The work of Robert MacPherson

along with a huge supporting cast of coauthors:

Paul Baum, Sasha Beilinson, Walter Borho, Tom Braden, Jean-Paul Brasselet, Jean-Luc Brylinski, Jeff Cheeger, Robert Coveyou, Corrado DeConcini, Pierre Deligne, Bill Fulton, Israel Gelfand, Sergei Gelfand, Mark Goresky, Dick Hain, Masaki Hanamura, Gunter Harder, Lizhen Ji, Steve Kleiman, Bob Kottwitz, George Lusztig, Mark MacConnell, Arvind Nair, Claudio Procesi, Vadim Schechtman, Vera Serganova, Frank Sottile, Berndt Sturmfels, and Kari Vilonen

and students:

Paul Dippolito (Brown 1974), Mark Goresky (B 1976), Thomas DeLio (B 1978—in music!), David Damiano (B 1980), Kari Vilonen (B 1983), Kevin Ryan (B 1984), Allen Shepard (B 1985), Mark McConnell (B 1987), Masaki Hanamura (B 1988), Wolfram Gerdes (MIT 1989), David Yavin (M 1991), Yi Hu (M 1991), Richard Scott (M 1993), Eric Babson (M 1994), Paul Gunnells (M 1994), Laura Anderson (M 1994), Tom Braden (M 1995), Mikhail Grinberg (Harvard 1997), Francis Fung (Princeton 1997), Jared Anderson (P 2000), David Nadler (P 2001), Julianna Tymoczko (P 2003), Brent Doran (P 2003), Aravind Asok (P 2004)

and I suppose a good bunch of these eager critics are sitting in this room today.

Born — Lakewood, Ohio — May 25, 1944

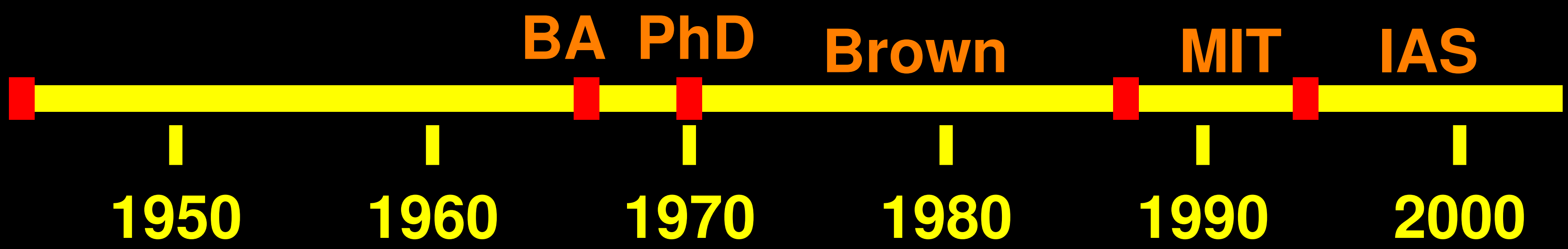
B.A. — Swarthmore College 1966

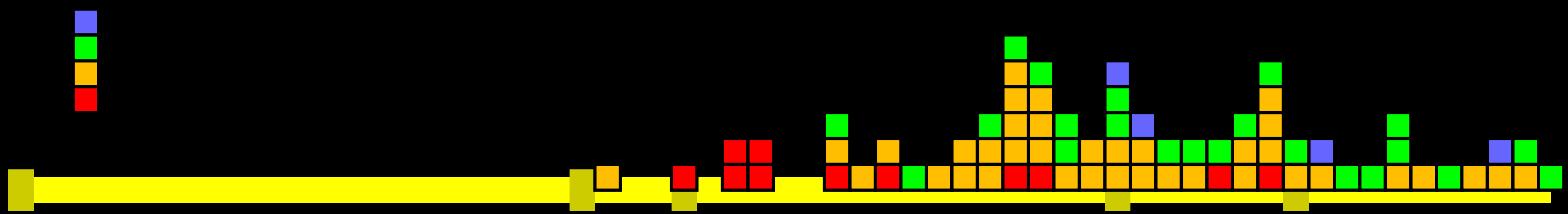
Ph.D. — Harvard University 1970

Brown University — 1970 – 1987

M.I.T. — 1987 – 1994

I.A.S. — 1994 –


BA
PhD
Brown
MIT
IAS
1950
1960
1970
1980
1990
2000



Publication record
(colour coded by the number of authors)

Part I. Test for random number generators

[105] ***Fourier Analysis of Uniform Random Number Generators***, with R. R. Coveyou, *Journal of Association for Computing Machinery*, 1967

Knuth in his book TAOCP: . . . *an especially important way to check the quality of linear congruential random number generators. Not only do good generators pass this test, all generators now known to be bad actually fail it. Thus it is by far the most powerful test known* . . .

Knuth in his editorial report:

. . . *almost too good to be true!*

Coveyou & MacPherson: *Known . . . tests so far have been for such weak statistical properties that generators designed to pass such tests have, more often than not, failed . . . it is a little ironic that the performance of [generators chosen randomly] is generally better than those chosen for plausible theoretical reasons.*

. . . generally better than those chosen with malice aforethought.

Knuth: the words with malice aforethought are not really appropriate . . . say instead chosen carefully from theoretical considerations.

Knuth in his editorial report: **there are places in which the text is a little too flippant, although in most cases I found the humor was well-managed.**

The generation of random numbers is too important to be left to chance. **Apparently first quoted by Martin Gardner in 1969.**

Internet obituary: *Robert Coveyou passed away February 19 [1996] in Oak Ridge. He was an original member of the small group of radiation protection specialists at the University of Chicago who first adopted the title of Health Physicist. First and foremost, Robert Coveyou was a mathematician . . . An avid chess player . . . and on a number of occasions an opponent of a young kid by the name of Bobby Fischer.*

(He was Tennesee State Chess Champion
1947 1954/55 55/56 58 59 61 63 68)

David Trubey, colleague at Oak Ridge: ***Bob Coveyou? He was quite a character . . . [In church one day in about 1960] I noticed that in the year 2000, Easter fell on my birthday. Not long afterward, I remarked to Coveyou that Easter fell on my birthday in the year 2000. He replied that that was the most useless information that he had ever been given!***

He originally came to Oak Ridge as a health physicist. . . . I remember one time when a senior member of our Division came into his office . . . and asked Coveyou what he was doing. [Coveyou] spent much time writing equations on a big pad of paper. He told the visitor, "You would not understand if I told you" . . . our Division Director called Coveyou and his protégées his 'Baker Street Irregulars' after the street urchins that kept Sherlock Holmes informed. [our director's] secretary said [the director] took pride in hiring eccentric people . . .

