Fixed-End Moment Equations for Continuous Prestressed Concrete Beams

by Duryl M. Bailey and Phil M. Ferguson*

SYNOPSIS

Fixed-end moments created by prestressing for constant cross-section beams are given. These fixed-end moments may be used with structural analysis methods to design continuous prestressed concrete beams. Fixed-end moments for cable profiles composed of straight line segments or parabolic segments are presented for both exterior and interior spans. Only symmetrical cable profiles are given for interior spans. Tables are presented for aiding in the solution of difficult equations.

CONTINUOUS BEAMS

In recent years there have been many excellent papers written about prestressed concrete including the design and analysis of both simple and continuous prestressed concrete beams. Some of these papers were written by Parme and Paris[1,2], Fiesenheiser[3], Moorman[4], and Lin[6].

Lin[6] has presented a very interesting and useful method in which he balances the dead load effect on the beam with the imposed load of the prestressing forces. This method eliminates the main difficulty in designing continuous prestressed concrete beams. Hence, the load balancing method is recommended in instances where it can be used. In cases where the dead load either is not or cannot be balanced by the prestressing force, the following equations and tables are submitted for use.

A brief description of continuously prestressed concrete beams will be given. The assumption is made that the reader is familiar with the design and analysis procedures of a simple prestressed concrete beam, namely, a prestressed concrete beam which is simply supported at each end, not continuous over a support. It is also assumed that the reader is familiar with the ordinary analysis of statically indeterminate beams.

Assumptions which are usually made in both simple prestressed concrete beams and continuous prestressed concrete beams are given in Moorman's paper[4]. These assumptions are

1. Hooke's Law is valid
2. The principle of superposition holds
3. The horizontal component of the tension in the cable is equal to the tension in the cable
4. The friction force is negligible
5. The lateral force from the

*Project Engineer, Soils Division, U.S. Army Waterways Experiment Station, Vicksburg, Mississippi and Professor of Civil Engineering, The University of Texas, Austin, Texas, respectively.

cable is either vertical or normal to the axis of the member

6. the loss of cable tension due to creep is negligible

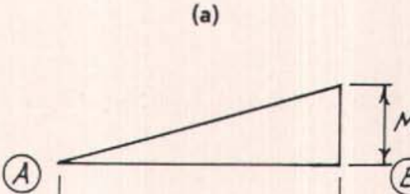
7. The reduction in cross-sectional area because of the cable ducts is negligible

In simple prestressed concrete beams, the moment induced by the prestressing force is equal to the horizontal force in the cable times the distance of the cable from the centroid of the beam. Elastic rotation at the supports and elastic deflection along the beam can occur without creating any moments at the supports because of the hinged ends of the beam. The term *hinged-end* refers to a beam whose end is free to rotate without moments being induced by this rotation. When the dead load and live load of the beam is not considered, the line of pressure from the prestressing force, or stress on the concrete, coincides with the cable profile.

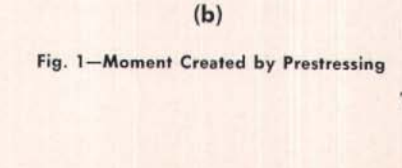
In a continuous prestressed concrete beam without dead load or live load, the line of pressure on the concrete may or may not coincide with the cable profile. If the line of pressure on the concrete does coincide with the cable, the cable is known as a concordant cable. In many cases the line of pressure on the concrete will not coincide with the cable profile. The reason for this is as follows:

Moments from the prestressing cable are induced along the continuous prestressed concrete beam as they are in the simple prestressed concrete beam. These moments along the beam cause deflections and in turn rotations at the supports. These rotations are not free to take place but are restrained due to the continuity of the beam. The resistance to these rotations creates moment at the interior supports of the beam. These moments at the support induced by the continuity of the structure cause the center of pressure on the concrete to deviate from the location of the cable profile.

Fig. 1(a) shows an unloaded end span of a continuous beam with M the moment created by the beam restraint at the interior support. The moment, M, creates an extra reaction at each end of the exterior span. This reaction is really due to the force in the cable and the continuity of the structure. The extra reaction gives a moment curve which varies linearly between supports as shown in Fig. 1(b). As a result the center of pressure on the concrete varies linearly from the cable profile between beam supports. However, the center of pressure on the concrete will follow the same intrinsic shape as the cable profile. Fig. 2 shows an assumed cable profile and an assumed center of pressure on the concrete for a continuous beam. At the end supports the cable profile and the center of pressure



(a)



(b)

Fig. 1—Moment Created by Prestressing

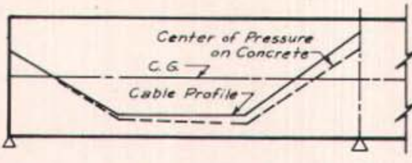


Fig. 2—Cable Profile and Center of Pressure on Concrete for End Span

on the concrete coincide. At the interior support, the cable profile and center of pressure on the concrete do not coincide. At the center line of the exterior span, the deviation between the center of pressure on the concrete and the cable profile is half the amount it is at the support.

In continuous prestressed concrete the line of pressure on the concrete is of the utmost importance in the elastic analysis of the beam. The line of pressure must fall within a limiting zone along the beam. The limiting zone is determined from the maximum and minimum moment curves which are created from external loads, cross-sectional properties of the beam, the allowable tension in the concrete used, and the force of the prestressing cable. Lin[5] discusses the location of the limiting zone in his book. The limiting zone will be briefly discussed here. Fig. 3 shows an elevation of

Fig. 3—Possible Limiting Zone for End Span

the end span of a continuous beam. The vertical scale of the beam is increased for convenience of plotting the limiting zone. The usual beam sign convention is used. A positive moment creates compression stress at the top fiber of the beam. M_{max} is the largest algebraic moment along the beam. M_{min} is the smallest algebraic moment along the beam. For clarification, M_{max} is associated with the maximum positive moment in a beam span and M_{min} is associated with the support moment for a continuous beam. F is the prestressing force of the cable. When no tension stress is allowed in the concrete, the limiting zone is determined by plotting from the kern lines of the beam. Fig. 3 shows a beam in which no tension stress is allowed in the concrete. The top portion of the limiting zone is determined by the M_{max} curve. Let $a_{max} = M_{max}/\text{F}$. Then a_{max} is plotted from the top kern line and varies along the beam. When M_{max} is positive, a_{max} is plotted below the kern line. When M_{max} is negative, a_{max} is plotted above the kern line. Let $a_{min} = M_{min}/\text{F}$. When M_{min} is positive, a_{min} is plotted below the bottom kern line. When M_{min} is negative, a_{min} is plotted above the bottom kern line. The zone between the curves obtained by plotting the values of a_{max} and a_{min} is the limiting zone of the beam.

The method used in this discussion is the method of equivalent loads. By this procedure, forces which are created by the action of the prestressing cable pressing against the concrete are determined. These equivalent loads have been determined previously by other writers[1,2,4,5]. For clarification, these loads will also be derived in this paper.

Fixed-end moments due to prestressing will be derived for various cable profiles. These fixed-end moments may be used in the Moment Distribution Method or Slope-Deflection Method to determine support moments in the beam due to the prestressing force. By using these support moments in the beam, with no live load or dead load considered, the location of a corresponding center of pressure on the concrete at the supports can be determined. The support moment due to prestressing when divided by the prestressing force in the cable locates the center of pressure on the concrete at the supports. Between supports the center of pressure on the concrete varies linearly from the cable profile. Hence, the center of pressure of the concrete from cable alone, can be located throughout the continuous beam.

CABLE PROFILES
STRAIGHT LINE SEGMENTS

Fig. 4 is a portion of a beam in which sharp bends in the cable occur at Section *A-A* and Section *B-B*. The cable between these sections is a straight line. The cable makes an angle, α, with the horizontal axis of the beam. Assume no friction loss in the cable between the two sections. Also, assume the horizontal component of the cable force is equal to the cable force. The vertical component of the cable force, F, at Section *A-A* and Section *B-B* is

$$P_A = P_B = F \tan \alpha \qquad (1)$$

However, since $\tan \alpha = t/kL$, Eq. 1 becomes

$$P_A = P_B = \frac{Ft}{kL} \qquad (2)$$

These are likewise the forces the concrete has to exert on the cable at A and B to keep the cable in the position shown in Fig. 4 when the cable is made horizontal to the left of A and to the right of B. Hence Eq. 2 gives the magnitude of the equivalent concentrated load due to this form of prestressing. From Eq. 2, note that the equivalent concen-

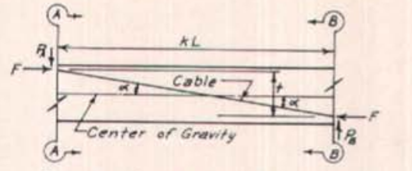
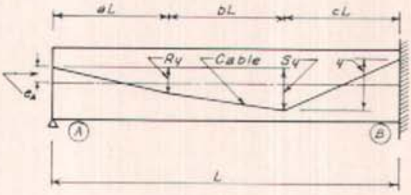


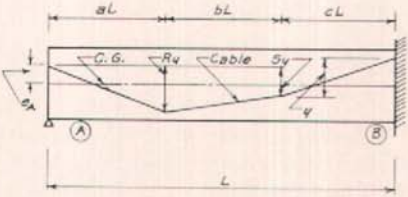
Fig. 4—Portion of Beam with Straight Cable

trated load is a function only of the force in the cable and the dimensions of the cable profile.

