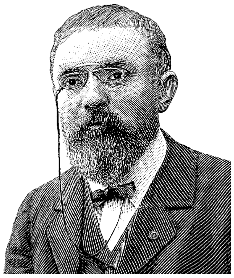

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Asymptotics for Strassen's optimal transport problem

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Abstract. In this paper, we consider Strassen's version of optimal transport (OT) problem, which concerns minimizing the excess-cost probability (i.e., the probability that the cost is larger than a given value) over all couplings of two given distributions. We derive large deviation, moderate deviation, and central limit theorems for this problem. Our proof is based on Strassen's dual formulation of the OT problem, Sanov's theorem on the large deviation principle (LDP) of empirical measures, as well as the moderate deviation principle (MDP) and central limit theorems (CLT) of empirical measures. In order to apply the LDP, MDP, and CLT to Strassen's OT problem, nested formulas for Strassen's OT problem are derived. Based on these nested formulas and using a splitting technique, we construct asymptotically optimal solutions to Strassen's OT problem and its dual formulation.

Résumé. Dans cet article, nous considérons la version de Strassen du problème de transport optimal (TO), qui porte sur la minimisation de la probabilité de surcoût (c'est-à-dire la probabilité que le coût soit supérieur à une valeur donnée) sur tous les couplages de deux distributions données. Nous obtenons des théorèmes de grande déviation, de déviation modérée et de limite centrale pour ce problème. Notre preuve est basée sur la formulation duale du problème TO introduite par Strassen, le théorème de Sanov sur le principe de grande déviation (PGD) des mesures empiriques, ainsi que le principe de déviation modérée (PDM) et les théorèmes centraux limites (TCL) des mesures empiriques. Afin d'appliquer les PGD, PDM et TLC au problème TO de Strassen, des formules imbriquées pour le problème TO de Strassen sont établies. Sur la base de ces formules imbriquées et en utilisant une technique de division, nous construisons des solutions asymptotiquement optimales au problème TO de Strassen et à sa formulation duale.

MSC2020 subject classifications: Primary 60F10; 60F05; secondary 49Q22

Keywords: Optimal transport; Large deviation; Moderate deviation; Central limit theorem

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Lipschitz continuity of probability kernels in the optimal transport framework

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Abstract. In Bayesian statistics, a continuity property of the posterior distribution with respect to the observable variable is crucial as it expresses well-posedness, i.e., stability with respect to errors in the measurement of data. Essentially, this requires analyzing the continuity of a probability kernel or, equivalently, of a conditional probability distribution with respect to the conditioning variable.

Here, we tackle this problem from a theoretical point of view. Let $(\mathbb{X}, \mathrm{d}_{\mathbb{X}})$ be a metric space, and let $\mathscr{B}(\mathbb{R}^d)$ denote the Borel σ-algebra on $\mathbb{R}^d$. Let $\pi(\cdot|\cdot) : \mathscr{B}(\mathbb{R}^d) \times \mathbb{X} \to [0, 1]$ be a dominated probability kernel, i.e. of the form $\pi(\mathrm{d}\theta|x) = g(x, \theta)\pi(\mathrm{d}\theta)$ for some suitable function $g : \mathbb{X} \times \mathbb{R}^d \to [0, +\infty)$. We provide general conditions ensuring the Lipschitz continuity of the mapping $\mathbb{X} \ni x \mapsto \pi(\cdot|x) \in \mathcal{P}(\mathbb{R}^d)$ when the space of probability measures $\mathcal{P}(\mathbb{R}^d)$ on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$ is endowed with a metric arising within the optimal transport framework, such as a Wasserstein metric. In particular, we prove explicit upper bounds for the Lipschitz constant in terms of Fisher-information functionals and weighted Poincaré constants, obtained by exploiting the dynamic formulation of the optimal transport.

Finally, we give some illustrations on noteworthy classes of probability kernels, and we apply the main results to improve on some open questions in Bayesian statistics, dealing with the approximation of posterior distributions by mixtures and posterior consistency.

Résumé. En statistique bayésienne, une propriété de continuité de la distribution a posteriori par rapport à la variable observée est cruciale puisque'elle exprime le caractère bien posé du problème, c'est-à-dire la stabilité par rapport aux erreurs de mesure dans les données. Cela nécessite essentiellement d'analyser la continuité d'un noyau de probabilité ou, de manière équivalente, d'une distribution de probabilité conditionnelle par rapport à la variable de conditionnement.