References

Coveyou, Studies in Applied Mathematics 3, 1969

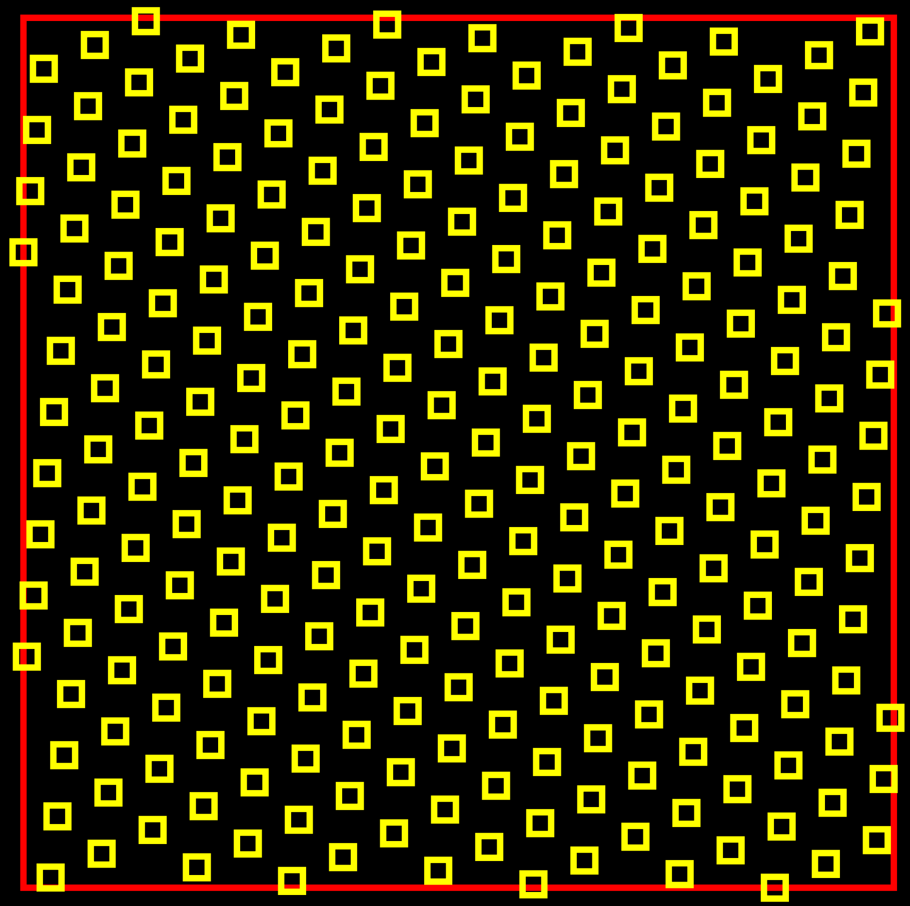
Knuth's The Art of Computer Programming, volume II. *The different editions are significantly different in their discussions of the 'spectral test'. The first edition is far closer to the original paper, but is somewhat less intuitive, without pictures. Perhaps more interesting.*

`http://random.mat.sbg.ac.at/tests/theory/spectral/`

Simple linear congruential generator:

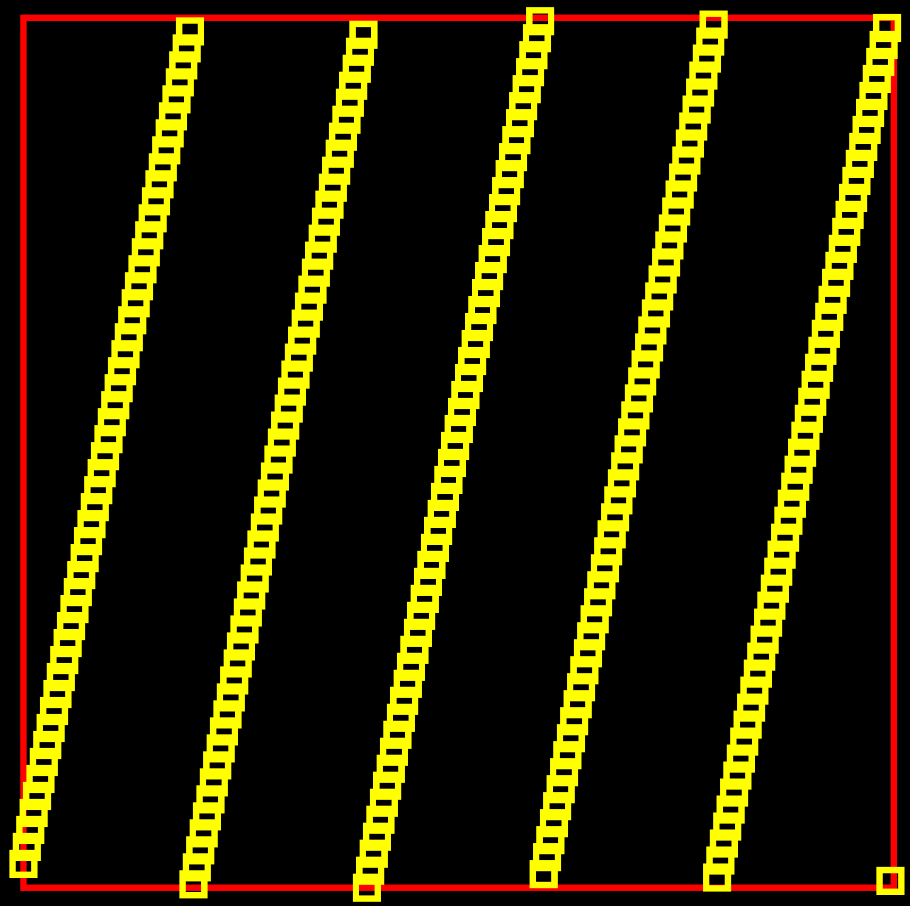
$$c_{n+1} = ac_n + b \pmod{m}$$
$$C_n = c_n/m$$

simulates numbers randomly distributed in $[0, 1)$. The first natural requirement is that this generate all numbers modulo m—i.e. that its period be m.