Figs. 5(a) and 5(b) show cable profiles for an end span. The cable profile is composed of straight line segments with sharp bends. The beams are shown with a hinged-end at A and a fixed-end at B. The fixed-end moment at B will be determined for the cable profile shown in Figs. 5(a) or 5(b) where R and S are fractional factors applied to the deflection y. Two equivalent concentrated loads due to prestressing will occur in the span. Two other concentrated loads due to prestressing will occur at the supports. However, the loads at the supports will not affect the fixed-end moment at B.



(a)



(b)

Fig. 5—Possible Straight Line Segment Cable Profile for End Spans

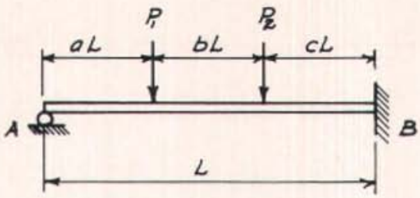


Fig. 6—Propped Cantilever Beam with Two Concentrated Loads

Fig. 6 shows a beam which is subjected to two concentrated loads. These concentrated loads are assumed positive when acting in a downward direction. The sign convention to be used for the fixed-end moments is the beam bending moment sign convention, that is, a positive moment is a moment which causes compression stress on the top fiber of the beam. By using Eq. 2 and the symbols and dimensions shown in Figs. 5(a) and 5(b), values for P_1 and P_2 shown in Fig. 6 can be obtained.

$$P_1 = -F\left[\frac{Ry}{aL} + \frac{Ry - Sy}{bL}\right] = \frac{Fy}{L}\left[\frac{R(a+b) - aS}{ab}\right] \qquad (3)$$

and

$$P_2 = -F\left[\frac{y}{cL} + \frac{Sy - Ry}{bL}\right] = -\frac{Fy}{L}\left[\frac{b + c(S-R)}{cb}\right] \qquad (4)$$

A statically determined moment, m, is applied at the hinged end A by the cable profiles in Figs. 5(a) and 5(b). The value of m is

$$m = Fe_A \qquad (5)$$

where F is the prestressing force and e_A is the distance of the cable from the centroid of the beam at support A. When the cable is above the centroid of the beam, e_A is positive.

Fig. 7 shows a propped cantilever beam with a concentrated load, P, at a distance x from the hinged-end. The equation for the moment, M_B, can be found in several engineering handbooks.

$$M_B = \frac{Px(L^2 - x^2)}{2L^2} \qquad (6)$$

If $P = P_1$ and $x = aL$ are substituted into Eq. 6 and if $P = P_2$ and $x = (1-c)L$ are substituted into Eq. 6 and these results are combined

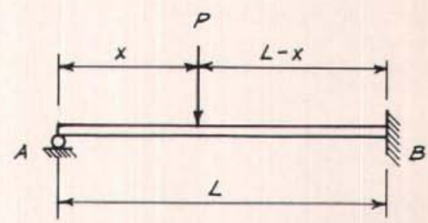


Fig. 7—Propped Cantilever Beam with One Concentrated Load

algebraically into one equation, the following result is obtained

$$M_B = -\frac{P_1 La(1-a^2)}{2} - \frac{P_2(1-c)L[L^2 - (1-c)^2L^2]}{2L^2} \qquad (7)$$

Substitute the values of P_1 and P_2 that are given in Eqs. 3 and 4 into Eq. 7.

$$M_B = \frac{Fy}{2b}\left\{[a(R-S)+bR](1-a^2) + [b+c(S-R)](1-c)(2-c)\right\} \qquad (8)$$

Fig. 8 shows a propped cantilever beam with an applied moment, m, at the hinged end. The fixed-end moment at B due to the applied moment at A is

$$M_B = -\frac{m}{2} - = \frac{Fe_A}{2} \qquad (9)$$

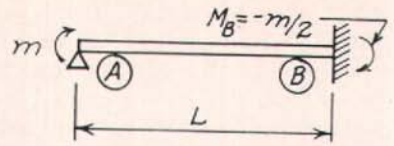


Fig. 8—Propped Cantilever Beam with Moment Applied at Hinged End

since the applied moment at A is given in Eq. 5.

If Eqs. 8 and 9 are added together algebraically, the fixed-end moments due to prestressing are determined for the cable profiles shown in Figs. 5(a) or 5(b).

$$M_B = \frac{Fy}{2b}\Big\{[a(R-S)+bR](1-a^2)+ [b+c(S-R)](1-c)(2-c)\Big\} - \frac{Fe_A}{2} \qquad (10)$$

Eq. 10 contains 7 independent variables and one dependent variable. The independent variables are F, y, e_A, a, b, R and S. Since $a+b+c=1$, c is not a variable after a and b are assigned values. The dependent variable of Eq. 10 is M_B.

One design chart that included all the variables of Eq. 10 would be very difficult to construct. However, three separate design tables are presented which will allow rapid calculation of M_B. Eq. 9 was added algebraically with Eq. 8 to obtain Eq. 10. Eq. 9 is a very simple equation. Therefore, it will not be included in the design tables. Eq. 9 reveals that the fixed-end moment, M_B, is equal to one-half the magnitude and opposite in sign to the applied moment created by the prestressing at support A.

The three design tables will be determined from Eq. 8.

Assume $R = S = 0$ and substitute these values in Eq. 8. The result is

$$M_B = \frac{Fy\,(1-c)\,(2-c)}{2} \qquad (11)$$

Divide each side of Eq. 11 by Fy.

$$\frac{M_B}{Fy} = \frac{(1-c)\,(2-c)}{2} \qquad (12)$$

The following equation is obtained from differential calculus:

$$\Delta M_B = \left(\frac{\partial M_B}{\partial R}\right)\Delta R \qquad (13)$$

Where ΔM_B is the change in the fixed-end moment M_B, ΔR is the change in the variable R, and $\frac{\partial M_B}{\partial R}$ is the partial derivative of M_B with respect to R. Take the partial derivative of Eq. 8,

$$\frac{\partial M_B}{\partial R} = \frac{Fy}{2b}\,[(a+b)(1-a^2)- c(1-c)(2-c)] \qquad (14)$$

Substitute Eq. 14 into Eq. 13,

$$\Delta M_B = \frac{Fy}{2b}[(a+b)(1-a^2)- c(1-c)(2-c)]\,(\Delta R) \qquad (15)$$

The following equation also comes from calculus:

$$\Delta M_B = \left(\frac{\partial M_B}{\partial S}\right)(\Delta S) \qquad (16)$$

Applying calculus to Eq. 8,

$$\frac{\partial M_B}{\partial S} = \frac{Fy}{2b}\,[-a(1-a^2)+ c(1-c)(2-c)]\ \Delta S \qquad (17)$$

Substitute Eq. 17 into Eq. 16,

$$\Delta M_B = \frac{Fy}{2b}\,[-a(1-a^2)+ c(1-c)(2-c)]\ \Delta S) \qquad (18)$$

Let $\Delta R = R$ in Eq. 15 and divide

both sides of Eq. 18 by FyS.

$$\frac{\Delta M_B}{FyR} = \frac{(a+b)(1-a^2)}{2b} + \frac{c(1-c)(2-c)}{2b} \qquad (19)$$

Let $\Delta S = S$ in Eq. 18 and divide both sides of Eq. 18 by FyS.

$$\frac{\Delta M_B}{FyS} = \frac{-a(1-a^2)}{2b} + \frac{c(1-c)(2-c)}{2b} \qquad (20)$$

The change of the fixed-end moment due to prestressing which is caused by the variables R and S can be calculated by Eqs. 19 and 20. Eq. 12 can be used to calculate M_B when $R = S = 0$. Hence, if Eqs. 12, 19 and 20 are added together algebraically, the fixed-end moment due to prestressing can be calculated.