Ici, nous abordons ce problème d'un point de vue théorique. Soit $(\mathbb{X}, \mathrm{d}_{\mathbb{X}})$ un espace métrique, et soit $\mathscr{B}(\mathbb{R}^d)$ la tribu borélienne sur $\mathbb{R}^d$. Soit $\pi(\cdot|\cdot) : \mathscr{B}(\mathbb{R}^d) \times \mathbb{X} \to [0, 1]$ un noyau de probabilité dominé, c'est-à-dire de la forme $\pi(\mathrm{d}\theta|x) = g(x, \theta)\pi(\mathrm{d}\theta)$ pour une fonction appropriée $g : \mathbb{X} \to [0, +\infty)$. Nous fournissons des conditions générales assurant la continuité lipschitzienne de l'application $x \in \mathbb{X} \mapsto \mathcal{P}(\mathbb{R}^d)$ lorsque que l'espace des mesures de probabilités $\mathcal{P}(\mathbb{R}^d)$ sur $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$ est muni d'une métrique issue d'un cadre de transport optimal, telle qu'une métrique de Wasserstein. En particulier, nous prouvons des bornes supérieures explicites pour la constante de Lipschitz en termes de fonctionnelles d'information de Fisher et de constantes de Poincaré pondérées, obtenues en exploitant la formulation dynamique du transport optimal.

Enfin, nous donnons quelques illustrations sur des classes remarquables de noyaux de probabilité, et nous appliquons nos résultats principaux pour améliorer certaines questions ouvertes en statistique bayésienne, traitant de l'approximation de distributions a posteriori par des mélanges et la consistance a posteriori.

MSC2020 subject classifications: Primary 60F10; secondary 62F15

Keywords: Bayes formula; Bayesian consistency; Benamou–Brenier formula; Probability Kernel; Optimal transport; Wasserstein distance

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Reflecting Brownian motion in generalized parabolic domains: Explosion and superdiffusivity

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Abstract. For a multidimensional driftless diffusion in an unbounded, smooth, sub-linear generalized parabolic domain, with oblique reflection from the boundary, we give natural conditions under which either explosion occurs, if the domain narrows sufficiently fast at infinity, or else there is superdiffusive transience, which we quantify with a strong law of large numbers. For example, in the case of a planar domain, explosion occurs if and only if the area of the domain is finite. We develop and apply novel semimartingale criteria for studying explosions and establishing strong laws, which are of independent interest.

Résumé. Pour une diffusion multidimensionnelle sans dérive dans un domaine parabolique généralisé non borné, lisse, sous-linéaire, avec réflexion oblique à partir de la frontière, on donne des conditions naturelles dans lesquelles soit l'explosion se produit, si le domaine se rétrécit suffisamment vite envers l'infini, soit il y a la transience superdiffusive, que nous quantifions avec une loi forte des grands nombres. Par exemple, dans le cas d'un domaine planaire, l'explosion se produit si et seulement si la surface du domaine est finie. Nous développons et appliquons de nouveaux critères de semimartingale pour étudier les explosions et établir des lois fortes, qui présentent un intérêt indépendant.

MSC2020 subject classifications: Primary 60J60; secondary 60J55; 60J65; 60F15; 60K50

Keywords: Reflected diffusion; Oblique reflection; Horn-shaped domain; Explosion; Transience; Semimartingale criteria; Law of large numbers; Anomalous diffusion

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Time reversal of diffusion processes under a finite entropy condition

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Abstract. Motivated by entropic optimal transport, time reversal of diffusion processes is revisited. An integration by parts formula is derived for the carré du champ of a Markov process in an abstract space. It leads to a time reversal formula for a wide class of diffusion processes in $\mathbb{R}^n$ possibly with singular drifts, extending the already known results in this domain.

The proof of the integration by parts formula relies on stochastic derivatives. This formula is applied to compute the semimartingale characteristics of the time-reversed P^* of a diffusion measure P provided that the relative entropy of P with respect to another diffusion measure R is finite, and the semimartingale characteristics of the time-reversed R^* are known (for instance when the reference path measure R is reversible).

As an illustration of the robustness of this method, the integration by parts formula is also employed to derive a time-reversal formula for a random walk on a graph.

Résumé. L'étude du retournement du temps des processus de diffusion est reprise en vue d'applications au transport entropique optimal. On prouve une formule d'intégration par parties pour le carré du champ d'un processus de Markov dans un espace abstrait qui nous permet d'obtenir une formule de retournement du temps pour une grande classe de processus de diffusion à valeurs dans $\mathbb{R}^n$ dont les coefficients de dérive peuvent présenter des singularités, étendant en cela les résultats antérieurs sur le sujet.

