Part 2: FIR Filters – Weighted-Chebyshev Method

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Victoria, BC, Canada
Email: aantoniou@ieee.org

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Introduction

The weighted-Chebyshev method for the design of FIR filters is an iterative multi-variable optimization method based on the *Remez Exchange Algorithm*.

It can be used to design optimal FIR (nonrecursive) filters with arbitrary amplitude responses.

Note: The material for this module is taken from Antoniou, *Digital Signal Processing: Signals, Systems, and Filters,* Chap. 15.

Introduction – Historical Evolution

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- These developments led in 1975 to the well-known McClellan-Parks-Rabiner computer program for the design of FIR filters, which has found widespread applications.
- Enhancements to the weighted-Chebyshev method were proposed by Antoniou during the early eighties.

Problem Formulation

Consider an FIR filter characterized by the transfer function

$$H(z) = \sum_{n=0}^{N-1} h(nT) z^{-n}$$

and assume that

- N is odd,
- the impulse response is symmetrical, and
- the sampling frequency is $\omega_s = 2\pi$.

Problem Formulation

The frequency response of the filter can be expressed as

$$H(e^{j\omega}) = e^{-jc\omega} P_c(\omega)$$

where

$$P_c(\omega) = \sum_{k=0}^{c} a_k \cos k\omega \qquad \text{(A)}$$

is the *gain function* and

$$\begin{aligned} a_0 &= h(c) \\ a_k &= 2h(c-k) \quad \text{for } k = 1, 2, \ldots, c \\ c &= (N-1)/2 \end{aligned}$$

Note that $P_c(\omega)$ is the frequency response of a noncausal version of the required filter.

Error Function

- An error function $E(\omega)$ can be constructed as

$$E(\omega) = W(\omega)[D(\omega) - P_c(\omega)]$$

where $e^{-jc\omega}D(\omega)$ is the idealized frequency response of the desired filter, $W(\omega)$ is a weighting function, and

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- If $|E(\omega)|$ is minimized such that

$$|E(\omega)| = |W(\omega)[D(\omega) - P_c(\omega)]| \leq \delta_p \quad \text{for} \quad \omega \in \Omega \qquad \text{(B)}$$

with respect a set of frequencies in the interval $[0, \pi]$, say Ω, a filter can be obtained in which

$$|E_0(\omega)| = |D(\omega) - P_c(\omega)| \leq \frac{\delta_p}{|W(\omega)|} \quad \text{for} \quad \omega \in \Omega \qquad \text{(C)}$$

Lowpass Filters

- In the case of a lowpass filter, the minimization of $|E(\omega)|$ will force the inequality

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$$D(\omega) = \begin{cases} 1 & \text{for } 0 \leq \omega \leq \omega_p \\ 0 & \text{for } \omega_a \leq \omega \leq \pi \end{cases}$$

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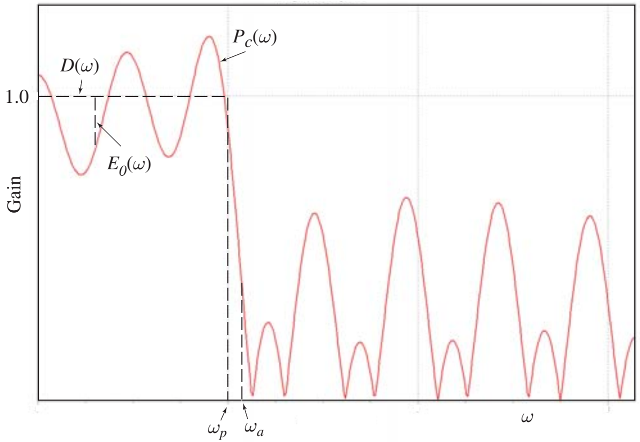
- In effect, a minimization algorithm will force the actual gain function $P_c(\omega)$ to approach the ideal gain function $D(\omega)$.

Lowpass Filters *Cont'd*



- If we choose the weighting function

$$W(\omega) = \begin{cases} 1 & \text{for } 0 \leq \omega \leq \omega_p \\ \frac{\delta_p}{\delta_a} & \text{for } \omega_a \leq \omega \leq \pi \end{cases}$$

then from Eq. (C), i.e.,

$$|E_0(\omega)| = |D(\omega) - P_c(\omega)| \leq \frac{\delta_p}{|W(\omega)|} \quad \text{for} \quad \omega \in \Omega \qquad \text{(C)}$$

we get

$$|E_0(\omega)| \leq \begin{cases} \delta_p & \text{for } 0 \leq \omega \leq \omega_p \\ \delta_a & \text{for } \omega_a \leq \omega \leq \pi \end{cases}$$

Minimax Problem

- The most appropriate approach for the solution of the optimization problem just described is to solve the minimax problem

$$\underset{\boldsymbol{x}}{\text{minimize}} \ \{\max_{\omega} |E(\omega)|\}$$

where

$$\boldsymbol{x} = [a_0 \quad a_1 \quad \cdots \quad a_c]^T$$

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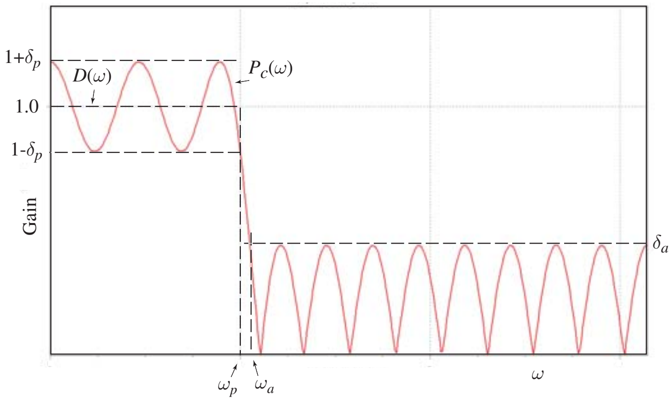
- By virtue of the so-called *alternation theorem*, there is a *unique equiripple* solution of the above minimax problem.
- *Note* that weighted-Chebyshev filters are so called because they have an *equiripple* amplitude response just like Chebyshev filters but are not related to Chebyshev filters in any other way.

