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INDECOMPOSABLE GENERALIZED COHEN-MACAULAY MODULES

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ABSTRACT. The aim of this paper is to study the indecomposable modules which are Cohen-Macaulay on the punctured spectrum of a local ring, and to describe some of their invariants such as their local cohomology groups and ranks. One of our main concerns is to find indecomposable quasi-Buchsbaum modules of high rank with prescribed cohomology over a regular local ring.

INTRODUCTION

The study of indecomposable finite modules over Artinian rings was extended in recent years to the study of indecomposable maximal Cohen-Macaulay modules over Cohen-Macaulay local rings (see, e.g., [2, 4, 9, 12, 19, 27, 32]). So representation theory has achieved a remarkable progress in the attempt to generalize the methods and results known for Artinian rings to higher dimensional local rings. It is very natural to consider the class of maximal Cohen-Macaulay modules in higher dimensions, because from a homological point of view, maximal Cohen-Macaulay modules correspond best to finite modules over Artinian local rings, and also the class of maximal Cohen-Macaulay modules is small enough in order to obtain finite representation type for interesting classes of Cohen-Macaulay local rings. On the other hand, Goto has shown [13–15] that the representation theory of the somewhat larger class of Buchsbaum modules also yields very satisfactory results. In particular he has shown (cf. [13, (3.1)]) that the only indecomposable Buchbaum modules over a regular local ring $(A, \mathfrak{m}, k)$ are just the syzygy-modules of k, and this implies that a regular local ring is of finite Buchsbaum representation type.

The category of Buchsbaum modules is contained in the larger category of the so-called generalized Cohen-Macaulay modules. Let $(A, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring. A finite A-module M is a generalized Cohen-Macaulay module if there exists an integer t such that $\mathfrak{m}^t H^i_\mathfrak{m}(M) = 0$ for $i \neq \dim M$, $H^i_\mathfrak{m}(M)$ denoting the ith local cohomology module of M.

We denote by $\mathscr{C}_t(A)$ the full subcategory of the category all finite A-modules M satisfying $\mathfrak{m}^t H^i_\mathfrak{m}(M) = 0$ for $i \neq \dim A$, and set $\mathscr{C}(A) = \bigcup_{t\geq 0} \mathscr{C}_t(A)$. It is

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clear that $\mathscr{C}(A)$ contains all maximal generalized Cohen-Macaulay A-modules, but also any A-module of finite length. Though we are mainly interested in modules of maximal dimension it turns out to be more natural to add the modules of finite length to the category $\mathscr{C}(A)$ as well. Indeed, if A is regular, then $\mathscr{C}(A)$ is exactly the category of finite A-modules which are locally free on the punctured spectrum of A (including those of rank 0). Note that the sheafication of such a module defines a vector bundle on the punctured spectrum of A. One central problem in the theory of bundles is to find bundles of low rank. We refer to [11] for a survey of the algebraic aspects of this problem. In this paper we are concerned with the opposite problem of finding indecomposable vector bundles of high rank but with prescribed cohomology. Another reason to add modules of finite length to $\mathscr{C}(A)$ is that several functors we are considering (e.g., the Auslander-Bridger dual) are only defined in $\mathscr{C}(A)$.

Note that the category of maximal Buchsbaum modules is a subcategory of $\mathscr{C}_1(A)$, the category of quasi-Buchsbaum modules (QB-modules for short) of maximal dimension or of dimension zero. The first part of the paper is mainly concerned with the study of $\mathscr{C}_1(A)$ if A is a regular local ring.

In dimension ≤ 2 all QB-modules in $\mathscr{C}_1(A)$ are actually Buchsbaum except one module E which is defined by the (unique) nontrivial extension

$$0 \to k \to E \to \mathfrak{m} \to 0$$

if $\dim A = 2$. Quite contrary to the result of Goto we show however that A is of infinite QB-representation type, provided $\dim A > 2$. Even more is true: The category $\mathscr{C}_1(A)$ satisfies the second Brauer-Thrall conjecture, that is, if $\mathscr{C}_1(A)$ is not of finite representation type (this is the case if $\dim A > 2$), then there exists a strictly increasing sequence (t_i) of positive integers such that for all i there are infinitely many isomorphism classes of indecomposable QB-modules of rank t_i. Note that the first Brauer-Thrall conjecture, which only predicts the existence of indecomposable modules of arbitrary high rank, is trivial for the category $\mathscr{C}(A)$. Indeed, the rank of the second syzygy module $\Omega^2(A/\mathfrak{m}^t)$ (which is indecomposable) is increasing with t, but $\Omega^2(A/\mathfrak{m}^t) \in \mathscr{C}_t(A)\backslash\mathscr{C}_{t-1}(A)$; and so for a fixed t one has to resort to a less obvious method of constructing indecomposable modules of higher rank in $\mathscr{C}_t(A)$.

It seems to us that the sequence $(h_0, \dots, h_{d-1})$ of integers attached with a module $M \in \mathscr{C}_1(A)$, $h_i = \dim_k H^i_{\mathfrak{m}}(M)$ for $1 \leq i < d$, is an even more significant invariant of M as its rank. In this paper we call a sequence $(h_0, \dots, h_{d-1})$ admissible if there exists an indecomposable module $M \in \mathscr{C}_1(A)$ with $h_i = \dim_k H^i_{\mathfrak{m}}(M)$. At this moment it is a mystery to us which sequences are admissible. If all $h_i = 0$ except one, say h_j, then $h_j = 1$. This is easy to see, and shown in Proposition 2.2. The next case is when exactly two h_i do not vanish. The subcase in which $h_i = 0$ for $i \neq 0$, $d-1$ is exceptionally, see 2.3(b). Otherwise it suffices to consider the case when $h_j \neq 0$, $j > 1$, and all $h_i = 0$ for $i \neq 1$, j as shown in 2.8. We then define a functor from the category $\mathscr{A}$ of these modules to the category $\operatorname{mod}(B)$ of finite modules over a certain Artinian local ring $(B, \mathfrak{n})$ with $\mathfrak{n}^2 = 0$. This functor establishes a bijection between the indecomposable modules in $\mathscr{A}$ and $\operatorname{mod}(B)$, and it turns out that under this bijection the modules with invariants h_1 and h_j correspond to modules over B whose minimal number of generators is h_j and whose socle dimension is h_1 (cf. Theorem 2.13). Thus our problem of determining all admissible sequences $(h_0, \dots, h_{d-1})$ with $h_i \neq 0$ for $i \neq 1, j$ boils down to determining all pairs of numbers $(\mu(M), r(M))$, M indecomposable, where μ resp. r denote the minimal number of generators resp. the type of M. We do not have a general answer to this problem, but it can be easily shown that μ and r may exceed any given pair of numbers. This result suggests the following conjecture: Let $d > 2$; then for any given sequence of numbers $(h_0, \dots, h_{d-1})$ there exists an indecomposable module $M \in \mathscr{C}_1(A)$ such that $\dim_k H^i_{\mathfrak{m}}(M) \geq h_i$ for $1 \leq i < d$.

In the second part of the paper we employ the theory of reduction ideals to study maximal generalized Cohen-Macaulay modules over Cohen-Macaulay local rings, a method which was successfully introduced by Dieterich and Yoshino in order to prove the first Brauer-Thrall conjecture for maximal Cohen-Macaulay modules. This technique works very well for the category of maximal Cohen-Macaulay modules over a Cohen-Macaulay local ring A with isolated singularity (see, [9, 24, 25, 32]), the proof using the following facts:

(1) the existence of a reduction ideal (a power $\mathfrak{m}^r$ of $\mathfrak{m}$) which allows to embed (by base change) the set of isomorphism classes of indecomposable maximal Cohen-Macaulay A-modules in the set of isomorphism classes of indecomposable $A/\mathfrak{m}^r$-modules (this line parallels to Maranda's approach for lattices over orders and allows to show a Harada-Sai Lemma for maximal Cohen-Macaulay modules),

(2) the existence of Auslander-Reiten sequences in the category of maximal Cohen-Macaulay A-modules (cf. [2]).

We are able to extend partly these methods to generalized Cohen-Macaulay modules. Actually we prove (1) (see 3.15) for the category $\mathscr{C}_t(\mathfrak{a}, A)$ of modules M satisfying $\mathfrak{a}^t H^i_{\mathfrak{m}}(M) = 0$ for $i \neq \dim A$, if A is a reduced excellent Henselian local ring containing a field, and the residue class field k of A is either perfect or finite over k^p if $\operatorname{char} k = p > 0$. Also our Corollary 3.20 gives a form of the Harada-Sai Lemma for a suitable subcategory of $\mathscr{C}_t(A)$. Unfortunately we cannot prove (2)—the existence of Auslander-Reiten sequences—for this category of modules. However as a consequence of our results we deduce that the $\mathfrak{a}$-adic completion induces a bijection between the indecomposable modules in $\mathscr{C}_t(\mathfrak{a}, A)$ and the indecomposable modules in $\mathscr{C}_t(\mathfrak{a}A', A')$ where A' denotes the $\mathfrak{a}$-adic completion of A.

1. Generalized Cohen-Macaulay modules over regular rings

The aim of this section is to give some preliminaries on generalized Cohen-Macaulay modules. We start with some results concerning ideals annihilating Ext as, e.g., 1.2 and 1.6 which will be useful in the last section. We also include here some simple facts concerning duality of generalized Cohen-Macaulay modules over regular local rings.

Let $(A, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring of dimension d and $\mathfrak{a} \subset A$ a proper ideal. Given a positive integer t, let $\mathscr{C}_t(\mathfrak{a}, A)$ be the class of finite A-modules E such that $\mathfrak{a}^t H^i_{\mathfrak{m}}(E) = 0$ for all $i \neq d$. Then $\mathscr{C}(\mathfrak{a}, A) = \bigcup_{t \in \mathbb{N}} \mathscr{C}_t(\mathfrak{a}, A)$ is exactly the class of all modules which are maximal Cohen-Macaulay on the open set $D(\mathfrak{a})$ of all prime ideals of A which do not contain $\mathfrak{a}$. If $\operatorname{Reg} A = \{\mathfrak{q} \in \operatorname{Spec} A \colon A_{\mathfrak{q}} \text{ is regular}\}$ is open (for example if A is an isolated singularity or is excellent) then the modules in $\mathscr{C}(\mathfrak{a}, A)$ are free on the open set $D(\mathfrak{a}) \cap \operatorname{Reg} A$. If $\mathfrak{a} = \mathfrak{m}$, we write $\mathscr{C}_t(A)$ (resp. $\mathscr{C}(A)$) instead of $\mathscr{C}_t(\mathfrak{m}, A)$ (resp. $\mathscr{C}(\mathfrak{m}, A)$). $\mathscr{C}(A)$ is the full subcategory of the category $\mathscr{G}$ of all generalized Cohen-Macaulay A-modules which are either of maximal dimension or of dimension 0. Note further that $\mathscr{C}_0(A)$ is the category of maximal Cohen-Macaulay modules, and that $\mathscr{C}_1(A)$ is a full subcategory of the category of quasi-Buchsbaum (shortly QB) A-modules. Again $\mathscr{C}_1(A)$ contains exactly the QB-modules which are of dimension d or 0. It is worthwhile to mention that $\mathscr{C}(A)$ is exactly the class of finite A-modules which are free on the punctured spectrum if A is an isolated singularity.

For the rest of this section we assume that $(A, \mathfrak{m}, k)$ is regular. Let $I(k)$ be the injective hull of k. For any A-module N we denote by N' the module $\operatorname{Hom}_A(N, I(k))$ and by $\widehat{N}$ the $\mathfrak{m}$-adic completion of N. Then by local duality we have

$$H^i_{\mathfrak{m}}(E) \cong \operatorname{Ext}^{d-i}_A(E, A)' \quad \text{and} \quad \operatorname{Ext}^{d-i}_A(E, A)^\wedge \cong H^i_{\mathfrak{m}}(E)',$$

for any finite A-module E. In particular, a finite A-module E belongs to $\mathscr{C}_t(\mathfrak{a}, A)$ if and only if $\mathfrak{a}^t \operatorname{Ext}^i_A(E, A) = 0$ for $i \geq 1$.