The solutions of Eqs. 12, 19 and 20 are given in Tables 1, 2 and 3 respectively. The increments of the tabulated values are such that straight line interpolation may be used while creating a maximum error of 10 in the fifth decimal place.

Fig. 9(a) shows a propped cantilever beam with a cable profile. The magnitude of the prestressing force is assumed constant along the entire span. The design tables, Tables 1, 2 and 3, will be used to calculate the fixed-end moment. Eq. 9 also will be used to calculate the fixed-end moment for $F = 100$ kips.

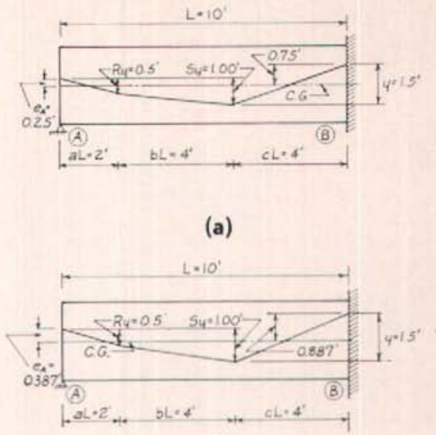


Fig. 9—Straight Line Segment Profile for End Span

$$M_{B1} = -\frac{Fe_A}{2} = -\frac{100\,(0.25)}{2} = -12.5 \text{ kip-ft.}$$

When $a = 0.2$ and $c = 0.4$, the fixed-end moment due to Table 1 is

$$M_{B2} = +0.48\,Fy = +0.48\,(100)\,(1.5) = +72 \text{ kip-ft.}$$

When $a = 0.2$, $c = 0.4$ and $R = 1/3$,

Table 1—Values of $\left.\frac{M_B}{Fy}\right|_1 = \frac{(1-c)(2-c)}{2}$

c	Value	c	Value	c	Value	c	Value
0.00	1.0000	0.20	0.7200	0.40	0.4800	0.60	0.2800
0.02	0.9702	0.22	0.6942	0.42	0.4582	0.62	0.2622
0.04	0.9408	0.24	0.6688	0.44	0.4368	0.64	0.2448
0.06	0.9118	0.26	0.6438	0.46	0.4158	0.66	0.2278
0.08	0.8832	0.28	0.6192	0.48	0.3952	0.68	0.2112
0.10	0.8550	0.30	0.5950	0.50	0.3750	0.70	0.1950
0.12	0.8272	0.32	0.5712	0.52	0.3552	0.72	0.1792
0.14	0.7998	0.34	0.5478	0.54	0.3358	0.74	0.1638
0.16	0.7728	0.36	0.5248	0.56	0.3168	0.76	0.1488
0.18	0.7462	0.38	0.5022	0.58	0.2982	0.78	0.1342

Table 2—Values of $\left.\frac{M_B}{F\,y\,R}\right|_2 = \frac{(a+b)(1-a^2) + c(1-c)(2-c)}{2b}$

c \ a	0.20	0.22	0.24	0.26	0.28	0.30	0.32	0.34	0.36	0.38	0.40	0.42	0.44	0.46	0.48	0.50	0.52	0.54	0.56	0.58	0.60
0.00	6000	6100	6200	6300	6400	6500	6600	6700	6800	6900	7000	7100	7200	7200	7400	7500	7600	7700	7800	7900	8000
0.02	5782	5880	5978	6076	6174	6272	6370	6468	6566	6664	6762	6860	6958	7056	7154	7252	7350	7448	7546	7644	7742
0.04	5568	5664	5760	5856	5952	6048	6144	6240	6336	6432	6528	6624	6720	6816	6912	7008	7104	7200	7296	7392	7488
0.06	5358	5452	5546	5640	5734	5828	5922	6016	6110	6204	6298	6392	6486	6580	6674	6768	6862	6956	7050	7144	7238
0.08	5152	5244	5336	5428	5520	5612	5704	5796	5888	5980	6072	6164	6256	6348	6440	6532	6624	6716	6808	6900	6992
0.10	4950	5040	5130	5220	5310	5400	5490	5580	5670	5760	5850	5940	6030	6120	6210	6300	6390	6480	6570	6660	6750
0.12	4752	4840	4928	5016	5104	5192	5280	5368	5456	5544	5632	5720	5808	5896	5984	6072	6160	6248	6336	6424	6512
0.14	4558	4644	4730	4816	4902	4988	5074	5160	5246	5332	5418	5504	5590	5676	5762	5848	5934	6020	6106	6192	6278
0.16	4368	4452	4536	4620	4704	4788	4872	4956	5040	5124	5208	5292	5376	5460	5544	5428	5712	5796	5880	5964	6048
0.18	4182	4264	4346	4428	4510	4592	4674	4756	4838	4920	5002	5084	5166	5248	5330	5412	5494	5576	5658	5740	5822
0.20	4000	4080	4160	4240	4320	4400	4480	4560	4640	4720	4800	4880	4960	5040	5120	5200	5280	5360	5440	5520	5600
0.22	3822	3900	3978	4056	4134	4212	4290	4368	4446	4524	4602	4680	4758	4836	4914	4992	5070	5148	5226	5304	5382
0.24	3648	3724	3800	3876	3952	4028	4104	4180	4256	4332	4408	4484	4560	4636	4712	4788	4864	4940	5016	5092	5168
0.26	3478	3552	3626	3700	3774	3848	3922	3996	4070	4144	4218	4292	4366	4440	4514	4588	4662	4736	4810	4884	4958
0.28	3312	3384	3456	3528	3600	3672	3744	3816	3888	3960	4032	4104	4176	4248	4320	4392	4464	4536	4608	4680	4752
0.30	3150	3220	3290	3360	3430	3500	3570	3640	3710	3780	3850	3920	3990	4060	4130	4200	4270	4340	4410	4480	4550
0.32	2992	3060	3128	3196	3264	3332	3400	3468	3536	3604	3672	3740	3808	3876	3944	4012	4080	4148	4216	4284	4352
0.34	2838	2904	2970	3036	3102	3168	3234	3300	3366	3432	3498	3564	3630	3696	3762	3828	3894	3960	4026	4092	4158
0.36	2688	2752	2816	2880	2944	3008	3072	3136	3200	3264	3328	3392	3456	3520	3584	3648	3712	3776	3840	3904	3968
0.38	2542	2604	2666	2728	2790	2852	2914	2976	3038	3100	3162	3224	3286	3348	3410	3472	3534	3596	3658	3720	
0.40	2400	2460	2520	2580	2640	2700	2760	2820	2880	2940	3000	3060	3120	3180	3240	3300	3360	3420	3480		
0.42	2262	2320	2378	2436	2494	2552	2610	2668	2726	2786	2842	2900	2958	3016	3074	3132	3190	3248			
0.44	2128	2184	2240	2296	2352	2408	2464	2520	2576	2632	2688	2744	2800	2856	2912	2968	3024				
0.46	1998	2052	2106	2160	2214	2268	2322	2376	2430	2484	2538	2592	2646	2700	2754	2808					
0.48	1872	1924	1976	2028	2080	2132	2184	2236	2288	2340	2392	2444	2496	2548	2600						
0.50	1750	1800	1850	1900	1950	2000	2050	2100	2150	2200	2250	2300	2350	2400							
0.52	1632	1680	1728	1776	1824	1872	1920	1968	2016	2064	2112	2160	2208								
0.54	1518	1564	1610	1656	1702	1748	1794	1840	1886	1932	1978	2024									
0.56	1408	1452	1496	1540	1584	1628	1672	1716	1760	1804	1848										
0.58	1302	1344	1386	1428	1470	1512	1554	1596	1638	1680											
0.60	1200	1240	1280	1320	1360	1400	1440	1480	1520												
0.62	1102	1140	1178	1216	1254	1292	1330	1368													
0.64	1008	1044	1080	1116	1152	1188	1224														
0.66	0918	0952	0986	1020	1054	1088															
0.68	0832	0864	0896	0928	0960																
0.70	0750	0780	0810	0840																	
0.72	0672	0700	0728																		
0.74	0598	0624																			
0.76	0528																				