Minimax Problem *Cont'd*



If $P_c(\omega)$ is a linear combination of $r = c + 1$ cosine functions of the form

$$P_c(\omega) = \sum_{k=0}^{c} a_k \cos k\omega$$

then a necessary and sufficient condition that $P_c(\omega)$ be the unique, best, weighted-Chebyshev approximation to a continuous function $D(\omega)$ on Ω, where Ω is a dense and compact subset of the frequency interval $[0,\ \pi]$, is that the weighted error function $E(\omega)$ exhibit at least $r + 1$ extremal frequencies $\hat{\omega}_i$ in Ω such that

$$\hat{\omega}_0 < \hat{\omega}_1 < \cdots < \hat{\omega}_r$$

$$E(\hat{\omega}_i) = -E(\hat{\omega}_{i+1}) \quad \text{for } \ i = 0,\ 1,\ \ldots,\ r - 1$$

and

$$|E(\hat{\omega}_i)| = \max_{\omega \in \Omega} |E(\omega)| \quad \text{for } \ i = 0,\ 1,\ \ldots,\ r$$

- From the alternation theorem and Eq. (B), i.e.,

$$E(\omega) = W(\omega)[D(\omega) - P_c(\omega)] \qquad \text{(B)}$$

we can write

$$E(\hat{\omega}_i) = W(\hat{\omega}_i)[D(\hat{\omega}_i) - P_c(\hat{\omega}_i)] = (-1)^i\delta$$

for $i = 0, 1, \ldots, r$, where δ is a constant.

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for $i = 0,\ 1,\ \ldots,\ r$, where δ is a constant.

- The above system of equations can be put in matrix form as

$$\begin{bmatrix} 1 & \cos\hat{\omega}_0 & \cos\hat{\omega}_0 & \cdots & \cos\hat{\omega}_0 & \frac{1}{W(\hat{\omega}_0)} \\ 1 & \cos\hat{\omega}_1 & \cos\hat{\omega}_1 & \cdots & \cos\hat{\omega}_1 & \frac{-1}{W(\hat{\omega}_1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \cos\hat{\omega}_r & \cos\hat{\omega}_r & \cdots & \cos\hat{\omega}_r & \frac{(-1)^r}{W(\hat{\omega}_r)} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_c \\ \delta \end{bmatrix} = \begin{bmatrix} D(\hat{\omega}_0) \\ D(\hat{\omega}_1) \\ \vdots \\ D(\hat{\omega}_{r-1}) \\ D(\hat{\omega}_r) \end{bmatrix}$$

- If the extremal frequencies (or extremals for short) were known, coefficients a_k and, in turn, the frequency response of the filter could be computed using Eq. (A), i.e.,

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- The solution of this system exists since the above $(r+1) \times (r+1)$ matrix is known to be nonsingular.

Remez Exchange Algorithm *Cont'd*

- The Remez exchange algorithm is an *iterative multivariable algorithm* that is naturally suited for the solution of the minimax problem just described.
- It is based on the second optimization method of Remez.

1. Initialize extremal frequencies $\hat{\omega}_0, \hat{\omega}_1, \ldots, \hat{\omega}_r$ and ensure that an extremal is assigned at each band edge.

Basic Remez Exchange Algorithm *Cont'd*

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2. Solve the system of equations to get δ and the coefficients $a_0, a_1, \ldots, a_c$.
3. Using the coefficients $a_0, a_1, \ldots, a_c$, calculate $P_c(\omega)$ and the magnitude of the error

$$|E(\omega)| = |W(\omega)[D(\omega) - P_c(\omega)]|$$

Basic Remez Exchange Algorithm *Cont'd*

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3. Using the coefficients $a_0, a_1, \ldots, a_c$, calculate $P_c(\omega)$ and the magnitude of the error

$$|E(\omega)| = |W(\omega)[D(\omega) - P_c(\omega)]|$$

4. Locate the frequencies $\widehat{\omega}_0, \widehat{\omega}_1, \ldots, \widehat{\omega}_\rho$ at which $|E(\omega)|$ is maximum and $|E(\widehat{\omega}_i)| \geq \delta$ (these frequencies are *potential extremals* for the next iteration).

5. Compute the convergence parameter

$$Q = \frac{\max |E(\widehat{\omega}_i)| - \min |E(\widehat{\omega}_i)|}{\max |E(\widehat{\omega}_i)|}$$

where $i = 0,\ 1,\ \ldots,\ \rho$.

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   where $i = 0,\ 1,\ \ldots,\ \rho$.

6. Reject $\rho - r$ *superfluous potential extremals* $\widehat{\omega}_i$ according to an appropriate rejection criterion and renumber the remaining $\widehat{\omega}_i$ by setting $\hat{\omega}_i = \widehat{\omega}_i$ for $i = 0,\ 1,\ \ldots,\ r$.

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7. If $Q > \varepsilon$, where ε is a convergence tolerance (say $\varepsilon = 0.01$), repeat from step 2; otherwise continue to step 8.

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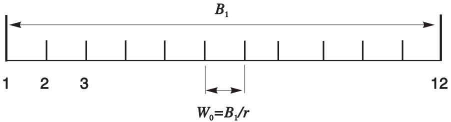
7. If $Q > \varepsilon$, where ε is a convergence tolerance (say $\varepsilon = 0.01$), repeat from step 2; otherwise continue to step 8.

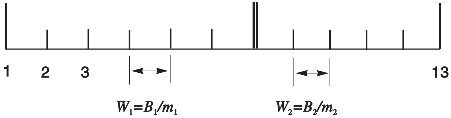
8. Compute $P_c(\omega)$ using the last set of extremal frequencies; then deduce $h(n)$, the impulse response of the required filter, and stop.

Initialization of Extremal Frequencies

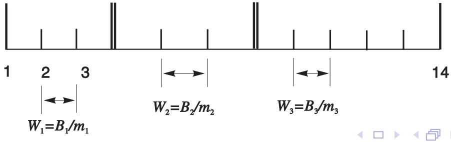
Bands: 1
Extremals: r+1 (12)
Intervals: r (11)



Bands: 2
Extremals: r+1 (13)
Intervals: r-1 (11)



Bands: 3
Extremals: r+1 (14)
Intervals: r-2 (11)



Initialization of Extremal Frequencies *Cont'd*

For a filter with J bands with bandwidths $B_1,\ B_2,\ \ldots,\ B_J$, the number of extremals and interval between extremals for each band can be calculated by using the following formulas:

$$W_0 = \frac{1}{r+1-J}\sum_{j=1}^{J} B_j$$

$$m_j = \left(\frac{B_j}{W_0}+0.5\right) \quad \text{for } j = 1,\ 2,\ \ldots,\ J-1$$

$$\text{and} \quad m_J = r - \sum_{j=1}^{J-1}(m_j+1)$$

$$W_j = \frac{B_j}{m_j} \quad \text{for } j = 1,\ 2,\ \ldots,\ J$$

where $r = (N+1)/2$ and N is the filter length.