Note that $\operatorname{Ext}^i_A(E, A)$ is of finite length for all modules $E \in \mathscr{C}(A)$ and all $i > 0$, which implies that $\operatorname{Ext}^i_A(E, A)^\wedge \cong \operatorname{Ext}^i_A(E, A)$ in this range.

Lemma 1.1. *Let R be a Noetherian ring, $\mathfrak{b} \subset R$ an ideal, and M a finite R-module such that $\mathfrak{b} \operatorname{Ext}^i_R(M, R) = 0$ for all $i \geq 1$. Then for every finite R-module N with $r = \operatorname{proj\,dim} N < \infty$ we have*

$$\mathfrak{b}^{r+1} \operatorname{Ext}^i_R(M, N) = 0 \quad \textit{for all } i \geq 1. \tag{1}$$

Proof. Let N be as above. We will use induction on $r = \operatorname{proj\,dim} N$. If $r = 0$ then N is a direct summand of a free module, and the assertion follows from the hypothesis. Suppose that $r \geq 1$. Consider an exact sequence of finite R-modules $0 \to N' \to F \to N \to 0$, where F is a free R-module and $\operatorname{proj\,dim} N' = r - 1$. We get the following exact sequence:

$$\operatorname{Ext}^i_R(M, F) \to \operatorname{Ext}^i_R(M, N) \to \operatorname{Ext}^{i+1}_R(M, N'). \tag{2}$$

Since by assumption we have $\mathfrak{b} \operatorname{Ext}^i_R(M, F) = 0$, $\mathfrak{b}_r \operatorname{Ext}^i_R(M, N') = 0$ for all $i \geq 1$, the conclusion follows from (2). □

Corollary 1.2. *Let t be a positive integer and $E \in \mathscr{C}_t(\mathfrak{a}, A)$. Then*

$$\mathfrak{a}^{t(d+1)} \operatorname{Ext}^i_A(E, N) = 0$$

for all $i \geq 1$ and every finite A-module N (A is regular !).

Lemma 1.3. *Let R be a Noetherian ring and $\mathfrak{b} \subset R$ an ideal such that*

$$\mathfrak{b} \operatorname{Ext}^1_R(M, \Omega^1(M)) = 0,$$

where $\Omega^i(M)$ denotes the ith *syzygy module of M. Then*

$$\mathfrak{b} \operatorname{Tor}^R_1(M, N) = 0$$

for all R-modules N.

Proof. Let

$$0 \to \Omega^1(M) \to F \xrightarrow{f} M \to 0$$

be the exact sequence defining $\Omega^1(M)$, F being a free finite R-module. Then we get the following exact sequence

$$0 \to \mathrm{Hom}_R(M, \Omega^1(M)) \to \mathrm{Hom}_R(M, F) \xrightarrow{f^*} \mathrm{Hom}_R(M, M) \to \mathrm{Ext}^1_R(M, \Omega^1(M)).$$

Let $u \in \mathfrak{b}$. We have $u\,\mathrm{Hom}_R(M, M) \subset \mathrm{Im}\, f^*$, and so there exists $g_u \colon M \to F$ such that

$$fg_u = u\,\mathrm{id}_M. \tag{2}$$

Apply $\mathrm{Tor}^R_1(-, N)$ to (1) for an R-module N. It follows that $\mathrm{Tor}^R_1(u\,\mathrm{id}_M, N)$ factorizes through $\mathrm{Tor}^R_1(F, N) = 0$. Then $\mathrm{Tor}^R_1(u\,\mathrm{id}_M, N) = 0$, and so $u\,\mathrm{Tor}^R_1(M, N) = 0$. □

Lemma 1.4. *Let R be a Noetherian ring, $\mathfrak{b}, \mathfrak{c} \subset R$ two ideals, M a finite R-module and y an element from R which is not a zero-divisor on $R/\mathfrak{c}$. Suppose that*

$$\mathfrak{b}\,\mathrm{Tor}^R_1(M, N) = 0 \tag{1}$$

for all R-module N. Then for all $i \geq 1$ we have

$$(\mathfrak{c}M : y^i)_M \subset (\mathfrak{c}M : \mathfrak{b})_M = \{z \in M : \mathfrak{b}z \subset \mathfrak{c}M\}.$$

Proof. By hypothesis we have the exact sequence

$$0 \to R/\mathfrak{c} \xrightarrow{y^i} R/\mathfrak{c} \to R/(\mathfrak{c}, y^i) \to 0.$$

Tensorizing the exact sequence with M we get the following exact sequence

$$\mathrm{Tor}^R_1(M, R/(\mathfrak{c}, y^i)) \to M/\mathfrak{c}M \xrightarrow{y^i} M/\mathfrak{c}M.$$

Using (1) it follows that $\mathfrak{b}(\mathfrak{c}M : y^i)_M \subset \mathfrak{c}M$, and this is what we wanted to show. □

Remark 1.5. Certainly our lemma is not too far from the results concerning weak sequences (see, e.g., [8, (3.3)] or [30, Proposition 13, p. 257]).

Proposition 1.6. *Let R be a Noetherian ring, $\mathfrak{b}, \mathfrak{c} \subset R$ two ideals, $\mathscr{C}$ any class of finite R-modules and y an element of* $\mathrm{Rad}\,\mathfrak{b}$. *Suppose that*

$$\mathfrak{b}\,\mathrm{Ext}^1_R(M, N) = 0 \tag{1}$$

for every R-module $M \in \mathscr{C}$ and every finite R-module N. Then there exists a positive integer e such that

$$(\mathfrak{c}M : y^e)_M = (\mathfrak{c}M : y^{e+1})_M$$

for every R-module $M \in \mathscr{C}$.

Proof. First we suppose that y is not a zero-divisor on $R/\mathfrak{c}$. Let e be a positive integer such that $y^e \in \mathfrak{b}$; then e works. Indeed, let M be from $\mathscr{C}$. By Lemma 1.3 and Lemma 1.4 we get

$$(\mathfrak{c}M : y^i)_M \subset (\mathfrak{c}M : \mathfrak{b})_M,$$

because of (1). Since $y^e \in \mathfrak{b}$ it follows that

$$(\mathfrak{c}M : \mathfrak{b})_M \subset (\mathfrak{c}M : y^e)_M,$$

and so

$$(\mathfrak{c}M : y^i)_M = (\mathfrak{c}M : y^e)_M$$

for all $i \geq e$.

Suppose now that y is a zero-divisor on $R/\mathfrak{c}$. As R is Noetherian we have $(\mathfrak{c}: y^s)_R = (\mathfrak{c}: y^{s+1})_R$ for a certain $s \in \mathbb{N}$. Then y is not a zero-divisor on $R/\mathfrak{d}$ for $\mathfrak{d} = (\mathfrak{c}: y^s)_R$. Let e' be given as above for $\mathfrak{d}$. Then $e = e' + s$ works. Indeed, let M be in $\mathscr{C}$ and $z \in M$ an element such that $y^{e+1}z \in \mathfrak{c}M \subset \mathfrak{d}M$. As above we get $y^{e'}z \in \mathfrak{d}M$, and so $y^e z \in \mathfrak{c}M$. Thus $(\mathfrak{c}M : y^{e+1})_M \subset (\mathfrak{c}M : y^e)_M$, the other inequality being trivial. □

Corollary 1.7. *Let t be a positive integer, $\mathfrak{c} \subset A$ a proper ideal and y an element of* Rad $\mathfrak{a}$. *Then there exists a positive integer e such that*

$$(\mathfrak{c}E : y^e)_E = (\mathfrak{c}E : y^{e+1})_E$$

for all A-modules $E \in \mathscr{C}_t(\mathfrak{a}, A)$.

Now we are going to study how local cohomology behaves under certain duality operators, and assume for the rest of the section that $d \geq 2$, $\mathfrak{a} = \mathfrak{m}$ and E is an A-module of the category $\mathscr{C}_t(A)$. Then $H^i_{\mathfrak{m}}(E)$ is a finite Artinian A-module for $0 \leq i < d$, and we set $h_i(E) = l(H^i_{\mathfrak{m}}(E))$. Let $E^* = \operatorname{Hom}_A(E, A)$ be the A-dual of E.

The following result can be found in [11]. For the reader's convenience we give its proof here.

Proposition 1.8. *The A-dual E^* of E satisfies:*

(a) $H^i_{\mathfrak{m}}(E^*) = 0$ *for* $i = 0, 1$.
(b) *(Duality)* $H^{i+1}_{\mathfrak{m}}(E^*) = H^{d-i}_{\mathfrak{m}}(E)'$ *for* $0 < i \leq d - 2$.
In particular

$$h_i(E^*) = \begin{cases} 0 & \text{if } i = 0 \text{ or } 1, \\ h_{d-i+1}(E) & \text{if } 2 \leq i < d. \end{cases}$$

Proof. Since depth $E^* \geq 2$, (a) follows.

(b) Let

$$F_\cdot : 0 \to F_p \xrightarrow{\partial_p} F_{p-1} \to \cdots \xrightarrow{\partial_1} F_0 \to 0$$

be a minimal free resolution of E, and let $F_\cdot^*$ be the dual complex. Then $H^i(F_\cdot^*) \cong \operatorname{Ext}^i_A(E, A) \cong H^{d-i}_{\mathfrak{m}}(E)'$ by local duality. Hence it suffices to show that $H^{i+1}_{\mathfrak{m}}(E^*) \cong H^i(F^*)$ for $1 < i \leq d-2$. We set $Z_{i-1} = \operatorname{Ker} \partial_i^*$, $B_i = \operatorname{Im} \partial_i^*$. Clearly we have

(1) $Z_0 = E^*$,
(2) for $0 \leq i \leq p$ the sequence $0 \to B_i \to Z_i \to H^i(F_\cdot^*) \to 0$ is exact,
(3) for $0 \leq i \leq p$ the sequence $0 \to Z_i \to F_i^* \to B_{i+1} \to 0$ is exact.

Since $H^i(F_\cdot^*)$ has finite length ($E \in \mathscr{C}_t(A)$) we have $H^j_{\mathfrak{m}}(H^i(F_\cdot^*)) = 0$ for $j > 0$. Note that depth $B_i \geq 1$ and depth $Z_i \geq 2$. Using these facts and the exact sequences (2), the long exact cohomology sequence yields

(4) $$H^i(F_\cdot^*) \cong H^1_{\mathfrak{m}}(B_i) \quad \text{for } i > 0 \qquad \text{and} \qquad H^j_{\mathfrak{m}}(Z_i) \cong H^j_{\mathfrak{m}}(B_i)$$

for $i > 0$, $j \geq 2$.

Since $H^j_{\mathfrak{m}}(F_i^*) = 0$ for $j < d$, the long exact cohomology sequence derived from (3) gives

$$H^j_{\mathfrak{m}}(B_{i+1}) \cong H^{j+1}_{\mathfrak{m}}(Z_i) \tag{5}$$

for $j \le d - 2$.

Now (4) and (5) yield the required isomorphism. Indeed, let $1 \le i \le d - 2$. Then

$$H^i(F.^*) \cong H^1_{\mathfrak{m}}(B_i) \cong H^2_{\mathfrak{m}}(Z_{i-1}) \cong H^2_{\mathfrak{m}}(B_{i-1}) \cong \cdots \cong H^i_{\mathfrak{m}}(B_1) \cong H^{i+1}_{\mathfrak{m}}(Z_0),$$

and the assertion follows from (1). □

Remark 1.9. Let $X = \operatorname{Spec} A \backslash \{\mathfrak{m}\}$ and $\mathscr{E} = \widetilde{E}$ the bundle on X associated to E. One may assume that E is reflexive since $\widetilde{E} = \widetilde{E}^{**}$. Then $H^i(X, \mathscr{E}) \cong H^{i+1}_{\mathfrak{m}}(E)$ when $i > 0$, and 1.8(b) yields the duality

$$H^i(X, \mathscr{E}) \cong H^{d-i-1}(X, \mathscr{E}^*)',$$

$(\dim X = d - 1)$.

Corollary 1.10. *If* $E \in \mathscr{C}_t(A)$ *then* $E^* \in \mathscr{C}_t(A)$.

