows from (3.21) that

$$\left(\frac{1}{h_{\hat{E}}} \sum_{i=1}^{5} \hat{a}_i^2\right)^{\frac{1}{2}} \sim \sup_{\substack{\hat{q}\in H^1(\hat{E}), \|\nabla \hat{q}\|_{L^2(\hat{E})}=1 \\ \int_{\hat{E}} \hat{q}=0}} \int_{\partial \hat{E}} g|J|\hat{q}\, d\hat{S}, \tag{3.22}$$

where the constants in this equivalence depend only on the aspect ratio of E and so, since $\hat{E}$ can be considered with $h_{\hat{E}} \sim 1$, equation (3.22) together with (3.18) and (3.19) give

$$\frac{1}{h_E} \sum_{i=1}^{5} \hat{a}_i^2 \sim \|\boldsymbol{v}\|_{L^2(E)}^2.$$

Using (3.15) and (3.20),

$$\hat{a}_i = a_i \frac{|J|_{f_i}|}{|f_i|} |\hat{f}_i| \sim a_i, \qquad 1 \leqslant i \leqslant 5,$$

and then

$$\frac{1}{h_E} \sum_{i=1}^{5} a_i^2 \sim \|\boldsymbol{v}\|_{L^2(E)}^2$$

as we wanted, since the constants in this equivalence also depend only on the aspect ratio of E. □

Proposition (3.1.10) yields the next corollary.

3.1.11 Corollary. *For all $\boldsymbol{v} \in V_h(E)$ and all pyramidal $E \in \mathcal{T}_h$ it holds*

$$c_E\, a^E(\boldsymbol{v}, \boldsymbol{v}) \leqslant h_E^{-1} \langle (\boldsymbol{v})_B, (\boldsymbol{v})_B \rangle \leqslant C_E\, a^E(\boldsymbol{v}, \boldsymbol{v})$$

where c_E and C_E depend only on the shape regularity of E.

3.2 Discrete Problem

The discrete problem in the Finite–Virtual Element scheme is stated as follows.

3.2.1 Problem. *To find* $\boldsymbol{u}_{\mathcal{T}_h} \in V_{\mathcal{T}_h}$ *and* $p_{\mathcal{T}_h} \in Q_{\mathcal{T}_h}$ *such that* $\forall \boldsymbol{v} \in V_{\mathcal{T}_h} \forall q \in Q_{\mathcal{T}_h}$

$$\begin{aligned} a_{\mathcal{T}_h}(\boldsymbol{u}_{\mathcal{T}_h}, \boldsymbol{v}) + b_{\mathcal{T}_h}(\boldsymbol{v}, p_{\mathcal{T}_h}) &= 0 \\ - b_{\mathcal{T}_h}(\boldsymbol{u}_{\mathcal{T}_h}, q) &= (f, q). \end{aligned}$$

In Section 5.1 we will be concerned in proving existence and uniqueness for Problem 3.2.1. It will be done showing coercivity for one of the bilinear forms and a discrete version of the inf–sup condition for the other.

3.2.2 Lemma. *For all* $E \in \mathcal{T}_h$*, all* $\boldsymbol{w} \in W(E)$ *and all* $\boldsymbol{v} \in V_h(E)$

$$a_h^E(\boldsymbol{w}, \boldsymbol{v}) = a^E(\boldsymbol{w}, \boldsymbol{v}) \tag{3.23}$$

$$c_E\, a^E(\boldsymbol{v}, \boldsymbol{v}) \leqslant a_h^E(\boldsymbol{v}, \boldsymbol{v}) \leqslant C_E\, a^E(\boldsymbol{v}, \boldsymbol{v}) \tag{3.24}$$

with $c_E = C_E = 1$ *when* E *is tetrahedral or prismatic and* c_E, C_E *depending only on the shape regularity of* E *when* E *is pyramidal.*

Proof. The relation $\boldsymbol{w} = \pi^E \boldsymbol{w}$ for all $\boldsymbol{w} \in W(E)$, condition (3.8) and the symmetry of a^E imply

$$a_h^E(\boldsymbol{w}, \boldsymbol{v}) = a^E(\pi^E \boldsymbol{w}, \pi^E \boldsymbol{v}) = a^E(\pi^E \boldsymbol{w}, \boldsymbol{v}) = a^E(\boldsymbol{w}, \boldsymbol{v})$$

which proves (3.23). To prove (3.24), if E is not a Pyramid this property is already a consequence of Remark 3.1.9. In the case of a Pyramid, firstly a simple computation yields

$$a^E(\boldsymbol{v}, \boldsymbol{v}) = a^E(\boldsymbol{v} - \pi^E \boldsymbol{v}, \boldsymbol{v} - \pi^E \boldsymbol{v}) + a^E(\pi^E \boldsymbol{v}, \pi^E \boldsymbol{v}). \tag{3.25}$$

Corollary 3.1.11 and (3.25) give

$$\begin{aligned} a_h^E(\boldsymbol{v}, \boldsymbol{v}) &\leqslant a^E(\pi^E \boldsymbol{v}, \pi^E \boldsymbol{v}) + C_E\, a^E(\boldsymbol{v} - \pi^E \boldsymbol{v}, \boldsymbol{v} - \pi^E \boldsymbol{v}) \\ &\leqslant \max\{1, C_E\}\, a^E(\boldsymbol{v}, \boldsymbol{v}). \end{aligned}$$

In a similar manner, it holds

$$a_h^E(\boldsymbol{v}, \boldsymbol{v}) \geqslant \min\{1, c_E\}\, a^E(\boldsymbol{v}, \boldsymbol{v}).$$

□

3.3 A Projection Space $W(E)$

Since it holds clearly that $W(E) = V_h(E)$ when E is a tetrahedron or a prism, the purpose of this Section is to characterize $W(E)$ only when E is a pyramid. The Section finishes with some computational insights.

3.3.1 Lemma. *Let $\hat{E}$ be the reference pyramid of Definition 1.1.18 and recall the notation for the faces in Table 6.1. If $\hat{\boldsymbol{v}} \in P_1(\hat{E})^3$ verifies $\hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{n}} = 0$ on $\hat{f}_1$, $\hat{f}_2$, $\hat{f}_3$ and $\hat{f}_5$, then $\hat{\boldsymbol{v}}(\hat{\boldsymbol{x}}) = (0, c\hat{x}_2, 0)$ for a constant c.*

Proof. The conditions of $\hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{n}}$ over the faces $\hat{f}_1$, $\hat{f}_2$ and $\hat{f}_5$ yield

$$\hat{\boldsymbol{v}}(\hat{\boldsymbol{x}}) = (b_1\hat{x}_1, c_2\hat{x}_2, d_3\hat{x}_3)'.$$

Using that $\hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{n}} = 0$ on $\hat{f}_3$ we have $b_1 = d_3 = 0$. Then $\hat{\boldsymbol{v}} = (0, c_2\hat{x}_2, 0)'$ as we wanted to prove. □

3.3.2 Lemma. *Let P be any pyramid on a physical mesh. Then $\dim W(P) \leqslant 4$.*

Proof. As $W(P) \subseteq V_h(P)$, we have $\dim W(P) \leqslant 5$. We will prove that $W(P) \neq V_h(P)$ by showing that there exists no field $\boldsymbol{v} = \nabla q_2$ for $q_2 \in P_2(P)$ if we impose the restriccion that the normal component of $\boldsymbol{v}$ vanishes on four different faces of P while is different from zero on the remaining face.

Let $\hat{E}$ be the reference pyramid of Definition 1.1.18 and let us use the same notation for the faces. Let $F(\hat{\boldsymbol{x}}) = M_P\hat{\boldsymbol{x}} + \boldsymbol{x}_P$ be an affine map from $\hat{E}$ onto P and let $f_i = F(\hat{f}_i)$ denote the faces of $\hat{E}$ and P. Suppose that $\boldsymbol{v} = \nabla q_2$ for a $q_2 \in P_2(P)$ and that $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ on f_1, f_2, f_3 and f_5, while $\boldsymbol{v} \cdot \boldsymbol{n} = 1$ on f_4. Now we transform it with (2.33) to get

$$\boldsymbol{v}(\boldsymbol{x}) = \frac{1}{|M_P|} M_P \hat{\boldsymbol{v}}(\hat{\boldsymbol{x}}), \qquad \boldsymbol{x} = F(\hat{\boldsymbol{x}}), \tag{3.26}$$

where $\hat{\boldsymbol{v}}$ is in $P_1(\hat{E})^3$. If $\phi \in P_1(f_i)$, put $\hat{\phi} = \phi \circ F$. Using the properties of the transformed degrees of freedom we explained in Section 2.3 we have, for $i = 1, 2, 3, 5$,

$$\iint_{\hat{f}_i} \hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{n}}\, \hat{\phi}\, d\hat{S} = \iint_{f_i} \boldsymbol{v} \cdot \boldsymbol{n}\, \phi\, dS = 0$$

for all $\phi \in P_1(f_i)$. Since $\hat{\boldsymbol{v}}|_{\hat{f}_i} \cdot \hat{\boldsymbol{n}}$ is itself a $P_1(\hat{f}_i)$ polynomial, this implies $\hat{\boldsymbol{v}}|_{\hat{f}_i} \cdot \hat{\boldsymbol{n}}$ vanishes identically over $\hat{f}_i$. By Lemma 3.3.1 we have $\hat{\boldsymbol{v}}(\hat{\boldsymbol{x}}) = (0, c\hat{x}_2, 0)'$. Then

$$\boldsymbol{v}(\boldsymbol{x}) = \frac{1}{|M_P|} M_P (0, c\hat{x}_2, 0)' = \frac{c\,\hat{x}_2}{|M_P|} \boldsymbol{m}_2$$

where we write $M_P = [\boldsymbol{m}_1\ \boldsymbol{m}_2\ \boldsymbol{m}_3]$ columnwise. So on f_4 we have

$$\boldsymbol{v}(\boldsymbol{x}) \cdot \boldsymbol{n} = \frac{c\,\hat{x}_2}{|M_P|} \boldsymbol{m}_2 \cdot \boldsymbol{n}$$

but $\hat{x}_2$ is not a constant on f_4 since it ranges from 0 to 1. Moreover, $\boldsymbol{m}_2 \cdot \boldsymbol{n}$ is not zero because $\boldsymbol{m}_2$ is a vector whose direction is not contained in f_4. Then $\boldsymbol{v}(\boldsymbol{x}) \cdot \boldsymbol{n}$ is not a constant on f_4, which contradicts our definition of $\boldsymbol{v}$. □

Now we explicit the pyramidal projection space.

3.3.3 Proposition. *Let P be a pyramid in the mesh. Then $W(P) = P_0^3(P) + \boldsymbol{x}P_0(P)$.*

Proof. First $P_0^3(P) + \boldsymbol{x}P_0(P) \subseteq W(P)$. Since by Lemma 3.3.2 both spaces have the same dimension the assertion concludes. □

If P is a physical pyramid, given a field $\boldsymbol{v} \in V_h(P)$ we can construct $\pi^P\boldsymbol{v}$ as follows. We choose a basis $\{\boldsymbol{w}_i\}$ of $W(P)$, for example,

$$\{(1,0,0)', (0,1,0)', (0,0,1)', (x,y,z)'\} =: \{\boldsymbol{w}_i : 1 \leqslant i \leqslant 4\}$$

with $\boldsymbol{w}_i = \nabla q_i$, $i = 1,2,3,4$. Then $a^P(\boldsymbol{v}, \boldsymbol{w}_i)$ is calculable from the degrees of freedom of $\boldsymbol{v}$ by

$$a^P(\boldsymbol{v}, \boldsymbol{w}_i) = \int_P \boldsymbol{v} \cdot \nabla q_i \, d\boldsymbol{x} = -\int_P \operatorname{div} \boldsymbol{v} \, q_i \, d\boldsymbol{x} + \iint_{\partial P} \boldsymbol{v} \cdot \boldsymbol{n} \, q_i \, dS.$$

Then, if $\pi^P\boldsymbol{v} = \sum_{j=1}^4 \alpha_j \boldsymbol{w}_j$ we can compute the coefficients α_j by solving the linear system

$$\sum_{j=1}^4 \alpha_j a^P(\boldsymbol{w}_j, \boldsymbol{w}_i) = a^P(\boldsymbol{v}, \boldsymbol{w}_i), \qquad 1 \leqslant i \leqslant 4.$$

In order to compute the stabilization part of the discrete bilinear form a_h^P we need to write $\pi^P\boldsymbol{v}$ in terms of the basis $\{\boldsymbol{v}_i\}$ of $V_h(E)$ associated with the degrees of freedom. That is (always in the pyramidal case)

$$\iint_{f_i} \boldsymbol{v}_j \cdot \boldsymbol{n} \, dS = \delta_{ij}, \qquad 1 \leqslant i, j \leqslant 5.$$

In this case, we have $\pi^P\boldsymbol{v} = \sum_{i=1}^5 \beta_i \boldsymbol{v}_i$, with

$$\beta_i = \sum_{j=1}^4 \alpha_j \iint_{f_i} \boldsymbol{w}_j \cdot \boldsymbol{n} \, dS.$$

Chapter 4

Local Interpolation

Introducción al capítulo

Probamos estimaciones anisótropas de error de interpolación local para el operador de interpolación $H(\text{div})$–conforme y usamos este resultado para estimar el error global de aproximación más adelante en el Capítulo 5. Recordamos usar la notación para los espacios polinomiales dada en Notación 2.0.2. Además nos remitiremos a la notación en indexación de la Tabla 2.1. Más adelante, en el Capítulo 6 estableceremos los resultados análogos en $H(\textbf{curl})$.

Introduction to the chapter

We prove anisotropic local interpolation error estimates for the div–conforming interpolation operator and use this result to estimate the global approximation error later in Chapter 5. Please recall the notation for the polynomial spaces given in Notation 2.0.2. We will alse stick to the notation and indices of Table 2.1. Later in Chapter 6 we will state the $H(\textbf{curl})$ analogues.

4.1 Prismatic Finite Elements

4.1.1 Anisotropic Stability Estimates for $H(\textbf{div})$–Conforming Finite Elements on Prisms

In the present subsection $\hat{\boldsymbol{u}}$ will be an element of $W^{1,1}(\hat{E})$, a space in which the elements have well defined normal traces over the faces of $\hat{E}$, another possibility being, as mentioned in [33], Lemma 5.15, page 120, to assume there is a positive δ such that $\hat{\boldsymbol{u}}$ belongs to $H^{1/2+\delta}(\hat{E})^3$, but the latter does not suffice to obtain our results. For the whole subsection, $\hat{\boldsymbol{r}}_{\hat{E}}$ will be the k–th order face interpolation operator on the reference Prism determined by the finite element of Definition 2.1.1.

In [23] there are also local anisotropic estimates for prismatic elements conforming in $H(\mathrm{div})$, which are made with a different technique, and which are only for the least order case. Here we prove local anisotropic estimates for arbitrary order interpolation.

Recall that $\hat{u}_i$ and $(\boldsymbol{r}_{\hat{E}}\hat{\boldsymbol{u}})_i$ denote the components of $\hat{\boldsymbol{u}}$ and $\boldsymbol{r}_{\hat{E}}\hat{\boldsymbol{u}}$ respectively.

4.1.1 Lemma. *$(\boldsymbol{r}_{\hat{E}}\hat{\boldsymbol{u}})_3$ is linearly and univocally determined by $\hat{u}_3$.*

Proof. By the unisolvence of the finite element in Definition 2.1.1, $(\boldsymbol{r}_{\hat{E}}\hat{\boldsymbol{u}})_3$ is determined by the following linear equations.

$$\hat{\rho}_{\hat{f}_j,\hat{q}}(\boldsymbol{r}_{\hat{E}}\hat{\boldsymbol{u}}) = \hat{\rho}_{\hat{f}_j,\hat{q}}(\hat{\boldsymbol{u}}) \quad \text{for } j = 3,4 \text{ and } \hat{q} \in P_{k-1}(\hat{f}_j) \tag{4.1}$$

$$\hat{\rho}_{\hat{\boldsymbol{r}}}(\boldsymbol{r}_{\hat{E}}\hat{\boldsymbol{u}}) = \hat{\rho}_{\hat{\boldsymbol{r}}}(\hat{\boldsymbol{u}}) \quad \text{for } \hat{\boldsymbol{r}} = (0,0,\hat{r}_3),\ \hat{r}_3 \in P_{k-1,k-2}. \tag{4.2}$$

We have ${}^{k(k+1)^2}/{}_2$ independent equations, which is the number of independent coefficients in $(\boldsymbol{r}_{\hat{E}}\hat{\boldsymbol{u}})_3$. Now set $\hat{u}_3 = 0$, which makes the right hand side of all the equations in (4.1) and (4.2) equal to zero. We will prove that the present square homogeneous linear system has a unique trivial solution, and the Lemma will follow. Take $\hat{f}$ to be either $\hat{f}_3$ or $\hat{f}_4$. Put $\hat{q} = (\boldsymbol{r}_{\hat{E}}\hat{\boldsymbol{u}})_3|_{\hat{f}}$ in (4.1) recalling (2.2). It yields

$$\iint_{\hat{f}} \{(\boldsymbol{r}_{\hat{E}}\hat{\boldsymbol{u}})_3\}^2 \, d\hat{S} = 0.$$

so there is a polinomial $\hat{q}_3 \in P_{k-1,k-1}$ such that $(\boldsymbol{r}_{\hat{E}}\hat{\boldsymbol{u}})_3(\hat{x}_1,\hat{x}_2,\hat{x}_3) = \hat{x}_3(\hat{x}_3 - 1)\,\hat{q}_3(\hat{x}_1,\hat{x}_2,\hat{x}_3)$ and the Lemma will follow once we apply the degrees of freedom (4.2) with the test polynomial $\hat{\boldsymbol{r}}$ set equal to $(0,0,\hat{q}_3)'$. □

4.1.2 Lemma. *Take an element $\hat{\boldsymbol{u}}$ of the form $\hat{\boldsymbol{u}}(\hat{x}_1,\hat{x}_2,\hat{x}_3) = (0,0,\hat{u}_3(\hat{x}_1,\hat{x}_2,\hat{x}_3))'$. Then $\boldsymbol{r}_{\hat{E}}\hat{\boldsymbol{u}}(\hat{x}_1,\hat{x}_2,\hat{x}_3) = (0,0,\hat{\xi}_3(\hat{x}_1,\hat{x}_2,\hat{x}_3))'$ for some $\hat{\xi}_3 \in P_{k-1}(\hat{f}_3) \otimes P_k(\hat{x}_3)$.*

Proof. First take an explicit expression for $\boldsymbol{r}_{\hat{E}}\hat{\boldsymbol{u}} = (\hat{p}_1,\hat{p}_2,\hat{p}_3)' \in P_{\hat{E}}$ as

$$\begin{aligned}
\hat{p}_1(\hat{x}_1,\hat{x}_2,\hat{x}_3) &= \hat{q}_1(\hat{x}_1,\hat{x}_2,\hat{x}_3) + \hat{x}_1\,\hat{h}(\hat{x}_1,\hat{x}_2,\hat{x}_3) \\
\hat{p}_2(\hat{x}_1,\hat{x}_2,\hat{x}_3) &= \hat{q}_2(\hat{x}_1,\hat{x}_2,\hat{x}_3) + \hat{x}_2\,\hat{h}(\hat{x}_1,\hat{x}_2,\hat{x}_3) \\
\hat{p}_3(\hat{x}_1,\hat{x}_2,\hat{x}_3) &= \hat{q}_3(\hat{x}_1,\hat{x}_2,\hat{x}_3)
\end{aligned} \tag{4.3}$$

for unique $\hat{q}_1,\hat{q}_2 \in P_{k-1}(\hat{f}_3) \otimes P_{k-1}(\hat{x}_3)$, $\hat{q}_3 \in P_{k-1}(\hat{f}_3) \otimes P_k(\hat{x}_3)$, and $\hat{h} \in \tilde{P}_{k-1}(\hat{f}_3) \otimes P_{k-1}(\hat{x}_3)$. Next, take an arbitrary $\hat{q} \in P_{k-1}(\hat{f}_1) \otimes P_{k-1}(\hat{z})$. The trick will be to apply Green's Theorem to the field $(\hat{p}_1,\hat{p}_2,0)'$ and the scalar $\hat{q}$. By the surface degrees of freedom (2.4) and the volume degrees of freedom (2.5) we have

$$\begin{aligned}
\int_{\hat{E}} \mathrm{div}(\hat{p}_1,\hat{p}_2,0)'\,\hat{q}\,d\hat{\boldsymbol{x}} &= \int_{\partial\hat{E}} (\hat{p}_1,\hat{p}_2,0)' \cdot \hat{\boldsymbol{n}}\,\hat{q}\,d\hat{S} - \int_{\hat{E}} (\hat{p}_1,\hat{p}_2,0)' \cdot \nabla\hat{q}\,d\hat{\boldsymbol{x}} \\
&= \int_{\hat{f}_5} (\hat{u}_1 + \hat{u}_2)\hat{q}_{|_{\hat{f}_5=1}}\,d\hat{S} - \int_{\hat{f}_1} \hat{u}_1\hat{q}_{|_{\hat{f}_1}}\,d\hat{S} \\
&\quad - \int_{\hat{f}_2} \hat{u}_2\hat{q}_{|_{\hat{f}_2}}\,d\hat{S} - \int_{\hat{E}} \hat{u}_1\tfrac{\partial\hat{q}}{\partial\hat{x}_1} + \hat{u}_2\tfrac{\partial\hat{q}}{\partial\hat{x}_2}\,d\hat{\boldsymbol{x}} \qquad = 0.
\end{aligned}$$

And since $\operatorname{div}(\hat{p}_1, \hat{p}_2, 0)'$ also belongs to $P_{k-1,k-1}$, we just established it vanishes on all $\hat{E}$.

Now the nullity of $\operatorname{div}(\hat{p}_1, \hat{p}_2, 0)'$ implies at once that $\hat{h} \equiv 0$ in expression (4.3) (it can be deduced directly derivating the polynomials and observing the degrees of the terms; cfr. proof of Lemma 6.1.2 onwards). This means we may assume that $\hat{p}_1$ and $\hat{p}_2$ belong to $P_{k-1}(\hat{f}_3) \otimes P_{k-1}(\hat{x}_3)$. Now it is convenient to use again the degrees of freedom on the faces normal to $(-1, 0, 0)'$ and $(0, -1, 0)'$. The conditions

$$\hat{\rho}_{\hat{f}_i \, \hat{q}}(\boldsymbol{r}_{\hat{E}} \hat{\boldsymbol{u}}) = \iint_{\hat{f}_i} \hat{p}_i \hat{q} \, d\hat{S} = 0 \qquad \text{for all } \hat{q} \in P_{k-1}(\hat{f}_i),\ 1 \leqslant i \leqslant 2$$

ensure that $\hat{x}_i$ divides $\hat{p}_i$ for both $i = 1$ and 2. But finally, if we evaluate the degrees of freedom (2.5), we see that $\hat{p}_1 = (\boldsymbol{r}_{\hat{E}} \hat{\boldsymbol{u}})_1$ and $\hat{p}_2 = (\boldsymbol{r}_{\hat{E}} \hat{\boldsymbol{u}})_2$ can be no other that constantly null over all $\hat{E}$. □

4.1.3 Lemma.

(a) If $\hat{\boldsymbol{u}}(\hat{x}_1, \hat{x}_2, \hat{x}_3) = (0, \hat{u}_2(\hat{x}_1, \hat{x}_3), 0)'$, then $\boldsymbol{r}_{\hat{E}} \hat{\boldsymbol{u}}(\hat{x}_1, \hat{x}_2, \hat{x}_3) = (0, \hat{\xi}_2(\hat{x}_1, \hat{x}_3), 0)'$ for some $\hat{\xi}_2 \in P_{k-1}(\hat{x}_2) \otimes P_k(\hat{x}_3)$.

(b) If $\hat{\boldsymbol{u}}(\hat{x}_1, \hat{x}_2, \hat{x}_3) = (\hat{u}_1(\hat{x}_2, \hat{x}_3), 0, 0)'$ then $\boldsymbol{r}_{\hat{E}} \hat{\boldsymbol{u}}(\hat{x}_1, \hat{x}_2, \hat{x}_3) = (\hat{\xi}_1(\hat{x}_2, \hat{x}_3), 0, 0)'$ for some $\hat{\xi}_1 \in P_{k-1}(\hat{x}_1) \otimes P_k(\hat{x}_3)$.

Proof. Let us prove the first one of the two claims. The second one has an analogous proof. By Lemma 4.1.1 we get $(\boldsymbol{r}_{\hat{E}} \hat{\boldsymbol{u}})_3 = 0$. The nullity of $\operatorname{div} \hat{\boldsymbol{u}}$ and the commutative diagram property (2.39) give us $\operatorname{div} \boldsymbol{r}_{\hat{E}} \hat{\boldsymbol{u}} = 0$. This last fact, together with the result of the evaluation of the degrees of freedom (2.4) on the face $\hat{f}_1 \subseteq \{x_1 = 0\}$ and (2.5) over $\hat{E}$, implies, as we have seen in Lemma 4.1.2, that $(\boldsymbol{r}_{\hat{E}} \hat{\boldsymbol{u}})_1 = 0$. And if we look again at $\operatorname{div} \boldsymbol{r}_{\hat{E}} \hat{\boldsymbol{u}} = \partial \boldsymbol{r}_{\hat{E}} \hat{\boldsymbol{u}} / \partial \hat{x}_2 = 0$ we have that $(\boldsymbol{r}_{\hat{E}} \hat{\boldsymbol{u}})_2$ does not depend on $\hat{x}_2$. □

It is time to state and prove the Theorem that was the purpose of this section.

4.1.4 Theorem. *Given $\hat{\boldsymbol{u}} \in W^{1,1}(\hat{E})$*

$$\|(\boldsymbol{r}_{\hat{E}} \hat{\boldsymbol{u}})_1\|_{L^\infty(\hat{E})} \lesssim \|\hat{u}_1\|_{W^{1,1}(\hat{E})} + \|\operatorname{div}(\hat{u}_1, \hat{u}_2, 0)\|_{L^1(\hat{E})} \tag{4.4}$$

$$\|(\boldsymbol{r}_{\hat{E}} \hat{\boldsymbol{u}})_2\|_{L^\infty(\hat{E})} \lesssim \|\hat{u}_2\|_{W^{1,1}(\hat{E})} + \|\operatorname{div}(\hat{u}_1, \hat{u}_2, 0)\|_{L^1(\hat{E})} \tag{4.5}$$

$$\|(\boldsymbol{r}_{\hat{E}} \hat{\boldsymbol{u}})_3\|_{L^\infty(\hat{E})} \lesssim \|\hat{u}_3\|_{W^{1,1}(\hat{E})} \tag{4.6}$$

where the constants in the inequalities depend olny on $\hat{E}$.

Proof. The proof is based on the last three Lemmas. By Proposition 1.1.14 we can state the estimate for a smooth field $\hat{\boldsymbol{u}}$ and then finish the proof with a density argument.

Take $\hat{\boldsymbol{u}} \in \mathcal{C}^\infty(\bar{\hat{E}})^3$. For the first inequality. Set $\hat{\boldsymbol{v}} = (\hat{u}_1, \hat{u}_2 - \hat{u}_2(\hat{x}_1, 0, \hat{x}_3), 0)$. Then $(\boldsymbol{r}_{\hat{E}} \hat{\boldsymbol{v}})_1 = (\boldsymbol{r}_{\hat{E}} \hat{\boldsymbol{u}})_1$. By evaluating the degrees of freedom of $\hat{\boldsymbol{v}}$ we observe that some of them vanish or depend exclusively on $\hat{u}_1$ in the way we need them to depend on $\hat{u}_1$. As for the others, pick first $\hat{q}_0 \in P_{k-1}(\hat{x}_1) \otimes P_{k-1}(\hat{x}_3)$ and extend it as the same polynomial $\hat{q}$ to $Q_{k-1,k-1,k-1}$.

$$\begin{aligned}\hat{\rho}_{\hat{f}_5,\hat{q}_0}(\hat{\boldsymbol{v}}) &= \iint_{\hat{f}_5} \hat{u}_1\, \hat{q}_0\, d\hat{S} + \sqrt{2} \iint_{[0,1]^2} \hat{q}_0(\hat{x}_1, \hat{x}_3) \int_0^{1-\hat{x}_1} \tfrac{\partial \hat{v}_2}{\partial \hat{x}_2}(\hat{x}_1, \hat{t}, \hat{x}_3)\, d\hat{t} d\hat{x}_1 d\hat{x}_3 \\ &= \iint_{\hat{f}_5} \hat{u}_1\, \hat{q}_0\, d\hat{S} + \sqrt{2} \int_{\hat{E}} \hat{q}\, \tfrac{\partial \hat{v}_2}{\partial \hat{x}_2}\, d\hat{\boldsymbol{x}} \\ &= \iint_{\hat{f}_5} \hat{u}_1\, \hat{q}_0\, d\hat{S} + \sqrt{2} \int_{\hat{E}} \hat{q}\, \{ \operatorname{div}(\hat{u}_1, \hat{u}_2, 0)' - \tfrac{\partial \hat{u}_1}{\partial \hat{x}_1} \}\, d\hat{\boldsymbol{x}}.\end{aligned}$$

For the volume degrees of freedom (2.5) take $\hat{q}_2 \in P_{k-2,k-1}$. Write

$$\hat{v}_2(\hat{x}_1, \hat{x}_2, \hat{x}_3) = \int_0^{\hat{x}_2} \tfrac{\partial \hat{u}_2}{\partial \hat{x}_2}(\hat{x}_1, \hat{t}, \hat{x}_3)\, d\hat{t}$$

and do

$$\begin{aligned}\int_{\hat{E}} \hat{v}_2\, \hat{q}_2 &= \int_0^1 \int_0^1 \int_0^{1-\hat{x}_1} \int_0^{\hat{x}_2} \tfrac{\partial \hat{u}_2}{\partial \hat{x}_2}(\hat{x}_1, \hat{t}, \hat{x}_3)\, d\hat{t}\, \hat{q}_2(\hat{x}_1, \hat{x}_2, \hat{x}_3)\, d\hat{x}_2\, d\hat{x}_1\, d\hat{x}_3 \\ &= \int_0^1 \int_0^1 \int_0^{1-\hat{x}_1} \int_0^{\hat{x}_2} \tfrac{\partial \hat{u}_2}{\partial \hat{x}_2}(\hat{x}_1, \hat{t}, \hat{x}_3)\, \hat{q}_2(\hat{x}_1, \hat{x}_2, \hat{x}_3)\, d\hat{t}\, d\hat{x}_2\, d\hat{x}_1\, d\hat{x}_3 \\ &= \int_0^1 \int_0^1 \int_0^{1-\hat{x}_1} \int_{\hat{t}}^{1-\hat{x}_1} \tfrac{\partial \hat{u}_2}{\partial \hat{x}_2}(\hat{x}_1, \hat{t}, \hat{x}_3)\, \hat{q}_2(\hat{x}_1, \hat{x}_2, \hat{x}_3)\, d\hat{x}_2\, d\hat{t}\, d\hat{x}_1\, d\hat{x}_3 \\ &= \int_0^1 \int_0^1 \int_0^{1-\hat{x}_1} \tfrac{\partial \hat{u}_2}{\partial \hat{x}_2}(\hat{x}_1, \hat{t}, \hat{x}_3) \int_{\hat{t}}^{1-\hat{x}_1} \hat{q}_2(\hat{x}_1, \hat{x}_2, \hat{x}_3)\, d\hat{x}_2\, d\hat{t}\, d\hat{x}_1\, d\hat{x}_3 \\ &= \int_0^1 \int_0^1 \int_0^{1-\hat{x}_1} \tfrac{\partial \hat{u}_2}{\partial \hat{x}_2}(\hat{x}_1, \hat{t}, \hat{x}_3)\, \hat{\phi}(\hat{x}_1, \hat{t}, \hat{x}_3)\, d\hat{t}\, d\hat{x}_1\, d\hat{x}_3 \\ &= \int_{\hat{E}} \operatorname{div}(\hat{u}_1, \hat{u}_2, 0)'\, \hat{\phi}\, d\hat{\boldsymbol{x}} - \int_{\hat{E}} \tfrac{\partial \hat{u}_1}{\partial \hat{x}_1}\, \hat{\phi}\, d\hat{\boldsymbol{x}}\end{aligned}$$

(for some $\hat{\phi} \in P_{k-1}(\hat{f}_3) \otimes P_{k-1}(\hat{x}_3)$), which is what we needed. The inequality (4.5) is proved in the same way. For inequality (4.6) we can do

$$\begin{aligned}(\boldsymbol{r}_{\hat{E}} \hat{\boldsymbol{u}})_3 &= (\hat{\boldsymbol{r}}_k (0, 0, \hat{u}_3)')_3 \\ &= \sum_{i=3,4;\, \hat{\boldsymbol{q}}} \iint_{\hat{f}_i} \hat{u}_3 \hat{q}_3\, d\hat{S}\, (\hat{\boldsymbol{v}}_{\hat{f}_i, \hat{\boldsymbol{q}}})_3 + \sum_{\hat{\boldsymbol{r}}} \int_{\hat{E}} \hat{u}_3\, \hat{r}_3\, d\hat{\boldsymbol{x}}\, (\hat{\boldsymbol{v}}_{\hat{\boldsymbol{r}}})_3.\end{aligned}$$

Then, by standard results for traces in Sobolev spaces,

$$\begin{aligned}\|(\boldsymbol{r}_{\hat{E}}\hat{\boldsymbol{u}})_3\|_{L^\infty(\hat{E})} &\leqslant C(\hat{E})\left\{\sum_{i=3,4}\int_{\hat{f}_i}|\hat{u}_3|\,d\hat{\gamma}+\int_{\hat{E}}|\hat{u}_3|\,d\hat{\boldsymbol{x}}\right\}\\ &\leqslant C(\hat{E})(\|\hat{u}_3|_{\partial\hat{E}}\|_{L^1(\partial\hat{E})}+\|\hat{u}_3\|_{L^1(\hat{E})})\\ &\leqslant C(\hat{E})\|\hat{u}_3\|_{W^{1,1}(\hat{E})}.\end{aligned}$$

□

Theorem 4.1.4 shows that the interpolation determined by the finite element in Definition 2.1.1 is anisotropically stable, in the sense that the image of a field $\hat{\boldsymbol{u}}$ under the linear operator depends not only continously on $\hat{\boldsymbol{u}}$, but also with a *componentwise* bound, with perhaps and additional divergence term. We refer the reader to Theorem 6.1.4 onwards to note the same property for the **curl**–conforming case.

The next step is to estimate the stability in an anisotropically rescaled prism. Given three positive numbers h_1, h_2 and h_3 we denote

$$\tilde{E}=\tilde{T}\times\tilde{I} \tag{4.7}$$

where

$$\begin{aligned}\tilde{T}&=\{0<{}^{\tilde{x}_1}/_{h_1}+{}^{\tilde{x}_2}/_{h_2}<1\}\\ \tilde{I}&=\{0<{}^{\tilde{x}_3}/_{h_3}<1\}.\end{aligned}$$

Of course $\tilde{E}=F(\hat{E})$ where F is the linear $\mathbb{R}^3\to\mathbb{R}^3$ transformation such that

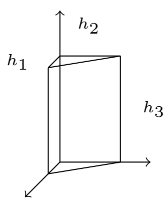


Figure 4.1 – Rescaled Prism

$$F\hat{\boldsymbol{x}}=\begin{pmatrix}h_1&0&0\\0&h_2&0\\0&0&h_3\end{pmatrix}\hat{\boldsymbol{x}}=\tilde{\boldsymbol{x}}. \tag{4.8}$$

4.1.5 Theorem. *There is $C>0$, independent of h_1, h_2 and h_3, s.t. for all $p\geqslant 1$ and $\tilde{\boldsymbol{u}}\in W^{1,p}(\tilde{E})$*

$$\begin{aligned}\|\boldsymbol{r}_{\tilde{E}}\tilde{\boldsymbol{u}}\|_{L^p(\tilde{E})}\leqslant C\Big(\|\tilde{\boldsymbol{u}}\|_{L^p(\tilde{E})}+\sum_{i=1}^{3}h_i\|\tfrac{\partial\tilde{\boldsymbol{u}}}{\partial\tilde{x}_i}\|_{L^p(\tilde{E})}\\ +\max\{h_1,h_2\}\|div\,(\tilde{u}_1,\tilde{u}_2,0)\,\|_{L^p(\tilde{E})}\Big).\end{aligned}$$

4.1.6 Remark. *When it comes to estimate in terms of the data f of a problem we will assume $h_3 \geqslant C\max\{h_1, h_2\}$ so that Theorem 4.1.5 implies*

$$\|\boldsymbol{r}_{\tilde{E}}\tilde{\boldsymbol{u}}\|_{L^p(\tilde{E})} \leqslant C\left(\|\tilde{\boldsymbol{u}}\|_{L^p(\tilde{E})} + \sum_{i=1}^{3} h_i\|\tfrac{\partial\tilde{\boldsymbol{u}}}{\partial\tilde{x}_i}\|_{L^p(\tilde{E})} + h_{\tilde{E}}\,\|div\,\tilde{\boldsymbol{u}}\|_{L^p(\tilde{E})}\right).$$

This is not a restriction since we needed the prisms to be elongated exactly along the direction paralell to the cuadrilateral faces.

Proof of Theorem 4.1.5. Pick $p \geqslant 1$. If we pull $\tilde{\boldsymbol{u}}$ back to $\hat{E}$ we get the relation

$$\hat{\boldsymbol{u}}(\hat{\boldsymbol{x}}) = (det\,DF)DF^{-1}\tilde{\boldsymbol{u}}(F\hat{\boldsymbol{x}}) \tag{4.9}$$

$$D\hat{\boldsymbol{u}}(\hat{\boldsymbol{x}}) = \begin{pmatrix} h_2\,h_3 & 0 & 0 \\ 0 & h_1\,h_3 & 0 \\ 0 & 0 & h_1\,h_2 \end{pmatrix} \cdot D\tilde{\boldsymbol{u}}(F\hat{\boldsymbol{x}}) \cdot \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix} \tag{4.10}$$

and by (2.55)

$$(\det DF)DF^{-1}\boldsymbol{r}_{\tilde{E}}\tilde{\boldsymbol{u}}(F(\hat{\boldsymbol{x}})) = \boldsymbol{r}_{\hat{E}}\hat{\boldsymbol{u}}(\hat{\boldsymbol{x}}). \tag{4.11}$$

With expressions (4.9) and (4.11) and stability inequality (4.4) plus Hölder's inequality we obtain

$$\begin{aligned}
\|(\boldsymbol{r}_{\tilde{E}}\tilde{\boldsymbol{u}})_1\|_{L^\infty(\tilde{E})} &= (h_2\,h_3)^{-1}\|(\boldsymbol{r}_{\hat{E}}\hat{\boldsymbol{u}})_1\|_{L^\infty(\hat{E})} \\
&\leqslant (h_2\,h_3)^{-1}\left(\|\hat{u}_1\|_{W^{1,1}(\hat{E})} + \|\text{div}\,(\hat{u}_1,\hat{u}_2,0)'\|_{L^1(\hat{E})}\right) \\
&= (h_2\,h_3)^{-1}\left(\int_{\hat{E}}|\hat{u}_1|\,d\hat{\boldsymbol{x}} + \sum_{i=1}^{3}\int_{\hat{E}}\left|\tfrac{\partial\hat{u}_1}{\partial\hat{x}_i}\right|\,d\hat{\boldsymbol{x}} + \int_{\hat{E}}\left|\tfrac{\partial\hat{u}_1}{\partial\hat{x}_1} + \tfrac{\partial\hat{u}_2}{\partial\hat{x}_2}\right|\,d\hat{\boldsymbol{x}}\right) \\
&= (\det DF)^{-1}\left(\int_{\tilde{E}}|\tilde{u}_1|\,d\tilde{\boldsymbol{x}} + \sum_{i=1}^{3}h_i\int_{\tilde{E}}|\tfrac{\partial\tilde{u}_1}{\partial\tilde{x}_i}|\,d\tilde{\boldsymbol{x}} + h_1\int_{\tilde{E}}|\tfrac{\partial\tilde{u}_1}{\partial\tilde{x}_1} + \tfrac{\partial\tilde{u}_2}{\partial\tilde{x}_2}|\,d\tilde{\boldsymbol{x}}\right) \\
&= (2|\tilde{E}|)^{-1}\left(\|\tilde{u}_1\|_{L^1(\tilde{E})} + \sum_{i=1}^{3}h_i\|\tfrac{\partial\tilde{u}_1}{\partial\tilde{x}_i}\|_{L^1(\tilde{E})} + h_1\|\text{div}\,(\tilde{u}_1,\tilde{u}_2,0)\|_{L^1(\tilde{E})}\right) \\
&\leqslant (2|\tilde{E}|^{1/p})^{-1}\left(\|\tilde{u}_1\|_{L^p(\tilde{E})} + \sum_{i=1}^{3}h_i\|\tfrac{\partial\tilde{u}_1}{\partial\tilde{x}_i}\|_{L^p(\tilde{E})} + h_1\|\text{div}\,(\tilde{u}_1,\tilde{u}_2,0)\|_{L^p(\tilde{E})}\right).
\end{aligned} \tag{4.12}$$

Now

$$\begin{aligned}
\|(\boldsymbol{r}_{\tilde{E}}\tilde{\boldsymbol{u}})_1\|_{L^p(\tilde{E})} &\leqslant |\tilde{E}|^{1/p}\|(\tilde{\boldsymbol{r}}_k\tilde{\boldsymbol{u}})_1\|_{L^\infty(\tilde{E})} \\
&\lesssim \|\tilde{u}_1\|_{L^p(\tilde{E})} + \sum_{i=1}^{3}h_i\|\tfrac{\partial\tilde{u}_1}{\partial\tilde{x}_i}\|_{L^p(\tilde{E})} + h_1\|\text{div}\,(\tilde{u}_1,\tilde{u}_2,0)\|_{L^p(\tilde{E})},
\end{aligned}$$

and again, the symmetric inequality holds for component two. For component three, stability inequality (4.6) gives us

$$\begin{aligned}\|(\boldsymbol{r}_{\tilde{E}}\tilde{\boldsymbol{u}})_3\|_{L^\infty(\tilde{E})} &= (h_1h_2)^{-1}\|(\boldsymbol{r}_{\hat{E}}\hat{\boldsymbol{u}})_3\|_{L^\infty(\hat{E})}\\ &\leqslant C(h_1h_2)^{-1}\,\|\hat{u}_3\|_{W^{1,1}(\hat{E})}\\ &= C\,|\tilde{E}|^{-1}\left[\|\tilde{u}_3\|_{L^1(\tilde{E})} + \sum_{i=1,2,3} h_i\,\|\tfrac{\partial\tilde{u}_3}{\partial\tilde{x}_i}\|_{L^1(\tilde{E})}\right]\\ &\leqslant C\,|\tilde{E}|^{-1/p}\left[\|\tilde{u}_3\|_{L^p(\tilde{E})} + \sum_{i=1,2,3} h_i\,\|\tfrac{\partial\tilde{u}_3}{\partial\tilde{x}_i}\|_{L^p(\tilde{E})}\right]\end{aligned}$$

so, immediately,

$$\|(\boldsymbol{r}_{\tilde{E}}\tilde{\boldsymbol{u}})_3\|_{L^p(\tilde{E})} \leqslant C\left(\|\tilde{u}_3\|_{L^p(\tilde{E})} + \sum_{i=1,2,3} h_i\,\|\tfrac{\partial\tilde{u}_3}{\partial\tilde{x}_i}\|_{L^p(\tilde{E})}\right) \tag{4.13}$$

and the sum of the three estimates yields the Theorem. □

4.1.2 Local Interpolation Estimates for Prismatic Elements

We will state scaling consequences of inequalities (1.21). Some of them will be used here and the rest will be used in Chapter 6.