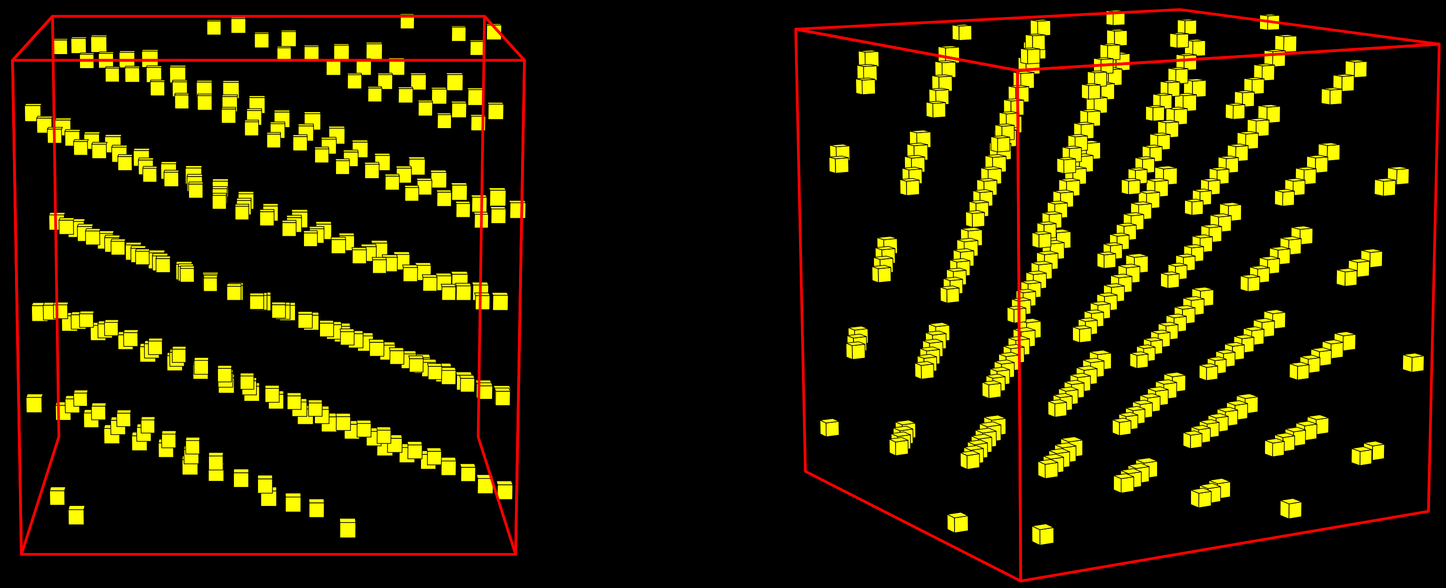
$$c_{n+1} = 137c_n + 187 \pmod{256}$$

Plotting pairs (c_n, c_{n+1})**.**



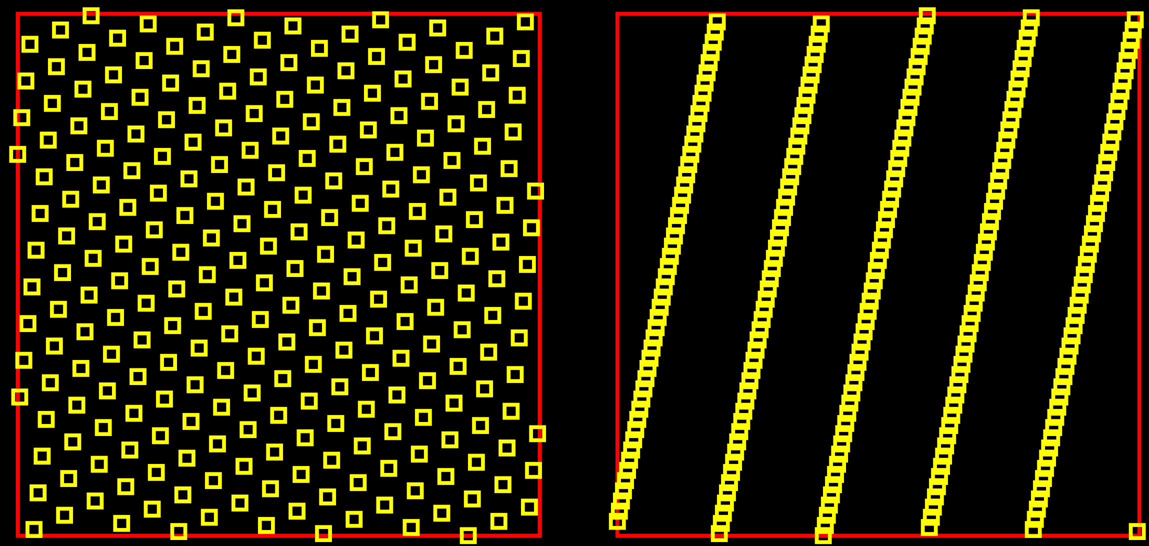
$$c_{n+1} = 5c_n + 7$$

A bad generator.



$$c_{n+1} = 137c_n + 187$$

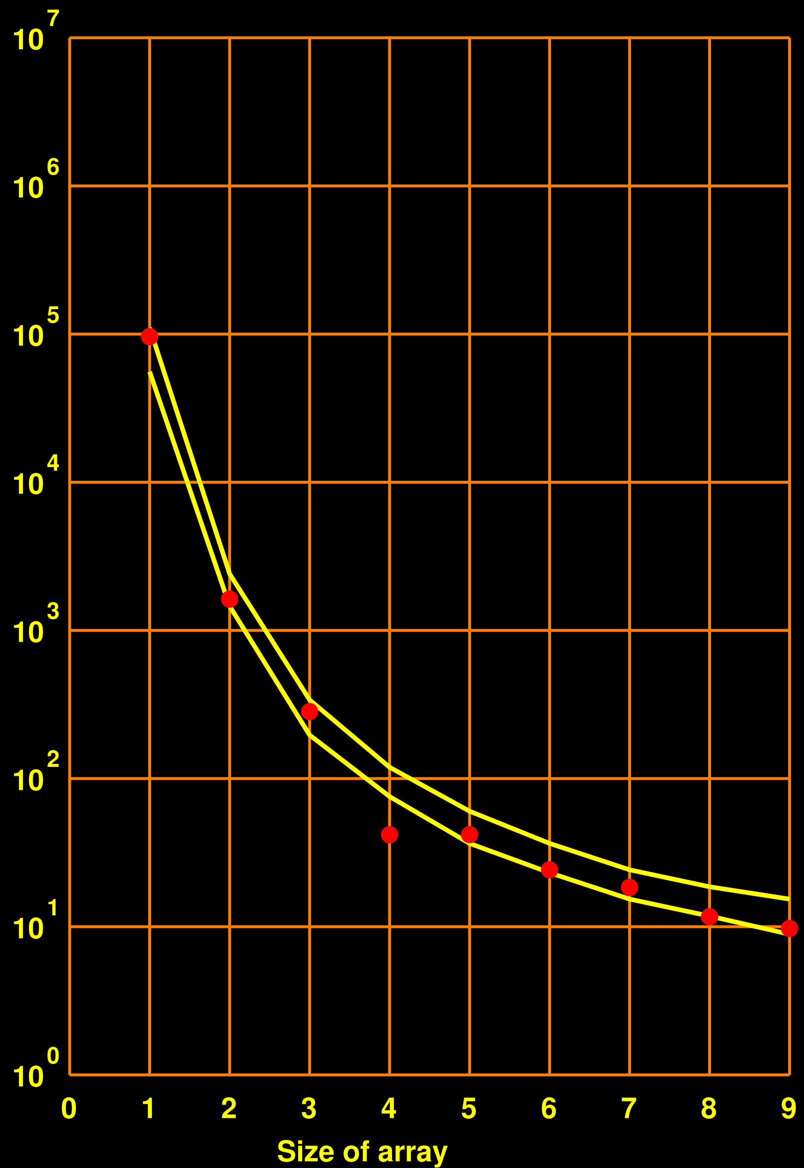
Plotting triples (c_n, c_{n+1}, c_{n+2}). A fixed number of points becomes more sparse as the dimension goes up.