Corollary 1.11. *If* E *is not free then* $\operatorname{depth} E + \operatorname{depth} E^* \le d + 1$.

Proof. We may assume that $2 \le \operatorname{depth} E = s < d$ since E is not free. Using 1.8(b) we get

$$H^{d-s+1}_{\mathfrak{m}}(E^*) \cong H^s_{\mathfrak{m}}(E)'. \tag{1}$$

Since for a finite A-module N one has $\operatorname{depth} N = \min\{j : H^j_{\mathfrak{m}}(N) \ne 0\}$ it follows from (1) that $H^s_{\mathfrak{m}}(E) \ne 0$, and so $\operatorname{depth} E^* \le d - s + 1$. □

The modules E for which the inequality 1.11 becomes an equality will be characterized in the next proposition. We set $E_i = \Omega^i(k)$ for $1 \le i \le d$. According to a theorem of Goto [13, (3.1)], these modules are the only maximal indecomposable Buchsbaum A-modules.

Proposition 1.12. *Suppose that* E *is a nonfree QB-module. The following conditions are equivalent*:

(a) $\operatorname{depth} E + \operatorname{depth} E^* = d + 1$.
(b) $\operatorname{depth} E > 0$ *and* $H^i_{\mathfrak{m}}(E) = 0$ *for* $i \ne d$, $i \ne \operatorname{depth} E$.
(c) *There exist a free* A*-module* F *and positive integers* r *and* j *such that* $E \cong E_j^r \oplus F$.

Proof. (a) ⇒ (b). We set $s = \operatorname{depth} E = d + 1 - \operatorname{depth} E^* > 0$. Assume that $H^u_{\mathfrak{m}}(E) \ne 0$ for $u \ne d, s$. We have $s < u < d$ since $s = \min\{j : H^j_{\mathfrak{m}}(E) \ne 0\}$. Thus duality implies $H^{d-u+1}_{\mathfrak{m}}(E^*) \cong H^u_{\mathfrak{m}}(E)' \ne 0$ $(u \ge 2)$, and so $\operatorname{depth} E^* \le d - u + 1$. It follows that

$$\operatorname{depth} E^* + \operatorname{depth} E \le d - u + 1 + s < d + 1,$$

which contradicts (a).

(b) ⇒ (a). Since $s > 0$, duality implies $\operatorname{depth} E^* = d - s + 1$, as required.

(c) ⇒ (b). Clearly we may assume that $F = 0$ and $r = 1$. As $E_j = \Omega^j(k)$, it follows that $H^i_{\mathfrak{m}}(E_j) \cong H^{i-j}_{\mathfrak{m}}(k)$ for $i < d$, hence the conclusion.

(b) ⇒ (c). Let

$$0 \to F_{d-s} \to \cdots \to F_0 \to E \to 0$$

be a minimal free resolution of E. Then

$$\mathrm{Ext}_A^{d-i}(E, A) \cong H^i_{\mathfrak{m}}(E)' = 0$$

for $i \neq d$, $i \neq s$ by our assumption. Thus our condition implies that

$$(1) \qquad 0 \to E^* \to F_0^* \to \cdots \to F_{d-s}^* \to 0$$

is acyclic. Hence if $\mathrm{Ext}_A^{d-s}(E, A) \cong k^r$ for a certain $r \geq 1$, then $E^* \cong \Omega^{d-s}(k^r) \oplus F^* \cong E^r_{d-s} \oplus F^*$ for some free A-module F. In case E is reflexive it follows that

$$E \cong E^{**} \cong (E^r_{d-s})^* \oplus F^{**} \cong E^r_s \oplus F.$$

If E is not reflexive, then $s = 1$, and (1) implies that E^* is free. In 1.19 we will show that there exists an exact sequence

$$0 \to E \to E^{**} \to k^r \to 0$$

for some $r \geq 1$. Since E^{**} is free, it follows that $E \cong \mathfrak{m}^r \oplus F$ for some free A-module F. □

Note that the equivalence of (b) and (c) also follows from a more general result of Stückrad and Vogel [29, Corollary 1.1] and Goto's theorem.

Let R be a Noetherian ring and M a finite R-module with the following minimal presentation:

$$F_1 \xrightarrow{\varphi} F_0 \to M \to 0.$$

The module $D(M) = \mathrm{Coker}\,\varphi^*$ is called the *Auslander-Bridger dual* or the *transpose* of M (see, e.g., [3, 7, (16.E)]). Clearly, $D(D(M)) \cong M$.

Lemma 1.13. *$D(M)$ has no free nonzero direct summands.*

Proof. We have the following exact sequence:

$$(1) \qquad 0 \to M^* \to F_0^* \xrightarrow{\varphi^*} F_1^* \xrightarrow{\pi} D(M) \to 0.$$

Dualizing (1) we get

$$0 \to D(M)^* \xrightarrow{\pi^*} F_1^{**} \xrightarrow{\varphi^{**}} F_0^{**}.$$

If L is a free direct summand of $D(M)$ then there exist two R-linear maps $p: D(M) \to L$ and $u: L \to D(M)$ such that $pu = \mathrm{id}_L$. Clearly we can lift u to an R-linear map $u': L \to F_1^*$ such that $\pi u' = u$. Thus $u'^*(p\pi)^* = \mathrm{id}_{L^*}$, and so L^* is a direct summand of F_1^{**} which is contained in $\mathrm{Ker}\,\varphi^{**}$. Note that $\mathrm{Ker}\,\varphi^{**} \subset \mathfrak{m}F_1^{**}$ because φ (and so φ^{**}) is minimal. But (0) is the only direct summand of F_1^{**} contained in $\mathfrak{m}F_1^{**}$, and so $L^* = 0$. Hence $L = 0$. □

Remark 1.14. Note that in (1) the map π induces an isomorphism $F_1^*/\mathfrak{m}F_1^* \to D(M)/\mathfrak{m}D(M)$ since $\mathrm{Im}\,\varphi^* \subset \mathfrak{m}F_1^*$. Thus we have

$$\mu(D(M)) = \mathrm{rank}\,F_1^* = \mathrm{rank}\,F_1 = \mu(\Omega^1(M)),$$

where $\mu(N)$ denotes the minimal number of a system of generators of N. In particular, $D(M) = 0$ if and only if M is free.

Proposition 1.15. *M is indecomposable if and only if $D(M)$ is indecomposable.*

Proof. If $D(M) = G \oplus H$ is a decomposition of $D(M)$ then $M \cong D(D(M)) \cong D(G) \oplus D(H)$, and so $D(G) = 0$ or $D(H) = 0$. Hence G or H is free by 1.14. But $D(M)$ has no free nonzero direct summands by 1.13. Thus $G = 0$ or $H = 0$, and so $D(M)$ is indecomposable. To prove the converse one uses again the isomorphism $M = D(D(M))$. □

Proposition 1.16 ([3] or [7, (16.32)]). *There exists an exact sequence*

$$0 \to \operatorname{Ext}^1_R(D(M), R) \to M \to M^{**} \to \operatorname{Ext}^2_R(D(M), R) \to 0.$$

Combining 1.16 and 1.8 we obtain another duality theorem.

Corollary 1.17. *Suppose $d \geq 2$; then*

$$H^i_{\mathfrak{m}}(D(E)) \cong H^{d-i-1}_{\mathfrak{m}}(E)' \quad \text{for } 0 \leq i < d.$$

Proof. Consider the exact sequence

(1) $$0 \to E^* \to F_0^* \xrightarrow{\varphi^*} F_1^* \to D(E) \to 0.$$

Let $i \leq d-3$. Using duality and (1) we get

$$H^i_{\mathfrak{m}}(D(E)) \cong H^{i+2}_{\mathfrak{m}}(E^*) \cong H^{d-i-1}_{\mathfrak{m}}(E)'.$$

To obtain the result for $i = d-1$ and $i = d-2$, we split the exact sequence 1.16 (where M is replaced by E) into short exact sequences

(2) $$0 \to \operatorname{Ext}^1_A(D(E), A) \to E \to U \to 0,$$

(3) $$0 \to U \to E^{**} \to \operatorname{Ext}^2_A(D(E), A) \to 0.$$

As $d \geq 2$, we have $\operatorname{depth} E^{**} \geq 2$ and $\operatorname{depth} U \geq 1$. So the sequence (2) yields

(4) $$\operatorname{Ext}^1_A(D(E), A) \cong H^0_{\mathfrak{m}}(\operatorname{Ext}^1_A(D(E), A)) \cong H^0_{\mathfrak{m}}(E),$$

and

(5) $$H^1_{\mathfrak{m}}(E) \cong H^1_{\mathfrak{m}}(U),$$

because the kernel of $E \to E^{**}$ has finite length, E being free on the punctured spectrum. Similarly the sequence (3) yields

(6) $$\operatorname{Ext}^2_A(D(E), A) \cong H^0_{\mathfrak{m}}(\operatorname{Ext}^2_A(D(E), A)) \cong H^1_{\mathfrak{m}}(U).$$

The assertion follows now from (4)–(6) using Grothendieck's local duality theorem. □

Corollary 1.18. *Let M be a finite A-module. Then $M \in \mathscr{C}_t(A)$ if and only if $D(M) \in \mathscr{C}_t(A)$.*

Corollary 1.19. *Let $E \in \mathscr{C}_1(A)$. There exists an exact sequence*

$$0 \to k^u \to E \to E^{**} \to k^r \to 0.$$

If $\operatorname{depth} E = 1$ *(resp.* ≥ 2*) then* $u = 0$ *(resp.* $u = r = 0$*).*

For the proof apply 1.16 with $M = E$, Grothendieck's local duality theorem and 1.18.

2. Indecomposable quasi-Buchsbaum modules

In this section we assume that $(A, \mathfrak{m}, k)$ is a regular Henselian local ring of dimension $d \geq 2$. Note that the local cohomology modules $H^i_{\mathfrak{m}}(E)$, $0 \leq i < d$, of a module $E \in \mathscr{C}_1(A)$ are k-vector spaces, and we set $h_i(E) = \dim_k H^i_{\mathfrak{m}}(E)$.

Definition 2.1. A sequence $(h_0, \dots, h_{d-1})$ of nonnegative integers is called *admissible* if there exists an indecomposable nonfree module $E \in \mathscr{C}_1(A)$ with $h_i(E) = h_i$ for $0 \le i < d$.

Our results from §1 yield the following

Proposition 2.2. (a) *The sequence* $(0, \dots, 0, h_j, 0, \dots, 0)$ *is admissible if and only if* $h_j = 1$.

(b) *If* $(h_0, \dots, h_{d-1})$ *is admissible then* $(h_{d-1}, \dots, h_0)$ *is admissible as well.*

(a) is a consequence of 1.12, and (b) follows from 1.15 and 1.17.

The following theorem is one of the main results of this section and a first attempt to throw some light on admissible sequences.

Theorem 2.3. (a) *Given integers* $\alpha, \beta > 0$ *and integers* $0 \le i < j < d$, $d \ge 3$, *with* $i \ne 0$ *or* $j \ne d-1$, *there exists an admissible sequence* $(h_0, \dots, h_{d-1})$ *with* $h_i \ge \alpha$, $h_j \ge \beta$.

(b) $(\alpha, 0, \dots, 0, \beta)$ *is admissible if and only if* $\alpha = \beta = 1$.

For the proof of 2.3 we need some preparations.

Lemma 2.4. *Let* M *be a finite* A*-module, and* $j \ge 1$ *the smallest integer for which* $H_{\mathfrak{m}}^{d-j}(M) \ne 0$. *Suppose that* M *has no free direct summands and* $\Omega^e(M)$ *is indecomposable for a certain* e *with* $0 \le e < j$. *Then* $\Omega^i(M)$ *is indecomposable for all* $i < j$.