La preuve de la formule d'intégration par parties se fait à l'aide de dérivées stochastiques. Cette formule nous permet de calculer les caractéristiques de la semi-martingale de loi P^* retournée temporelle de la loi P d'une diffusion, sous l'hypothèse que l'entropie relative de P par rapport à une mesure de chemins R de référence dont on connait les caractéristiques de la semi-martingale de loi retournée R^*, par exemple lorsque R est réversible.

Pour illustrer la flexibilité de cette méthode, la formule d'intégration par parties est aussi utilisée pour prouver une formule de retournement du temps pour des marches aléatoires sur des graphes.

MSC2020 subject classifications: 60J60; 60J25

Keywords: Time-reversal; Diffusion process; Stochastic derivative; Relative entropy; Random walk; Entropic optimal transport

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Stable limit theorems for additive functionals of one-dimensional diffusion processes

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Abstract. We consider a positive recurrent one-dimensional diffusion process with continuous coefficients and we establish stable central limit theorems for a certain type of additive functionals of this diffusion. In other words we find some explicit conditions on the additive functional so that its fluctuations behave like some α-stable process in large time for $\alpha \in (0, 2]$.

Résumé. On considère une diffusion unidimensionnelle, récurrente positive à coefficients continus et on établit des théorèmes central limite pour un certain type de fonctionnelles additives de la diffusion. En d'autres termes, on donne des conditions explicites sur la fonctionnelle additive pour que ses fluctuations se comportent comme un processus α-stable en temps grand, avec $\alpha \in (0, 2]$.

MSC2020 subject classifications: 60J60; 60F05

Keywords: One-dimensional diffusion processes; Stable central limit theorem; Stable processes; Local times

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Doubly stochastic Yule cascades (part II): The explosion problem in the non-reversible case

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Abstract. We analyze the explosion problem for a class of stochastic models introduced in (Dascaliuc et al. (2021)), referred to as doubly stochastic Yule cascades. These models arise naturally in the construction of solutions to evolutionary PDEs as well as in purely probabilistic first passage percolation phenomena having a Markov-type statistical dependence, new for this context. Using cut-set arguments and a greedy algorithm, we respectively establish criteria for non-explosion and explosion without requiring the time-reversibility of the underlying branching Markov chain (a condition required in Dascaliuc et al. 2021). Notable applications include the explosion of the self-similar cascade of the Navier–Stokes equations in dimension $d = 3$ and non-explosion in dimensions $d \geq 12$.

Résumé. Nous étudions le problème d'explosion pour une classe de modèles stochastiques introduites dans (Dascaliuc et al. (2021)), appelées cascades de Yule doublement stochastiques. Ces modèles interviennent naturellement dans la construction de solutions des équations aux dérivées partielles évolutives ainsi que dans les phénomènes de percolation de premier passage purement probabilistes ayant une dépendance statistique de type Markov, nouvelle dans ce contexte. À l'aide d'arguments d'ensembles séparateurs et d'algorithme glouton, nous établissons respectivement des critères de non-explosion et d'explosion sans exiger la réversibilité temporelle de la chaîne de Markov branchante sous-jacente (une condition requise dans Dascaliuc et al. 2021). Les applications notables incluent l'explosion de la cascade auto-similaire des équations de Navier–Stokes en dimension $d = 3$ et la non-explosion en dimension $d \geq 12$.

MSC2020 subject classifications: 60H30; 60J80

Keywords: Yule cascade; Doubly stochastic Yule cascade; Stochastic explosion; Navier–Stokes equations; KPP equation

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Asymptotics of the distribution and harmonic moments for a supercritical branching process in a random environment

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Abstract. Let (Z_n) be a supercritical branching process in an independent and identically distributed random environment ξ. We deduce the exact decay rate of the probability $\mathbb{P}(Z_n = j|Z_0 = k)$ as $n \to \infty$, for each $j \geq k$, assuming that $\mathbb{P}(Z_1 = 0) = 0$. We also study the existence of harmonic moments of the random variable $W = \lim_{n\to\infty} \frac{Z_n}{\mathbb{E}(Z_n|\xi)}$ under a simple moment condition.

Résumé. Soit (Z_n) un processus de branchement surcritique en environnement aléatoire ξ indépendant et identiquement distribué. Nous donnons un équivalent de la probabilité $\mathbb{P}(Z_n = j|Z_0 = k)$ lorsque $n \to \infty$, pour tout $j \geq k$, sous la condition $\mathbb{P}(Z_1 = 0) = 0$. Nous étudions également l'existence des moments harmoniques de la variable aléatoire limite $W = \lim_{n\to\infty} \frac{Z_n}{\mathbb{E}(Z_n|\xi)}$, sous une hypothèse simple d'existence de moments.