A generator is measured by the maximum distance between systems of ‘planes’ containing the plots, or equivalently by the length of shortest vectors in the dual lattice. Shorter dual vectors means greater ‘planar’ spacing, so we want the shortest vector not to be too short.

Coveyou & MacPherson quote classic results from the geometry of numbers (Hermite, Mordell, . . .) that give a bound on how well you can do.

Hermite ***An integral quadratic form of dimension p and determinant D takes on some integral vector a value at most $(4/3)^{(p-1)/2} D^{1/p}$.***



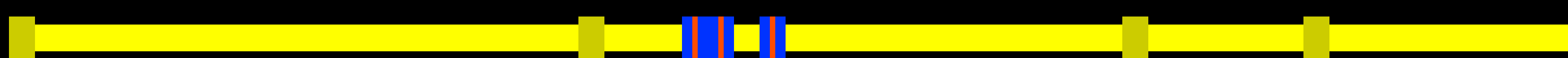
Three steps: (1) For each array size, estimate the best possible, on the basis of the classical results; (2) from this set a reasonable lower bound; (3) for a given generator, find the shortest vector.

Part II.(a) Thesis

Bob's thesis was about characteristic classes and maps between smooth vector bundles on a smooth manifold.

[2] Singularities of Vector Bundle Maps, Proceedings of Liverpool Singularities 1971

[3] Generic Vector Bundle Maps 1973



. . . ***at sentimental occasions such as birthdays, personal lies are still to be preferred to quite impersonal verities.*** **Raoul Bott**

Knuth says somewhere that comprehensible lies are often preferable to incomprehensible truths.

Keep in mind the wisdom the advice of these honest men. as this talk proceeds.

complex bundle map $f\colon E \to F/X$
Whitney stratification $\Sigma_i =$ **points** x, $\dim K_x = i$
$\dim I_x = n - i$

(1) Desingularizations

$$\phi_i\colon \widetilde{\Sigma}_i \to \overline{\Sigma}_i$$

and vector bundles K^i, I^{n-i} **extending** K **and** I **on** Σ_i.

(2) $\xi_i =$ **canonical bundle over**

$$\mathrm{Gr}_i(K^i \oplus \mathrm{Cok}^{p-n+i}) \ .$$

$$\begin{array}{l} \xi_i \oplus I^{n-i} \\ \quad\downarrow \\ \mathrm{Gr}_i \xrightarrow{\ \pi_i\ } \widetilde{\Sigma}_i \xrightarrow{\ \phi_i\ } \Sigma_i \hookrightarrow X \end{array}$$

(3) Then

$$\mathrm{cl}(E) = \sum_i (\phi_i \pi_i)_* \mathrm{cl}(\xi^i \oplus I^{n-i})$$

Graph construction:

$$\mathrm{Hom}(K^i, C) \hookrightarrow \mathrm{Gr}_i(K^i \oplus C)$$

Another situation: $f\colon X \to Y$ **a map of non-singular varieties, giving rise to bundle maps** $T_f\colon TX \to TY$**, the Grassmann graph of** T_f **in** $\mathrm{Gr}(TX \oplus TY)$**. Also of** $T_{\lambda f}$**, and let** $\lambda \to \infty$**. The limit is a union of varieties that give information about the singularity of** f**. Used in 'Chern classes' and in intersection theory in a deformation of something to its normal cone bundle.**

Part II.(a) 'Classical' algebraic geometry

[118] Chern Classes for Singular Algebraic Varieties, Annals of Math. 1974

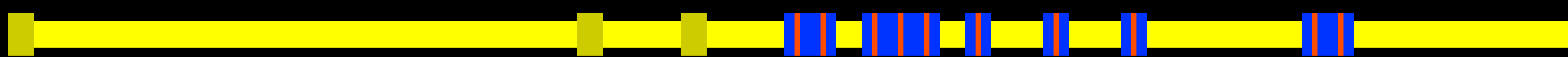
[93] Riemann-Roch for Singular Varieties, with P. Baum and W. Fulton, 1976

[14] Intersecting Cycles on an Algebraic Variety, with W. Fulton, Oslo 1976

[] Defining Algebraic Intersections, with W. Fulton, Tromso 1978

[] Categorical Framework for the Study of Singular Spaces, with W. Fulton, Memoirs A.M.S. 1981

[47] A Compactification of Configuration Space, with W. Fulton, 1994



Steve Kleiman in MR of the Tromso survey article:

. . . around 1974 the ground began to shake and, although the dust is still flying, a magnificent new structure has emerged. . . . a radically new point of view . . . the article is in fact another foreshock portending an enormous upheaval.

The major reference now is Bill Fulton's book.

Part III. Intersection homology

The development of intersection homology was somewhat like that of quantum mechanics in the mid-twenties - a supercooled liquid in the process of freezing. I suppose Steve Kleiman might have also called this an “enormous upheaval.” It has become a permanent feature of the landscape.

Steve Kleiman wrote a valuable historical account in 1988, but I felt it would be worthwhile to explore the origins a bit more carefully.

After all, nobody started out saying, “Hey! Let’s invent intersection homology.” The development had to take place in small steps.