4.1.7 Remark. *Recall the vector averaged Taylor polynomial $\mathbf{Q}_{m,E}(\cdot)$ defined in (1.18). If $(\,\cdot\,)\hat{}$ denotes any of transformations (2.29) and (2.33) then it holds*

$$\mathbf{Q}_{m,\hat{E}}\hat{\boldsymbol{w}} = (\mathbf{Q}_{m,E}\,\boldsymbol{w})\hat{}.$$

4.1.8 Lemma. *Let $\tilde{E}$ be the rescaled reference prism in Figure 4.1. Given $p \geqslant 1$, $m \geqslant 0$ and $\tilde{\boldsymbol{u}} \in W^{m+1,p}(\tilde{E})$, then for $m \geqslant 0$ and $p \geqslant 1$ the following items hold.*

1. *For any component $1 \leqslant i \leqslant 3$, for any $\boldsymbol{\beta}$ with $|\boldsymbol{\beta}| \leqslant m+1$*

$$\|\partial^{\boldsymbol{\beta}}(\tilde{u}_i - \tilde{\mathcal{Q}}_{m,\tilde{E}}\,\tilde{u}_i)\|_{L^p(\tilde{E})} \leqslant C \sum_{|\boldsymbol{\alpha}|=m-|\boldsymbol{\beta}|+1} \boldsymbol{h}^{\boldsymbol{\alpha}}\left\|\partial^{\boldsymbol{\alpha}+\boldsymbol{\beta}}\tilde{u}_i\right\|_{L^p(\tilde{E})} \tag{4.14}$$

2. *For any component $1 \leqslant i \leqslant 3$, if $m \geqslant 1$ and $p \geqslant 1$*

$$\|\boldsymbol{curl}\,(\tilde{\boldsymbol{u}} - \tilde{\mathbf{Q}}_{m,\tilde{E}}\tilde{\boldsymbol{u}})_i\|_{L^p(\tilde{E})} \leqslant C \sum_{j+k+l=m} h_1^j h_2^k h_3^l \left\|\frac{\tilde{\partial}^m(\boldsymbol{curl}\,\tilde{\boldsymbol{u}})_i}{\partial\tilde{x}_1^j\partial\tilde{x}_2^k\partial\tilde{x}_3^l}\right\|_{L^p(\tilde{E})} \tag{4.15}$$

3. *For any component $1 \leqslant i \leqslant 3$ and any $1 \leqslant j \leqslant 3$, if $m \geqslant 1$*

$$\left\|\frac{\partial\boldsymbol{curl}\,(\tilde{\boldsymbol{u}} - \tilde{\mathbf{Q}}_{m,\tilde{E}}\tilde{\boldsymbol{u}})_i}{\partial\tilde{x}_j}\right\|_{L^p(\tilde{E})} \leqslant C \sum_{j+k+l=m-1} h_1^j h_2^k h_3^l \left\|\frac{\tilde{\partial}^{m-1}\tilde{\partial}(\widetilde{\boldsymbol{curl}\,\boldsymbol{u}})_i}{\partial\tilde{x}_1^j\partial\tilde{x}_2^k\partial\tilde{x}_3^l\partial\tilde{x}_j}\right\|_{L^p(\tilde{E})} \tag{4.16}$$

4. *For the divergence it holds*

$$\|\tilde{div}\,(\tilde{\boldsymbol{u}} - \tilde{\mathbf{Q}}_{m,\tilde{E}}\tilde{\boldsymbol{u}})\|_{L^p(\tilde{E})} \leqslant C \sum_{j+k+l=m} h_1^j h_2^k h_3^l \left\| \frac{\tilde{\partial}^m \tilde{div}\tilde{\boldsymbol{u}}}{\partial\tilde{x}_1^j \partial\tilde{x}_2^k \partial\tilde{x}_3^l} \right\|_{L^p(\tilde{E})} \tag{4.17}$$

where C depends only on m, σ (cfr. Theorem 1.3.6) and the reference element.

Proof of Lemma 4.1.8. We will use Lemma 1.3.5 rescaling $\hat{E}$ so that $d = 1$ and then pushing forward to $\tilde{E}$. In fact, by (2.31) for any multi–index $\boldsymbol{\alpha}$ we have

$$h_1^{\alpha_1} h_2^{\alpha_2} h_3^{\alpha_3} \tilde{\partial}^\alpha \tilde{u}_i(\tilde{\boldsymbol{x}}) = ({}^1\!/\!{}_{h_i} \hat{\partial}^\alpha \hat{u}_i)(F_{\tilde{E}}^{-1}\tilde{\boldsymbol{x}}) \tag{4.18}$$

so, to prove (4.14),

$$\begin{aligned}
\|\partial^{\boldsymbol{\beta}}(\tilde{u}_i - \tilde{\mathcal{Q}}_{m,\tilde{E}}\,\tilde{u}_i)\|^p_{L^p(\tilde{E})} &= \|\frac{1}{\boldsymbol{h}^{\boldsymbol{\beta}}}\frac{1}{h_i}(\partial^{\boldsymbol{\beta}} u_i - \mathcal{Q}_{m-|\boldsymbol{\beta}|,E}\,\partial^{\boldsymbol{\beta}} u_i)\circ F_{\tilde{E}}^{-1}\|^p_{L^p(\tilde{E})} \\
&= \frac{|\det M_E|}{h_i^p}\frac{1}{(\boldsymbol{h}^{\boldsymbol{\beta}})^p}\|\partial^{\boldsymbol{\beta}} u_i - \mathcal{Q}_{m-|\boldsymbol{\beta}|,E}\,\partial^{\boldsymbol{\beta}} u_i\|^p_{L^p(\hat{E})} \\
&\leqslant \frac{C|\det M_E|}{h_i^p}\frac{1}{(\boldsymbol{h}^{\boldsymbol{\beta}})^p}|\partial^{\boldsymbol{\beta}} u_i|^p_{m-|\boldsymbol{\beta}|+1,p,\hat{E}} \\
&\simeq |\det M_E| \sum_{|\boldsymbol{\alpha}|=m-|\boldsymbol{\beta}|+1} \frac{1}{(\boldsymbol{h}^{\boldsymbol{\beta}})^p} \left\| \frac{1}{h_i}\partial^{\boldsymbol{\alpha}+\boldsymbol{\beta}} u_i \right\|^p_{L^p(\hat{E})} \\
\text{(by (4.18))} \qquad &= C \sum_{|\boldsymbol{\alpha}|=m-|\boldsymbol{\beta}|+1} (\boldsymbol{h}^{\boldsymbol{\alpha}})^p \left\| \tilde{\partial}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} \tilde{u}_i \right\|^p_{L^p(\tilde{E})}.
\end{aligned}$$

To prove (4.15), by a straightforward manipulation we have

$$\tilde{\mathbf{curl}}\,\tilde{\mathbf{Q}}_{m,\tilde{E}}\tilde{\boldsymbol{u}} = \tilde{\mathbf{Q}}_{m-1,\tilde{E}}\tilde{\mathbf{curl}}\,\tilde{\boldsymbol{u}}.$$

Then, componentwise, by (4.14),

$$\begin{aligned}
\|\tilde{\mathbf{curl}}\,(\tilde{\boldsymbol{u}} - \tilde{\mathbf{Q}}_{m,\tilde{E}}\tilde{\boldsymbol{u}})_i\|_{L^p(\tilde{E})} &= \|(\tilde{\mathbf{curl}}\,\tilde{\boldsymbol{u}})_i - \tilde{\mathcal{Q}}_{m-1}(\tilde{\mathbf{curl}}\,\tilde{\boldsymbol{u}})_i\|_{L^p(\tilde{E})} \\
&\leqslant C \sum_{j+k+l=m} h_1^j h_2^k h_3^l \left\| \frac{\tilde{\partial}^m (\tilde{\mathbf{curl}}\,\tilde{\boldsymbol{u}})_3}{\partial\tilde{x}_1^j \partial\tilde{x}_2^k \partial\tilde{x}_3^l} \right\|_{L^p(\tilde{E})}.
\end{aligned}$$

Now (4.16) is an easy consecuence because

$$\begin{aligned}
\left\| \frac{\partial \mathbf{curl}\,(\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{q}})_i}{\partial\tilde{x}_j} \right\|_{L^p(\tilde{E})} &= \left\| \frac{\partial(\tilde{\mathbf{curl}}\,\tilde{\boldsymbol{u}})_i}{\partial\tilde{x}_j} - \frac{\partial\tilde{\mathcal{Q}}_{m-1,\tilde{E}}(\tilde{\mathbf{curl}}\,\tilde{\boldsymbol{u}})_i}{\partial\tilde{x}_j} \right\|_{L^p(\tilde{E})} \\
&= \left\| \frac{\partial(\tilde{\mathbf{curl}}\,\tilde{\boldsymbol{u}})_i}{\partial\tilde{x}_j} - \tilde{\mathcal{Q}}_{m-2,\tilde{E}}\frac{\partial(\tilde{\mathbf{curl}}\,\tilde{\boldsymbol{u}})_i}{\partial\tilde{x}_j} \right\|_{L^p(\tilde{E})} \\
&\leqslant C \sum_{j+k+l=m-1} h_1^j h_2^k h_3^l \left\| \frac{\tilde{\partial}^{m-1}\tilde{\partial}(\tilde{\mathbf{curl}}\,\tilde{\boldsymbol{u}})_i}{\partial\tilde{x}_1^j \partial\tilde{x}_2^k \partial\tilde{x}_3^l \partial\tilde{x}_j} \right\|_{L^p(\tilde{E})}.
\end{aligned}$$

Equality (4.17) follows in a simpler way. □

We arrived at one of the main results in this Thesis which is the anisotropic local interpolation error estimates for div–conforming elements on prisms of any order. As we said earlier, in Chapter 6 we will arrive at the **curl**–conforming analogue.

We will adopt the following notations. For a prism E, h will denote its diameter and, for $1 \leqslant i \leqslant 3$, ξ_i will denote unitary vectors with the directions of the three edges $\boldsymbol{e}_i$ sharing a vertex $\boldsymbol{x}_E$ of E whose lengths are h_i and ξ_3 will be the particular direction of the edge which is common to two quadrilateral faces. Recall that, for $\boldsymbol{h} = (h_1, h_2, h_3)'$, $\boldsymbol{h}^{\boldsymbol{\alpha}}$ means $h_1^{\alpha_1} h_2^{\alpha_2} h_3^{\alpha_3}$ and $\partial^{\boldsymbol{\alpha}} = \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial\xi_1^{\alpha_1}\xi_2^{\alpha_2}\xi_3^{\alpha_3}}$.

Following the proof of Theorem 6.2 in [1] and adding the information of (4.17), from Remark 4.1.6 we derive the following Theorem.

4.1.9 Theorem. *Let $k \in \mathbb{N}_0$ and $p \geqslant 1$. Let E be an oblique prism (cfr. Figure 4.2). There exists $C > 0$, which depends on the greatest angle of the triangular faces and the angles on the cuadrilateral faces of E, and three edges $\boldsymbol{e}_i$ of E incident to a common vertex $\boldsymbol{x}_E$ such that for all $\boldsymbol{u} \in W^{m+1,p}(E)^3$ and $m \leqslant k$,*

$$\|\boldsymbol{u} - \boldsymbol{r}_E\boldsymbol{u}\|_{L^p(E)} \leqslant C\left\{\sum_{|\boldsymbol{\alpha}|=m+1} \boldsymbol{h}^{\boldsymbol{\alpha}}\|\partial^{\boldsymbol{\alpha}}\boldsymbol{u}\|_{L^p(E)} + h_E \sum_{|\boldsymbol{\alpha}|=m} \boldsymbol{h}^{\boldsymbol{\alpha}}\|\partial^{\boldsymbol{\alpha}} div\, \boldsymbol{u}\|_{L^p(E)}\right\}. \tag{4.19}$$

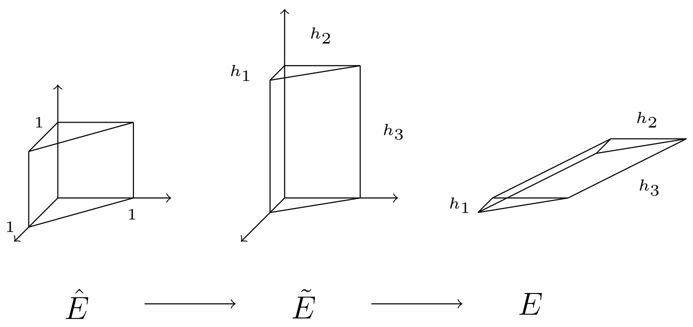


Figure 4.2 – Affine transformations of the reference prism.

4.2 Local Interpolation Estimates for Virtual Elements on Pyramids

In the present Section we develop some of the needed results to work with pyramids following a virtual element framework. In Section 6.3 we put the definition of a version of conforming pyramidal finite elements and in Section 6.4 we have included anisotropic local interpolation estimates found for them to show that we could build a full FE framework as a variant of our final work.

By a pyramidal element we mean every affine image of the element in Definition 1.1.18.

4.2.1 Lemma. *If E is a pyramidal element, $\boldsymbol{u}$ is an element of $W^{1,p}(E)$ with $1 \leqslant p \leqslant 2$ and $I\boldsymbol{u}$ is its low order $H(div)$ interpolant defined in Corollary 3.1.2, then*

$$\|I\boldsymbol{u}\|_{L^p(E)} \leqslant C\left(\|\boldsymbol{u}\|_{L^p(E)} + h|\boldsymbol{u}|_{W^{1,p}(E)}\right) \tag{4.20}$$

with h being the diameter of E and the constant C depending only on the shape regularity of E.

Proof. Denote by f_j, $1 \leqslant j \leqslant 5$ the faces of E and by $\{\boldsymbol{v}_i\}_{i=1}^5$ the basis of $V_h(E)$ which is dual to the degrees of freedom (3.3). We start with the estimate of the L^2-norm of any $\boldsymbol{v}_i$. Pick $1 \leqslant i \leqslant 5$. We saw in the proof of Lemm 3.1.1 that $\boldsymbol{v}_i = \nabla\psi$ where ψ is the solution of the auxiliary problem

$$\begin{aligned} \Delta\psi &= \tfrac{1}{|E|} \qquad \text{in } E \\ \tfrac{\partial\psi}{\partial\boldsymbol{n}} &= g \qquad \text{on } \partial E \end{aligned}$$

where g was defined as

$$g|_{f_j} = \begin{cases} \frac{1}{|f_i|} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} . \tag{4.21}$$

Fix a solution requiring that $\int_E \psi\, d\boldsymbol{x} = 0$. Integration by parts gives

$$\begin{aligned} \|\nabla\psi\|^2_{L^2(E)} &= \iint_{\partial E} g\psi\, dS \\ &\leqslant \|g\|_{L^2(\partial E)}\|\psi\|_{L^2(\partial E)}. \end{aligned}$$

Using Poincare's and traces inequalities we have, for a positive constant C depending only on the aspect ratio of E,

$$\|\nabla\psi\|^2_{L^2(E)} \leqslant C\, h_E^{1/2}\|g\|_{L^2(\partial E)}\|\nabla\psi\|_{L^2(E)}. \tag{4.22}$$

But, by (4.21), it holds $\|g\|_{L^2(\partial E)} \leqslant Ch_E^{-1}$ and so from (4.22) we obtain

$$\|\boldsymbol{v}_i\|_{L^2(E)} = \|\nabla\psi\|_{L^2(E)} \leqslant Ch_E^{-1/2}. \tag{4.23}$$

Now, for $1 \leqslant p \leqslant 2$, using Hölder's inequality we have

$$\begin{aligned} \|I\boldsymbol{u}\|_{L^p(E)} &\leqslant |E|^{1/p-1/2}\|I\boldsymbol{u}\|_{L^2(E)} \\ &\leqslant |E|^{1/p-1/2}\sum_{i=1}^5 \left|\iint_{f_i} \boldsymbol{u}\cdot\boldsymbol{n}\, dS\right| \|\boldsymbol{v}_i\|_{L^2(E)}. \end{aligned}$$

By using (4.23) and traces inequalities we obtain

$$\begin{aligned} \|I\boldsymbol{u}\|_{L^p(E)} &\leqslant C|E|^{1/p-1/2}h_E^{-1/p}\left(\|\boldsymbol{u}\|_{L^p(E)} + h_E\|\nabla\boldsymbol{u}\|_{L^p(E)}\right)\cdot \\ &\qquad\qquad |\partial E|^{1-1/p}\|\boldsymbol{v}_i\|_{L^2(E)} \\ &\leqslant C\left(\|\boldsymbol{u}\|_{L^p(E)} + h_E\|\nabla\boldsymbol{u}\|_{L^p(E)}\right) \end{aligned}$$

as we wanted. □

4.2.2 Proposition. *Let E be a pyramid satisfying the shape-regularity property with constant σ. Then there exists a constant C depending only on σ such that for all $\boldsymbol{u} \in H^1(E)$*

$$\|\boldsymbol{u} - I\boldsymbol{u}\|_{L^2(E)} \leqslant Ch_E|\boldsymbol{u}|_{H^1(E)}.$$

Proof. Let $\mathbf{Q}\boldsymbol{u}$ be the $L^2(E)$ projection of $\boldsymbol{u}$ onto the constant fields. Then we have

$$\boldsymbol{u} - I\boldsymbol{u} = (\boldsymbol{u} - \mathbf{Q}\boldsymbol{u}) + I(\mathbf{Q}\boldsymbol{u} - \boldsymbol{u})$$

and using the previous Lemma and a clasical estimate for the $L^2(E)$ projection error we have

$$\begin{aligned} \|\boldsymbol{u} - I\boldsymbol{u}\|_{L^2(E)} &\leqslant \|\boldsymbol{u} - \mathbf{Q}\boldsymbol{u}\|_{L^2(E)} + \|I(\boldsymbol{u} - \mathbf{Q}\boldsymbol{u})\|_{L^2(E)} \\ &\leqslant \|\boldsymbol{u} - \mathbf{Q}\boldsymbol{u}\|_{L^2(E)} \\ &\quad + C\left(\|\boldsymbol{u} - \mathbf{Q}\boldsymbol{u}\|_{L^2(E)} + h_E\|\nabla(\boldsymbol{u} - \mathbf{Q}\boldsymbol{u})\|_{L^2(E)}\right) \\ &= C\left(\|\boldsymbol{u} - \mathbf{Q}\boldsymbol{u}\|_{L^2(E)} + h_E\|\nabla\boldsymbol{u}\|_{L^2(E)}\right) \\ &\leqslant Ch_E\|\nabla\boldsymbol{u}\|_{L^2(E)} \end{aligned}$$

as we wanted to prove. □

By repeating the last step of the previous proof we can derive the following Proposition whose utility will be apparent in Chapter 5.

4.2.3 Proposition. *Let E be a pyramid satisfying a shape regularity property with constant σ, and $\boldsymbol{u} \in H^1(E)$. There is a field $\boldsymbol{w_u} \in W(E)$ such that*

$$\|\boldsymbol{u} - \boldsymbol{w_u}\|_{L^2(E)} \leqslant Ch_E|\boldsymbol{u}|_{H^1(E)}.$$

Proof. Take $\boldsymbol{w_u}$ as the $L^2(E)$ projection of $\boldsymbol{u}$ onto the space of constants fields. □

Chapter 5

Approximation

Introducción al capítulo

En el presente capítulo probamos existencia y unicidad para un problema discreto elíptico modelo y establecemos los errores de aproximación. Primero tratamos el caso convexo y después el no convexo. Para este último presentamos nuestro proceso de mallado en términos de prismas, tetraedros y pirámides, que produce mallas adecuadamente graduadas para problemas en dominios poliedrales singulares y escribimos y demostramos nuestro teorema de error global de interpolación en estas mallas, el cual es probado usando los espacios de Sobolev con pesos presentados en el Capítulo 1, y con este probamos nuestro teorema de error de aproximación.

Ahora recordamos brevemente la comparación hecha en la Introducción entre nuestro método y aquellos presentados en [23]. En ese artículo los autores proponen un método con elementos prismáticos anisótropos solamente para dominios cilíndricos y aquí proponemos un método que admite dominios polihedrales arbitrarios en dimensión 3 mediante una combinación de elementos de geometrías diferentes (prismas, pirámides y tetraedros).

Además los autores en [23] incluyen un resultado que usa mallas de tetraedros subdividiendo cada prisma en tres, que requieren del uso de tetraedros que no verifican una condición $\mathcal{RVP}$ uniforme (cfr. Figura 2) con la consecuencia de requerir regularidad adicional al dato f del problema. En nuestro método, la construcción de una malla híbrida evita el uso de los mencionados tetraedros y por esto recuperamos el orden óptimo de convergencia del error de aproximación con dato f en L^2.

Introduction to the chapter

In the present chapter we prove discrete well posedness of a model elliptic problem and then we state the approximation errors. First we deal with the convex case and then with the non–convex one. For the latter, we present our general meshing procedure in terms of prisms, tetrahedra and pyramids, and which yields suitable graded meshes for singular polihedral domains, and state first our main global interpolation result in those meshes, which is proved using the weighted Sobolev spaces presented in Chapter 1, and with this theorem we prove our approximation error theorem.

Now we recall briefly the comparison, that was made in the Introduction of this thesis, between our method and the ones presented in [23]. There the authors propose a method with anisotropic prismatic elements only for cartesian product type (cylindrical) domains in $\mathbb{R}^3$ and here we propose a method that allows for arbitrary polihedral domains $\mathbb{R}^3$ by means of a combination of elements of three different geometries: prisms, pyramids and tetrahedra.

Furthermore the authors in [23] include a result using tetrahedral meshes by splitting each prism into three tetrahedra, which require the use of tetrahedra that don't fulfill a uniform $\mathcal{RVP}$ property (cfr. Figure 2) and the price paid is that they require more regularity to the datum f of the problem. In our method, the construction of a hybrid mesh avoids the use of the mentioned tetrahedra and therefore we recover the optimal order of convergence for the approximation error with datum f in L^2.

5.1 Discrete Well Posedness of a Model Elliptic Problem

If we consider the following kernel

$$\begin{aligned}\mathcal{K}_{\mathcal{T}_h} &= \{\boldsymbol{v}_h \in V_h : b(\boldsymbol{v}_h, q) = 0 \text{ for all } q \in Q_h\} \\ &= V_h \cap \ker \operatorname{div}\end{aligned}$$

then Lemma 3.2.2 implies the following Proposition.

5.1.1 Proposition. *The form a_h in (3.11) is coercive over $\mathcal{K}_{\mathcal{T}_h}$ and the coercivity constant depends only on the aspect ratio of the pyramids of the mesh.*

5.1.2 Proposition. *For every $E \in \mathcal{T}_h$ the local discrete bilinear form a_h^E in (3.10) is continuous en $L^2(E)$. That is, for all $\boldsymbol{u}$,$\boldsymbol{v} \in V_h(E)$*

$$a_h^E(\boldsymbol{u}, \boldsymbol{v}) \leqslant C\|\boldsymbol{u}\|_{L^2(E)}\|\boldsymbol{v}\|_{L^2(E)},$$

where C equals 1 when E is a right prism or tetrahedron, and depends only on the aspect ratio of E in the case of pyramids.

Proof. When E is a prism or tetrahedron, $a_h^E(\boldsymbol{u}, \boldsymbol{v}) = a^E(\boldsymbol{u}, \boldsymbol{v})$ for $\boldsymbol{u}$ and $\boldsymbol{v}$ in $V_h(E)$, and the result is immediate. If E is a pyramid, as a_h^E is symmetric, the coercivity arising from (3.24) implies that a_h^E defines an inner product. Hence by Lemma 3.2.2 we have

$$\begin{aligned}\left|a_h^E(\boldsymbol{u}, \boldsymbol{v})\right| &\leqslant a_h^E(\boldsymbol{u}, \boldsymbol{u})^{\frac{1}{2}} a_h^E(\boldsymbol{v}, \boldsymbol{v})^{\frac{1}{2}} \qquad \leqslant \\ &\leqslant C_E\, a^E(\boldsymbol{u}, \boldsymbol{u})^{\frac{1}{2}} a^E(\boldsymbol{v}, \boldsymbol{v})^{\frac{1}{2}} \leqslant C_E\|\boldsymbol{u}\|_{L^2(E)}\|\boldsymbol{v}\|_{L^2(E)}.\end{aligned}$$

□

5.1.3 Lemma. *There exists a constant $\beta^* > 0$ depending only on Ω and the maximum aspect ratio of the pyramids of $\mathcal{T}_h$ such that for all $q^* \in Q_h$ there exists $\boldsymbol{w}_h^* \in V_h$ such that*

$$\textit{div}\, \boldsymbol{w}_h^* = q^* \quad \textit{and} \quad \beta^* \|\boldsymbol{w}_h^*\|_Q \leqslant \|q^*\|_Q.$$

Proof. Start considering the infinite dimensional version of the statement. There is, in fact, a constant β^* depending only on Ω such that for every $q^* \in Q_h \subseteq Q$ there exists $\boldsymbol{w}^* \in H_0^1(\Omega)^3$ with div $\boldsymbol{w}^* = q^*$ such that

$$\beta^* \|\boldsymbol{w}^*\|_{H^1(\Omega)} \leqslant \|q^*\|_Q. \tag{5.1}$$

We refer the reader to [22]. Now, for each $q^* \in Q_h$ take $\boldsymbol{w}_h^*$ such that, in every $E \in \mathcal{T}_h$, $\boldsymbol{w}_h^*|_E := I(\boldsymbol{w}^*|_E)$, as defined in Corollary 3.1.2. As a consequence of Proposition 3.1.4 and because we are considering the exact same degrees of freedom, this interpolation operator coincides, in the lowest order case, with the $H(\text{div})$–conforming operators determined by the finite elements in Definitions 2.1.1 and 2.2.3 for prisms and tetraedra.

For prismatic elements we use the estimate in Remark 4.1.6 which, together with the anisotropic rescalings used in Theorem 4.1.9, applies to any right prism, and Theorem 3.1 in page 149 of [1] for tetrahedral elements. In the case of a pyramidal element we draw upon estimate (4.20). With all these together and (5.1) for all cases we get

$$\|\boldsymbol{w}_h^*\|_Q = \|I\boldsymbol{w}^*\|_Q \leqslant C(1+h)\|\boldsymbol{w}^*\|_{H^1(\Omega)} \leqslant \frac{C(1+h)}{\beta^*}\|q^*\|_Q.$$

Besides, since $q^* \in Q_h$, by Lemma 3.1.3 we have

$$\text{div}\, I\boldsymbol{w}^* = P_0\, \text{div}\, \boldsymbol{w}^* = P_0\, q^* = q^*.$$

□

Now we can prove the discrete inf-sup condition for the b_h form in (3.7).

5.1.4 Theorem. *Consider the bilinear form b_h in (3.7). There exists $\beta > 0$ such that for all $q^* \in Q_h$*

$$\sup_{0 \neq \boldsymbol{v} \in V_h} \frac{b_h(\boldsymbol{v}, q^*)}{\|\boldsymbol{v}\|_{V_h}} \geqslant \beta \|q^*\|_{Q_h}. \tag{5.2}$$

Proof. By Lemma 5.1.3 for $q^* \in Q_h$ there exists $\boldsymbol{w}_h^* \in V_h$ such that div $\boldsymbol{w}_h^* = q^*$ and $\beta^* \|\boldsymbol{w}_h^*\|_{L^2(\Omega)} \leqslant \|q^*\|_{Q_h}$ (the constant β^* is independent of q^*). Then

$$\|\boldsymbol{w}_h^*\|_{V_h}^2 = \|\boldsymbol{w}_h^*\|_{L^2(\Omega)}^2 + \|q^*\|_{Q_h}^2 \leqslant \left(\frac{1}{(\beta^*)^2} + 1\right) \|q^*\|_{Q_h}^2$$

and

$$\sup_{0 \neq \boldsymbol{v} \in V_h} \frac{b_h(\boldsymbol{v}, q^*)}{\|\boldsymbol{v}\|_{V_h}} \geqslant \frac{b_h(\boldsymbol{w}_h^*, q^*)}{\|\boldsymbol{w}_h^*\|_{V_h}} \geqslant \frac{1}{\sqrt{\frac{1}{(\beta^*)^2} + 1}} \|q^*\|_{Q_h}$$

as we wanted. □

5.1.5 Theorem. *Problem 3.2.1 has a unique solution.*

Proof. Theorem 5.2 of [22] states that Problem 3.2.1 has a unique solution provided (5.2) holds and that there exists some $\alpha > 0$ such that for all $\boldsymbol{u} \in \mathcal{K}_{\mathcal{T}_h}$

$$\sup_{0\neq \boldsymbol{v}\in\mathcal{K}_{\mathcal{T}_h}} \frac{a_h(\boldsymbol{u},\boldsymbol{v})}{\|\boldsymbol{v}\|_{V_h}} \geqslant \alpha\|\boldsymbol{u}\|_{V_h},$$

but this is implied by the coercivity of a_h since

$$\sup_{0\neq \boldsymbol{v}\in\mathcal{K}_{\mathcal{T}_h}} \frac{a_h(\boldsymbol{u},\boldsymbol{v})}{\|\boldsymbol{v}\|_{V_h}} \geqslant \frac{a_h(\boldsymbol{u},\boldsymbol{u})}{\|\boldsymbol{u}\|_{V_h}} \geqslant c\|\boldsymbol{u}\|_{V_h}.$$

□

We introduce the following global interpolation operator.

5.1.6 Definition. *Given a positive parameter h which tends to zero and is strictly decreasing and a conforming mesh $\mathcal{T}_h$ made up of prisms, pyramids and tetrahedra, let $\boldsymbol{r}_0\boldsymbol{u}$ be the* global interpolation operator

$$\boldsymbol{r}_0\boldsymbol{u} : W^{1,1}(\Omega) \to V_h \tag{5.3}$$

such that, for each $\{E : E \in \mathcal{T}_h\}$,

$$(\boldsymbol{r}_0\boldsymbol{u})|_E = \begin{cases} \boldsymbol{r}_E\left[\boldsymbol{u}|_E\right] & \text{as in Definition 2.1.3} \quad \text{if } E \text{ is a prism} \\ I\boldsymbol{u} & \text{as in Corollary 3.1.2} \quad \text{if } E \text{ is a pyramid} \\ \boldsymbol{r}_E\left[\boldsymbol{u}|_E\right] & \text{as in Definition 2.2.3} \quad \text{if } E \text{ is a tetrahedron,} \end{cases}$$

in all cases with the lowest interpolation order $k = 0$.

With regard to the pyarmids we remark here that we could perform an analysis similar to the one based on virtual elements using the finite elements on pyramids treated in Theorems 6.4.2 and 6.4.4.

5.1.7 Theorem. *The solution $(\boldsymbol{u}_h, p_h)$ of Problem 3.2.1 satisfies, for every approximation $\boldsymbol{w}$ of $\boldsymbol{u}$ in $W(\mathcal{T}_h)$,*

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^2(\Omega)^3} \leqslant C\{\|\boldsymbol{u}-\boldsymbol{r}_0\boldsymbol{u}\|_{L^2(\Omega)^3} + \|\boldsymbol{u}-\boldsymbol{w}\|_{L^2(\Omega)^3}\} \tag{5.4}$$

$$\|P_0\,p - p_h\|_{L^2(\Omega)} \leqslant C\{\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^2(\Omega)^3} + \|\boldsymbol{u}-\boldsymbol{w}\|_{L^2(\Omega)^3}\} \tag{5.5}$$

Proof. Cfr. proof of Theorem 5.1 in [15]. □

5.1.8 Definition. *Let P_{0,E_ℓ} be the projection onto the constants over E_ℓ and let $\boldsymbol{w}_{\boldsymbol{u}}$ be defined piecewise as*

$$(\boldsymbol{w}_{\boldsymbol{u}})|_{E_\ell} = \begin{cases} \boldsymbol{r}_{E_\ell}\left[\boldsymbol{u}|_{E_\ell}\right] & \text{if } E_\ell \text{ is a prism or a tetrahedron} \\ P_{0,E_\ell}\boldsymbol{u} & \text{if } E_\ell \text{ is a pyramid} \end{cases}$$

for every E_ℓ

5.1.9 Remark. *The one in Definition 5.1.8 is the approximation $\boldsymbol{w}$ of $\boldsymbol{u}$ piecewise in $W(E)$ that we will use on the right hand side of (5.4) and (5.5). Observe that $\boldsymbol{w}_{\boldsymbol{u}_r+\boldsymbol{u}_s} = \boldsymbol{w}_{\boldsymbol{u}_r} + \boldsymbol{w}_{\boldsymbol{u}_s}$.*

5.2 Approximation Error for the Convex Case

For the case in which Ω is convex and $f \in L^2(\Omega)$ we obtain from Theorem 5.1.7 the following Corollary.

5.2.1 Corollary.

$$\begin{aligned} \|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega)} &\leqslant Ch|p|_{H^2(\Omega)} \\ \|p - p_h\|_{L^2(\Omega)} &\leqslant Ch\|p\|_{H^2(\Omega)} \end{aligned}$$

where the constant C depends only on the aspect ratios of tetrahedra and pyramids and the maximum angle of the triangular faces of the right prisms on the mesh, provided that the tetrahedra fulfill a uniform maximum angle condition.

Proof. For a convex Ω and $f \in L^2(\Omega)$ the solution p of the problem belongs to $H^2(\Omega)$ and so $\boldsymbol{u} \in H^1(\Omega)^3$. In this case, using the interpolation error estimates we proved in Theorem 4.1.9 for general prisms and Proposition 4.2.2 for pyramids and the analogue for tetrahedra from [1], we have, for $\boldsymbol{r}_0\boldsymbol{u}$ as in Definition 5.1.6, that

$$\|\boldsymbol{u} - \boldsymbol{r}_0\boldsymbol{u}\|_{L^2(\Omega)^3} \leqslant Ch|p|_{H^2(\Omega)}.$$

Next again Theorem 4.1.9, Proposition 4.2.2 for pyramids and the analogue for tetrahedra from [1], and Proposition 4.2.3 for pyramids yield, for $\boldsymbol{w}_{\boldsymbol{u}}$ as in Definition 5.1.8,

$$\|\boldsymbol{u} - \boldsymbol{w}_{\boldsymbol{u}}\|_{L^2(\Omega)^3} \leqslant Ch|p|_{H^2(\Omega)}.$$

So joining the last two inequalities, from (5.4) in Theorem 5.1.7,

$$\begin{aligned} \|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega)^3} &\leqslant C\{\|\boldsymbol{u} - \boldsymbol{w}_{\boldsymbol{u}}\|_{L^2(\Omega)^3} + \|\boldsymbol{u} - \boldsymbol{r}_0\boldsymbol{u}\|_{L^2(\Omega)^3}\} \\ &\leqslant Ch|p|_{H^2(\Omega)} \\ &\leqslant Ch\,\|f\|_{L^2(\Omega)} \end{aligned}$$

as stated in [4], in expression after (3.42) on page 19. The bound for the scalar variable error in the convex case follows using (5.5) and the estimate $\|p - P_{0,\mathcal{T}_h} p\| \leqslant C\,h_E\,|p|_{1,E}$ for the orthogonal projection. □

Arbitrarily narrow right prisms can be used in the mesh without affecting this estimate. This fact can be further exploited when the domain Ω is not convex or f is not in $L^2(\Omega)$. Besides, we could also allow arbitrary narrow pyramids in the mesh (as long as they are not *flat*, that is, at least one side of the basis must be smaller than the height) and continue from Theorems 6.4.2 and 6.4.4, but in the meshes we designed, which appear in what follows, pyramids happened to be regular. The same can be said about the $\mathcal{MAC}$ condition mentioned in Corollary 5.2.1. Again, this is not a restriction, since our method required only the use of shape–regular tetrahedra. This will be made clear in Subsection 5.3.1.

The non–convex case including anisotropic prismatic elements is what follows. It is in this non–convex case where we will need our estimates of anisotropic type, in contrast with the previous convex case, where we didn't need them.

5.3 Approximation Error for the Non–Convex Case

Our main approximation error Theorem will be the following statement: There exists a family of anisotropic graded meshes $\{\mathcal{T}_h\}_{h\downarrow 0}$ made up of prisms, tetrahedra and pyramids for which

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega} \leqslant Ch\|f\|_{0,\Omega}$$
$$\|p - p_h\|_{0,\Omega} \leqslant Ch\|f\|_{0,\Omega}$$

with h being smaller than $CN_h^{-1/3}$, where N_h is the number of elements in $\mathcal{T}_h$.

The rest of the chapter contains its proof, which is summarized at the end of it. To avoid what would result in a proof causing weariness we decided to partition the proof in subsections, paragraphs and items. The first part of the proof will be the exhibition of the meshes and after that we will prove the estimates.

5.3.1 Graded Anisotropic Meshes and Meshing Procedure in Dimension 3

Our procedure starts taking a first partition of Ω into macro–elements, which may be prisms or tetrahedra, according to the main regularity result, Theorem 1.2.7. Read Section 1.2 again and recall Figures 1.3a and 1.3b. Here we recall it.

Let $\Omega = \cup_{\ell=1}^{N}\Lambda_\ell$ be a decomposition in macro–elements having, each one of them, at most a singular vertex $S^{(\ell)}$ and a singular edge $A_S^{(\ell)}$

Please recall also the notation for the distance $R(\boldsymbol{x})$ to the singular vertex and the angular distance $\theta(\boldsymbol{x})$ to the singular edge in Definition 1.2.4.