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Keywords: Branching processes; Random environment; Harmonic moments; Asymptotic distribution; Decay rate

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Limit distributions of branching Markov chains

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Abstract. We study branching Markov chains on a countable state space (space of types) $\mathcal{X}$ with the focus on the qualitative aspects of the limit behaviour of the evolving empirical population distributions. No conditions are imposed on the multitype offspring distributions at the points of $\mathcal{X}$ other than to have the same average and to satisfy a uniform $L \log L$ moment condition. We show that the arising population martingale is uniformly integrable. Convergence of population averages of the branching chain is then put in connection with stationary spaces of the associated ordinary Markov chain on $\mathcal{X}$ (assumed to be irreducible and transient). Our principal result is the almost sure convergence of the empirical distributions to a random probability measure on the boundary of an appropriate compactification of $\mathcal{X}$. Final considerations concern the general interplay between the measure theoretic boundaries of the branching chain and the associated ordinary chain.

Résumé. Nous étudions les chaînes de Markov branchantes sur un espace d'états (espace de types) dénombrable $\mathcal{X}$ en mettant l'accent sur les aspects qualitatifs du comportement limite de l'évolution des distributions empiriques de la population. Aucune condition n'est imposée sur les distributions multitypes des descendants des points de $\mathcal{X}$ autre que d'avoir la même moyenne et de satisfaire à une condition de moment de type $L \log L$. Nous montrons que la martingale de population résultante est uniformément intégrable. Ensuite, nous établissons le lien entre la convergence des moyennes empiriques de la chaîne branchante et les espaces stationnaires de la chaîne de Markov ordinaire associée sur $\mathcal{X}$ (supposée irréductible et transiente). Notre résultat principal est la convergence presque sûre des distributions empiriques vers une mesure de probabilité aléatoire sur le bord d'une compactification appropriée de $\mathcal{X}$. Les considérations finales portent sur l'interaction générale entre les bords mesurables de la chaîne branchante et de la chaîne ordinaire associée.

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Central limit theorems for the $(2 + 1)$-dimensional directed polymer in the weak disorder limit

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Abstract. In this article, we present an invariance principle for the paths of the directed random polymer in space dimension two in the subcritical intermediate disorder regime. More precisely, the distribution of diffusively rescaled polymer paths converges in probability to the law of Brownian motion when taking the weak disorder limit. So far analogous results have only been established for $d \neq 2$. Along the way, we prove a local limit theorem which allows us to factorise the point-to-point partition function of the directed polymer into a product of two point-to-plane partition functions.

Résumé. Dans cet article, nous présentons un principe d'invariance pour les chemins du polymère aléatoire dirigé dans l'espace de dimension deux et dans le régime de désordre intermédiaire sous-critique. Plus précisément, la distribution des chemins de polymères sous un changement d'échelle diffusif converge en probabilité vers la loi du mouvement brownien en prenant la limite de désordre faible. Jusqu'à présent, des résultats analogues n'ont été établis que pour $d \neq 2$. En cours de route, nous prouvons un théorème limite local qui nous permet de factoriser la fonction de partition point à point du polymère dirigé en un produit de deux fonctions de partition point à plan.

MSC2020 subject classifications: Primary 82D60; secondary 60F17; 60K37; 82B44

Keywords: Directed polymer model; Random environment; Weak disorder; Invariance principle; Local limit theorem; Functional central limit theorem

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Empirical measures and random walks on compact spaces in the quadratic Wasserstein metric

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Abstract. Estimating the rate of convergence of the empirical measure of an i.i.d. sample to the reference measure is a classical problem in probability theory. Extending recent results of Ambrosio, Stra and Trevisan on 2-dimensional manifolds, in this paper we prove sharp asymptotic and nonasymptotic upper bounds for the mean rate in the quadratic Wasserstein metric W_2 on a d-dimensional compact Riemannian manifold. Under a smoothness assumption on the reference measure, our bounds match the classical rate in the optimal matching problem on the unit cube due to Ajtai, Komlós, Tusnády and Talagrand. The i.i.d. condition is relaxed to stationary samples with a mixing condition. As an example of a nonstationary sample, we also consider the empirical measure of a random walk on a compact Lie group. Surprisingly, on semisimple groups random walks attain almost optimal rates even without a spectral gap assumption. The proofs are based on Fourier analysis, and in particular on a Berry–Esseen smoothing inequality for W_2 on compact manifolds, a result of independent interest with a wide range of applications.

