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AVOIDING TWO CONSECUTIVE BLOCKS OF SAME SIZE AND SAME SUM OVER $\mathbb{Z}^2$*

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Abstract. A long standing question asks whether $\mathbb{Z}$ is uniformly 2-repetitive, that is, whether or not there is an infinite sequence over a finite subset of $\mathbb{Z}$ avoiding two consecutive blocks of the same size and same sum [J. Justin, *J. Combin. Theory Ser. A*, 12 (1972), pp. 357–367], [G. Pirillo and S. Varricchio, *Semigroup Forum*, 49 (1994), pp. 125–129]. Cassaigne et al. [*J. ACM*, 61 (2014), 10] showed that $\mathbb{Z}$ is not uniformly 3-repetitive. We show that $\mathbb{Z}^2$ is not uniformly 2-repetitive. Moreover, this problem is related to a question from Mäkelä in combinatorics on words, and we answer a weak version of it.

Key words. combinatorics on words, avoidability of repetitions, abelian repetitions, repetitive groups

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1. Introduction.

Let $k \geq 2$ be an integer, and let $(G, +)$ be a group. An *additive kth power* is a nonempty word $w_1 \dots w_k$ over $\Sigma \subseteq G$ such that for every $i \in \{2, \dots, k\}$, $|w_i| = |w_1|$ and $\sum w_i = \sum w_1$ (where $\sum v = \sum_{i=1}^{|v|} v[i]$). Using the terminology of Pirillo and Varricchio [13], we say that a group $(G, +)$ is *k-uniformly repetitive* if every infinite word over a finite subset of G contains an additive kth power as a factor. It is a long standing question whether $\mathbb{Z}$ is uniformly 2-repetitive or not [8, 13]. Cassaigne et al. [3] showed that there is an infinite word over the finite alphabet $\{0, 1, 3, 4\} \subseteq \mathbb{Z}$ without additive 3rd powers, that is, $\mathbb{Z}$ is not uniformly 3-repetitive. In section 6 we show the following theorem.

THEOREM 8. *$\mathbb{Z}^2$ is not uniformly 2-repetitive.*

When $(G, +)$ is the abelian-free group generated by the elements of Σ we talk about abelian repetitions. The avoidability of abelian repetitions has been studied since a question from Erdős arose [6, 7]. An *abelian square* is any nonempty word uv where u and v are permutations of each other. Erdős asked whether there is an infinite abelian-square-free word over an alphabet of size 4. Keränen gave a positive answer to Erdős's question in 1992 by giving an 85-uniform morphism, found with the assistance of a computer, whose fixed point is abelian-square-free [10].

Erdős also asked if it is possible to construct a word over 2 letters which contains only small squares. Entringer, Jackson, and Schatz gave a positive answer to this question [5]. They also showed that every infinite word over 2 letters contains arbitrarily long abelian squares. This naturally leads to the following question from Mäkelä (see [11]).

PROBLEM 1. *Can you avoid abelian squares of the form uv where $|u| \geq 2$ over three letters? Computer experiments show that you can avoid these patterns at least in words of length 450.*

We show that the answer is positive if we replace 2 by 6.

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THEOREM 11. *There is an infinite word over* 3 *letters avoiding an abelian square of period more than* 5.

The proofs of Theorems 8 and 11 are close in spirit (in fact, both theorems imply independently that $\mathbb{Z}^3$ is not 2-repetitive). Moreover, the proofs are both based on explicit constructions using the following morphism:

$$h_6 : \begin{cases} a \to \ ace, & b \to \ adf, \\ c \to \ bdf, & d \to \ bdc, \\ e \to \ afe, & f \to \ bce. \end{cases}$$

First, we need to show the following theorem.

THEOREM 5. $h_6^\omega(a)$ *is abelian-square-free.*

In section 4 we describe an algorithm to decide if a morphic word avoids abelian powers, and use it to show Theorem 5. This algorithm generalizes the previously known ones [2, 4], and can decide on a wider class of morphisms which includes h_6. In section 5, we explain how to extend the decidability to additive and long abelian powers. Finally, in section 6, we give the results and the constructions.

2. Preliminaries. We use terminology and notation of Lothaire [12]. An *alphabet* Σ is a finite set of *letters*, and a *word* is a (finite or infinite) sequence of letters. The set of finite words is denoted by Σ^* and the empty word by ε. One can also view Σ^* equipped with the concatenation as the free monoid over Σ.

For any word w, we denote by $|w|$ the length of w, and for any letter $a \in \Sigma$, $|w|_a$ is the number of occurrences of a in w. The *Parikh vector* of a word $w \in \Sigma^*$, denoted by $\Psi(w)$, is the vector indexed by Σ such that for every $a \in \Sigma$, $\Psi(w)[a] = |w|_a$. Two words u and v are *abelian equivalent*, denoted by $u \approx_a v$, if they are permutations of each other, or equivalently if $\Psi(u) = \Psi(v)$. For any integer $k \geq 2$, an *abelian* k*th power* is a word w that can be written $w = w_1 w_2 \dots w_k$ with $\forall\, i \in \{2, \dots, k\}$, $w_i \approx_a w_1$. Its *period* is $|w_i|$. An *abelian square* (resp., *cube*) is an abelian 2nd power (resp., abelian 3rd power). A word is *abelian-*k*th-power-free*, or *avoids abelian* k*th powers*, if none of its nonempty factors are an abelian kth power.

Let $(G, +)$ be a group, and let $\Phi : (\Sigma^*, .) \to (G, +)$ be a morphism. Two words u and v are Φ*-equivalent*, denoted $u \approx_\Phi v$, if $\Phi(u) = \Phi(v)$. For any $k \geq 2$, a k*th power modulo* Φ is a word $w = w_1 w_2 \dots w_k$ with $\forall\, i \in \{2, \dots, k\}$, $w_i \approx_\Phi w_1$. If, moreover, $|w_1| = |w_2| = \cdots = |w_k|$, then it is a *uniform* k*th power modulo* Φ. A *square modulo* Φ (resp., *cube modulo* Φ) is a 2nd power (resp., 3rd power) modulo Φ. In this article, we only consider groups $(G, +) = (\mathbb{Z}^d, +)$ for some $d > 0$. We say that $(G, +)$ is k*-repetitive* (resp., *uniformly* k*-repetitive*) if for any alphabet Σ and any morphism $\Phi : (\Sigma^*, .) \to (G, +)$ every infinite word over Σ contains a k-power modulo Φ (resp., a uniform k-power modulo Φ). Note that, for any integers n and k, if $(\mathbb{Z}^{n+1}, +)$ is k-repetitive, then $(\mathbb{Z}^n, +)$ is uniformly k-repetitive. Uniform kth powers modulo Φ are sometimes called *additive* k*th powers*, without mention of the morphism Φ, if the value of $\Phi(a)$ is clear from the context. Φ can be seen as a linear map from the Parikh vector of a word to $\mathbb{Z}^d$. Therefore, we can associate to Φ the matrix F_Φ such that $\forall w \in \Sigma^*$, $\Phi(w) = F_\Phi \Psi(w)$. Note that if $d = |\Sigma|$ and F_Φ is invertible, then two words are abelian-equivalent if and only if they are Φ-equivalent. An application of Szemerédi's theorem shows that for $d = 1$, for any finite alphabet Σ, and for $k \in \mathbb{N}$, it is not possible to avoid kth power modulo Φ over Σ, that is, $(\mathbb{Z}, +)$ is k-repetitive for any k. On the other hand, whether $\mathbb{Z}$ is uniformly 2-repetitive or not is a long standing open question [8, 13], and Cassaigne et al. showed that $\mathbb{Z}$ is not uniformly 3-repetitive [3]. We show on Theorem 8 that $\mathbb{Z}^2$ is not uniformly 2-repetitive.

Let $\mathrm{Suff}(w)$ (resp., $\mathrm{Pref}(w)$, $\mathrm{Fact}(w)$) be the set of suffixes (resp., prefixes, factors) of w. For any morphism h, let $\mathrm{Suff}(h) = \cup_{a\in\Sigma} \mathrm{Suff}(h(a))$, $\mathrm{Pref}(h) = \cup_{a\in\Sigma} \mathrm{Pref}(h(a))$, and $\mathrm{Fact}(h) = \cup_{a\in\Sigma} \mathrm{Fact}(h(a))$.

A morphism h is *nonerasing* if there is no a such that $h(a) = \varepsilon$. A morphism $h : \Sigma^* \mapsto \Sigma^*$ is *prolongable* at $a \in \Sigma$ if $h(a) = as$ for some $s \in \Sigma^+$. In this case the sequence $(h^i(a))_{i\in\mathbb{N}}$ converges toward the infinite word $w = ash(s)h^2(s)h^3(s)\dots$. If the morphism is nonerasing, w is infinite, and we say that w is a *pure morphic word generated by* h, denoted by $h^\omega(a)$. Note that every pure morphic word generated by a morphism h is a fixed point of h. A *morphic word* is the image of a pure morphic word by a second morphism.

To a morphism h on Σ^*, we associate a matrix M_h on $\Sigma \times \Sigma$ such that $(M_h)_{a,b} = |h(b)|_a$. The *eigenvalues of* h are the eigenvalues of M_h.