We will show that, in each prismatic or tetrahedral macro–element, our meshes fulfill the grading in the following paragraph (cfr. [7, 3, 4, 5, 38]). We refer the reader to [6, 8] to see other techniques to the treatment of singularities, alternative to mesh grading.

Given a macro–element Λ_ℓ with just a singular edge A, for any element $E \subseteq \Lambda_\ell$ it holds

$$\begin{aligned} h_{E,1}, h_{E,2} &\sim \begin{cases} h^{\frac{1}{\mu}} & \text{if } d(E,A) = 0 \\ h\, d(E,A)^{1-\mu} & \text{if } 0 < d(E,A) < 1 \\ h & \text{if } d(E,A) \sim 1 \end{cases} \\ h_{E,3} &\sim h \end{aligned} \tag{5.6}$$

Given a macro–element Λ_ℓ with just a singular vertex S, for any element $E \subseteq \Lambda_\ell$ it holds

$$h_{E,1}, h_{E,2}, h_{E,3} \sim \begin{cases} h^{\frac{1}{\nu}} & \text{if } d(E,S) = 0 \\ h\, d(E,S)^{1-\nu} & \text{if } 0 < d(E,S) < 1 \\ h & \text{if } d(E,S) \sim 1. \end{cases} \tag{5.7}$$

Given a macro–element Λ_ℓ with a singular edge A_S and a singular vertex S at an endpoint of A_S, for any element $E \subseteq \Lambda_\ell$ it holds

$$\begin{aligned} h_{E,1}, h_{E,2} &\sim \begin{cases} h^{\frac{1}{\mu}} & \text{if } d(E, A_S) = 0 \\ h\, d(E, A_S)^{1-\mu} & \text{if } 0 < d(E, A_S) < 1 \\ h & \text{if } d(E, A_S) \sim 1 \end{cases} \\ h_{E,3} &\sim \begin{cases} h^{\frac{1}{\nu}} & \text{if } d(E, S) = 0 \\ h\, d(E, S)^{1-\nu} & \text{if } 0 < d(E, S) < 1 \\ h & \text{if } d(E, S) \sim 1 \end{cases} \end{aligned} \tag{5.8}$$

with $\mu \leqslant 1 - \delta$ for $\delta > 1 - \lambda_{\boldsymbol{e},S}$ and $\nu \leqslant 1 - \beta$ for $\beta > \frac{1}{2} - \lambda_{\boldsymbol{v},S}$ for all cases, respectively, and with $\mu \leqslant \nu < 1$ in the presence of both singularities.

5.3.1.1 Macro–element with singular edge and singular vertex

Let Λ be a tetrahedral macro–element with vertices P_0, P_1, P_2 and P_3. We suppose that P_0 is the singular vertex and that the singular edge is P_0P_1. The mesh $\mathcal{T}_h$ on Λ will contain tetrahedra, prisms and pyramids. In Tables 5.1–5.4 we explicit the elements in terms of the barycentric coordinates of the vertices of each element corresponding to the ordered vertices P_0, P_1, P_2 and P_3.

Table 5.1 – Barycentric coordinates of prismatic mesh points in Λ_ℓ
$0 \leqslant l \leqslant n-2,\ i+j \leqslant n-l-2$

p_0	$1-\left(\frac{n-l}{n}\right)^{1/\mu}$	$\left(\frac{n-l}{n}\right)^{1/\mu}-\left(\frac{i+j}{n}\right)^{1/\mu}$	$\frac{i}{n}\left(\frac{i+j}{n}\right)^{1/\mu-1}$	$\frac{j}{n}\left(\frac{i+j}{n}\right)^{1/\mu-1}$
p_1	$1-\left(\frac{n-l}{n}\right)^{1/\mu}$	$\left(\frac{n-l}{n}\right)^{1/\mu}-\left(\frac{i+j+1}{n}\right)^{1/\mu}$	$\frac{i+1}{n}\left(\frac{i+1+j}{n}\right)^{1/\mu-1}$	$\frac{j}{n}\left(\frac{i+1+j}{n}\right)^{1/\mu-1}$
p_2	$1-\left(\frac{n-l}{n}\right)^{1/\mu}$	$\left(\frac{n-l}{n}\right)^{1/\mu}-\left(\frac{i+j+1}{n}\right)^{1/\mu}$	$\frac{i}{n}\left(\frac{i+j+1}{n}\right)^{1/\mu-1}$	$\frac{j+1}{n}\left(\frac{i+1+j}{n}\right)^{1/\mu-1}$
p_3	$1-\left(\frac{n-l-1}{n}\right)^{1/\mu}$	$\left(\frac{n-l-1}{n}\right)^{1/\mu}-\left(\frac{i+j}{n}\right)^{1/\mu}$	$\frac{i}{n}\left(\frac{i+j}{n}\right)^{1/\mu-1}$	$\frac{j}{n}\left(\frac{i+j}{n}\right)^{1/\mu-1}$
p_4	$1-\left(\frac{n-l-1}{n}\right)^{1/\mu}$	$\left(\frac{n-l-1}{n}\right)^{1/\mu}-\left(\frac{i+1+j}{n}\right)^{1/\mu}$	$\frac{i+1}{n}\left(\frac{i+1+j}{n}\right)^{1/\mu-1}$	$\frac{j}{n}\left(\frac{i+1+j}{n}\right)^{1/\mu-1}$
p_5	$1-\left(\frac{n-l-1}{n}\right)^{1/\mu}$	$\left(\frac{n-l-1}{n}\right)^{1/\mu}-\left(\frac{i+j+1}{n}\right)^{1/\mu}$	$\frac{i}{n}\left(\frac{i+j+1}{n}\right)^{1/\mu-1}$	$\frac{j+1}{n}\left(\frac{i+1+j}{n}\right)^{1/\mu-1}$

In Figure 5.1 there are a couple of examples.

Table 5.2 – Barycentric coordinates of prismatic mesh points in Λ_ℓ
$0 \leqslant l \leqslant n-2, \qquad i \geq 1, \qquad \text{and} \qquad i+j \leqslant n-l-2$

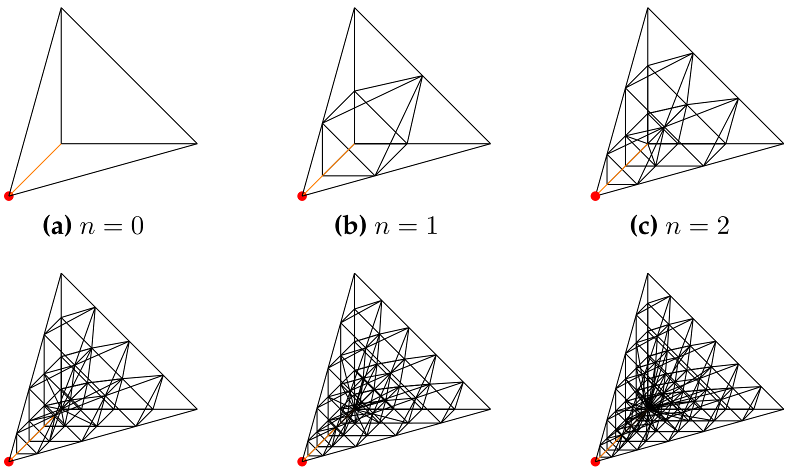
p_0	$1-\left(\frac{n-l}{n}\right)^{1/\mu}$	$\left(\frac{n-l}{n}\right)^{1/\mu}-\left(\frac{i+j}{n}\right)^{1/\mu}$	$\frac{i}{n}\left(\frac{i+j}{n}\right)^{1/\mu-1}$	$\frac{j}{n}\left(\frac{i+j}{n}\right)^{1/\mu-1}$
p_1	$1-\left(\frac{n-l}{n}\right)^{1/\mu}$	$\left(\frac{n-l}{n}\right)^{1/\mu}-\left(\frac{i+j+1}{n}\right)^{1/\mu}$	$\frac{i}{n}\left(\frac{i+j+1}{n}\right)^{1/\mu-1}$	$\frac{j+1}{n}\left(\frac{i+j+1}{n}\right)^{1/\mu-1}$
p_2	$1-\left(\frac{n-l}{n}\right)^{1/\mu}$	$\left(\frac{n-l}{n}\right)^{1/\mu}-\left(\frac{i+j}{n}\right)^{1/\mu}$	$\frac{i-1}{n}\left(\frac{i+j}{n}\right)^{1/\mu-1}$	$\frac{j+1}{n}\left(\frac{i+j}{n}\right)^{1/\mu-1}$
p_3	$1-\left(\frac{n-l-1}{n}\right)^{1/\mu}$	$\left(\frac{n-l-1}{n}\right)^{1/\mu}-\left(\frac{i+j}{n}\right)^{1/\mu}$	$\frac{i}{n}\left(\frac{i+j}{n}\right)^{1/\mu-1}$	$\frac{j}{n}\left(\frac{i+j}{n}\right)^{1/\mu-1}$
p_4	$1-\left(\frac{n-l-1}{n}\right)^{1/\mu}$	$\left(\frac{n-l-1}{n}\right)^{1/\mu}-\left(\frac{i+j+1}{n}\right)^{1/\mu}$	$\frac{i}{n}\left(\frac{i+j+1}{n}\right)^{1/\mu-1}$	$\frac{j+1}{n}\left(\frac{i+j+1}{n}\right)^{1/\mu-1}$
p_5	$1-\left(\frac{n-l-1}{n}\right)^{1/\mu}$	$\left(\frac{n-l-1}{n}\right)^{1/\mu}-\left(\frac{i+j}{n}\right)^{1/\mu}$	$\frac{i-1}{n}\left(\frac{i+j}{n}\right)^{1/\mu-1}$	$\frac{j+1}{n}\left(\frac{i+j}{n}\right)^{1/\mu-1}$

Table 5.3 – Barycentric coordinates of pyramidal mesh points in Λ_ℓ.
$0 \leqslant l \leqslant n-2 \qquad \text{and} \qquad 1 \leqslant i \leqslant n-l-1.$

p_0	$1-\left(\frac{n-l}{n}\right)^{1/\mu}$	$\left(\frac{n-l}{n}\right)^{1/\mu}-\left(\frac{n-l-1}{n}\right)^{1/\mu}$	$\frac{i}{n}\left(\frac{n-l-1}{n}\right)^{1/\mu-1}$	$\frac{n-l-i-1}{n}\left(\frac{n-l-1}{n}\right)^{1/\mu-1}$
p_1	$1-\left(\frac{n-l}{n}\right)^{1/\mu}$	$\left(\frac{n-l}{n}\right)^{1/\mu}-\left(\frac{n-l-1}{n}\right)^{1/\mu}$	$\frac{i-1}{n}\left(\frac{n-l-1}{n}\right)^{1/\mu-1}$	$\frac{n-l-i}{n}\left(\frac{n-l-1}{n}\right)^{1/\mu-1}$
p_2	$1-\left(\frac{n-l-1}{n}\right)^{1/\mu}$	0	$\frac{i}{n}\left(\frac{n-l-1}{n}\right)^{1/\mu-1}$	$\frac{n-l-i-1}{n}\left(\frac{n-l-1}{n}\right)^{1/\mu-1}$
p_3	$1-\left(\frac{n-l-1}{n}\right)^{1/\mu}$	0	$\frac{i-1}{n}\left(\frac{n-l-1}{n}\right)^{1/\mu-1}$	$\frac{n-l-i}{n}\left(\frac{n-l-1}{n}\right)^{1/\mu-1}$
p_4	$1-\left(\frac{n-l}{n}\right)^{1/\mu}$	0	$\frac{i}{n}\left(\frac{n-l}{n}\right)^{1/\mu-1}$	$\frac{n-l-i}{n}\left(\frac{n-l}{n}\right)^{1/\mu-1}$

Table 5.4 – Barycentric coordinates of tetrahedral mesh points in Λ_ℓ.
$0 \leqslant l \leqslant n-1 \qquad \text{and} \qquad 1 \leqslant i \leqslant n-l-1$

p_0	$1-\left(\frac{n-l}{n}\right)^{1/\mu}$	$\left(\frac{n-l}{n}\right)^{1/\mu}-\left(\frac{n-l-1}{n}\right)^{1/\mu}$	$\frac{i}{n}\left(\frac{n-l-1}{n}\right)^{1/\mu-1}$	$\frac{n-l-i-1}{n}\left(\frac{n-l-1}{n}\right)^{1/\mu-1}$
p_1	$1-\left(\frac{n-l-1}{n}\right)^{1/\mu}$	0	$\frac{i}{n}\left(\frac{n-l-1}{n}\right)^{1/\mu-1}$	$\frac{n-l-i-1}{n}\left(\frac{n-l-1}{n}\right)^{1/\mu-1}$
p_2	$1-\left(\frac{n-l}{n}\right)^{1/\mu}$	0	$\frac{i}{n}\left(\frac{n-l}{n}\right)^{1/\mu-1}$	$\frac{n-l-i}{n}\left(\frac{n-l}{n}\right)^{1/\mu-1}$
p_3	$1-\left(\frac{n-l}{n}\right)^{1/\mu}$	0	$\frac{i+1}{n}\left(\frac{n-l}{n}\right)^{1/\mu-1}$	$\frac{n-l-i-1}{n}\left(\frac{n-l}{n}\right)^{1/\mu-1}$



(a) $n = 0$ **(b)** $n = 1$ **(c)** $n = 2$

(d) $n = 3$ **(e)** $n = 4$ **(f)** $n = 5$

Figure 5.1 – Tetrahedral macro–elements with singularities of both types.

5.3.1.2 Macro–element with a singular edge

This case corresponds to the prismatic macro–element as well as the tetrahedral macro–element of Subsubsection 5.3.1.1 but only with a singular edge. Therefore, we only need to present the meshing for the former case. A mesh in the prismatic macro–elements is constructed as the cartesian product between a graded mesh in a triangle and a quasi–uniform mesh along the singular edge. For $0 \leqslant i \leqslant n$ and $0 \leqslant j \leqslant n-1$ let $p_{i,j}$ be the point with barycentric coordinates, with respect to P_0, P_1, P_2, equal to

$$\lambda_0(i,j) = 1 - \lambda_1(i,j) - \lambda_2(i,j),$$
$$\lambda_1(i,j) = \frac{i}{n}\left(\frac{i+j}{n}\right)^{1/\mu - 1}, \quad \lambda_2(i,j) = \frac{j}{n}\left(\frac{i+j}{n}\right)^{1/\mu - 1},$$

Now let $\mathcal{T}$ be the family of triangles with vertices

$$\begin{array}{lll} p_{i,j}p_{i+1,j} \quad p_{i,j+1} & 0 \leqslant i < n, & 0 \leqslant j < n-i \\ p_{i,j}p_{i+1,j-1}p_{i+1,j} & 0 \leqslant i < n-1, & 1 \leqslant j < n-i. \end{array}$$

The elements in Λ_ℓ will be $\tau \times (P_{0,3} + \frac{k}{n}h_{\Lambda_\ell,3}, P_{0,3} + \frac{k+1}{n}h_{\Lambda_\ell,3})$ for each $\tau \in \mathcal{T}$ and each $0 \leqslant k < n$. See Figure 5.2.

5.3.1.3 Macro–element with a singular vertex

We consider again a tetrahedral macro–element T with vertices P_0, P_1, P_2 and P_3, assuming that it has a singular vertex at P_0 and no singular edge. We construct a triangulation of T made of tetrahedra describing the barycentric coordinates of the vertices of each tetrahedron with respect to P_0, P_1, P_2 and P_3.

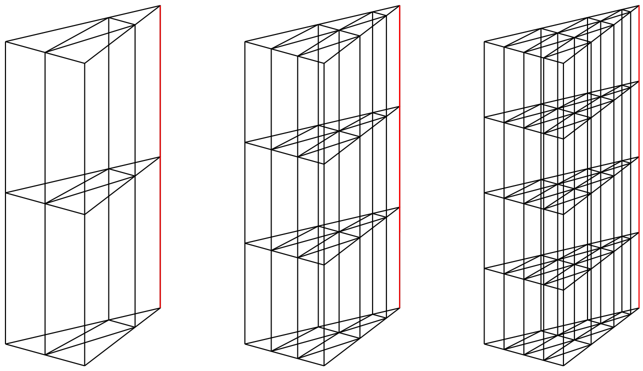


Figure 5.2 – Elements of the family of meshes restricted to a prismatic macro–element.$\mu = .63$

Let $p_{i,j,k}$ be the points with barycentric coordinates

$$\lambda_0 = 1 - \lambda_1 - \lambda_2 - \lambda_3,$$
$$\lambda_1 = \tfrac{i}{n}\left(\tfrac{i+j+k}{n}\right)^{1/\mu-1}, \quad \lambda_2 = \tfrac{j}{n}\left(\tfrac{i+j+k}{n}\right)^{1/\mu-1}, \quad \lambda_3 = \tfrac{k}{n}\left(\tfrac{i+j+k}{n}\right)^{1/\mu-1}$$

for $0 \leqslant i \leqslant n$, $0 \leqslant j \leqslant n-i$, $0 \leqslant k \leqslant n-i-j$. Then, the tetrahedra are the n^3 ones with the following vertices

$$\begin{aligned}
&p_{i,j,k}, p_{i+1,j,k}, p_{i,j+1,k}, p_{i,j,k+1}, && 0 \leqslant i+j+k \leqslant n-1,\\
&p_{i+1,j,k}, p_{i,j+1,k}, p_{i,j,k+1}, p_{i+1,j,k+1}, && 0 \leqslant i+j+k \leqslant n-2,\\
&p_{i,j+1,k}, p_{i,j,k+1}, p_{i+1,j,k+1}, p_{i,j+1,k+1}, && 0 \leqslant i+j+k \leqslant n-2,\\
&p_{i+1,j,k}, p_{i,j+1,k}, p_{i+1,j+1,k}, p_{i+1,j,k+1}, && 0 \leqslant i+j+k \leqslant n-2,\\
&p_{i,j+1,k}, p_{i+1,j+1,k}, p_{i+1,j,k+1}, p_{i,j+1,k+1}, && 0 \leqslant i+j+k \leqslant n-2,\\
&p_{i+1,j+1,k}, p_{i+1,j,k+1}, p_{i,j+1,k+1}, p_{i+1,j+1,k+1}, && 0 \leqslant i+j+k \leqslant n-3.
\end{aligned}$$

5.3.1 Proposition. *The pyramids and tetrahedra in the meshes defined in Subsection 5.3.1 are isotropic.*

Proof. It is enough to prove the proposition for pyramids, as the proof for tetrahedra is analogue (each tetrahedra in a macroelement with hybrid mesh shares two faces with pyramids). Consider a pyramid with vertices $p_0, \ldots, p_4$ in a macro–element of vertices P_0, P_1, P_2 and P_3 as in Subsection 5.3.1.1. With $|p_i - p_j|$ or $|P_i - P_j|$ we mean euclidean distance. Note that the basis of the pyramid is the paralelogram $p_0p_1p_3p_2$ with

$$p_1 - p_0 = p_3 - p_2 = \tfrac{1}{n}\left(\tfrac{n-l-1}{n}\right)^{1/\mu-1}(P_3 - P_2) \tag{5.9}$$
$$p_2 - p_0 = p_3 - p_1 = \left[\left(\tfrac{n-l}{n}\right)^{1/\mu} - \left(\tfrac{n-l-1}{n}\right)^{1/\mu}\right](P_0 - P_1). \tag{5.10}$$

So there hold

$$\begin{aligned}
\tfrac{1/\mu}{n}\left(\tfrac{n-l-1}{n}\right)^{1/\mu-1}|P_0 - P_1| \leqslant |p_2 - p_0| &= |p_3 - p_1|\\
&\leqslant \tfrac{1/\mu}{n}\left(\tfrac{n-l}{n}\right)^{1/\mu-1}|P_0 - P_1|,
\end{aligned}$$

and

$$\mu\left(\tfrac{1}{2}\right)^{1/\mu-1} \leqslant \mu\left(\tfrac{n-l-1}{n-l}\right)^{1/\mu-1} \leqslant \tfrac{|p_1-p_0|}{|p_2-p_0|} \leqslant \mu.$$

Then the parallelogram $p_0p_1p_3p_2$ is shape-regular since the angle between $P_0 - P_1$ and $P_3 - P_2$ depends only on the macro–element, and so it is away from 0 and π.

Now we prove that there exists constants c_0 and c_1 depending only on $1/\mu$ and the vertices of the macro–element such that

$$c_0 \leqslant \tfrac{|p_4-p_2|}{|p_2-p_0|} \leqslant c_1 \tag{5.11}$$

and

$$c_0 \leqslant \tfrac{|p_4-p_3|}{|p_2-p_0|} \leqslant c_1. \tag{5.12}$$

After simple computations we obtain

$$\begin{aligned} p_4 - p_2 &= \left[\left(\tfrac{n-l}{n}\right)^{1/\mu} - \left(\tfrac{n-l-1}{n}\right)^{1/\mu}\right](P_3 - P_0) \\ &\quad + \tfrac{i}{n}\left[\left(\tfrac{n-l}{n}\right)^{1/\mu-1} - \left(\tfrac{n-l-1}{n}\right)^{1/\mu-1}\right](P_2 - P_3) \\ &= \left[\left(\tfrac{n-l}{n}\right)^{1/\mu} - \left(\tfrac{n-l-1}{n}\right)^{1/\mu}\right]\Bigg\{P_3 - P_0 + \\ &\quad + \frac{\tfrac{i}{n}\left[\left(\tfrac{n-l}{n}\right)^{1/\mu-1} - \left(\tfrac{n-l-1}{n}\right)^{1/\mu-1}\right]}{\left[\left(\tfrac{n-l}{n}\right)^{1/\mu} - \left(\tfrac{n-l-1}{n}\right)^{1/\mu}\right]}(P_2 - P_3)\Bigg\} \\ &\sim \left[\left(\tfrac{n-l}{n}\right)^{1/\mu} - \left(\tfrac{n-l-1}{n}\right)^{1/\mu}\right](P_3 - P_0) \\ &\sim p_3 - p_1 \end{aligned}$$

where we used the following relation

$$\frac{\tfrac{i}{n}\left[\left(\tfrac{n-l}{n}\right)^{1/\mu-1} - \left(\tfrac{n-l-1}{n}\right)^{1/\mu-1}\right]}{\left[\left(\tfrac{n-l}{n}\right)^{1/\mu} - \left(\tfrac{n-l-1}{n}\right)^{1/\mu}\right]} \leqslant \tfrac{1/\mu-1}{1/\mu},$$

and also that the angle between $P_3 - P_0$ and $P_2 - P_3$ is fixed (and depends only on the macro–element) and equation (5.10). This proves (5.11). Inequalities in (5.12) follow analogously. To finish the proof of the isotropy of the pyramids we have to observe that the basis $p_0p_1p_3p_2$ is contained in a plane Π_1 that is parallel to the one generated by the directions $P_1 - P_0$ and $P_3 - P_2$, and the face $p_2p_3p_4$ is in a plane parallel to the plane Π_2 in which lies the triangle $P_0P_2P_3$ (this is a face of the macro–element), and the angle between planes Π_1 and Π_2 depends only on the macro–element. □

5.3.2 Remark. *Our meshes sastify the property that for each ℓ and each h there exists an index set $I_{h,\ell}$ such that*

$$\Lambda_\ell = \cup_{i \in I_{h,\ell}} E_{h,i}.$$

In this case we are denoting $\mathcal{T}_h = \{E_{h,i} \,:\, 1 \leqslant i \leqslant N_h\}$.

This means the meshes resolve the macro–elements in a conforming way and refinement is done within the macro–elements of the initial mesh.

5.3.3 Remark. *If a mesh satisfies conditions (5.8) for $\mu = \mu_0$ and $\nu = \nu_0$, then it satisfies the same for $\mu > \mu_0$ and $\nu > \nu_0$. Thus, the mesh can be constructed for $\mu_0 = \nu_0 = 1 - \gamma$ with $1 > \gamma > \max\{1 - \lambda_{\boldsymbol{e},S}, \frac{1}{2} - \lambda_{\boldsymbol{v},S}\}$, and it still verifies the former conditions. The possibility to use $\mu_0 = \nu_0$ for the construction of the mesh allows us to validate the assumptions at the beginning of Section 3.1, which are that the tetrahedra and pyramids in $\mathcal{T}_h$ don't have to be necessarily anisotropic.*

Therefore, we assume that the meshes $\{\mathcal{T}_h\}_h$ restricted to a macroelement Λ_ℓ with a singular edge A_S and a singular vertex S satisfy conditions (5.8) for $\mu, \nu \geqslant 1 - \gamma$. And in particular, those conditions hold for $\mu = 1 - \delta$ for some $\delta > 1 - \lambda_{\boldsymbol{e}}$ and for $\nu = 1 - \beta$ for some $\beta > \frac{1}{2} - \lambda_{\boldsymbol{v},S}$.

For the case of a tetrahedral macroelement with just a singular edge, which is assumed to have a face f perpendicular to the singular edge, we assume that it is meshed as in the case in which the opposite vertex to the face f is singular, satisfying conditions (5.8) with $\mu = \nu = 1 - \delta$. This assumption does not affect the asymptotic relation between h and the number of elements.

5.3.4 Remark. *If the diameter $\delta(\Lambda)$ of a macro–element Λ were greater than 1, then*

$$R_K(\boldsymbol{x}) \leqslant \delta(\Lambda) \min\{R_K(\boldsymbol{x}), 1\}$$

so we will assume $R_K(\boldsymbol{x}) \leqslant 1$.

5.3.5 Remark. $\mu < 1 \Rightarrow \mu \leqslant \nu$.

5.3.6 Remark. *The meshing procedure we propose fulfills conditions (5.6)–(5.8), as can be seen using the ideas of [3, 5, 38] and Section 8.4 of [26].*

5.4 Main Global Interpolation Error Theorem

In the present section we state the anisotropic interpolation error estimates over the meshes we constructed in the previous section. We leave the proof for the whole next Section since it has many parts.

Then, to estimate the approximation error of Theorem 5.6.1, we will take the inequalities of Theorem 5.1.7 and use each term in the right–hand sides (cfr. Section 5.6).

As in the meshing algorithm we proposed, the pyramids turned out to be isotropic, we will use Proposition 4.2.2 as the local interpolation estimate. Nevertheless, as we will show later, we proved anisotropic pyramidal finite element stability estimates in Theorems 6.4.2 and 6.4.4, and with them we obtained the results in Theorems 6.4.5 and 6.4.3 for the interpolation operators determined by the pyramidal finite elements in Definitions 6.3.1 and 6.3.4. So, we could use those anisotropic estimates from Chapter 6, and try to make the estimates in terms of weighted norms exactly as were made for the anisotropic prisms in the method we present here. These observations are made to point out that we may as well propose a full Finite Elements Method as an alternative.

The following is the main interpolation error theorem.

5.4.1 Theorem. *Let Ω and f be the data and $(\boldsymbol{u}, p)$ be the solution of Problem 1.2.2. Let $\boldsymbol{r}_0$ be the operator (5.3) corresponding to the mesh constructed in Subsection 5.3.1, then*

$$\|\boldsymbol{u} - \boldsymbol{r}_0\boldsymbol{u}\|_{0,\Omega} \leqslant Ch\|f\|_{0,\Omega} \tag{5.13}$$

$$\|p - P_{\mathcal{T}_h}p\|_{0,\Omega} \leqslant Ch\|\nabla p\|_{0,\Omega}. \tag{5.14}$$

5.4.2 Remark. *The operator (5.3) is well defined because, for any polyhedral element considered, for any of its faces, if a field $\boldsymbol{v}$ of the discrete global space has all the degrees of freedom with respect to that face equal to zero, then the restriction of $\boldsymbol{v}$ to that face vanishes identically.*

5.5 Proof of Theorem 5.4.1

For the scalar variable the estimate follows from Lemma 1.3.5. For the vectorial variable start splitting the field $\boldsymbol{u}$ as in (1.12). By Lemma 5.5.1, the field $\boldsymbol{r}_0\boldsymbol{u} = \boldsymbol{r}_0\boldsymbol{u}_s + \boldsymbol{r}_0\boldsymbol{u}_r$ is well defined, so we will work with the regular part $\boldsymbol{u}_r - \boldsymbol{r}_0\boldsymbol{u}_r$ and the singular part $\boldsymbol{u}_s - \boldsymbol{r}_0\boldsymbol{u}_s$ separately. The bound for the error of the interpolation of the singular part will be performed in each type of macro–element.

First we'd like to remark that, for a given macro–element in $\mathcal{T}_{h_0}$ the case when there is no weight with respect to the distance $R(\boldsymbol{x})$ to the vertex is also available by simply putting $R \equiv 1$ in the computations we show. The case when there is no weight with respect to the angular distance $\theta(\boldsymbol{x})$ is also available by simply putting $\theta \equiv 1$. Additionally, if $\boldsymbol{e}$ is not singular, the condition for δ in (5.8) turns $\delta > -\infty$ and if $\boldsymbol{v}$ is not singular, the condition for β turns $\beta > -\infty$.

Now it is the time to make a remark of crucial importance. If we take the solution $(\boldsymbol{u}, p)$ of Problem 1.2.1, the singular part of the vectorial variable, $\boldsymbol{u}_s$, has a well defined $H(div, \Omega)$–conforming interpolate. This is implied by the next result, since the normal traces on faces of the elements in $W^{1,1}(\Lambda_\ell)$ are well defined.

5.5.1 Lemma. *If $\beta, \delta \in [0, 1)$ and $\beta + \delta \leqslant 1$ then*

$$V^{1,2}_{\beta,\delta}(\Lambda_\ell) \subseteq W^{1,1}(\Lambda_\ell)$$

for all macro–element Λ_ℓ.

Proof. Every function $\boldsymbol{v} \in V^{1,2}_{\beta,\delta}(\Lambda_\ell)$ has a finite $L^1(\Lambda_\ell)$ norm. On the other hand,

$$R^{-\beta}\theta^{-\delta} \leqslant \left(\max_{\boldsymbol{x}\in\Lambda_\ell} R(\boldsymbol{x})^\delta\right) r^{-\beta-\delta} \leqslant Cr^{-\beta-\delta} \in L^2(\Lambda_\ell)$$

which implies $\partial^\alpha \boldsymbol{v} \in L^1(\Lambda_\ell)$ for all $\boldsymbol{v} \in V^{1,2}_{\beta,\delta}(\Lambda_\ell)$. □

5.5.1 Bound for the Regular Part in (5.13)

Take any macro–element Λ_ℓ. Given an element $E \subset \Lambda_\ell$, let $\boldsymbol{h} = (h_1, h_2, h_3)'$. By Theorem 4.1.9

$$\begin{aligned} \|\boldsymbol{u}_r - \boldsymbol{r}_0\boldsymbol{u}_r\|^2_{L^2(\Omega)} &= \sum_{E\in\mathcal{T}_h} \|\boldsymbol{u}_r - \boldsymbol{r}_0\boldsymbol{u}_r\|^2_{L^2(E)} \\ &\leqslant C \sum_{E\in\mathcal{T}_h} \left(\sum_{|\boldsymbol{\alpha}|=1} \boldsymbol{h}^{\boldsymbol{\alpha}} \|\partial^{\boldsymbol{\alpha}}\boldsymbol{u}_r\|_{L^2(E)} + h_E \|\mathrm{div}\, \boldsymbol{u}_r\|_{L^2(E)} \right)^2 \\ &\leqslant \sum_{E\in\mathcal{T}_h} \left\{ \sum_{|\boldsymbol{\alpha}|=1} \boldsymbol{h}^{2\boldsymbol{\alpha}} + h_E^2 \right\} \left\{ \sum_{|\boldsymbol{\alpha}|=1} \|\partial^{\boldsymbol{\alpha}}\boldsymbol{u}_r\|^2_{L^2(E)} + \|\mathrm{div}\, \boldsymbol{u}_r\|^2_{L^2(E)} \right\}. \end{aligned}$$

Now following the grading (5.8) it holds $\boldsymbol{h}^{\boldsymbol{\alpha}} \leqslant C\,h$ and $h_E \leqslant C\,h_3 \sim h$ for all the elements of the mesh, so the last expression is bounded above by

$$\begin{aligned} C^2 \sum_{E\in\mathcal{T}_h} 4\,h^2 \left\{ \sum_{|\boldsymbol{\alpha}|=1} \|\partial^{\boldsymbol{\alpha}}\boldsymbol{u}_r\|^2_{L^2(E)} + \|\mathrm{div}\boldsymbol{u}_r\|^2_{L^2(E)} \right\} = \\ = C\,h^2 \left\{ \sum_{|\boldsymbol{\alpha}|=1} \|\partial^{\boldsymbol{\alpha}}\boldsymbol{u}_r\|^2_{L^2(\Omega)} + \|\mathrm{div}\boldsymbol{u}_r\|^2_{L^2(\Omega)} \right\}. \end{aligned}$$

Then, finally

$$\begin{aligned} \|\boldsymbol{u}_r - \boldsymbol{r}_0\boldsymbol{u}_r\|_{L^2(\Omega)} &\leqslant C\,h\,|\boldsymbol{u}_r|_{1,\Omega} \\ \text{by (1.13)} \qquad &\leqslant C\,h\,\|f\|_{0,\Omega}. \end{aligned}$$

5.5.2 Bound for the Singular Part in (5.13) in a prismatic macro–element with a singular Edge

Let Λ_ℓ be a prismatic element of $\mathcal{T}_{h_0}$. Let $\boldsymbol{e}$ be the singular edge of Λ_ℓ and let $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)'$ be it's local coordinates, all as in the hypotheses of Theorem 1.2.7. In this case we need to distinguish between the elements E with $d(E, \boldsymbol{e}) > 0$ and those with $d(E, \boldsymbol{e}) = 0$.

$$\begin{aligned} \|\boldsymbol{u}_s - \boldsymbol{r}_0\boldsymbol{u}_s\|^2_{L^2(\Lambda_\ell)} \leqslant &\sum_{d(E,\boldsymbol{e})=0} \left(\|\boldsymbol{u}_s\|_{L^2(E)} + \|\boldsymbol{r}_0\boldsymbol{u}_s\|_{L^2(E)} \right)^2 \\ &+ \sum_{d(E,\boldsymbol{e})>0} \left(\sum_{|\boldsymbol{\alpha}|=1} \boldsymbol{h}^{\boldsymbol{\alpha}} \|\partial^{\boldsymbol{\alpha}}\boldsymbol{u}_s\|_{L^2(E)} + h_E \|\mathrm{div}\boldsymbol{u}_s\|_{L^2(E)} \right)^2. \end{aligned} \tag{5.15}$$

The objective of the following paragraphs and items is to bound each one of the terms involved in the right hand side of (5.15) by a constant times h times the norm of f, to sum everything up afterwards.

Elements with $d(E,\boldsymbol{e})>0$ in (5.15).

1. Bound for $\|\partial_{\xi_1}\boldsymbol{u}_s\|_{L^2(E)}$. Pick any weight $w(\boldsymbol{x})\geqslant h_1$.

$$h_1\|\partial_{\xi_1}\boldsymbol{u}_s\|_{L^2(E)}=\sum_{i=1,2}\|h_1\partial_{\xi_1}u_{s,i}\|_{L^2(E)}+h_1\|\partial_{\xi_1}u_{s,3}\|_{L^2(E)}$$
$$\leqslant\sum_{i=1,2}\|w(\boldsymbol{x})\partial_{\xi_1}u_{s,i}\|_{L^2(E)}+h_1\|u_{s,3}\|_{V_0^{1,2}(E)}.$$

Now it turns out that if we could choose $w(\boldsymbol{x})=h\,r(\boldsymbol{x})^\delta\geqslant h_1$ for some δ, then we would have

$$h_1\|\partial_{\xi_1}\boldsymbol{u}_s\|_{L^2(E)}\leqslant h\sum_{i=1,2}\|u_{s,i}\|_{V_\delta^{1,2}(E)}+h\|u_{s,3}\|_{V_0^{1,2}(E)}$$
$$\leqslant C\,h\|\boldsymbol{u}_s\|_{\mathcal{V}_{\beta,\delta}(E)}. \tag{5.16}$$

Let's look for δ. We have $r(\boldsymbol{x})\geqslant d(E,\boldsymbol{e})$ so, according to the grading (5.8) , in the case $0<d(E,\boldsymbol{e})<1$, it holds

$$h\,r(\boldsymbol{x})^{1-\mu}\geqslant h\,d(E,\boldsymbol{e})^{1-\mu}\sim h_1.$$

In the case $d(E,\boldsymbol{e})\sim 1$

$$h\,r(\boldsymbol{x})^{1-\mu}\gtrsim h\sim h_1.$$

In both cases we have to take $\delta\sim 1-\mu>1-\frac{\pi}{\omega_\ell}$ to get the estimate (5.16). The bound for the term $\|\partial_{\xi_2 2}\boldsymbol{u}_s\|$ is done similarly.

2. Bound for $\|\partial_{\xi_3}\boldsymbol{u}_s\|_{L^2(E)}$. If $i=1,2$ or 3, $\partial_{\xi_3}u_i$ equals $\partial_{\xi_i}u_3$, so

$$h_3^2\|\partial_{\xi_3}\boldsymbol{u}_s\|_{L^2(E)}^2=h_3^2\sum_{i=1,2,3}\|\partial_{\xi_3}u_{s,i}\|_{L^2(E)}^2$$
$$\lesssim h^2\,\|u_3\|_{V_0^{1,2}(E)}^2.$$

3. Bound for the divergence. By the grading in the present macro–element and the triangle inequality we have

$$h_E\,\|\operatorname{div}\boldsymbol{u}_s\|_{L^2(E)}\lesssim h_3\,\|\operatorname{div}\boldsymbol{u}_s\|_{L^2(E)}\sim h\,\|\operatorname{div}\boldsymbol{u}_s\|_{L^2(E)}$$
$$\leqslant h\,\left\{\|\operatorname{div}\boldsymbol{u}\|_{L^2(E)}+\|\operatorname{div}\boldsymbol{u}_r\|_{L^2(E)}\right\}. \tag{5.17}$$

Elements with $d(E,\boldsymbol{e})=0$ in (5.15). Take $\delta\sim 1-\mu$ in this paragraph again.

1. Bound for $\|\boldsymbol{u}_s\|_{L^2(E)}$.

$$
\begin{aligned}
\|\boldsymbol{u}_s\|^2_{L^2(E)} &= \sum_{i=1,2} \|u_{s,i}\|^2_{L^2(E)} + \|u_{s,3}\|^2_{L^2(E)} \\
&= \sum_{i=1,2} \|r(\boldsymbol{x})^{1-\delta} r(\boldsymbol{x})^{\delta-1} u_{s,i}\|^2_{L^2(E)} + \|r\, r^{-1} u_{s,3}\|^2_{L^2(E)} \\
&\leqslant \max r(\boldsymbol{x})^{2(1-\delta)} \sum_{i=1,2} \|u_{s,i}\|^2_{V^{1,2}_\delta(E)} + \max_{\boldsymbol{x}\in E} r(\boldsymbol{x})^2 \|u_{s,3}\|^2_{V^{1,2}_0(E)} \\
&\lesssim h_1^{2(1-\delta)} \sum_{i=1,2} \|u_{s,i}\|^2_{V^{1,2}_\delta(E)} + h_1 \|u_{s,3}\|^2_{V^{1,2}_0(E)} \\
&\sim (h^{2/\mu})^{\mu} \sum_{i=1,2} \|u_{s,i}\|^2_{V^{1,2}_\delta(E)} + (h^{1/\mu})^2 \|u_{s,3}\|^2_{V^{1,2}_0(E)} \\
&\leqslant C\, h^2 \|\boldsymbol{u}_s\|^2_{\mathcal{V}^{1,2}_\delta(E)}
\end{aligned}
$$

(because $\mu \leqslant 1$).