Résumé. Estimer la vitesse de convergence de la mesure empirique d'un échantillon i.i.d. vers la mesure de référence est un problème classique en théorie des probabilités. Dans cet article, nous étendons des résultats récents d'Ambrosio, Stra et Trevisan sur les variétés riemanniennnes de dimension 2, et prouvons des bornes supérieures optimales, à la fois asymptotiques et non-asymptotiques, pour la vitesse moyenne selon la distance de Wasserstein quadratique W_2 sur une variété riemannienne compacte de dimension d. En supposant que la mesure de référence est suffisamment lisse, nos bornes coïncident avec la vitesse de convergence classique pour le problème d'appariement optimal sur le cube unité, dû à Ajtai, Komlós, Tusnády et Talagrand. Nous remplaçons l'hypothèse i.i.d. par celle plus faible d'échantillons stationnaires satisfaisant une condition de mélange. Comme exemple d'échantillon non-stationnaire, nous considérons aussi la mesure empirique d'une marche aléatoire sur un groupe de Lie compact. Étonnamment, pour les groupes semi-simples, les marches aléatoires atteignent des vitesses de convergence presque optimales, même sans hypothèse de trou spectral. Les preuves sont basées sur de l'analyse de Fourier, et en particulier sur une inégalité de lissage de Berry–Esseen pour W_2 sur des variétés riemanniennes compactes, un résultat qui est intéressant en lui-même et possède un grand nombre d'applications.

MSC2020 subject classifications: 60B05; 60B15; 60G10; 49Q22

Keywords: Riemannian manifold; Lie group; Optimal transportation; Berry–Esseen inequality; Heat kernel; Occupation measure

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Variability of paths and differential equations with BV-coefficients

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Abstract. We define compositions $\varphi(X)$ of Hölder paths X in $\mathbb{R}^n$ and functions of bounded variation φ under a relative condition involving the path and the gradient measure of φ. We show the existence and properties of generalized Lebesgue–Stieltjes integrals of compositions $\varphi(X)$ with respect to a given Hölder path Y. These results are then used, together with Doss' transform, to obtain existence and, in a certain sense, uniqueness results for differential equations in $\mathbb{R}^n$ driven by Hölder paths and involving coefficients of bounded variation. Examples include equations with discontinuous coefficients driven by paths of two-dimensional fractional Brownian motions.

Résumé. Nous définissons les compositions $\varphi(X)$ de trajectoires Hölder X dans $\mathbb{R}^n$ et les fonctions de variation bornée φ sous une condition relative qui fait intervenir la trajectoire et la mesure de gradient de φ. Nous montrons l'existence et les propriétés des intégrales généralisées de Lebesgue–Stieltjes des compositions de $\varphi(X)$ par rapport à un trajectoire donnée de Hölder Y. Ces résultats sont ensuite utilisés, ensemble avec la transformation de Doss, pour obtenir des résultats d'existence et d'unicité pour des équations différentielles dans $\mathbb{R}^n$ conduites par des trajectoires Hölder et avec des coefficients de variation bornée. Les exemples incluent des équations avec des coefficients discontinus conduits par des trajectoires de mouvement brownien fractionnaire à deux dimensions.

MSC2020 subject classifications: Primary 31B10; 34A12; 34A34; secondary 26A33; 26A42; 26B30; 26B35; 28A78; 31B99; 60G22

Keywords: Functions of bounded variation; Generalized Lebesgue–Stieltjes integrals; Occupation measure; Hölder path; Riesz potential; Systems of nonlinear differential equations

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Functional CLT for non-Hermitian random matrices

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Abstract. For large dimensional non-Hermitian random matrices X with real or complex independent, identically distributed, centered entries, we consider the fluctuations of $f(X)$ as a matrix where f is an analytic function around the spectrum of X. We prove that for a generic bounded square matrix A, the quantity $\operatorname{Tr} f(X)A$ exhibits Gaussian fluctuations as the matrix size grows to infinity, which consists of two independent modes corresponding to the tracial and traceless parts of A. We find a new formula for the variance of the traceless part that involves the Frobenius norm of A and the L^2-norm of f on the boundary of the limiting spectrum.