Updating of Extremals

- In each iteration, the extremals need to be updated. This is done by finding the maxima of the error function

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- This could be done by solving the system

$$\begin{bmatrix} 1 & \cos\hat{\omega}_0 & \cos\hat{\omega}_0 & \cdots & \cos\hat{\omega}_0 & \frac{1}{W(\hat{\omega}_0)} \\ 1 & \cos\hat{\omega}_1 & \cos\hat{\omega}_1 & \cdots & \cos\hat{\omega}_1 & \frac{-1}{W(\hat{\omega}_1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \cos\hat{\omega}_r & \cos\hat{\omega}_r & \cdots & \cos\hat{\omega}_r & \frac{(-1)^r}{W(\hat{\omega}_r)} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_c \\ \delta \end{bmatrix} = \begin{bmatrix} D(\hat{\omega}_0) \\ D(\hat{\omega}_1) \\ \vdots \\ D(\hat{\omega}_{r-1}) \\ D(\hat{\omega}_r) \end{bmatrix}$$

for the coefficients a_k and then calculating

$$P_c(\omega) = \sum_{k=0}^{c} a_k \cos k\omega$$

and in turn $E(\omega)$.

- This approach is inefficient and may be subject to numerical ill-conditioning, in particular if δ is small and N is large.

  Note: A 50×50 matrix is quite typical.

Updating of Extremals *Cont'd*

- An alternative and more efficient approach is to deduce δ analytically (by using Cramer's rule) and then interpolate $P_c(\omega)$ on the r frequency points using the barycentric form of the Lagrange interpolation formula, as follows:

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- Calculate parameter δ as

$$\delta = \sum_{k=0}^{r} \frac{\alpha_k D(\hat{\omega}_k)}{\frac{\sum_{k=0}^{r} (-1)^k \alpha_k}{W(\hat{\omega})}}$$

Updating of Extremals *Cont'd*

- With δ known, $P_c(\omega)$ can be obtained as

$$P_c(\omega) = \begin{cases} C_k & \text{for } \omega = \hat{\omega}_0,\ \hat{\omega}_1,\ \ldots,\ \hat{\omega}_{r-1} \\ \dfrac{\displaystyle\sum_{k=0}^{r-1} \frac{\beta_k C_k}{x - x_k}}{\displaystyle\sum_{k=0}^{r-1} \frac{\beta_k}{x - x_k}} & \text{otherwise} \end{cases}$$

where $\alpha_k = \prod_{i=0,\ i\neq k}^{r} \frac{1}{x_k - x_i}, \quad \beta_k = \prod_{i=0,\ i\neq k}^{r-1} \frac{1}{x_k - x_i}$

and $C_k = D(\hat{\omega}_k) - (-1)^k \frac{\delta}{W(\hat{\omega}_k)}$

with $x = \cos\omega$ and $x_i = \cos\hat{\omega}_i$ for $i = 0,\ 1,\ \ldots,\ r$

Rejection of Superfluous Potential Extremals

- The problem formulation is such that *there must be exactly $r+1$ extremals* in each iteration.

Rejection of Superfluous Potential Extremals

- The problem formulation is such that *there must be exactly $r + 1$ extremals* in each iteration.
- Analysis will show that $|E(\omega)|$ can have as many as $r + 2J - 1$ maxima where J is the number of bands.

  If in any iteration the number of maxima exceeds $r + 1$, then the iteration is said to have generated *superfluous potential extremals*.

Rejection of Superfluous Potential Extremals *Cont'd*

- In the standard McClellan, Rabiner, and Parks algorithm, this difficulty is circumvented by rejecting the $\rho - r$ potential extremals $\widehat{\omega}_i$ that yield the lowest error $|E(\omega)|$.

Computation of Impulse Response

- The impulse response in Step 8 of the algorithm can be determined by recalling that function $P_c(\omega)$ is the frequency response of a noncausal version of the required filter.

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- The impulse response in Step 8 of the algorithm can be determined by recalling that function $P_c(\omega)$ is the frequency response of a noncausal version of the required filter.
- The impulse response of the noncausal filter, denoted as $h_0(n)$ for $-c \leq n \leq c$, can be determined by computing $P_c(k\Omega)$ for $k = 0, 1, \ldots, c$ where $\Omega = 2\pi/N$, and then using the inverse discrete Fourier transform.

- It can be shown that

$$h_0(n) = h_0(-n) = \frac{1}{N}\left\{P_c(0) + \sum_{k=1}^{c} 2P_c(k\Omega)\cos\left(\frac{2\pi kn}{N}\right)\right\}$$

for $n = 0,\ 1,\ \ldots,\ c$.

Computation of Impulse Response *Cont'd*

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  for $n = 0, 1, \ldots, c$.

- The impulse response of the required causal filter is given by

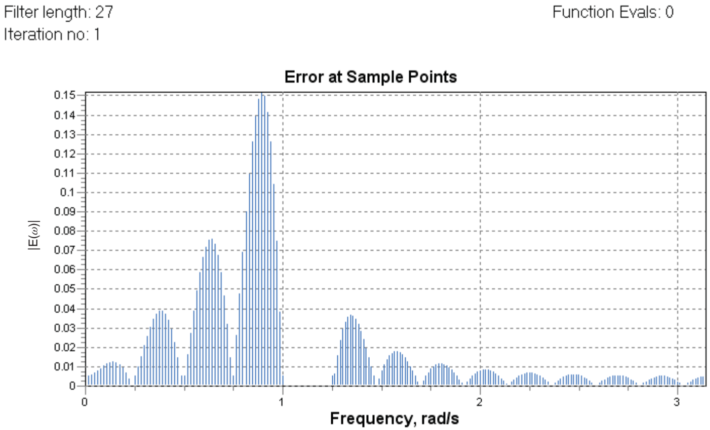
$$h(n) = h_0(n-c)$$

  for $n = 0, 1, \ldots, c$.

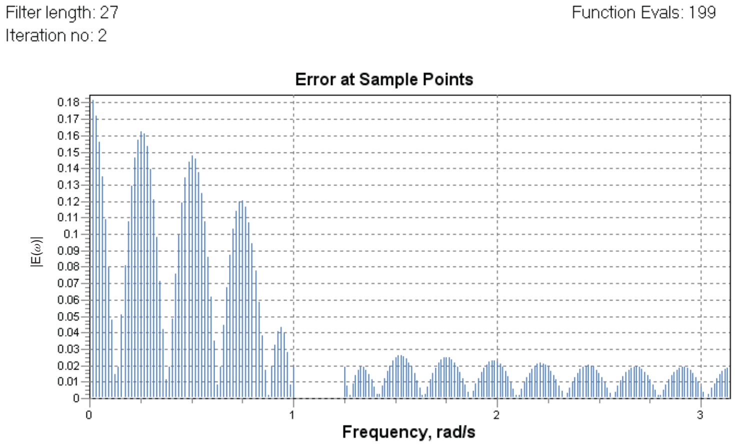
Example

Band	$D(\omega)$	$W(\omega)$	Left band edge	Right band edge
1	1	1	0	1.0
2	0	0.4	1.25	π
Sampling frequency: 2π				