For any morphism $h : \Sigma^* \mapsto \Sigma^*$, let $\mathrm{Fact}^\infty(h) = \cup_{i=1}^\infty \mathrm{Fact}(h^i)$. We say that h is *primitive* if there exists $k \in \mathbb{N}$ such that $\forall\, a \in \Sigma$, $h^k(a)$ contains all of the letters of Σ (that is, M_h is primitive). If h is primitive, then for any letter $a \in \Sigma$, $\mathrm{Fact}^\infty(h) = \cup_{i=1}^\infty \mathrm{Fact}(h^i(a))$, and we can use that fact to show the following property.

PROPOSITION 1. *Let h be a primitive morphism on Σ^* prolongable at a; then* $\mathrm{Fact}(h^\omega(a)) = \mathrm{Fact}^\infty(h)$.

Proof. Since h is prolongable at a there is, by definition, a nonempty word $s \in \Sigma^+$ such that $h(a) = as$ and $h^\omega(a) = h(a)h(s)h^2(s)\dots$. Remark that $\forall\, i$, $h(a)h(s)h^2(s)\dots h^i(s) = h^{i+1}(a)$. Thus, by primitivity of h, $\mathrm{Fact}(h^\omega(a)) = \cup_{i=1}^\infty \mathrm{Fact}(h^i(a)) = \mathrm{Fact}^\infty(h)$. ∎

In the rest of this section we recall some classical notions from linear algebra.

Jordan decomposition. A *Jordan block* $J_n(\lambda)$ is an $n \times n$ matrix with $\lambda \in \mathbb{C}$ on the diagonal, 1 on top of the diagonal, and 0 elsewhere:

$$J_n(\lambda) = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & 1 & \text{\Large 0} \\ \text{\Large 0} & & \ddots & 1 \\ & & & \lambda \end{pmatrix}.$$

We recall the following well-known proposition (see [1]).

PROPOSITION 2 (Jordan decomposition). *For any $n \times n$ matrix M on $\mathbb{C}$, there is an invertible $n \times n$ matrix P and an $n \times n$ matrix J such that $M = PJP^{-1}$, and the matrix J is as follows:*

$$\begin{pmatrix} J_{n_1}(\lambda_1) & & & \\ & J_{n_2}(\lambda_2) & & 0 \\ 0 & & \ddots & \\ & & & J_{n_p}(\lambda_p) \end{pmatrix},$$

where the $J_{n_i}(\lambda_i)$ are Jordan blocks on the diagonal. PJP^{-1} is a Jordan decomposition *of M.*

The λ_i, $i \in \{1, \dots, p\}$, are the (not necessarily distinct) eigenvalues of M. The set of columns from P are *generalized eigenvectors* of M.

Note that for every $k \geq 0$, $(J_n(\lambda))^k$ is the $n \times n$ matrix M with $M_{i,j} = \binom{k}{j-i}\lambda^{k-j+i}$, with $\binom{a}{b} = 0$ if $a < b$ or $b < 0$. Thus, if $|\lambda| < 1$, $\sum_{k=0}^{\infty}(J_n(\lambda))^k$ is the matrix N where $N_{i,j} = (1-\lambda)^{i-j-1}$ if $j \geq i$, and 0 otherwise. We can easily deduce from these observations the series of kth powers of a matrix in Jordan normal form, and its sum.

Smith decomposition. The Smith decomposition is useful to solve systems of linear Diophantine equations.

PROPOSITION 3 (Smith decomposition). *For any matrix $M \in \mathbb{Z}^{n\times m}$, there are $U \in \mathbb{Z}^{n\times n}$, $D \in \mathbb{Z}^{n\times m}$, and $V \in \mathbb{Z}^{m\times m}$ such that*
- *D is diagonal, i.e., $D_{i,j} = 0$ if $i \neq j$),*
- *U and V are unimodular, i.e., their determinant is 1 or -1),*
- *$M = UDV$.*

Since U and V are unimodular, they are invertible over the integers. If one wants to find integer solutions $\mathbf{x}$ of the equation $M\mathbf{x} = \mathbf{y}$, where M is an integer matrix and $\mathbf{y}$ an integer vector, one can use the Smith decomposition UDV of M. One can suppose without loss of generality (w.l.o.g.) that $n = m$. Otherwise, one can fill in with zeros. Then $DV\mathbf{x} = U^{-1}\mathbf{y}$. Integer vectors in $\ker(M)$ form a lattice Λ. The set of columns i in V^{-1} such that $D_{i,i} = 0$ gives a basis of Λ. Let $\mathbf{y'} = U^{-1}\mathbf{y}$, which is also an integer vector. Finding the solution $\mathbf{x'}$ of $D\mathbf{x'} = \mathbf{y'}$ is easy, since D is diagonal. The set of solutions is nonempty if and only if for every i, $\mathbf{y'}_i$ is a multiple of $D_{i,i}$. One can take $\mathbf{x_0} = V^{-1}\mathbf{x'_0}$ as a particular solution to $M\mathbf{x_0} = \mathbf{y}$, with $(\mathbf{x'_0})_i = 0$ if $D_{i,i} = 0$, and $(\mathbf{x'_0})_i = \mathbf{y'}_i/D_{i,i}$ otherwise. The set of solutions is given by $\mathbf{x_0} + \Lambda$.

For any vector $\mathbf{x}$ we denote by $||\mathbf{x}||$ its Euclidean norm. For any complex matrix M, let $||M||$ be its norm induced by the Euclidean norm, that is, $||M|| = \sup\left\{\frac{||M\mathbf{x}||}{||\mathbf{x}||} : \mathbf{x} \neq \overrightarrow{0}\right\}$. Let M^* be the conjugate transpose of the matrix M. We will use the following classical proposition from linear algebra (see [1]).

PROPOSITION 4. *Let M be a matrix, and let μ_{min} (resp., μ_{max}) be the minimum (resp., maximum) over the eigenvalues of M^*M (which are all real and nonnegative). Then for any $\mathbf{x}$,*

$$\mu_{min}||\mathbf{x}||^2 \leq ||M\mathbf{x}||^2 \leq \mu_{max}||\mathbf{x}||^2.$$

For any vector $\mathbf{x}$, we also denote by $||\mathbf{x}||_1$ its L_1 *norm*, that is, the sum of the absolute value of its coordinates. The L_1 norm is useful for us because of the following property: for any $w \in \Sigma^*$, $|w| = ||\Psi(w)||_1$.

3. Templates. The notion of templates was first introduced by Currie and Rampersad for their decision algorithm [4]. A *k-template* is a $(2k)$-tuple of the form $t = [a_1, \ldots, a_{k+1}, \mathbf{d}_1, \ldots, \mathbf{d}_{k-1}]$ where $\forall\, i,\ a_i \in \Sigma \cup \{\varepsilon\}$ and $\mathbf{d}_i \in \mathbb{Z}^n$. A word $w = a_1w_1a_2w_2\ldots w_ka_{k+1}$, where $w_i \in \Sigma^*$, is a *realization* of (or *realizes*) the template t if $\forall\, i \in \{1,\ldots,k-1\}$, $\Psi(w_{i+1}) - \Psi(w_i) = \mathbf{d}_i$. A template t is *realizable by h* if there is a word in $\mathrm{Fact}^\infty(h)$ which realizes t.

Using the notion of k-templates, we can give another equivalent definition of abelian kth powers.

PROPOSITION 5. *Let $k \geq 2$ be an integer. A nonempty word is an abelian kth power if and only if it realizes the k-template $[\varepsilon, \ldots, \varepsilon, \overrightarrow{0}, \ldots, \overrightarrow{0}]$.*

Let $t' = [a'_1, \ldots, a'_{k+1}, \mathbf{d'}_1, \ldots, \mathbf{d'}_{k-1}]$ and $t = [a_1, \ldots, a_{k+1}, \mathbf{d}_1, \ldots, \mathbf{d}_{k-1}]$ be two k-templates, and let h be a morphism. We say that t' *is a parent by h of t* if there are $p_1, s_1, \ldots, p_{k+1}, s_{k+1} \in \Sigma^*$ such that
- $\forall i \in \{1,\ldots,k+1\}$, $h(a'_i) = p_ia_is_i$,

- $\forall i \in \{1,\dots,k-1\}$, $\mathbf{d}_i = M_h\mathbf{d'}_i + \Psi(s_{i+1}p_{i+2}) - \Psi(s_ip_{i+1})$.

We denote by $\mathrm{Par}_h(t)$ the set of parents by h of t. We will show in Proposition 1 that for any $t' \in \mathrm{Par}_h(t)$, if t' is realized by a word w, then t is realized by a factor of $h(w)$. In Proposition 2 we show that if t is realized by a long enough word from $\mathrm{Fact}^\infty(h)$, then there is a realizable template $t' \in \mathrm{Par}_h(t)$.

A template t' is an *ancestor by* h of a template t if there exists $n \geq 1$ and a sequence of templates $t = t_1, t_2, \dots, t_n = t'$ such that for any i, t_{i+1} is a parent by h of t_i. A template t' is a *realizable ancestor by* h of a template t if t' is an ancestor by h of t and if t' is realizable by h. For a template t, we denote by $\mathrm{Anc}_h(t)$ (resp., $\mathrm{Ranc}_h(t)$) the set of all the ancestors (resp., realizable ancestors) by h of t. We may omit "by h" when the morphism is clear from the context.

4. The decision algorithm.

In this section, we show the following theorem.

THEOREM 1. *For any primitive morphism h with no eigenvalue of absolute value 1 and any template t_0, it is possible to decide whether $\mathrm{Fact}^\infty(h)$ realizes t_0.*

Together with Proposition 1, Theorem 1 implies the following corollary.

