The symmetry of M.C. Escher's *Circle Limit IV* pattern and related patterns.

Douglas Dunham
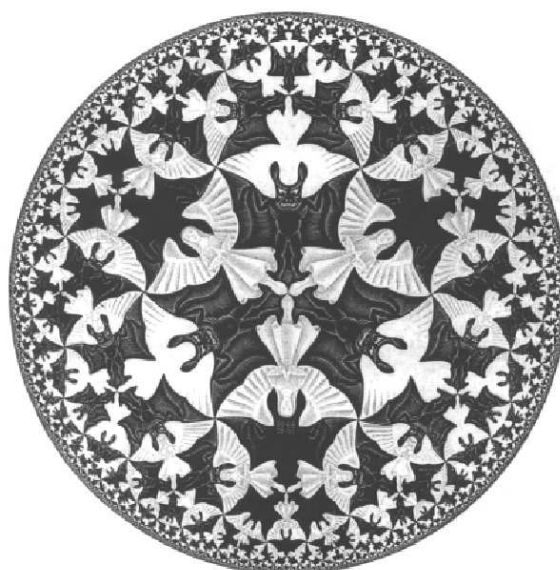
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Two Questions:

- **What is the correct orientation for the *Circle Limit IV* pattern?**

- **What is the symmetry group of the *Circle Limit IV* pattern?**

What is the correct orientation? Answer:

A Sub-question

Why aren't there signatures in all three of the blank devils' faces next to the central angels?
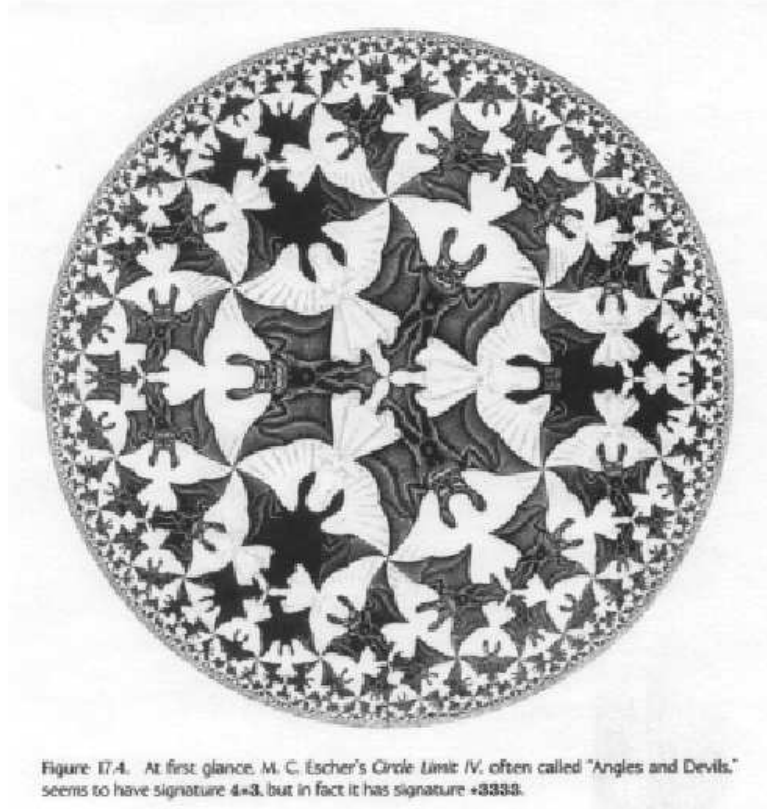
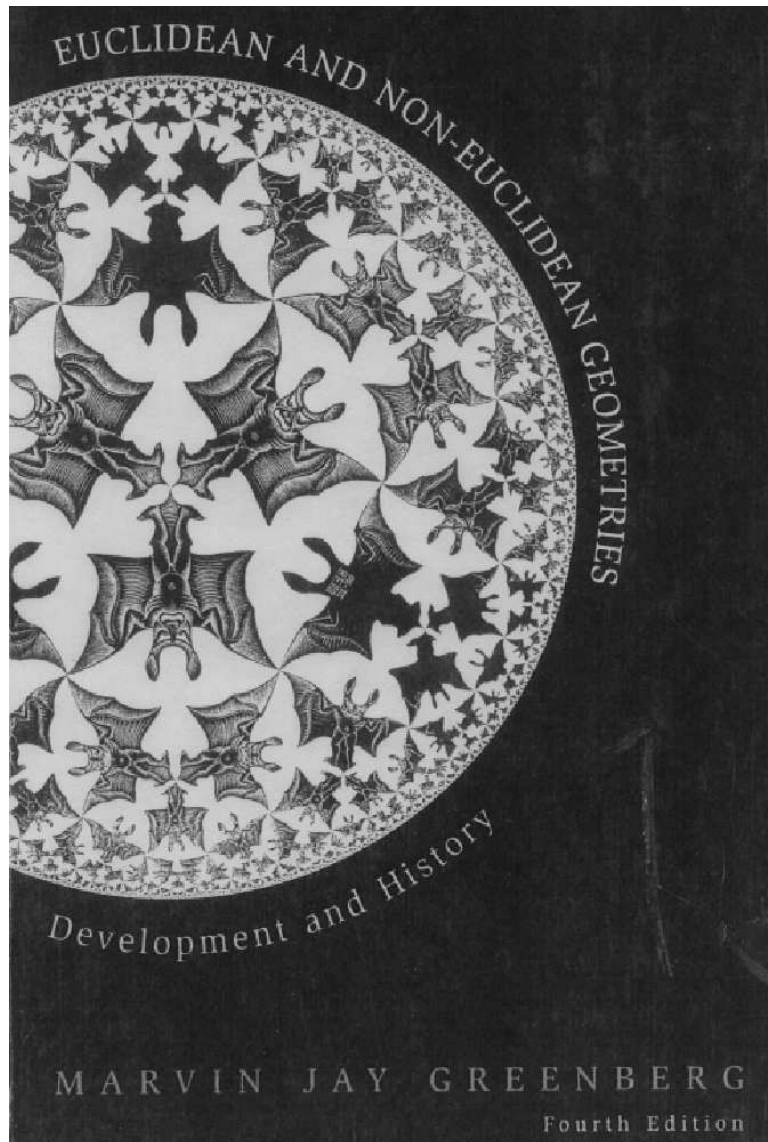# Escher's Signature (graphic)