Note: Above values must be multiplied by 10^{-4}

Table 3—Values of $\frac{M_B}{F\,y\,S}\Big|_3 = \frac{-a\,(1-a^2)+c\,(1-c)\,(2-c)}{2\,b}$

c \ a	0.20	0.22	0.24	0.26	0.28	0.30	0.32	0.34	0.36	0.38	0.40	0.42	0.44	0.46	0.48	0.50	0.52	0.54	0.56	0.58	0.60
0.00	-1200	-1342	-1488	-1638	-1792	-1950	-2112	-2278	-2448	-2622	-2800	-2982	-3168	-3358	-3552	-3750	-3952	-4158	-4368	-4582	-4800
0.02	-0982	-1122	-1266	-1414	-1566	-1722	-1882	-2046	-2214	-2386	-2562	-2742	-2926	-3114	-3306	-2502	-3702	-3906	-4114	-4326	-4542
0.04	-0768	-0906	-1048	-1194	-1344	-1498	-1656	-1818	-1984	-2154	-2328	-2506	-2688	-2874	-3064	-3258	-3456	-3658	-3864	-4074	-4288
0.06	-0558	-0694	-0834	-0978	-1126	-1278	-1434	-1594	-1758	-1926	-2098	-2274	-2454	-2638	-2826	-2018	-3214	-3414	-3618	-3826	-4038
0.08	-0352	-1486	-0624	-0766	-0912	-1062	-1216	-1374	-1536	-1702	-1872	-2046	-2224	-2406	-2592	-2782	-2976	-3174	-3376	-3582	-3792
0.10	-0150	-0282	-0418	-0558	-0702	-0850	-1002	-1158	-1318	-1482	-1650	-1822	-1998	-2178	-2362	-2550	-2742	-2938	-3138	-3342	-3550
0.12	0048	-0082	-0216	-0354	-0496	-0642	-0792	-0946	-1104	-1266	-1432	-1602	-1776	-1954	-2136	-2322	-2512	-2706	-2904	-3106	-3312
0.14	0242	0114	-0018	-0154	-0294	-0438	-0586	-0738	-0894	-1054	-1218	-1386	-1558	-1734	-1914	-2098	-2286	-2478	-2674	-2874	-3078
0.16	0432	0306	0176	0042	-0096	-0238	-0384	-0534	-0688	-0846	-1008	-1174	-1344	-1518	-1696	-1878	-2064	-2252	-2448	-2646	-2848
0.18	0618	0494	0366	0234	0098	-0042	-0186	-0334	-0486	-0642	-0802	-0966	-1134	-1306	-1483	-1662	-1846	-2034	-2226	-2422	-2622
0.20	0800	0678	0552	0422	0288	0150	0008	-0138	-0288	-0442	-0600	-0762	-0938	-1098	-1272	-1450	-1632	-1818	-2008	-2202	-2400
0.22	0978	0858	0734	0606	0474	0338	0198	0054	-0094	-0246	-0402	-0562	-0726	-0894	-1066	-1242	-1422	-1606	-1794	-1986	-2182
0.24	1152	1034	0912	0786	0656	0522	0384	0242	0096	-0054	-0208	-0366	-0528	-0694	-0864	-1038	-1216	-1398	-1584	-1774	-1968
0.26	1322	1206	1086	0962	0834	0702	0566	0426	0282	0134	-0018	-0174	-0334	-0498	-0666	-0838	-1014	-1194	-1378	-1566	-1758
0.28	1488	1374	1256	1134	1008	0878	0744	0606	0464	0318	0168	0014	-0144	-0306	-0472	-0642	-0816	-0994	-1176	-1362	-1552
0.30	1650	1538	1422	1302	1178	1050	0918	0782	0642	0498	0350	0198	0042	-0118	-0782	-0450	-0622	-0798	-0978	-1162	-1350
0.32	1808	1698	1584	1466	1344	1218	1088	0954	0816	0674	0528	0378	0224	0066	-0096	-0262	-0432	-0606	-0784	-0966	-1152
0.34	1962	1854	1742	1626	1506	1382	1254	1122	0986	0846	0702	0554	0402	0246	0086	-0078	-0246	-0418	-0594	-0774	-0958
0.36	2112	2006	1896	1782	1664	1542	1416	1286	1152	1014	0872	0726	0576	0422	0264	0102	-0064	-0234	-0408	-0586	-0768
0.38	2258	2154	2046	1934	1818	1698	1574	1446	1314	1178	1038	0894	0746	0594	0438	0278	0114	-0054	-0226	-0402	
0.40	2400	2298	2192	2082	1968	1850	1728	1602	1472	1138	1200	1058	0912	0762	0608	0450	0288	0122	-0048		
0.42	2538	2438	2334	2226	2114	1998	1878	1754	1626	1494	1358	1218	1074	0926	0774	0618	0458	0294			
0.44	2672	2574	2472	2366	2256	2142	2024	1902	1776	1646	1512	1374	1232	1086	0936	0782	0624				
0.46	2802	2706	2606	2502	2394	2282	2166	2046	1922	1794	1662	1526	1386	1242	1094	0942					
0.48	2928	2834	2736	2634	2528	2418	2304	2186	2064	1938	1808	1674	1536	1394	1248						
0.50	3050	2958	2862	2762	2658	2550	2438	2322	2202	2078	1950	1818	1682	1542							
0.52	3168	3078	2984	2886	2784	2678	2568	2454	2336	2214	2088	1958	1824								
0.54	3282	3194	3102	3006	2906	2802	2694	2582	2466	2346	2222	2094									
0.56	3392	3306	3216	3122	3024	2922	2816	2706	2592	2474	2352										
0.58	3498	3414	3326	3234	3138	3038	2934	2826	2714	2598											
0.60	3600	3518	3432	3342	3248	3150	3048	2942	2832												
0.62	3698	3618	3534	3446	3354	3258	3158	3054													
0.64	3792	3714	3632	3546	3456	3362	3264														
0.66	3882	3806	3726	3642	3554	3462															
0.68	3968	3894	3816	3734	3648																
0.70	4050	3978	3902	3822																	
0.72	4128	4058	3984																		
0.74	4202	4134																			
0.76	4272																				

Note: Above values must be multiplied by 10^{-4}

the fixed-end moment due to Table 2 is

$$\begin{aligned} M_{B3} &= +0.24\, FyR \\ &= +0.24\,(100)\,(1.5)\,(1/3) \\ &= +12 \text{ kip-ft.} \end{aligned}$$

For $a = 0.2$, $c = 0.4$ and $S = 2/3$, the fixed-end moment given by Table 3 is

$$\begin{aligned} M_{B4} &= +0.24\, FyS \\ &= +0.24\,(100)\,(1.5)\,(2/3) \\ &= +24 \text{ kip-ft.} \end{aligned}$$

Hence, the fixed-end moment for the beam shown in Fig. 9(a) is obtained by the following algebraic summation.

$$M_B = -12.5 + 72 + 12 + 24 = +95.5 \text{ kip-ft.}$$

If Fig. 9(a) is one span of a two-span continuous beam with equal spans, a concordant cable can be obtained in the following manner. Assume the cable profile can be raised or lowered. Also assume that the cable must keep the same deflected shape as in 9(a). Thus, if only e_A may be varied and R and S must be retained, the values from the tables remain the same as in the previous calculations. The algebraic summation from these three charts is $M_B = 72 + 24 + 12 = 108$ kip-ft. The fixed-end moment when e_A is a variable is

$$M_B = 108 - \frac{Fe_A}{2}$$

$$108 - 50\, e_A \qquad (20a)$$

For the cable to be concordant, the fixed-end moment must be equal to the force in the cable times the distance of the cable from the centroid of the beam. Hence,

$$\begin{aligned} M_B &= +F\,(e_A - 1.00 + 1.50) \\ &= +100\,(e_A + 0.50) \qquad (21) \end{aligned}$$

Set Eq. 20a equal to Eq. 21 and solve for e_A.

$$100\,(e_A + 0.50) = 108 - 50e_A$$

$$e_A = \frac{+58}{150} = 0.387 \text{ ft.}$$

Hence, Fig 9(b) shows a cable profile which is a concordant cable for a two span continuous beam of equal spans. The beam and cable profile is symmetrical about support B.

Fig. 10 shows a symmetrical cable profile for an interior span composed of straight line segments with sharp bends.

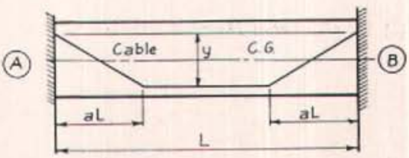
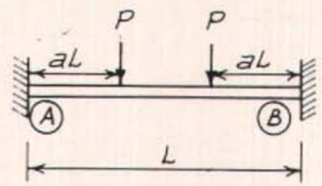


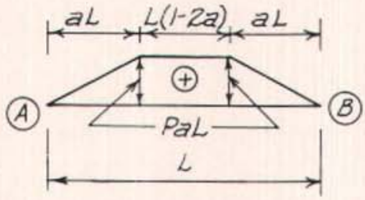
Fig. 10—Symmetrical Straight Line Segment Profile for Interior Span

In a three-span continuous unit, the moment curves will usually be symmetrical about the center-line of the middle span. In a four-span continuous unit, the point of maximum positive moment will not occur exactly at the centerline of the interior span. However, the maximum positive moment will occur closer to the centerline of the interior span than it will in the exterior span. Only a cable profile which is symmetrical about the centerline of span will be considered.