Proof. We first note that all $\Omega^i(M)$, $0 \le i < j$, have no free direct summands (by assumption this holds for $i = 0$). Indeed, if $0 < i < j$ then there is an exact sequence

$$(1) \qquad 0 \to \Omega^i(M) \xrightarrow{\varphi} F \xrightarrow{\psi} \Omega^{i-1}(M) \to 0$$

with F free and $\operatorname{Im} \varphi \subset \mathfrak{m}F$. The dual sequence

$$(2) \qquad 0 \to \Omega^{i-1}(M)^* \xrightarrow{\psi^*} F^* \xrightarrow{\varphi^*} \Omega^i(M)^* \to 0$$

is exact since $\operatorname{Ext}_A^1(\Omega^{i-1}(M), A) \cong \operatorname{Ext}_A^i(M, A) \cong H_{\mathfrak{m}}^{d-i}(M)' = 0$ by assumption.

Now assume that $L \ne 0$ is a free direct summand of $\Omega^i(M)$. Then L^* is a free direct summand of $\Omega^i(M)^*$, and there exists a surjection $\pi: \Omega^i(M)^* \to L^*$. The composition $\pi\varphi^*: F^* \to L^*$ is split surjective since L^* is free. It follows that $\varphi\pi^*: L \to F$ is split injective, contradicting our assumption $\operatorname{Im}\varphi \subset \mathfrak{m}F$. Note that we have also proved here that $\Omega^i(M)^*$ has no free direct summands.

Next we show by induction on r that $\Omega^{e-r}(M)$, $0 \le r \le e$, is indecomposable. For $r = 0$ this holds by the hypothesis. Now let $0 < r \le e$, and assume that $\Omega^{e-r}(M) = G \oplus H$ is a decomposition. Then $\Omega^{e-(r-1)}(M) \cong \Omega^1(G) \oplus \Omega^1(H)$ is indecomposable by the induction hypothesis, and so $\Omega^1(G) = 0$ or $\Omega^1(H) = 0$. Hence either G or H is free, and so either $G = 0$ or $H = 0$ since $\Omega^{e-(r-1)}(M)$ has no free direct summands.

Finally we show by induction on s that $\Omega^{e+s}(M)$, $0 \le s < j-e$, is indecomposable. This is true for $s = 0$. Let $0 < s < j-e$. For $i = e+s$ the sequence (2) becomes

$$(3) \qquad 0 \to \Omega^{e+s-1}(M)^* \xrightarrow{\psi^*} F^* \xrightarrow{\varphi^*} \Omega^{e+s}(M)^* \to 0.$$

We have $\operatorname{Im}\psi^* \subset \mathfrak{m}F^*$ since otherwise $\Omega^{e+s-1}(M)^*$ would have a free direct summand. It follows that $\Omega^{e+s-1}(M)^* = \Omega^1(\Omega^{e+s}(M)^*)$, and we conclude as before that $\Omega^{e+s}(M)^*$ is indecomposable. □

Definition 2.5. Let M be a finite A-module with $\operatorname{depth} M \geq 1$. Then we define negative syzygies of M as follows. First we choose a minimal free presentation

$$0 \to \Omega^1(M^*) \xrightarrow{u} F \xrightarrow{\varepsilon} M^* \to 0$$

of M^*. Then the composed map $M \xrightarrow{w} M^{**} \xrightarrow{\varepsilon^*} F^*$ gives a monomorphism from M into a free module (the canonical homomorphism into the bidual is a monomorphism since $\operatorname{depth} M \geq 1$). We set $\Omega^{-1}(M) = \operatorname{Coker}(\varepsilon^* w)$. Then:

(a) M is free if and only if $\Omega^{-1}(M) = 0$,

(b) $\Omega^{-1}(M_1 \oplus M_2) \cong \Omega^{-1}(M_1) \oplus \Omega^{-1}(M_2)$ for every decomposition $M = M_1 \oplus M_2$.

Lemma 2.6. *Suppose that $E \in \mathscr{C}_1(A)$ has no free direct summands. If* $\operatorname{depth} E \geq 1$, *then*:

(a) $\Omega^1(\Omega^{-1}(E)) \cong E$.

(b) *There exists an exact sequence*

$$0 \to H^1_{\mathfrak{m}}(E) \to \Omega^{-1}(E) \to \Omega^1(E^*)^* \to H^2_{\mathfrak{m}}(E) \to 0.$$

(c) $H^{i+1}_{\mathfrak{m}}(\Omega^1(E)) \cong H^i_{\mathfrak{m}}(E)$ *for* $0 \leq i \leq d-2$.

(d) $H^i_{\mathfrak{m}}(\Omega^{-1}(E)) \cong H^{i+1}_{\mathfrak{m}}(E)$ *for* $0 \leq i \leq d-2$.

Proof. (a) We only have to show that $\operatorname{Im}(\varepsilon^* w) \subset \mathfrak{m}F^*$. Suppose this inclusion does not hold. Then there exists a projection $\pi\colon F^* \to A$ such that $\pi\varepsilon^* w$ is surjective, and so A is a direct summand of E, a contradiction.

(b) The exact sequence 2.5 gives the following exact sequence

$$0 \to E^{**} \xrightarrow{\varepsilon^*} F^* \xrightarrow{u^*} \Omega^1(E^*)^* \to \operatorname{Ext}^1_A(E^*, A) \to 0.$$

By 1.8(b) and Grothendieck's local duality it follows that

$$\operatorname{Ext}^1_A(E^*, A) \cong (H^{d-1}_{\mathfrak{m}}(E^*))' \cong H^2_{\mathfrak{m}}(E),$$

and so we get the following exact sequences

$$0 \to E^{**} \xrightarrow{\varepsilon^*} F^* \to \operatorname{Im} u^* \to 0, \tag{1}$$

$$0 \to \operatorname{Im} u^* \to \Omega^1(E^*)^* \to H^2_{\mathfrak{m}}(E) \to 0. \tag{2}$$

Clearly there exists an A-linear map $v\colon \Omega^{-1}(E) \to \operatorname{Im} u^*$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \to & E & \xrightarrow{\varepsilon^* w} & F^* & \to & \Omega^{-1}(E) & \to & 0 \\ & & w\downarrow & & \| & & \downarrow v & & \\ 0 & \to & E^{**} & \xrightarrow{\varepsilon^*} & F^* & \to & \operatorname{Im} u^* & \to & 0 \end{array}$$

Applying the Snake Lemma to the above diagram we get $\operatorname{Coker} w \cong \operatorname{Ker} v$. From 1.16, local duality and 1.17 we get

$$\operatorname{Coker} w \cong \operatorname{Ext}^2_A(D(E), A) \cong H^{d-2}_{\mathfrak{m}}(D(E))' \cong H^1_{\mathfrak{m}}(E),$$

and so $\operatorname{Ker} v \cong H^1_{\mathfrak{m}}(E)$. The assertion follows now from (2). □

Lemma 2.7. *Suppose that* $E \in \mathscr{C}_1(A)$ *has no free direct summands,* $\operatorname{depth} E \geq 1$ *and* $d \geq 3$. *Then the following statements hold*:

(a) $H_{\mathfrak{m}}^{d-1}(\Omega^{-1}(E)) = 0$.

(b) $\Omega^{-1}(E) \in \mathscr{C}_1(A)$.

(c) *If* E^* *is not free, then* E *is indecomposable if and only if* $\Omega^{-1}(E)$ *is indecomposable.*

Proof. (a) By 2.6(b) we have the following exact sequence:

$$0 \to H_{\mathfrak{m}}^1(E) \to \Omega^{-1}(E) \to \Omega^1(E^*)^* \to H_{\mathfrak{m}}^2(E) \to 0$$

which yields the following exact sequences

$$(1) \qquad 0 \to H_{\mathfrak{m}}^1(E) \to \Omega^{-1}(E) \to T \to 0,$$

and

$$(2) \qquad 0 \to T \to \Omega^1(E^*)^* \to H_{\mathfrak{m}}^2(E) \to 0.$$

Applying the local cohomology functor we get the following exact sequences:

$$(3) \qquad H_{\mathfrak{m}}^{d-1}(H_{\mathfrak{m}}^1(E)) \to H_{\mathfrak{m}}^{d-1}(\Omega^{-1}(E)) \to H_{\mathfrak{m}}^{d-1}(T),$$

and

$$(4) \qquad H_{\mathfrak{m}}^{d-2}(H_{\mathfrak{m}}^2(E)) \to H_{\mathfrak{m}}^{d-1}(T) \to H_{\mathfrak{m}}^{d-1}(\Omega^1(E^*)^*).$$

Since $E \in \mathscr{C}_1(A)$ we get $H_{\mathfrak{m}}^i(H_{\mathfrak{m}}^j(E)) = 0$ for $j < d$ and $i > 0$, and so the left ends are zero in (3) and (4) $(d \geq 3)$. On the other hand we have

$$H_{\mathfrak{m}}^{d-1}(\Omega^1(E^*)^*) \cong H_{\mathfrak{m}}^{d-(d-2)}(\Omega^1(E^*))' \cong H_{\mathfrak{m}}^1(E^*)' = 0,$$

see 1.8, and 2.6(c). Thus from (4) we get $H_{\mathfrak{m}}^{d-1}(T) = 0$. Now our assertion follows from (3).

(b) follows from (a) and 2.6(d).

(c) Let $N = \Omega^{-1}(E)$. If N is decomposable then $E = \Omega^1(N)$ (see 2.6(a)) is also decomposable. Conversely, suppose that N is indecomposable. From (a) and (b) we conclude that $N \in \mathscr{C}_1(A)$ and $H_{\mathfrak{m}}^{d-1}(N) = 0$. Let $j \geq 1$ be the smallest integer for which $H_{\mathfrak{m}}^{d-j}(N) \neq 0$. Then $j \geq 2$, and applying Lemma 2.4 for $e = 0$ we get that $E = \Omega^1(N)$ indecomposable. □

Corollary 2.8 (The shifting of admissible sequences). $(h_0, h_1, \dots, h_{d-2}, 0)$ *is an admissible sequence, if and only if* $(0, h_0, h_1, \dots, h_{d-2})$ *is an admissible sequence.*

Proof. Let $E \in \mathscr{C}_1(A)$ be indecomposable such that $h_i(E) = h_i$ for $0 \leq i \leq d-1$ and $h_{d-1} = 0$. We may assume tht E is not free. The previous results imply that $\Omega^1(E)$ yields the admissible sequence $(0, h_1, h_2, \dots, h_{d-2})$. The converse is proved similarly applying Ω^{-1} to E. □

Lemma 2.9. *Let* (α, β) *be a pair of positive integers, and* j *an integer with* $1 < j < d$.

(a) *Given an indecomposable maximal QB A-module* U *such that*

$$h_i(U) = \begin{cases} \alpha & \text{if } i = 1, \\ \beta & \text{if } i = j, \\ 0 & \text{otherwise}, \end{cases}$$

there exists an exact sequence

$$0 \to U \to E_j^\beta \to k^\alpha \to 0.$$

(b) *Given an indecomposable maximal QB A-module E such that*

$$h_i(E) = \begin{cases} \alpha & \text{if } i = 0, \\ \beta & \text{if } i = j, \\ 0 & \text{otherwise}, \end{cases}$$

there exists an exact sequence

$$0 \to k^\alpha \to E \to E_j^\beta \to 0.$$

Proof. (a) By 1.19 we have an exact sequence

$$0 \to U \to U^{**} \to k^\lambda \to 0$$

($\operatorname{depth} U = 1$). Noting that $\operatorname{depth} U^{**} \geq 2$, the long exact cohomology sequence of the above sequence yields $\lambda = \alpha$ and $h_i(U^{**}) = \beta\delta_{ij}$, where δ_{ij} denotes the Kronecker symbol. Thus it follows from 1.12 that $U^{**} \cong E_j^\beta \oplus F$, where F is free. If $F \neq 0$, the module U splits off a direct summand. Indeed, then there exists a split surjection $\pi\colon U^{**} \to A$ and we have $\pi(U) \supset \mathfrak{m}$ since $U \supset \mathfrak{m}U^{**}$ (see the above exact sequence). If $\pi(U) = A$ then π induces a split surjection $U \to A$. Otherwise π induces a surjection $\rho\colon U \to \mathfrak{m}$. Since the section of π maps $\mathfrak{m}$ in $\mathfrak{m}U^{**} \subset U$, it follows that ρ splits too. But U is indecomposable and so we have either $U \cong A$ or $U \cong \mathfrak{m}$. Both cases cannot happen because $h_j(U) = \beta > 0$. Hence $U^{**} \cong E_j^\beta$.