Escher's Signature (text)

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MCE
VII-'60
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Why Examine Orientation Now?

- *The Symmetries of Things*
  Conway, Burgiel, Goodman␣Strass
  A.K. Peters, 2008

- *Euclidean and Non-Euclidean Geometries: Development and History*
  Marvin Greenberg
  W.H. Freeman, 2008

Figure 17.4. At first glance, M. C. Escher's *Circle Limit IV*, often called "Angles and Devils," seems to have signature 4*3, but in fact it has signature *3333.
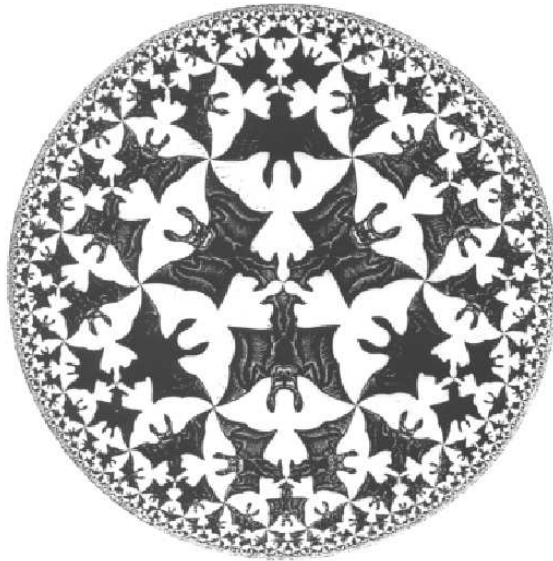
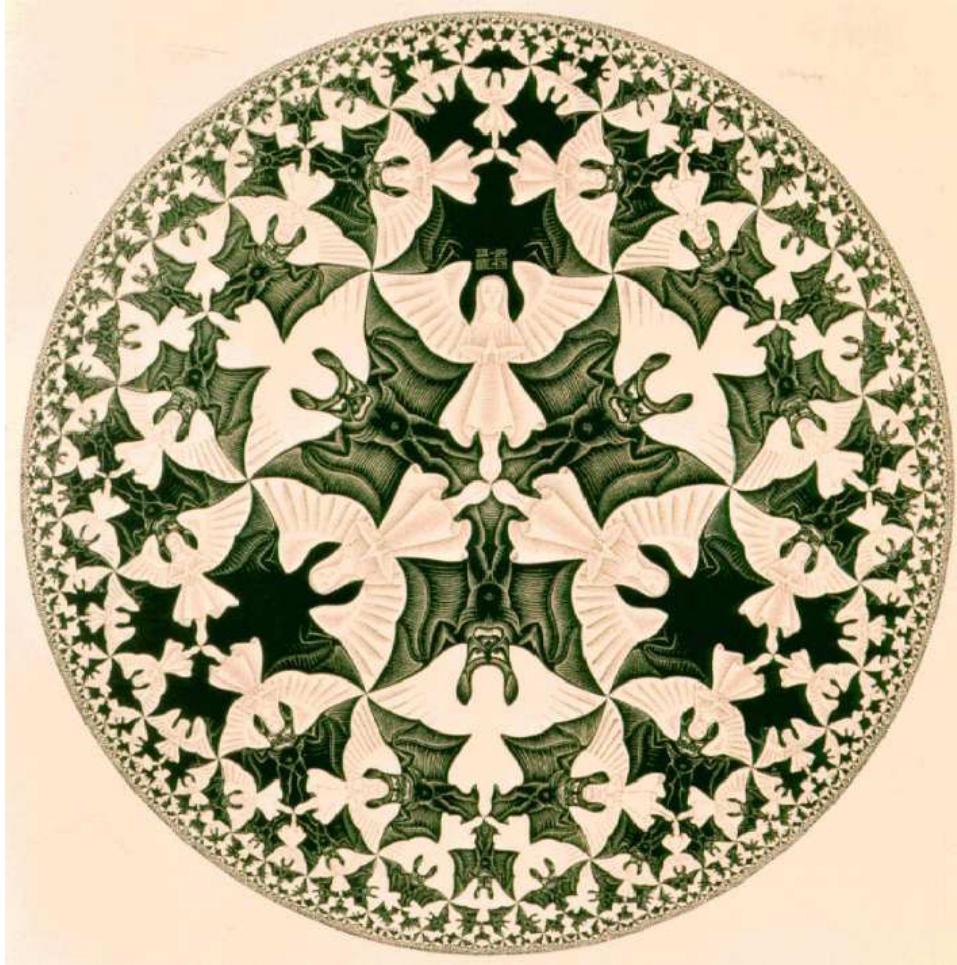# *Euclidean and Non-Euclidean Geometries*

The World of M.C. Escher His Life and Complete Graphic Work
(Harry N. Abrams, 1981)