Résumé. On étudie les fluctuations de $f(X)$, où X est une matrice aléatoire non-hermitienne de grande taille à coefficients i.i.d. (réels ou complexes), et f une fonction analytique sur un domaine qui contient le spectre de X. On prouve que, pour une matrice carrée générique et bornée A, les fluctuations de la quantité $\operatorname{tr} f(X)A$ sont asymptotiquement gaussiennes et comportent deux modes indépendants, correspondant aux composantes traciale et de trace nulle de A. Une nouvelle formule est établie pour la variance de la composante de trace nulle, qui fait intervenir la norme de Frobenius de A et la norme L^2 de f sur la frontière du spectre limite.

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Tightness of discrete Gibbsian line ensembles with exponential interaction Hamiltonians

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Abstract. In this paper we introduce a framework to prove the tightness of a sequence of discrete Gibbsian line ensembles $\mathcal{L}^N = \{\mathcal{L}_k^1(u), \mathcal{L}_k^2(u), \dots\}$, a countable collection of random curves. The sequence of discrete line ensembles $\mathcal{L}^N$ we consider enjoys a resampling invariance property, which we call $(\mathbf{H}^N, \mathrm{H}^{\mathrm{RW},N})$-Gibbs property. We assume that $\mathcal{L}^N$ satisfies technical Assumptions A1–A4 on $(\mathbf{H}^N, \mathrm{H}^{\mathrm{RW},N})$ and that the lowest labeled curve with a parabolic shift, $\mathcal{L}_1^N(u) + u^2/2$, converges weakly to a stationary process in the topology of uniform convergence on compact sets. Under these assumptions, we prove our main result Theorem 2.18 that $\mathcal{L}^N$ is tight and the $\mathbf{H}$-Brownian Gibbs property holds for all subsequential limit line ensembles with $\mathbf{H}(x) = e^x$. Together with the characterization result in Dimitrov (2021), this proves the convergence to the KPZ line ensemble. As an application of Theorem 2.18, under the weak noise scaling, we show that the scaled log-gamma line ensemble $\overline{\mathcal{L}}^N$ converge to the KPZ line ensemble.

Résumé. Dans cet article, nous introduisons un cadre de travail pour prouver la tension d'une suite d'ensembles discrets de droites gibbsiennes $\mathcal{L}^N = \{\mathcal{L}_k^1(u), \mathcal{L}_k^2(u), \dots\}$, une collection dénombrable de courbes aléatoires. La suite d'ensembles de lignes discrètes $\mathcal{L}^N$ que nous considérons jouit d'une propriété d'invariance par rééchantillonnage, que nous appelons propriété $(\mathbf{H}^N, \mathrm{H}^{\mathrm{RW},N})$-Gibbs. Nous supposons que $\mathcal{L}^N$ satisfait les hypothèses techniques A1–A4 sur $(\mathbf{H}^N, \mathrm{H}^{\mathrm{RW},N})$ et que la courbe étiquetée la plus basse avec un décalage parabolique, $\mathcal{L}_1^N(u) + u^2/2$, converge faiblement vers un processus stationnaire dans la topologie de la convergence uniforme sur les ensembles compacts. Sous ces hypothèses, nous prouvons notre résultat principal, Theorem 2.18, que $\mathcal{L}^N$ est tendu et que la propriété $\mathbf{H}$-Brownienne de Gibbs est vraie pour toutes les limites de sous-suites des ensembles de lignes avec $\mathbf{H}(x) = e^x$. Avec le résultat de la caractérisation dans Dimitrov (2021), cela prouve la convergence vers l'ensemble de lignes KPZ. En application du théorème 2.18 nous montrons que, dans la limite de bruit faible, l'ensemble de lignes log-gamma mis à l'échelle $\overline{\mathcal{L}}^N$ converge vers l'ensemble de lignes KPZ.

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Core size of a random partition for the Plancherel measure

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Abstract. We prove that the size of the e-core of a partition chosen under the Poissonised Plancherel measure converges in distribution to, as the Poisson parameter goes to $+\infty$ and after a suitable renormalisation, a sum of $e-1$ mutually independent Gamma distributions with explicit parameters. Such a result already exists for the uniform measure on the set of partitions of n as n goes to $+\infty$, the parameters of the Gamma distributions being all equal. We rely on the fact that the descent set of a partition is a determinantal point process under the Poissonised Plancherel measure and on a central limit theorem for such processes.

Résumé. Nous montrons que la taille du e-cœur d'une partition tirée selon la mesure de Plancherel poissonisée converge en distribution, quand le paramètre de Poisson tend vers $+\infty$ et après renormalisation, vers une somme de $e-1$ variables Gamma indépendentes avec des paramètres explicites. Un tel résultat existe dans la littérature pour la loi uniforme sur les partitions de n quand n tend vers $+\infty$, les paramètres des lois Gamma étant tous égaux. La preuve repose sur le fait que l'ensemble de descentes d'une partition est un processus ponctuel déterminantal sous la mesure de Plancherel poissonisée et sur l'utilisation d'un théorème central limite pour de tels processus.