(b) Since $H_\mathfrak{m}^0(E) \cong k^\alpha$, we get the following exact sequence:

$$0 \to k^\alpha \to E \to \overline{E} = E/H_\mathfrak{m}^0(E) \to 0$$

which gives

$$H_\mathfrak{m}^i(\overline{E}) = \begin{cases} 0 & \text{for } i = 0, \\ H_\mathfrak{m}^i(E) & \text{for } i > 0. \end{cases}$$

Hence $h_i(\overline{E}) = \beta\delta_{ij}$, and so $\overline{E} \cong E_j^\beta \oplus F$ by 1.12 for a certain free A-module F. Since E is indecomposable it follows that $F = 0$, and so $\overline{E} \cong E_j^\beta$. □

A first application of the shifting property of admissible sequences is the folllowing:

Corollary 2.10. *Let (α, β) be a pair of positive integers and let j be an integer with $1 < j < d$. The following statements are equivalent:*

(a) *There exists an exact sequence*

$$0 \to U \to E_j^\beta \to k^\alpha \to 0,$$

where U is indecomposable.

(b) *There exists an exact sequence*

$$0 \to k^\alpha \to E \to E_{j-1}^\beta \to 0,$$

where E is indecomposable.

The corollary follows from 2.8 and 2.9.

Proof of 2.3(b). If $(\alpha, 0, \dots, 0, \beta)$ is an admissible sequence then there exists an exact sequence

$$0 \to k^\alpha \to E \to E_{d-1}^\beta \to 0, \tag{1}$$

where E is indecomposable (see 2.9(b)).

Let $K.$ be the Koszul complex of a regular system of parameters of A, then

$$0 \to K_d \xrightarrow{\partial} K_{d-1} \xrightarrow{\varepsilon} E_{d-1} \to 0, \tag{2}$$

is the minimal free resolution of E_{d-1}. Note that k^α maps into $\mathfrak{m}E$ because otherwise k is a direct summand of E as shown in the next remark. But E is indecomposable.

Remark 2.11. Let $x \in E \backslash \mathfrak{m}E$ be an element such that $\operatorname{Ann}_A x = \mathfrak{m}$. Then $k \cong Ax$ is a direct summand of E. Indeed, choose a system of elements $y = (y_1, \dots, y_s)$ of E such that $\{x, y\}$ induces a basis in $E/\mathfrak{m}E$. Then $Ax + Ay = E$, and if $z \in Ax \cap Ay$, then $z = \lambda x = \sum_{i=1}^s \mu_i y_i$ for some elements $\lambda, \mu_i \in A$. Since $\{x, y\}$ induces a basis in $E/\mathfrak{m}E$ we get $\lambda, \mu_i \in \mathfrak{m}$, and so $z = 0$.

Continuing with the proof of 2.3(b), we note that the identity map of E_{d-1}^β can be lifted to a morphism of complexes as shown in the following diagram:

$$\begin{array}{ccccccccc}
0 & \to & K_d^\beta & \xrightarrow{\partial^\beta} & K_{d-1}^\beta & \xrightarrow{\varepsilon^\beta} & E_{d-1}^\beta & \to & 0 \\
 & & \psi \downarrow & & \varphi \downarrow & & \| & & \\
0 & \to & k^\alpha & \to & E & \to & E_{d-1}^\beta & \to & 0
\end{array}$$

where the upper row is a direct sum of β-copies of (2) and the bottom row is the exact sequence (1). Since k^α maps into $\mathfrak{m}E$ the maps φ and ψ are necessarily surjective. Thus $\beta \geq \alpha$ since $K_d \cong A$. Then by 2.2 we have $\alpha \geq \beta$ too, and so $\alpha = \beta$.

Now note that $\operatorname{Ker}\psi = \mathfrak{m}K_d^\beta$. Applying the Snake Lemma to the above diagram, it follows that ∂^β induces an isomorphism $\operatorname{Ker}\varphi \cong \operatorname{Ker}\psi$, and so we get an exact sequence

$$0 \to (\mathfrak{m}K_d)^\beta \xrightarrow{(\partial u)^\beta} K_{d-1}^\beta \xrightarrow{\varphi} E \to 0,$$

where u denotes the inclusion $\mathfrak{m}K_d \subset K_d$. Thus $E \cong T^\beta$, where $T = \operatorname{Coker}\partial u$, and since E is indecomposable it follows that $\alpha = \beta = 1$. Conversely, we claim that T is an indecomposable of module $\mathscr{C}_1(A)$ with

$$h_i(T) = \begin{cases} 1 & \text{for } i = 1, d-1, \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, by definition we have the following exact sequences:

$$0 \to E_1 = \mathfrak{m} \cong \mathfrak{m}K_d \xrightarrow{\partial u} K_{d-1} \to T \to 0, \tag{3}$$

$$0 \to k \cong K_d/\mathfrak{m}K_d \to T \to \operatorname{Coker}\partial = E_{d-1} \to 0. \tag{4}$$

If T has a free direct summand then so does E_{d-1}, a contradiction. If $T = G \oplus H$ is a decomposition of T, then as in the proof of 2.4 we get from (3) that $E_1 \cong \Omega^1(T) = \Omega^1(G) \oplus \Omega^1(H)$ and so $\Omega^1(G) = 0$ or $\Omega^1(H) = 0$, E_1 being indecomposable. Thus G or H is free, and this is not possible as seen above.

The other statements follow easily from (3) and (4). Thus $(1, 0, \ldots, 0, 1)$ is admissible. □

In view of 2.8 and 2.10, Theorem 2.3(a) follows immediately from

Proposition 2.12. *For all integers $a, b > 0$ and all integers j such that $1 < j < d$, there exist two integers $\alpha \geq a$, $\beta \geq b$ and an exact sequence*

$$0 \to U \to E_j^{\beta} \to k^{\alpha} \to 0,$$

where U is indecomposable.

The proposition in turn is an immediate consequence of the next theorem.

We introduce a few notations. Let $1 < j < d$ be an integer and denote by $\mathscr{A}$ the full additive subcategory of $\mathscr{C}_1(A)$ whose objects $U \neq 0$ have no direct summands isomorphic to $\mathfrak{m}$ or $\mathscr{A}$ and satisfy $h_j(U) \neq 0$ and $h_i(U) = 0$ for $i \neq 1, j, d$. Let B be the trivial algebra extension (Nagata extension principle) of k by $\overline{E}_j = E_j/\mathfrak{m}E_j$. Note that $B = k * \overline{E}_j$ is an Artinian local ring with $\mathfrak{n} = 0 * \overline{E}_j$ as maximal ideal. We have $\mathfrak{n}^2 = 0$ and $e = \operatorname{emb}\dim B = \dim_k \overline{E}_j$. Let $\operatorname{mod}(B)$ be the category of finite B-modules. If $M \in \operatorname{mod}(B)$ then $r(M)$ denotes the type (the socle dimension) of M and $\mu(M) = \dim_k M/\mathfrak{n}M$ the minimal number of generators of M.

Theorem 2.13. *There is an additive functor $F: \mathscr{A} \to \operatorname{mod}(B)$ inducing a bijection between the isomorphism classes of indecomposable modules of $\mathscr{A}$ and $\operatorname{mod}(B)$. Moreover, if $M = F(U)$, $U \in \mathscr{A}$, is indecomposable and $M \ncong k$, then*

$$\mu(M) = h_j(U) \quad \text{and} \quad r(M) = h_1(U).$$

Let us first show how Proposition 2.12 follows from Theorem 2.13. With the help of the preceding theorem the statement of 2.12 can be translated as follows:

"Given integers $a, b \geq 0$, there exists an indecomposable B-module M with $\mu(M) \geq b$ and $r(M) \geq a$."

We will prove this well-known fact (which also follows directly from the first Brauer-Thrall conjecture valid for all Artinian rings) by constructing a series of indecomposable B-modules $(M_i)_{i \in \mathbb{N}}$ such that

$$(*) \qquad \mu(M_{i+1}) > \mu(M_i), \quad r(M_{i+1}) > r(M_i), \quad \text{and} \quad \mu(M_i) > r(M_i)$$

for all $i \geq 1$. Let $M_1 = \omega_B$ be the canonical module of B. We have $\mu(\omega_B) = e$, $r(\omega_B) = 1$.

Suppose that $M_1, \ldots, M_i$ have already been constructed. Then we set $M_{i+1} = \operatorname{Hom}_B(D(M_i), \omega_B)$ which is the Auslander-Reiten translation of M_i. By 1.15 the module $D(M_i)$ is indecomposable, and so M_{i+1} is indecomposable too. Note further that for an arbitrary indecomposable B-module $N \neq 0$,

$$(1) \qquad \mu(D(N)) = e\mu(N) - r(N) \quad \text{if } N \ncong k,$$

$$(2) \qquad r(D(N)) = (e^2 - 1)\mu(N) - er(N) \quad \text{if } N \ncong k, D(k),$$

$$(3) \qquad \mu(\operatorname{Hom}_B(N, \omega_B)) = r(N),$$

$$(4) \qquad r(\operatorname{Hom}_B(N, \omega_B)) = \mu(N).$$

Indeed, consider the exact sequence

$$(5) \qquad 0 \to \Omega^1(N) \to B^{\mu(N)} \to N \to 0.$$

B has a structure of graded ring: $B_0 = k$, $B_1 = \mathfrak{n}$, and $B_i = 0$ for $i > 1$. Since $\Omega^1(N) \subset \mathfrak{n}B^{\mu(N)}$ and $\mathfrak{n}^2 = 0$, the exact sequence (5) induces a natural grading on N: $N_0 = B_0^{\mu(N)} = k^{\mu(N)}$, and $N_1 = B_1^{\mu(N)}/\Omega^1(N) = \mathfrak{n}^{\mu(N)}/\Omega^1(N)$.

Notice that $\mu(N) = \dim_k N_0$, and that

$$r(N) = \mu(N_1) = e\mu(N) - \dim_k \Omega^1(N) = e\mu(N) - \mu(D(N)),$$

provided $N \ncong k$, and N is indecomposable. (If there would exist a socle element in N_0 it would split off: see 2.11). Hence we have proved formula (1). Replacing N by $D(N)$ in (1) we get

$$\mu(N) = \mu(D(D(N))) = e\mu(D(N)) - r(D(N)) = e^2\mu(N) - er(N) - r(D(N)),$$

i.e., (2) holds. The equalities (3), (4) are well known (see, e.g., [18]). The formulas (1)–(4) yield

$$\mu(M_{i+1}) = (e^2 - 1)\mu(M_i) - er(M_i), \quad r(M_{i+1}) = e\mu(M_i) - r(M_i).$$

Since $e \geq 3$ (because $d \geq 3$), these equalities imply the inequalities $(*)$.

For the proof of Theorem 2.13 we will need the following result.