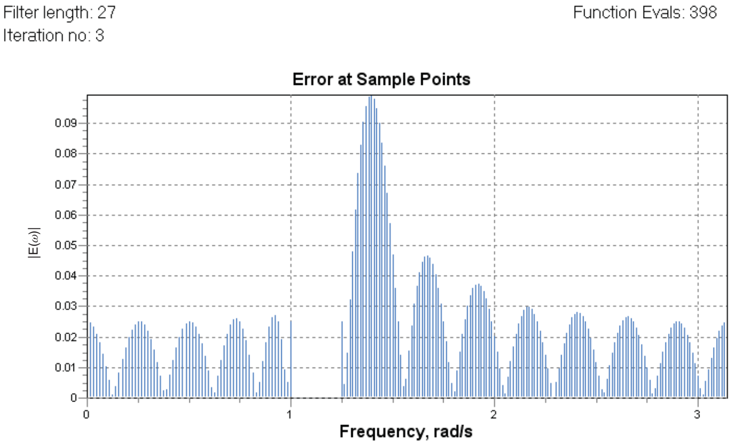
Example *Cont'd*



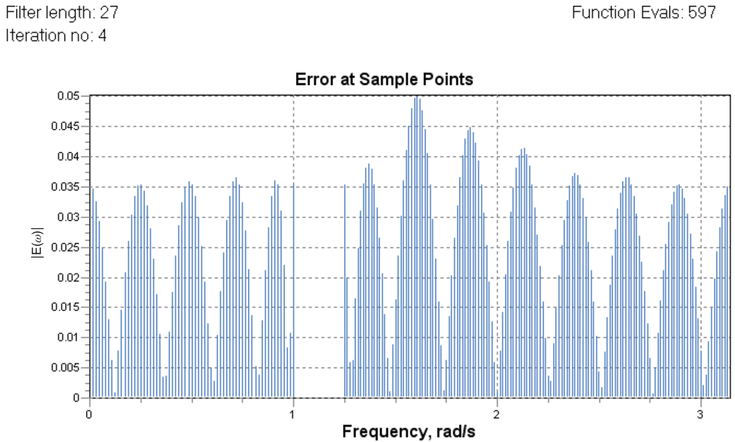
Example *Cont'd*

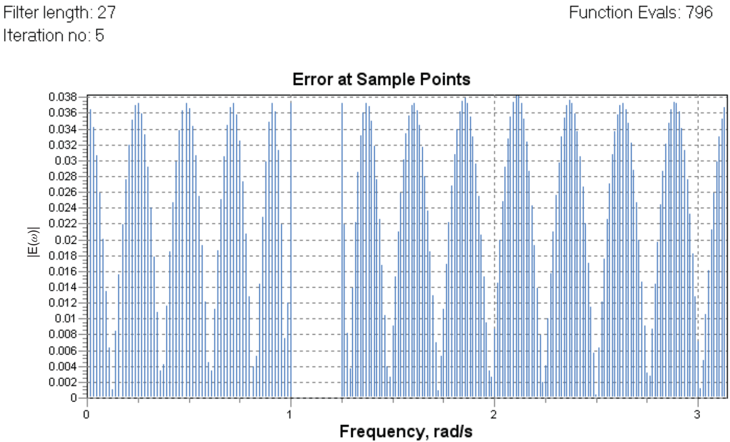


Example *Cont'd*

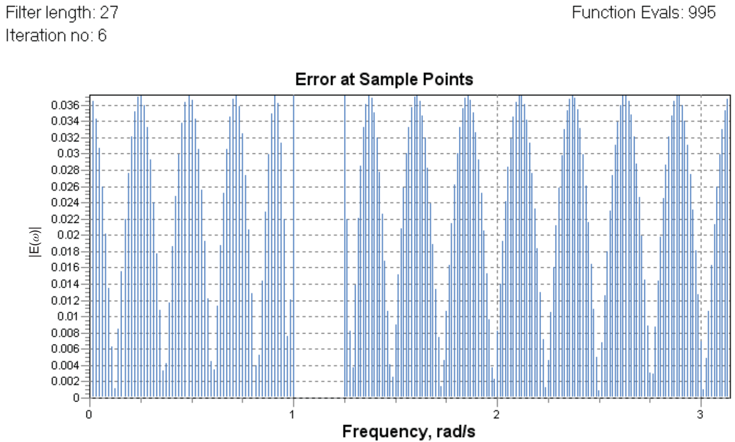


Example *Cont'd*




Filter length: 27
Iteration no: 5
Function Evals: 796
Error at Sample Points
0.038
0.036
0.034
0.032
0.03
0.028
0.026
0.024
0.022
0.02
0.018
0.016
0.014
0.012
0.01
0.008
0.006
0.004
0.002
0
|E(ω)|
0
1
2
3
Frequency, rad/s

Example *Cont'd*



Selective Step-by-Step Search

- When the system of equations

$$\begin{bmatrix} 1 & \cos\hat{\omega}_0 & \cos\hat{\omega}_0 & \cdots & \cos\hat{\omega}_0 & \frac{1}{W(\hat{\omega}_0)} \\ 1 & \cos\hat{\omega}_1 & \cos\hat{\omega}_1 & \cdots & \cos\hat{\omega}_1 & \frac{-1}{W(\hat{\omega}_1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \cos\hat{\omega}_r & \cos\hat{\omega}_r & \cdots & \cos\hat{\omega}_r & \frac{(-1)^r}{W(\hat{\omega}_r)} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_c \\ \delta \end{bmatrix} = \begin{bmatrix} D(\hat{\omega}_0) \\ D(\hat{\omega}_1) \\ \vdots \\ D(\hat{\omega}_{r-1}) \\ D(\hat{\omega}_r) \end{bmatrix}$$

is solved, the error function $|E(\omega)|$ is forced to satisfy the relation

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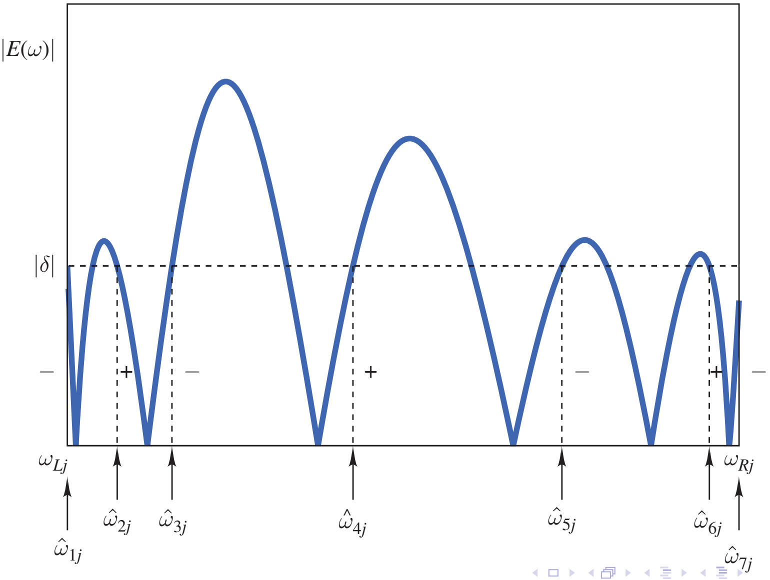
- This relation can be satisfied in a number of ways but the most likely possibility for the jth band is illustrated in the next slide where ω_{Lj} and ω_{Rj} are the left-hand and right-hand edges, respectively.