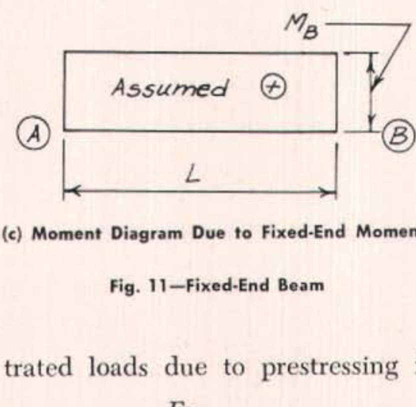
Two equivalent concentrated loads due to prestressing will be created by the cable profile shown in Fig. 10. These loads will be located at a distance aL from each support. Fig. 11(a) shows equal positive, i.e., downward, loads acting on the beam. Applying the dimensions shown in Fig. 10 with Eq. 2, the magnitude of the equivalent concen-



(a) Fixed-End Beam with Symmetrical Concentrated Loads



(b) Simple Beam Moment Diagram



(c) Moment Diagram Due to Fixed-End Moments

Fig. 11—Fixed-End Beam

trated loads due to prestressing is

$$P = -\frac{Fy}{aL} \qquad (22)$$

Fig. 11(b) shows the moment diagram for a simple beam with two equal concentrated loads at a distance aL from each end. Fig. 11(c) shows the moment diagram which is created by equal end moments. When a beam is fixed at each end and subjected to the symmetrical loads shown in Fig. 11(a), the total moment diagram for the beam can be obtained by combining the moment diagrams shown in Figs. 11(b) and 11(c) to make the total area equal to zero (zero angle change from A to B). The fixed-end moments are then

$$M_A = M_B = -PaL(1-a) \qquad (23)$$

Substitute Eq. 22 into Eq. 23,

$$M_A = M_B = +Fy(1-a) \qquad (24)$$

Eq. 24 gives the fixed-end moments due to prestressing for the cable profile shown in Fig. 10.

CABLE PROFILE PARABOLIC CURVES

Fig. 12 shows a portion of a beam with a constant cross-section that has a parabolic shaped cable profile. Section B-B is passed through the beam where the slope of the parabola is horizontal. Section A-A is a section which is a distance kL from Section B-B. The equation of the parabola with respect to its horizontal tangent at Section B-B is $z = cx^2$.

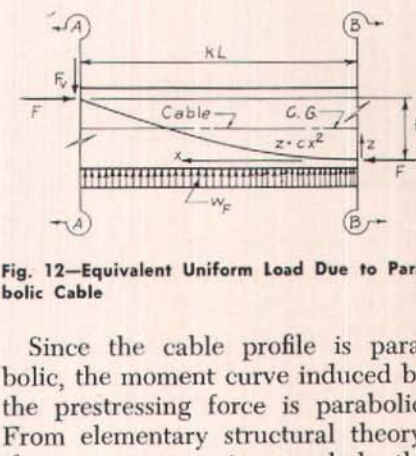


Fig. 12—Equivalent Uniform Load Due to Parabolic Cable

Since the cable profile is parabolic, the moment curve induced by the prestressing force is parabolic. From elementary structural theory, if a moment curve is a parabola, the load condition which produces it is a uniform load. Again it is assumed that the loss due to friction is negligible and the horizontal component of the cable force taken is equal to the total force on the cable. Section B-B is a point of zero shear because the slope of the moment curve, like the cable, is horizontal when the beam is considered under the action of the prestressing force alone. Let w_F be the equivalent uniform load due to prestressing. F is the force in the cable, t is the vertical rise of the parabola, and kL is the horizontal length of the parabola being considered. Take moments at Section A-A at the point of the cable and set this moment summation equal to zero.

$$\Sigma M = Ft - \frac{w_F k^2 L^2}{2} = 0 \qquad (25)$$

Solve Eq. 25 for w_F.

$$w_F = \frac{2\,Ft}{k^2 L^2} \qquad (26)$$

Fig. 13 shows an end span with a cable profile composed of three different parabolas. The cable pro-

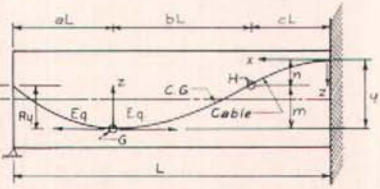


Fig. 13—Possible Parabolic Profile for End Span

file is continuous and the parabolas have a common tangent at their points of intersection at G and H. The equations for these curves are

$$z = \frac{x^2\,Ry}{a^2 L^2} \qquad (27)$$

$$z = \frac{x^2\,m}{b^2 L^2} \qquad (28)$$

and

$$z = \frac{x^2\,n}{c^2 L^2} \qquad (29)$$

Eq. 27 and 28 have horizontal tangents at their origins, i.e., $x = 0$, which is the common point between the two curves. Eq. 29 has a horizontal tangent at the fixed end of the beam. For Eq. 28 and 29 to have a common tangent at their intersection, the derivatives of each curve must be equal at point H. The derivative of Eq. 28 at $x = bL$ is

$$\frac{dz}{dx} = \frac{2mx}{b^2 L^2} = \frac{2m}{bL} \qquad (30)$$

The derivative of Eq. 29 at $x = cL$ is

$$\frac{dz}{dx} = \frac{2nx}{c^2 L^2} = \frac{2\,n}{c\,L} \qquad (31)$$

Equate Eqs. 30 and 31 and combine terms,

$$\frac{m}{n} = \frac{b}{c} \qquad (32)$$

From Fig. 13, it can be seen that $a + b + c = 1$ and $m + n = y$. Thus with these two equations and Eq. 32, m and n can be expressed in terms of c, b, a and y.

$$n = y - m = y - \frac{nb}{c} =$$

$$\frac{yc}{a + b} = \frac{yc}{(1 - c)} \qquad (33)$$

and

$$m = y - n = y - \frac{cm}{b} =$$

$$\frac{yb}{c + b} = \frac{yb}{(1 - a)} \qquad (34)$$

Let w_a be the equivalent uniform load over the aL portion of the span, w_b the equivalent uniform load over the bL portion of the span, and w_c the equivalent uniform load over the cL portion of the span. The uniform loads are considered positive when acting in a downward direction. Using the above load notation and the dimensions shown in Fig. 13, the Eqs. 26, 33 and 34, the equations for the equivalent uniform loads are

$$w_a = -\frac{2 F R y}{a^2 L^2} \quad (35)$$

$$w_b = -\frac{2 F y}{b(1-a) L^2} \quad (36)$$

and

$$w_c = +\frac{2 F y}{c(1-a) L^2} \quad (37)$$

Equations for the fixed-end moments of a propped cantilever beam with a uniform load over portions of the span will now be derived. Fig. 7 shows a propped cantilever beam with a concentrated load at x distance from the hinged-end. The equation for the fixed-end moment for the beam in Fig. 7 is

$$M_B = -\frac{Px(L^2 - x^2)}{2 L^2} \quad (6)$$

Fig. 14 shows a method of expressing a uniform load as an infinite number of concentrated loads along a beam. The magnitude of these con-

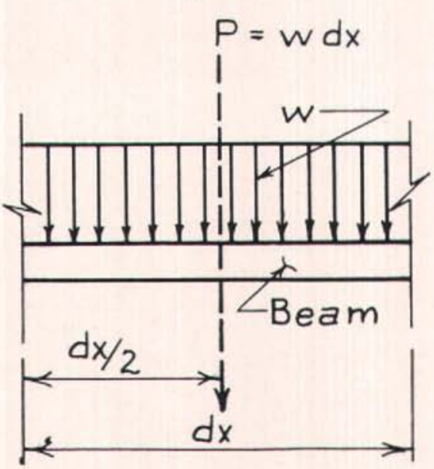


Fig. 14—Method of Expressing Uniform Load as Infinite Number of Concentrated Loads

centrated loads is $P = w\,dx$, where w is the uniform load on the beam and dx is an infinitely small length of the beam. Substitute this value of the concentrated load into Eq. 6.

$$M_B = -\frac{(wdx)\,x(L^2 - x^2)}{2 L^2} \quad (38)$$

Eq. 38 is the equation for the fixed-end moment of a propped cantilever beam due to a uniform load over an infinitely small length of beam at a distance x from the hinged-end. The equation for fixed-end moment due to a uniform load over a portion of the span can be obtained by integrating Eq. 38 between the limits of the load.