Proposition 2.14. *For all integers j the image of the canonical map $\mathrm{End}_A(E_j) \to \mathrm{End}_k(\overline{E}_j)$, $\overline{E}_j = E_j/\mathfrak{m}E_j$, consists only of maps which are defined by scalar multiplication.*

Proof. We may assume that $1 \leq j < d$. As before we denote by $K.$ the Koszul complex of a regular system of parameters of A. Then

$$0 \to K_d \xrightarrow{\partial_d} K_{d-1} \to \cdots \to K_{j+1} \xrightarrow{\partial_{j+1}} K_j \xrightarrow{\varepsilon} E_j \to 0$$

is a minimal free A-resolution of E_j. Let $\varphi \in \mathrm{End}_A(E_j)$; then there exists a morphism $\varphi.: K. \to K.$ of complexes extending φ. Since $\mathrm{Ext}^i_A(E_j, A) = 0$ for $i \neq d - j$, we get after dualizing a commutative diagram

$$\begin{array}{ccccccccccc} 0 & \to & E_j^* & \xrightarrow{\varepsilon^*} & K_j^* & \xrightarrow{\partial_{j+1}^*} \cdots \xrightarrow{\partial_d^*} & K_d^* & \to & k & \to & 0 \\ & & \varphi^* \downarrow & & \varphi_j^* \downarrow & & \varphi_d^* \downarrow & & & & \\ 0 & \to & E_j^* & \xrightarrow{\varepsilon^*} & K_j^* & \xrightarrow{\partial_{j+1}^*} \cdots \xrightarrow{\partial_d^*} & K_d^* & \to & k & \to & 0 \end{array}$$

with exact rows. But φ_d^* induces an endomorphism of k which is induced by a multiplication $\mu_a: k \to k$ for a certain $a \in A$. It follows that the two complex homomorphisms

$$\begin{array}{ccccccccc} 0 & \to & E_j^* & \xrightarrow{\varepsilon^*} & K_j^* & \xrightarrow{\partial_{j+1}^*} \cdots \xrightarrow{\partial_d^*} & K_d^* & \to & 0 \\ & & \mu_a \downarrow\downarrow \varphi^* & & \mu_a \downarrow\downarrow \varphi_j^* & & \mu_a \downarrow\downarrow \varphi_d^* & & \\ 0 & \to & E_j^* & \xrightarrow{\varepsilon^*} & K_j^* & \xrightarrow{\partial_{j+1}^*} \cdots \xrightarrow{\partial_d^*} & K_d^* & \to & 0 \end{array}$$

are homotopic. Let us say $f.$ is the homotopy. Then we have

$$f_{j+1}\partial_{j+1}^* + \varepsilon^* f_j = \varphi_j^* - \mu_a.$$

Dualizing we obtain the equation

$$\varphi_j - \mu_a = \partial_{j+1} f_{j+1}^* + f_j^* \varepsilon.$$

The map $f_j^*\varepsilon: K_j \to K_j$ is the dual of $\varepsilon^* f_j: K_j^* \to K_j^*$ whose image lies in $\mathfrak{m}K_j^*$. It follows that $\mathrm{Im}(f_j^*\varepsilon) \subset \mathfrak{m}K_j$. Also $\mathrm{Im}(\partial_{j+1} f_{j+1}^*) \subset \mathfrak{m}K_j$ because $\mathrm{Im}\,\partial_{j+1} \subset \mathfrak{m}K_j$. Hence $\mathrm{Im}(\varphi_j - \mu_a) \subset \mathfrak{m}K_j$, and so $\mathrm{Im}(\varphi - \mu_a) \subset \mathfrak{m}E_j$. □

Proof of Theorem 2.13. Let U be an object of $\mathscr{A}$ with $h_1(U) = \alpha$ and $h_j(U) = \beta$. By 1.19 (see the proof of 2.9) there exists an exact sequence

$$0 \to U \overset{u_U}{\to} E_j^\beta \to k^\alpha \to 0,$$

where u_U is the composed map $U \overset{u'}{\to} U^{**} \overset{\psi_U}{\to} E_j^\beta$, u' being canonically given, and ψ_U being a certain isomorphism which we will fix now for every U. Our functor F will depend on the choice of the isomorphisms (ψ_U). Note that $\operatorname{Im} u_U \supset \mathfrak{m}E_j^\beta$, and so there is a natural embedding

(1) $$V(U) = \operatorname{Im} u_U / \mathfrak{m}E_j^\beta \subset B^\beta,$$

and we set $F(U) = B^\beta / V(U)$. Then $\mu(F(U)) = \beta$ since $V(U) \subset \mathfrak{n}B^\beta$, and if $F(U)$ is indecomposable and $F(U) \ncong k$, then

$$r(F(U)) = e\beta - \dim_k V(U) = \dim_k E_j^\beta / \operatorname{Im} u_U = \dim_k k^\alpha = \alpha.$$

This proves the second part of the theorem.

Let $\varphi\colon U_1 \to U_2$ be a morphism in $\mathscr{A}$, $\psi_{U_i}\colon U_i^{**} \to E_j^{\beta_i}$, $i = 1, 2$, the given isomorphism. Clearly the homomorphism $\psi = \psi_{U_2}\varphi^{**}\psi_{U_1}^{-1}$ makes the following diagram commutative:

$$\begin{array}{ccc} U_1 & \overset{u_{U_1}}{\to} & E_j^{\beta_1} \\ \varphi \downarrow & & \downarrow \psi \\ U_2 & \underset{u_{U_2}}{\to} & E_j^{\beta_2} \end{array}$$

This diagram induces a commutative diagram

$$\begin{array}{ccc} V(U_1) & \to & \mathfrak{n}B^{\beta_1} \\ \overline{\varphi} \downarrow & & \downarrow \overline{\psi} \\ V(U_2) & \to & \mathfrak{n}B^{\beta_2}. \end{array}$$

We claim that $\overline{\psi}$ can be canonically extended to a B-linear map $\widetilde{\psi}\colon B^{\beta_1} \to B^{\beta_2}$. Indeed, ψ is given by a matrix $(\psi_{\lambda,\nu})$, $1 \le \lambda \le \beta_1$, $1 \le \nu \le \beta_2$, of elements from $\operatorname{End} E_j$ in the following way:

(2) $$\psi(z_1, \ldots, z_{\beta_1}) = \left(\sum_{\lambda=1}^{\beta_1} \psi_{\lambda\nu}(z_\lambda) \right)_{1 \le \nu \le \beta_2}.$$

But modulo $\mathfrak{m}$ the endomorphism $\psi_{\lambda\nu}$ is a multiplication of E_j by an element $a_{\lambda\nu} \in A$ (see Proposition 2.14). Thus $\overline{\psi}$ is given by the matrix $(\overline{a}_{\lambda\nu})$, $\overline{a}_{\lambda\nu}$ being the residue class of $a_{\lambda\nu}$. Clearly the map $\widetilde{\psi}\colon B^{\beta_1} \to B^{\beta_2}$ given by $(\overline{a}_{\lambda\nu})$ is an extension of $\overline{\psi}$. In particular, $\widetilde{\psi}$ induces a morphism $\varphi'\colon F(U_1) \to F(U_2)$ and we set $F(\varphi) = \varphi'$.

It is obvious that F is indeed an additive functor. In the next step we show that F induces a bijection between the isomorphism classes of the modules of $\mathscr{A}$ and of the finite B-modules.

Surjectivity. For a given $M \in \operatorname{mod} B$ we choose a minimal presentation

(3) $$0 \to V \to B^\beta \to M \to 0.$$

Then $V \subset E_j^\beta/\mathfrak{m}E_j^\beta$. Let U be the preimage of V under the canonical epimorphism $E_j^\beta \to E_j^\beta/\mathfrak{m}E_j^\beta$. Then $F(U) \cong M$ (the isomorphism depends on ψ_U and on (3)).

Injectivity. Let $U' \in \mathscr{A}$ be another module with $F(U') \cong M \cong F(U)$. Choose an isomorphism $\eta': F(U) \to F(U')$. Then there exist two homomorphisms $\tilde{\eta}: B^\beta \to B^\beta$, $\overline{\rho}: V(U) \to V(U')$ such that the following diagram

$$\begin{array}{ccccccccc} 0 & \to & V(U) & \to & B^\beta & \to & F(U) & \to & 0 \\ & & \overline{\rho}\downarrow & & \tilde{\eta}\downarrow & & \downarrow\eta' & & \\ 0 & \to & V(U') & \to & B^\beta & \to & F(U') & \to & 0 \end{array}$$

is commutative. Since $B/\mathfrak{n} \otimes \tilde{\eta} \cong B/\mathfrak{n} \otimes \eta'$ is an isomorphism, Nakayama's Lemma implies that $\tilde{\eta}$ is surjective. But then it follows that both $\tilde{\eta}$ and $\overline{\rho}$ are isomorphisms. The above diagram induces the following commutative diagram:

$$\begin{array}{ccc} U/\mathfrak{m}E_j^\beta & \to & \mathfrak{n}B^\beta \\ \overline{\rho}\downarrow & & \downarrow\overline{\eta} \\ U'/\mathfrak{m}E_j^\beta & \to & \mathfrak{n}B^\beta \end{array}$$

because $V(U) \subset \mathfrak{n}B^\beta$. Since $\tilde{\eta}$ is given by an invertible matrix $\tilde{a}$ with coefficients in B and $\mathfrak{n}B^\beta$ is a k-vector space it follows that $\overline{\eta}$ is given by a matrix $\overline{a}$ which is induced by $\tilde{a}$ modulo $\mathfrak{n}$. Let $a = (a_{\lambda\nu})$ be an invertible matrix over A lifting $\overline{a}$ and $\eta: E_j^\beta \to E_j^\beta$ the A-linear map given by

$$\eta(z_1, \dots, z_\beta) = \left(\sum_{\lambda=1}^{\beta} a_{\lambda\nu} z_\lambda\right)_{1\le\nu\le\beta}.$$

Then η is an isomorphism lifting $\overline{\eta}$ with $\eta(U) = U'$, i.e., $U \cong U'$.

It remains to show that $M = F(U)$ is indecomposable if and only if U is indecomposable. Since F is additive, U is indecomposable if M is so. Conversely, suppose that M is decomposable say $M = M_1 \oplus M_2$ with $M_i \neq 0$, $i = 1, 2$. Choose $U_i \in \mathscr{A}$ with $F(U_i) = M_i$. Then $U_i \neq 0$, $i = 1, 2$, and $F(U_1 \oplus U_2) \cong M$. By the above "injectivity" it follows that $U \cong U_1 \oplus U_2$, and so U is decomposable. □

Before we can get the first corollary of 2.13 we need the following result about indecomposable B-modules.

Lemma 2.15. *Let α, β be two positive integers such that there exists an indecomposable B-module M with $\mu(M) = \beta$, $r(M) = \alpha$. Then*

(a) *$M \cong B$ if and only if $e\beta = \alpha$.*
(b) *$M \cong \omega_B$ if and only if $e\alpha = \beta$.*
(c) *$M \cong T = D(k)$ if and only if $(e^2 - 1)\beta = e\alpha$.*
(d) *$M \cong T' = \mathrm{Hom}_B(T, \omega_B)$ if and only if $(e^2 - 1)\alpha = e\beta$.*
(e) *M is not isomorphic with any module of $\{B, \omega_B, T, T'\}$ if and only if $(e^2 - 1)\alpha > e\beta$, $(e^2 - 1)\beta > e\alpha$.*

Proof. In this proof we will refer to the formulas (1)–(4) in the proof of Proposition 2.13. (a) $M \cong B$ if and only if $D(M) = 0$, i.e., if and only if $\mu(D(M)) = e\beta - \alpha = 0$ (see (1)).

(c) If $M \cong D(k)$, then $\beta = e$ and $\alpha = e^2 - 1$, and so $(e^2 - 1)\beta = e\alpha$. If $M \ncong k$, $D(k)$, then by (2) we have $r(D(M)) = (e^2 - 1)\beta - e\alpha > 0$.

Similarly we get (b) and (d) from (a) and (c) using (3) and (4). If M is not isomorphic with any module of $\{B, \omega_B, T, T'\}$, then $r(D(M)) = (e^2-1)\beta - e\alpha > 0$, and $r(D(\mathrm{Hom}_B(M, \omega_B))) = (e^2-1)\alpha - e\beta > 0$ by (2) and (4). □

Corollary 2.16. *Let* (α, β) *be a pair of positive integers,* i *and* j *integers such that* $0 \leq i < j < d$, *and* $(i, j) \neq (0, d-1)$. *Suppose that the sequence* $h = (h_0, \dots, h_{d-1})$ *with*

$$h_\lambda = \begin{cases} \alpha & \text{for } \lambda = i, \\ \beta & \text{for } \lambda = j, \\ 0 & \text{otherwise}, \end{cases}$$

is admissible, and let $\mathscr{F} = \{(1, e), (e, 1), (e, e^2-1), (e^2-1, e)\}$, *where*

$$e = \binom{d}{j-i+1}.$$

Then, either $(\alpha, \beta) \in \mathscr{F}$ *in which case there exists (up to an isomorphism) a unique indecomposable* $U \in \mathscr{C}_1(A)$ *with* $h_\lambda(U) = h_\lambda$ *for all* λ, *or else* $(e^2-1)\alpha > e\beta$, *and* $(e^2-1)\beta > e\alpha$.