Selective Step-by-Step Search *Cont'd*



Selective Step-by-Step Search *Cont'd*

Because of the special nature of the error function

(a) the maxima of $|E(\omega)|$ can be easily found by searching in the vicinity of the extremals;

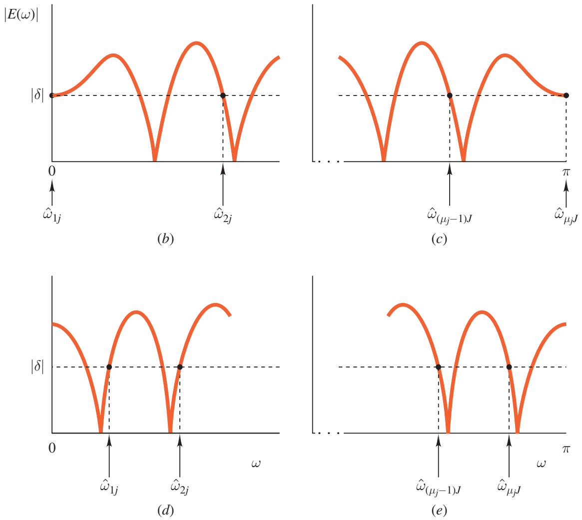
(b) gradient information can be used to expedite the search for the maxima of $|E(\omega)|$; and

(c) the closer we get to the solution, the closer are the maxima of the error function to the extremals.

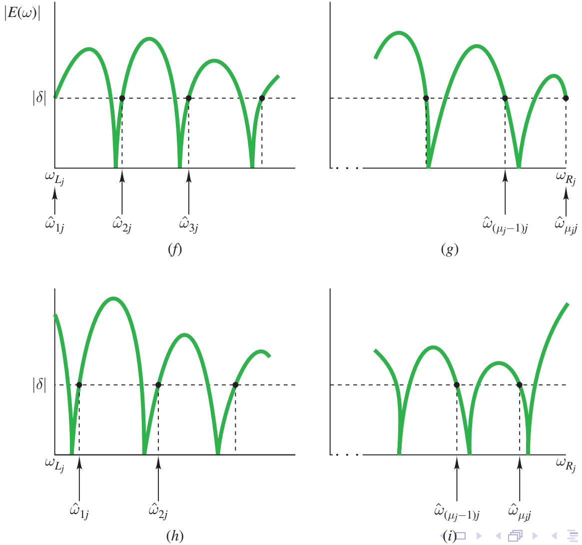
By using a *selective step-by-step search*, a large amount of computation can be eliminated.

Selective Step-by-Step Search *Cont'd*

Extra ripples can arise in the first and last bands.:



Also in interior bands:



Cubic Interpolation Search

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Cubic Interpolation Search

- Increased computational efficiency can be achieved by using a search based on cubic interpolation.
- Assuming that the error function shown in the figure can be represented by the third-order polynomial

$$|E(\omega)| = M = a + b\omega + c\omega^2 + d\omega^3$$

where $a,\ b,\ c$, and d are constants then

$$\frac{dM}{d\omega} = G = b + 2c\omega + 3d\omega^2$$

Hence, the frequencies at which M has stationary points are given by

$$\bar{\omega} = \frac{1}{3d}\left[-c \pm \sqrt{(c^2 - 3bd)}\right]$$

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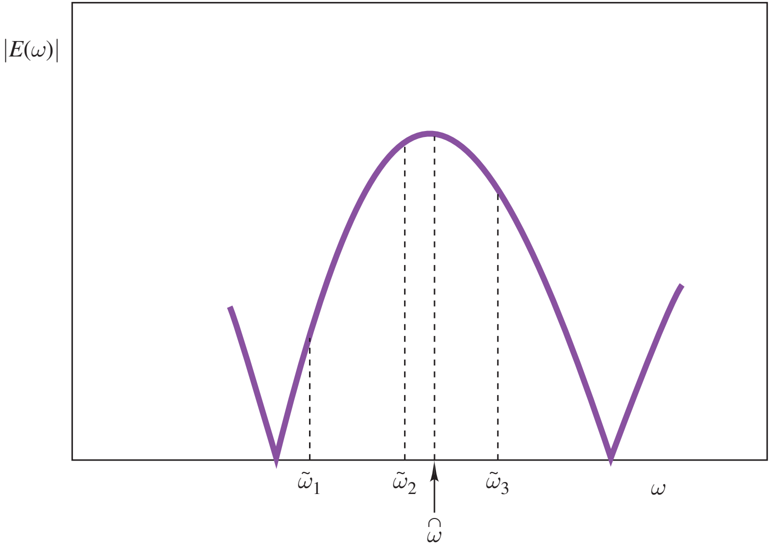
- Therefore, $|E(\omega)|$ has a maximum if

$$\frac{d^2M}{d\omega^2} = 2c + 6d\widehat{\omega} < 0 \quad \text{or} \quad \widehat{\omega} < -\frac{c}{3d}$$

Cubic Interpolation Search *Cont'd*



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- The selective step-by-step search may require as many as 8 function evaluations per potential extremal in the first two or three iterations but as the solution is approached only two or three function evaluations are required.

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- By using the cubic interpolation to start with and then switching over to the step-by-step search, an very efficient algorithm can be constructed.
- The decision to switch from cubic to selective can be based on the value of the convergence parameter Q (see Step 5).

  Switching from the cubic to the selective when Q is reduced below 0.65 works well.

Improved Rejection Scheme for Superfluous Potential Extremals

- If an extremal does not move from one iteration to the next, then the minimum value of $E(\widehat{\omega}_i)$ is simply δ, as can be easily shown, and this happens quite often even in the first or second iteration of the Remez algorithm.

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- As a consequence, rejecting potential extremals on the basis of the individual values of $E(\widehat{\omega}_i)$ tends to become random and this can slow the Remez algorithm quite significantly particularly for multiband filters.
- An improved scheme for the rejection of superfluous extremals based the rejection on the lowest average band error as well as the individual values of $E(\widehat{\omega}_i)$ is described in the next transparency.