MSC2020 subject classifications: 60C05

Keywords: Plancherel measure; Partition; Core; Determinantal point process; Central limit theorem

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Determinantal point processes conditioned on randomly incomplete configurations

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Abstract. For a broad class of point processes, including determinantal point processes, we construct associated marked and conditional ensembles, which allow to study a random configuration in the point process, based on information about a randomly incomplete part of the configuration. We show that our construction yields a well behaving transformation of sufficiently regular point processes. In the case of determinantal point processes, we explain that special cases of the conditional ensembles already appear implicitly in the literature, namely in the study of unitary invariant random matrix ensembles, in the Its–Izergin–Korepin–Slavnov method to analyze Fredholm determinants, and in the study of number rigidity. As applications of our construction, we show that a class of determinantal point processes induced by orthogonal projection operators, including the sine, Airy, and Bessel point processes, satisfies a strengthened notion of number rigidity, and we give a probabilistic interpretation of the Its–Izergin–Korepin–Slavnov method.

Résumé. Pour une large classe de processus de points, y compris les processus de points déterminantaux, nous construisons les ensembles marqués et conditionnés associés qui permettent d'étudier une configuration aléatoire dans le processus de points, à partir d'information sur une partie aléatoirement incomplète de cette configuration. Nous montrons que notre construction produit une transformation qui se comporte bien pour des processus de points suffisamment réguliers. Dans le cas des processus de points déterminantaux, nous expliquons que certains cas particuliers de ces ensembles conditionnés sont déjà apparus implicitement dans la littérature, à savoir dans l'étude des ensembles de matrices aléatoires unitairement invariant, dans la méthode de Its, Izergin, Korepin et Slavnov pour analyser les déterminants de Fredholm, et dans l'étude de la rigidité des nombres. Comme application de notre construction, nous montrons qu'une classe de processus de points déterminantaux induits par des projections orthogonales, comprenant les processus de points sinus, Airy et Bessel, satisfait une propriété plus forte que la rigidité des nombres, et nous donnons une interprétation probabiliste de la méthode de Its, Izergin, Korepin et Slavnov.

MSC2020 subject classifications: 60K35; 60B20; 35Q15

Keywords: Determinantal point processes; Riemann-Hilbert problems

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Uniform convergence to the Airy line ensemble

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Abstract. We show that classical integrable models of last passage percolation and the related nonintersecting random walks converge uniformly on compact sets to the Airy line ensemble. Our core approach is to show convergence of nonintersecting Bernoulli random walks in all feasible directions in the parameter space. We then use coupling arguments to extend convergence to other models.

Résumé. Nous montrons que les modèles intégrables classiques de percolation de dernier passage, et les marches aléatoires non-intersectantes associées, convergent uniformément sur tout compact vers l'ensemble de lignes d'Airy. Le coeur de notre approche est de montrer la convergence de marches aléatoires de Bernoulli non-intersectantes, dans toutes les directions possibles de l'espace des paramètres. Nous utilisons ensuite des arguments de couplage afin d'étendre la convergence à d'autres modèles.

MSC2020 subject classifications: 60K35

Keywords: Nonintersecting processes; Integrable probability; KPZ universality; Last passage percolation

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From the asymmetric simple exclusion processes to the stationary measures of the KPZ fixed point on an interval

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Abstract. Barraquand and Le Doussal (*Europhys. Lett.* **137** (2022) 61003) introduced a family of stationary measures for the (conjectural) KPZ fixed point on an interval with Neumann boundary conditions, and predicted that they arise as scaling limits of stationary measures of all models in the KPZ universality class on an interval. In this paper, we show that the stationary measures for KPZ fixed point on an interval arise as the scaling limits of the height increment processes for the open asymmetric simple exclusion process in the steady state, with parameters changing appropriately as the size of the system tends to infinity.

Résumé. Barraquand et Le Doussal (*Europhys. Lett.* **137** (2022) 61003) ont introduit une famille de mesures stationnaires pour le point fixe KPZ (conjectural) sur un intervalle avec conditions de Neumann aux bords. Ils ont prédit qu'elles apparaissent comme limites d'échelle des mesures stationnaires dans tous les modèles de la classe d'universalité KPZ sur un intervalle. Dans cet article, nous montrons que les mesures stationnaires pour le point fixe KPZ sur un intervalle apparaissent comme limites d'échelle du processus des accroissements de la hauteur du processus d'exclusion simple asymétrique ouvert à l'état stable, avec des paramètres changeants de façon appropriée lorsque la taille du système tend vers l'infini.