COROLLARY 1. *For any primitive morphism h with no eigenvalue of absolute value 1 it is possible to decide whether the fixed points of h are abelian kth power-free.*

The main difference compared with the algorithm from Currie and Rampersad [4] is that we allow h to have eigenvalues of absolute value less than 1.

We first show that for any set S such that $\mathrm{Ranc}_h(t_0) \subseteq S \subseteq \mathrm{Anc}_h(t_0)$, $\mathrm{Fact}^\infty(h)$ realizes t_0 if and only if there is a small factor of $\mathrm{Fact}^\infty(h)$ which realizes a template in S. Then we explain how to compute such a finite set S. Since S is finite we can check for any k-template $t \in S$ whether a small factor realizes t and we can conclude.

4.1. Parents and preimages.

The next two lemmas tell that the realizations of the parents of a template t form the set of preimages by h of the realizations of h up to finitely many missing factors.

LEMMA 1. *Let t' be a parent of a k-template t_0, and let $w \in \Sigma^*$. If w realizes t', $h(w)$ contains a factor that realizes t_0.*

Proof. Let $t_0 = [a_1,\dots,a_{k+1},\mathbf{d}_1,\dots,\mathbf{d}_{k-1}]$ and $t' = [a'_1,\dots,a'_{k+1},\mathbf{d'}_1,\dots,\mathbf{d'}_{k-1}]$. Since w realizes t', there are $w_1,\dots,w_k \in \Sigma^*$ such that $w = a'_1w_1a'_2\dots w_ka'_{k+1}$, and $\forall\, i \in \{1,\dots,k-1\},\ \Psi(w_{i+1}) - \Psi(w_i) = \mathbf{d}'_i$.

Since t' is a parent of t_0, there are $p_1, s_1, \dots, p_{k+1}, s_{k+1} \in \Sigma^*$ such that

- $\forall i \in \{1,\dots,k+1\}$, $h(a'_i) = p_ia_is_i$,
- $\forall i \in \{1,\dots,k-1\}$, $\mathbf{d}_i = M_h\mathbf{d'}_i + \Psi(s_{i+1}p_{i+2}) - \Psi(s_ip_{i+1})$.

Thus $h(w) = p_1a_1s_1h(w_1)p_2a_2s_2h(w_2)\dots h(w_k)p_{k+1}a_{k+1}s_{k+1}$. Now let $\forall i, u_i = s_ih(w_i)p_{i+1}$; then the word $u = a_1u_1a_2u_2\dots u_ka_{k+1}$ is a factor of $h(w)$. Moreover, $\forall\, i$,

$$\begin{aligned}
\Psi(u_{i+1}) - \Psi(u_i) &= \Psi(s_{i+1}h(w_{i+1})p_{i+2}) - \Psi(s_ih(w_i)p_{i+1})\\
&= \Psi(h(w_{i+1})) - \Psi(h(w_i)) + \Psi(s_{i+1}p_{i+2}) - \Psi(s_ip_{i+1})\\
&= M_h\left(\Psi(w_{i+1}) - \Psi(w_i)\right) + \Psi(s_{i+1}p_{i+2}) - \Psi(s_ip_{i+1})\\
&= M_h\mathbf{d}'_i + \Psi(s_{i+1}p_{i+2}) - \Psi(s_ip_{i+1}),\\
\Psi(u_{i+1}) - \Psi(u_i) &= \mathbf{d}_i.
\end{aligned}$$

Thus u realizes t_0. ∎

Let $\delta = \max_{a\in\Sigma} |h(a)|$ and $\Delta(t) = \max_{i=1}^{k-1} ||\mathbf{d}_i||_1$ for any k-template $t = [a_1, \dots, a_{k+1}, \mathbf{d}_1, \dots, \mathbf{d}_{k-1}]$.

LEMMA 2. *Let t be a k-template, and let $w \in \Sigma^*$ be a word which realizes t. If $|w| > k\left(\frac{(k-1)\Delta(t)}{2} + \delta + 1\right) + 1$, then for every w' such that $w \in \text{Fact}(h(w'))$ there is a parent t' of t such that a factor of w' realizes t'.*

The idea is that if the realization is long enough, then the part corresponding to each vector is longer than δ. This implies that the a_i are images of different letters, and we can then unfold the definitions.

Proof of Lemma 2. Let $t = [a_1, \dots, a_{k+1}, \mathbf{d}_1, \dots, \mathbf{d}_{k-1}]$ be a k-template, and let $w \in \text{Fact}(h(w'))$ such that $|w| > k\left(\frac{(k-1)\Delta(t)}{2} + \delta + 1\right) + 1$ and w realizes t. Then there are $w_1, \dots, w_n \in \Sigma^*$ such that $w = a_1 w_1 a_2 w_2 \dots w_k a_{k+1}$ and $\forall i \in \{1, \dots, k-1\}$, $\Psi(w_{i+1}) - \Psi(w_i) = \mathbf{d}_i$. Thus for any $i, j \in \{1, \dots, k\}$ such that $j < i$, $\Psi(w_i) = \Psi(w_j) + \sum_{m=j}^{i-1} \mathbf{d}_m$ and, by triangular inequality, we have

$$\begin{aligned} \big||w_i| - |w_j|\big| &= \big|||\Psi(w_i)||_1 - ||\Psi(w_j)||_1\big| \\ &\le ||\Psi(w_i) - \Psi(w_j)||_1 \\ &\le \left|\left|\sum_{m=j}^{i-1} \mathbf{d}_m\right|\right|_1 \\ &\le \sum_{m=j}^{i-1} ||\mathbf{d}_m||_1 \\ &\le (i-j)\Delta(t). \end{aligned}$$

Therefore, for any $i, j \in \{1, \dots, k\}$, $|w_j| \le |i-j|\Delta(t) + |w_i|$. Combining this inequality with $|w| = k+1+\sum_{m=1}^{k} |w_m|$, we deduce that for any $i \in \{1, \dots, k\}$, $|w| \le \sum_{m=1}^{k}(|i-m|\Delta(t)+|w_i|)+k+1 \le \frac{k(k-1)}{2}\Delta(t)+k|w_i|+k+1$. Then, by hypothesis, $k\left(\frac{(k-1)\Delta(t)}{2} + |w_i| + 1\right) + 1 \ge |w| > k\left(\frac{(k-1)\Delta(t)}{2} + \delta + 1\right) + 1$, and consequently $\forall i,\ |w_i| > \delta = \max_{a\in\Sigma} |h(a)|$. We also know that $w \in \text{Fact}(h(w'))$ so there are $a'_1, \dots, a'_{k+1} \in \Sigma$, $w'_1, \dots, w'_k \in \Sigma^*$, $p_1, \dots, p_{k+1} \in \text{Pref}(h)$, and $s_1, \dots, s_{k+1} \in \text{Suff}(h)$ such that

- $w'' = a'_1 w'_1 a'_2 \dots a'_k w'_k a'_{k+1}$ is a factor of w',
- $\forall i,\ h(a'_i) = p_i a_i s_i$,
- $\forall i,\ w_i = s_i h(w'_i) p_{i+1}$.

Then w'' realizes $t' = [a'_1, \dots, a'_{k+1}, \Psi(w'_2) - \Psi(w'_1), \dots, \Psi(w'_k) - \Psi(w'_{k-1})]$. Moreover, $\forall i$:

$$\begin{aligned} \mathbf{d}_i &= \Psi(w_{i+1}) - \Psi(w_i), \\ \mathbf{d}_i &= \Psi(s_{i+1} h(w'_{i+1}) p_{i+2}) - \Psi(s_i h(w'_i) p_{i+1}), \\ \mathbf{d}_i &= M_h \Psi(w'_i) - M_h \Psi(w'_i) + \Psi(s_{i+1} p_{i+2}) - \Psi(s_i p_{i+1}), \\ \mathbf{d}_i &= M_h (\Psi(w'_i) - \Psi(w'_i)) + \Psi(s_{i+1} p_{i+2}) - \Psi(s_i p_{i+1}). \end{aligned}$$

Thus t' is a parent of t, and t' is realized by w'' as a factor of w'. ☐

A *small realization* of a k-template t is a realization w of t such that $|w| < k\left(\frac{(k-1)\Delta(t)}{2}+\delta+1\right)+1$. Using Lemmas 1 and 2 we can show the following proposition.

PROPOSITION 6. *Let h be a primitive morphism, and let t_0 be a k-template. Then* *the following conditions are equivalent:*

1. $\mathrm{Fact}^\infty(h)$ *contains no realization* t_0.
2. $\mathrm{Fact}^\infty(h)$ *contains no small realizations of any elements of* $\mathrm{Anc}_h(t_0)$.
3. $\mathrm{Fact}^\infty(h)$ *contains no small realizations of any elements of* $\mathrm{Ranc}_h(t_0)$.

Proof. **2**. $\iff$ **3**. If a template $t \in \mathrm{Anc}_h(t_0)$ is realized, then by definition $t \in \mathrm{Ranc}_h(t_0)$ so 3 $\implies$ 2. The other direction is clear from $\mathrm{Ranc}_h(t_0) \subseteq \mathrm{Anc}_h(t_0)$.

1. $\implies$ **2**. Assume that $\mathrm{Fact}^\infty(h)$ contains a small realization w of $t \in \mathrm{Anc}_h(t_0)$. By definition there are $t_n = t, t_{n-1}, t_{n-2}, \dots, t_1 \in \mathrm{Anc}_h(t_0)$ such that $\forall i \in [0, n-1]$, $t_{i+1} \in \mathrm{Par}_h(t_i)$. Now by applying inductively Lemma 1 we get that $\forall i$, t_{n-i} is realized by a factor of $h^i(w) \in \mathrm{Fact}^\infty(h)$. So, in particular, $\mathrm{Fact}^\infty(h)$ contains a realization of t_0.