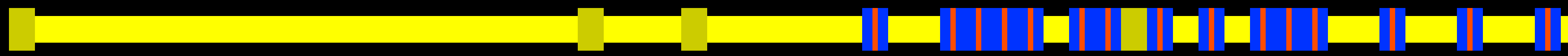
[13] La Dualité de Poincaré pour les Espaces Singulières, with M. Goresky, Comptes Rendus 1977

[132] Intersection Homology Theory, with M. Goresky, 1980

[49] Représentations des Groupes de Weyl et Homologie d'Intersection pour les Variétés Nilpotents, with W. Borho, 1981

[] Schubert cells and Verma modules—a dictionary, with Sergei Gelfand, 1983

[182] Intersection Homology II, with M. Goresky, 1983



1971 Goresky graduates from U.B.C. and goes to Brown

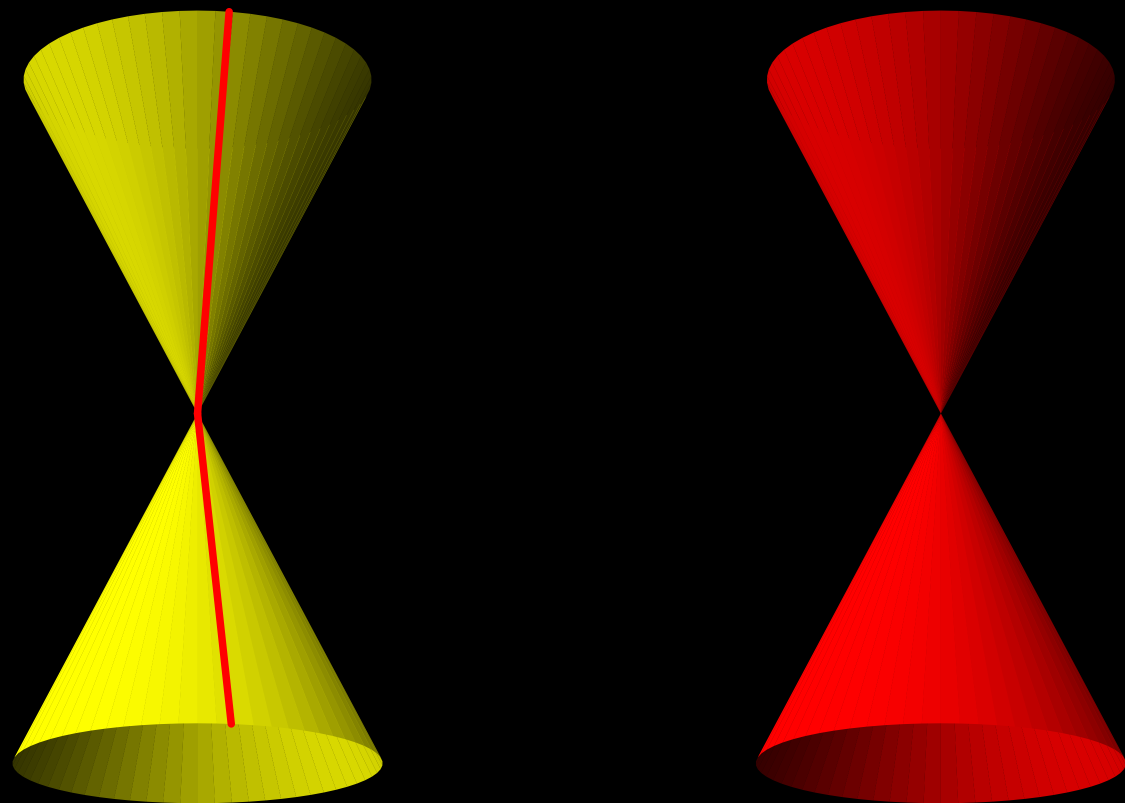
1972 He starts work on a thesis, with MacPherson as advisor

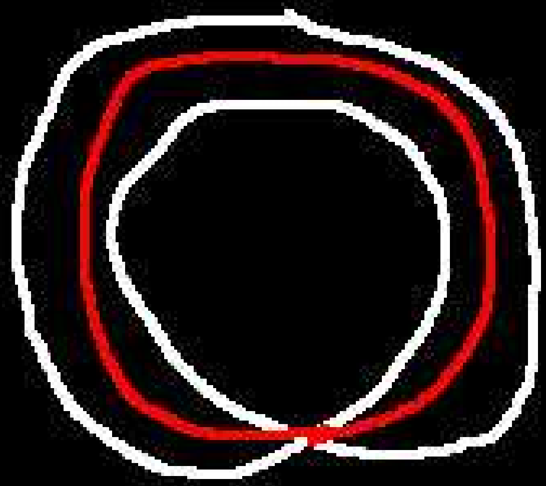
Problem: how to represent cohomology
by cycles on a stratified space?

Answer: cycles meeting a stratum
must contain full transverse fibres

1973 A moving lemma allows intersections to be defined

1974





1971

1972

1973 Goresky continues writing this up

1974 G & M go to IHES for the year
With secondary operations in mind, they try to intersect cycles

1974

With secondary operations in mind, they try to intersect cycles satisfying dimensional restrictions on lower strata.

1975

1976

1977

1974

With secondary operations in mind, they try to intersect cycles satisfying dimensional restrictions on lower strata.

As a temporary expedient (!) they require that cycles lie mostly in open strata.

1975

1976

1977

1974 With secondary operations in mind, they try to intersect cycles
satisfying dimensional restrictions on lower strata.
As a temporary expedient (!) they require that
cycles lie mostly in open strata.
They arrive at the notion of perverse cycles.
1975 Perversity measures how far from a cohomology cycle.

1976

1977

1974
1975
1976
1977

With secondary operations in mind, they try to intersect cycles satisfying dimensional restrictions on lower strata.

As a temporary expedient (!) they require that cycles lie mostly in open strata.

They arrive at the notion of perverse cycles.

Perversity measures how far from a cohomology cycle.

The most important moment is when they realize perversity conditions define a complex.

1974

With secondary operations in mind, they try to intersect cycles satisfying dimensional restrictions on lower strata.

As a temporary expedient (!) they require that cycles lie mostly in open strata.

They arrive at the notion of perverse cycles.

1975

Perversity measures how far from a cohomology cycle.

The most important moment is when they realize perversity conditions define a complex.

They take up a question of Dennis Sullivan about a signature as a cobordism invariant.

1976

1977

1974 With secondary operations in mind, they try to intersect cycles
satisfying dimensional restrictions on lower strata.
As a temporary expedient (!) they require that
cycles lie mostly in open strata.
They arrive at the notion of perverse cycles.
1975 Perversity measures how far from a cohomology cycle.
The most important moment is when
they realize perversity conditions define a complex.
They take up a question of Dennis Sullivan
about a signature as a cobordism invariant.
1976 Realizing that Poincare duality is crucial for this,
they observe that dual perversities satisfy it.
Their first proof, messy, is never published.