Proof. By the Shifting Lemma we may assume that $h_1 = \alpha$, $h_s = \beta$, $s = j-i+1$, and $h_\lambda = 0$ for $\lambda \neq 1, s$. Now it is enough to apply 2.13 and 2.15 to $B = k * \overline{E}_s$. Note also that in 2.15(c) we have $\mu(T) = e$ and $r(T) = e^2-1$. □

Corollary 2.17. *If* $d = 2$, *then the only indecomposable maximal QB-modules are* $\mathfrak{m}$, A, *and* E, *the unique module determined by the nonsplit extension*

$$0 \to k \to E \to \mathfrak{m} \to 0.$$

Proof. The only admissible sequences are $(0, 1)$, $(1, 0)$, and $(1, 1)$ by 2.2(a) and 2.3(b). But $(1, 0)$ corresponds to k which is not maximal. We are left to show that E is unique up to an isomorphism. This is clear since $\mathrm{Ext}^1_A(\mathfrak{m}, k) \cong k$. □

Corollary 2.18. (a) *If* $d = 3$, *then* E_2 *is the only maximal nonfree indecomposable QB-module which is reflexive.*

(b) *If* $d > 3$, *there exist maximal indecomposable reflexive QB-modules of arbitrarily high rank.*

Proof. (a) If E is a maximal indecomposable reflexive QB module then $h_0(E) = h_1(E) = 0$. So only $h_2(E)$ is possibly different from zero. If $h_2(E) = 0$ then $E \cong A$, if $h_2(E) \neq 0$ then $h_2(E) = 1$ and $E \cong E_2$ by 2.2(a).

(b) Pick j, $1 < j < d-1$, and α, β for which there exists an indecomposable maximal QB module U such that

$$h_\lambda(U) = \begin{cases} \alpha & \text{for } i = 1, \\ \beta & \text{for } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Then there exists an exact sequence

$$0 \to U \to E_j^\beta \to k^\alpha \to 0$$

by 2.9, and so

$$\mathrm{rank}(U) = \beta\, \mathrm{rank}(E_j) = \beta \binom{d-1}{j-1}$$

and

$$\mu(U) \geq \mu(E_j^\beta) - \mu(k^\alpha) = \binom{d}{j}\beta - \alpha.$$

Hence

$$\operatorname{rank} \Omega^1(U) = \mu(U) - \operatorname{rank}(U) \geq \binom{d-1}{j}\beta - \alpha.$$

We have depth $\Omega^1(U) = 2$ since depth $U = 1$. Applying 2.4 for U with $e = 0$ we get $\Omega^1(U)$ is indecomposable because $H_{\mathfrak{m}}^{d-1}(U) = 0$. By 2.13 and by (2), (4) in the proof of 2.12, we may assume that $\beta \geq \alpha$ and α is arbitrarily large. Hence

$$\operatorname{rank} \Omega^1(U) \geq \binom{d-1}{j}\beta - \alpha \geq \left(\binom{d-1}{j} - 1\right)\alpha. \quad \square$$

Using a result from representation theory of Artinian algebras, statement (b) of the preceding corollary can be strengthened essentially.

Corollary 2.19. *The category $\mathscr{C}_1(A)$ satisfies the second Brauer-Thrall conjecture; that is, if $\mathscr{C}_1(A)$ is of infinite representation type (which is the case for* $\dim A > 2$*), then there exists a strictly increasing sequence (t_i) of integers such that for all $i \in \mathbb{N}$ the set of isomorphism classes of indecomposable modules of rank t_i is infinite.*

Proof. It is known that the Artinian algebra B (as any other finite-dimensional algebra) satisfies the second Brauer-Thrall conjecture, cf. [22 and 28]. Here, of course, one has to replace "rank" by "length". So let (t_i) be a strictly increasing sequence of integers for which there exist infinitely many isomorphism classes of indecomposable B-modules of length t_i. For each i there exist only $t_i - 1$ pairs of positive integers (α, β) with $\alpha + \beta = t_i$. From this we conclude that there is a sequence of pairs of integers (α_i, β_i) such that for all i there exist infinitely many isomorphism classes of modules $M \in \operatorname{mod} B$ with $\mu(M) = \beta_i$ and $r(M) = \alpha_i$. Note that $\lim_{i\to\infty} \alpha_i = \infty$ and $\lim_{i\to\infty} \beta_i = \infty$ because of 2.15(e). As in the proof of the preceding corollary it follows now that the rank of the modules $N \in \mathscr{A}$ (see 2.13) with $h_1(N) = \alpha_i$ and $h_j(N) = \beta_i$ tends to infinity with $i \to \infty$. Hence the conclusion follows from 2.13. $\square$

Remark 2.20. (a) The cardinality of the set of isomorphism classes of indecomposable modules $E \in \mathscr{C}_1(A)$ for which just two integers of the sequence $h(E) = (h_i(E))_{0\leq i<d}$ are nonzero depends only on d and k, as follows from 2.13 and the proof of 2.3. But by 2.2(a) there exist only d isomorphism classes of indecomposable modules $E \in \mathscr{C}_1(A)$ for which just one integer of $h(E)$ in nonzero. In particular, the cardinality of these last isomorphism classes depends only on d.

(b) Theorem 2.13 shows how important it is to know all indecomposable modules over our Artinian local ring $(B, \mathfrak{n})$. In the following we present an idea for a computational approach. Let s_1, s_2 be two nonnegative integers, β_1, β_2 two positive integers such that $s_i \leq e\beta_i$, $e = \dim_k \mathfrak{n}$, $\beta = \beta_1 + \beta_2$, $s = s_1 + s_2$, and $z_\lambda = (z_{\lambda,1}, \ldots, z_{\lambda,\beta})$, $1 \leq \lambda \leq s$, some elements of $\mathfrak{n}B^\beta$ which are k-linearly independent. Then the B-module $M = B^\beta / \sum_{\lambda=1}^{s} B z_\lambda$ has a decomposition $M = M_1 \oplus M_2$ with $\mu(M_i) = \beta_i$, $r(M_i) = e\beta_i - s_i$, $i = 1, 2$, if and only if there exist two invertible square matrices H_1, H_2 with coefficients in k such that

$$H_1(z_{\lambda\nu})H_2 = \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix},$$

where G_i is a $s_i \times \beta_i$-matrix with entries in $\mathfrak{n}$, $i = 1, 2$. Let $e = 3$, and y_1, y_2, y_3 a basis of $\mathfrak{n}$. Then, for example, the following matrices define indecomposable modules:

$$[y_1], \quad [y_1 y_2], \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \begin{bmatrix} y_1 & 0 \\ y_2 & y_3 \end{bmatrix}, \quad \begin{bmatrix} y_1 & 0 \\ y_2 & y_1 \\ y_3 & y_2 \end{bmatrix}, \quad \begin{bmatrix} y_1 & 0 \\ y_2 & y_1 \\ y_3 & y_2 \\ 0 & y_3 \end{bmatrix}, \quad \begin{bmatrix} y_1 & 0 \\ y_2 & 0 \\ y_3 & y_1 \\ 0 & y_2 \\ 0 & y_3 \end{bmatrix}.$$

3. Reduction ideals for generalized Cohen-Macaulay modules

We assume in this section that $(A, \mathfrak{m}, k)$ is a local Cohen-Macaulay ring of dimension d whose regular locus $\operatorname{Reg} A$ is open, $\mathfrak{a} \subset A$ is a proper ideal and t is a nonnegative integer.

Definition 3.1. A *proper ideal* $\mathfrak{b} \subset A$ is a $\mathscr{C}_t(\mathfrak{a}, A)$-reduction ideal if the following statements hold:

(a) A module $M \in \mathscr{C}_t(\mathfrak{a}, A)$ is indecomposable if and only if $M/\mathfrak{b}M$ is indecomposable.

(b) Two indecomposable modules $M, N \in \mathscr{C}_t(\mathfrak{a}, A)$ are isomorphic if and only if $M/\mathfrak{b}M$ and $N/\mathfrak{b}N$ are isomorphic over $A/\mathfrak{b}$.

The ideal

$$\mathscr{I}_s(A) = \bigcap_{q \in \operatorname{Sing} A} \mathfrak{q},$$

where $\operatorname{Sing} A = \operatorname{Spec} A \setminus \operatorname{Reg} A$, defines the nonregular locus of A, that is, $V(\mathscr{I}_s(A)) = \operatorname{Spec} A \setminus \operatorname{Reg} A$. The aim of this section is to show that in special situations a power of $\mathfrak{a}\mathscr{I}_s(A)$ is a $\mathscr{C}_t(\mathfrak{a}, A)$-reduction ideal.

Definition 3.2. Let $\mathfrak{b} \subset A$ be a proper ideal. The couple $(A, \mathfrak{b})$ is a $\mathscr{C}_t(\mathfrak{a}, A)$-approximation if there exists a function $\nu\colon \mathbb{N} \to \mathbb{N}$, $\nu \geq \mathrm{id}_{\mathbb{N}}$, such that for all $s \in \mathbb{N}$, all modules $M, N \in \mathscr{C}_t(\mathfrak{a}, A)$ and every A-linear map $\varphi\colon M \to N/\mathfrak{b}^{\nu(s)}N$ there exists an A-linear map $\psi\colon M \to N$ such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N/\mathfrak{b}^{\nu(s)}N \\ \psi\downarrow & & \downarrow \\ N & \rightarrow & N/\mathfrak{n}b^s N. \end{array}$$

In other words $A/\mathfrak{b}^s \otimes_A \psi \cong A/\mathfrak{b}^s \otimes_A \varphi$.

The importance of this concept is given by the following

Lemma 3.3. *Let $\mathfrak{b} \subset A$ be a proper ideal. Suppose that A is Henselian and $(A, \mathfrak{b})$ is a $\mathscr{C}_t(\mathfrak{a}, A)$-approximation. Then there exists a positive integer $u \in \mathbb{N}$ such that $\mathfrak{b}^u$ is a $\mathscr{C}_t(\mathfrak{a}, A)$-reduction ideal.*

The proof is given in [24, (4.5), and (4.6)] (the arguments there apply to any class of modules); we note here only that if ν is the function associated to $(A, \mathfrak{b})$, then we may choose $u = \nu(1)$.

Proposition 3.4. *Suppose that* A *satisfies the following bound condition:*
$(\Omega_{\mathfrak{a}})$ *There exists a positive integer* r *such that*

$$(\mathfrak{a}\mathscr{I}_s(A))^r \operatorname{Ext}^1_A(M, W) = 0$$

for all modules $M \in \mathscr{C}_t(\mathfrak{a}, A)$ *and all finite* A*-modules* W.
Then $(A, \mathfrak{a}\mathscr{I}_s(A))$ *is a* $\mathscr{C}_t(\mathfrak{a}, A)$*-approximation.*

This proposition follows from our next lemma which is stronger as is needed for the proof of 3.4 but suits better for induction.

Lemma 3.5. *Suppose that* A *satisfies* $(\Omega_{\mathfrak{a}})$, *and let* $\mathfrak{c} \subset \mathfrak{a}\mathscr{I}_s(A)$, $\mathfrak{d} \subset A$ *be two ideals. Then there exists a function* $\nu: \mathbb{N} \to \mathbb{N}$, $\nu \geq \mathrm{id}_{\mathbb{N}}$, *such that for all* $s \in \mathbb{N}$, *all modules* $M, N \in \mathscr{C}_t(\mathfrak{a}, A)$ *and all homomorphisms* $\varphi: M \to N/(\mathfrak{d}, \mathfrak{c}^{\nu(s)})N$, *there is a homomorphism* $\psi: M \to N/\mathfrak{d}N$ *such that the following diagram commutes:*

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N/(\mathfrak{d}, \mathfrak{c}^{\nu(s)})N \\ \psi\downarrow & & \downarrow \\ N/\mathfrak{d}N & \to & N/(\mathfrak{d}, \mathfrak{c}^s)N. \end{array}$$

Proof. By $(\Omega_{\mathfrak{a}})$ there is a positive integer $r \in \mathbb{N}$ such that

$$(1) \qquad \mathfrak{c}^r \operatorname{Ext}^1_A(M, W) = 0$$

for every module $M \in \mathscr{C}_t(\mathfrak{a}, A)$ and every finite A-module W. Let $y_1, \dots, y_u$ be a system of generators of $\mathfrak{c}$ and apply induction on u. If $u = 1$, we apply 1.6. Then there exists a positive integer e such that

$$(2) \qquad (\mathfrak{d}N : y_1^e)_N = (\mathfrak{d}N : y_1^{e+1})_N$$

for all modules $N \in \mathscr{C}_t(\mathfrak{a}, A)$. We need the following result (cf. [25, (2.1)]).