Improved Rejection Scheme *Cont'd*

- Compute the average band errors

$$E_j = \frac{1}{\nu_j} \sum_{\widehat{\omega}_i \in \Omega_j} |E(\widehat{\omega}_i)| \quad \text{for } \; j = 1, 2, \ldots, J$$

where Ω_j is the set of extremals in band j given by

$$\Omega_j = \{\widehat{\omega}_i : \omega_{Lj} \leq \widehat{\omega}_i \leq \omega_{Rj}\}$$

ν_j is the number of potential extremals in band j, and J is the number of bands.

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- Rank the J bands in the order of lowest average error and let $l_1, l_2, \ldots, l_J$ be the ranked list obtained, i.e., l_1 and l_J are the bands with the lowest and highest average error, respectively.

Improved Rejection Scheme *Cont'd*

- Reject one $\widehat{\omega}_i$ in each of bands $I_1,\ I_2,\ \ldots,\ I_{J-1},\ I_1,\ I_2,$ ... until $\rho - r$ superfluous $\widehat{\omega}_i$ are rejected.

  In each case, reject the $\widehat{\omega}_i$, other than a band edge, that yields the lowest $|E(\widehat{\omega}_i)|$ in the band.

Example:

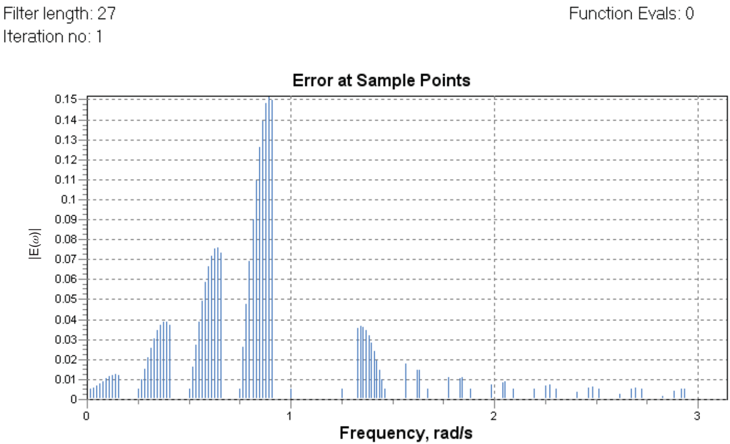
If $J = 3$, $\rho - r = 3$, and the average errors for bands 1, 2, and 3 are 0.05, 0.08, and 0.02, then $\widehat{\omega}_i$ are rejected in bands 3, 1, and 3.

Note: The potential extremals are not rejected in band 2 which is the band of highest average error.

Example

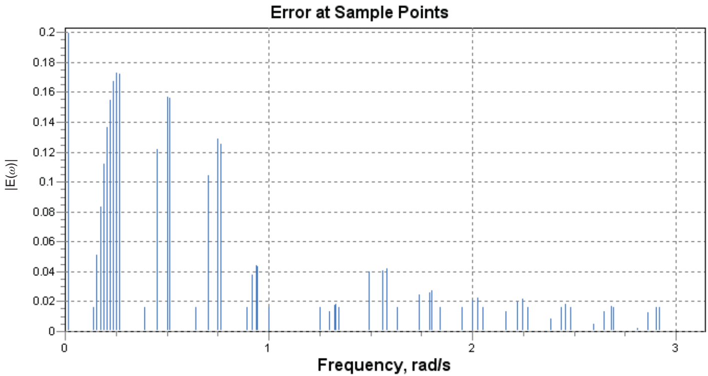
Band	$D(\omega)$	$W(\omega)$	Left band edge	Right band edge
1	1	1	0	1.0
2	0	0.4	1.25	π
Sampling frequency: 2π				