1977

1974 With secondary operations in mind, they try to intersect cycles
satisfying dimensional restrictions on lower strata.
As a temporary expedient (!) they require that
cycles lie mostly in open strata.
They arrive at the notion of perverse cycles.
1975 Perversity measures how far from a cohomology cycle.
The most important moment is when
they realize perversity conditions define a complex.
They take up a question of Dennis Sullivan
about a signature as a cobordism invariant.
1976 Realizing that Poincare duality is crucial for this,
they observe that dual perversities satisfy it.
Their first proof, messy, is never published.
Sullivan convinces them that
the name ‘perverse homology’ won’t sell,
1977 and suggests ‘intersection homology’.

So by the summer of 1975, G. & M. have defined the various homology groups for different perversities on a stratified topological space non-singular in codimension two. The extremal ones are (with a mild restriction on the space) ordinary homology and cohomology.

If the stratification is

$$X = X_n \supseteq X_{n-2} \supseteq \ldots \supseteq X_1 \supseteq X_0\ ,$$

the perversity $\overline{p}$ is an array of integers $(\overline{p}(2), \ldots\ , \overline{p}(n))$ with

$$\overline{p}(2) = 0,\ \overline{p}(i) \leq \overline{p}(i+1) \leq \overline{p}(i) + 1\ .$$

Perverse chains of dimension i intersect X_{n-k} in dimension at most $i - k + \overline{p}(k)$. The chain complex is named $IC^{\overline{p}}(X)$.

If $\overline{p} = (0, \ldots, 0)$ for example then the intersection of an i-chain with X_{n-k} must be of dimension at most $i - k$—it fills up as much of the open stratum as possible.

The dual of $\overline{p}$ is $\overline{q}(k) = k - 2 - \overline{p}(k)$. The dual of cohomology is homology.

Dual perversities satisfy Poincaré duality. Sullivan's conjecture about the cobordism signature invariant is true.

Are their groups are a topological invariant? They have not considered algebraic varieties as anything special.

1975

G. finishes his thesis

M. works on intersection theory with Fulton

1976

M. talks to Cheeger

M. goes to Paris for the year
and G. to MIT

1977

1978

1976

M. meets Deligne at a Halloween party

1977

1978

1979

1976

M. meets Deligne at a Halloween party

M. tells D. about IC

D. tells M. about extending local coefficient systems

defined on X - Y, Y = divisor with normal crossings

1977

inspired by Zucker's variation of Hodge structure

extension defined by truncation

1978

1979

1976

1977

1978

1979

M. meets Deligne at a Halloween party

M. tells D. about IC

D. tells M. about extending local coefficient systems

defined on X - Y, Y = divisor with normal crossings

inspired by Zucker's variation of Hodge structure

extension defined by truncation

They compare the case of isolated singularities

1976

M. meets Deligne at a Halloween party

- M. tells D. about IC
- D. tells M. about extending local coefficient systems
  - defined on X - Y, Y = divisor with normal crossings
  - inspired by Zucker's variation of Hodge structure
  - extension defined by truncation
- They compare the case of isolated singularities
- **Deligne conjectures: his construction and M.'s are the same.**

1977

1978

1979

Deligne uses sheaves and derived categories for his definition

$$\mathbf{IC}_{\overline{p}}(X) = \ldots\ \tau_{\leq p(3)} \mathbf{R}_{j,2,*} \tau_{\leq p(2)} \mathbf{R}_{j,0,*} \mathbb{C}_{X-X_2}$$

and conjecture

$$IH_i^{\overline{p}}(X) = H^{2d-i}(\mathbf{IC}_{\overline{p}}(X)\ .$$

The proof of Poincaré duality in this scheme follows from Verdier duality.

MacPherson learns more about these things from Verdier. He is persuaded by I. M. Gelfand, visiting Paris, to write up an announcement about IH for the *Comptes Rendues*. It makes no big splash in MR.

Back at Brown in 1977/78 G. & M. hold a seminar on derived categories. As an exercise they prove Verdier duality, and in addition they prove independence of stratification. The start writing IH I. and IH II.

Tales not told: Kazhdan and Lusztig are trying to give a new construction of Springer's representations of Weyl groups. They are interested in Poincaré duality; talk to Bott, who refers them to MacPherson. MacPherson explains about IH and they eventually write to Deligne.

March 11, 1979—they submit KL I. to the *Inventiones*.

April 20, 1979—Deligne writes them in English about his results and conjectures:

Dear Lusztig and Kazhdan,

I am very far to be able to do what you want. $\ldots$

He explains what the theory should be in characteristic p. KL use a version of this to prove that their polynomials, the $P_{x,y}(q)$ appearing their W-graphs and in conjectures about Verma modules, are local IH Poincaré polynomials. An embarassing moment occurs at an Arbeitstagung about IH.

MacPherson and Gelfand conjecture the decomposition theorem. The category of perverse sheaves is introduced, purity is proven, the decomposition theorem is proven (all through characteristic p). Later, Saito proves this by purely analytic techniques.

The later developments had at least one wholly unexpected side effect, a modest reconciliation between two well known mathematicians:

Serre to Grothendieck, July 23, 1985: . . . *ce terme de « perverse » me déplait tout autant qu'à toi: voilà un point où nous sommes d'accord* . . .

Part IV(a). Automorphic forms - Hecke operators

The goal here is admirable—to try to see how the theory of automorphic forms fits in with broader subjects in mathematics. Eichler started the interaction between fixed-point theorems and automorphic forms. Shimura played an important role in one development by explaining this to Bott at the Woods Hole conference in 1964.

The technical difficulty in fixed point theorems is IH is not functorial.

[7] Lefschetz Fixed Point Theorem for Intersection Homology, with M. Goresky 1985

[6] Lefschetz Numbers of Hecke Correspondences, with Mark Goresky, in Zeta Functions to Picard Modular Surfaces 1992

[15] Local Contribution to the Lefschetz Fixed Point Formula, with M. Goresky 1993

[17] Weighted Cohomology, with Gunter Harder and M. Goresky 1994

[5] **Discrete Series Characters and the Lefschetz Formula for Hecke Operators**, with R. Kottwitz and M. Goresky, 1997

Local intersection cohomology of Baily-Borel compactifications, with A. Nair, M. Goresky and G. Harder, 2002

The topological trace formula, with Mark Goresky, 2003

Part IV(b). Automorphic forms - other stuff

[31] Equivariant Cohomology, Koszul Duality, and the Localization Theorem, with Goresky and Kottwitz 1998

Purity of equivalued affine Springer fibers, with Goresky and Kottwitz 2003

Homology of affine Springer fibers in the unramified case, with Goresky and Kottwitz 2004

Part V. Reduction theory

[4] *Classical Projective Geometry and Modular Varieties*, with Mark McConnell, 1989

[10] *Explicit reduction Theory for Siegel Modular Threefolds*, with Mark McConnell, 1993

Geometry of compactifications of locally symmetric spaces, with Lizhen Ji, 2002

Let $X = G/K$ where $G = G(\mathbb{R})$, with the group G defined over $\mathbb{Q}$, Γ an arithmetic subgroup. Li and MacPherson show that various compactifications of $\Gamma\backslash X$ are the same, answering some old questions. Some are purely geometric in nature, and the MR emphasizes that the paper will likely be of interest to differential geometers.