- **Page 98 (large image): correctly oriented**
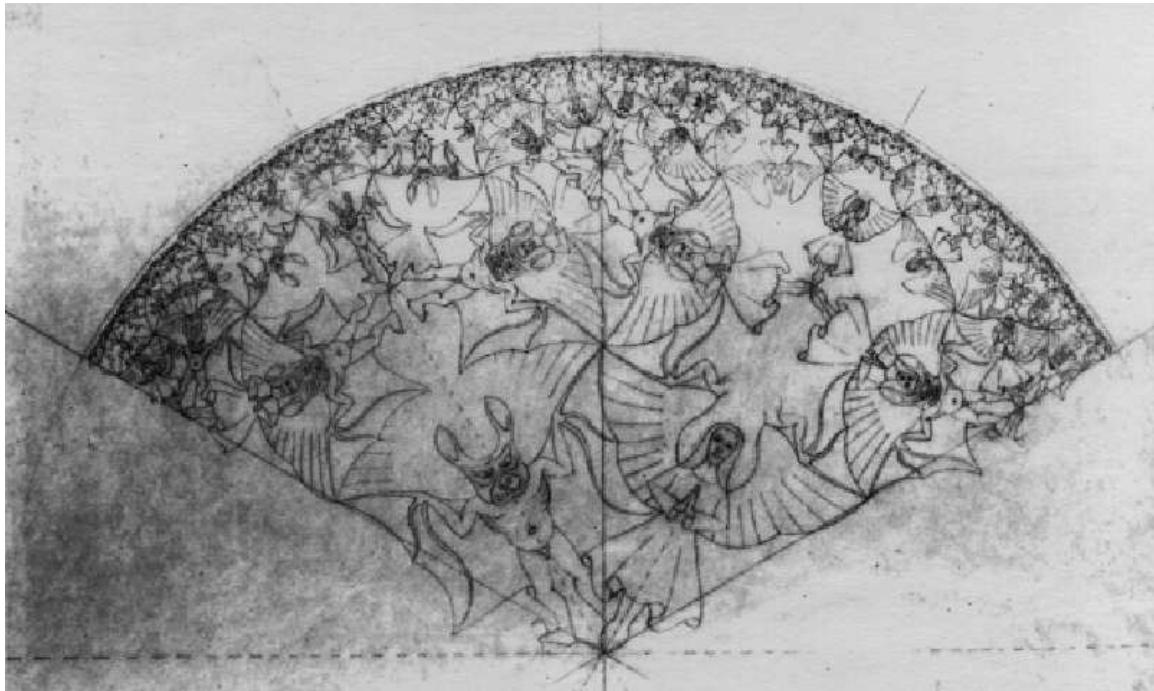
- **Page 322 (small image — Catalogue number 436):**

M.C. Escher web site

The Signature

Second Question: What is the Symmetry group of *Circle Limit IV* ? C_3?

It seems to be D_3

Escher (in *M.C. Escher The Graphic Work* Barnes & Noble/TASCHEN 2007 ISBN 0-7607-9669-6, page 10):

Here too, we have the components diminishing in size as they move outwards. The six largest (three white angels and three black devils) are arranged about the centre and radiate from it. The disc is divided into six sections in which, turn and turn about, the angels on a black background and then the devils on a white one gain the upper hand. In this way, heaven and hell change place six times. In the intermediate, "earthly" stages, they are equivalent.

The "Heaven and Hell" Maple Sphere

"Heaven and Hell" carved sphere, 1942. Maple, stained in two colors, diameter 235 mm.

Escher's Description (in a letter to C.V.S. Roosevelt, *M.C. Escher: Visions of Symmetry* Harry N. Abrams, 2004 ISBN 0-8109-4308-5, page 245):

It has two poles and an equator. One pole represents "heaven," with only white angels on a black background, which I carved much deeper than the angle figures. The other pole shows "hell," with only black devils on a deeply carved white background. At the equator both angels and devils are visible and equivalent, carved at the same sphere-level.