2. $\implies$ **1**. Let $w \in \mathrm{Fact}^\infty(h)$ be a realization of t_0. By definition, there is an integer i and a letter $a \in \Sigma$ such that $w \in \mathrm{Fact}(h^i(a))$. If w is a small realization of t_0, then we are done since $t_0 \in \mathrm{Anc}_h(t_0)$. If w is not a small realization, we can apply Lemma 2, and we know that there is a parent t_1 of t_0 and $w_1 \in \mathrm{Fact}(h^{i-1}(a))$ such that w_1 realizes t_1. By Lemma 2, if w_1 is not a small realization of t_1 there is a parent t_2 of t_1 and $w_2 \in \mathrm{Fact}(h^{i-2}(a))$ such that w_2 realizes t_2.

We can apply this reasoning inductively until we get a w_k, which is a small realization of t_k. This happens eventually since $\forall\, j \in [1, i-1]$, $|w_j| \le |h^{i-j}(a)|$. By construction t_k is an ancestor of t_0, so we have a small realization of an ancestor of t_0. ∎

We get the following corollary.

Corollary 2. *Let h be a primitive morphism prolongable at a, and let t_0 be a k-template. Let S be a set of k-template such that* $\mathrm{Ranc}_h(t_0) \subseteq S \subseteq \mathrm{Anc}_h(t_0)$. *Then the following conditions are equivalent:*

1. $h^\omega(a)$ *avoids* t_0.
2. $h^\omega(a)$ *avoids every small realizations of every element of* S.

Any given template only has finitely many small realizations, and we need only compute small factors of $h^\omega(a)$ to compute them. If we can compute a finite set S such that $\mathrm{Ranc}_h(t_0) \subseteq S \subseteq \mathrm{Anc}_h(t_0)$, then we can decide if $h^\omega(a)$ avoids t_0.

In particular, Currie and Rampersad showed that if M_h^{-1} is defined and has an induced Euclidean norm smaller than 1, then $\mathrm{Anc}_h(t_0)$ is finite and computable [4]. They deduced a result really similar to the following theorem.

Theorem 2. *For any primitive morphism h, if M_h^{-1} is defined and has induced Euclidean norm smaller than* 1, *then, for any template t_0, it is possible to decide whether* $\mathrm{Fact}^\infty(h)$ *realizes* t_0.

In the setting of Theorem 1 M_h is not necessarily invertible, which implies that t_0 could have infinitely many parents and ancestors. Thus we need to find a way to discard many elements of $\mathrm{Anc}_h(t_0)$. In fact, using the Jordan normal form of M_h, we can find conditions on the vectors of the templates of $\mathrm{Ranc}_h(t_0)$.

4.2. Finding the set $\mathrm{Ranc}_h(t_0) \subseteq S \subseteq \mathrm{Anc}_h(t_0)$. Let $M = M_h$ be the matrix associated to h, i.e., $\forall i, j$, $M_{i,j} = |h(j)|_i$. We recall that we have the following equality:

$$\forall w \in \Sigma^*, \quad \Psi(h(w)) = M\Psi(w).$$

We assume that M has no eigenvalue of absolute value 1. Moreover, since it is primitive, it has at least one eigenvalue of absolute value greater than 1. From Proposition 2, there is an invertible matrix P and a Jordan matrix J such that $M = PJP^{-1}$.

Thus $P^{-1}M = JP^{-1}$, and for any vector $\mathbf{x}$, $P^{-1}M\mathbf{x} = JP^{-1}\mathbf{x}$. We define the map r such that $r(\mathbf{x}) = P^{-1}\mathbf{x}$ and its projections $\forall i,\ r_i(\mathbf{x}) = (P^{-1}\mathbf{x})_i$. Using this notation we have for any w, $r(\Psi(h(w))) = r(M\Psi(w)) = Jr(\Psi(w))$. Recall that J is as follows:

$$\begin{pmatrix} J_{n_1}(\lambda_1) & & & \\ & J_{n_2}(\lambda_2) & & 0 \\ 0 & & \ddots & \\ & & & J_{n_p}(\lambda_p) \end{pmatrix},$$

where the $J_{n_i}(\lambda_i)$ are Jordan blocks on the diagonal. That is, $J_n(\lambda)$ is a $n \times n$ matrix with $\lambda \in \mathbb{C}$ on the diagonal, 1 on top of the diagonal, and 0 elsewhere. Note that it may happen that for $i \neq j$, $\lambda_i = \lambda_j$.

Bounds on the P basis. We introduce some additional notation used in Propositions 7 and 8. Given a square matrix M and PJP^{-1} a Jordan decomposition of M, let $b : \{1, \ldots, n\} \to \{1, \ldots, p\}$ be the function that associates to an index i of M the number corresponding to its Jordan block in the matrix J; thus $\forall i \in \{1, \ldots, n\}$, $\lambda_{b(i)} = J_{i,i}$. Let B be the map that associates to an index i the submatrix corresponding to the Jordan block containing this index, $\forall i \in \{1, \ldots, n\}$, $B(i) = J_{n_{b(i)}}(\lambda_{b(i)})$. For any vector $\mathbf{x}$ and $1 \leq i_s \leq i_e \leq n$ such that i_s is the index of the first row of a Jordan block and i_e is the index of the last row of the same block, we denote by $\mathbf{x}_{[i_s,i_e]}$ the subvector of $\mathbf{x}$ starting at index i_s and ending at index i_e, and then $(J\mathbf{x})_{[i_s,i_e]} = B(i)\mathbf{x}_{[i_s,i_e]}$. Let $E_c(M)$ be the *contracting eigenspace* of M, that is, the subspace generated by columns i of P such that $|\lambda_{b(i)}| < 1$. Similarly, let $E_e(M)$ be the *expanding eigenspace* of M, that is, the subspace generated by columns i of P such that $|\lambda_{b(i)}| > 1$. Note that $E_c(M)$ and $E_e(M)$ are independent from the Jordan decomposition we chose.

We show that for any vector $\mathbf{x}$ appearing on a realizable ancestor of any template t_0 and any i, $|r_i(\mathbf{x})|$ is bounded, handling separately generalized eigenvectors of eigenvalues of absolute value less and more than 1. It implies that there are finitely many such integer vectors, since columns of P form a basis of $\mathbb{C}^n$.

Proposition 7. *For any i such that $|\lambda_{b(i)}| < 1$, $\{|r_i(\Psi(w))| : w \in \mathrm{Fact}^\infty(h)\}$ is bounded.*

Proof. Take i such that $|\lambda_{b(i)}| < 1$, and let i_s (resp., i_e) be the index that starts (resp., ends) the Jordan block $b(i)$ (thus $i_s \leq i \leq i_e$). Let w be a factor of $\mathrm{Fact}^\infty(h)$. Then there is a factor $w' \in \mathrm{Fact}(h)$, an integer l, and for every $j \in \{0, \ldots, l-1\}$, a pair of words $(s_j, p_j) \in (\mathrm{Suff}(h), \mathrm{Pref}(h))$ such that

$$w = \left(\prod_{j=0}^{l-1} h^j(s_j)\right) h^l(w') \left(\prod_{j=l-1}^{0} h^j(p_j)\right).$$

Thus

$$r(\Psi(w)) = \sum_{j=0}^{l-1} J^j r(\Psi(s_j)) + J^l r(\Psi(w')) + \sum_{j=0}^{l-1} J^j r(\Psi(p_j))$$

and

$$r(\Psi(w))_{[i_s,i_e]} = \sum_{j=0}^{l-1} B(i)^j r(\Psi(s_j p_j))_{[i_s,i_e]} + B(i)^l r(\Psi(w'))_{[i_s,i_e]}.$$

Since $\lim_{l\to\infty}\left(\sum_{j=0}^{l} B(i)^j\right)$ exists, $|r_i(\Psi(w))|$ is bounded.

More precisely, a bound for $|r_i(\Psi(w))|$ can be found in the following way. Let $\Sigma^{-1} = \{a^{-1} : a \in \Sigma\}$ be the set of inverses of the letters of Σ. Recall that the free group generated by Σ is the group made of the set of words over $\Sigma \cup \Sigma^{-1}$, where the only nontrivial equalities can be deduced from the fact that $\forall\, a \in \Sigma$, $aa^{-1} = a^{-1}a = \varepsilon$. We can also extend the notion of Parikh vector such that the Parikh vector of the inverse of a letter counts as a negative occurrence of the letter. Now for any $a \in \Sigma \cup \Sigma^{-1}$ and word s, p, and f such that $h(a) = pfs$ we have $fsh(a^{-1})pf = f$. For all $a \in \Sigma$, $a \in \mathrm{Fact}(h)$, since h is primitive. This implies that for every $l' > l$ one can find $a \in \Sigma \cup \Sigma^{-1}$ and extend the sequence $(s_j, p_j)_{j\in\{0,\dots,l-1\}}$ to the sequence $(s_j, p_j)_{j\in\{0,\dots,l'-1\}}$ such that

$$w = \left(\prod_{j=0}^{l'-1} h^j(s_j)\right) h^{l'}(a) \left(\prod_{j=l'-1}^{0} h^j(p_j)\right).$$

Thus there is an infinite sequence $(s_j, p_j)_{j\in\mathbb{N}}$ of elements in $(\mathrm{Suff}(h), \mathrm{Pref}(h))$ such that

$$r(\Psi(w))_{[i_s,i_e]} = \sum_{j=0}^{\infty} B(i)^j r(\Psi(s_j p_j))_{[i_s,i_e]}.$$

For any i such that $|\lambda_{b(i)}| < 1$, $r_i(\Psi(w))$ is bounded by $\mathbf{u}\cdot\mathbf{v}$, where

- $\mathbf{u}$ is the vector such that $\mathbf{u}_j = \max\{|r_j(\Psi(sp))| : (s,p) \in (\mathrm{Suff}(h), \mathrm{Pref}(h))\}$,
- $\mathbf{v}$ is the vector such that $\mathbf{v}_j = (1 - |\lambda_{b(i)}|)^{i-j-1}$ if $j \in \{i, \dots, i_e\}$ and zero otherwise. ∎

Let $r_i^* = 2\times\max\{|r_i(\Psi(w))| : w \in \mathrm{Fact}^\infty(h)\}$. Let $\mathcal{R}_B$ be the set of templates $t = [a_1, \dots, a_{k+1}, \mathbf{d}_1, \dots, \mathbf{d}_{k-1}]$ such that for every i with $|\lambda_{b(i)}| < 1$ and $j \in \{1, \dots, k-1\}$, $|r_i(\mathbf{d}_j)| \le r_i^*$.