Example *Cont'd*

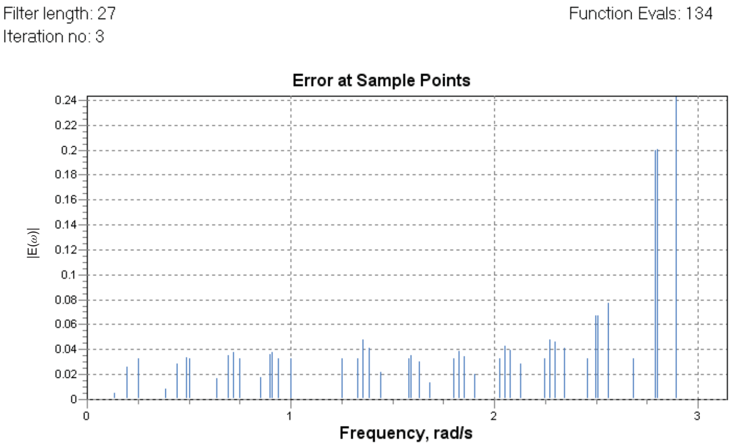


Filter length: 27
Iteration no: 2

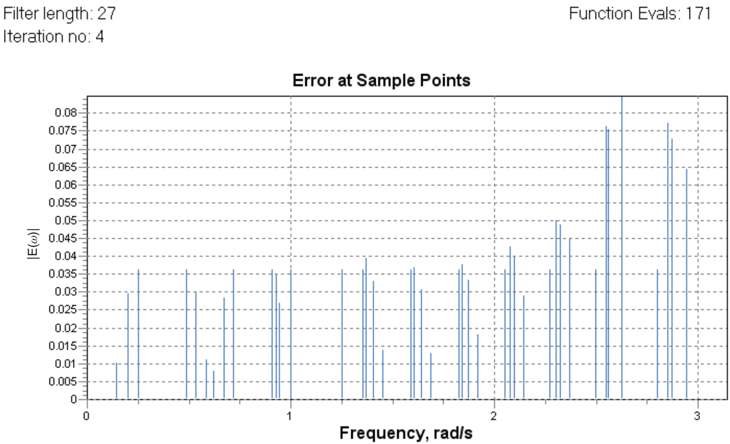
Function Evals: 87

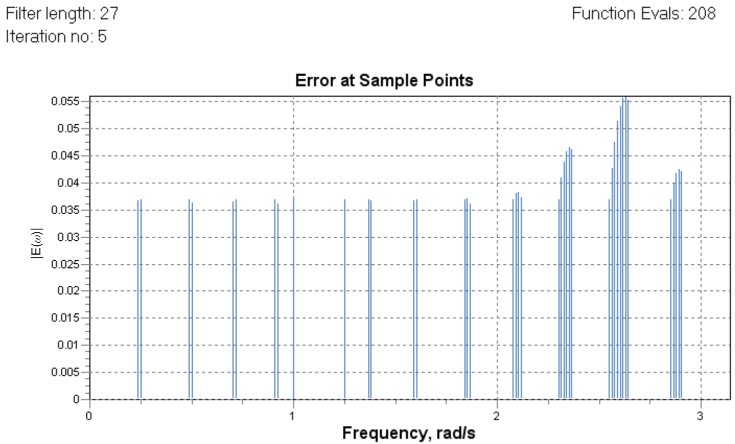


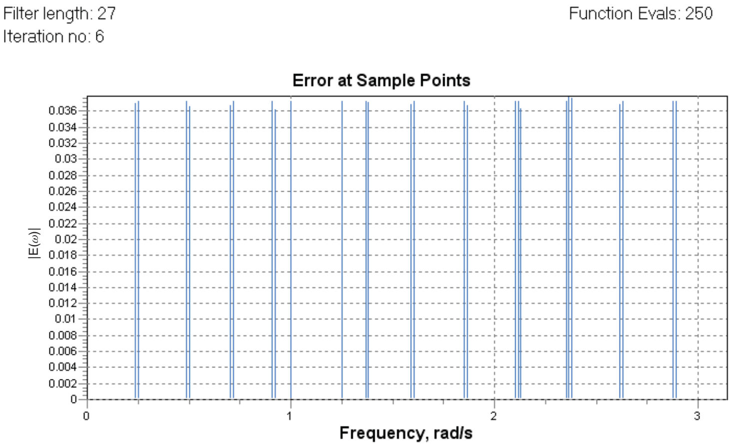
Example *Cont'd*



Example *Cont'd*




Filter length: 27
Iteration no: 5
Function Evals: 208
Error at Sample Points
0.055
0.05
0.045
0.04
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
|E(ω)|
0
1
2
3
Frequency, rad/s

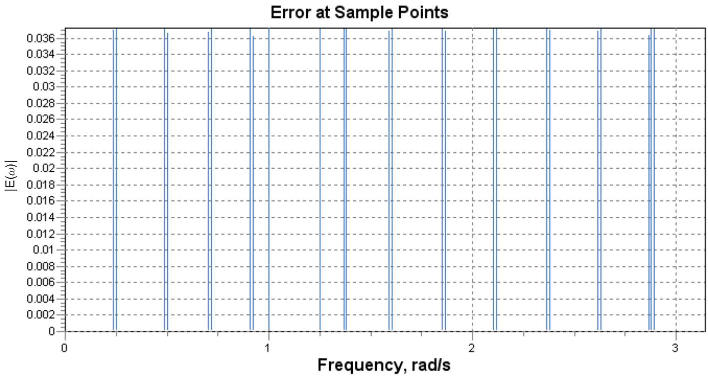

Filter length: 27
Iteration no: 6
Function Evals: 250
Error at Sample Points
0.036
0.034
0.032
0.03
0.028
0.026
0.024
0.022
0.02
0.018
0.016
0.014
0.012
0.01
0.008
0.006
0.004
0.002
0
|E(ω)|
0
1
2
3
Frequency, rad/s

Example *Cont'd*

Filter length: 27

Iteration no: 7

Function Evals: 278



Comparisons — Amount of Computation

Type of Filter	No. of Examples	Range of N	Ave. Funct. Evals.			Saving, %	
			A	B	C	C v B	C v A
LP	45	9-101	2691	722	372	48.9	86.3
HP	42	9-101	2774	710	356	49.9	87.2
BP	44	21-89	2777	667	338	49.3	87.8
BS	35	21-91	2720	639	336	47.4	87.6

A: Exhaustive search

B: Selective search

C: Selective plus cubic search

Comparisons — Robustness

Type of Filter	No. of Examples	No. Failures		
		A	B	C
LP	46	1	0	0
HP	43	1	0	0
BP	50	3	2	5
BS	45	6	8	8

A: Exhaustive search

B: Selective search

C: Selective plus cubic search

Prescribed Specifications

- Given a filter length N, a set of passband and stopband edges, and a ratio δ_p/δ_a, an FIR filter with approximately piecewise-constant amplitude-response specifications can be readily designed.

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- While the filter obtained will have passband and stopband edges at the correct locations and the ratio δ_p/δ_a will be exactly as required, the amplitudes of the passband and stopband ripples are highly unlikely to have the specified values.
- An acceptable design can be obtained by predicting the value of N on the basis of the required specifications and then designing filters for increasing or decreasing values of N until the lowest value of N that satisfies the specifications is found.

Filter Length Prediction

- A reasonably accurate empirical formula for the prediction of the required filter length, N, for the case of lowpass and highpass filters, due to Herrmann, Rabiner, and Chan, is

$$N = \text{int}\left[\frac{(D - FB^2)}{B} + 1.5\right]$$

where

$$\begin{aligned}
B &= |\omega_a - \omega_p|/2\pi \\
D &= [0.005309(\log_{10}\delta_p)^2 + 0.07114\log_{10}\delta_p - 0.4761]\log_{10}\delta_a \\
&\quad -[0.00266(\log_{10}\delta_p)^2 + 0.5941\log_{10}\delta_p + 0.4278] \\
F &= 0.51244(\log_{10}\delta_p - \log_{10}\delta_a) + 11.012
\end{aligned}$$

- The formula of Herrmann et al. can also be used to predict the filter length in the design of bandpass, bandstop, and multiband filters in general.

Filter Length Prediction

- The formula of Herrmann et al. can also be used to predict the filter length in the design of bandpass, bandstop, and multiband filters in general.
- In these filters, a value of N is computed for each transition band between a passband and stopband or a stopband and passband and the largest value of N so obtained is taken to be the predicted filter length.

Algorithm

1. Compute N using the prediction formula of Herrmann et al.; if N is even, set $N = N + 1$.
2. Design a filter of length N using the Remez algorithm and determine the minimum value of δ, say $\breve{\delta}$.
   - (A) If $\breve{\delta} > \delta_p$, then do:
     - (a) Set $N = N + 2$, design a filter of length N using the Remez algorithm, and find $\breve{\delta}$;
     - (b) If $\breve{\delta} \leq \delta_p$, then go to step 3; else, go to step $2(A)(a)$.
   - (B) If $\breve{\delta} < \delta_p$, then do:
     - (a) Set $N = N - 2$, design a filter of length N using the Remez algorithm, and find $\breve{\delta}$;
     - (b) If $\breve{\delta} > \delta_p$, then go to step 4; else, go to step $2(B)(a)$.

3. If part A of the algorithm was executed, use the last set of extremals and the corresponding value of N to obtain the impulse response of the required filter and stop.
4. If part B of the algorithm was executed, use the last but one set of extremals and the corresponding value of N to obtain the impulse response of the required filter and stop.

Example

In an application, an FIR equiripple bandstop filter is required which should satisfy the following specifications:

- Odd filter length
- Maximum passband ripple A_p: 0.5 dB
- Minimum stopband attenuation A_a: 50.0 dB
- Lower passband edge ω_{p1}: 0.8 rad/s
- Upper passband edge ω_{p2}: 2.2 rad/s
- Lower stopband edge ω_{a1}: 1.2 rad/s
- Upper stopband edge ω_{a2}: 1.8 rad/s
- Sampling frequency ω_s: 2π rad/s

Design the lowest-order filter that will satisfy the specifications.