2. Bound for $\|\boldsymbol{r}_E \boldsymbol{u}_s\|_{L^2(E)^3}$. First recall that if ϕ is a scalar polynomial defined on a physical element E, then

$$
\|\phi\|_{L^2(E)} \leqslant C\, |E|^{-1/2}\, \|\phi\|_{L^1(E)}. \tag{5.18}
$$

Now we estimate the L^1 norms, starting with the stability estimate in the rescaled element $\tilde{E}$ of Figure (4.1). As in our mesh we are considering right prisms and local coordinates in Λ_ℓ such that ξ_3 is the direction containing the singular edge $\boldsymbol{e}_\ell$, there is a matrix M_E of the form

$$
M_E = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.19}
$$

for which an affine transform F_E with matrix M_E maps $\tilde{E}$ onto E. The infinity norm of $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is bounded by a quantity c_E depending only on the maximum angle of the projection of E onto the (ξ_1, ξ_2) plane and the infinity norm of its inverse is bounded by one. So changing variables and pulling back to $\tilde{E}$ we get

$$
\begin{aligned}
\|(\boldsymbol{r}_E \boldsymbol{u}_s)_1\|_{L^1(E)} &= \int_E |(\boldsymbol{r}_E \boldsymbol{u}_s)_1|\, d\boldsymbol{x} = \int_{\tilde{E}} |(M_E\, \tilde{\boldsymbol{r}}_{\tilde{E}} \tilde{\boldsymbol{u}}_s)_1|\, d\tilde{\boldsymbol{x}} \\
&\leqslant c_E \Big\{ \sum_{i=1,2} \|\tilde{u}_{s,i}\|_{L^1(\tilde{E})} + \sum_{i=1}^{3} h_i \left(\|\partial_i \tilde{u}_{s,1}\|_{L^1(\tilde{E})} + \|\partial_i \tilde{u}_{s,2}\|_{L^1(\tilde{E})} \right) \\
&\qquad + 2h_1 \|\mathrm{div}\,(\tilde{u}_{s,1}, \tilde{u}_{s,2}, 0)\|_{L^1(\tilde{E})} \Big\}.
\end{aligned} \tag{5.20}
$$

Now the work comes to rewrite and bound each term in (5.20). Recalling the properties (2.33) and (2.34) about the derivatives with coordinate changes,

$$\begin{aligned}\|\tilde{u}_{s,1}\|_{L^1(\tilde{E})} &= \frac{1}{|\det(M_E)|}\int_E \left|\tilde{u}_{s,1}(F_E^{-1}(\boldsymbol{x}))\right| \, d\boldsymbol{x} \\ &= \int_E |(M_E^{-1})_{\mathrm{row}_1}\boldsymbol{u}_s(\boldsymbol{x})| \, d\boldsymbol{x} \\ &\leqslant \|M_E^{-1}\|_\infty \sum_{i=1,2}\|u_{s,i}\|_{L^1(E)} \\ &\leqslant \sum_{i=1,2}\|u_{s,i}\|_{L^1(E)}\end{aligned}$$

and by the same reasons

$$\|\tilde{u}_{s,2}\|_{L^1(\tilde{E})} \leqslant \sum_{i=1,2}\|u_{s,i}\|_{L^1(E)}.$$

For the first derivatives, let $k, l = 1$ or 2.

$$\begin{aligned}\|\partial_{\tilde{x}_l}\tilde{u}_{s,k}\|_{L^1(\tilde{E})} &= \frac{1}{|\det(M_E)|}\int_E \left|\det(M_E)\,(M_E^{-1})_{\mathrm{row}_k}\, D\boldsymbol{u}_s(\boldsymbol{x})\,(M_E)_{\mathrm{col}_l}\right| \, d\boldsymbol{x} \\ &\leqslant \sum_{i=1,2}\|\partial_{\xi_l}u_{s,i}\|_{L^1(E)}\end{aligned}$$

and, similarly, if $k = 1, 2$

$$\|\partial_{\tilde{x}_3}\tilde{u}_{s,k}\|_{L^1(\tilde{E})} \leqslant \|M_E^{-1}\|_\infty \sum_{i=1,2}\|\partial_{\xi_3}u_{s,i}\|_{L^1(E)}.$$

To cope with the divergence term, because of the blocks of the matrix M_E we have

$$\begin{aligned}(u_1(F_E\tilde{\boldsymbol{x}}), u_2(F_E\tilde{\boldsymbol{x}}), 0)' &= \frac{1}{\det M_E} M_E\,(\tilde{u}_1(\tilde{\boldsymbol{x}}), \tilde{u}_2(\tilde{\boldsymbol{x}}), 0)', \\ \mathrm{div}\,(u_1, u_2, 0)' &= \frac{1}{\det M_E}\mathrm{div}\,(\tilde{u}_1, \tilde{u}_2, 0)'.\end{aligned}$$

Then, changing variables,

$$\|\mathrm{div}\,(\tilde{u}_{s,1}, \tilde{u}_{s,2}, 0)'\|_{L^1(\tilde{E})} = \|\mathrm{div}\,(u_{s,1}, u_{s,2}, 0)'\|_{L^1(E)}.$$

Finally joining everything,

$$\begin{aligned}\|(\boldsymbol{r}_E\boldsymbol{u}_s)_1\|_{L^1(E)} \leqslant C\Bigg\{&\sum_{i=1,2}\Bigg\{\|u_{s,i}\|_{L^1(E)} \\ &+ h_i\sum_{j=1,2}\|\partial_{\xi_i}u_{s,j}\|_{L^1(E)^3} + h_3\|\partial_{\xi_3}u_{s,i}\|_{L^1(E)}\Bigg\} \\ &+ h_1\|\mathrm{div}\,(u_{s,1}, u_{s,2}, 0)'\|_{L^1(E)}\Bigg\}. \qquad (5.21)\end{aligned}$$

Now we enumerate the bound for each term on the right of (5.21).

(a) For $\|u_{s,i}\|$ we have

$$\begin{aligned}\|u_{s,i}\|_{L^1(E)} &= \|r(\boldsymbol{x})^{1-\delta} r(\boldsymbol{x})^{\delta-1} u_{s,i}\|_{L^1(E)} \\ &\leqslant \|r^{1-\delta}\|_{L^2(E)} \|u_{s,i}\|_{V_\delta^{1,2}(E)} \\ &\leqslant C\, h_1^{1-\delta}\, |E|^{1/2}\, \|u_{s,i}\|_{V_\delta^{1,2}(E)} \\ &\leqslant C\, h\, |E|^{1/2}\, \|u_{s,i}\|_{V_\delta^{1,2}(E)}.\end{aligned}$$

(b) With respect to the derivatives orthogonal to the singular edge, take $i,j = 1,2$ and, by Hölder's inequality,

$$h_1 \|\partial_{\xi_j} u_{s,i}\|_{L^1(E)} \leqslant h_1\, \|r^{-\delta}\|_{L^2(E)}\, \|u_{s,i}\|_{V_\delta^{1,2}(E)}. \tag{5.22}$$

Integrating the radial weight $r^{-\delta}$ we get

$$\begin{aligned}\|r^{-\delta}\|_{L^2(E)} &\leqslant C\, h_1^{-\delta}\, h_1\, h_3^{1/2} \\ &\sim h_1^{-\delta} (h_1 h_2 h_3)^{1/2} \\ &\leqslant C\, h_1^{-\delta} |E|^{1/2},\end{aligned}$$

(where we used $h_1 \sim h_2$ and the minimun angle condition to bound the area of a triangular face from below with $h_1 h_2$), so in (5.22) we have

$$\begin{aligned}h_1 \|\partial_{\xi_j} u_{s,i}\|_{L^1(E)} &\leqslant C\, h_1^{1-\delta} |E|^{1/2} \|u_{s,i}\|_{V_\delta^{1,2}(E)} \\ &\sim (h^{1/\mu})^{\mu} |E|^{1/2} \|u_{s,i}\|_{V_\delta^{1,2}(E)} \\ &= |E|^{1/2}\, h\, \|u_{s,i}\|_{V_\delta^{1,2}(E)}. \end{aligned} \tag{5.23}$$

(c) For the derivatives along the singular edge we use $\boldsymbol{u} = \nabla p$ to commute the indices of the derivatives and Hölder's inequality. We have

$$\begin{aligned}h_3\, \|\partial_{\xi_3} u_{s,i}\|_{L^1(E)} &= h_3\, \|\partial_{\xi_i} u_{s,3}\|_{L^1(E)} \\ &\leqslant h_3\, |E|^{1/2}\, \|\partial_{\xi_i} u_{s,3}\|_{L^2(E)} \\ &\leqslant C\, |E|^{1/2}\, h\, \|u_{s,3}\|_{V_0^{1,2}(E)}. \end{aligned} \tag{5.24}$$

(d) For the divergence we can recall the grading condition $h_1 \sim h_2$ and observe that

$$h_1 \|\mathrm{div}(u_{s,1}, u_{s,2}, 0)\|_{L^1(E)} \leqslant h_1 \sum_{j=1,2} \|\partial_{\xi_j} u_{s,j}\|_{L^1(E)}$$

and reuse the estimates (5.23).

The estimate for $\|(\boldsymbol{r}_0 \boldsymbol{u}_s)_2\|_{L^2(E)}$ is the same. The estimate for component $(\boldsymbol{r}_0 \boldsymbol{u}_s)_3$ is as follows. By the commutativity of the local interpolator and the coordinate change in Corollary 2.3.17 and the estimate (4.13) in the proof Theorem 4.1.5 we have

$$\begin{aligned}\|(\boldsymbol{r}_E\boldsymbol{u}_s)_3\|_{L^1(E)} &= \|(\boldsymbol{r}_{\tilde{E}}\tilde{\boldsymbol{u}}_s)_3\|_{L^1(\tilde{E})}\\ &\leqslant C\,\Big\{\,\|\tilde{u}_{s,3}\|_{L^1(\tilde{E})} + \sum_{j=1}^{3} h_j\,\|\partial_{\tilde{x}_j}\tilde{u}_{s,3}\|_{L^1(\tilde{E})}\,\Big\}\end{aligned}$$

which by (5.19) and the considerations made there is bounded by

$$C\,(\,\|u_{s,3}\|_{L^1(E)} + \sum_{j=1}^{3} h_j\|\partial_{\xi_j}u_{s,3}\|_{L^1(E)}).$$

Now

$$\begin{aligned}\|u_{s,3}\|_{L^1(E)} &\leqslant \|r\|_{L^2(E)}\,\|r^{-1}\,u_{s,3}\|_{L^2(E)}\\ &\leqslant C\,h_1\,|E|^{1/2}\,\|u_{s,3}\|_{V_0^{1,2}(E)}\\ &\leqslant C\,h\,|E|^{1/2}\,\|u_{s,3}\|_{V_0^{1,2}(E)}.\end{aligned}$$

We have also

$$(h_1+h_2)\sum_{i=1,2}\|\partial_{x_i}u_{s,3}\|_{L^1(E)} \leqslant C\,|E|^{1/2}\,h\,\|u_{s,3}\|_{V_0^{1,2}(E)}$$

and by (5.24)

$$h_3\,\|\partial_{\xi_3}u_{s,3}\|_{L^1(E)} \leqslant C\,|E|^{1/2}\,h\,\|u_{s,3}\|_{V_0^{1,2}(E)}.$$

At last, having the bound for each of the three terms in $\|\boldsymbol{r}_E\boldsymbol{u}_s\|_{L^2(E)^3}$, (5.18) yields

$$\|\boldsymbol{r}_E\boldsymbol{u}_s\|_{L^2(E)^3} \leqslant C\,h\,\|\boldsymbol{u}_s\|_{\mathcal{V}_\delta(E)}.$$

5.5.3 Bound for the Singular Part of (5.13) in a Tetrahedral Macro–Element with Singular Edge and Vertex

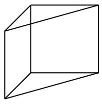
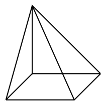
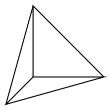
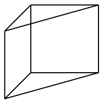
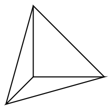
Again we organize separating cases in terms of distance to the singular edge and then, for this type of macro–element, the terms with $d(E,\boldsymbol{e})=0$ will be separated in terms of distance to the singular vertex.

Table 5.5 shows what has to be done for each type of element. The rest of the needed estimates are contained in the proof of the former case.

Take a tetrahedral macro–element Λ_ℓ in $\mathcal{T}_{h_0}$ as in the title of the subsection. Let us start with

$$\|\boldsymbol{u}_s-\boldsymbol{r}_0\boldsymbol{u}_s\|^2_{L^2(\Lambda_\ell)^3} = \sum_{d(E,\boldsymbol{e})=0}\|\boldsymbol{u}_s-\boldsymbol{r}_0\boldsymbol{u}_s\|^2_{L^2(E)^3} + \sum_{d(E,\boldsymbol{e})>0}\|\boldsymbol{u}_s-\boldsymbol{r}_0\boldsymbol{u}_s\|^2_{L^2(E)^3}. \tag{5.25}$$

Table 5.5 – Singular Part.

$d(E, \boldsymbol{e}) > 0$			
$d(E, \boldsymbol{e}) = 0$	 $d(E,\boldsymbol{v})>0$		 $d(E,\boldsymbol{v})=0$

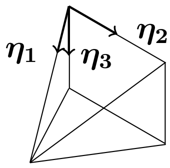


Figure 5.3 – Macro–element directions in a pyramid

Terms in (5.25) such that $d(E, \boldsymbol{e}) > 0$**.**

1. Case E is a pyramid. Remember the local ordering for the directions of the edges of sizes h_1, h_2 and h_3 for pyramids in our setting (cfr. Figure 5.3).

   As pyramids in $\mathcal{T}_h$ don't touch singularities and are regular, and $1-\mu \geqslant 1-\nu$, we have

$$\begin{aligned}
\|\boldsymbol{u}_s - \boldsymbol{r}_0\boldsymbol{u}_s\|^2_{L^2(E)} &\leqslant h_E^2|\boldsymbol{u}_s|^2_{1,E} \leqslant h^2 d(E,\boldsymbol{e})^{2(1-\mu)}\sum_{i=1}^{3}|u_{s,i}|^2_{1,E} \\
&\leqslant h^2\sum_{i=1}^{3}\sum_{j=1}^{3}\|r^{1-\mu}\partial_{\eta_j}u_{s,i}\|^2_{L^2(E)} \\
&\leqslant h^2\left(\sum_{i=1}^{2}\sum_{j=1}^{3}\|R^{1-\mu}\theta^{1-\mu}\partial_{\eta_j}u_{s,i}\|^2_{L^2(E)} + \sum_{j=1}^{3}\|R^{1-\mu}\partial_{\eta_j}u_{s,3}\|^2_{L^2(E)}\right) \\
&\leqslant h^2\left(\sum_{i=1}^{2}\sum_{j=1}^{3}\|R^{1-\nu}\theta^{1-\mu}\partial_{\eta_j}u_{s,i}\|^2_{L^2(E)} + \sum_{j=1}^{3}\|R^{1-\nu}\partial_{\eta_j}u_{s,3}\|^2_{L^2(E)}\right) \\
&\leqslant h^2\left(\sum_{i=1}^{2}\sum_{j=1}^{3}\|R^{\beta}\theta^{\delta}\partial_{\eta_j}u_{s,i}\|^2_{L^2(E)} + \sum_{j=1}^{3}\|R^{\beta}\partial_{\eta_j}u_{s,3}\|^2_{L^2(E)}\right) \\
&\leqslant h^2\|\boldsymbol{u}_s\|^2_{\mathcal{V}_{\beta,\delta}(E)}. \qquad (5.26)
\end{aligned}$$

2. Case E is a prism (with $d(E, \boldsymbol{e}) > 0$) or a tetrahedron. In the case of prisms we use the result of Theorem 4.1.9 and in the case of tetrahedra we use

Theorem 6.2 of [1] to get

$$\|\boldsymbol{u}_s - \boldsymbol{r}_0\boldsymbol{u}_s\|^2_{L^2(K)} \leqslant C\left\{\sum_{i=1}^{3} h_i^2\|\partial_{\xi_i}\boldsymbol{u}_s\|^2 + h_K^2\|\mathrm{div}\,\boldsymbol{u}_s\|^2\right\}. \tag{5.27}$$

Now the treatment is a slight variation of the previous item including a few small tricks. In fact,

$$\begin{aligned}
h_1^2\|\partial_{\xi_1}\boldsymbol{u}_s\|^2_{L^2(E)^3} &= h_1^2\sum_{i=1}^{3}\|\partial_{\xi_1}u_{s,i}\|^2_{L^2(E)} \leqslant h^2\sum_{i=1}^{3}\|r^{1-\mu}\partial_{\xi_1}u_{s,i}\|^2_{L^2(E)}\\
&\leqslant h^2\left(\sum_{i=1}^{2}\|R^{1-\mu}\theta^{1-\mu}\partial_{\xi_1}u_{s,i}\|^2_{L^2(E)} + \|R^{1-\mu}\partial_{\xi_1}u_{s,3}\|^2_{L^2(E)}\right)\\
&\leqslant h^2\left(\sum_{i=1}^{2}\|R^{1-\nu}\theta^{1-\mu}\partial_{\xi_1}u_{s,i}\|^2_{L^2(E)} + \|R^{1-\nu}\partial_{\xi_1}u_{s,3}\|^2_{L^2(E)}\right)\\
&\leqslant h^2\left(\sum_{i=1}^{2}\|R^{\beta}\theta^{\delta}\partial_{\xi_1}u_{s,i}\|^2_{L^2(E)} + \|R^{\beta}\partial_{\xi_1}u_{s,3}\|^2_{L^2(E)}\right)\\
&\leqslant h^2\|\boldsymbol{u}_s\|^2_{\mathcal{V}_{\beta,\delta}(E)}.
\end{aligned}$$

Now, since $h_2 \sim h_1$, with the same argument,

$$h_2\|\partial_{\xi_2}\boldsymbol{u}_s\|_{L^2(E)^3} = h_2\sum_{i=1}^{3}\|\partial_{\xi_2}u_{s,i}\| \leqslant h\|\boldsymbol{u}_s\|_{\mathcal{V}_{\beta,\delta}(E)}.$$

For the derivative with respect to ξ_3 in (5.27),

$$\begin{aligned}
h_3^2\|\partial_{\xi_3}\boldsymbol{u}_s\|^2_{L^2(E)^3} &= h_3^2\sum_{i=1}^{3}\|\partial_{\xi_3}u_{s,i}\|^2_{L^2(E)}\\
&= h_3^2\left(\sum_{i=1}^{2}\|\partial_{\xi_i}u_{s,3}\|^2_{L^2(E)} + \|\partial_{\xi_3}u_{s,3}\|^2_{L^2(E)}\right)\\
&\leqslant h^2\left(\sum_{i=1}^{2}\|d(E,v)^{1-\nu}\partial_{\xi_i}u_{s,3}\|^2_{L^2(E)} + \|d(E,v)^{1-\nu}\partial_{\xi_3}u_{s,3}\|^2_{L^2(E)}\right)\\
&\leqslant h^2\left(\sum_{i=1}^{2}\|R^{\beta}\partial_{\xi_i}u_{s,3}\|^2_{L^2(E)} + \|R^{\beta}\partial_{\xi_3}u_{s,3}\|^2_{L^2(E)}\right)\\
&= h^2\sum_{i=1}^{3}\|R^{\beta}\partial_{\xi_i}u_{s,3}\|^2_{L^2(E)}\\
&\leqslant 3h^2\|u_{s,3}\|^2_{V^{1,2}_{\beta,0}(E)}.
\end{aligned}$$

The divergence term in (5.27) goes like in (5.17).

Terms in (5.25) such that $d(E, \boldsymbol{e}_\ell) = 0$. In this case we have anisotropic prisms and tetrahedra from an isotropic subfamily, one in each tetrahedral macro–element of the present type.

From (5.25), after triangle inequality, we write

$$\sum_{d(E,\boldsymbol{e})=0} \|\boldsymbol{u}_s - \boldsymbol{r}_0\boldsymbol{u}_s\|^2_{L^2(E)^3} \lesssim$$
$$\lesssim \|\boldsymbol{r}_0\boldsymbol{u}_s\|^2_{L^2(T_\ell)^3} + \sum_{\substack{\text{prisms } P\\ d(P,\boldsymbol{e})=0}} \|\boldsymbol{r}_0\boldsymbol{u}_s\|^2_{L^2(P)^3} + \sum_{d(E,\boldsymbol{e})=0} \|\boldsymbol{u}_s\|^2_{L^2(E)^3}. \tag{5.28}$$

Here we put T_ℓ for the mentioned tetrahedron and E for generic elements in Λ_ℓ. Now let us bound each term.

1. For the last term on the right of (5.28), let E be any of both the tetrahedron with the singular vertex or a prism. If E is a prism, recall in this case we have $d(E, \boldsymbol{e}) = 0$ and $d(E, \boldsymbol{v}) > 0$. So

$$\begin{aligned}\|\boldsymbol{u}_s\|^2_{L^2(E)^3} &= \sum_{i=1}^{2} \|R^\nu\theta^\mu R^{-\nu}\theta^{-\mu}u_{s,i}\|^2_{0,E} + \|R^\nu\theta^{-1}R^{-\nu}\theta u_{s,3}\|^2_{0,E}\\ &\leqslant \|R^\nu\theta^\mu\|^2_{L^\infty(E)} \sum_{i=1}^{2} \|R^{-\nu}\theta^{-\mu}u_{s,i}\|^2_{0,E}\\ &\qquad + \|R^\nu\theta\|^2_{\infty,E}\|R^{-\nu}\theta^{-1}u_{s,3}\|^2_{0,E}.\end{aligned}$$

   Since $\theta < 1$ and $\mu \leqslant \nu < 1$, then

$$R(\boldsymbol{x})^\nu\theta(\boldsymbol{x}) \leqslant R(\boldsymbol{x})^\nu\theta(\boldsymbol{x})^\mu \leqslant R(\boldsymbol{x})^\mu\theta(\boldsymbol{x})^\mu = r(\boldsymbol{x})^\mu. \tag{5.29}$$

   Using this we get

$$\begin{aligned}\|\boldsymbol{u}_s\|^2_{0,E} &\leqslant \|r^\mu\|^2_{\infty,E}\left(\sum_{i=1}^{2} \|R^{-\nu}\theta^{-\mu}u_{s,i}\|^2_{0,E} + \|R^{-\nu}\theta^{-1}u_{s,3}\|^2_{0,E}\right)\\ &\leqslant \max\{h_1,h_2\}^{2\mu}\left(\sum_{i=1}^{2} \|R^{-\nu}\theta^{-\mu}u_{s,i}\|^2_{0,E} + \|R^{-\nu}\theta^{-1}u_{s,3}\|^2_{0,E}\right)\\ &\lesssim h^2\|\boldsymbol{u}_s\|^2_{\mathcal{V}_{\beta,\delta}(E)}\end{aligned}$$

   provided we take $\beta \sim 1-\nu$ and $\delta \sim 1-\mu$.

2. For the first term in (5.28), although, as we said, the tetrahedra touching the singular vertex, one for each subsequent mesh restricted to Λ_ℓ, form a regular subfamily, we estimate the term with the weighted norms anisotropically for the sake of completeness and generality. Recall T_ℓ also has an edge

contained in the singular edge of Λ_ℓ. We work for the first component of the local interpolate in the $L^1(T_\ell)$ norm using (5.18).

$$\|(\boldsymbol{r}_0\boldsymbol{u}_s)_1\|_{L^2(T_\ell)} \leqslant C|T_\ell|^{-1/2}\|(\boldsymbol{r}_0\boldsymbol{u}_s)_1\|_{L^1(T_\ell)}.$$

Remember that by Lemma 5.5.1 we know $\boldsymbol{u}_s$ is in $W^{1,1}(\Lambda_\ell)^3$. We start pulling $\boldsymbol{r}_0\boldsymbol{u}_s$ back to a rescaled reference tetrahedron $\tilde{T}$ with the mapping $F_{T_\ell}(\tilde{\boldsymbol{x}}) = M_{T_\ell}\tilde{\boldsymbol{x}} + \boldsymbol{x}_{T_\ell}$ (cfr. Figure 5.4).

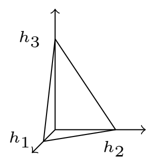


Figure 5.4 – Rescaled Tetrahedron

Using Lemma 5.22 in page 123 of [33], which gives the analog of our statement in (2.55) for tetrahedra, and then using Proposition 3.4 of [1], we get

$$\begin{aligned}
\|(\boldsymbol{r}_0\boldsymbol{u}_s)_1\|_{L^1(T_\ell)} &= \int_{\tilde{T}} |(M_{T_\ell}\tilde{\boldsymbol{r}}_0\tilde{\boldsymbol{u}}_s)_1|\, d\tilde{\boldsymbol{x}} \\
&\leqslant C\Bigg\{ \sum_{1\leqslant i\leqslant 3} \|\tilde{u}_{s,i}\|_{L^1(\tilde{T})} + \sum_{1\leqslant j\leqslant 3} h_j\|\partial_j\tilde{u}_{s,i}\|_{L^1(\tilde{T})} \\
&\qquad + h_i\|\mathrm{div}\,\tilde{\boldsymbol{u}}_s\|_{L^1(\tilde{T})}\Bigg\} \\
&\leqslant C\Bigg\{ \|\boldsymbol{u}_s\|_{L^1(T_\ell)^3} + \sum_{1\leqslant j\leqslant 3} h_j\|\partial_{\xi_j}\boldsymbol{u}_s\|_{L^1(T_\ell)^3} \\
&\qquad + (h_1+h_2+h_3)\|\mathrm{div}\,\boldsymbol{u}_s\|_{L^1(T_\ell)}\Bigg\}
\end{aligned} \tag{5.30}$$

where C depends only on the maximum angle of T_ℓ. Now we estimate each term on the right hand side of (5.30).

(2a) By Hölder's inequality

$$\begin{aligned}
\|\boldsymbol{u}_s\|_{L^1(T_\ell)^3} \leqslant \sum_{i=1,2} &\|R^\nu\theta^\mu\|_{L^2(T_\ell)}\|R^{-\nu}\theta^{-\mu}u_{s,i}\|_{L^2(T_\ell)} \\
&+ \|R^\nu\theta\|_{L^2(T_\ell)}\|R^{-\nu}\theta^{-1}u_{s,3}\|_{L^2(T_\ell)}
\end{aligned}$$

and by (5.29) and the grading criterion (5.8) we have

$$\begin{aligned}
\|\boldsymbol{u}_s\|_{L^1(T_\ell)^3} &\leqslant (h^{1/\mu})^\mu|T_\ell|^{1/2}\|\boldsymbol{u}_s\|_{V^{1,2}_{\beta,\delta}(T_\ell)^2\times V^{1,2}_{\beta,0}(T_\ell)} \\
&= h\,|T_\ell|^{1/2}\|\boldsymbol{u}_s\|_{\mathcal{V}_{\beta,\delta}(T_\ell)}.
\end{aligned}$$

(2b) Take $1 \leqslant j, i \leqslant 3$. Then

$$h_j \|\partial_{\xi_j} u_{s,i}\|_{L^1(T_\ell)} \leqslant h_j \|R^{\nu-1}\theta^{\mu-1}\|_{0,T_\ell} \|R^{1-\nu}\theta^{1-\mu}\partial_{\xi_j} u_{s,i}\|_{0,T_\ell}.$$

As it holds $0 \leqslant 1-\nu \leqslant 1-\mu < 1$, we get

$$R(\boldsymbol{x})^{1-\nu} \geqslant R(\boldsymbol{x})^{1-\mu}$$

and

$$R(\boldsymbol{x})^{\nu-1}\theta(\boldsymbol{x})^{\mu-1} \leqslant R(\boldsymbol{x})^{\mu-1}\theta(\boldsymbol{x})^{\mu-1} = r(\boldsymbol{x})^{\mu-1}.$$

From this we will use that

$$\|R^{\nu-1}\theta^{\mu-1}\|_{L^2(T_\ell)} \leqslant \|r^{\mu-1}\|_{L^2(T_\ell)}.$$

Now, calculating the right hand side integrating in a cylindrical section,

$$\|r^{\mu-1}\|_{L^2(T_\ell)} = \sqrt{\tfrac{\pi}{4\mu}}\sqrt{h_3}\, h_1^{\mu} \sim \sqrt{h_3}\, h.$$

Then

$$\begin{aligned} h_j \|r^{\mu-1}\|_{L^2} &\lesssim \sqrt{h_1 h_2 h_3}\, h \\ &\sim \sqrt{|T_\ell|}\, h \end{aligned}$$

and then

$$\begin{aligned} h_j \|\partial_{\xi_j} u_{s,i}\|_{L^1(T_\ell)} &\lesssim h\sqrt{|T_\ell|}\, \|R^{1-\nu}\theta^{1-\mu}\partial_{\xi_j} u_{s,i}\|_{L^2(T_\ell)} \\ &\leqslant h\sqrt{|T_\ell|}\, \|u_{s,i}\|_{V^{1,2}_{\beta,\delta}(T_\ell)}. \end{aligned} \tag{5.31}$$

(2c) For the divergence term in (5.30):

$$\begin{aligned} (h_1+h_2+h_3)\|\mathrm{div}\, \boldsymbol{u}_s\|_{L^1(T_\ell)} &\leqslant (2h^{1/\mu} + h^{1/\nu})\|\mathrm{div}\, \boldsymbol{u}_s\|_{L^1(T_\ell)} \\ &\leqslant 3h\sqrt{|T_\ell|}\, \|\mathrm{div}\, \boldsymbol{u}_s\|_{L^2(T_\ell)} \\ &\leqslant 3h\sqrt{|T_\ell|}\, \left\{\|\mathrm{div}\, \boldsymbol{u}\|_{L^2(T_\ell)} + \|\mathrm{div}\, \boldsymbol{u}_r\|_{L^2(T_\ell)}\right\}. \end{aligned} \tag{5.32}$$

3. Now prisms, the middle term on the right hand of (5.28). Remember the difference with the tetrahedron is that $d(P, \boldsymbol{v}_\ell) \gtrsim h_1$. We start pulling back from P to the rescaled prism in Figure 4.1 called here $\tilde{P}$. By the stability estimate in Theorem 4.1.5,

$$\begin{aligned} \|\boldsymbol{r}_0 \boldsymbol{u}_s\|_{L^1(P)^3} &\leqslant \|M_P\|_\infty \|\tilde{\boldsymbol{r}}_0 \tilde{\boldsymbol{u}}_s\|_{L^1(\tilde{P})^3} \\ &\lesssim \|\tilde{\boldsymbol{u}}_s\|_{L^1(\tilde{P})^3} + \sum_{j=1}^{3} h_j \left\|\partial_{\tilde{x}_j} \tilde{\boldsymbol{u}}_s\right\|_{L^1(\tilde{P})^3} + h_{\tilde{P}} \left\|\mathrm{div}\,(\tilde{u}_{s,1}, \tilde{u}_{s,2}, 0)\right\|_{L^1(\tilde{P})} \\ &\lesssim \|\boldsymbol{u}_s\|_{L^1(P)^3} + \sum_{j=1}^{3} h_j \left\|\partial_{\xi_j} \boldsymbol{u}_s\right\|_{L^1(P)^3} + h_P \left\|\mathrm{div}\,(u_{s,1}, u_{s,2}, 0)\right\|_{L^1(P)}, \end{aligned} \tag{5.33}$$

by the same argument as in (5.21) for elements touching the singular edge in the prismatic macro–element. Now to estimate each term, again for $i = 1$ or 2 we have

$$\|u_{s,i}\|_{L^1(P)} \leqslant \|R^{\nu}\theta^{\mu}\|_{0,P}\|R^{-\nu}\theta^{-\mu}u_{s,i}\|_{0,P} \leqslant h\,|P|^{1/2}\|R^{-\nu}\theta^{-\mu}u_{s,i}\|_{0,P}$$

and

$$\|u_{s,3}\|_{L^1(P)} \leqslant \|R^{\nu}\theta\|_{0,P}\|R^{-\nu}\theta^{-1}u_{s,3}\|_{0,P} \leqslant h\,|P|^{1/2}\|R^{-\nu}\theta^{-1}u_{s,3}\|_{0,P}.$$

Next fix $j = 1$ or 2. If $i = 1$ or 2, with the same argument as in (5.31), we state that

$$h_j\|\partial_{\xi_j}u_{s,i}\|_{L^1(P)} \lesssim h\,\sqrt{|P|}\,\|R^{1-\nu}\theta^{1-\mu}\partial_{\xi_j}u_{s,i}\|_{0,P}. \tag{5.34}$$

For the derivatives of $u_{s,3}$ it holds that

$$\begin{aligned} h_j\|\partial_{\xi_j}u_{s,3}\|_{L^1(P)} &\leqslant h_j\|R^{\nu-1}\|_{0,P}\|R^{1-\nu}\partial_{\xi_j}u_{s,3}\|_{0,P} \\ &\leqslant h_j\|r^{\mu-1}\|_{0,P}\|R^{1-\nu}\partial_{\xi_j}u_{s,3}\|_{0,P} \\ &\lesssim h\,\sqrt{|P|}\,\|R^{1-\nu}\partial_{\xi_j}u_{s,3}\|_{0,P}. \end{aligned}$$

Now, for the derivative along the direction of the singular edge $\partial_{\xi_3}(\cdot)$, remember in this case it is $h_3 \sim h\,d(P,\boldsymbol{v})^{1-\nu}$. If we take first and second components,

$$\begin{aligned} h_3\|\partial_{\xi_3}u_{s,i}\|_{L^1(P)} &= h_3\|\partial_{\xi_i}u_{s,3}\|_{L^1(P)} \\ &\leqslant Ch\|R^{1-\nu}\partial_{\xi_i}u_{s,3}\|_{L^1(P)} \\ &\leqslant Ch|P|^{1/2}\|R^{1-\nu}\partial_{\xi_i}u_{s,3}\|_{L^2(P)}. \end{aligned}$$

And if we take the third component,

$$\begin{aligned} h_3\|\partial_{\xi_3}u_{s,3}\|_{L^1(P)} &\sim h\|d(P,\boldsymbol{v})^{1-\nu}\partial_{\xi_3}u_{s,3}\|_{L^1(P)} \\ &\leqslant Ch\|R^{1-\nu}\partial_{\xi_3}u_{s,3}\|_{L^1(P)} \\ &\leqslant Ch|P|^{1/2}\|R^{1-\nu}\partial_{\xi_3}u_{s,3}\|_{L^2(P)}. \end{aligned}$$

For the divergence term in (5.33),

$$\begin{aligned} h_P\|\mathrm{div}\,(u_{s,1},u_{s,1},0)\|_{L^1(P)} &\leqslant h_P\|\mathrm{div}\,\boldsymbol{u}_s\|_{L^1(P)} + h_P\|\partial_{\xi_3}u_{s,3}\|_{L^1(P)} \\ &\lesssim h_P\|\mathrm{div}\,\boldsymbol{u}_s\|_{L^1(P)} + h_3\|\partial_{\xi_3}u_{s,3}\|_{L^1(P)}; \end{aligned}$$

the estimate for $h_P\|\mathrm{div}\,\boldsymbol{u}_s\|_{L^1(P)}$ follows as in (5.32), and the estimate for $h_3\|\partial_{\xi_3}u_{s,3}\|_{L^1(P)}$ should be

$$\begin{aligned} h_3\|\partial_{\xi_3}u_{s,3}\|_{L^1(P)} &\lesssim h|P|^{1/2}\|R^{1-\nu}\partial_{\xi_3}u_{s,3}\|_{L^2(P)} \\ &\lesssim h|P|^{1/2}\|u_{s,3}\|_{V_{\beta,0}(P)}. \end{aligned}$$

Collecting all the estimates in items 2. and 3. above, using (5.18) we bound the first and second term in (5.28), belonging to the case with zero distance to the edge, in the L^2 norm.

5.5.4 Bound for the Singular Part in a Tetrahedral Macro–Element with Singular Vertex and no Singular Edge

This macroelement is made up with a graded subfamily of isotropic tetrahedra. The grading fulfills the relations (5.7) and the estimates follow repeating (in fact, simpler) arguments used in Subsection 5.5.3.

5.6 Main Approximation Error Theorem

5.6.1 Theorem. *Let Ω be a polihedral domain in $\mathbb{R}^3$, let $f \in L^2(\Omega)$ and let $(\boldsymbol{u}, p)$ and $(\boldsymbol{u}_h, p_h)$ be the solutions of Problems 1.2.2 and 3.2.1 respectively. There exists a family of anisotropic meshes $\{\mathcal{T}_h\}_{h\to 0}$ made up of prisms, tetrahedra and pyramids, graded according to (5.6)–(5.8) for which*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega} \leqslant Ch\|f\|_{0,\Omega}$$
$$\|p - p_h\|_{0,\Omega} \leqslant Ch\|f\|_{0,\Omega}$$

with h being smaller than $CN_h^{-1/3}$, where N_h is the number of elements in $\mathcal{T}_h$.

Proof. Recall from inequality (5.4) for every $\boldsymbol{w} \in W(\mathcal{T}_h)$ it holds

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega)^3} \leqslant C\{\|\boldsymbol{u} - \boldsymbol{r}_0\boldsymbol{u}\|_{L^2(\Omega)^3} + \|\boldsymbol{u} - \boldsymbol{w}\|_{L^2(\Omega)^3}\}.$$

For each $\boldsymbol{w}$ we decompose the norm with the macro–elements as follows

$$\begin{aligned}\|\boldsymbol{u} - \boldsymbol{w}\|_{0,\Omega}^2 &= \sum_{\ell=1}^{N_{h_0}} \|\boldsymbol{u} - \boldsymbol{w}\|_{0,\Lambda_\ell}^2 \\ &= \sum_{\ell=1}^{N_{h_0}} \left\{ \sum_{\substack{\text{prisms } E_\ell \\ E_\ell \subseteq \Lambda_\ell}} \|\boldsymbol{u} - \boldsymbol{w}\|_{0,E_\ell}^2 + \sum_{\substack{\text{tetrahedra } E_\ell \\ E_\ell \subseteq \Lambda_\ell}} \|\boldsymbol{u} - \boldsymbol{w}\|_{0,E_\ell}^2 + \sum_{\substack{\text{pyramids } E_\ell \\ E_\ell \subseteq \Lambda_\ell}} \|\boldsymbol{u} - \boldsymbol{w}\|_{0,E_\ell}^2 \right\}\end{aligned}$$

and then take $\boldsymbol{w}_{\boldsymbol{u}}$ as in Definition 5.1.8. For the sum term over the pyramids (remember pyramids do not touch singularities) we have, restricting $\boldsymbol{w}_{\boldsymbol{u}}$, by the inequality we proved in (5.26) for pyramids,

$$\begin{aligned}\|\boldsymbol{u} - \boldsymbol{w}_{\boldsymbol{u}}\|_{0,E_\ell}^2 &= \|\boldsymbol{u} - P_{0,E_\ell}\boldsymbol{u}\|_{0,E_\ell}^2 \\ &\leqslant C\left\{h_E|\boldsymbol{u}_r|_{H^1(E_\ell)^3} + h_E|\boldsymbol{u}_s|_{H^1(E_\ell)^3}\right\} \\ &\leqslant Ch\left\{|\boldsymbol{u}_r|_{H^1(E_\ell)^3} + \|\boldsymbol{u}_s\|_{\mathcal{V}_{\beta,\delta}(E)}\right\}.\end{aligned}$$

With this and all the others for prisms and tetrahedra within the proof of the interpolation Theorem 5.4.1 we have

$$\|\boldsymbol{u}-\boldsymbol{w}_{\boldsymbol{u}}\|_{0,\Omega}^2 = \sum_{\ell=1}^{N_{h_0}} \left\{ \sum_{\substack{\text{prisms } E_\ell \\ E_\ell \subseteq \Lambda_\ell}} \|\boldsymbol{u}-\boldsymbol{r}_0\boldsymbol{u}\|_{0,E_\ell}^2 + \sum_{\substack{\text{tetrahedra } E_\ell \\ E_\ell \subseteq \Lambda_\ell}} \|\boldsymbol{u}-\boldsymbol{r}_0\boldsymbol{u}\|_{0,E_\ell}^2 \right.$$
$$\left. + \sum_{\substack{\text{pyramids } E_\ell \\ E_\ell \subseteq \Lambda_\ell}} \|\boldsymbol{u}-P_{0,E_\ell}\boldsymbol{u}\|_{0,E_\ell}^2 \right\}$$
$$\leqslant C\sum_{\ell=1}^{N_{h_0}} h^2\|f\|_{0,\Omega}^2 \leqslant C\,h^2\|f\|_{0,\Omega}^2. \tag{5.35}$$

Note C depends on the cardinal of the initial mesh. Now by Theorem 5.4.1 and (5.35) we have, for the vectorial variable, $\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^2(\Omega)^3} \leqslant Ch\|f\|_{L^2(\Omega)}$. For the scalar variable estimate we may do

$$\|p-p_h\|_{0,\Omega} \leqslant \|p-P_0p\|_{0,\Omega} + \|P_0p-p_h\|_{0,\Omega}$$

and then apply Theorem 5.4.1 to the first term and Theorem 5.1.7 to the second and reuse the error estimate for the vectorial variable. □

Chapter 6

Further Results on Finite Elements

Introducción al capítulo

Probamos aquí estimaciones anisótropas de error de interpolación local para el operador **curl**–conforme de orden arbitrario sobre elementos prismáticos. Con técnicas similares a las del Capítulo 4 podemos realizar el mismo análisis para estos elementos $H(\mathbf{curl})$–conformes en prismas que el que fue hecho para los elementos $H(\mathrm{div})$–conformes.