Corollary 3. *Every k-template which is realized by h is in $\mathcal{R}_B$.*

We need a tight upper bound on r_i^* for the algorithm corresponding to Theorem 1 to be efficient. The bound from the last proposition could be too loose, but we can reach better bounds by considering the fact that (since h is primitive) for any $l > 1$, h^l has the same factors as h. For example, for the abelian square-free morphism h_8 (section 6.1) the bound for the eigenvalue $\lambda \sim 0.33292, +0.67077i$ is 5.9633 and becomes 1.4394 for the eigenvalue λ^{20} of $(h_8)^{20}$, while the observed bound on the prefix of size approximately 1 million of a fixed point of $(h_8)^2$ is 1.4341.

For any k-template t_0, we denote by $\mathbf{X}_{t_0}$ the set of all the vectors that appear on an ancestor of t_0.

Proposition 8. *For every i such that $|\lambda_{b(i)}| > 1$, for every k-template t_0, $\{|r_i(\mathbf{x})| : \mathbf{x} \in \mathbf{X}_{t_0}\}$ is bounded.*

Proof. The proof is close to the proof of Proposition 7. Let $\mathbf{x}$ be a vector of $\mathbf{X}_{t_0}$. If it is not a vector of t_0, then it appears on a template t which is a parent of an ancestor t' of t_0. If x' is the vector at the corresponding position in t', then, by definition of parent, there are $s, s', p, p' \in (\mathrm{Suff}(h), \mathrm{Suff}(h), \mathrm{Pref}(h), \mathrm{Pref}(h))$ such that $x' = Mx + \Psi(sp) - \Psi(s'p')$.

By induction there is a vector $\mathbf{x}_0$ of t_0, an integer l, and a sequence of 4-tuple of words $(s_j, s'_j, p_j, p'_j)_{0\leq i\leq l-1} \in (\mathrm{Suff}(h), \mathrm{Suff}(h), \mathrm{Pref}(h), \mathrm{Pref}(h))^{0\leq i\leq l-1}$ such that

$$\mathbf{x}_0 = \sum_{j=0}^{l-1} M^j \Psi(s_j p_j) + M^l \mathbf{x} - \sum_{j=0}^{l-1} M^j \Psi(s'_j p'_j).$$

Thus

$$r(\mathbf{x}_0) = \sum_{j=0}^{l-1} J^j r(\Psi(s_j p_j) - \Psi(s'_j p'_j)) + J^l r(\mathbf{x}).$$

Let i_s (resp., i_e) be the starting (resp., ending) index of the block $b(i)$. Thus

$$B(i)^l r(\mathbf{x})_{[i_s,i_e]} = r(\mathbf{x}_0)_{[i_s,i_e]} + \sum_{j=0}^{l-1} B(i)^j r(\Psi(s'_j p'_j) - \Psi(s_j p_j))_{[i_s,i_e]}.$$

Moreover, we know that $B(i)$ is invertible so

$$r(\mathbf{x})_{[i_s,i_e]} = B(i)^{-l} r(\mathbf{x}_0)_{[i_s,i_e]} + \sum_{j=0}^{l-1} B(i)^{j-l} r(\Psi(s'_j p'_j) - \Psi(s_j p_j))_{[i_s,i_e]}.$$

The only eigenvalue of $B(i)^{-1}$ is $\lambda_{b(i)}^{-1}$, which has absolute value less than 1, and thus $\sum_{j=1}^{\infty} ||B(i)^{-j}||$ converges. Hence $||r(\mathbf{x})_{[i_s,i_e]}||$ can be bounded by a constant depending only on h, P, J, and i. Thus there is a constant r^*_{i,t_0} such that $\forall\, \mathbf{x} \in \mathbf{X}_{t_0}$, $|r_i(\mathbf{x})| \leq r^*_{i,t_0}$. $\square$

In the subsection *Computing S efficiently*, we explain why we do not need to compute a value for the bound r^*_{i,t_0}. Since the columns of P form a basis, Propositions 7 and 8 imply that the norm of any vector of a k-template from $\mathcal{R}_B \cap \mathrm{Anc}_h(t_0)$ is bounded, and thus $\mathcal{R}_B \cap \mathrm{Anc}_h(t_0)$ is finite. We sum up all of the interesting properties about $\mathcal{R}_B \cap \mathrm{Anc}_h(t_0)$ in the next corollary.

Corollary 4. *For any template t_0 and any morphism h whose matrix has no eigenvalue of absolute value* 1*, we have*

- $\mathrm{Ranc}_h(t_0) \subseteq \mathcal{R}_B \cap \mathrm{Anc}_h(t_0) \subseteq \mathrm{Anc}_h(t_0)$,
- $\mathcal{R}_B \cap \mathrm{Anc}_h(t_0)$ *is finite.*

From Corollaries 2 and 4, we know that if we can compute $\mathcal{R}_B \cap \mathrm{Anc}_h(t_0)$, then we can decide whether $h_\omega(a)$ avoids abelian kth powers.

We can deduce from Propositions 7 and 8 a naive algorithm to compute a set S of templates such that $\mathrm{Ranc}_h(t_0) \subseteq S \subseteq \mathrm{Anc}_h(t_0)$. We first compute a set of templates T_{t_0} whose vectors' coordinates in basis P are bounded by r^*_i or r^*_{i,t_0}, then we compute the parent relation inside T_{t_0} and we select the parents that are accessible from t_0. This naive algorithm is not efficient. We explain at the end of this section a more efficient way to compute such a set S, based on the fact that for morphisms whose fixed points avoid abelian powers, the set of ancestors $\mathcal{R}_B \cap \mathrm{Anc}_h(t_0)$ is usually very small relative to T_{t_0}.

We summarize the proof of Theorem 1. We know from Corollary 4 that one can compute a set S such that $\mathrm{Ranc}_h(t_0) \subseteq S \subseteq \mathrm{Anc}_h(t_0)$. Moreover, from Corollary 2 we know that the following are equivalent:

1. $h^\omega(a)$ avoids t_0.
2. $h^\omega(a)$ avoids every small realizations of every elements of S.

For any integer l, we can compute every factor of $h^\omega(a)$ of bounded size l. Moreover, S is finite so we can check every template of S one by one. Therefore, we can check condition 2 with a computer. Hence one can decide whether $h^\omega(a)$ avoids t_0.

Computing S efficiently. The following algorithm does not necessarily compute $\mathcal{R}_B \cap \mathrm{Anc}_h(t_0)$, but a set S such that $\mathrm{Ranc}_h(t_0) \subseteq S \subseteq \mathcal{R}_B \cap \mathrm{Anc}_h(t_0)$. We compute recursively a set of templates A_{t_0} that we initialize at $\{t_0\}$, and each time that we add a new template t, we compute the set of parents of t which are in $\mathcal{R}_B$ and add them to A_{t_0}. At any time we have $A_{t_0} \subseteq \mathcal{R}_B \cap \mathrm{Anc}_h(t_0)$, which is finite so this algorithm terminates. Moreover, if a parent of a template is realizable, then this template also is realizable. It implies that, in the end, $\mathrm{Ranc}_h(t_0) \subseteq A_{t_0}$.

We need to be able to compute a finite superset of the set of realizable parents of a template. Let $t = [a_1, \ldots, a_{k+1}, \mathbf{d}_1, \ldots, \mathbf{d}_{k-1}]$ be a template, and assume that $t' = [a'_1, \ldots, a'_{k+1}, \mathbf{d'}_1, \ldots, \mathbf{d'}_{k-1}]$ is a parent of t, and t' is realizable by h. Then there are $p_1, s_1, \ldots, p_{k+1}, s_{k+1} \in \Sigma^*$ such that

- $\forall i \in \{1, \ldots, k+1\}$, $h(a'_i) = p_i a_i s_i$,
- $\forall i \in \{1, \ldots, k-1\}$, $\mathbf{d}_i = M\mathbf{d'}_i + \Psi(s_{i+1}p_{i+2}) - \Psi(s_i p_{i+1})$.