Escher's Notebook Drawing # 45
## (*M.C. Escher: Visions of Symmetry* page 150)
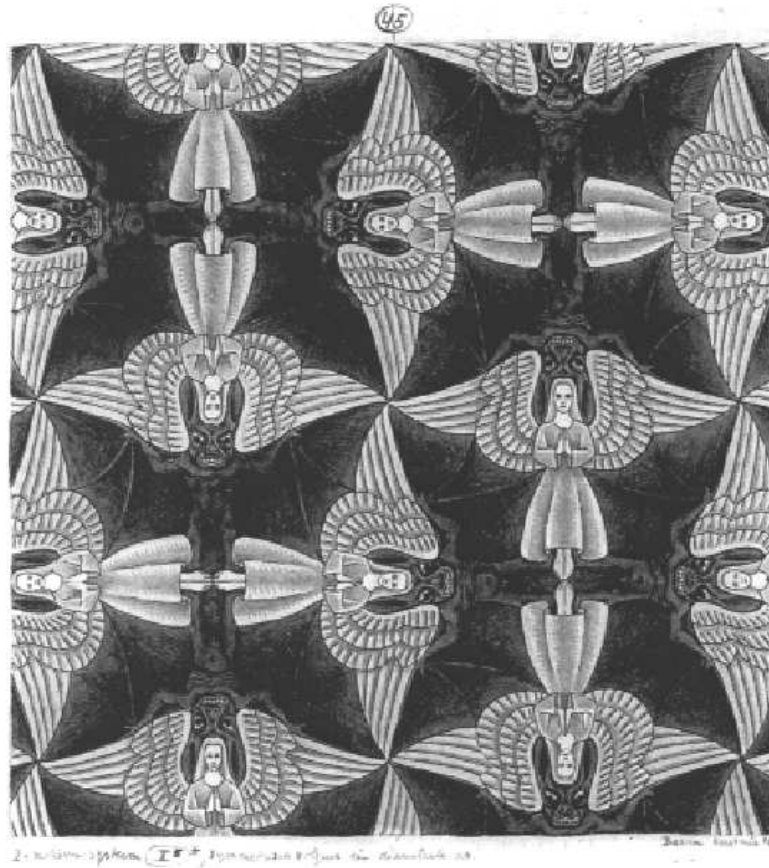
Symmetry Groups

The symmetry group of Notebook Drawing 45 is denoted $[4^+, 4]$ in H.S.M. Coxeter's notation, and 4*2 in orbifold notation.

If all the angels and devils were on the same sphere-level on the maple ball, the symmetry group would be $[3^+, 4]$ (Coxeter's notation) or 3*2 (orbifold notation).