Con respecto a las aplicaciones, teniendo en cuenta nuestros resultados para el espacio $H(\mathrm{div})$ del Capítulo 4 más los que presentaremos ahora para el espacio $H(\mathbf{curl})$, se podría intentar realizar un análisis para las ecuaciones de Maxwell armónicas en tiempo como el que se encuentra en [17], pero ahora usando mallas que contengan prismas alargados – en [17] los autores consideran mallas exclusivamente de tetraedros –, y también para el problema armónico en tiempo en una cavidad, para el problema del resonador de cavidad, para el problemma de la dispersión de un objeto acotado y para el problema de la dispersión de un objeto enterrado, todos los cuales son algunos de los principales problemas de contorno para las ecuaciones de Maxwell (ver la Sección 1.4 de [33]), y con la misma consideración recién hecha acerca de las mallas.

En [35] se encuentran estimaciones anisótropas para el caso de grado más bajo de edge–elements en prismas (y para tetraedros resultantes de la subdivisión de cada prisma), aplicados al mallado de dominios cilíndricos. Nuevamente, algunas de las estimaciones que hallamos en el presente capítulo extienden a las anteriores a grado arbitrario, y además podríamos intentar aplicarlas a dominios poliedrales generales, como hicimos con nuestras estimaciones $H(\mathrm{div})$.

Recordar la Notación 2.0.2 para los espacios polinomiales. También nos sujetaremos a la notación e indexación de la Tabla 2.2.

Desde la Sección 6.3 en adelante definimos familias de elementos finitos **curl**–conformes y div–conformes en pirámides. Estos elementos se encuentran en [25, 27]. Allí los autores realizan una construcción de formas diferenciales discretas mediante la resolución de un problema de interpolación local en cierta pirámide de referencia. El elemento finito en una pirámide arbitraria de una malla es obtenida haciendo *push–forward* a los campos vectoriales *proxies* de dichas formas discretas. A continuación probamos estimaciones anisótropas de error de interpolación local para los correspondientes operadores de orden más bajo.

Introduction to the chapter

We prove anisotropic local interpolation error estimates for the **curl**–conforming operator of arbitrary order over a prismatic element. With techniques similar to those of Chapter 4 we can perform the same analysis for these $H(\mathbf{curl})$–conforming elements as the one performed for the $H(\mathrm{div})$ elements.

With respect to the applications, taking into account our results from Chapter 4 for the $H(\mathrm{div})$ space together with the ones we are presenting now for $H(\mathbf{curl})$, we could try an analysis for the time–harmonic Maxwell's equations analogue to the one found in [17], but now using meshes including elongated prisms – in [17] the authors consider meshes of tetrahedra –, and also for the time–harmonic cavity problem, for the cavity resonator problem, for the problem of the scattering from a bounded object, and for the problem of the scattering from a buried object, all of which are some of the principal boundary value problems for the Maxwell's equations (see Section 1.4 of [33]), and with the same consideration as above concerning the meshes.

In [35] there are anisotropic estimates for the least order case of the edge–elements on prisms (and for the tetrahedra resulting from dividing each prism into three), applied to cylindrical domains. Again, some of the estimates found in this chapter extend those estimates to abitrary degree, and moreover we could try to apply them to general polihedral domains, as we did with our $H(\mathrm{div})$ estimates.

Please recall the notation for the polynomial spaces given in Notation 2.0.2. We will also stick to the notation and indices of Table 2.2.

Later, from Section 6.3 on, we define families of **curl**–conforming and div–conforming finite elements on pyramids. The finite elements defined here are the ones found in [25]. There the authors perform a construction of discrete differential forms by solving a local interpolation problem on a reference pyramid. The finite element in an arbitrary mesh pyramid is obtained by pushing forward the vector proxies of those discrete forms. Then we prove local anisotropic interpolation error estimates for the corresponding interpolation operators of lowest order.

6.1 Anisotropic Stability Estimates for $H(\mathbf{curl})$ Conforming Finite Elements on Prisms

We are writing the following three Lemmas which state some special behavior of the interpolation operator, mostly related to the preservation of null components of the fields, and whose proofs consist in a smart use of the degrees of freedom and the very definition of the operator. These Lemmas, although just with technical purpose, exhibit nice properties of the interpolators.

In the present subsection $\hat{\boldsymbol{u}}$ is an element in $W^{1,p}(\hat{E})$ for $p > 2$ which is a space whose elements have well defined tangential traces on each edge of the prism $\hat{E}$. Another possibility, as stated in Lemma 5.38 in the page 134 of [33], is to assume there are a positive δ and a $p > 2$ such that $\hat{\boldsymbol{u}}$ belongs to $H^{1/2+\delta}(\hat{E})^3$ and $\mathbf{curl}\,\boldsymbol{u}$ belongs to $L^p(\hat{E})^3$. For the whole section, $\hat{\boldsymbol{w}}_k$ will be the k–th order edge interpolation operator on the reference Prism determined by the element of *Definition* 2.1.8.

6.1.1 Lemma. $(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_3$ *is linearly and univocally determined by* $\hat{u}_3$.

Proof. If we pay attention to the directions of the unit tangents and normals to the edges and faces, respectively, of $\hat{E}$, we realize that the degrees of freedom which involve $(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_3$ give rise only to the following linear equations

$$\varphi_{\hat{\mathbf{e}}_i,p}\left(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}}\right) = \varphi_{\hat{\mathbf{e}}_i,p}\left(\hat{\boldsymbol{u}}\right) \quad \text{as in (2.13) for } i = 3, 6, 7, \tag{6.1}$$

$$\varphi_{f_1,\boldsymbol{q}}\left(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}}\right) = \varphi_{f_1,\boldsymbol{q}}\left(\hat{\boldsymbol{u}}\right) \quad \text{as in (2.15)} \tag{6.2}$$

$$\varphi_{f_2,\boldsymbol{q}}\left(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}}\right) = \varphi_{f_2,\boldsymbol{q}}\left(\hat{\boldsymbol{u}}\right) \quad \text{as in (2.16)} \tag{6.3}$$

$$\varphi_{f_5,\boldsymbol{q}}\left(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}}\right) = \varphi_{f_5,\boldsymbol{q}}\left(\hat{\boldsymbol{u}}\right) \quad \text{as in (2.17)} \tag{6.4}$$

$$\varphi_{\boldsymbol{r}}\left(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}}\right) = \varphi_{\boldsymbol{r}}\left(\hat{\boldsymbol{u}}\right) \quad \text{as in (2.18).} \tag{6.5}$$

These are $3k$+$3k(k-1)$+$k(k-1)(k-2)/2 = k(k+1)(k+2)/2$ equations, just the dimension of $P_k(\hat{T})\otimes P_{k-1}(\hat{I})$, which is the space $(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_3$ belongs to by definition. Now set all those equations equal to zero (that is, pick $u_3 = 0$) and see that the unique solution is $(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_3 = 0$. A little more explicitly, we have:

$$\int_{\hat{\mathbf{e}}_i} (\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_3\,\hat{q}\,d\alpha \quad = 0 \qquad \text{for } i = 3, 6 \text{ and } 7,\ q \in P_{k-1}(\hat{\boldsymbol{e}}_i) \tag{6.6}$$

$$\iint_{\hat{f}_j} (\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_3\,\hat{q}\,d\hat{S} = 0 \qquad \text{for } j = 1, 2, \text{ and } 5,\ \hat{q} \in Q_{k-2,k-1}(\hat{f}_j) \tag{6.7}$$

$$\int_{\hat{E}} (\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_3\,\hat{q}_3\,d\boldsymbol{x} \quad = 0 \qquad \text{for } \hat{q}_3 \in P_{k-3,k-1}. \tag{6.8}$$

Start considering the face $\hat{f}_2$. The restriction of $(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_3$ to $\boldsymbol{e}_3$ is itself an element in $P_{k-1}(\boldsymbol{e}_3)$, and the same holds for $\boldsymbol{e}_6$, so equations (6.6), for $i = 3, 6$, say that $(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_3$ is identically null on those edges by which the restriction $(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_3|_{f_2}$, which is an element of $P_k(\hat{x}_1) \otimes P_{k-1}(\hat{x}_3)$ may be se factorized as $(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_3|_{f_2}(\hat{x}_1, 0, \hat{x}_3) = \hat{x}_1\,(1-\hat{x}_1)\,w_0(\hat{x}_1, \hat{x}_3)$, with w_0 equal to some polynomial in $P_{k-2}(\hat{x}_1) \otimes P_{k-1}(\hat{x}_3)$. Now choose $\hat{q} = w_0$ in the degrees of freedom (6.7) for the face $\hat{f}_2$ and it holds

$$\iint_{\hat{f}_2} \hat{x}_1(1-\hat{x}_1)w_0(\hat{x}_1, \hat{x}_3)^2\,d\hat{S} = 0. \tag{6.9}$$

But $\hat{x}_1\,(1-\hat{x}_1)$ is almost everywhere positive over the closure of $\hat{f}_2$, so $(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_3|_{\hat{f}_2}$ vanishes identically. By a completely symmetric observation we can prove $(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_3|_{\hat{f}_1} \equiv 0$.
One more time, if we use (6.6) for $i = 6, 7$, $\hat{x}_1\,(1-\hat{x}_1)$ divides the restriction $(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_3|_{\hat{f}_5}$, so that there is a $w_1 \in P_{k-2}(\hat{x}_1)\otimes P_{k-1}(\hat{x}_3)$ for which $(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_3|_{\hat{f}_5}(\hat{x}_1,\hat{x}_2,\hat{x}_3) = \hat{x}_1\,(1-\hat{x}_1)\,w_1(\hat{x}_1,\hat{x}_3)$. Now equality (6.7) for $j = 5$ implies $(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_3|_{f_5} \equiv 0$.
Next, since $(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_3$ vanishes when restricted $\hat{f}_1$, $\hat{f}_2$ and $\hat{f}_5$ we get to factorize it on $\hat{E}$ as

$$
\begin{aligned}
(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_3(\hat{x}_1,\hat{x}_2,\hat{x}_3) &= \hat{x}_1\,\hat{x}_2\,(1-\hat{x}_1-\hat{x}_2)\,w_3(\hat{x}_1,\hat{x}_2,\hat{x}_3), \\
w_3 &\in P_{k-3}(\hat{x}_1)\otimes P_{k-1}(\hat{x}_3).
\end{aligned}
$$

And now we evaluate the degrees of freedom (6.8) choosing $\hat{q}_3 \equiv w_3$ and conclude immediately $(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_3 \equiv 0$ on $\hat{E}$. □

6.1.2 Lemma. *Let $\hat{\boldsymbol{u}} \in W^{1,p}(\hat{E})$ for $p > 2$.*

(a) If $\hat{\boldsymbol{u}}(\hat{x}_1,\hat{x}_2,\hat{x}_3) = (0,\hat{u}_2(\hat{x}_2,\hat{x}_3),0)'$, then

$$
\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}}(\hat{x}_1,\hat{x}_2,\hat{x}_3) = (0,\hat{\xi}_2(\hat{x}_2,\hat{x}_3),0)'
$$

for some $\hat{\xi}_2 \in P_{k-1}(\hat{x}_2)\otimes P_k(\hat{x}_3)$.

(b) If $\hat{\boldsymbol{u}}(\hat{x}_1,\hat{x}_2,\hat{x}_3) = (\hat{u}_1(\hat{x}_1,\hat{x}_3),0,0)'$ then

$$
\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}}(\hat{x}_1,\hat{x}_2,\hat{x}_3) = (\hat{\xi}_1(\hat{x}_1,\hat{x}_3),0,0)'
$$

for some $\hat{\xi}_1 \in P_{k-1}(\hat{x}_1)\otimes P_k(\hat{x}_3)$.

Proof. We will prove the first inequality, as the second follows with the same ideas. In Subsection 2.1.2 we found expression (2.19) which states

$$
\begin{aligned}
\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}}(\hat{x}_1,\hat{x}_2,\hat{x}_3) &= (p_1(\hat{x}_1,\hat{x}_2,\hat{x}_3),p_2(\hat{x}_1,\hat{x}_2,\hat{x}_3),p_3(\hat{x}_1,\hat{x}_2,\hat{x}_3))^t \\
&= \begin{pmatrix} \xi_1(\hat{x}_1,\hat{x}_2,\hat{x}_3) + \hat{x}_2\,h(\hat{x}_1,\hat{x}_2,\hat{x}_3) \\ \xi_2(\hat{x}_1,\hat{x}_2,\hat{x}_3) - \hat{x}_1\,h(\hat{x}_1,\hat{x}_2,\hat{x}_3) \\ \xi_3(\hat{x}_1,\hat{x}_2,\hat{x}_3) \end{pmatrix}
\end{aligned}
\tag{6.10}
$$

for ξ_1 and ξ_2 in $P_{k-1}(\hat{f}_3)\otimes P_k(\hat{x}_3)$, ξ_3 in $P_k(\hat{f}_3)\otimes P_{k-1}(\hat{x}_3)$, and h in $\tilde{P}_{k-1}(\hat{f}_3)\otimes P_k(\hat{x}_3)$. Thanks to Lemma 6.1.1 we already know that $\xi_3 \equiv 0$, so we are going to show that $h \equiv 0$, $\xi_1 \equiv 0$ and that ξ_2 does not depend on $\hat{x}_1$. First, if $\hat{f}$ is either $\hat{f}_3$ o $\hat{f}_4$, then with a direct calculation we see that $(\mathbf{curl}\,\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_3|_{\hat{f}}$ belongs to $P_{k-1}(\hat{f})$. By the commutative diagram property expressed in (2.38), the definition of the degrees of freedom (2.3) and the interpolation operator $\boldsymbol{r}_{\hat{E}}$ in Definition 2.1.3, it holds that, if $\hat{f}$ is either $\hat{f}_3$ or $\hat{f}_4$, then for every $q \in P_{k-1}(\hat{f})$,

$$
\begin{aligned}
\hat{\rho}_{\hat{f},q}\,(\mathbf{curl}\,\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}}) &= \hat{\rho}_{\hat{f},q}(\boldsymbol{r}_{\hat{E}}\,\mathbf{curl}\,\hat{\boldsymbol{u}}) \\
&= \iint_{\hat{f}} (\mathbf{curl}\,\hat{\boldsymbol{u}})_3\,q\,d\hat{S} = 0.
\end{aligned}
$$

As is for time being expected, the choice $q = (\mathbf{curl}\, \boldsymbol{w}_{\hat{E}} \hat{\boldsymbol{u}})_3|_{\hat{f}}$ yields

$$(\mathbf{curl}\, \boldsymbol{w}_{\hat{E}} \hat{\boldsymbol{u}})_3|_{\hat{f}} \equiv 0$$

so we may write again $(\mathbf{curl}\, \boldsymbol{w}_{\hat{E}} \hat{\boldsymbol{u}})_3 = \hat{x}_3\,(\hat{x}_3 - 1)\,\hat{\psi}$ for a $\hat{\psi} \in P_{k-1}(\hat{f}) \otimes P_{k-2}(\hat{x}_3)$. We choose now $q = \hat{\psi}$ in the degrees of freedom (2.6). By the commutative diagram property and the definition of $\boldsymbol{r}_{\hat{E}}$, we have

$$\begin{aligned}
\int_{\hat{E}} \hat{x}_3\,(\hat{x}_3 - 1)\,\hat{\psi}^2\, d\hat{\boldsymbol{x}} &= \int_{\hat{E}} (\mathbf{curl}\, \boldsymbol{w}_{\hat{E}} \hat{\boldsymbol{u}})_3\, \hat{\psi}\, d\hat{\boldsymbol{x}} \\
&= \int_{\hat{E}} (\boldsymbol{r}_{\hat{E}} \mathbf{curl}\, \hat{\boldsymbol{u}})_3\, \hat{\psi}\, d\hat{\boldsymbol{x}} \\
&= \int_{\hat{E}} (\mathbf{curl}\, \hat{\boldsymbol{u}})_3\, \hat{\psi}\, d\hat{\boldsymbol{x}} \quad = 0
\end{aligned}$$

and it follows that

$$(\mathbf{curl}\, \boldsymbol{w}_{\hat{E}} \hat{\boldsymbol{u}})_3 \equiv 0. \tag{6.11}$$

Now if we explore $(\mathbf{curl}\, \boldsymbol{w}_{\hat{E}} \hat{\boldsymbol{u}})_3$ taking derivatives in expression (6.10) we get

$$\begin{aligned}
(\mathbf{curl}\, \boldsymbol{w}_{\hat{E}} \hat{\boldsymbol{u}})_3 &= \frac{\partial}{\partial \hat{x}_1} (\boldsymbol{w}_{\hat{E}} \hat{\boldsymbol{u}})_2 - \frac{\partial}{\partial \hat{x}_2} (\boldsymbol{w}_{\hat{E}} \hat{\boldsymbol{u}})_1 \\
&= -(2\,h + \hat{x}_2\, \frac{\partial h}{\partial \hat{x}_2} + \hat{x}_1\, \frac{\partial h}{\partial \hat{x}_1}) + \frac{\partial \hat{\xi}_2}{\partial \hat{x}_1} - \frac{\partial \hat{\xi}_1}{\partial \hat{x}_2}.
\end{aligned} \tag{6.12}$$

Observing the degrees in each term, there hold

1. $g \,:=\, 2\,h + \hat{x}_2\, \dfrac{\partial h}{\partial \hat{x}_2} + \hat{x}_1\, \dfrac{\partial h}{\partial \hat{x}_1}$ belongs to $\tilde{P}_{k-1}(\hat{f}_3) \otimes P_k(\hat{x}_3)$

2. $\dfrac{\partial \hat{\xi}_2}{\partial \hat{x}_1} - \dfrac{\partial \hat{\xi}_1}{\partial \hat{x}_2}$ belongs to $P_{k-2}(\hat{f}_3) \otimes P_k(\hat{x}_3)$,

but from this it follows necessarily that $g \equiv 0$. Now, how do the terms of g look like? Let us put

$$h(\hat{x}_1, \hat{x}_2, \hat{x}_3) = \sum_{\substack{i+j\,=\,k-1 \\ l\,\leqslant\, k}} \alpha_{i,j,l}\, \hat{x}_1^i \hat{x}_2^j \hat{x}_3^l.$$

Then

$$\begin{aligned}
g(\hat{x}_1, \hat{x}_2, \hat{x}_3) &= \sum_{\substack{i+j\,=\,k-1 \\ l\,\leqslant\, k}} (2\alpha_{i,j,l} + j\,\alpha_{i,j,l} + i\,\alpha_{i,j,l}) \hat{x}_1^i \hat{x}_2^j \hat{x}_3^l \\
&= (k+1)\, h(\hat{x}_1, \hat{x}_2, \hat{x}_3) \;=\; 0,
\end{aligned} \tag{6.13}$$

so $h \equiv 0$ too and, for now,

$$\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}}(\hat{x}_1, \hat{x}_2, \hat{x}_3) = (\hat{\xi}_1(\hat{x}_1, \hat{x}_2, \hat{x}_3), \hat{\xi}_2(\hat{x}_1, \hat{x}_2, \hat{x}_3), 0)'.$$

The second–to–last task is to see that $\hat{\xi}_1$ vanishes identically. We turn back to de edge degrees of freedom. Set $\boldsymbol{e}$ equal to $\hat{\boldsymbol{e}}_1$ or $\hat{\boldsymbol{e}}_4$. Then the restriction $\hat{\xi}_1|_{\boldsymbol{e}}$ belongs to $P_{k-1}(\boldsymbol{e})$, so letting $\hat{q} = \hat{\xi}_1|_{\boldsymbol{e}}$ in (2.13) we obtain

$$0 = \hat{\varphi}_{\boldsymbol{e},\hat{\xi}_1}(\hat{\boldsymbol{u}}) = \hat{\varphi}_{\boldsymbol{e},\hat{\xi}_1}(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}}) = \int_{\boldsymbol{e}} (\hat{\xi}_1)^2\, d\alpha,$$

so, for some $\hat{p} \in P_{k-1}(\hat{x}_1) \otimes P_{k-2}(\hat{x}_3)$ we have

$$\hat{\xi}_1|_{\hat{f}_2}(\hat{x}_1, \hat{x}_3) = \hat{x}_3\,(\hat{x}_3 - 1)\,\hat{p}(\hat{x}_1, \hat{x}_3).$$

Next choose $\hat{f} = \hat{f}_2$ and $\boldsymbol{q} = (0, 0, \hat{p})'$ in (2.16).

$$0 = \hat{\varphi}_{\hat{f}_2,\boldsymbol{q}}(\hat{\boldsymbol{u}}) = \hat{\varphi}_{\hat{f}_2,\boldsymbol{q}}(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}}) = \iint_{\hat{f}_2} \hat{x}_3\,(\hat{x}_3 - 1)\hat{p}^2\, d\hat{S}.$$

It follows that $\hat{\xi}_1|_{\hat{f}_2} \equiv 0$, by which we know it exists certain $\zeta \in P_{k-2}(\hat{f}_3) \otimes P_k(\hat{x}_3)$ satisfying

$$\hat{\xi}_1(\hat{x}_1, \hat{x}_2, \hat{x}_3) = \hat{x}_2\,\zeta(\hat{x}_1, \hat{x}_2, \hat{x}_3).$$

Now we switch to the faces $\hat{f} = \hat{f}_3$ or $\hat{f}_4$. Take $\hat{\boldsymbol{q}} = (\zeta|_{\hat{f}}, 0, 0)'$ in (2.14)

$$0 = \varphi_{\hat{f},\hat{\boldsymbol{q}}}(\hat{\boldsymbol{u}}) = \varphi_{\hat{f},\hat{\boldsymbol{q}}}(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}}) = \iint_{\hat{f}} \hat{x}_2\zeta^2\, d\hat{S},$$

and it follows that $\hat{x}_3\,(\hat{x}_3 - 1)$ divides ζ. So putting this together with the previous factorization, there is some $r \in P_{k-2}(\hat{f}_3) \otimes P_{k-2}(\hat{x}_3)$ which satisfies.

$$\hat{\xi}_1(\hat{x}_1, \hat{x}_2, \hat{x}_3) = \hat{x}_2\,\hat{x}_3\,(\hat{x}_3 - 1)\,r(\hat{x}_1, \hat{x}_2, \hat{x}_3).$$

It remains to use the volume degrees of freedom. We could choose $\hat{\boldsymbol{r}} := (r, 0, 0)'$ in degree of freedom (2.18) to get

$$0 = \hat{\varphi}_{\hat{\boldsymbol{r}}}(\hat{\boldsymbol{u}}) = \hat{\varphi}_{\hat{\boldsymbol{r}}}(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}}) = \int_{\hat{E}} \hat{x}_2\,\hat{x}_3\,(\hat{x}_3 - 1)\,r(\hat{x}_1, \hat{x}_2, \hat{x}_3)^2\, d\hat{\boldsymbol{x}},$$

which yields, over all $\hat{E}$, $\hat{\xi}_1 \equiv 0$. Finally, if we combine this last property with (6.11) we prove that $\hat{\xi}_2$ does not depend on $\hat{x}_2$. □

6.1.3 Lemma. *If $\hat{\boldsymbol{u}}$ is of the form*

$$\hat{\boldsymbol{u}}(\hat{x}_1, \hat{x}_2, \hat{x}_3) = (0, 0, \hat{u}_3(\hat{x}_1, \hat{x}_2, \hat{x}_3))^t,$$

then $\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}}$ is of the form

$$\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}}(\hat{x}_1, \hat{x}_2, \hat{x}_3) = (0, 0, \hat{\xi}_3(\hat{x}_1, \hat{x}_2, \hat{x}_3))^t$$

for some $\hat{\xi}_3 \in P_k(\hat{f}_3) \otimes P_{k-1}(\hat{x}_3)$.

Proof. We will work again with expression (6.10). By expression (6.12) and the commutativity in equation (2.38), if we apply degrees of freedom (2.3) to $\mathbf{curl}\,\hat{\boldsymbol{u}}$ we obtain that $(\mathbf{curl}\,\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_3$ vanishes on any of the horizontal faces $\hat{f}_3$ or $\hat{f}_4$ in Table 2.1. In other words, $(\mathbf{curl}\,\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_3 = \hat{x}_3\,(\hat{x}_3 - 1)\,\hat{\psi}$, $(\hat{\psi} \in P_{k-1}(\hat{f}_3) \otimes P_{k-2}(\hat{x}_3))$ and if we set $\hat{\boldsymbol{r}} := (0, 0, \hat{\psi})'$ in the $H(\text{div})$ degrees of freedom (2.6) we have

$$0 = \int_{\hat{E}} (\mathbf{curl}\,\hat{\boldsymbol{u}})_3\,\hat{\psi} = \hat{\rho}_{\boldsymbol{r}}(\mathbf{curl}\,\hat{\boldsymbol{u}}) = \hat{\rho}_{\boldsymbol{r}}(\hat{\boldsymbol{r}}_k\mathbf{curl}\,\hat{\boldsymbol{u}}) = \hat{\rho}_{\boldsymbol{r}}(\mathbf{curl}\,\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})$$
$$= \int_{\hat{E}} \hat{x}_3(1 - \hat{x}_3)\hat{\psi}^2\,d\boldsymbol{x}$$

yielding that $\hat{\psi}$ is identically zero, and also $(\mathbf{curl}\,\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_3$ is identically zero.

From this point, if we copy the argument in the proof of Lemma 6.1.2 starting with equation (6.12) we arrive at $h \equiv 0$, so we may rewrite (6.10) for the present case as

$$\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}} = (\hat{\xi}_1, \hat{\xi}_2, \hat{\xi}_3)^t. \tag{6.14}$$

We claim that $\hat{\xi}_1 \equiv \hat{\xi}_2 \equiv 0$. To see this, first observe that the evaluation of the degrees of freedom for the edges $\hat{\boldsymbol{e}}_1$ and $\hat{\boldsymbol{e}}_2$ yields $\hat{\xi}_1|_{\hat{\boldsymbol{e}}_1} \equiv \hat{\xi}_2|_{\hat{\boldsymbol{e}}_2} \equiv 0$, hence, evaluating the degree of freedom (2.14) tangent to the face $\hat{f}_3$ two times we have $\hat{\xi}_1|_{\hat{f}_3} \equiv \hat{\xi}_2|_{\hat{f}_3} \equiv 0$. In equal manner, if we pick $\hat{\boldsymbol{e}}_4$ and $\hat{\boldsymbol{e}}_5$, and then the degree of freedom tangent to $\hat{f}_4$ we obtain $\hat{\xi}_1|_{\hat{f}_3} \equiv \hat{\xi}_2|_{\hat{f}_3} \equiv 0$. So we proved there are polynomials p_1 and p_2 in $P_{k-1}(\hat{f}_3) \otimes P_{k-2}(\hat{x}_3)$ which allow us to write

$$\hat{\xi}_1(\hat{x}_1, \hat{x}_2, \hat{x}_3) = \hat{x}_3(1 - \hat{x}_3)p_1(\hat{x}_1, \hat{x}_2, \hat{x}_3)$$
$$\hat{\xi}_2(\hat{x}_1, \hat{x}_2, \hat{x}_3) = \hat{x}_3(1 - \hat{x}_3)p_2(\hat{x}_1, \hat{x}_2, \hat{x}_3).$$

Take $\hat{\boldsymbol{q}} := (0, 0, \hat{q}_2|_{\hat{f}_2})'$ and evaluate the degree of freedom (2.15). We have

$$0 = \hat{\varphi}_{\hat{f}_1,\hat{\boldsymbol{q}}}(\hat{\boldsymbol{u}}) = \hat{\varphi}_{\hat{f}_1,\hat{\boldsymbol{q}}}(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}}) = \iint_{\hat{f}_1} \hat{x}_3(1 - \hat{x}_3)\hat{q}_2^2\,d\hat{S}.$$

Hence, there is some $\hat{r}_2 \in P_{k-2}(\hat{f}_3) \otimes P_{k-2}(\hat{x}_3)$ such that $\hat{\xi}_2 = \hat{x}_1\hat{x}_3(1 - \hat{x}_3)\hat{r}_2$. Now choose $\boldsymbol{r} = (0, \hat{r}_2, 0)'$ and use degree of freedom (2.18) to obtain $\int_{\hat{E}} \hat{x}_1\hat{x}_3(1 - \hat{x}_3)\hat{r}_2^2\,d\hat{\boldsymbol{x}} = 0$. Since $\hat{x}_1\hat{x}_3(1 - \hat{x}_3)\hat{r}_2^2$ is almost everywhere greater than zero, this implies $\hat{\xi}_2 = 0$. With the simmetric procedure starting with face $\hat{f}_2$ we get to prove $\hat{\xi}_1 = 0$. □

Now here is our first important result.

6.1.4 Theorem. *Given $p > 2$, $\hat{\boldsymbol{u}} \in W^p(\boldsymbol{curl}, \hat{E})$,*

$$\|(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_1\|_{L^\infty(\hat{E})} \lesssim \|\hat{u}_1\|_{W^{1,p}(\hat{E})} + \|(\boldsymbol{curl}\,\hat{\boldsymbol{u}})_3\|_{W^{1,1}(\hat{E})} \tag{6.15}$$
$$\|(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_2\|_{L^\infty(\hat{E})} \lesssim \|\hat{u}_2\|_{W^{1,p}(\hat{E})} + \|(\boldsymbol{curl}\,\hat{\boldsymbol{u}})_3\|_{W^{1,1}(\hat{E})} \tag{6.16}$$
$$\|(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_3\|_{L^\infty(\hat{E})} \lesssim \|\hat{u}_3\|_{W^{1,p}(\hat{E})} \tag{6.17}$$

where the constants in the inequalities depend only on $\hat{E}$.

Proof. The proof will rely on the three previous Lemmas, the triangular inequality applied on each component of expression (2.23) and traces inequalities or, more precisely, the proof of Lemma 5.38 in the page 134 of [33] and Theorem 3.9 (*Trace Theorem*) in page 43 of the same book. First we will take a smooth field $\hat{\boldsymbol{u}}$ defined on $\hat{E}$ and, by Proposition 1.1.14, we will conclude the Theorem with a density argumentation.

To prove (6.15) the idea will be to take another function $\hat{\boldsymbol{w}}$ such that its interpolate has the same first component as the one of $\hat{\boldsymbol{u}}$ and such that its degrees of freedom are more easily bounded in terms of $\hat{u}_1$ and $\mathbf{curl}\,(\hat{\boldsymbol{u}})_3$.

Let us define, for a given $\hat{\boldsymbol{u}} \in C^\infty(\hat{\bar{E}})^3$, $\hat{\boldsymbol{v}} : \hat{E} \to \mathbb{R}^3$ with

$$\hat{\boldsymbol{v}}(\hat{x}_1, \hat{x}_2, \hat{x}_3) = (\hat{u}_1(\hat{x}_1, \hat{x}_2, \hat{x}_3), \hat{u}_2(\hat{x}_1, \hat{x}_2, \hat{x}_3) - \hat{u}_2(0, \hat{x}_2, \hat{x}_3), 0)'. \tag{6.18}$$

Thanks to the Lemmas 6.1.2 and 6.1.3 it holds

$$\begin{aligned}(\hat{\boldsymbol{w}}_{\hat{E}}\hat{\boldsymbol{v}})_1 &= (\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_1 - \hat{\boldsymbol{w}}_{\hat{E}}(0, \hat{u}_2(0, \hat{x}_2, \hat{x}_3), 0)_1 - \hat{\boldsymbol{w}}_{\hat{E}}(0, 0, \hat{u}_3)_1 \\ &= (\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_1,\end{aligned}$$

and we also have $(\mathbf{curl}\,\hat{\boldsymbol{u}})_3 = (\mathbf{curl}\,\hat{\boldsymbol{v}})_3$. Now let us explore one by one the degrees of freedom that define $\hat{\boldsymbol{w}}_{\hat{E}}\hat{\boldsymbol{v}}$. The only edge degrees that do not vanish directly or depend explicitly just on $\hat{u}_1$ are

$$\begin{aligned}\int_{\hat{\mathbf{e}}_8} q\,\hat{\boldsymbol{v}} \cdot d\hat{\boldsymbol{\alpha}} &= \tfrac{1}{\sqrt{2}} \int_{\hat{\mathbf{e}}_8} (\hat{v}_1 - \hat{v}_2)\, q\, d\alpha \\ \int_{\hat{\mathbf{e}}_9} q\,\hat{\boldsymbol{v}} \cdot d\hat{\boldsymbol{\alpha}} &= \tfrac{1}{\sqrt{2}} \int_{\hat{\mathbf{e}}_9} (\hat{v}_1 - \hat{v}_2)\, q\, d\alpha\end{aligned}$$

for q in $\mathcal{P}_{k-1}(\hat{\boldsymbol{e}}_8)$ or $\mathcal{P}_{k-1}(\hat{\boldsymbol{e}}_9)$ respectively. Pick a polynomial $q \in P_{k-1}(\hat{\boldsymbol{e}}_8)$. Since on $\hat{\mathbf{e}}_8$ it is $\hat{x}_1 = 1 - \hat{x}_2$, we evaluate q as $q(\hat{x}_2)$, with $0 \leqslant \hat{x}_2 \leqslant 1$. Integration by parts over the face $\hat{f}_4$ yields

$$\begin{aligned}\iint_{\hat{f}_4} (\mathbf{curl}\,\hat{\boldsymbol{v}})_3\, q\, d\hat{S} &= -\iint_{\hat{f}_4} (\hat{v}_2\, \partial_{\hat{x}_1} q - \hat{v}_1\, \partial_{\hat{x}_2} q)\; d\hat{S} + \int_{\partial \hat{f}_4} (\hat{v}_2\, \hat{\nu}_1 - \hat{v}_1\, \hat{\nu}_2)\; q\, d\hat{\alpha} \\ &= \iint_{\hat{f}_4} \hat{v}_1\, \partial_{\hat{x}_2} q\, d\hat{S} + \int_{\hat{\mathbf{e}}_8} (\hat{v}_2 - \hat{v}_1)\; q\, d\hat{\alpha} + \int_{\hat{\mathbf{e}}_4} \hat{v}_1\, q\, d\hat{\alpha},\end{aligned}$$

hence

$$\begin{aligned}\hat{\varphi}_{\hat{\mathbf{e}}_8, q}(\hat{\boldsymbol{v}}) &= \frac{1}{\sqrt{2}} \int_{\hat{\mathbf{e}}_8} (\hat{v}_1 - \hat{v}_2)\, q\, d\hat{\alpha} \\ &= \tfrac{1}{\sqrt{2}} \int_{\hat{\mathbf{e}}_4} \hat{u}_1\, q\, d\hat{\alpha} - \tfrac{1}{\sqrt{2}} \iint_{\hat{f}_4} (\mathbf{curl}\,\hat{\boldsymbol{u}})_3\, q\, d\hat{S} + \iint_{\hat{f}_3} \hat{u}_1\, \partial_{\hat{x}_2} q\, d\hat{S}.\end{aligned} \tag{6.19}$$

In a similar manner if we integrate over $\hat{f}_3 \subseteq \{\hat{x}_3 = 0\}$ we get

$$\begin{aligned}\hat{\varphi}_{\hat{\mathbf{e}}_9, q}(\hat{\boldsymbol{v}}) &= \tfrac{1}{\sqrt{2}} \int_{\hat{\mathbf{e}}_9} (\hat{v}_1 - \hat{v}_2)\, q\, d\hat{\alpha} \\ &= \tfrac{1}{\sqrt{2}} \int_{\hat{\mathbf{e}}_1} \hat{u}_1\, q\, d\hat{\alpha} - \tfrac{1}{\sqrt{2}} \iint_{\hat{f}_3} (\mathbf{curl}\,\hat{\boldsymbol{u}})_3\, q\, d\hat{S} + \iint_{\hat{f}_3} \hat{u}_1\, \partial_{\hat{x}_2} q\, d\hat{S}.\end{aligned} \tag{6.20}$$

If we evaluate now the face degrees of freedom, we only have to bound those corresponding to $\hat{f}_3$, $\hat{f}_4$ and $\hat{f}_5$. Take $\hat{q}_1, \hat{q}_2 \in P_{k-2}(\hat{f}_3)$ and consider $\hat{\boldsymbol{q}} := (\hat{q}_1, \hat{q}_2, 0)$.

$$\iint_{\hat{f}_3} \hat{\boldsymbol{v}} \times \hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{q}} \, d\hat{S} = \iint_{\hat{f}_3} \hat{u}_1 \, \hat{q}_2 \, d\hat{S} - \iint_{\hat{f}_3} \hat{v}_2 \, \hat{q}_1 \, d\hat{S}. \tag{6.21}$$

Observe that $\hat{v}_2$ vanishes over the face $\hat{f}_1 \subseteq \{\hat{x}_1 = 0\}$. Now we need a polynomial $\hat{\zeta} \in P_{k-1}(\hat{f}_3)$ such that $\partial_{\hat{x}_1} \hat{\zeta} = \hat{q}_1$ and $\hat{\zeta}|_{\hat{\mathbf{e}}_9} = 0$; take for instance $\hat{\zeta}(\hat{x}_1, \hat{x}_2) = -\int_{\hat{x}_1}^{1-\hat{x}_2} \hat{q}_1(t, \hat{x}_2) \, dt$. Then

$$\iint_{\hat{f}_3} (\mathbf{curl}\, \hat{\boldsymbol{v}})_3 \, \hat{\zeta} \, d\hat{S} = -\iint_{\hat{f}_3} \left(\hat{v}_2 \, \hat{q}_1 - \hat{v}_1 \, \partial_{\hat{x}_2} \hat{\zeta} \right) d\hat{S} - \int_{\hat{\mathbf{e}}_1} \hat{v}_1 \, \hat{n}_2 \, \hat{\zeta} \, d\hat{\alpha},$$

which, together with (6.21) implies

$$\begin{aligned} \hat{\varphi}_{\hat{f}_3, \hat{\boldsymbol{q}}}(\hat{\boldsymbol{v}}) = & \iint_{\hat{f}_3} \hat{u}_1 \, \hat{q}_2 \, d\hat{S} + \iint_{\hat{f}_3} (\mathbf{curl}\, \hat{\boldsymbol{u}})_3 \, \hat{\zeta} \, d\hat{S} \\ & - \iint_{\hat{f}_3} \hat{u}_1 \, \partial_{\hat{x}_2} \hat{\zeta} \, d\hat{S} + \int_{\hat{\mathbf{e}}_1} \hat{u}_1 \, \hat{n}_2 \, \hat{\zeta} \, d\hat{\alpha}. \end{aligned} \tag{6.22}$$

If we repeated the procedure for the degree of freedom on $\hat{f}_4$, for a given $\hat{\boldsymbol{p}} = (\hat{p}_1, \hat{p}_2, 0) \in P_{k-2}(\hat{f}_4)^2 \times \{0\}$ we would set $\hat{\psi}(\hat{x}_1, \hat{x}_2) = \int_{1-\hat{x}_2}^{\hat{x}_1} \hat{p}_1(t, \hat{x}_2) \, dt$ and had

$$\begin{aligned} \hat{\varphi}_{\hat{f}_4, \hat{\boldsymbol{p}}}(\hat{\boldsymbol{v}}) = & -\iint_{\hat{f}_4} \hat{u}_1 \, \hat{p}_2 \, d\hat{S} - \iint_{\hat{f}_4} (\mathbf{curl}\, \hat{\boldsymbol{u}})_3 \, \hat{\psi} \, d\hat{S} \\ & + \iint_{\hat{f}_4} \hat{u}_1 \, \partial_{\hat{x}_2} \hat{\psi} \, d\hat{S} - \int_{\hat{\mathbf{e}}_4} \hat{u}_1 \, \hat{n}_2 \, \hat{\psi} \, d\hat{\alpha}. \end{aligned} \tag{6.23}$$

For the degree of freedom (2.17) corresponding to $\hat{f}_5$, given $\hat{\boldsymbol{q}} = (0, \hat{q}_3, \hat{q}_1) \in \{0\} \times Q_{k-2,k-1} \times Q_{k-1,k-2}$ observe

$$\iint_{\hat{f}_5} \hat{\boldsymbol{v}} \times \boldsymbol{n} \cdot \hat{\boldsymbol{q}} \, d\hat{S} = \iint_{\hat{f}_5} (\hat{v}_1 - \hat{v}_2) \, \hat{q}_1 \, d\hat{S} \tag{6.24}$$

Now, if $\hat{q}$ is the extension of $\hat{q}_1$ to the whole prism, then

$$\begin{aligned} \iint_{\hat{f}_5} \hat{v}_2 \, \hat{q}_1 \, d\hat{S} &= \sqrt{2} \iint\limits_{[0,1]^2} \hat{v}_2(1 - \hat{x}_2, \hat{x}_2, \hat{x}_3) \hat{q}_1(\hat{x}_2, \hat{x}_3) \, d\hat{x}_2 d\hat{x}_3 \\ &= \sqrt{2} \iint\limits_{[0,1]^2} \int_0^{1-\hat{x}_2} \tfrac{\partial \hat{v}_2}{\partial \hat{x}_1}(\hat{t}, \hat{x}_2, \hat{x}_3) \hat{q}(\hat{t}, \hat{x}_2, \hat{x}_3) \, d\hat{t} d\hat{x}_2 d\hat{x}_3 \\ &= \sqrt{2} \int_{\hat{E}} (\mathbf{curl}\, \hat{\boldsymbol{v}})_3 \hat{q} \, d\hat{\boldsymbol{x}} + \sqrt{2} \int_{\hat{E}} \tfrac{\partial \hat{v}_1}{\partial \hat{x}_2} \hat{q} \, d\hat{\boldsymbol{x}}. \end{aligned} \tag{6.25}$$

Joining (6.24) and (6.25) and using the expression for $\hat{\boldsymbol{v}}$ in (6.18) we get

$$\varphi_{\hat{f}_5,\hat{\boldsymbol{q}}}(\hat{\boldsymbol{v}}) = \iint_{\hat{f}_5} \hat{u}_1\,\hat{q}_1\,d\hat{S} - \sqrt{2}\int_{\hat{E}} \tfrac{\partial \hat{u}_1}{\partial \hat{x}_2}\hat{q}\,d\hat{\boldsymbol{x}} - \sqrt{2}\int_{\hat{E}} (\mathbf{curl}\,\hat{\boldsymbol{u}})_3\,\hat{q}\,d\hat{\boldsymbol{x}}. \tag{6.26}$$

At last, we study the volume degrees of freedom. Pick $\hat{\boldsymbol{r}} = (\hat{r}_1, \hat{r}_2, \hat{r}_3)'$ belonging to the space

$$(P_{k-2}(\hat{f}_3) \otimes P_{k-2}(\hat{x}_3))^2 \times P_{k-3}(\hat{f}_3) \otimes P_{k-1}(\hat{x}_3)$$

(cfr. degree of freedom (2.18)) and let $\hat{\varphi}_2$ be defined by

$$\varphi_2(\hat{x}_1, \hat{x}_2, \hat{x}_3) = \int_{1-\hat{x}_2}^{\hat{x}_1} \hat{r}_2(\hat{t}, \hat{x}_2, \hat{x}_3)\,d\hat{t}.$$

Green's Theorem and the fact that $\varphi_2|_{\hat{f}_5} \equiv 0$ give

$$\int_{\hat{E}} \hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{r}}\,d\hat{\boldsymbol{x}} = \int_{\hat{E}} \hat{u}_1\,\hat{r}_1\,d\hat{\boldsymbol{x}} - \int_{\hat{E}} (\mathbf{curl}\,\hat{\boldsymbol{u}})_3\,\hat{\varphi}_2\,d\hat{\boldsymbol{x}} - \int_{\hat{E}} \tfrac{\partial \hat{u}_1}{\partial \hat{x}_2}\,\hat{\varphi}_2\,d\hat{\boldsymbol{x}}. \tag{6.27}$$

Now we collect what has been said so far. For the edge degrees of freedom we use an inequality in page 135 of [33], in the proof of Lemma 5.38, which states, for $\hat{\boldsymbol{u}}$ in the present conditions,

$$\left| \int_{\hat{\boldsymbol{e}}} \hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{\tau}}\,q\,d\hat{\alpha} \right| \leqslant C(q)\,\{\,\|\mathbf{curl}\,\hat{\boldsymbol{u}}\|_{L^p(\hat{E})^3} + \|\mathrm{Tr}\,\hat{\boldsymbol{u}}\|_{L^p(\partial\hat{E})^3}\}. \tag{6.28}$$

The details needed to the proof of inequality (6.28) can be completed from Theorem 3.14 of [16].