There are finitely many ways of choosing the a'_i in t' and finitely many ways of choosing the s_i and the p_i, so we need only be able to compute the possible values of the $\mathbf{d'}_i$ of a template with fixed $a'_1, \ldots, a'_{k+1}$ and $s_1, p_1, \ldots, s_{k+1}, p_{k+1}$. (Note that this is easy if M is invertible.)

Suppose we want to compute $\mathbf{d'}_m$ for some m. That is, we want to compute all of the integer solutions $\mathbf{x}$ of $M\mathbf{x} = \mathbf{v}$, where $\mathbf{v} = \mathbf{d}_m - \Psi(s_{m+1}p_{m+2}) + \Psi(s_m p_{m+1})$. Moreover, since we are interested in realizable parents we can restrict ourselves to solutions that respect the bounds from Proposition 7. The rest is only linear algebra.

First, we can use the Smith decomposition of M, as explained after Proposition 3, in order to find a particular solution $\mathbf{x_0}$ and a basis $(\beta_1, \ldots, \beta_\kappa)$ (where $\kappa = \dim\ker(M)$) of the lattice $\Lambda = \ker(M) \cap \mathbb{Z}^n$. If this equation has no integer solution, then the template t has no parents with this choice of a_i, p_i, and s_i. We are only interested in parents realizable by h, so we want to compute the set $\mathbf{X} = \{\mathbf{x} \in \mathbf{x_0} + \Lambda : \ \forall i \text{ s.t. } |\lambda_{b(i)}| < 1,\ |r_i(\mathbf{x})| \le r_i^*\}$. Since Λ is included in the union of the generalized eigenspaces of eigenvalue 0, we know by Proposition 7 that $\mathbf{X}$ is finite. Let $\mathcal{B}$ be the matrix whose columns are the elements of the basis $(\beta_1, \ldots, \beta_\kappa)$, and let $\mathbf{X}_\mathcal{B} = \{\mathbf{x} \in \mathbb{Z}^\kappa : \ \mathbf{x_0} + \mathcal{B}\mathbf{x} \in \mathbf{X}\}$. $\ker(M)$ is generated by $\mathcal{B}$ but also by the generalized eigenvectors corresponding to a null eigenvalue which are columns of P. So there is a matrix Q made of rows of P^{-1} such that $Q\mathcal{B}$ is invertible. All of the rows of Q are rows of P^{-1}; thus from Proposition 7 there are $c_1, \ldots, c_\kappa \in \mathbb{R}$ such that for any $\mathbf{x} \in \mathbf{X}_\mathcal{B}$ and $i \in \{1, \ldots, \kappa\}$, $|(Q(\mathcal{B}\mathbf{x}+\mathbf{x_0}))_i| \le c_i$ and thus $|(Q\mathcal{B}\mathbf{x})_i| \le c_i + |(Q\mathbf{x_0})_i|$. Then

$$||Q\mathcal{B}\mathbf{x}||^2 \le \sum_{i=1}^{\kappa} (c_i + |(Q\mathbf{x_0})_i|)^2.$$

Let $c = \sum_{i=1}^{\kappa} (c_i + |(Q\mathbf{x_0})_i|)^2$. From Proposition 4, if μ_{min} is the smallest eigenvalue of $(Q\mathcal{B})^*(Q\mathcal{B})$, then $\mu_{min}||\mathbf{x}||^2 \le ||Q\mathcal{B}\mathbf{x}||^2 \le c$. Moreover, $Q\mathcal{B}$ is invertible, thus $\mu_{min} \neq 0$, and $\mathbf{X}_\mathcal{B}$ contains only integer points in the ball of radius $\sqrt{\frac{c}{\mu_{min}}}$. We can easily compute a finite superset of $\mathbf{X}_\mathcal{B}$, and thus of $\mathbf{X}$, and then we can select the elements that are actually in $\mathbf{X}$. The choice of $\mathbf{x_0}$ is significant for the sharpness of the bound c; it is preferable to take an $\mathbf{x_0}$ nearly orthogonal to $\ker(M)$.

5. Applications. If a morphism h has k eigenvalues of absolute value less than 1 (counting their algebraic multiplicities), then Proposition 7 tells us that the Parikh vectors of the factors of $\mathrm{Fact}^\infty(h)$ are close to the subspace $E_e(M_h)$ of dimension $n - k$. This can be useful to avoid patterns in images of $\mathrm{Fact}^\infty(h)$.

If one tries to avoid a template t in a morphic word $g(h^\infty)$, with $g : \Sigma \to \Sigma'$ and

$|\Sigma'| < |\Sigma|$, then the set of parents of t is generally infinite: the set of the vectors in the parents is close to the subspace $\ker(M_g)$ of dimension $|\Sigma| - |\Sigma'|$ (if M_g has full rank). But if the intersection of $\ker(M_g)$ with $E_e(M_h)$ is of dimension 0, then we can generate a finite superset of the realizable parents, and decide with the algorithm from section 4.

We can use the same idea to avoid additive powers. This is a generalization of the method used in [3] to show that we can avoid additive cubes in a word over $\{0, 1, 3, 4\}$.

We present here two applications of this method: decide if a morphic word does not contain large abelian powers and decide if a pure morphic word avoids additive powers. Other possible applications, such as deciding if a morphic word avoids k-abelian powers, are not explained here, but the method can be easily generalized.

5.1. Deciding if a morphic word contains large abelian power. In this subsection, we explain how to decide whether a morphic word $g(h^\infty(a))$ avoids large abelian kth powers.

PROPOSITION 9. *Let $h : \Sigma^* \mapsto \Sigma^*$ and $g : \Sigma^* \mapsto \Sigma'^*$ be two morphisms, and let M_h and M_g be the matrices associated to those morphisms. If M_h has no eigenvalue of absolute value 1 and $E_e(M_h) \cap \ker(M_g) = \{\vec{0}\}$, then for any template t' one can compute a finite set S that contains any template realizable by h and the parent of t' by g.*

Proof. The proof is similar to the computation of parents in section 4. Let $M_h = PJP^{-1}$ be a Jordan decomposition of M_h. Let $\kappa = \dim\ker(M_g)$, and let $\Lambda = \ker(M_g) \cap \mathbb{Z}^\kappa$. We use the Smith decomposition of M_g to get the matrix B, whose columns form an integral basis of Λ. Assume $t = [a_1, \dots, a_{k+1}, \mathbf{d}_1, \dots, \mathbf{d}_{k-1}]$ is realizable by h and the parent of $t' = [a'_1, \dots, a'_{k+1}, \mathbf{d'}_1, \dots, \mathbf{d'}_{k-1}]$ by g. Then there are $p_1, s_1, \dots, p_{k+1}, s_{k+1} \in \Sigma^*$ such that

- $\forall i,\ g(a_i) = p_i a'_i s_i$,
- $\forall i,\ \mathbf{d'}_i = M_g \mathbf{d}_i + \Psi(s_{i+1}p_{i+2}) - \Psi(s_i p_{i+1})$.

There are finitely many choices for the a_i, s_i and p_i. We need to be able to compute all the possible values for $\mathbf{d}_m$ for some m with fixed $a_1, \dots, a_{k+1}$ and $p_1, s_1, \dots, p_{k+1}, s_{k+1}$. Then $\mathbf{d}_m$ is an integer solution of $M_g\mathbf{x} = \mathbf{v}$, with $\mathbf{v} = \mathbf{d'}_m + \Psi(s_m p_{m+1}) - \Psi(s_{m+1}p_{m+2})$. We will see that we have only finitely many choices for $\mathbf{d}_m$. As already explained in section 3, if such a solution exists, then $\mathbf{d}_m \in \mathbf{x_0} + \Lambda$, and $\mathbf{x_0}$ can be found with the Smith decomposition of M_g.

Let Q be the rectangular submatrix of P^{-1} such that the ith line of P^{-1} is a line of Q if and only if $|\lambda_{b(i)}| < 1$. For every $\mathbf{x} \in \mathbb{C}^\kappa \setminus \{\vec{0}\}$, $B\mathbf{x} \in \ker(M_g)$ by definition of B. Then, by hypothesis, $B\mathbf{x} \notin E_e(M_h)$ and $QB\mathbf{x} \neq \vec{0}$ since the lines of Q generate the subspace orthogonal to $E_e(M_h)$. Thus we have $\operatorname{rank}(QB) = \kappa$, which implies that there is a submatrix Q' of Q such that $Q'B$ is invertible.

From Proposition 7, $\forall i \in \{1, \dots, \kappa\}$, there is $c_i \in \mathbb{R}$ such that for any two factors u and v of $\mathrm{Fact}^\infty(h)$, $|(Q'(\Psi(u) - \Psi(v)))_i| \le c_i$.

Let $\mathbf{X} = \{\mathbf{x} \in \mathbf{x_0} + \Lambda : \forall i \in \{1, \dots, \kappa\},\ |(Q'\mathbf{x})_i| \le c_i\}$. Since we are only interested in realizable solutions, $\mathbf{d}_m$ has to be in $\mathbf{X}$. Let $\mathbf{X}_B = \{\mathbf{x} \in \mathbb{Z}^\kappa :\ (\mathbf{x_0} + B\mathbf{x}) \in \mathbf{X}\}$ and $\mathbf{x} \in \mathbf{X}_B$. Then $\forall\, i$, $|(Q'(B\mathbf{x} + \mathbf{x_0}))_i| \le c_i$ and thus $|(Q'(B\mathbf{x}))_i| \le c_i + |(Q'\mathbf{x_0})_i|$. Then $||Q'B\mathbf{x}||^2 \le \sum_{i=1}^{l} (c_i + |(Q'\mathbf{x_0})_i|)^2 = c$. From Proposition 4, if μ_{min} is the smallest eigenvalue of $(Q'B)^*(Q'B)$, we have $\mu_{min}||\mathbf{x}||^2 \le ||Q'B\mathbf{x}||^2 \le c$. Since $Q'B$ is invertible, $\mu_{min} \neq 0$ and $||\mathbf{x}|| \le \sqrt{\frac{c}{\mu_{min}}}$. Then $\mathbf{X}_B$ and $\mathbf{X}$ are finite, and we can easily compute them. ∎

We can easily adapt the proof of Lemma 2 to get the following proposition.