If the details of all or none of the angels and devils were filled in for the *Circle Limit IV* pattern, the resulting pattern would have symmetry group $[4^+, 6]$ or 4*3.
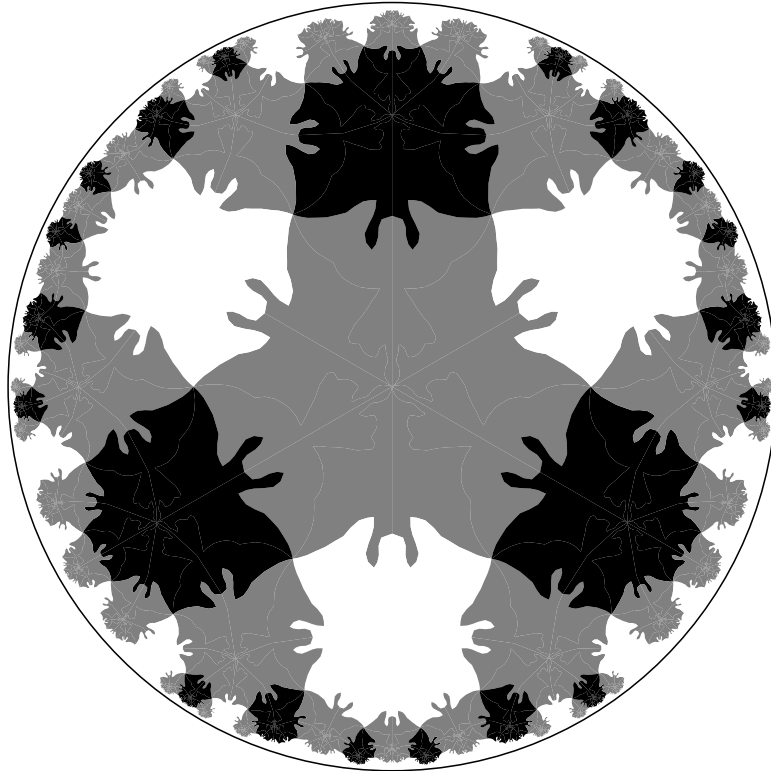
A Pattern with Symmetry Group $[4^+, 6]$ or 4*3
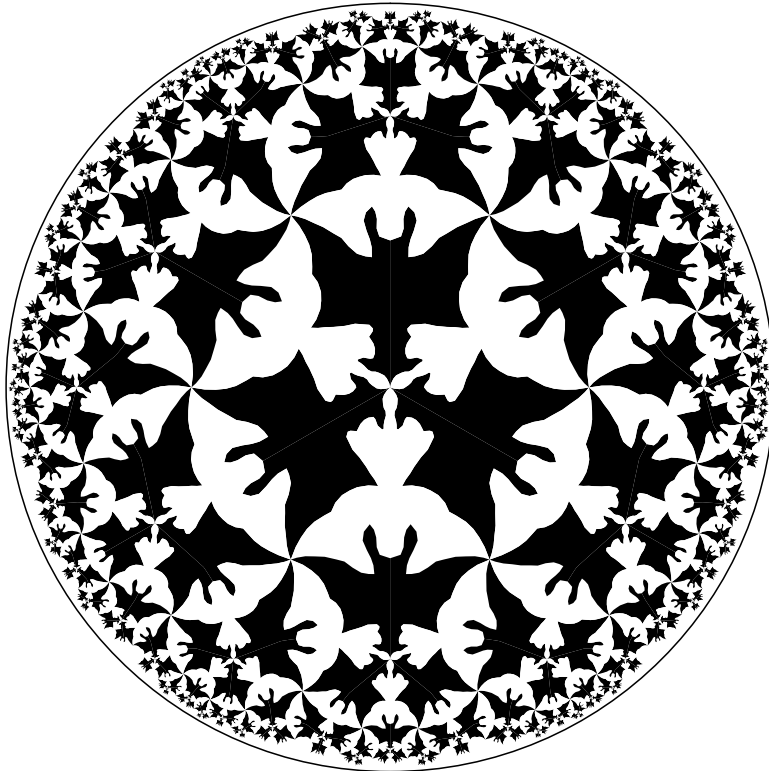
A Pattern with Symmetry Group 2*33