Now if we put the field $(\hat{u}_1, 0, 0)'$ in inequality (6.28) then by, Hölder's Inequality and standard traces inequalities, equation (6.19) and (6.20) yield, for $i = 8$ and 9,

$$\begin{aligned} |\varphi_{\hat{\mathbf{e}}_i,\hat{q}}(\hat{\boldsymbol{v}})| &\leqslant c(\hat{q})\,\{\,\|\hat{u}_1\|_{W^{1,p}(\hat{E})} + \|\mathrm{Tr}\,(\mathbf{curl}\,\hat{\boldsymbol{u}})_3\|_{L^1(\partial\hat{E})} + \|\mathrm{Tr}\,\hat{u}_1\|_{L^p(\partial\hat{E})}\,\} \\ &\leqslant c(\hat{q})\,\{\,\|\hat{u}_1\|_{W^{1,p}(\hat{E})} + \|(\mathbf{curl}\,\hat{\boldsymbol{u}})_3\|_{W^{1,1}(\hat{E})}\,\}. \end{aligned} \tag{6.29}$$

If we repeat the argument for the line integral terms in (6.22) and (6.23) we get, for $j = 3$ and 4,

$$\left|\varphi_{\hat{f}_j,\hat{\boldsymbol{q}}}(\hat{\boldsymbol{v}})\right| \leqslant c(\hat{\boldsymbol{q}})\,\{\,\|\mathrm{Tr}\,\hat{u}_1\|_{L^p(\partial\hat{E})} + \|\mathrm{Tr}\,(\mathbf{curl}\,\hat{\boldsymbol{u}})_3\|_{L^1(\partial\hat{E})} + |\hat{u}_1|_{W^{1,p}(\hat{E})}\,\}. \tag{6.30}$$

And finally, by estimates (6.24)–(6.30) and one more time Hölder's and traces inequalities,

$$\begin{aligned} \|(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_1\|_{L^\infty(\hat{E})} = \|(\hat{\boldsymbol{w}}_{\hat{E}}\hat{\boldsymbol{v}})_1\|_{L^\infty(\hat{E})} &\lesssim \sum_{i=8,9,\,\hat{p}} |\hat{\varphi}_{\hat{\mathbf{e}}_i,\hat{p}}(\hat{\boldsymbol{v}})|\,\|(\hat{\boldsymbol{v}}_{\hat{\mathbf{e}}_i,\hat{p}})_1\|_{L^\infty(\hat{E})} \\ &+ \sum_{j=3,4,\,\hat{\boldsymbol{q}}} |\hat{\varphi}_{\hat{f}_j,\hat{\boldsymbol{q}}}(\hat{\boldsymbol{v}})|\,\|(\hat{\boldsymbol{v}}_{\hat{f}_j,\hat{\boldsymbol{q}}})_1\|_{L^\infty(\hat{E})} + \sum_{\hat{\boldsymbol{r}}} |\hat{\varphi}_{\hat{\boldsymbol{r}}}(\hat{\boldsymbol{v}})|\,\|(\hat{\boldsymbol{v}}_{\hat{\boldsymbol{r}}})_1\|_{L^\infty(\hat{E})} \\ &\leqslant c(\hat{E})\,\{\,\|\hat{u}_1\|_{W^{1,p}(\hat{E})} + \|(\mathbf{curl}\,\hat{\boldsymbol{u}})_3\|_{W^{1,1}(\hat{E})}\,\} \end{aligned}$$

which is the bound we wanted to prove. The summation indices with polynomials $\hat{p}$, $\hat{\boldsymbol{q}}$ and $\hat{\boldsymbol{r}}$ mean that we use the way of writing the interpolator stated in (2.23). The same proving procedure applies to inequality (6.16).

To prove (6.17), given $\hat{\boldsymbol{u}} \in W^{1,p}(\hat{E})^3$, define $\hat{\boldsymbol{v}} = (0, 0, \hat{u}_3)'$. Thanks to Lemma 6.1.1 we have $(\hat{\boldsymbol{w}}_{\hat{E}}\hat{\boldsymbol{v}})_3 = (\hat{\boldsymbol{w}}_{\hat{E}}\hat{\boldsymbol{u}})_3 - (\hat{\boldsymbol{w}}_{\hat{E}}(\hat{u}_1, \hat{u}_2, 0))_3 = (\hat{\boldsymbol{w}}_{\hat{E}}\hat{\boldsymbol{u}})_3$. By expression (2.23), taking another look at the unit tangent vector of the edges and unit normal vectors to the faces, we have

$$(\hat{\boldsymbol{w}}_{\hat{E}}\hat{\boldsymbol{v}})_3 = \sum_{j=3,6,7;\,\hat{\boldsymbol{p}}} \int_{\hat{e}_j} \hat{u}_3 \hat{p}_3 \, d\alpha \, (\hat{\boldsymbol{v}}_{\hat{\boldsymbol{e}}_j,\hat{\boldsymbol{p}}})_3 + \sum_{i=1,2,4;\,\hat{q}} \int_{\hat{f}_i} \hat{u}_3 \hat{q} \, d\hat{S} \, (\hat{\boldsymbol{v}}_{\hat{f}_i,\hat{q}})_3$$
$$+ \sum_{\hat{\boldsymbol{r}}} \int_{\hat{E}} \hat{u}_3 \hat{r}_3 \, d\hat{\boldsymbol{x}} \, (\hat{\boldsymbol{v}}_{\hat{\boldsymbol{r}}})_3.$$

This implies, by traces inequalities and (6.28), that

$$\|(\hat{\boldsymbol{w}}_{\hat{E}}\hat{\boldsymbol{u}})_3\|_{L^\infty(\hat{E})} \leqslant C(\hat{E})\Big\{ \sum_{j=3,6,7;\hat{\boldsymbol{p}}} \left| \int_{\hat{e}_j} \hat{u}_3 \, \hat{p}_3 \, d\alpha \right|$$
$$+ \sum_{i=1,2,4} \iint_{\hat{f}_i} |\hat{u}_3|^p \, d\hat{S} + \int_{\hat{E}} |\hat{u}_3|^p \, d\hat{\boldsymbol{x}} \Big\}$$
$$\leqslant C(\hat{E}) \|\hat{u}_3\|_{W^{1,p}(\hat{E})}.$$

The constants in the three inequalities of this Theorem depend only on the choice of the bases of the test polynomials for the degrees of freedom. ☐

As in the div–conforming case, the next step is to estimate the stability in an anisotropically rescaled prism. Consider again the element $\tilde{E}$ defined in (4.7). Given a natural number k, denote with $\boldsymbol{w}_{\tilde{E}}$ the k–th order **curl**–conforming interpolation operator over $\tilde{E}$ defined as in Corollary 2.3.8. For the rest of the Subsection, $\tilde{\boldsymbol{u}}$ will be an element with a well defined **curl**–conforming interpolate. Write the diameter of $\tilde{E}$ as $h_{\tilde{E}}$ and as $\tilde{x}_i$, $1 \leqslant i \leqslant 3$, the coordinates along the axes in $\mathbb{R}^3$.

6.1.5 Lemma. *There exists a positive C, independent of h_i, $1 \leqslant i \leqslant 3$ such that for all $p > 2$ and $\tilde{\boldsymbol{u}} \in W^p(\boldsymbol{curl}, \tilde{E})$*

$$\|\boldsymbol{w}_{\tilde{E}}\tilde{\boldsymbol{u}}\|_{L^\infty(\tilde{E})^3} \leqslant C \Bigg[|\tilde{E}|^{-1/p} \left(\|\tilde{\boldsymbol{u}}\|_{L^p(\tilde{E})^3} + \sum_{i=1}^{3} h_i \, \|\partial_{\tilde{x}_i} \tilde{\boldsymbol{u}}\|_{L^p(\tilde{E})^3} \right)$$
$$+ (h_1 + h_2) \, |\tilde{E}|^{-1} \left(\|(\boldsymbol{curl}\ \tilde{\boldsymbol{u}})_3\|_{L^1(\tilde{E})} + \sum_{i=1}^{3} h_i \, \|\partial_{\tilde{x}_i} (\boldsymbol{curl}\ \tilde{\boldsymbol{u}})_3\|_{L^1(\tilde{E})} \right) \Bigg].$$

Proof. The proof of this estimate will be made componentwise using the inequalities of Theorem 6.1.4 and the vectorial bound will hold immediately. Bounds for $(\boldsymbol{w}_{\tilde{E}}\tilde{\boldsymbol{u}})_1$ and $(\boldsymbol{w}_{\tilde{E}}\tilde{\boldsymbol{u}})_3$ will be established, as the bounding for $(\boldsymbol{w}_{\tilde{E}}\tilde{\boldsymbol{u}})_2$ is the same as the first one. Pulling $\boldsymbol{w}_{\tilde{E}}\tilde{\boldsymbol{u}}$ back to $\hat{E}$ we get by (2.48) that $(\boldsymbol{w}_{\tilde{E}}\tilde{\boldsymbol{u}})_i =$

${}^1/_{h_i}(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_i$, $1 \leqslant i \leqslant 3$. By inequality (6.15) and a suitable, though elementary, change of variables dictated by (4.8) we do

$$\begin{aligned}\|(\boldsymbol{w}_{\tilde{E}}\tilde{\boldsymbol{u}})_1\|_{L^\infty(\tilde{E})} &= \frac{1}{h_1}\|(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_1\|_{L^\infty(\hat{E})} \\ &\leqslant \frac{c(\hat{E})}{h_1}\left[\|\hat{u}_1\|_{W^{1,p}(\hat{E})} + \|(\mathbf{curl}\ \hat{\boldsymbol{u}})_3\|_{W^{1,1}(\hat{E})}\right] \\ &\leqslant c(\hat{E})\left[|\tilde{E}|^{-1/p}\{\|\tilde{u}_1\|_{L^p(\tilde{E})} + \sum_{i=1}^3 h_i\|\tfrac{\partial\tilde{u}_1}{\partial\tilde{x}_i}\|_{L^p(\tilde{E})}\}\right. \\ &\left. + h_2|\tilde{E}|^{-1}\{\|(\mathbf{curl}\ \tilde{\boldsymbol{u}})_3\|_{L^1(\tilde{E})} + \sum_{i=1}^3 h_i\|\partial_{\tilde{x}_i}(\mathbf{curl}\ \tilde{\boldsymbol{u}})_3\|_{L^1(\tilde{E})}\}\right].\end{aligned} \tag{6.31}$$

With respect to component number three, from (6.17) we write

$$\|(\boldsymbol{w}_{\tilde{E}}\tilde{\boldsymbol{u}})_3\|_{L^\infty(\tilde{E})} \leqslant C|\tilde{E}|^{-1/p}\left(\|\tilde{u}_3\|_{L^p(\tilde{E})} + \sum_{i=1}^3 h_i\|\partial_{\tilde{x}_i}\tilde{u}_3\|_{L^p(\tilde{E})}\right). \tag{6.32}$$

□

With the previous bound we deduce the following anisotropic stability estimate for the rescaled prismatic element $\tilde{E}$.

6.1.6 Theorem. *There is a $C > 0$ independent of h_i such that for all $\tilde{\boldsymbol{u}} \in W^p(\boldsymbol{curl}, \tilde{E})$ and $p > 2$.*

$$\begin{aligned}\|\boldsymbol{w}_{\tilde{E}}\tilde{\boldsymbol{u}}\|_{L^p(\tilde{E})} \leqslant C &\left[\|\tilde{\boldsymbol{u}}\|_{L^p(\tilde{E})} + \sum_{i=1}^3 h_i\,\|\partial_{\tilde{x}_i}\tilde{\boldsymbol{u}}\|_{L^p(\tilde{E})}\right. \\ &\left. + (h_1+h_2)\left(\|(\boldsymbol{curl}\ \tilde{\boldsymbol{u}})_3\|_{L^p(\tilde{E})} + \sum_{i=1}^3 h_i\,\|\partial_{\tilde{x}_i}(\boldsymbol{curl}\ \tilde{\boldsymbol{u}})_3\|_{L^p(\tilde{E})}\right)\right].\end{aligned}$$

Proof. From Lemma 6.1.5, since $|\tilde{E}|$ is finite measured, the Hölder inequality tells us that, for any real $q \geqslant 1$,

$$\begin{aligned}\|(\mathbf{curl}\ \tilde{\boldsymbol{u}})_3\|_{L^1(\tilde{E})} &\leqslant |\tilde{E}|^{1-\frac{1}{q}}\,\|(\mathbf{curl}\ \tilde{\boldsymbol{u}})_3\|_{L^q(\tilde{E})} \\ \|\partial_{\tilde{x}_i}(\mathbf{curl}\ \tilde{\boldsymbol{u}})_3\|_{L^1(\tilde{E})} &\leqslant |\tilde{E}|^{1-\frac{1}{q}}\,\|\partial_{\tilde{x}_i}(\mathbf{curl}\ \tilde{\boldsymbol{u}})_3\|_{L^q(\tilde{E})}.\end{aligned}$$

So we get to

$$\begin{aligned}\|(\tilde{\boldsymbol{w}}_{\tilde{E}}\tilde{\boldsymbol{u}})_1\|_{L^p(\tilde{E})} &\leqslant |\tilde{E}|^{1/p}\,\|(\tilde{\boldsymbol{w}}_{\tilde{E}}\tilde{\boldsymbol{u}})_1\|_{L^\infty(\tilde{E})} \\ \text{(by (6.31))}\quad &\leqslant C\left[\|\tilde{u}_1\|_{L^p(\tilde{E})} + \sum_{i=1}^3 h_i\|\tfrac{\partial\tilde{u}_1}{\partial\tilde{x}_i}\|_{L^p(\tilde{E})}\right. \\ &\left. + h_2\left(\|(\mathbf{curl}\ \tilde{\boldsymbol{u}})_3\|_{L^p(\tilde{E})} + \sum_{i=1}^3 h_i\|\partial_{\tilde{x}_i}(\mathbf{curl}\ \tilde{\boldsymbol{u}})_3\|_{L^p(\tilde{E})}\right)\right].\end{aligned}$$

Now combine this with an entirely analogous argument for component two and with (6.32) and the Theorem follows. □

6.2 Local Interpolation Estimates for $H(\mathbf{curl})$ Conforming Prismatic Elements

6.2.1 Theorem. *Let $k \in \mathbb{N}$ and $p > 2$ and let E be a prism whose triangular faces have greatest angle less than c_0. There exists $C > 0$ and three edges $\boldsymbol{e}_i$ of E incident to a common vertex $\boldsymbol{x}_E$ such that for all $\boldsymbol{u} \in W^{m+1,p}(E)^3$ and $m \leqslant k-1$,*

$$\|\boldsymbol{u} - \boldsymbol{w}_E \boldsymbol{u}\|_{L^p(E)} \leqslant C \left\{ \sum_{|\boldsymbol{\alpha}|=m+1} \boldsymbol{h}^{\boldsymbol{\alpha}} \|\partial^{\boldsymbol{\alpha}} \boldsymbol{u}\|_{L^p(E)} + h_E \sum_{|\boldsymbol{\alpha}|=m} \boldsymbol{h}^{\boldsymbol{\alpha}} \|\partial^{\boldsymbol{\alpha}} (\mathbf{curl}\, \boldsymbol{u})_3\|_{L^p(E)} \right\}. \tag{6.33}$$

C depends only on c_0. C can be chosen so that, if M_E is the matrix made with ξ_i as columns, then $\|M\|_\infty \leqslant C$ and $\|M^{-1}\|_\infty \leqslant C$.

Notice the anisotropic character of the inequality in (6.33). Notice only the component of the **curl** corresponding to the direction that is orthogonal to the triangular faces.

Proof of Theorem 6.2.1. Since $W^{m+1,p}(E) \hookrightarrow W^{1,p}(E)$ and p is greater than 2, the edge interpolator $\boldsymbol{w}_E$ is well–defined via Corollary 2.3.8. Consider the prism $\tilde{E}$ as in (4.7). By the argument in the proof of Theorem 2.2 in [1], there is an affine map $\tilde{\boldsymbol{x}} \mapsto \boldsymbol{x} = M_E \tilde{\boldsymbol{x}} + \boldsymbol{x}_E = F_E \tilde{\boldsymbol{x}}$ from $\tilde{E}$ onto E, such that $\|M_E\|$, $\|M_E\|^{-1} \leqslant C(c_0)$. Notice that this is the only place where the dependence on c_0 is found. The matrix M_E is made up of vectors ξ_i, $i = 1, 2, 3$ as its columns. First we take $\boldsymbol{q} := \mathbf{Q}_{m,E}\, \boldsymbol{u}$ and do

$$\|\boldsymbol{u} - \boldsymbol{w}_E \boldsymbol{u}\|_{L^p(E)} \leqslant \|\boldsymbol{u} - \boldsymbol{q}\|_{L^p(E)} + \|\boldsymbol{w}_E(\boldsymbol{u} - \boldsymbol{q})\|_{L^p(E)}$$

For the first term we may simply do, by Remark 4.1.7 and the transformation (2.29),

$$\begin{aligned} \|\boldsymbol{u} - \boldsymbol{q}\|_{L^p(E)} &= \|M_E^{-t}(\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{q}}) \circ F_E^{-1}\|_{L^p(E)} \\ &\leqslant \|M^{-1}\| |\det M_E|^{1/p} \|\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{q}}\|_{L^p(\tilde{E})}. \end{aligned} \tag{6.34}$$

With regard to the second term, by the commutativity property (2.48) and again the coordinate transformation,

$$\|\boldsymbol{w}_E(\boldsymbol{u} - \boldsymbol{q})\|_{L^p(E)} \leqslant |M|^{1/p} \|M_E^{-1}\| \|\tilde{\boldsymbol{w}}_{\tilde{E}}(\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{q}})\|_{L^p(\tilde{E})}.$$

Theorem 6.1.6 implies

$$
\begin{aligned}
&\|\boldsymbol{w}_E(\boldsymbol{u}-\boldsymbol{q})\|_{L^p(E)} \leqslant \\
&\quad C\|M^{-1}\|\,|\det M_E|^{1/p}\Bigg[\|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{q}}\|_{L^p(\tilde{E})} + \sum_{i=1}^{3} h_i \|\partial_{\tilde{x}_i}(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{q}})\|_{L^p(\tilde{E})} \\
&\quad + h\left(\|(\mathbf{curl}\,(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{q}}))_3\|_{L^p(\tilde{E})} + \sum_{i=1}^{3} h_i\,\|\partial_{\tilde{x}_i}(\mathbf{curl}\,(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{q}}))_3\|_{L^p(\tilde{E})}\right)\Bigg].
\end{aligned}
\tag{6.35}
$$

By expressions (4.14), (4.15), (4.16) the last expression is bounded by a constant times $\|M_E^{-1}\|\,|\det M_E|^{1/p}$ times the following sum

$$
\begin{aligned}
&\sum_{i+j+k=m+1} h_1^i h_2^j h_3^k \left\|\frac{\partial^{m+1}\tilde{\boldsymbol{u}}}{\partial\tilde{x}_1^i\partial\tilde{x}_2^j\partial\tilde{x}_3^k}\right\|_{0,\tilde{E}} + h \sum_{j+k+l=m} h_1^j h_2^k h_3^l \left\|\frac{\tilde{\partial}^m(\mathbf{curl}\,\tilde{\boldsymbol{u}})_3}{\partial\tilde{x}_1^j\partial\tilde{x}_2^k\partial\tilde{x}_3^l}\right\|_{0,\tilde{E}} \\
&\qquad + h\sum_{i=1}^{3} h_i \sum_{j+k+l=m-1} h_1^j h_2^k h_3^l \left\|\frac{\tilde{\partial}^{m-1}\tilde{\partial}(\widetilde{\mathbf{curl}}\,\tilde{\boldsymbol{u}})_3}{\partial\tilde{x}_1^j\partial\tilde{x}_2^k\partial\tilde{x}_3^l\partial\tilde{x}_j}\right\|_{0,\tilde{E}} \\
&\lesssim \sum_{i+j+k=m+1} h_1^i h_2^j h_3^k \left\|\frac{\partial^{m+1}\tilde{\boldsymbol{u}}}{\partial\tilde{x}_1^i\partial\tilde{x}_2^j\partial\tilde{x}_3^k}\right\|_{0,\tilde{E}} + h \sum_{j+k+l=m} h_1^j h_2^k h_3^l \left\|\frac{\tilde{\partial}^m(\mathbf{curl}\,\tilde{\boldsymbol{u}})_3}{\partial\tilde{x}_1^j\partial\tilde{x}_2^k\partial\tilde{x}_3^l}\right\|_{0,\tilde{E}}.
\end{aligned}
\tag{6.36}
$$

From equality (2.31), for every $\boldsymbol{\alpha}$ of order $m+1$ it holds

$$
\|\tilde{\partial}^{\boldsymbol{\alpha}}\tilde{\boldsymbol{u}}\|_{L^p(\tilde{E})} \leqslant \|M_E\|\,|\det M_E|^{-1/p}\|\partial^{\boldsymbol{\alpha}}\boldsymbol{u}\|_{L^p(E)}. \tag{6.37}
$$

Lastly, adapting Lemma 3.57 in page 77 of [33], we observe

$$
\begin{pmatrix} 0 & -(\widetilde{\mathbf{curl}}\,\tilde{\boldsymbol{u}})_3 & 0 \\ (\widetilde{\mathbf{curl}}\,\tilde{\boldsymbol{u}})_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} M_E^{-1} = M_E^t \begin{pmatrix} 0 & -(\mathbf{curl}\,\boldsymbol{u})_3 & 0 \\ (\mathbf{curl}\,\boldsymbol{u})_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \circ F_E
$$

which implies, for every $\boldsymbol{\alpha}$ of order m,

$$
\|\tilde{\partial}^{\boldsymbol{\alpha}}(\widetilde{\mathbf{curl}}\,\tilde{\boldsymbol{u}})_3\|_{L^p(\tilde{E})} \leqslant C|\det M_E|^{-1/p}\,\|M_E\|^2\|\partial^{\boldsymbol{\alpha}}(\mathbf{curl}\,\boldsymbol{u})_3\|_{L^p(E)}. \tag{6.38}
$$

Now combine (6.36), (6.37) and (6.38) with (6.35) and (6.34) to obtain the Theorem. □

6.3 Pyramidal Finite Elements

Here we state and prove least order anisotropic stability inequalities and anisotropic local interpolation inequalities for the finite elements on pyramids defined in [25] and [36]. This estimates could be used to build a variant of our method presented in this Thesis, using finite elements in all the types of elements. As we said in the introduction of the Thesis, one of the ideas of our method was the combination of finite elements in prisms and tetrahedra with virtual elements on pyramids.

Table 6.1 – Notation for the faces and positive normals of the reference pyramid.

$\hat{f}_1 \subseteq \{\hat{x}_2 = 0\}$	$\hat{\boldsymbol{n}}_1 = (0, -1, 0)'$
$\hat{f}_2 \subseteq \{\hat{x}_1 = 0\}$	$\hat{\boldsymbol{n}}_2 = (-1, 0, 0)'$
$\hat{f}_3 \subseteq \{\hat{x}_1 + \hat{x}_3 = 1\}$	$\hat{\boldsymbol{n}}_3 = 2^{-1/2}(1, 0, 1)'$
$\hat{f}_4 \subseteq \{\hat{x}_2 + \hat{x}_3 = 1\}$	$\hat{\boldsymbol{n}}_4 = 2^{-1/2}(0, 1, 1)'$
$\hat{f}_5 \subseteq \{\hat{x}_3 = 0\}$	$\hat{\boldsymbol{n}}_5 = (0, 0, -1)'$

Table 6.2 – Notation for the edges and positive tangents of the reference pyramid.

$\hat{\boldsymbol{e}}_1 = \{(\hat{x}_1, 0, 0)^t : 0 \leqslant \hat{x}_1 \leqslant 1\}$	$\hat{\boldsymbol{\tau}}_1 = (1, 0, 0)'$
$\hat{\boldsymbol{e}}_2 = \{(1, \hat{x}_2, 0)^t : 0 \leqslant \hat{x}_2 \leqslant 1\}$	$\hat{\boldsymbol{\tau}}_2 = (0, 1, 0)'$
$\hat{\boldsymbol{e}}_3 = \{(\hat{x}_1, 1, 0)^t : 0 \leqslant \hat{x}_1 \leqslant 1\}$	$\hat{\boldsymbol{\tau}}_3 = (-1, 0, 0)'$
$\hat{\boldsymbol{e}}_4 = \{(0, \hat{x}_2, 0)^t : 0 \leqslant \hat{x}_2 \leqslant 1\}$	$\hat{\boldsymbol{\tau}}_4 = (0, 1, 0)'$
$\hat{\boldsymbol{e}}_5 = \{(0, 0, \hat{x}_3)^t : 0 \leqslant \hat{x}_3 \leqslant 1\}$	$\hat{\boldsymbol{\tau}}_5 = (0, 0, 1)'$
$\hat{\boldsymbol{e}}_6 = \{(1 - \hat{x}_3, 0, \hat{x}_3)^t : 0 \leqslant \hat{x}_3 \leqslant 1\}$	$\hat{\boldsymbol{\tau}}_6 = 2^{-1/2}(-1, 0, 1)'$
$\hat{\boldsymbol{e}}_7 = \{(0, 1 - \hat{x}_3, \hat{x}_3)^t : 0 \leqslant \hat{x}_3 \leqslant 1\}$	$\hat{\boldsymbol{\tau}}_7 = 2^{-1/2}(0, -1, 1)'$
$\hat{\boldsymbol{e}}_8 = \{(1 - \hat{x}_3, 1 - \hat{x}_3, \hat{x}_3)^t : 0 \leqslant \hat{x}_3 \leqslant 1\}$	$\hat{\boldsymbol{\tau}}_8 = 3^{-1/2}(-1, -1, 1)$

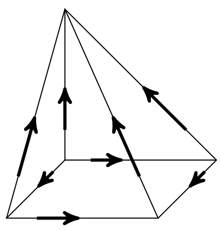


Figure 6.1 – Directions of the positive unit tangents (cfr. Table 6.2).

6.3.1 $H(\mathbf{curl})$–Conforming Element on Pyramids

6.3.1 Definition. *The following items define a least order* ***curl****–conforming finite element on the reference Pyramid.*

1. *$\hat{E}$ is the reference Pyramid in Definition 1.1.18.*
2. *The rational space $P_{\hat{E}}$ is the span of $\{\hat{\boldsymbol{\gamma}}_1, \ldots, \hat{\boldsymbol{\gamma}}_8\}$ with $\hat{\boldsymbol{\gamma}}_i$ as in Table 6.3.*
3. *The degrees of freedom are the line integrals*

$$\int_{\hat{\mathbf{e}}_j} \hat{\boldsymbol{u}} \cdot d\hat{\boldsymbol{\alpha}}$$

for every edge $\hat{\mathbf{e}}_j$ of $\hat{E}$, $1 \leqslant j \leqslant 8$.

Table 6.3 – Edge shape functions on the reference pyramid

$$\hat{\boldsymbol{\gamma}}_1 = \begin{pmatrix} 1-z-y \\ 0 \\ x - \frac{xy}{1-z} \end{pmatrix} \quad \hat{\boldsymbol{\gamma}}_2 = \begin{pmatrix} 0 \\ x \\ \frac{xy}{1-z} \end{pmatrix} \quad \hat{\boldsymbol{\gamma}}_3 = \begin{pmatrix} y \\ 0 \\ \frac{xy}{1-z} \end{pmatrix} \quad \hat{\boldsymbol{\gamma}}_4 = \begin{pmatrix} 0 \\ 1-z-x \\ y - \frac{xy}{1-z} \end{pmatrix}$$

$$\hat{\boldsymbol{\gamma}}_5 = \begin{pmatrix} z - \frac{yz}{1-z} \\ z - \frac{xz}{1-z} \\ 1 - x - y + \frac{xy}{1-z} - \frac{xyz}{(1-z)^2} \end{pmatrix} \quad \hat{\boldsymbol{\gamma}}_6 = \begin{pmatrix} -z + \frac{yz}{1-z} \\ \frac{xz}{1-z} \\ x - \frac{xy}{1-z} + \frac{xyz}{(1-z)^2} \end{pmatrix}$$

$$\hat{\boldsymbol{\gamma}}_7 = \begin{pmatrix} \frac{yz}{1-z} \\ -z + \frac{xz}{1-z} \\ y - \frac{xy}{1-z} + \frac{xyz}{(1-z)^2} \end{pmatrix} \quad \hat{\boldsymbol{\gamma}}_8 = \begin{pmatrix} -\frac{yz}{1-z} \\ -\frac{xz}{1-z} \\ \frac{xy}{1-z} - \frac{xyz}{(1-z)^2} \end{pmatrix}$$

A direct computation yields the following Lemma.

6.3.2 Lemma. *For $1 \leqslant i, j \leqslant 8$, $\int_{\hat{\mathbf{e}}_j} \hat{\boldsymbol{\gamma}}_i \cdot d\hat{\boldsymbol{\alpha}} = \delta_{ij}$ which implies immediately that the finite element in Definition 6.3.1 is unisolvent in $\hat{E}$.*

6.3.3 Lemma. $P_0(\hat{E})^3 \subseteq P_{\hat{E}}$.

Proof. Cfr. Lemma 7.3 of [36]. □

6.3.2 $H(\mathbf{div})$–Conforming Element on Pyramids

6.3.4 Definition. *The following items define a least order div–conforming finite element on the reference Pyramid.*

1. *$\hat{E}$ is the reference pyramid of Figure 6.1.*
2. *The space $P_{\hat{E}}$ is the span of $\{\hat{\boldsymbol{\zeta}}_1, \ldots, \hat{\boldsymbol{\zeta}}_5\}$ with $\hat{\boldsymbol{\zeta}}_i$ as in Table 6.4.*
3. *The degrees of freedom are the surface integrals*

$$\iint_{\hat{f}_j} \hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{n}} \, d\hat{S}$$

for every face $\hat{f}_j$ of $\hat{E}$, $1 \leqslant j \leqslant 5$.

Table 6.4 – Face shape functions on the reference pyramid

$$\hat{\zeta}_1 = \begin{pmatrix} -\frac{xz}{1-z} \\ y - 2 + \frac{y}{1-z} \\ z \end{pmatrix} \quad \hat{\zeta}_2 = \begin{pmatrix} x - 2 + \frac{x}{1-z} \\ -\frac{yz}{1-z} \\ z \end{pmatrix}$$

$$\hat{\zeta}_3 = \begin{pmatrix} x + \frac{x}{1-z} \\ -\frac{yz}{1-z} \\ z \end{pmatrix} \quad \hat{\zeta}_4 = \begin{pmatrix} -\frac{xz}{1-z} \\ y + \frac{y}{1-z} \\ z \end{pmatrix} \quad \hat{\zeta}_5 = \begin{pmatrix} x \\ y \\ z - 1 \end{pmatrix}$$

A direct computation yields the following Lemma.

6.3.5 Lemma. *For $1 \leqslant i, j \leqslant 5$, $\iint_{f_j} \hat{\boldsymbol{\zeta}}_i \cdot \hat{\boldsymbol{n}} \, d\hat{S} = \delta_{ij}$ which implies immediately that the finite element in Definition 6.3.4 is unisolvent in $\hat{E}$.*

6.3.6 Lemma. $P_0(\hat{E})^3 \subseteq P_{\hat{E}}$.

Proof. Cfr. Lemma 7.2 of [36]. □

6.4 Anisotropic Local interpolation Estimates for Pyramidal Finite Elements

$\hat{E}$ will be the reference pyramid in Figure 6.1. Anisotropic interpolation error estimates for pyramidal **curl**–conforming and div–conforming finite elements of least order will we established.

6.4.1 Lemma. *The shape functions in Tables 6.3 and 6.4 are bounded.*

Proof. Observe that the pyramid is contained in the region $\{ x + z \leqslant 1, \; y + z \leqslant 1 \}$ which yields

$$\tfrac{x}{1-z} \leqslant 1, \quad \tfrac{xy}{1-z} \leqslant y, \quad \tfrac{xyz}{(1-z)^2} \leqslant z.$$

□

6.4.1 Anisotropic Interpolation Estimates for $H(\mathbf{curl})$ conforming Elements on Pyramids

Here we state an anisotropic stability bound for the least order **curl** – conforming operator on pyramids. As always, we write it componentwise to make the anisotropy of the estimate clearer.