But the most interesting aspect is the construction of a skeleton of the quotient, which summarizes a good deal of classical reduction theory in one convenient package. It completes work of Jim Arthur, Gunter Harder, Ulrich Stuhler, Dan Grayson, Erich

Leuzinger, Leslie Saper, Mischa Gromov, and Toshiaki Hattori.

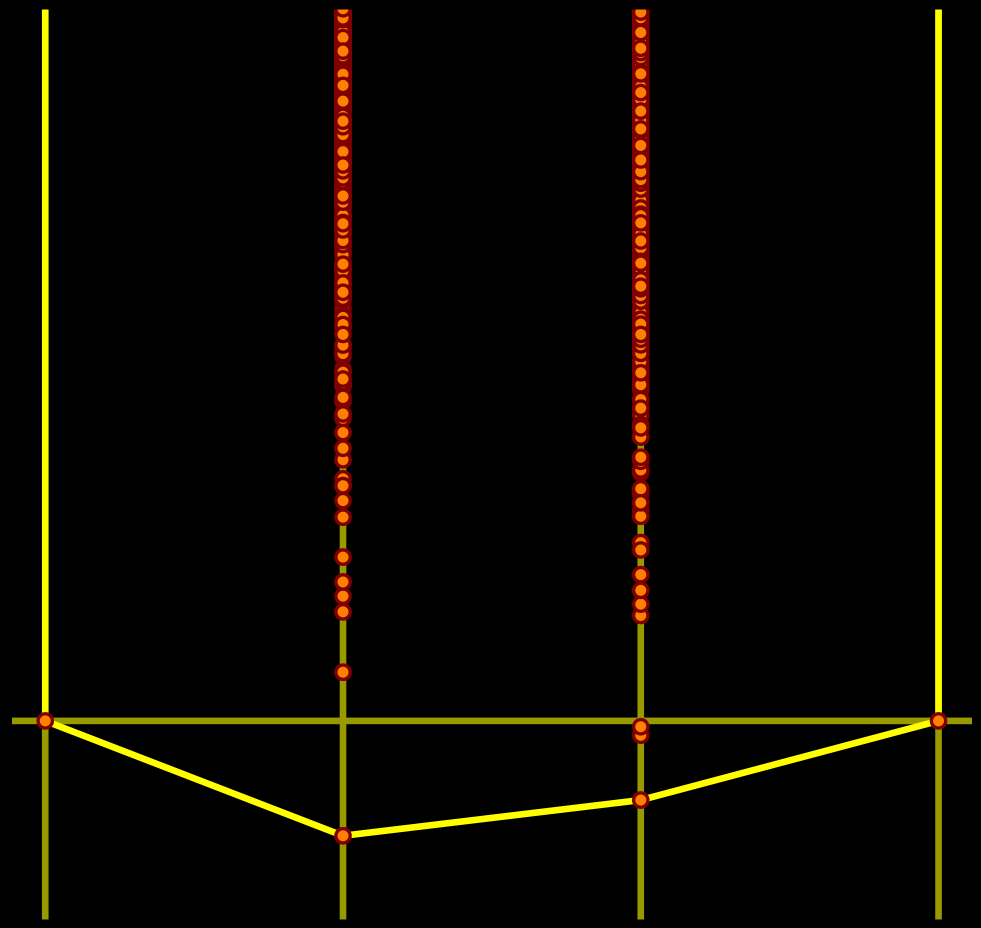
Ji and MacPherson were not apparently aware of the connection with the work of Harder, Stuhler, and Grayson, but I think it gives the best approach to their construction.

Suppose $G = GL_n$, $\Gamma = GL_n(\mathbb{Z})$. The space $\Gamma\backslash X$ parametrizes isomorphism classes of lattices L, or *free Abelian groups of rank* n + *Euclidean metric*. Grayson associates to each lattice its plot, its profile, its canonical flag, and from this in turn can be defined a slope vector.

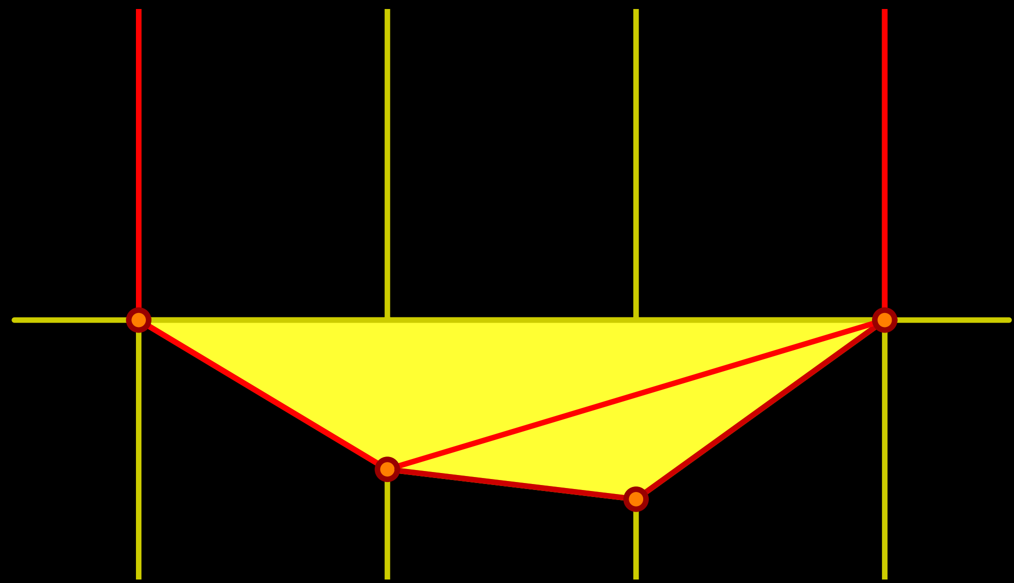
Plot: $(i, \log \mathrm{vol}\ M)$ with $M \subseteq L$, $\dim M = i$
Profile: convex hull of the plot
Slope: (s_i), the array of slopes of the profile.