PROPOSITION 10. *If no parent of the k-template $[\varepsilon, \ldots, \varepsilon, \overrightarrow{0}, \ldots, \overrightarrow{0}]$ by g is realizable by h, then $g(\mathrm{Fact}^{\infty}(h))$ avoids abelian kth powers of a period larger than* $\max_{a\in\Sigma} |g(a)|$.

The condition of Proposition 10 can be easily checked by a computer using Proposition 9 and Theorem 1. If one wants to decide whether $g(\mathrm{Fact}^{\infty}(h))$ avoids abelian kth powers of period at least $p \le \max_{a\in\Sigma} |g(a)|$, then one can use Proposition 10 and check if $g(\mathrm{Fact}^{\infty}(h))$ does not contain an abelian kth power of period l for every $p \le l < \max_{a\in\Sigma} |g(a)|$. If $p > \max_{a\in\Sigma} |g(a)|$, then one can take a large enough integer k such that $p \le \max_{a\in\Sigma} |g(h^k(a))|$ and do the computation on $g\circ h^k$ instead of g. Note that if $E_e(M_h) \cap \ker(M_g) = \{\overrightarrow{0}\}$, then for every $k \in \mathbb{N}$, $E_e(M_h) \cap \ker(M_{g\circ h^k}) = \{\overrightarrow{0}\}$. Otherwise, for the sake of contradiction let $\mathbf{x} \in (E_e(M_h) \cap \ker(M_{g\circ h^k})) \setminus \{\overrightarrow{0}\}$. Then $M_h^k\mathbf{x} \in \ker(M_g)$. Moreover, $\mathbf{x} \in E_e(M_h) \setminus \{\overrightarrow{0}\}$, so $M_h^k\mathbf{x} \in E_e(M_h)$ and $M_h^k\mathbf{x} \neq \overrightarrow{0}$. Thus $M_h^k\mathbf{x} \in E_e(M_h) \cap \ker(M_g) \setminus \{\overrightarrow{0}\}$, and we have a contradiction.

Consequently we have the following theorem.

THEOREM 3. *Let $h : \Sigma^* \to \Sigma^*$ be a primitive morphism with no eigenvalue of absolute value* 1, *let $g : \Sigma^* \to \Sigma'^*$ be a morphism, and let $p, k \in \mathbb{N}$. If $E_e(M_h) \cap \ker(M_g) = \{\overrightarrow{0}\}$, then one can decide whether $g(h^{\infty}(a))$ avoids abelian kth powers of period larger than p.*

In section 6.4, we present a morphic word over 3 letters which avoids abelian squares of period more than 5.

5.2. Deciding if a pure morphic word avoids additive powers on $\mathbb{Z}^d$. In this part we consider the morphism $\Phi : (\Sigma^*, .) \to (\mathbb{Z}^d, +)$ with $d \in \mathbb{N}$. Let the matrix F_Φ be such that $\forall w$, $\Phi(w) = F_\Phi \Psi(w)$.

PROPOSITION 11. *If M_h has no eigenvalue of absolute value* 1 *and $E_e(M_h) \cap \ker(F_\Phi) = \{\overrightarrow{0}\}$, then one can compute a finite set of templates S such that each kth power modulo Φ in* $\mathrm{Fact}^{\infty}(\mathrm{h})$ *is a realization of a template in S.*

Proof. Let $\kappa = \dim \ker(F_\Phi)$, and let $\Lambda = \ker(F_\Phi) \cap \mathbb{Z}^d$. By definition any kth power modulo Φ realizes at least one template of the form $t = [\varepsilon, \ldots, \varepsilon, \mathbf{d}_1, \ldots, \mathbf{d}_{k-1}]$, where $\forall\, i$, $\mathbf{d}_i \in \Lambda$. We use the Smith decomposition of F_Φ, as explained after Proposition 3, to get the matrix B, whose columns form an integral basis of Λ.

Let Q be the rectangular submatrix of P^{-1} such that the ith line of P^{-1} is a line of Q if and only if $|\lambda_{b(i)}| < 1$. By definition of B, for every $\mathbf{x} \in \mathbb{C}^\kappa \setminus \{\overrightarrow{0}\}$, $B\mathbf{x} \in \ker(F_\Phi)$; then, by hypothesis, $B\mathbf{x} \notin E_e(M_h)$. Since the lines of Q generate the subspace orthogonal to $E_e(M_h)$, $QB\mathbf{x} \neq \overrightarrow{0}$. Thus we have $\mathrm{rank}(QB) = \kappa$, which implies that there is a submatrix Q' of Q such that $Q'B$ is invertible.

For all $i \in \{1, \ldots, \kappa\}$, let p_i be the function such that $\forall$ vector $\mathbf{x}$, $p_i(\mathbf{x}) = (Q'\mathbf{x})_i$. From Proposition 7, $\forall\, i \in \{1, \ldots, \kappa\}$, there is $c_i \in \mathbb{R}$ such that for any two factors u and v of $\mathrm{Fact}^{\infty}(h)$, $|p_i(\Psi(u) - \Psi(v))| \le c_i$.

Let $\mathbf{X} = \{\mathbf{x} \in \Lambda : \forall i \in \{1, \ldots, \kappa\},\ |p_i(\mathbf{x})| \le c_i\}$. Since we are only interested in realizable templates for S, we can add the following condition: $\forall\, i$, $\mathbf{d}_i \in \mathbf{X}$.

Let $\mathbf{X}_B = \{\mathbf{x} \in \mathbb{Z}^\kappa :\ B\mathbf{x} \in \mathbf{X}\}$ and $\mathbf{x} \in \mathbf{X}_B$. Then $\forall\, i$, $|p_i(B\mathbf{x})| \le c_i$, and then $||Q'B\mathbf{x}||^2 \le \sum_{i=1}^{l} c_i^2 = c$. From Proposition 4, if μ_{min} is the smallest eigenvalue of $(Q'B)^*(Q'B)$, we have $\mu_{min}||\mathbf{x}||^2 \le ||Q'B\mathbf{x}||^2 \le c$. Since $Q'B$ is invertible, $\mu_{min} \neq 0$ and $||\mathbf{x}|| \le \sqrt{\frac{c}{\mu_{min}}}$. Then $\mathbf{X}_B$ and $\mathbf{X}$ are finite, and we can easily compute them.

Therefore, we can compute $S = \{[\varepsilon, \ldots, \varepsilon, \mathbf{d}_1, \ldots, \mathbf{d}_{k-1}] : \forall i, \mathbf{d}_i \in \mathbf{X}\}$. $\square$

From Theorem 1 we know that for any given template we can decide whether it is avoided by a word generated by a primitive morphism with no eigenvalue of absolute value 1. We can deduce the following result.

THEOREM 4. *Let $h : \Sigma^* \to \Sigma^*$ be a primitive morphism with no eigenvalue of absolute value* 1*, and let $\Phi : \Sigma^* \to \mathbb{Z}^d$ be a morphism. If $E_e(M_h) \cap \ker(\Phi) = \{\vec{0}\}$, then one can decide whether every word in* $\mathrm{Fact}^\infty(h)$ *avoids kth powers modulo* Φ.

The conditions from Theorem 4 seem restrictive, but we can apply this theorem to every morphic word avoiding additive powers that we found. It seems reasonable to think that the condition $E_e(M_h) \cap \ker(\Phi) = \{\vec{0}\}$ is necessary in order to generate a word avoiding kth power modulo Φ. But for the sake of completeness, we ask the following question.

PROBLEM 2. *Is there an algorithm deciding kth power modulo Φ freeness of (pure) morphic words?*

6. Results. In this section we use the algorithms described in sections 4 and 5 to show that additive squares are avoidable over $\mathbb{Z}^2$, and that abelian squares of period more than 5 are avoidable over the ternary alphabet. We also give some other new results about additive power avoidability and long 2-abelian power avoidability.

6.1. Abelian-square-free pure morphic words. Let h_6 be the following morphism:

$$h_6 : \begin{cases} a \to & ace, \qquad\qquad b \to \quad adf, \\ c \to & bdf, \qquad\qquad d \to \quad bdc, \\ e \to & afe, \qquad\qquad f \to \quad bce. \end{cases}$$

THEOREM 5. *$h_6^\omega(a)$ is abelian square-free.*

We provide a computer program[1] that applies the algorithm described in the previous section in order to show Theorem 5.