6.4.2 Theorem. *Let $\hat{E}$ be the reference pyramid and let $p > 2$. Let $\boldsymbol{w}_{\hat{E}}(\cdot)$ denote the interpolation operator determined by the degrees of freedom in Definition 6.3.1. There is $C > 0$ such that, for all $\hat{\boldsymbol{u}} \in W^{1,p}(\hat{E})$ with first derivatives in $W^{1,1}(\hat{E})$, there hold*

$$\|(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_1\|_{L^\infty(\hat{E})} \lesssim \|\hat{u}_1\|_{W^{1,p}(\hat{E})} + \|(\nabla\times\hat{\boldsymbol{u}})_2\|_{W^{1,1}(\hat{E})} + \|(\nabla\times\hat{\boldsymbol{u}})_3\|_{W^{1,1}(\hat{E})} + \left\|\frac{\partial\hat{u}_1}{\partial\hat{x}_2}\right\|_{W^{1,1}(\hat{E})} + \left\|\frac{\partial^2\hat{u}_3}{\partial\hat{x}_2\partial\hat{x}_1}\right\|_{L^1(\hat{E})}. \tag{6.39}$$

$$\|(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_2\|_{L^\infty(\hat{E})} \lesssim \|\hat{u}_2\|_{W^{1,p}(\hat{E})} + \|(\nabla\times\hat{\boldsymbol{u}})_1\|_{W^{1,1}(\hat{E})} + \|(\nabla\times\hat{\boldsymbol{u}})_3\|_{W^{1,1}(\hat{E})} + \left\|\frac{\partial\hat{u}_2}{\partial\hat{x}_1}\right\|_{W^{1,1}(\hat{E})} + \left\|\frac{\partial^2\hat{u}_3}{\partial\hat{x}_2\partial\hat{x}_1}\right\|_{L^1(\hat{E})}. \tag{6.40}$$

$$\|(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_3\|_{L^\infty(\hat{E})} \lesssim \|\hat{u}_3\|_{W^{1,p}(\hat{E})} + \|(\nabla\times\hat{\boldsymbol{u}})_2\|_{W^{1,1}(\hat{E})} + \|(\nabla\times\hat{\boldsymbol{u}})_1\|_{W^{1,1}(\hat{E})} + \left\|\frac{\partial\hat{u}_3}{\partial\hat{x}_1}\right\|_{W^{1,1}(\hat{E})} + \left\|\frac{\partial\hat{u}_2}{\partial\hat{x}_1}\right\|_{W^{1,1}(\hat{E})} + \left\|\frac{\partial\hat{u}_1}{\partial\hat{x}_2}\right\|_{W^{1,1}(\hat{E})} + \left\|\frac{\partial^2\hat{u}_3}{\partial\hat{x}_2\partial\hat{x}_1}\right\|_{L^1(\hat{E})}. \tag{6.41}$$

Proof. Take an element $\hat{\boldsymbol{u}}$ of $W^{1,p}(\hat{E})$ for a $p > 2$. Let us recall the shape funtions in Table 6.3. For the variables of the shape functions in upcoming computations we will write x,y and z instead of $\hat{x}_i$ to get a cleaner reading. Start with $\hat{\boldsymbol{u}}$ of the form $(\hat{u}_1, 0, 0)'$. After calculating we have

$$\begin{aligned}\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}} &= [\textstyle\int_{\hat{\mathbf{e}}_1}\hat{\boldsymbol{u}}\cdot d\hat{\boldsymbol{\alpha}}_1]\hat{\boldsymbol{\gamma}}_1 + [\int_{\hat{\mathbf{e}}_3}\hat{\boldsymbol{u}}\cdot d\hat{\boldsymbol{\alpha}}_3]\hat{\boldsymbol{\gamma}}_3 + [\int_{\hat{\mathbf{e}}_6}\hat{\boldsymbol{u}}\cdot d\hat{\boldsymbol{\alpha}}_6]\hat{\boldsymbol{\gamma}}_6 + [\int_{\hat{\mathbf{e}}_8}\hat{\boldsymbol{u}}\cdot d\hat{\boldsymbol{\alpha}}_8]\hat{\boldsymbol{\gamma}}_8\\ &=: \varphi_1(\hat{\boldsymbol{u}})\hat{\boldsymbol{\gamma}}_1 + \varphi_3(\hat{\boldsymbol{u}})\hat{\boldsymbol{\gamma}}_3 + \varphi_6(\hat{\boldsymbol{u}})\hat{\boldsymbol{\gamma}}_6 + \varphi_8(\hat{\boldsymbol{u}})\hat{\boldsymbol{\gamma}}_8.\end{aligned}$$

$$\begin{aligned}(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_1(x,y,z) &= \varphi_1(\hat{\boldsymbol{u}})(1-z-y) + \varphi_3(\hat{\boldsymbol{u}})y + \varphi_6(\hat{\boldsymbol{u}})(-z+\frac{yz}{1-z})\\ &\quad + \varphi_8(\hat{\boldsymbol{u}})(-\frac{yz}{1-z})\\ &= \varphi_1(\hat{\boldsymbol{u}}) - (\varphi_1+\varphi_6)(\hat{\boldsymbol{u}})\,z + (\varphi_3-\varphi_1)(\hat{\boldsymbol{u}})\,y\\ &\quad + (\varphi_6-\varphi_8)(\hat{\boldsymbol{u}})\,\frac{yz}{1-z}.\end{aligned}$$

Now we explore the new coefficients separately. As the tangential component of $\hat{\boldsymbol{u}}$ along $\hat{\mathbf{e}}_5$ equals zero, and this is an argument we are using repeatedly in the forthcoming computations, we may write, by Stokes' Theorem,

$$\begin{aligned}(\varphi_1+\varphi_6)(\hat{\boldsymbol{u}}) &= \int_{\hat{\mathbf{e}}_1} \hat{\boldsymbol{u}}\cdot d\hat{\boldsymbol{\alpha}}_1 + \int_{\hat{\mathbf{e}}_6} \hat{\boldsymbol{u}}\cdot d\hat{\boldsymbol{\alpha}}_6 - \int_{\hat{\mathbf{e}}_5} \hat{\boldsymbol{u}}\cdot d\hat{\boldsymbol{\alpha}}_5 \\ &= \iint_{\hat{f}_1} \nabla\times\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}\, d\hat{S} \\ &= -\iint_{\hat{f}_1} \tfrac{\partial \hat{u}_1}{\partial \hat{x}_3}\, d\hat{S}.\end{aligned}$$

Next,

$$\begin{aligned}(\varphi_3-\varphi_1)(\hat{\boldsymbol{u}}) &= (\varphi_3-\varphi_2-\varphi_1+\varphi_4)(\hat{\boldsymbol{u}}) \\ &= -\int_{\partial\hat{f}_5} \hat{\boldsymbol{u}}\cdot d\hat{\boldsymbol{\alpha}} \\ &= -\iint_{\hat{f}_5} \nabla\times\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}_5\, d\hat{S} \\ &= \iint_{\hat{f}_5} \tfrac{\partial \hat{u}_1}{\partial \hat{x}_2}\, d\hat{S}.\end{aligned}$$

And

$$\begin{aligned}(\varphi_6-\varphi_8)(\hat{\boldsymbol{u}}) &= \int_{\hat{\mathbf{e}}_6} \hat{\boldsymbol{u}}\cdot d\hat{\boldsymbol{\alpha}}_6 - \int_{\hat{\mathbf{e}}_8} \hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{\tau}}_6\, d\hat{s} - \int_{\hat{\mathbf{e}}_2} \hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{\tau}}_6\, d\hat{s} \\ &= -\int_{\partial\hat{f}_3} \hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{\tau}}\, d\hat{s} = -\iint_{\hat{f}_3} \nabla\times\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}_3\, d\hat{S} = \tfrac{1}{\sqrt{2}}\iint_{\hat{f}_3} \tfrac{\partial \hat{u}_1}{\partial \hat{x}_2}\, d\hat{S}.\end{aligned}$$

So in this case in which $\hat{\boldsymbol{u}}$ has null first and second components, it holds

$$\begin{aligned}(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_1 = \int_{\hat{\mathbf{e}}_1} \hat{u}_1\, d\hat{\alpha}_1 + z\iint_{\hat{f}_1} \tfrac{\partial \hat{u}_1}{\partial \hat{x}_3}\, d\hat{S} + y\iint_{\hat{f}_5} \tfrac{\partial \hat{u}_1}{\partial \hat{x}_2}\, d\hat{S} \\ +\frac{yz}{1-z}\, 2^{-1/2}\iint_{\hat{f}_3} \tfrac{\partial \hat{u}_1}{\partial \hat{x}_2}\, d\hat{S}.\end{aligned} \tag{6.42}$$

By exactly the last computation,

$$(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_2 = (\varphi_6-\varphi_8)(\hat{\boldsymbol{u}})\frac{xz}{1-z} = \iint_{\hat{f}_3} \tfrac{\partial \hat{u}_1}{\partial \hat{x}_2}\, d\hat{S}\,\frac{xz}{1-z}. \tag{6.43}$$

Next,

$$\begin{aligned}(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_3 &= \varphi_1(\hat{\boldsymbol{u}})\left(x-\frac{xy}{1-z}\right) + \varphi_3(\hat{\boldsymbol{u}})\frac{xy}{1-z} \\ &\quad + \varphi_6(\hat{\boldsymbol{u}})\left(x-\frac{xy}{1-z}+\frac{xyz}{(1-z)^2}\right) + \varphi_8(\hat{\boldsymbol{u}})\left(\frac{xy}{1-z}-\frac{xyz}{(1-z)^2}\right) \\ &= (\varphi_1+\varphi_6)(\hat{\boldsymbol{u}})\, x + (\varphi_3-\varphi_1+\varphi_8-\varphi_6)(\hat{\boldsymbol{u}})\,\frac{xy}{1-z} \\ &\quad +(\varphi_6-\varphi_8)(\hat{\boldsymbol{u}})\,\frac{xyz}{(1-z)^2}.\end{aligned}$$

As $\hat{\boldsymbol{u}}$ has zero tangential component along $\hat{\boldsymbol{e}}_5$ and $\hat{\boldsymbol{e}}_7$,

$$\begin{aligned}(\varphi_3-\varphi_1+\varphi_8-\varphi_6)(\hat{\boldsymbol{u}}) &= (\varphi_3-\varphi_7+\varphi_8)(\hat{\boldsymbol{u}})+(\varphi_5-\varphi_6-\varphi_1)(\hat{\boldsymbol{u}}) \\ &= -\int_{\partial\hat{f}_4}\hat{\boldsymbol{u}}\cdot d\hat{\boldsymbol{\alpha}}-\int_{\partial\hat{f}_1}\hat{\boldsymbol{u}}\cdot d\hat{\boldsymbol{\alpha}} \\ &= -\iint_{\hat{f}_1}\nabla\times\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}\,d\hat{S}-\iint_{\hat{f}_4}\nabla\times\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}\,d\hat{S} \\ &= \iint_{\hat{f}_1}(\nabla\times\hat{\boldsymbol{u}})_2\,d\hat{S} \\ &\qquad -2^{-1/2}\iint_{\hat{f}_4}[(\nabla\times\hat{\boldsymbol{u}})_2+(\nabla\times\hat{\boldsymbol{u}})_3]\,d\hat{S}. \\ &= \iint_{\hat{f}_1}\tfrac{\partial\hat{u}_1}{\partial\hat{x}_3}\,d\hat{S}-2^{-1/2}\iint_{\hat{f}_4}[\tfrac{\partial\hat{u}_1}{\partial\hat{x}_3}+\tfrac{\partial\hat{u}_1}{\partial\hat{x}_2}]\,d\hat{S}.\end{aligned}$$

We write down this component:

$$\begin{aligned}(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_3 &= -x\iint_{\hat{f}_1}\tfrac{\partial\hat{u}_1}{\partial\hat{x}_3}\,d\hat{S}+\frac{xyz}{(1-z)^2}\,2^{-1/2}\iint_{\hat{f}_3}\tfrac{\partial\hat{u}_1}{\partial\hat{x}_2}\,d\hat{S} \\ &\quad +\frac{xy}{1-z}\left\{\iint_{\hat{f}_1}\tfrac{\partial\hat{u}_1}{\partial\hat{x}_3}\,d\hat{S}-2^{-1/2}\iint_{\hat{f}_4}[\tfrac{\partial\hat{u}_1}{\partial\hat{x}_3}+\tfrac{\partial\hat{u}_1}{\partial\hat{x}_2}]\,d\hat{S}\right\} \qquad (6.44)\end{aligned}$$

Now, as expected, we switch to $\hat{\boldsymbol{u}}=(0,\hat{u}_2,0)'$. In this case we have

$$\begin{aligned}\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}} &= \varphi_2(\hat{\boldsymbol{u}})\,\hat{\boldsymbol{\gamma}}_2+\varphi_4(\hat{\boldsymbol{u}})\,\hat{\boldsymbol{\gamma}}_4+\varphi_7(\hat{\boldsymbol{u}})\,\hat{\boldsymbol{\gamma}}_7+\varphi_8(\hat{\boldsymbol{u}})\,\hat{\boldsymbol{\gamma}}_8. \\ (\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_1 &= (\varphi_7-\varphi_8)(\hat{\boldsymbol{u}})\,\frac{yz}{1-z} \\ &= (\varphi_7-\varphi_8-\varphi_3)(\hat{\boldsymbol{u}})\,\frac{yz}{1-z} \\ &= \int_{\partial\hat{f}_4}\hat{\boldsymbol{u}}\cdot d\hat{\boldsymbol{\alpha}}\,\frac{yz}{1-z} \\ &= \iint_{\hat{f}_4}\nabla\times\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}_4\,d\hat{S}\,\frac{yz}{1-z} \;=\; 2^{-1/2}\iint_{\hat{f}_4}\tfrac{\partial\hat{u}_2}{\partial\hat{x}_1}\,d\hat{S}\,\frac{yz}{1-z}. \qquad (6.45)\end{aligned}$$

For the next component,

$$
\begin{aligned}
(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_2 &= \varphi_4(\hat{\boldsymbol{u}}) + (\varphi_2 - \varphi_4)(\hat{\boldsymbol{u}})\, x - (\varphi_4 + \varphi_7)(\hat{\boldsymbol{u}})\, z \\
&\quad + (\varphi_7 - \varphi_8)(\hat{\boldsymbol{u}})\, \frac{xz}{1-z}. \\
(\varphi_2 - \varphi_4)(\hat{\boldsymbol{u}}) &= (\varphi_2 - \varphi_3 - \varphi_4 + \varphi_1)(\hat{\boldsymbol{u}}) \\
&= -\int_{\partial \hat{f}_5} \hat{\boldsymbol{u}} \cdot d\hat{\boldsymbol{\alpha}} \\
&= -\iint_{\hat{f}_5} \nabla \times \hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{n}}_5 \, d\hat{S} = \iint_{\hat{f}_5} \tfrac{\partial \hat{u}_2}{\partial \hat{x}_1}\, d\hat{S}. \\
(\varphi_4 + \varphi_7)(\hat{\boldsymbol{u}}) &= (\varphi_4 + \varphi_7 - \varphi_5)(\hat{\boldsymbol{u}}) \\
&= -\int_{\partial \hat{f}_2} \hat{\boldsymbol{u}} \cdot d\hat{\boldsymbol{\alpha}} \\
&= -\iint_{\hat{f}_2} \nabla \times \hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{n}} \, d\hat{S} \;=\; -\iint_{\hat{f}_2} \tfrac{\partial \hat{u}_2}{\partial \hat{x}_3}\, d\hat{S},
\end{aligned}
$$

and we write down this second component of the interpolate

$$
\begin{aligned}
(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_2 = \int_{\hat{\boldsymbol{e}}_4} \hat{u}_2 \, d\hat{\alpha}_4 + x \iint_{\hat{f}_5} \tfrac{\partial \hat{u}_2}{\partial \hat{x}_1}\, d\hat{S}. + z \iint_{\hat{f}_2} \tfrac{\partial \hat{u}_2}{\partial \hat{x}_3}\, d\hat{S} \\
+ \frac{xz}{1-z}\, 2^{-1/2} \iint_{\hat{f}_4} \tfrac{\partial \hat{u}_2}{\partial \hat{x}_1}\, d\hat{S}.
\end{aligned} \tag{6.46}
$$

And for the third one,

$$
\begin{aligned}
(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_3 &= (\varphi_4 + \varphi_7)(\hat{\boldsymbol{u}})\, y + (\varphi_2 - \varphi_4 - \varphi_7 + \varphi_8)(\hat{\boldsymbol{u}})\, \frac{xy}{1-z} \\
&\quad + (\varphi_7 - \varphi_8)(\hat{\boldsymbol{u}})\, \frac{xyz}{(1-z)^2}. \\
&= (\varphi_2 - \varphi_4)(\hat{\boldsymbol{u}})\, \frac{xy}{1-z} + (\varphi_4 + \varphi_7)(\hat{\boldsymbol{u}})\, y \\
&\quad - (\varphi_7 - \varphi_8)(\hat{\boldsymbol{u}})\, \frac{xy}{(1-z)^2}.
\end{aligned}
$$

But expressions for $(\varphi_2 - \varphi_4)(\cdot)$, $(\varphi_4 + \varphi_7)(\cdot)$ and $(\varphi_7 - \varphi_8)(\cdot)$ were already stated above, so we have

$$
\begin{aligned}
(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_3 = \frac{xy}{1-z} \iint_{\hat{f}_5} \tfrac{\partial \hat{u}_2}{\partial \hat{x}_1}\, d\hat{S} - y \iint_{\hat{f}_2} \tfrac{\partial \hat{u}_2}{\partial \hat{x}_3}\, d\hat{S} \\
-2^{-1/2} \iint_{\hat{f}_4} \tfrac{\partial \hat{u}_2}{\partial \hat{x}_1}\, d\hat{S}\, \frac{xy}{(1-z)^2}.
\end{aligned} \tag{6.47}
$$

Finally for $\hat{\boldsymbol{u}} = (0, 0, \hat{u}_3)'$ it is $\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}} = \varphi_5(\hat{\boldsymbol{u}})\hat{\boldsymbol{\gamma}}_5 + \varphi_6(\hat{\boldsymbol{u}})\hat{\boldsymbol{\gamma}}_6 + \varphi_7(\hat{\boldsymbol{u}})\hat{\boldsymbol{\gamma}}_7 + \varphi_8(\hat{\boldsymbol{u}})\hat{\boldsymbol{\gamma}}_8$ and $\nabla \times \hat{\boldsymbol{u}} = (\partial_2 \hat{u}_3, -\partial_1 \hat{u}_3, 0)'$.

First component of the interpolate:

$$(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_1 = (\varphi_5 - \varphi_6)(\hat{\boldsymbol{u}})z + (-\varphi_5 + \varphi_6 + \varphi_7 - \varphi_8)(\hat{\boldsymbol{u}})\,\frac{yz}{1-z}.$$

On one hand,

$$\begin{aligned}(\varphi_5 - \varphi_6)(\hat{\boldsymbol{u}}) &= (\varphi_5 - \varphi_6 - \varphi_1)(\hat{\boldsymbol{u}}) \\ &= -\iint_{\hat{f}_1} \nabla \times \hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{n}}\, d\hat{S}\end{aligned}$$

On the other hand and analogously

$$(\varphi_7 - \varphi_8)(\hat{\boldsymbol{u}}) = \iint_{\hat{f}_4} \nabla \times \hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{n}}\, d\hat{S}$$

so it follows

$$\begin{aligned}(-\varphi_5 + \varphi_6 + \varphi_7 - \varphi_8)(\hat{\boldsymbol{u}}) &= \iint_{\hat{f}_1} \nabla \times \hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{n}}\, d\hat{S} + \iint_{\hat{f}_4} \nabla \times \hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{n}}\, d\hat{S} \\ &= \int_{\mathbb{D}_{\hat{f}_1}} \tfrac{\partial \hat{u}_3}{\partial \hat{x}_1}(\Phi_{\hat{f}_1}(t_1,t_2))\, dt_1 dt_2 - \int_{\mathbb{D}_{\hat{f}_4}} \tfrac{\partial \hat{u}_3}{\partial \hat{x}_1}(\Phi_{\hat{f}_4}(t_1,t_2))\, dt_1 dt_2 \\ &= \int_0^1 \int_0^{1-t_2} \left[\tfrac{\partial \hat{u}_3}{\partial \hat{x}_1}(t_1,0,t_2) - \tfrac{\partial \hat{u}_3}{\partial \hat{x}_1}(t_1,1-t_2,t_2)\right] dt_1 dt_2 \\ &= -\int_0^1 \int_0^{1-t_2} \int_0^{1-t_2} \tfrac{\partial^2 \hat{u}_3}{\partial \hat{x}_2 \partial \hat{x}_1}(t_1,s,t_2)\, ds dt_1 dt_2 \\ &= -\int_{\hat{E}} \tfrac{\partial^2 \hat{u}_3}{\partial \hat{x}_2 \partial \hat{x}_1}\, d\hat{\boldsymbol{x}}.\end{aligned}$$

For now we obtained

$$(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_1 = -z \iint_{\hat{f}_1} \tfrac{\partial \hat{u}_3}{\partial \hat{x}_1}\, d\hat{S} - \frac{yz}{1-z} \int_{\hat{E}} \tfrac{\partial^2 \hat{u}_3}{\partial \hat{x}_2 \partial \hat{x}_1}\, d\hat{\boldsymbol{x}}. \tag{6.48}$$

Regarding the second component, it is the symmetrical case, so we write

$$\begin{aligned}(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_2 &= (\varphi_5 - \varphi_7)(\hat{\boldsymbol{u}})z + (-\varphi_5 + \varphi_6 + \varphi_7 - \varphi_8)(\hat{\boldsymbol{u}})\frac{xz}{1-z} \\ &= z \iint_{\hat{f}_2} \nabla \times \hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{n}}\, d\hat{S} - \frac{xz}{1-z} \int_{\hat{E}} \tfrac{\partial^2 \hat{u}_3}{\partial \hat{x}_2 \partial \hat{x}_1}\, d\hat{\boldsymbol{x}} \\ &= -z \iint_{\hat{f}_2} \tfrac{\partial \hat{u}_3}{\partial \hat{x}_2}\, d\hat{S} - \frac{xz}{1-z} \int_{\hat{E}} \tfrac{\partial^2 \hat{u}_3}{\partial \hat{x}_2 \partial \hat{x}_1}\, d\hat{\boldsymbol{x}}.\end{aligned} \tag{6.49}$$

For the third component let us denote $\xi(x,y,z) = \frac{xyz}{(1-z)^2} - \frac{xz}{1-z}$. Then

$$\begin{aligned}(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_3 &= \varphi_5(\hat{\boldsymbol{u}}) + x\,(\varphi_6 - \varphi_5)(\hat{\boldsymbol{u}}) + y\,(\varphi_7 - \varphi_5)(\hat{\boldsymbol{u}}) \\ &\quad + \xi(x,y,z)\,(\varphi_6 - \varphi_5 + \varphi_7 - \varphi_8)(\hat{\boldsymbol{u}}) \\ &= \int_{\hat{\boldsymbol{e}}_5} \hat{\boldsymbol{u}} \cdot d\hat{\boldsymbol{\alpha}}_5 + y \iint_{\hat{f}_2} \tfrac{\partial \hat{u}_3}{\partial x_2}\, d\hat{S} + x \iint_{\hat{f}_1} \tfrac{\partial \hat{u}_3}{\partial x_1}\, d\hat{S} \\ &\quad - \xi(x,y,z) \int_{\hat{E}} \tfrac{\partial^2 \hat{u}_3}{\partial \hat{x}_2 \partial \hat{x}_1}\, d\hat{\boldsymbol{x}}.\end{aligned} \tag{6.50}$$

All together, for a $\hat{\boldsymbol{u}} = (\hat{u}_1, \hat{u}_2, \hat{u}_3)'$, if we combine what was obtained in (6.42)–(6.50) then it holds

$$\begin{aligned}
(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_1 = & \int_{\hat{\mathbf{e}}_1} \hat{u}_1\, d\hat{\alpha} + z \iint_{\hat{f}_1} (\nabla \times \hat{\boldsymbol{u}})_2\, d\hat{S} + y \iint_{\hat{f}_5} \tfrac{\partial \hat{u}_1}{\partial \hat{x}_2}\, d\hat{S} \\
& + \frac{yz}{1-z} \iint_{\hat{f}_3} \tfrac{\partial \hat{u}_1}{\partial \hat{x}_2}\, d\hat{S} + \frac{yz}{1-z} \iint_{\hat{f}_4} \tfrac{\partial \hat{u}_2}{\partial \hat{x}_1}\, d\hat{S} \\
& - \frac{yz}{1-z} \int_{\hat{E}} \tfrac{\partial^2 \hat{u}_3}{\partial \hat{x}_2 \partial \hat{x}_1}\, d\hat{\boldsymbol{x}}. \\
(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_2 = & \int_{\hat{\mathbf{e}}_4} \hat{u}_2\, d\hat{\alpha} - z \iint_{\hat{f}_2} (\nabla \times \hat{\boldsymbol{u}})_1\, d\hat{S} + x \iint_{\hat{f}_5} \tfrac{\partial \hat{u}_2}{\partial \hat{x}_1}\, d\hat{S} \\
& + \frac{xz}{1-z} \iint_{\hat{f}_3} \tfrac{\partial \hat{u}_1}{\partial \hat{x}_2}\, d\hat{S} + \frac{xz}{1-z} \iint_{\hat{f}_4} \tfrac{\partial \hat{u}_2}{\partial \hat{x}_1}\, d\hat{S} \\
& + \frac{xz}{1-z} \int_{\hat{E}} \tfrac{\partial^2 \hat{u}_3}{\partial \hat{x}_2 \partial \hat{x}_1}\, d\hat{\boldsymbol{x}}. \\
(\boldsymbol{w}_{\hat{E}}\hat{\boldsymbol{u}})_3 = & \int_{\hat{\mathbf{e}}_5} \hat{u}_3\, d\hat{\alpha} - x \iint_{\hat{f}_1} (\nabla \times \hat{\boldsymbol{u}})_2\, d\hat{S} + y \iint_{\hat{f}_2} (\nabla \times \hat{\boldsymbol{u}})_1\, d\hat{S} \\
& + \frac{xy}{1-z} \left\{ \iint_{\hat{f}_5} \tfrac{\partial \hat{u}_2}{\partial \hat{x}_1}\, d\hat{S} + \iint_{\hat{f}_1} \tfrac{\partial \hat{u}_1}{\partial \hat{x}_3}\, d\hat{S} - 2^{-1/2} \iint_{\hat{f}_4} \tfrac{\partial \hat{u}_1}{\partial \hat{x}_3}\, d\hat{S} \right. \\
& \left. - 2^{-1/2} \iint_{\hat{f}_4} \tfrac{\partial \hat{u}_1}{\partial \hat{x}_2}\, d\hat{S} \right\} - \frac{xy}{(1-z)^2} 2^{-1/2} \iint_{\hat{f}_4} \tfrac{\partial \hat{u}_2}{\partial \hat{x}_1}\, d\hat{S} \\
& + \frac{xyz}{(1-z)^2} 2^{-1/2} \iint_{\hat{f}_3} \tfrac{\partial \hat{u}_1}{\partial \hat{x}_2}\, d\hat{S} + \xi(x,y,z) \int_{\hat{E}} \tfrac{\partial^2 u_3}{\partial \hat{x}_1 \partial \hat{x}_2}\, d\hat{\boldsymbol{x}}.
\end{aligned} \tag{6.51}$$

From here we apply Lemma 6.4.1 and the result follows. □

We continue with the local interpolation error estimate.

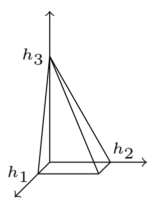


Figure 6.2 – Rescaled Pyramid

6.4.3 Theorem. *Let E be any pyramid which is a non degenerate affine image of the reference pyramid $\hat{E}$. We fix a positively oriented local system of coordinates $(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3)$ with origin in a vertex $\boldsymbol{x}_E$ of the parallelogram basis, for which $(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2)$ correspond to the two basis edges incident to $\boldsymbol{x}_E$ and $\boldsymbol{\xi}_3$ is parallel to the edge joining $\boldsymbol{x}_E$ with the top of the* *pyramid. Let h_1, h_2, h_3 be the corresponding edge lengths. With $\partial^{\boldsymbol{\alpha}}$ we denote $\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial_{\boldsymbol{\xi}_1}^{\alpha_1}\partial_{\boldsymbol{\xi}_2}^{\alpha_2}\partial_{\boldsymbol{\xi}_3}^{\alpha_3}}$. Suppose that $h_3 \geqslant \min\{h_1, h_2\}$ and let $p > 2$. For all $\boldsymbol{u} \in W^{2,p}(E)^3$*

$$\begin{aligned}\|\boldsymbol{u} - \boldsymbol{w}_E\boldsymbol{u}\|_{L^p(E)} \lesssim \sum_{|\boldsymbol{\alpha}|=1} &\boldsymbol{h}^{\boldsymbol{\alpha}}\|\partial^{\boldsymbol{\alpha}}\boldsymbol{u}\|_{L^p(E)}+ \\ &+ h_E\big\{\|\boldsymbol{curl}\,\boldsymbol{u}\|_{L^p(E)} + \sum_{|\boldsymbol{\alpha}|=1} \boldsymbol{h}^{\boldsymbol{\alpha}}\|\partial^{\boldsymbol{\alpha}}\boldsymbol{curl}\,\boldsymbol{u}\|_{L^p(E)}\big\} \\ &+ h_E^2|\boldsymbol{u}|_{2,p,E} \\ &+ \max\{h_1, h_2\}\big\{\|\partial_{\boldsymbol{\xi}_1}u_2\|_{L^p(E)} + \|\partial_{\boldsymbol{\xi}_2}u_1\|_{L^p(E)}\big\}.\end{aligned}$$

Proof. Consider the matrix $M_{\tilde{E}}$ with coefficients $h_i\delta_{i,j}$, $1 \leqslant i,j \leqslant 3$ and take $\tilde{E}$ as the rescaled reference pyramid, that is, $\tilde{E} = M_{\tilde{E}}\hat{E}$. In Figure 6.2 we have illustrated the scaling. Let us start with a stability estimate in $\tilde{E}$. Given a field $\tilde{\boldsymbol{u}}$ in $\tilde{E}$, pulling $\tilde{\boldsymbol{u}}$ back to $\hat{E}$, using (6.39) and pushing forward to $\tilde{E}$ we get

$$\begin{aligned}\|(\boldsymbol{w}_{\tilde{E}}\tilde{\boldsymbol{u}})_1\|_{L^\infty(\tilde{E})} &\lesssim |\tilde{E}|^{-1/p}\big\{\|\tilde{u}_1\|_{L^p(\tilde{E})} + \sum_{i=1}^{3} h_i\|\partial_{\tilde{x}_i}\tilde{u}_1\|_{L^p(\tilde{E})}\big\} \\ &+ |\tilde{E}|^{-1}h_2\big\{\|(\nabla\times\tilde{\boldsymbol{u}})_3\|_{L^1(\tilde{E})} + \sum_{i=1}^{3} h_i(\|\partial_{\tilde{x}_i}(\nabla\times\tilde{\boldsymbol{u}})_3\|_{L^1(\tilde{E})} + \|\tfrac{\partial^2\tilde{u}_1}{\partial\tilde{x}_i\partial\tilde{x}_2}\|_{L^1(\tilde{E})})\big\} \\ &+ |\tilde{E}|^{-1}h_3\big\{\|(\nabla\times\tilde{\boldsymbol{u}})_2\|_{L^1(\tilde{E})} + \sum_{i=1}^{3} h_i(\|\partial_{\tilde{x}_i}(\nabla\times\tilde{\boldsymbol{u}})_2\|_{L^1(\tilde{E})}\big\} \\ &+ |\tilde{E}|^{-1}h_2h_3\|\tfrac{\partial^2\tilde{u}_3}{\partial\tilde{x}_1\partial\tilde{x}_2}\|_{L^1(\tilde{E})}.\end{aligned}$$

Estimate for component number two yields the analogue and now we write something similar to the third component. Note that in some cases we group terms using

$$|\tilde{E}|^{-\frac{1}{q}}\|g\|_{L^q(\tilde{E})} \leqslant |\tilde{E}|^{-\frac{1}{p}}\|g\|_{L^p(\tilde{E})}, \tag{6.52}$$

whenever $q < p$, for scalar functions in L^p. From (6.41)

$$\begin{aligned}
\|(\boldsymbol{w}_{\tilde{E}}\tilde{\boldsymbol{u}})_3\|_{L^\infty(\tilde{E})} \lesssim |\tilde{E}|^{-1/p}\big\{\|\tilde{u}_3\|_{L^p(\tilde{E})} + \sum_{i=1}^{3} h_i \|\tfrac{\partial \tilde{u}_3}{\partial \tilde{x}_i}\|_{L^p(\tilde{E})}\big\} \\
+ |\tilde{E}|^{-1} h_1 \big\{\|(\nabla \times \tilde{\boldsymbol{u}})_2\|_{L^1(\tilde{E})} + \\
+ \sum_{i=1}^{3} h_i (\|\partial_{\tilde{x}_i}(\nabla \times \tilde{\boldsymbol{u}})_2\|_{L^1(\tilde{E})} + \|\tfrac{\partial^2 \tilde{u}_3}{\partial \tilde{x}_i \partial \tilde{x}_1}\|_{L^1(\tilde{E})})\big\} \\
+ |\tilde{E}|^{-1} h_2 \big\{\|(\nabla \times \tilde{\boldsymbol{u}})_1\|_{L^1(\tilde{E})} + \sum_{i=1}^{3} h_i (\|\partial_{\tilde{x}_i}(\nabla \times \tilde{\boldsymbol{u}})_1\|_{L^1(\tilde{E})}\big\} \\
+ |\tilde{E}|^{-1} \tfrac{h_1 h_2}{h_3} \big\{\|\partial_{\tilde{x}_1}\tilde{u}_2\|_{L^1(\tilde{E})} + \|\partial_{\tilde{x}_2}\tilde{u}_1\|_{L^1(\tilde{E})} \\
+ \sum_{i=1}^{3} h_i (\|\tfrac{\partial^2 \tilde{u}_1}{\partial \tilde{x}_i \partial \tilde{x}_2}\|_{L^1(\tilde{E})} + \|\tfrac{\partial^2 \tilde{u}_2}{\partial \tilde{x}_i \partial \tilde{x}_1}\|_{L^1(\tilde{E})})\big\} \\
+ |\tilde{E}|^{-1} h_1 h_2 \|\tfrac{\partial^2 \tilde{u}_3}{\partial \tilde{x}_1 \partial \tilde{x}_2}\|_{L^1(\tilde{E})}.
\end{aligned}$$

Proceeding as in the proof of Theorem 6.1.6 we obtain the following vectorial stability estimate in $\tilde{E}$:

$$\begin{aligned}
\|\boldsymbol{w}_{\tilde{E}}\tilde{\boldsymbol{u}}\|_{L^p(\tilde{E})} \leqslant \|\tilde{\boldsymbol{u}}\|_{L^p(\tilde{E})} + \sum_{i=1}^{3} h_i \|\partial_{\tilde{x}_i}\tilde{\boldsymbol{u}}\|_{L^p(\tilde{E})} \\
+ \max\{h_i\} \left(\|\nabla \times \tilde{\boldsymbol{u}}\|_{L^p(\tilde{E})} + \sum_{i=1}^{3} h_i \|\partial_{\tilde{x}_i} \nabla \times \tilde{\boldsymbol{u}}\|_{L^p(\tilde{E})} \right) \\
+ \max\{h_i\}^2 |\tilde{\boldsymbol{u}}|_{2,p,\tilde{E}} \\
+ \max\{h_1, h_2\} \big(\|\partial_{\tilde{x}_1}\tilde{u}_2\|_{L^p(\tilde{E})} + \|\partial_{\tilde{x}_2}\tilde{u}_1\|_{L^p(\tilde{E})}\big)
\end{aligned}$$

And now we proceed as in the proof of Theorem 6.2.1. First we transform from a physical pyramidal element E to $\tilde{E}$. In Section 5 of [25] the approximation property of the finite element is stated and then we add the estimate (4.14) for the rescaled pyramid $\tilde{E}$ in the case with multi–indices of order two, to use in the corresponding terms of the averaged Taylor polynomial approximation and the result follows. □

6.4.2 Anisotropic Interpolation Estimates for $H(\mathbf{div})$ conforming Elements on Pyramids

Here we will work on the div–conforming analogue of Theorem 6.4.2.

6.4.4 Theorem. *Let $\hat{E}$ be the reference pyramid and let $p > 1$. Let $\boldsymbol{r}_{\hat{E}}(\cdot)$ denote the interpolation operator determined by the degrees of freedom in Definition 6.3.4. There is $C > 0$ such that, for all $\hat{\boldsymbol{u}} \in W^{1,p}(\hat{E})$, there hold*

$$\|(\boldsymbol{r}_{\hat{E}}\hat{\boldsymbol{u}})_1\|_{L^\infty(\hat{E})} \lesssim \|\hat{u}_1\|_{W^{1,1}(\hat{E})} + \|div\, \hat{\boldsymbol{u}}\|_{L^1(\hat{E})} + \|\hat{u}_3\|_{W^{1,1}(\hat{E})}$$

$$\|(\boldsymbol{r}_{\hat{E}}\hat{\boldsymbol{u}})_2\|_{L^\infty(\hat{E})} \lesssim \|\hat{u}_2\|_{W^{1,1}(\hat{E})} + \|div\, \hat{\boldsymbol{u}}\|_{L^1(\hat{E})} + \|\hat{u}_3\|_{W^{1,1}(\hat{E})}$$

$$\|(\boldsymbol{r}_{\hat{E}}\hat{\boldsymbol{u}})_3\|_{L^\infty(\hat{E})} \lesssim \|\hat{u}_3\|_{W^{1,1}(\hat{E})} + \|div\, \hat{\boldsymbol{u}}\|_{L^1(\hat{E})}.$$

Proof. We will use the notation of Table 6.4 for the shape functions and Tables 6.1 and 6.2 for the boundary of the reference pyramid. This proof is based on explicit computation as well. The variables in the local coordinate system of $\hat{E}$ for the shape functions $\hat{\boldsymbol{\zeta}}_i$ are x, y and z instead of $\hat{x}_1$, $\hat{x}_2$ and $\hat{x}_3$.

Consider the case $\hat{\boldsymbol{u}} = (\hat{u}_1, 0, 0)'$ to start with and compute it's interpolate.

$$\begin{aligned}\boldsymbol{r}_{\hat{E}}\hat{\boldsymbol{u}} &= \left\{\textstyle\iint_{\hat{f}_2} \hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}_2\, d\hat{S}\right\} \hat{\boldsymbol{\zeta}}_2 + \left\{\textstyle\iint_{\hat{f}_3} \hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}_3\, d\hat{S}\right\} \hat{\boldsymbol{\zeta}}_3 \\ &=: \rho_2(\hat{\boldsymbol{u}})\, \hat{\boldsymbol{\zeta}}_2 + \rho_3(\hat{\boldsymbol{u}})\, \hat{\boldsymbol{\zeta}}_3.\end{aligned}$$

Then for the first two components of the interpolate it holds

$$\begin{aligned}(\boldsymbol{r}_{\hat{E}}\hat{\boldsymbol{u}})_1(x,y,z) &= -2\rho_2(\hat{\boldsymbol{u}}) + \{\rho_2(\hat{\boldsymbol{u}}) + \rho_3(\hat{\boldsymbol{u}})\}\, \tfrac{2x-xz}{1-z} \\ &= -2\iint_{\hat{f}_2} \hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}_2\, d\hat{S} \\ &\quad + \left\{\iint_{\hat{f}_2} \hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}_2\, d\hat{S} + \iint_{\hat{f}_3} \hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}_3\, d\hat{S}\right\} \tfrac{2x-xz}{1-z} \\ &= -2\iint_{\hat{f}_2} \hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}_2\, d\hat{S} + \iint_{\partial\hat{E}} \hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}\, d\hat{S}\, \tfrac{2x-xz}{1-z} \\ &= -2\iint_{\hat{f}_2} \hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}_2\, d\hat{S} + \int_{\hat{E}} \operatorname{div} \hat{\boldsymbol{u}}\, d\hat{\boldsymbol{x}}\, \tfrac{2x-xz}{1-z}\end{aligned}$$

and

$$\begin{aligned}(\boldsymbol{r}_{\hat{E}}\hat{\boldsymbol{u}})_2(\hat{x}_1,\hat{x}_2,\hat{x}_3) &= -(\rho_2(\hat{\boldsymbol{u}}) + \rho_3(\hat{\boldsymbol{u}}))\, \tfrac{yz}{1-z} \\ &= -\int_{\hat{E}} \operatorname{div} \hat{\boldsymbol{u}}\, d\hat{\boldsymbol{x}}\, \tfrac{yz}{1-z}.\end{aligned}$$

Switch to $\hat{\boldsymbol{u}}$ of the form $(0, \hat{u}_2, 0)'$.

$$\begin{aligned}\boldsymbol{r}_{\hat{E}}\hat{\boldsymbol{u}} &= \left(\textstyle\iint_{\hat{f}_1} \hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}_1\, d\hat{S}\right) \hat{\boldsymbol{\zeta}}_1 + \left(\textstyle\iint_{\hat{f}_4} \hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}_4\, d\hat{S}\right) \hat{\boldsymbol{\zeta}}_4 \\ &= \rho_1(\hat{\boldsymbol{u}})\, \hat{\boldsymbol{\zeta}}_1 + \rho_4(\hat{\boldsymbol{u}})\, \hat{\boldsymbol{\zeta}}_4.\end{aligned}$$

Then summing up yields, for now,

$$
\begin{aligned}
(\boldsymbol{r}_{\hat{E}}\hat{\boldsymbol{u}})_1(x,y,z) &= -(\rho_1(\hat{\boldsymbol{u}})+\rho_4(\hat{\boldsymbol{u}}))\,\tfrac{xz}{1-z}\\
&= -\int_{\hat{E}} \operatorname{div}\hat{\boldsymbol{u}}\,d\hat{\boldsymbol{x}}\,\tfrac{xz}{1-z}.\\
(\boldsymbol{r}_{\hat{E}}\hat{\boldsymbol{u}})_2(x,y,z) &= -2\rho_1(\hat{\boldsymbol{u}})+(\rho_1(\hat{\boldsymbol{u}})+\rho_4(\hat{\boldsymbol{u}}))\,\tfrac{2y-yz}{1-z}\\
&= -2\iint_{\hat{f}_1}\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}_1\,d\hat{S}+\iint_{\partial\hat{E}}\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}\,d\hat{S}\,\tfrac{2y-yz}{1-z}\\
&= -2\iint_{\hat{f}_1}\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}_1\,d\hat{S}+\int_{\hat{E}}\operatorname{div}\hat{\boldsymbol{u}}\,d\hat{\boldsymbol{x}}\,\tfrac{2y-yz}{1-z}.
\end{aligned}
$$

Now continue with $\hat{\boldsymbol{u}}$ of the form $(0,0,\hat{u}_3)'$.

$$
\begin{aligned}
\boldsymbol{r}_{\hat{E}}\hat{\boldsymbol{u}} \;&=\; \left(\textstyle\iint_{\hat{f}_3}\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}_3\,d\hat{S}\right)\hat{\boldsymbol{\zeta}}_3+\left(\textstyle\iint_{\hat{f}_4}\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}_4\,d\hat{S}\right)\hat{\boldsymbol{\zeta}}_4+\left(\textstyle\iint_{\hat{f}_5}\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}_5\,d\hat{S}\right)\hat{\boldsymbol{\zeta}}_5\\
&=: \rho_3(\hat{\boldsymbol{u}})\,\hat{\boldsymbol{\zeta}}_3+\rho_4(\hat{\boldsymbol{u}})\,\hat{\boldsymbol{\zeta}}_4+\rho_5(\hat{\boldsymbol{u}})\,\hat{\boldsymbol{\zeta}}_5.
\end{aligned}
$$

Then

$$
(\boldsymbol{r}_{\hat{E}}\hat{\boldsymbol{u}})_1(x,y,z) = \{\rho_3(\hat{\boldsymbol{u}})+\rho_5(\hat{\boldsymbol{u}})\}\,x+\rho_3(\hat{\boldsymbol{u}})\tfrac{x}{1-z}-\rho_4\tfrac{xz}{1-z}.
$$

Now observe that

$$
\begin{aligned}
(\rho_3+\rho_5)(\hat{\boldsymbol{u}}) &= \iint_{\partial\hat{E}}\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}\,d\hat{S}-\iint_{\hat{f}_4}\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}_4\,d\hat{S}\\
&= \int_{\hat{E}}\operatorname{div}\hat{\boldsymbol{u}}\,d\hat{\boldsymbol{x}}-\rho_4(\hat{\boldsymbol{u}})
\end{aligned}
$$

and, on the other hand,

$$
\begin{aligned}
(\rho_3-\rho_4)(\hat{\boldsymbol{u}}) &= \iint_{\hat{f}_3}\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}_3\,d\hat{S}-\iint_{\hat{f}_4}\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}_4\,d\hat{S}\\
&= \int_0^1\int_0^x \hat{u}_3(x,y,1-x)\,dydx-\int_0^1\int_0^y \hat{u}_3(x,y,1-y)\,dxdy,
\end{aligned}
$$

so

$$
\begin{aligned}
(\boldsymbol{r}_{\hat{E}}\hat{\boldsymbol{u}})_1(x,y,z) &= x\int_{\hat{E}}\operatorname{div}\hat{\boldsymbol{u}}\,d\hat{\boldsymbol{x}}+\\
&\left\{\int_0^1\int_0^x \hat{u}_3(x,y,1-x)\,dydx-\int_0^1\int_0^y \hat{u}_3(x,y,1-y)\,dxdy\right\}\tfrac{x}{1-z}.
\end{aligned}
$$

In a completely similar fashion we arrive at

$$
\begin{aligned}
(\boldsymbol{r}_{\hat{E}}\hat{\boldsymbol{u}})_2(x,y,z) &= y\int_{\hat{E}} \operatorname{div}\hat{\boldsymbol{u}}\,d\hat{\boldsymbol{x}} + \left\{\iint_{\hat{f}_3}\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}_3\,d\hat{S} - \iint_{\hat{f}_4}\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}_4\,d\hat{S}\right\}\tfrac{y}{1-z}.\\
&= y\int_{\hat{E}} \operatorname{div}\hat{\boldsymbol{u}}\,d\hat{\boldsymbol{x}} + \\
&\left\{\int_0^1\int_0^x \hat{u}_3(x,y,1-x)\,dydx - \int_0^1\int_0^y \hat{u}_3(x,y,1-y)\,dxdy\right\}\tfrac{y}{1-z}.
\end{aligned}
$$

We collect every term obtained so far for the first and second components in Table 6.5.