In Eq. 38, let $w = w_a$ and integrate between the limits of $x = 0$ and $x = aL$.

$$M_B = \int_o^{aL} \frac{w_a\,x(L^2 - x^2)\,dx}{2 L^2}$$

$$= -\frac{w_a}{2 L^2}\left[\frac{L^2 x^2}{2} - \frac{x^4}{4}\right]_o^{aL}$$

$$= -\frac{w_a L^2 a^2}{8}(2 - a^2) \quad (39)$$

Eq. 39 is the equation for the fixed-end moment at B for the beam in Fig. 15(a).

Let $w = w_b$ in Eq. 38. Integrate Eq. 38 between the limits of $x = aL$ and $x = (a + b)L$.

$$M_B = \int_{aL}^{(a+b)L} \frac{-w_b\,x(L^2 - x^2)\,dx}{2 L^2} =$$

$$\frac{w_b}{2 L^2}\left[\frac{L^2 x^2}{2} - \frac{x^4}{4}\right]_{aL}^{(a+b)L} =$$

$$-\frac{w_b}{2 L^2}\,[2L^4(a+b)^2 -$$

$$(a+b)^4 L^4 - 2a^2 L^4 + a^4 L^4] =$$

$$-\frac{w_b L^2}{8}[(a+b)^2[2-(a+b)^2] -$$

$$a^2(2-a^2)] \quad (40)$$

Eq. 40 is the fixed-end moment at B for the beam in Fig. 15(b).

The equation of the fixed-end moment for the beam in Fig. 15(c)

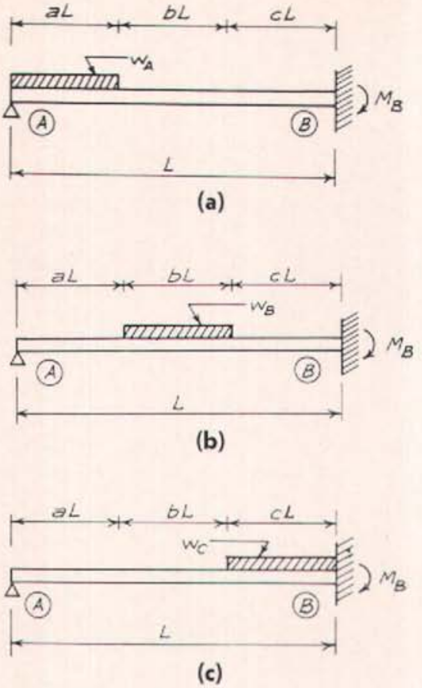


Fig. 15—Three Loading Conditions for Uniform Load Applied to a Propped Cantilever Beam

will now be derived. Let $w = w_c$ in Eq. 38. Integrate Eq. 38 between the the limits of $x = (1 - c)L$ and $x = L$.

$$M_B = \int_{(1-c)L}^{L} - \frac{w_c\, x\,(L^2 - x^2)\, dx}{2\, L^2}$$

$$= \frac{w_c}{2\, L^2}\left[\frac{L^2\, x^2}{2} - \frac{x^4}{4}\right]_{(1-c)L}^{L}$$

$$= -\frac{w_c\, L^2}{8}\left[1 - (1 - c)^2\right]^2$$

$$= -\frac{w_c\, L^2\, c^2\,(2 - c)^2}{8} \qquad (41)$$

The equation for the fixed-end moment of a propped cantilever beam due to an applied moment at the hinged-end has appeared previously. This equation was

$$M_B = -\frac{m}{2} = -\frac{Fe_A}{2} \qquad (9)$$

Add algebraically Eqs. 39, 40, 41 and 9.

$$M_B = \frac{w_a\, L^2\, a^2\,(2 - a^2)}{8} - \frac{w_b\, L^2}{8} \left[(a+b)^2[2-(a+b)^2] - a^2(2-a^2)\right] - \frac{w_c\, L^2\, c^2\,(2 - c)^2}{8} - \frac{m}{2} \qquad (42)$$

Substitute Eqs. 35, 36 and 37 into Eq. 42.

$$M_B = \frac{Fy}{2}\left[(2-a^2)\left(R - \frac{a^2}{b(1-a)}\right) + \frac{(a+b)^2[2-(a+b)^2]}{b(1-a)} - \frac{c(2-c)^2}{(1-a)}\right] - \frac{Fe_A}{2} \qquad (43)$$

If the cable location at A coincides with the centroid of the beam, $e_A = 0$. If $e_A = 0$, Eq. 43 reduces to

$$M_B = +\frac{Fy}{4}\left[(2-a^2)\left(R - \frac{a^2}{b(1-a)}\right) + \frac{(a+b)^2\,[2-(a+b)^2]}{b(1-a)} - \frac{c(2-c)^2}{(1-a)}\right] \qquad (44)$$

Substitute $a + b = 1 - c$ into the middle term of Eq. 44.

$$M_B = \frac{Fy}{4}\left[(2-a^2)\left(R - \frac{a^2}{b(1-a)}\right) + \frac{(1-c)^2\,(1+2c-c^2)}{b(1-a)} - \frac{c(2-c)^2}{(1-a)}\right] \qquad (45)$$

Eq. 44 or Eq. 45 will be used to develop design tables that will enable a rapid calculation of M_B. A correction can easily be made if e_A does not equal zero.

Design tables for Eq. 44 or Eq. 45 are Tables 4 and 5. In calculus it was shown that

$$\Delta M_B = \left(\frac{\partial M_B}{\partial R}\right)\Delta R \qquad (46)$$

In Eq. 46, ΔM_B is the change in the fixed-end moment at B, ΔR is the change in the variable R, and $\frac{\partial M_B}{\partial R}$ is the partial derivative of M_B with respect to R. The partial derivative of Eq. 45 is

$$\frac{\partial M_B}{\partial R} = \frac{Fy}{4}(2 - a^2) \quad (47)$$

Substitute Eq. 47 into Eq. 46,

$$\Delta M_B = \frac{+Fy}{4}(2 - a^2)(\Delta R) \quad (48)$$

If F, y and a are constant in Eq. 48, then the change in M_B is proportional to the change in R. Eq. 48 reveals that if the value of a is constant, R produces a linear effect on the fixed-end moment at B.

For Tables 4 and 5, straight line interpolation may be used with the same accuracy that occurs in Tables 1, 2 and 3. From Tables 4 and 5, it is observed that if the variables, R and a, are held constant, M_B decreases when c is increased. If the variables, R and c, are held constant, M_B increases when a is increased.

Parme and Paris[2] have derived a formula for the fixed-end moments due to prestressing for an interior span. The cable profile is made of parabolas and is symmetrical about the centerline of the interior span. The equation which they derived, expressed in the notation of Fig. 16 is

$$M_A = M_B = + \frac{Fy(1-a)}{1.5} \quad (49)$$

Beam sign conventions apply to Eq. 49.

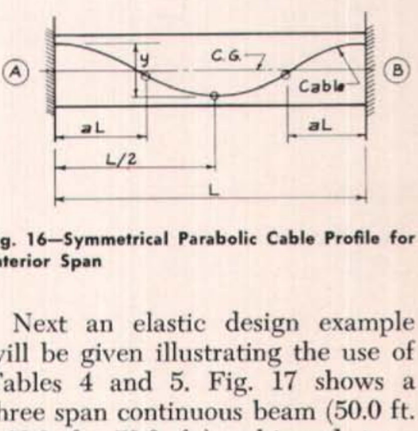


Fig. 16—Symmetrical Parabolic Cable Profile for Interior Span

Table 4

Solution of $\frac{\Delta M_B}{Fy(\Delta R)} = \frac{(2 - a^2)}{4}$

a	$\frac{2-a^2}{4}$
0.35	0.469375
0.36	0.467600
0.37	0.465775
0.38	0.463900
0.39	0.461975
0.40	0.460000
0.41	0.457975
0.42	0.455900
0.43	0.453775
0.44	0.451600
0.45	0.449375
0.46	0.447100
0.47	0.444775
0.48	0.442400
0.49	0.439975
0.50	0.437500

Next an elastic design example will be given illustrating the use of Tables 4 and 5. Fig. 17 shows a three span continuous beam (50.0 ft.–60.0 ft.–50.0 ft.) subjected to a uniform load of 600 lbs. per foot and a uniform live load of 1000 lbs. per foot. For simplicity, no partial span loading of the live load is used. Fig. 18 shows the maximum and minimum moment curves for this beam.