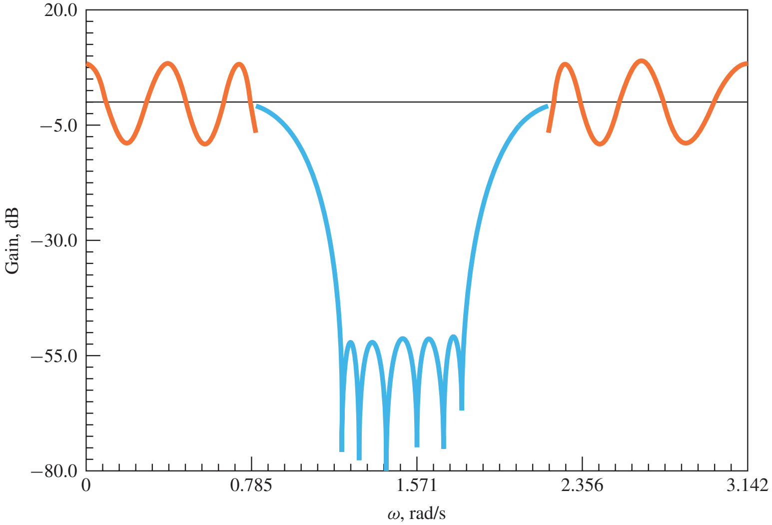
The design algorithm gave a filter with the following specifications:

- Passband ripple: 0.4342 dB
- Minimum stopband attenuation: 51.23 dB

Progress of Algorithm

N	Iters.	FE's	A_p, dB	A_a, dB
31	10	582	0.5055	49.91
33	7	376	0.5037	49.94
35	9	545	0.4342	51.23

Example *Cont'd*



Note: Passband errors multiplied by a factor of 40.

D-Filter

A DSP software package that incorporates the design techniques described in this presentation is *D-Filter*. Please see

http://www.d-filter.ece.uvic.ca

for more information.

Summary

- Three design techniques that bring about substantial improvements in the efficiency of the Remez algorithm have been described:
  - A step-by-step exhaustive search
  - A cubic interpolation search
  - An improved scheme for the rejection of superfluous potential extremals

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- Three design techniques that bring about substantial improvements in the efficiency of the Remez algorithm have been described:
  - A step-by-step exhaustive search
  - A cubic interpolation search
  - An improved scheme for the rejection of superfluous potential extremals
- These techniques are implemented in a DSP software package known as D-Filter.
- Extensive experimentation has shown that the selective and cubic interpolation searches reduce the amount of computation required by the Remez algorithm by almost 90% without degrading its robustness.

Summary *Cont'd*

- The rejection scheme described increases the efficiency and robustness of the Remez algorithm further but the scheme has not been compared with the original method of McClellan, Rabiner, and Parks.

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Summary *Cont'd*

- The rejection scheme described increases the efficiency and robustness of the Remez algorithm further but the scheme has not been compared with the original method of McClellan, Rabiner, and Parks.
- By using a prediction technique for the required filter length proposed by Herrmann, Rabiner, and Chan, filters that satisfy prescribed specifications can be designed.
- For off-line applications, the Remez algorithm continues to be the method of choice for the design of linear-phase filters, multiband filters, differentiators, Hilbert transformers.

- Despite the improvements described, the Remez algorithm continues to require a large amount of computation.

  For applications that need the filter to be designed on-line in real or quasi-real time, the window method is preferred although the filters obtained are suboptimal.

References

- A. Antoniou, *Digital Signal Processing: Signals, Systems, and Filters*, Chap. 15, McGraw-Hill, 2005.
- E. Ya. Remes, *General Computational Methods for Tchebycheff Approximation*, Kiev, 1957 (Atomic Energy Commission Translation 4491, pp. 1–85).
- O. Herrmann, “Design of nonrecursive digital filters with linear phase,” *Electron.Lett.*, vol. 6, pp. 182–184, May 1970.
- E. Hofstetter, A. Oppenheim, and J. Siegel, “A new technique for the design of non-recursive digital Filters,” *5th Annual Princeton Conf. Information Sciences and Systems*, pp. 64–72, Mar. 1971.
- T. W. Parks and J. H. McClellan, “Chebyshev approximation for nonrecursive digital filters with linear phase,” *IEEE Trans. Circuit Theory*, vol. CT-19, pp. 189–194, Mar. 1972.

- T. W. Parks and J. H. McClellan, "A program for the design of linear phase finite impulse response digital filters," *IEEE Trans. Audio Electroacoust.*, vol. AU-20, pp. 195–199, Aug. 1972.
- O. Herrmann, L. R. Rabiner, and D. S. K. Chan, "Practical design rules for optimum finite impulse response low-pass digital filters," *Bell Syst. Tech. J.*, vol. 52, pp. 769–799, Jul.-Aug. 1973.
- L. R. Rabiner and O. Herrmann, "On the design of optimum FIR low-pass filters with even impulse response duration," *IEEE Trans. Audio Electroacoust.*, vol. AU-21, pp. 329–336, Aug. 1973.
- J. H. McClellan and T. W. Parks, "A unified approach to the design of optimum FIR linear-phase digital filters," *IEEE Trans. Circuit Theory*, vol. CT-20, pp. 697–701, Nov. 1973.

- J. H. McClellan, T. W. Parks, and L. R. Rabiner, "A computer program for designing optimum FIR linear phase digital filters," *IEEE Trans. Audio Electroacoust.*, vol. AU-21, pp. 506–526, Dec. 1973.
- L. R. Rabiner, J. H. McClellan, and T. W. Parks, "FIR digital filter design techniques using weighted Chebyshev approximation," *Proc. IEEE*, vol. 63, pp. 595–610, Apr. 1975.
- J. H. McClellan, T. W. Parks, and L. R. Rabiner, "FIR linear phase filter design program," *Programs for digital signal processing*, IEEE Press, New York, pp. 5.1-1–5.1-13, 1979.
- A. Antoniou, "Accelerated procedure for the design of equiripple nonrecursive digital Filters," *IEE Proc.*, vol. 129, pt. G, pp. 1–10, Feb. 1982 (see *IEE Proc.*, vol. 129, pt. G, p. 107, Jun. 1982 for errata).

References *Cont'd*

- A. Antoniou, "New improved method for the design of weighted-Chebyshev, nonrecursive, digital filters," *IEEE Trans. Circuits Syst.*, vol. CAS-30, pp. 740–750, Oct. 1983.
- D. J. Shpak and A. Antoniou, "A generalized Reméz method for the design of FIR digital filters," *IEEE Trans. Circuits Syst.*, vol. CAS-37, pp. 161–174, Feb. 1990.

This slide concludes the presentation.
Thank you for your attention.