MSC2020 subject classifications: Primary 60K35; 60F05; secondary 82C22

Keywords: Asymmetric simple exclusion process; Scaling limit; KPZ fixed point on an interval

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The Wasserstein distance to the circular law

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Abstract. We investigate the Wasserstein distance between the empirical spectral distribution of non-Hermitian random matrices and the circular law. For Ginibre matrices, we obtain an optimal rate of convergence $n^{-1/2}$ in 1-Wasserstein distance. This shows that the expected transport cost of complex eigenvalues to the uniform measure on the unit disk decays faster (due to the repulsive behaviour) compared to that of i.i.d. points, which is known to include a logarithmic factor. For non-Gaussian entry distributions with finite moments, we also show that the rate of convergence nearly attains this optimal rate.

Résumé. Nous étudions la distance de Wasserstein entre la distribution spectrale empirique des matrices aléatoires non hermitiennes et la loi circulaire. Pour les matrices de Ginibre, nous obtenons un taux de convergence optimal $n^{-1/2}$ en distance 1-Wasserstein. Cela montre que d'espérance du coût de transport des valeurs propres complexes vers la mesure uniforme sur le disque unitaire décroît plus rapidement (en raison du comportement répulsif) par rapport à celui de points i.i.d. qui inclut un facteur logarithmique. Pour le cas des entrées avec loi non gaussienne à moments finis, nous montrons également que le taux de convergence atteint presque ce taux optimal.

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Keywords: Ginibre matrices; circular law; rate of convergence; Wasserstein distance; optimal transport

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Fluctuations and correlations for products of real asymmetric random matrices

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Abstract. We study the real eigenvalue statistics of products of independent real Ginibre random matrices. These are matrices all of whose entries are real i.i.d. standard Gaussian random variables. For such product ensembles, we demonstrate the asymptotic normality of suitably normalised linear statistics of the real eigenvalues and compute the limiting variance explicitly in both global and mesoscopic regimes. A key part of our proof establishes uniform decorrelation estimates for the related Pfaffian point process, thereby allowing us to exploit weak dependence of the real eigenvalues to give simple and quick proofs of the central limit theorems under quite general conditions. We also establish the universality of these point processes. We compute the asymptotic limit of all correlation functions of the real eigenvalues in the bulk, origin and spectral edge regimes. By a suitable strengthening of the convergence at the edge, we also obtain the limiting fluctuations of the largest real eigenvalue. Near the origin we find new limiting distributions characterising the smallest positive real eigenvalue.

Résumé. Nous étudions la distribution des valeurs propres du produit de matrices de Ginibre réelles indépendantes. Les coefficients de ces matrices sont des variables aléatoires i.i.d. réels Gaussiens. Pour de tels produits, on montre le caractère asymptotique Gaussien des statistiques linéaires des valeurs propres réelles et l'on calcule explicitement, en régime global et mésoscopique, les variances asymptotiques associées. Une partie clef de notre preuve établit des estimées de décorrélation pour le processus Pfaffien associé, ce qui permet d'exploiter la dépendance faible entre valeurs propres réelles pour donner des preuves simples et concises de théorèmes de la limite centrale sous des conditions générales. On établit également l'universalité de ces processus ponctuels. On calcule la limite des fonctions de corrélation des valeurs propres à l'intérieur et au bord du spectre limite. Grâce à un raffinement adéquat de la convergence au bord, on obtient les fluctuations limites de la plus grande valeur propre réelle. Près de l'origine, on trouve de nouvelles distributions limites caractérisant la plus petite valeur propre réelle.

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ANNALES DE L'INSTITUT HENRI POINCARÉ PROBABILITÉS ET STATISTIQUES

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Les articles acceptés pour publication sont considérés comme ne varietur. Les auteurs recevront une seule épreuve de leur article : celle-ci devra être retournée à l'éditeur dans le délai maximal d'une semaine. Toutes modifications ou corrections excessives autres que celles provenant d'erreurs typographiques peuvent être facturées aux auteurs. La publication des articles ou mémoires est gratuite, la facturation des pages est optionnelle. L'auteur correspondant recevra un fichier pdf de leur article final par courrier electronique.

Les auteurs sont encouragés à préparer leurs manuscrits au moyen de l'un des logiciels Plain TeX, LaTeX ou AMS TeX. Un support LATEX se trouve à l'adresse https://www.e-publications.org/ims/support/

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