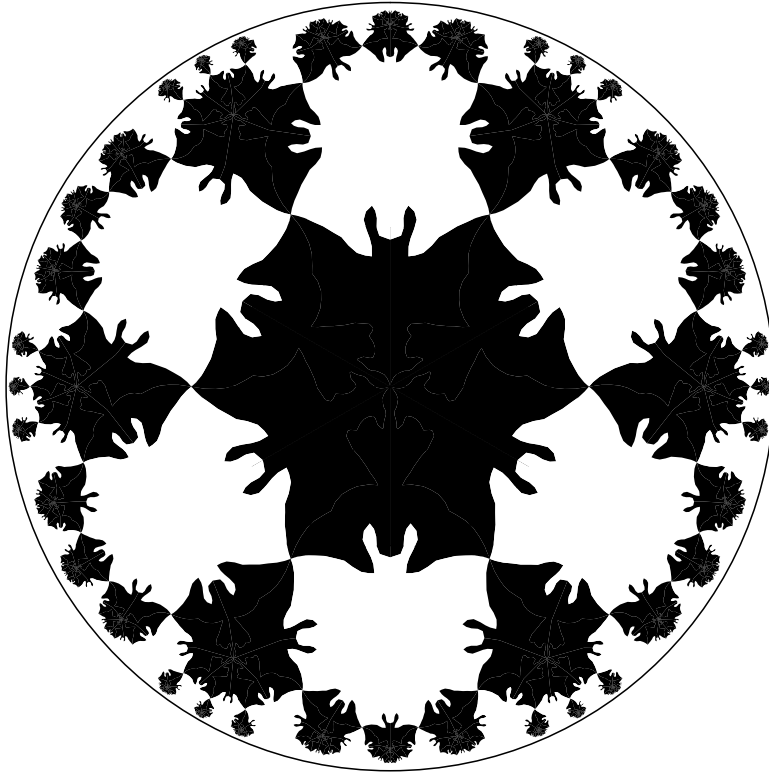
A Pattern with Symmetry Group *3333

Future Work

- Enhance the program to draw patterns with more complex symmetries.

- Use 3D printing technology to create a "heaven and hell" disk pattern with areas where angels are raised and devils are recessed, areas where the opposite is true, and intermediate areas where neither is dominant.