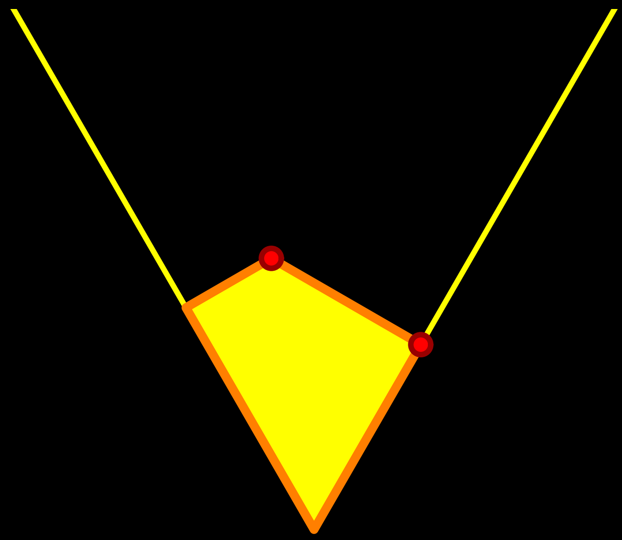
Convexity implies $s_i \leq s_{i+1}$.

Principal theorem: the break points of a profile correspond to a well defined flag.



Plot of the points $(i, \log \mathrm{vol}\ M^i)$ and its profile.





Profile and slope.

Every point of X gives rise to a rational parabolic subgroup P and a canonical point (s_i) of the positive Weyl chamber—a point of the open cone over the Tits building of G—defining canonically the skeletal projection

$$x \longmapsto \sigma(x) = (P_x, s_x) \in \Sigma_G$$

All rational groups possess similar structures.

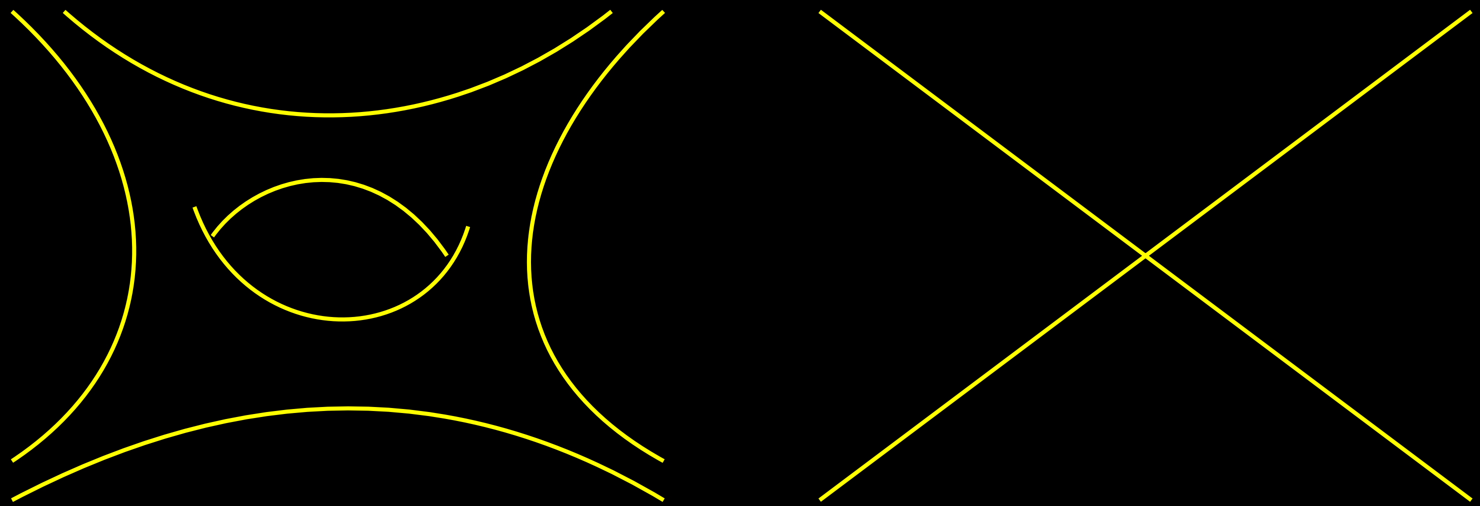
Reduction theory is conveniently formulated in terms of this.

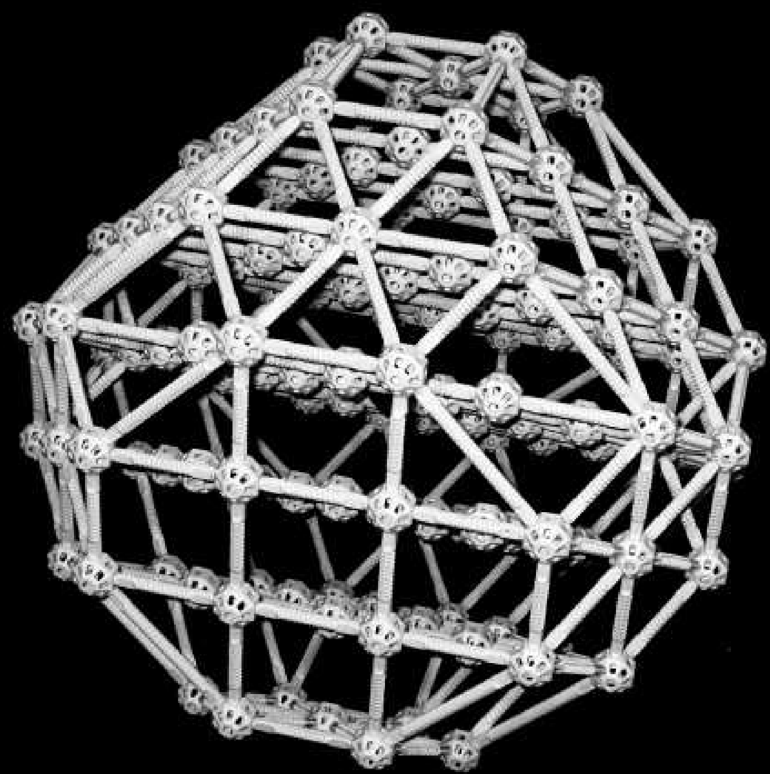
Fibres are compact.

Many results of Jim Arthur about truncation operators become transparent.

Compactifications give rise to maps into the closed cone.

For the upper half-plane:





My thanks to Mark Goresky as well as Bob MacPherson for tolerating my intrusions into the past.