Fig. 19 shows the selected beam which has a kern distance of 7.0 inches. The limiting zone for the center of pressure on concrete is

Table 5

$$\text{Solution of } \frac{M_B}{F\,y} = \frac{1}{4}\left[-\frac{a^2(2-a^2)}{b(1-a)} + \frac{(1-c)^2(1+2c-c^2)}{b(1-a)} - \frac{c(2-c)^2}{(1-a)}\right]$$

c	a															
	0.35	0.36	0.37	0.38	0.39	0.40	0.41	0.42	0.43	0.44	0.45	0.46	0.47	0.48	0.49	0.50
.00	4556	4624	4692	4761	4830	4900	4970	5041	5112	5184	5256	5329	5402	5476	5550	5625
.01	4473	4540	4608	4677	4746	4815	4885	4956	5027	5098	5170	5243	5316	5389	5463	5538
.02	4390	4457	4524	4593	4662	4731	4801	4871	4942	5013	5085	5157	5230	5303	5377	5451
.03	4307	4374	4442	4510	4578	4647	4717	4787	4857	4928	5000	5072	5144	5217	5291	5365
.04	4225	5292	4359	4427	4495	4565	4633	4703	4773	4844	4915	4987	5059	5132	5205	5279
.05	4144	4210	4277	4345	4413	4481	4550	5620	4690	4760	4831	4903	4975	5047	5120	5194
.06	4063	4129	4196	4263	4331	4399	4468	4537	4607	4677	4748	4819	4891	4963	5036	5109
.07	3982	4048	4115	4182	4249	4317	4386	4455	4524	4594	4665	4736	4807	4879	4952	5025
.08	3902	3968	4034	4101	4168	4236	4304	4373	4442	4512	4582	4653	4724	4796	4868	4941
.09	3823	3888	3954	4021	4088	4155	4223	4292	4361	4430	4500	4571	4642	4713	4785	4858
.10	3744	3809	3875	3941	4008	4075	4143	4211	4280	4349	4419	4489	4560	4631	4703	4775
.11	3665	3730	3796	3862	3928	3995	4063	4131	4199	4268	4338	4408	4478	4549	4621	4693
.12	3587	3652	3717	3783	3849	3916	3983	4051	4119	4188	4257	4327	4397	4468	4539	4611
.13	3510	3574	3639	3705	3771	3837	3904	3972	4040	4108	4179	4247	4316	4387	4458	4530
.14	3433	3497	3562	3627	3693	3759	3826	3893	3961	4029	4098	4167	4236	4307	4378	4449
.15	3356	3420	3485	3550	3615	3681	3748	3815	3882	3950	4019	4088	4157	4227	4298	4369
.16	3280	3344	3408	3473	3538	3604	3671	3737	3804	3872	3940	4009	4078	4148	4218	4289
.17	3205	5268	3332	3397	3462	3527	3593	3660	3727	3794	3862	3921	4000	4069	4139	4210
.18	3130	3193	3257	3321	3386	3451	3517	3583	3650	3717	3785	3853	3922	3991	4061	4131
.19	3055	3118	3182	3246	3310	3375	3441	3507	3573	3640	3708	3776	3844	3913	3983	4053
.20	3981	3044	3107	3171	3235	3300	3365	3431	3497	3564	3631	3699	3767	3836	3905	3975
.21	3908	3970	3033	3097	3161	3225	3290	3356	3422	3488	3555	3623	2691	3759	3828	3898
.22	2835	2897	2960	3023	3087	3151	3216	3281	3347	3413	3480	3547	3615	3683	3752	3821
.23	2762	2824	2887	2950	3013	3077	3142	3207	3272	3338	3405	3472	3539	3607	3676	3745
.24	2690	2752	2814	2877	2940	3004	3068	3133	3198	32y4	3330	3397	3464	3532	3600	3669
.25	2619	2680	2742	2805	2868	2931	2995	3060	3125	3190	3256	3323	3390	3457	3525	3594
.26	2548	2609	2671	2733	2796	3859	2923	2987	3052	3117	3183	3249	3316	3383	3451	3519
.27	2477	2538	2600	2662	2724	2787	2851	2915	2979	3044	3110	3176	3242	3309	3377	3445
.28	2407	2468	2529	2592	2653	2716	2779	3843	2907	2972	3037	3103	3169	3236	3303	3371
.29	2338	2398	2459	2521	2583	2645	2708	2772	2836	2900	2965	3031	3097	3163	3230	3298
.30	2269	2329	2390	2451	2513	2575	2638	2701	2765	2829	2894	2959	3025	3091	3158	3225
.31	2200	2260	2321	2382	2443	2505	2568	2631	2694	2758	2823	2888	2953	3019	3086	3153
.32	2132	2192	2252	2313	2374	2436	2498	2561	2624	2688	2752	2817	2882	2948	3014	3081
.33	2065	2124	2184	2245	2306	2367	2429	2492	2555	2618	2682	2747	2812	2877	2943	3010
.34	2000	2057	2117	2177	2239	2299	2361	2423	2486	2549	2613	2677	2742	2807	2873	2939
.35	1931	1990	2050	2110	2170	2231	2293	2355	2417	2480	2544	2608	2672	2737	2803	2869
.36	1865	1924	1983	2043	2103	2164	2225	2287	2349	2412	2475	2539	2603	2668	2733	2799
.37	1800	1858	1917	1977	2037	2097	2158	2210	2282	2344	2407	2471	2535	2599	2664	2730
.38	1735	1793	1852	1911	1971	2031	2092	2153	2215	2277	2340	2403	2467	2531	2596	2661
.39	1670	1728	1787	1846	1905	1965	2026	2087	2148	2210	2273	2336	2399	2463	2528	2593
.40	1606	1664	1722	1781	1840	1900	1960	2021	2082	2144	2206	2269	2332	2396	2460	2525
.41	1543	1600	1658	1717	1776	1835	1895	1956	2017	2078	2140	2203	2266	2329	2393	2458
.42	1480	1537	1595	1653	1712	1771	1831	1891	1952	2013	2075	2137	2200	2263	2327	2391
.43	1417	1474	1532	1590	1648	1707	1767	1827	1887	1948	2010	2072	2134	2197	2261	2325
.44	1355	1412	1469	1527	1585	1644	1703	1763	1823	1884	1945	2007	2069	2132	2195	2259
.45	1294	1350	1407	1465	1523	1581	1640	1700	1760	1820	1881	1943	2005	2067	2130	2194
.46	1233	1289	1346	1403	1461	1519	1578	1637	1697	1757	1818	1879	1941	2003	2066	
.47	1172	1228	1285	1342	1399	1457	1516	1575	1634	1694	1755	1816	1877	1939		
.48	1112	1168	1224	1281	1338	1396	1454	1513	1572	1632	1692	1753	1814			
.49	1053	1108	1164	1221	1278	1335	1393	1452	1511	1570	1630	1691				
.50	0994	1049	1105	1161	1219	1275	1333	1391	1450	1509	1569					
.51	0935	0990	1046	1102	1158	1215	1273	1331	1389	1448						
.52	0877	0932	0987	1043	1099	1156	1213	1271	1329							
.53	0820	0874	0929	0985	1041	1097	1154	1212								
.54	0763	0817	0872	0927	0983	1039	1096									
.55	0706	0760	0815	0870	0925	0981										
.56	0650	0704	0758	0813	0868											
.57	0595	0648	0702	0757												
.58	0540	0593	0647													
.59	0485	0538														
.60	0431															

Note: Above values must be multiplied by 10-4

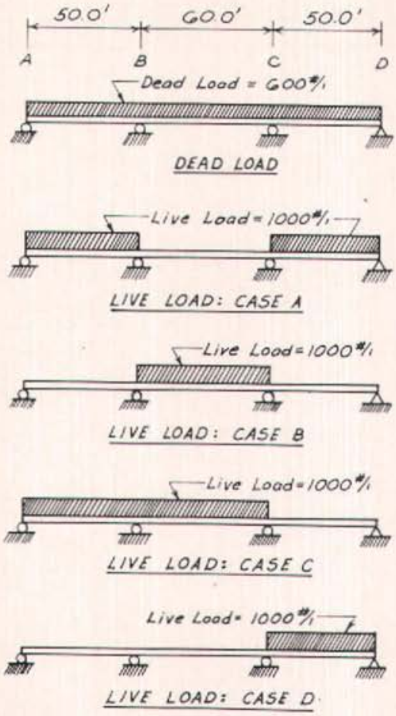


Fig. 17—Design Loads for Three Span Continuous Beam

also designated. No tension stress is permitted in this beam.