Table 6.5 – Terms in the proof of Theorem 6.4.4.
$q(s,t) = \frac{2s-st}{1-t}$, $r(s,t) = \frac{st}{1-t}$

	$(\boldsymbol{r}_{\hat{E}}\hat{\boldsymbol{u}})_1$	$(\boldsymbol{r}_{\hat{E}}\hat{\boldsymbol{u}})_2$
$(\hat{u}_1,0,0)'$	$-2\iint_{\hat{f}_2}\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}_2\,d\hat{S}$ $+q(x,z)\int_{\hat{E}}\operatorname{div}\hat{\boldsymbol{u}}\,d\hat{\boldsymbol{x}}$	$-r(y,z)\int_{\hat{E}}\operatorname{div}\hat{\boldsymbol{u}}\,d\hat{\boldsymbol{x}}$
$(0,\hat{u}_2,0)'$	$-r(x,z)\int_{\hat{E}}\operatorname{div}\hat{\boldsymbol{u}}\,d\hat{\boldsymbol{x}}$	$-2\iint_{\hat{f}_1}\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}_1\,d\hat{S}$ $+q(x,z)\int_{\hat{E}}\operatorname{div}\hat{\boldsymbol{u}}\,d\hat{\boldsymbol{x}}$
$(0,0,\hat{u}_3)'$	$x\int_{\hat{E}}\operatorname{div}\hat{\boldsymbol{u}}\,d\hat{\boldsymbol{x}}$ $+\left\{\iint_{\hat{f}_3}\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}_3\,d\hat{S}\right.$ $\left.-\iint_{\hat{f}_4}\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}_4\,d\hat{S}\right\}r(x,z)$	$y\int_{\hat{E}}\operatorname{div}\hat{\boldsymbol{u}}\,d\hat{\boldsymbol{x}}$ $+\left\{\iint_{\hat{f}_4}\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}_4\,d\hat{S}\right.$ $\left.-\iint_{\hat{f}_3}\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}_3\,d\hat{S}\right\}r(y,z)$

Lastly, the third component of $\boldsymbol{r}_{\hat{E}}\hat{\boldsymbol{u}}$ can be treated at once for any field $\hat{\boldsymbol{u}}$ as follows:

$$
\begin{aligned}
(\boldsymbol{r}_{\hat{E}}\hat{\boldsymbol{u}})_3(x,y,z) &= z\sum_{i=1}^{4}\iint_{\hat{f}_i}\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}_i\,d\hat{S} + (z-1)\iint_{\hat{f}_5}\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}_5\,d\hat{S}\\
&= z\iint_{\partial\hat{E}}\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}} - \iint_{f_5}\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}_5\,d\hat{S}\\
&= \hat{x}_3\int_{\hat{E}}\operatorname{div}\hat{\boldsymbol{u}}\,d\hat{\boldsymbol{x}} + \iint_{\hat{f}_5}\hat{u}_3\,d\hat{S}. \qquad (6.53)
\end{aligned}
$$

Now we bound each term in Table 6.5 and in expression (6.53).

$$
\begin{aligned}
(\boldsymbol{r}_{\hat{E}}\hat{\boldsymbol{u}})_1 &= (\boldsymbol{r}_{\hat{E}}(\hat{u}_1,0,0)')_1 + (\boldsymbol{r}_{\hat{E}}(0,\hat{u}_2,0)')_1 + (\boldsymbol{r}_{\hat{E}}(0,0,\hat{u}_3)')_1 \\
&= -2\iint_{\hat{f}_2} \hat{u}_1\, d\hat{S} + \tfrac{2x}{1-z}\int_{\hat{E}} \tfrac{\partial \hat{u}_1}{\partial \hat{x}_1}\, d\hat{\boldsymbol{x}} - \tfrac{xz}{1-z}\int_{\hat{E}} \tfrac{\partial \hat{u}_1}{\partial \hat{x}_1}\, d\hat{\boldsymbol{x}} \\
&\quad - \tfrac{xz}{1-z}\int_{\hat{E}} \tfrac{\partial \hat{u}_2}{\partial \hat{x}_2}\, d\hat{\boldsymbol{x}} + \left(x + \tfrac{xz}{1-z} - \tfrac{xz}{1-z}\right)\int_{\hat{E}} \tfrac{\partial \hat{u}_3}{\partial \hat{x}_3}\, d\hat{\boldsymbol{x}} \\
&\quad + \left(\iint_{\hat{f}_3} \hat{u}_3\, d\hat{S} - \iint_{\hat{f}_4} \hat{u}_3\, d\hat{S}\right)\tfrac{xz}{1-z} \\
&= -2\iint_{\hat{f}_2} \hat{u}_1\, d\hat{S} + \tfrac{2x}{1-z}\int_{\hat{E}} \tfrac{\partial \hat{u}_1}{\partial \hat{x}_1}\, d\hat{\boldsymbol{x}} - \tfrac{xz}{1-z}\int_{\hat{E}} \operatorname{div}\hat{\boldsymbol{u}}\, d\hat{\boldsymbol{x}} \\
&\quad + \tfrac{x}{1-z}\int_{\hat{E}} \tfrac{\partial \hat{u}_3}{\partial \hat{x}_3}\, d\hat{\boldsymbol{x}} + \left(\iint_{\hat{f}_3} \hat{u}_3\, d\hat{S} - \iint_{\hat{f}_4} \hat{u}_3\, d\hat{S}\right)\tfrac{xz}{1-z}.
\end{aligned} \tag{6.54}
$$

For the surface integrals in (6.54), by Lemma 5.15 in [33], page 120,

$$
\left|\iint_{\hat{f}_2} \hat{u}_1\, d\hat{S}\right| \leqslant C\,\|\hat{u}_1\|_{H^1(\hat{E})}
$$

and similarly

$$
\left|\iint_{\hat{f}_3} \hat{u}_3\, d\hat{S} - \iint_{\hat{f}_4} \hat{u}_3\, d\hat{S}\right| \leqslant C\,\|\hat{u}_3\|_{H^1(\hat{E})}
$$

all of which, together with Lemma 6.4.1, leads to

$$
\|(\boldsymbol{r}_{\hat{E}}\hat{\boldsymbol{u}})_1\|_{L^\infty(\hat{E})} \leqslant C_{\hat{E}}\left[\|\hat{u}_1\|_{H^1(\hat{E})} + \|\operatorname{div}\hat{\boldsymbol{u}}\|_{L^1(\hat{E})} + \|\hat{u}_3\|_{H^1(\hat{E})}\right].
$$

Copying the argument for the second component

$$
\|(\boldsymbol{r}_{\hat{E}}\hat{\boldsymbol{u}})_2\|_{L^\infty(\hat{E})} \leqslant C_{\hat{E}}\left[\|\hat{u}_2\|_{H^1(\hat{E})} + \|\operatorname{div}\hat{\boldsymbol{u}}\|_{L^1(\hat{E})} + \|\hat{u}_3\|_{H^1(\hat{E})}\right].
$$

Finally from (6.53) we deduce

$$
\|(\boldsymbol{r}_{\hat{E}}\hat{\boldsymbol{u}})_3\|_{L^\infty(\hat{E})} \leqslant C_{\hat{E}}\left[\|\hat{u}_3\|_{H^1(\hat{E})} + \|\operatorname{div}\hat{\boldsymbol{u}}\|_{L^1(\hat{E})}\right].
$$

The quantity $C_{\hat{E}}$ depends only on the supremum of the (fixed) basis shape functions of Table 6.4 over the pyramid. □

The next result is the div–conforming analogue of Theorem 6.4.3,

6.4.5 Theorem. *Let E be any pyramid which is a non degenerate affine image of the reference pyramid $\hat{E}$. We fix a positively oriented local system of coordinates $(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3)$ with origin in a vertex $\boldsymbol{x}_E$ of the parallelogram basis, for which $(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2)$ correspond to the two basis edges incident to $\boldsymbol{x}_E$ and $\boldsymbol{\xi}_3$ is parallel to the edge joining $\boldsymbol{x}_E$ with the top of the* *pyramid. Let h_1, h_2, h_3 be the corresponding edge lengths. With $\partial^{\boldsymbol{\alpha}}$ we denote $\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial^{\alpha_1}_{\boldsymbol{\xi}_1}\partial^{\alpha_2}_{\boldsymbol{\xi}_2}\partial^{\alpha_3}_{\boldsymbol{\xi}_3}}$. For all $\boldsymbol{u} \in W^{1,1}(E)^3$*

$$\|\boldsymbol{u} - \boldsymbol{r}_E\boldsymbol{u}\|_{L^p(E)} \lesssim \sum_{|\boldsymbol{\alpha}|=1} \boldsymbol{h}^{\boldsymbol{\alpha}} \|\partial^{\boldsymbol{\alpha}}\boldsymbol{u}\|_{L^p(E)} + h_E\|\mathit{div}\,\boldsymbol{u}\|_{L^p(E)} + \max\{h_1, h_2\} \sum_{|\boldsymbol{\alpha}|=1} \|\partial^{\boldsymbol{\alpha}} u_3\|_{L^p(E)}. \tag{6.55}$$

Proof. Using (4.9) again, using (2.34) and repeating the steps for (4.12)

$$\|(\boldsymbol{r}_{\tilde{E}}\tilde{\boldsymbol{u}})_1\|_{L^\infty(\tilde{E})} \lesssim \tfrac{1}{|\tilde{E}|}\{\|\tilde{u}_1\|_{L^1(\tilde{E})} + \sum_i h_i\|\partial_{\tilde{x}_i}\tilde{u}_1\|_{L^1(\tilde{E})} + h_1\|\mathrm{div}\,\tilde{\boldsymbol{u}}\|_{L^1(\tilde{E})} + \|\tilde{u}_3\|_{L^1(\tilde{E})} + h_1\|\partial_{\tilde{x}_1}\tilde{u}_3\|_{L^1(\tilde{E})} + h_2\|\partial_{\tilde{x}_2}\tilde{u}_3\|_{L^1(\tilde{E})} + h_1\|\partial_{\tilde{x}_3}\tilde{u}_3\|_{L^1(\tilde{E})}\}$$

The procedure for the second component is the same and the resulting estimate is symmetrical. For component three, the same procedure (with fewer terms) yields

$$\|(\boldsymbol{r}_{\tilde{E}}\tilde{\boldsymbol{u}})_3\|_{L^\infty(\tilde{E})} \lesssim |\tilde{E}|^{-1}\left(\|\tilde{u}_3\|_{L^1(\tilde{E})} + \sum_{i=1}^{3} h_i\|\partial_{\tilde{x}_i}\tilde{u}_3\|_{L^1(\tilde{E})} + h_3\|\mathrm{div}\,\tilde{\boldsymbol{u}}\|_{L^1(\tilde{E})}\right).$$

applying (6.52) and its analogue for the L^∞ norm we sum the component estimates to obtain the vectorial stability estimate in a physical pyramid:

$$\|\boldsymbol{r}_{\tilde{E}}\tilde{\boldsymbol{u}}\|_{L^p(\tilde{E})} \lesssim \|\tilde{\boldsymbol{u}}\|_{L^p(\tilde{E})} + \sum_i h_i\|\partial_{\tilde{x}_i}\tilde{\boldsymbol{u}}\|_{L^p(\tilde{E})} + h_{\tilde{E}}\|\mathrm{div}\,\tilde{\boldsymbol{u}}\|_{L^p(\tilde{E})} + \max\{h_1, h_2\}\|D\tilde{u}_3\|_{L^p(\tilde{E})}.$$

Now we apply (4.17) and (4.14) to the rescaled pyramid (Figure 6.2), making use of the approximation property for these finite elements (see Section 5 of [25]), and continue as in Theorem 4.1.9 or 6.2.1 and the result follows. □

Chapter 7

Implementation

Introducción al capítulo

En este capítulo mostramos algunos ejemplos obtenidos con programas que desarrollamos para esta tesis.

En la Sección 7.1 mostramos ejemplos de mallado obtenidos con la implementación de nuestro proceso propuesto en la Subsección 5.3.1. En este caso usamos el dominio de Fichera como ejemplo canónico de dominio singular en $\mathbb{R}^3$.

Esta primera versión del programa de mallado se puede ver en el siguiente repositorio [29].

El programa que implementa la resolución numérica con nuestro esquema FEM/VEM en las mallas híbridas se encuentra en desarrollo y será usado en experimentos numéricos en trabajos futuros.

En la Sección 7.2 mostramos resultados numéricos cuando aplicamos nuestro método a un dominio descompuesto en macro–elementos de un solo tipo, el macro–elemento prismático con una arista singular y sin vértices singulares.

Introduction to the chapter

In this chapter we show some examples obtained with programs we developed for the present thesis.

In Section 7.1 we show a meshing example made with the implementation of the procedure we proposed in Subsection 5.3.1. In this case we used the Fichera domain as a canonical example of a singular domain in $\mathbb{R}^3$.

This first version of the meshing program can be found in the following repository [29].

The program that implements the numerical solution with our FEM/VEM scheme over the hybrid meshes of Subsection 5.3.1 is being developed and will be used in numerical experiments of future works.

In Section 7.2 we show numerical results when we apply our method to a domain made exclusively of one kind of macro–elements, namely, the prismatic macro–element with a singular edge and no singular vertex.

7.1 Examples Of The Meshing Procedure In Dimension 3

Figure 7.1 shows the partition $\mathcal{T}_{h_0}$ (cfr. Remark 5.3.2) of the Fichera Domain $\Omega := (-1,1)^3 \setminus (0,1)^3$ from four different azimuthal angles.

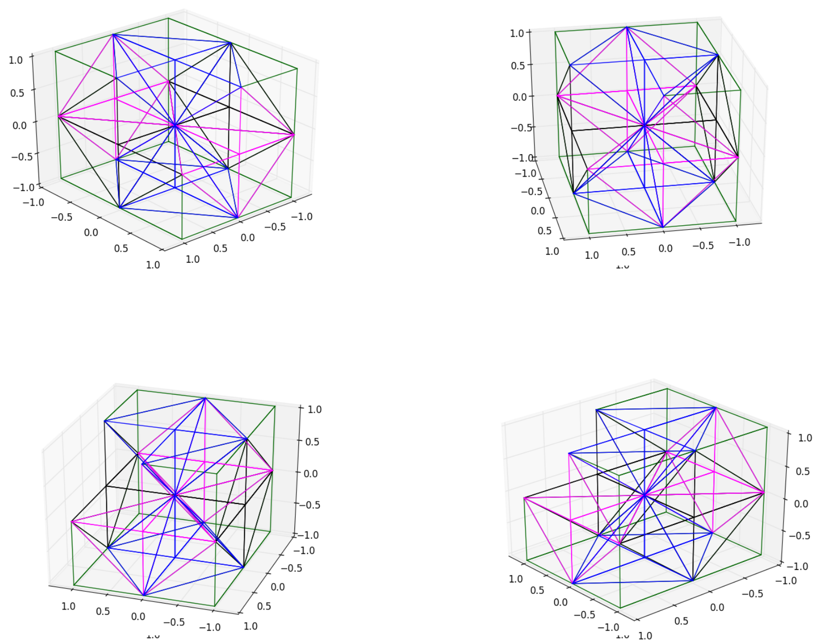


Figure 7.1 – Macroelements for the Fichera domain.

Here is a sample of the output of the program in the following instance: $\Omega = (-1,1)^3 \setminus (0,1)^3$ with 35 macro–elements, one singular vertex at $\mathbf{0}$ and three concurrent singular edges. Grading parameter $\mu = .65$ with 6 levels of refinement (7 nodes per macro–element edge).

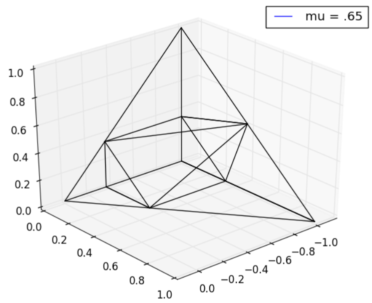
```
        vertices.txt                  faces.txt       elements_by_vertices.txt  elements_by_faces.txt

 1. -1.00000  1.00000  1.00000     1. 3 1 4 2           1. 6 1 4 2 7 9 8            1. 2 1 2 3 4 5
 2. -1.00000  1.00000  0.50000     2. 3 7 9 8           2. 4 2 5 3 8                2. 0 6 7 8 9 0
 3. -1.00000  1.00000  0.00000     3. 4 4 1 9 7         3. 4 4 6 5 9                3. 0 10 11 12 13 0
 4. -1.00000  0.50000  1.00000     4. 4 1 2 7 8         4. 5 2 4 9 8 5              4. 1 14 7 15 12 5
 5. -1.00000  0.50000  0.50000     5. 4 2 4 8 9         5. 4 7 9 8 10               5. 0 2 16 17 18 0
 6. -1.00000  0.00000  1.00000     6. 3 2 5 3           6. 6 11 14 12 17 19 18      6. 2 19 20 21 22 23
 7. -0.50000  1.00000  1.00000     7. 3 2 5 8           7. 4 12 5 6 18              7. 0 24 25 26 27 0
 8. -0.50000  1.00000  0.50000     8. 3 2 3 8           8. 4 14 3 5 19              8. 0 28 29 30 31 0
 9. -0.50000  0.50000  1.00000     9. 3 5 3 8           9. 5 12 14 19 18 5          9. 1 32 25 33 30 23
10.  0.00000  1.00000  1.00000    10. 3 4 6 5          10. 4 17 19 18 20           10. 0 20 34 35 36 0
11. -1.00000  0.00000  0.00000    11. 3 4 6 9          11. 4 20 18 19 24           11. 0 36 37 38 39 0
12. -1.00000  0.00000  0.34425    12. 3 4 5 9          12. 4 18 6 5 9              12. 0 27 40 41 13 0
13. -1.00000  0.00000  1.00000    13. 3 6 5 9          13. 4 19 5 3 8              13. 0 31 42 43 9 0
14. -1.00000  0.34425  0.00000    14. 3 2 4 5          14. 4 24 9 8 10             14. 0 44 45 46 18 0
15. -1.00000  0.50000  0.50000    15. 3 9 8 5          15. 4 18 19 24 9            15. 0 39 47 48 49 0
16. -1.00000  1.00000  0.00000    16. 3 7 9 10         16. 4 19 24 9 8             16. 0 49 50 51 44 0
17. -0.34425  0.00000  0.00000    17. 3 7 8 10         17. 4 18 19 5 9             17. 0 33 47 41 52 0
18. -0.34425  0.00000  0.34425    18. 3 9 8 10         18. 4 19 5 9 8              18. 0 52 42 51 15 0
19. -0.34425  0.34425  0.00000    19. 3 11 14 12                ...                          ...
20.  0.00000  0.00000  0.00000    20. 3 17 19 18
21.  0.00000  0.00000  0.00000    21. 4 14 11 19 17                                                      ...
22. -0.34425  0.00000  0.34425    22. 4 11 12 17 18
23. -0.34425  0.34425  0.00000    23. 4 12 14 18 19
24.  0.00000  0.34425  0.34425    24. 3 12 5 6
25. -0.34425  0.00000  0.34425    25. 3 12 5 18
26. -1.00000  0.00000  1.00000    26. 3 12 6 18
27. -1.00000  0.50000  0.50000    27. 3 5 6 18
28. -0.50000  0.50000  1.00000    28. 3 14 3 5
29. -0.34425  0.34425  0.00000    29. 3 14 3 19
30. -1.00000  0.50000  0.50000    30. 3 14 5 19
              ...                         ...
```

Then in Figure 7.2 we show how the hybrid mesh looks like in a tetrahedral macro–element as in Subsubsection 5.3.1.1.

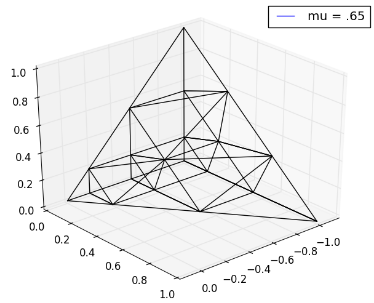
Figure 7.3 shows the detail of the decomposition into macro–elements of a cube defined as the intersection of Ω with an arbitrary octant. The cube consists of five tetrahedral macro–elements; a regular tetrahedron in the center of the cube meshed as a macro–element from Subsubsection 5.3.1.3 and four more macro–tetrahedra as in Subsubsection 5.3.1.1.

Finally in Figure 7.4 we can see the whole domain Ω meshed with our anisotropic grading procedure.

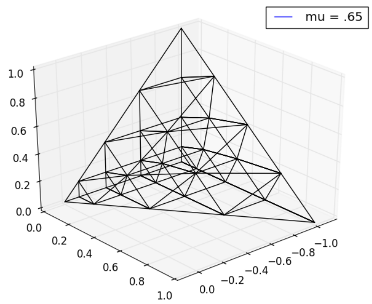
Additionally, we found interesting to illustrate some experiments that confirm the conformity of the meshes. These can be seen in Figure 7.5. In Subfigure 7.5a the idea is to show edge conformity from one macro–element to another, whereas in Subfigures 7.5b–7.5d we can see face conformity between macro–elements.



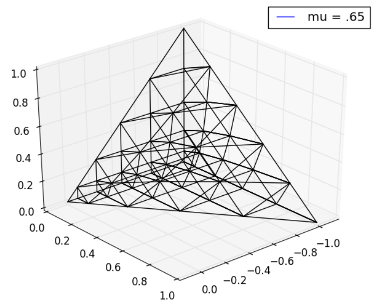
(a) n = 2, 3 nodes per edge



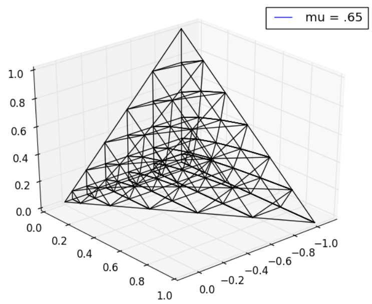
(b) n = 3



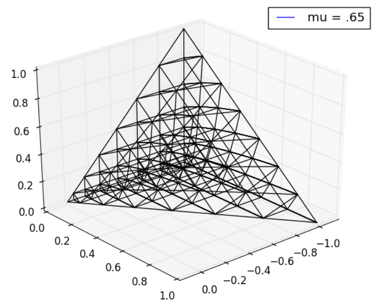
(c) n = 4



(d) n = 5

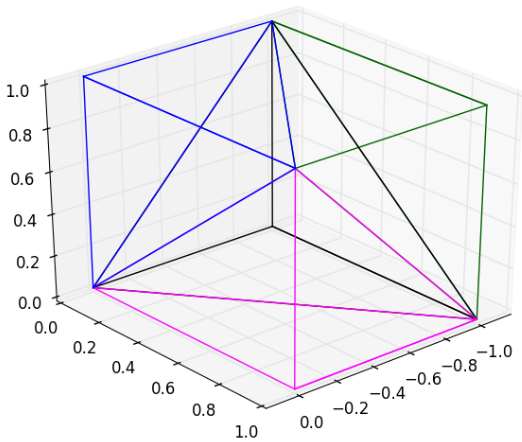


(e) n = 6

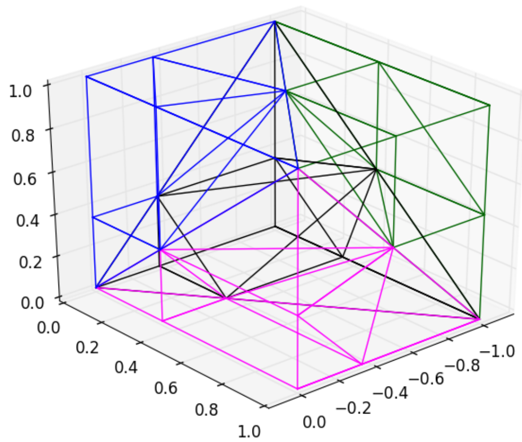


(f) n = 7

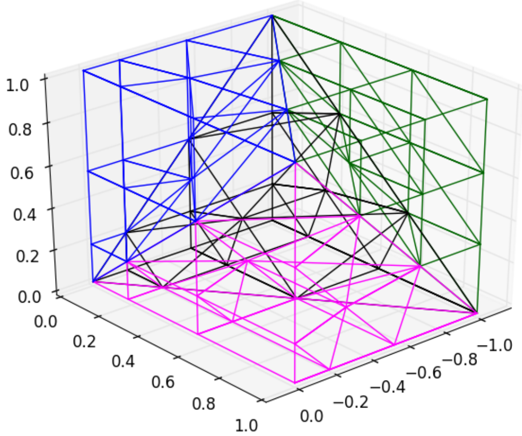
Figure 7.2 – Macroelement with the hybrid submesh.



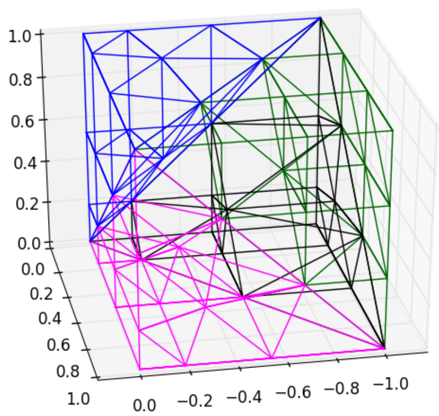
(a) n = 1



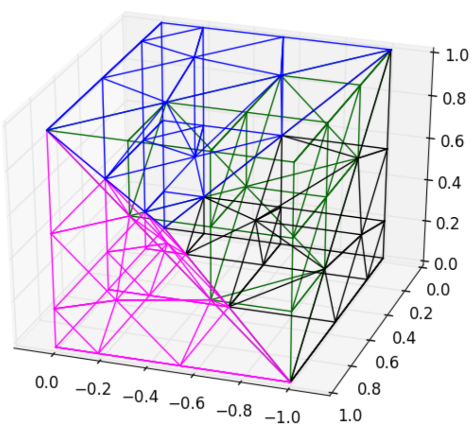
(b) n = 2



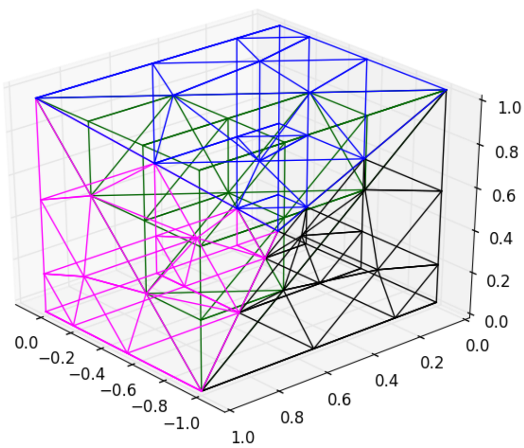
(c) n = 3



(d) n = 4

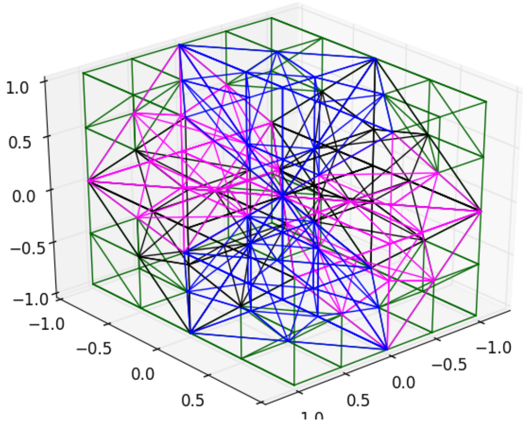


(e) n = 5

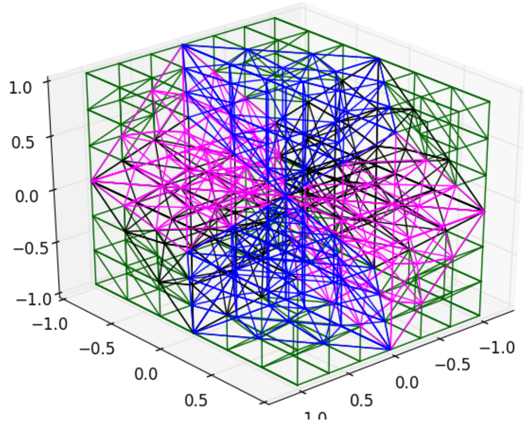


(f) n = 6

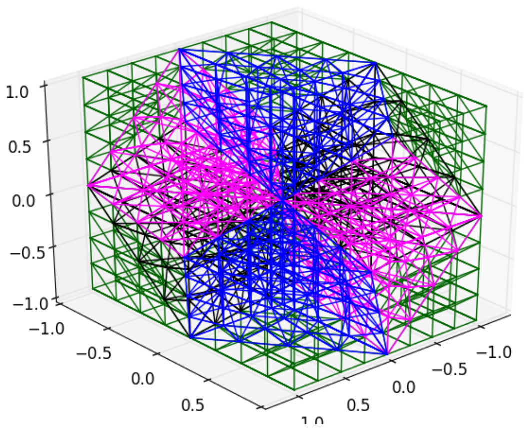
Figure 7.3 – Cube divided into five tetrahedra.



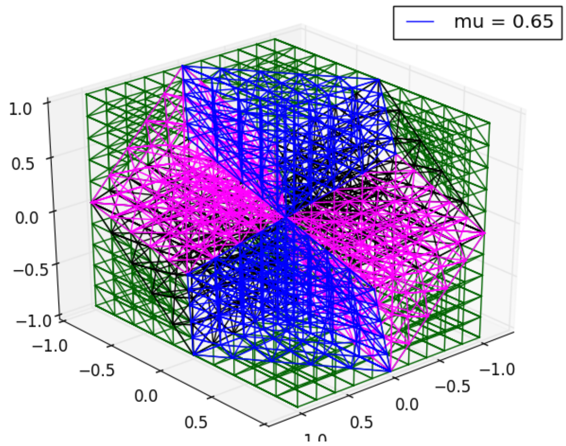
(a) n = 2



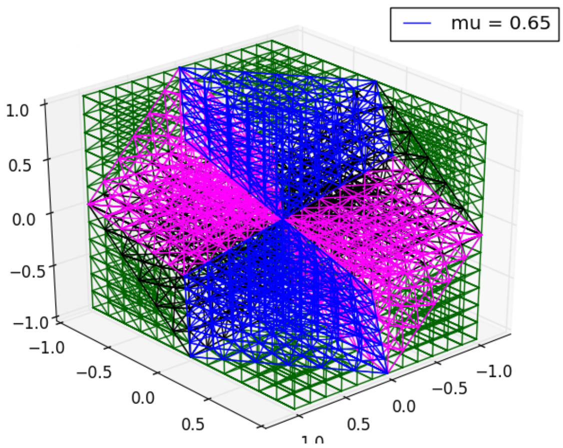
(b) n = 3



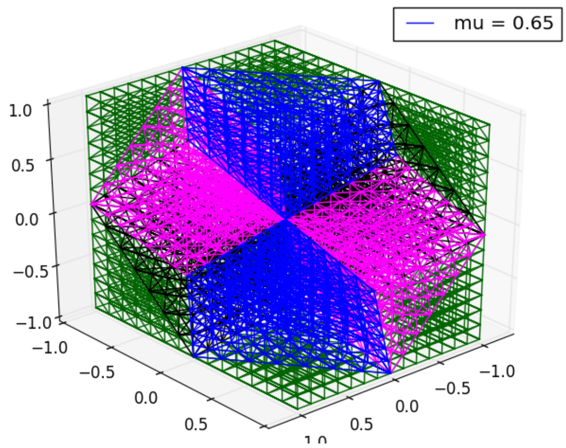
(c) n = 4



(d) n = 5



(e) n = 6



(f) n = 7

Figure 7.4 – The Fichera Domain meshed according to Subsection 5.3.1.

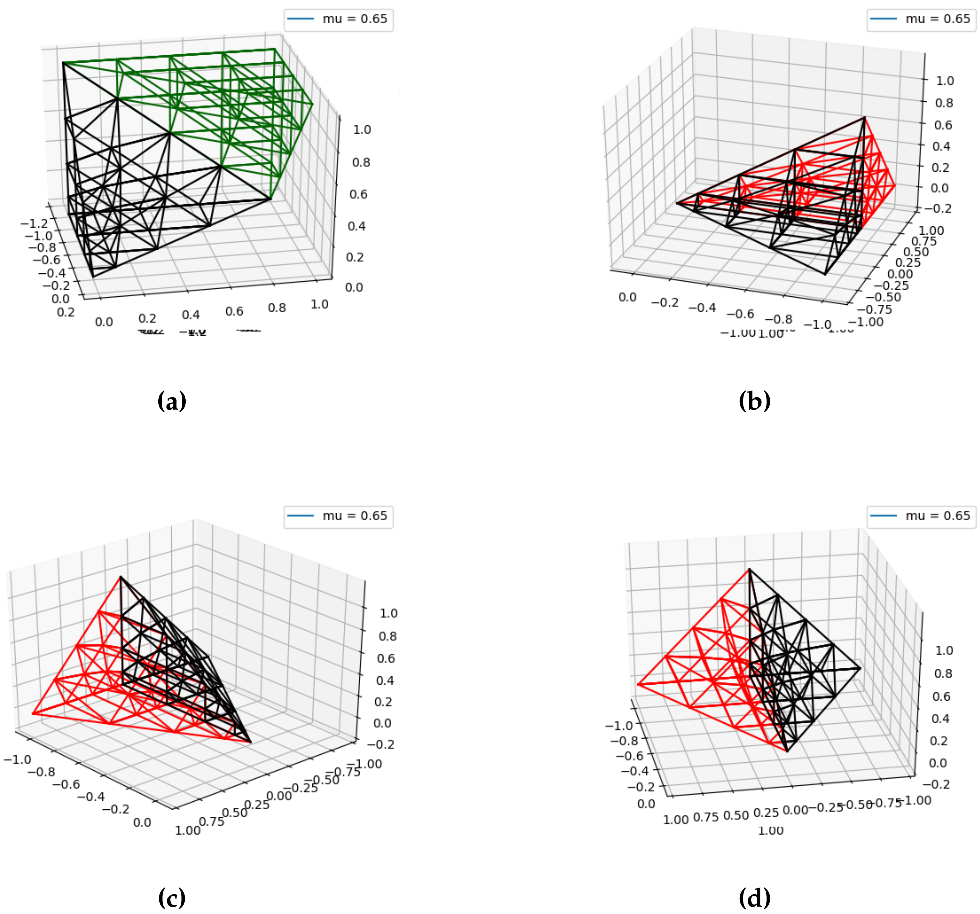
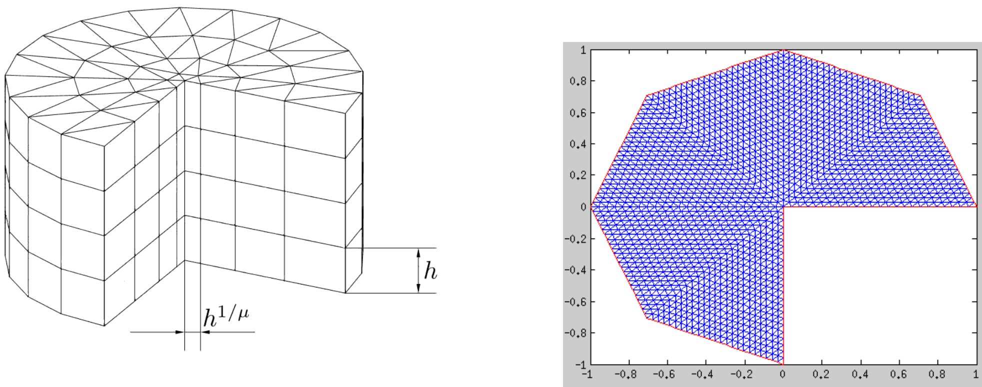


Figure 7.5 – Some experiments to show conformity.

7.2 Numerical Experiments in Cylindrical Domains with Edge Singularities

In Figure 7.6a we put a cylindrical singular domain to show the kind of problems we started to work with to develop our method. In Figures 7.6b and 7.7 we can compare the sectional views of a uniform mesh against a graded anisotropic refinement (the anisotropy becomes evident when we consider the third orthogonal direction). Finally, Tables 7.1 and 7.2 are presented to compare the approximation rate for the vectorial variable with quasi–uniform mesh against the one with an anisotropic cartesian product mesh. These experiments correspond to the instance of the problem which is also solved in [23], as pointed out before in this Thesis. As we wrote in the previous chapters, we focused on problems in general polyhedral domains, whith edge and vertex singularities, and proposed a more general meshing procedure, combining different types of elements, and as we said in the introduction of the present chapter, the program calculating the so-



(a) Domain Ω. **(b)** Sectional view of Ω.

Figure 7.6 – Cylindrical domain.

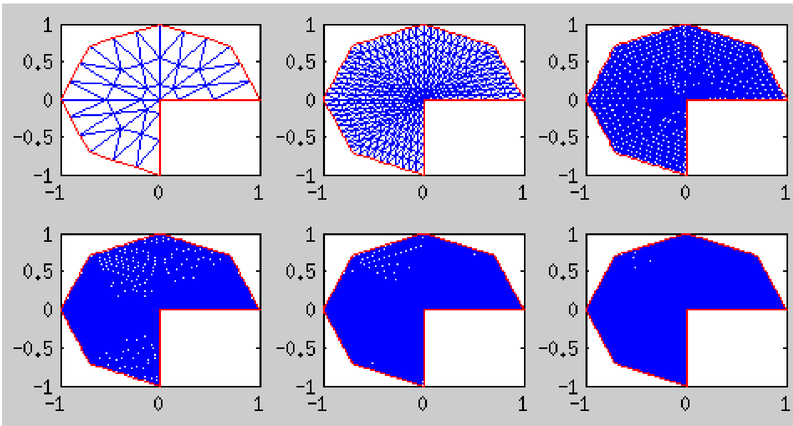


Figure 7.7 – Sectional view of an anisotropic graded mesh. Prismatic macro–elements.

lution with our FEM/VEM scheme over the hybrid meshes of Subsection 5.3.1 will be finished for future works.

Table 7.1 – $\omega = 3\pi/2$, $\gamma = 1$

n	Nel	$\boldsymbol{u}$	p	$\frac{\Delta_i \log(e_{\boldsymbol{u}})}{-\Delta_i \log(n)}$	$\frac{\Delta_i \log(e_p)}{-\Delta_i \log(n)}$
6	1512	0.632043	0.205469		
10	7000	0.324654	0.123666	1.304158	0.993902
20	56000	0.243846	0.064251	0.412932	0.944658
30	189000	0.130700	0.042238	1.538066	1.034557
40	448000	0.108696	0.031693	0.640810	0.998409
50	875000	0.091443	0.025360	0.774564	0.999010
60	1512000	0.079951	0.021128	0.736620	1.001384

Table 7.2 – $\omega = 3\pi/2$, $\gamma = 1.5$

n	Nel	$\boldsymbol{u}$	p	$\frac{\Delta_i \log(e_{\boldsymbol{u}})}{-\Delta_i \log(n)}$	$\frac{\Delta_i \log(e_p)}{-\Delta_i \log(n)}$
6	1512	0.612728	0.206872		
10	7000	0.292898	0.124441	1.444907	0.994997
20	56000	0.218083	0.064388	0.425517	0.950588
30	189000	0.094069	0.042391	2.073776	1.030897
40	448000	0.074568	0.031808	0.807546	0.998343
50	875000	0.057800	0.025453	1.141533	0.998845
60	1512000	0.047035	0.021210	1.130357	1.000286
70	2401000	0.040214	0.018179	1.016342	1.000308

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