The matrix associated has the following eigenvalues: 0 (with algebraic multiplicity 3), 3, $\sqrt{3}$, and $-\sqrt{3}$. A Jordan decomposition of M_{h_6} is PJP^{-1}, with

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{3} \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} -\frac{1}{2} & 0 & -1 & 1 & 2+\sqrt{3} & 2-\sqrt{3} \\ \frac{1}{2} & -1 & 0 & 1 & -2-\sqrt{3} & \sqrt{3}-2 \\ -\frac{1}{2} & 1 & -1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 & -3-2\sqrt{3} & 2\sqrt{3}-3 \\ 0 & \frac{1}{2} & 1 & 1 & 3+2\sqrt{3} & 3-2\sqrt{3} \\ \frac{1}{2} & -\frac{1}{2} & 0 & 1 & 1 & 1 \end{bmatrix}.$$

The bounds on r_i^*, $i \in \{1, 2, 3\}$, computed as explained in the proof of Proposition 7 on $(h_6)^2$, are, respectively, 4, $\frac{4}{3}$, and $\frac{4}{3}$. The template $[\varepsilon, \varepsilon, \varepsilon, \overrightarrow{0}]$ has 28514 parents with respect to those bounds, and it has 48459 different ancestors including itself. None of the factors of $h_6^\omega(a)$ is a small realization of a forbidden template so we can conclude that $h_6^\omega(a)$ avoids abelian squares.

From Proposition 7, the Parikh vectors of the factors of $h_6^\omega(a)$ are close to a subspace of dimension 3. The conditions from Theorem 3 explain why finding this morphism is the first step in showing that long abelian squares are avoidable over the ternary alphabet. It seems hard to find simpler morphisms with this property; in particular we are interested in the following question.

[1] The code can be found in the supplementary materials (M114937_01.zip [local/web 11.4KB]).

PROBLEM 3. *Is there an abelian square-free pure morphic word over* 4 *or* 5 *letters generated by a morphism with only* 3 *eigenvalues of norm at least* 1*?*

In fact, for similar reasons, a positive answer to the following question could help show that additive squares are avoidable over $\mathbb{Z}$.

PROBLEM 4. *Is there an abelian square-free pure morphic word generated by a morphism with only* 2 *eigenvalues of norm at least* 1*?*

Let h_8 be the following morphism:

$$h_8 : \begin{cases} a \to \ h, & b \to \ g, \\ c \to \ f, & d \to \ e, \\ e \to \ hc, & f \to \ ac, \\ g \to \ db, & h \to \ eb. \end{cases}$$

THEOREM 6. *Words in h_8^∞ (e.g., infinite fixed points of $(h_8)^2$) are abelian square-free.*

This morphism may also be interesting because it is a small morphism which gives an abelian square-free word, its matrix is invertible, and it has 4 eigenvalues of absolute value less than 1. In particular, such a morphism could be part of a simpler construction of an abelian square-free word over 4 letters.

It would be interesting for the sake of completeness to be able to decide the abelian kth power freeness for any morphism. We can get rid of the primitivity condition with some technicalities, but it seems much harder to deal with eigenvalues of absolute value exactly 1.

PROBLEM 5. *Is it decidable, for any morphism h, whether the fixed points of h are abelian kth power-free?*

In fact, we do not know of any example of interesting morphism with an eigenvalue of norm 1 generating an abelian kth power-free word.

6.2. Additive square-free words over $\mathbb{Z}^2$. Let Φ be the following morphism:

$$\Phi : \begin{cases} a \to \ (1,0,0), & b \to \ (1,1,1), \\ c \to \ (1,2,1), & d \to \ (1,0,1), \\ e \to \ (1,2,0), & f \to \ (1,1,0). \end{cases}$$

THEOREM 7. *$h_6^\omega(a)$ does not contain squares modulo Φ.*

We provide a computer program[2] that applies the algorithm described in the previous section to $\phi(h_6^\omega(a))$.

In other words, the fixed point $h_{\text{add}}^\omega(\binom{0}{0})$ of the following morphism does not contain any additive square:

$$h_{\text{add}} : \begin{cases} \begin{pmatrix}0\\0\end{pmatrix} \to \begin{pmatrix}0\\0\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix}\begin{pmatrix}2\\0\end{pmatrix}, & \begin{pmatrix}1\\1\end{pmatrix} \to \begin{pmatrix}0\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}, \\ \begin{pmatrix}2\\1\end{pmatrix} \to \begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}, & \begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix}, \\ \begin{pmatrix}2\\0\end{pmatrix} \to \begin{pmatrix}0\\0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}2\\0\end{pmatrix}, & \begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix}\begin{pmatrix}2\\0\end{pmatrix}. \end{cases}$$

This implies the following result.

[2]The code can be found in the supplementary materials (M114937_01.zip [local/web 11.4KB]).

THEOREM 8. *$\mathbb{Z}^2$ is not uniformly 2-repetitive.*

It seems rather natural to ask the following question.

PROBLEM 6. *What is the smallest alphabet $\Sigma \subseteq \mathbb{Z}^2$ over which we can avoid additive squares?*

6.3. Additive cubes-free words over $\mathbb{Z}$. Cassaigne et al. showed that the fixed point of $f : 0 \to 03, 1 \to 43, 3 \to 1, 4 \to 01$, avoids additive cubes [3]. Our algorithm is able to reach the same conclusion for this morphism. We can also use it to show that additive cubes are avoidable over some other alphabets of size 4. Let

$$h_4 : \begin{cases} 0 \to & 001, \\ 1 \to & 041, \\ 2 \to & 41, \\ 4 \to & 442, \end{cases} \quad h'_4 : \begin{cases} 0 \to & 03, \\ 2 \to & 53, \\ 3 \to & 2, \\ 5 \to & 02, \end{cases} \quad \text{and} \quad h''_4 : \begin{cases} 0 \to & 03, \\ 2 \to & 63, \\ 3 \to & 2, \\ 6 \to & 02. \end{cases}$$

THEOREM 9. *$h_4^\omega(0)$, $h_4'^\omega(0)$, and $h_4''^\omega(0)$ avoid additive cubes.*

In fact, it seems that $\{0, 1, 2, 3\}$ is the only alphabet of 4 integers over which additive cubes are hard to avoid.

PROBLEM 7. *Are additive cubes avoidable over $\{0, 1, 2, 3\}$?*

6.4. Mäkelä's problem. Let g_3 be the following morphism:

$$g_3 : \begin{cases} a \to & \texttt{bbbaabaaac}, \\ b \to & \texttt{bccacccbcc}, \\ c \to & \texttt{ccccbbbcbc}, \\ d \to & \texttt{cccccccaa}, \\ e \to & \texttt{bbbbbcabaa}, \\ f \to & \texttt{aaaaaaabaa}. \end{cases}$$

THEOREM 10. *The word obtained by applying g_3 to the fixed point of h_6, that is, $g_3(h_6^\omega(a))$, avoids abelian squares of period more than 5.*

The kernel of g_3 is of dimension 3, but using the bounds on the 3 null eigenvalues of h_6 we can compute that $[\varepsilon, \dots, \varepsilon, \overrightarrow{0}, \dots, \overrightarrow{0}]$ has at most 16214 parents by g_3 realizable by h_6. This is checked using Theorem 3. This gives an answer to a weak version of Problem 1.

THEOREM 11. *There is an infinite word over 3 letters avoiding abelian squares of period more than 5.*

The optimal value for this result is probably not 5, so we ask the following question.

PROBLEM 8. *What is the smallest $p \in \mathbb{N}$ such that one can avoid abelian squares of period more than p over 3 letters?*

The proof technique presented here could be helpful to solve this problem. Note that we know that $2 \leq p \leq 5$. In fact, $g_3(h_6^\omega(a))$ contains 34 different abelian squares. We could also ask to minimize the number of different abelian squares.

6.5. Avoidability of long 2-abelian squares. Recently, Karhumäki, Saarela, and Zamboni introduced the notion of *k-abelian equivalence* as a generalization of both abelian equivalence and equality of words [9]. Two words u and v are said to be *k-abelian equivalent* (for $k \geq 1$), denoted $u \approx_{a,k} v$, if for every $w \in \Sigma^*$ such that $|w| \leq k$, $|u|_w = |v|_w$. A word $u_1u_2 \dots u_n$ is a *k-abelian nth power* if it is nonempty, and $u_1 \approx_{a,k} u_2 \approx_{a,k} \ldots \approx_{a,k} u_n$. Its *period* is $|u_1|$. A word is said to be *k-abelian-nth-power-free* if none of its factors are a k-abelian nth power. Note that when $k=1$, the k-abelian equivalence is exactly the abelian equivalence.

The existence of the word from Theorem 10 allows us to answer the following question.

PROBLEM 9 (see [14, 15]). *Can we avoid* 2*-abelian squares of period at least* p *on the binary alphabet, for some* $p \in \mathbb{N}$*?*

Let h_2 be the following morphism:

$$h_2 : \begin{cases} a \to & 111111111000, \\ b \to & 101011110100, \\ c \to & 101011000000. \end{cases}$$

THEOREM 12. $h_2(g_3(h_6^\omega(a)))$ *does not contain any* 2*-abelian square of period more than* 63.

Using the same technique as in [15] we can show, by reasoning only on h_2, that any 2-abelian square of $h_2(g_3(h_6^\omega(a)))$ is small (shorter than 9) or has a parent realized by $g_3(h_6^\omega(a))$ which is an abelian square. Thus the largest 2-abelian squares of $h_2(g_3(h_6^\omega(a)))$ have a period of at most $12 \times 7 = 84$. The value 63 is then obtained by checking all of the factors of $h_2(g_3(h_6^\omega(a)))$ of size at most 168.

The value 63 is probably not optimal (the lower bound from [15] is 2). In fact, it is possible to reach 60 by using a simpler second morphism, but the proof is more complicated and requires adapting the notion of templates and parents to k-abelian powers. The easiest way to significantly improve this result would be to improve the upper bound on the period for Mäkelä's question.

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