A concordant cable consisting entirely of parabolas will be determined for this beam. For this design a prestressing force of 450 kips will be used. By inspecting Fig. 19, the following values have been chosen for span AB: $a = 0.42$, $c = 0.08$, $R = 0.4$, $y = 1.073$ ft.

From Table 4, when $a = 0.42$, the value is 0.455900 and from Table 5 when $a = 0.42$ and $c = 0.08$, the value is 0.4373. Hence the constant to calculate the fixed-end moment at end B of span AB is $0.4373 + 0.4$ (0.455900) 0.6197. Thus the fixed-end moment is

$$M = 0.6197\,Fy$$
$$= 0.6197\,(450)\,(1.073)$$
$$= 300 \text{ kip-ft.}$$

The stiffness of span AB at joint B is a $3\,EI/L$ member and due to symmetry of loading, the stiffness of span BC is a $2\,EI/L$ member. Hence, the distribution factors in the moment distribution analysis is 0.643 for member BA and 0.357 for member BC.

Fig. 19 shows that the cable is located 0.644 ft. above the centroid of the beam. For the cable to be concordant, the final moment must equal $450\ (0.644) = 290$ kip-ft. The final moment is now known and the fixed-end moment in span AB is known. Now a cable profile and a fixed-end moment in span BC must be chosen such that the cable will be concordant. Let v be the unknown fixed-end moment in span BC and solve for v using moment distribution (see Fig. 20).

Thus

$$+290 = +\,v + (300\text{-}v)0.357$$
$$v = 284 \text{ kip.-ft.}$$

From Eq. 23, the cable profile in span BC is determined.

$$v = 284 = \frac{Fy\,(1-a)}{1.5} = \frac{450\,y\,(1-a)}{1.5} = 300\,y\,(1-a)$$

The value of a is selected to be 0.08. Hence,

$$y = \frac{284}{300\,(0.92)} = 1.03 \text{ ft.}$$

Fig. 19 shows a concordant cable for the beam and presents an elastic design solution.

CONCLUSION

Data presented in Tables 1 and 4 are parabolic data. Also data presented in Tables 3 and 5 are parabolic in both the a and c directions. Data presented in Table 2 are parabolic in the c direction and linear in the a direction. When data are

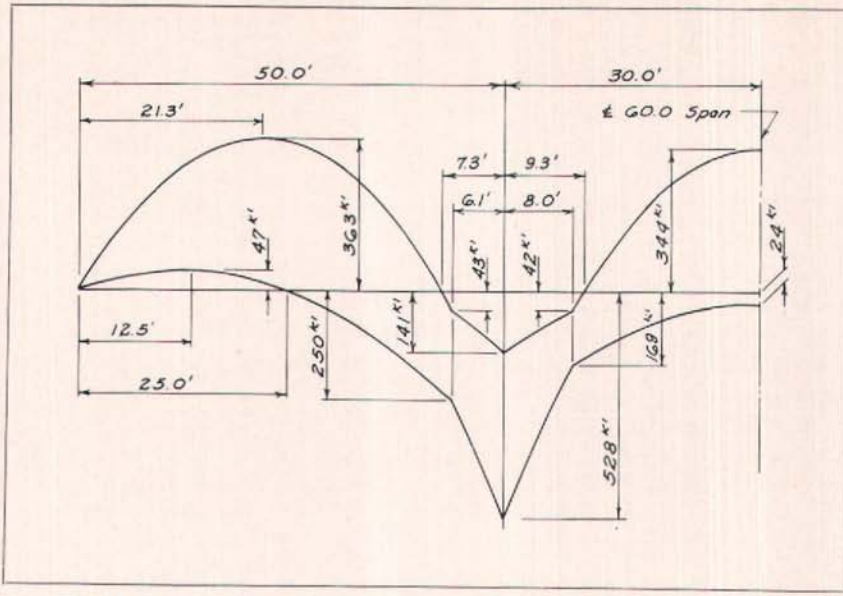


Fig. 18—Maximum and Minimum Moment Curves for Three Span Continuous Beam

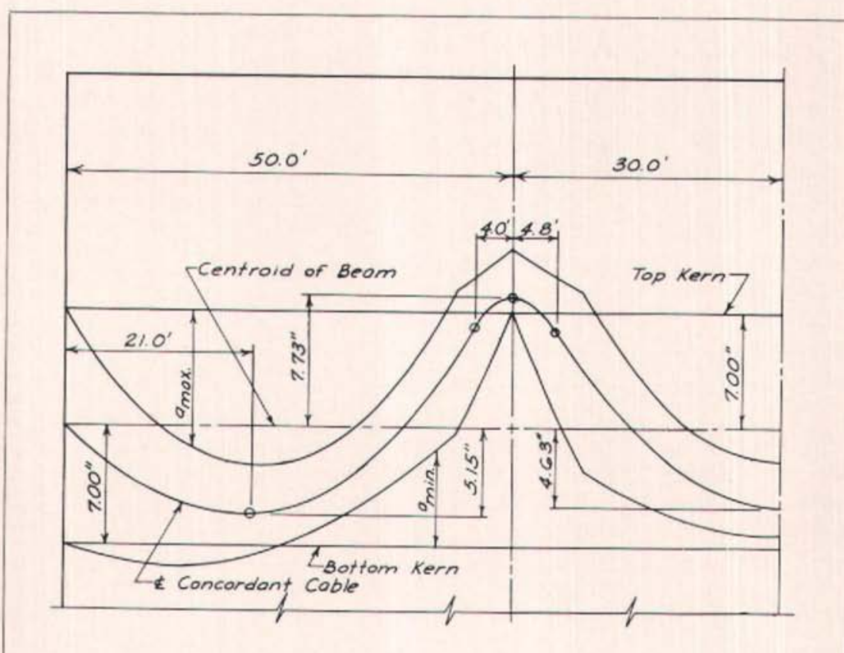


Fig. 19—Limiting Zone and Cable Profile for Three Span Continuous Beam

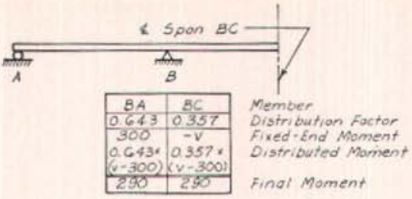


Fig. 20—Moment Distribution Calculation to Determine Concordant Cable

parabolic, exact values may be calculated by use of parabolic interpolation coefficients which may be developed by using Lagrange's Interpolation Polynominals. For those not familiar with this method and for those who do not feel that this refinement is necessary, straight line interpolation will give accuracy equal to or less than 1 in the fourth decimal place.

Equations and data for fixed-end moments for the end-spans of continuous units have been computed and derived for a beam fixed at one end and simply supported at the other end. These boundary conditions must be considered when using these fixed-end moments in the structural anlaysis of the beam.

The tables and equations presented in this paper offer the designer great flexibility in selecting his cable profile and a rapid method of calculating fixed-end moments due to prestressing for further use in structural analysis.

ACKNOWLEDGEMENT

The writers would like to acknowledge the Computation Center at the University of Florida, Gainesville, Florida for making possible the computation of the tables presented in this paper. Thanks is due to Dr. A. A. Toprac for reading this material and making constructive suggestions when it was submitted as a Master's Thesis.

BIBLIOGRAPHY

1. Parme, Alfred L. and Paris, George H., "Analysis of Continuous Prestressed Concrete Structures", *Structural and Railways Bureau*, Portland Cement Association, Chicago.
2. Parme, Alfred L. and Paris, George H., "Designing for Continuity in Prestressed Concrete Structures", *Journal of the American Concrete Institute*, Vol. 23, September 1951, p. 61.
3. Fiesenheiser, E. I., "Rapid Design of Continuous Prestressed Members", *Journal of the American Concrete Institute*, Vol. 50, April 1954, p. 673.
4. Moorman, Robert B., "Continuous Prestressing", *Transactions of the American Society of Civil Engineers*, Vol. 121, 1956, p. 815.
5. Lin, T. Y., *Design of Prestressed Concrete Structures*, John Wiley and Sons, Inc., New York, 1955, pp. 314-318.
6. Lin, T. Y., "Load-Balancing Method for Design and Analysis of Prestressed Concrete Structures", *Journal of the American Concrete Institute*, Vol. 60, June 1963, p. 719.

Portions of this paper were presented to The University of Texas in partial fulfillment of the requirements for the degree of Master of Science in Civil Engineering.

Discussion of this paper is invited. Please forward your discussion to PCI Headquarters before May 1 to permit publication in the August 1966 issue of the PCI JOURNAL.