Lemma 3.6. *Let* R *be a Noetherian ring,* P, T *two finite* R*-modules,* $x \in R$ *an element such that* $x \operatorname{Ext}^1_R(P, T') = 0$ *for every factor* R*-module* T' *of* T, w *a positive integer such that* $((0) : x^w)_T = ((0) : x^{w+1})_T$, *and* $s \in \mathbb{N}$. *Then for every* R*-linear map* $\mu: P \to T/x^{w+s+1}T$ *there exists an* R*-linear map* $\rho: P \to T$ *such that* $R/x^sR \otimes \rho \cong R/x^sR \otimes \mu$.

We may choose the function $\nu: \mathbb{N} \to \mathbb{N}$, $\nu(s) = r(e + s + 1)$. Indeed, let $M, N \in \mathscr{C}_t(\mathfrak{a}, A)$ and $\varphi: M \to N/(\mathfrak{d}, y_1^{\nu(s)})N$ be an A-linear map. Applying the above lemma for $P = M$, $T = N/\mathfrak{d}N$, $x = y_1^r$, $w = e$, and $\mu = \varphi$ we get ψ (by (1), (2) the numbers r and w do not depend on M, N).

Now suppose that $u > 1$. By the induction hypothesis there exists a function $\nu': \mathbb{N} \to \mathbb{N}$ for $\mathfrak{c}' = \langle y_1, \dots, y_{u-1}\rangle$ and $\mathfrak{d}$. For a positive integer s consider the function $\nu''_s: \mathbb{N} \to \mathbb{N}$ given as above for $\mathfrak{d}'_s = (\mathfrak{d}, (\mathfrak{c}')^{\nu'(s)}) \subset A$ and y_u. Then the function $\nu: \mathbb{N} \to \mathbb{N}$ given by $\nu(s) = \nu'(s) + \nu''_s(s)$ works. Indeed, let M, N be two modules of $\mathscr{C}_t(\mathfrak{a}, A)$ and $\varphi: M \to N/(\mathfrak{d}, \mathfrak{c}^{\nu(s)})N$ an A-linear map. As $\mathfrak{c}^{\nu(s)} \subset ((\mathfrak{c}')^{\nu'(s)}, y_u^{\nu''_s(s)})$, by the property of ν''_s there exists an A-linear map $\varphi': M \to N/\mathfrak{d}'_sN$ such that the following diagram commutes:

$$\begin{array}{ccccc} M & \xrightarrow{\varphi} & N/(\mathfrak{d}, \mathfrak{c}^{\nu(s)})N & \to & N/(\mathfrak{d}'_s, y_u^{\nu''_s(s)})N \\ \varphi'\downarrow & & & & \downarrow \\ N/(\mathfrak{d}, (\mathfrak{c}')^{\nu'(s)})N & \xrightarrow{=} & N/\mathfrak{d}'_sN & \to & N/(\mathfrak{d}'_s, y_u^s)N. \end{array}$$

Moreover, there exists by the property of ν' an A-linear map $\psi: M \to N/\mathfrak{d}N$ such that $A/\mathfrak{c}'^s \otimes \psi \cong A/\mathfrak{c}'^s \otimes \varphi'$. In particular, $A/\mathfrak{c}^s \otimes \psi \cong A/\mathfrak{c}^s \otimes \varphi'$ because $\mathfrak{c}' \subset \mathfrak{c}$, and using the above diagram we get $A/\mathfrak{c}^s \otimes \varphi' \cong A/\mathfrak{c}^s \otimes \varphi$, as required. □

Remark 3.7. The condition $(\Omega_\mathfrak{a})$ corresponds to the condition [25, (1.1)(1)] and the above condition (2) corresponds to the condition [25, (1.1)(2)]. The arguments in 1.6 may be adopted to prove the implication (1) ⇒ (2) in (1.1) of [25]. This fact would simplify the presentation of [25].

Theorem 3.8. *Suppose that A is Henselian and satisfies $(\Omega_\mathfrak{a})$. Then there exists a positive integer u such that $(\mathfrak{a}\mathcal{J}_s(A))^u$ is a $\mathscr{C}_t(a, A)$-reduction ideal.*

The proof follows from 3.4 and 3.3.

Corollary 3.9. *Suppose that A is a Henselian regular local ring. Then a power of $\mathfrak{a}$ is a $\mathscr{C}_t(\mathfrak{a}, A)$-reduction ideal.*

Note that A satisfies $(\Omega_\mathfrak{a})$ by 1.2. Hence the proof follows.

Next we are concerned with finding large classes of local rings A which satisfy $(\Omega_\mathfrak{a})$. We recall that given a finite ring homomorphism $u: B \to A$, the Noether different $\mathfrak{N}_{A/B}$ is defined by

$$\mathfrak{N}_{A/B} = \rho(((0): \operatorname{Ker} \rho)_{A\otimes_B A}),$$

where $\rho: A \otimes_B A \to A$ is the map given by $x \otimes x' \mapsto xx'$.

Lemma 3.10. *Let $(B, \mathfrak{n}) \subset (A, \mathfrak{m})$ be a finite extension of local rings, $\mathfrak{N}_{A/B}$ the Noether different of A over B, and $\mathfrak{b} = \mathfrak{a} \cap B$. Suppose that B is regular and let $\delta = 1 + d$. Then*

$$\mathfrak{b}^{t\delta}\mathfrak{N}_{A/B} \operatorname{Ext}^1_A(M, N) = 0$$

for all $M \in \mathscr{C}_t(\mathfrak{a}, A)$ and all finite A-modules N.

Proof. Let M, N be given as in the lemma. Then $\mathfrak{a}^t H^i_\mathfrak{m}(M) = 0$ for $i \neq d$. Since $H^i_\mathfrak{m}(M) = H^i_\mathfrak{n}(M)$ we get $\mathfrak{b}^t H^i_\mathfrak{n}(M) = 0$ for $i \neq d$. But $(B, \mathfrak{n})$ is regular, and so by local duality we get $\mathfrak{b}^t \operatorname{Ext}^i_B(M, B) = 0$ for every $i > 0$. In particular it follows from 1.2 that

$$\mathfrak{b}^{t\delta} \operatorname{Ext}^1_B(M, P) = 0 \tag{1}$$

for every finite B-module P.

Let $K = \Omega^1(M)$ and let $L = A^{\mu(M)}$. There exists an exact sequence

$$0 \to K \xrightarrow{w} L \xrightarrow{v} M \to 0.$$

Then we get the following exact sequences of B-modules:

$$0 \to E' = \operatorname{Hom}_B(M, N) \xrightarrow{\tilde{v}} E = \operatorname{Hom}_B(L, N) \xrightarrow{h} D \to 0, \tag{2}$$

$$0 \to D \xrightarrow{g} E'' = \operatorname{Hom}_B(K, N), \tag{3}$$

where D denotes the image of $\tilde{w} = \operatorname{Hom}_B(w, N)$, and h, g are given by $\tilde{w}$ in an obvious way.

Consider the commutative diagram

$$\begin{array}{ccccccc}
0 \to & E' & \xrightarrow{\tilde{v}} & E & \xrightarrow{h} & D & \to 0 \\
& \partial' \downarrow & & \partial \downarrow & & \downarrow \partial''_g & \\
0 \to & \operatorname{Hom}_B(A, E') & \to & \operatorname{Hom}_B(A, E) & \to & \operatorname{Hom}_B(A, E''), &
\end{array}$$

where $\partial'': E'' \to \mathrm{Hom}_B(A, E'')$ is given by $\partial''(f)(x) = xf - fx$ for $f \in E''$, $x \in A$ and ∂, ∂' are defined similarly. (The map $fx: K \to N$ is given by $z \to f(xz)$ and xf is given by $z \to xf(z)$.) Clearly the above diagram commutes and its rows are exact. Since $\mathrm{Ker}\,\partial' = \mathrm{Hom}_A(M, N)$ we get by the Snake Lemma the following exact sequence:

$$0 \to \mathrm{Hom}_A(M, N) \xrightarrow{v'} \mathrm{Hom}_A(L, N) \xrightarrow{h'} D_0 = D \cap \mathrm{Ker}\,\partial'' \to \mathrm{Coker}\,\partial'.$$

The last map actually has its image in the Hochschild cohomology module $H^1_B(A, E') \subset \mathrm{Coker}\,\partial'$ (see, e.g., [23, (11.2), (10.4)]). But we have

$$\mathfrak{N}_{A/B} H^1_B(A, P) = 0$$

for every A-bimodule P (see, e.g., [32, (2.2)], or [24, (2.12)]). Therefore

$$(4) \qquad \mathfrak{N}_{A/B} D_0 \subset \mathrm{Im}\, h'.$$

On the other hand, from (1)–(3) we get that $\mathfrak{b}^{t\delta} E'' \subset D$. Since $H^0_B(A, -)$ is a linear functor it follows that

$$(5) \qquad \mathfrak{b}^{t\delta}\,\mathrm{Hom}_A(K, N) = \mathfrak{b}^{t\delta} H^0_B(A, E'') \subset H^0_B(A, D) = D_0.$$

Combining (4) and (5) we get

$$\mathfrak{b}^{t\delta} \mathfrak{N}_{A/B}\,\mathrm{Hom}_A(K, N) \subset \mathrm{Im}\, h'.$$

As $\mathrm{Im}\, h' = \mathrm{Im}(\mathrm{Hom}_A(w, N))$, the lemma follows. □

Theorem 3.11. *Suppose that A is a reduced complete ring with perfect residue field k. If A contains a field then A satisfies $(\mathfrak{Q}_\mathfrak{a})$.*

For the proof of this theorem we shall need the following lemma.

Lemma 3.12. *Suppose that A contains its residue field k, and let $\mathfrak{q} \in \mathrm{Reg}\, A$ be such that $\mathfrak{q} \not\supset \mathfrak{a}$. Then there exists a system of parameters $x_\mathfrak{q}$ of A such that the ideals $\mathfrak{N}_{A/k[[x_\mathfrak{q}]]}$ and $\mathfrak{b}_\mathfrak{q} = \mathfrak{a} \cap k[[x_\mathfrak{q}]]$ are not contained in $\mathfrak{q}$.*

Proof. We choose first a system of elements $y_\mathfrak{q}$ in $\mathfrak{q}$ which form a regular system of parameters in $A_\mathfrak{q}$. By [33, (6.10)] or [24, (2.7)], $y_\mathfrak{q}$ can be completed to a system of parameters $x_\mathfrak{q}$ of A such that $\mathfrak{N}_{A/R_\mathfrak{q}} \not\subset \mathfrak{q}$, where $R_\mathfrak{q} = k[[x_\mathfrak{q}]]$. In order to have $\mathfrak{b}_\mathfrak{q} \not\subset \mathfrak{q}$ we will choose $y_\mathfrak{q}$ more carefully, namely such that

$$(1) \qquad \mathrm{height}(\mathfrak{a}, y_\mathfrak{q}) > u = \mathrm{height}\,\mathfrak{q}.$$

Let $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ be the minimal prime overideals of $\mathfrak{a}$ of height $\le u$. If $s = 0$ then height $\mathfrak{a} > u$ and there is nothing to prove. Otherwise, we have $\mathfrak{q} \not\subset \mathfrak{p}_i$ for all $1 \le i \le s$ because $\mathfrak{q} \not\supset \mathfrak{a}$. Then

$$\mathfrak{q} \not\subset \mathfrak{b} = \left(\bigcup_{i=1}^{s} \mathfrak{p}_i \right) \cup (\mathfrak{q}^2 A_\mathfrak{q} \cap A)$$

(see, e.g., [21, Example (1.6)]) and we may choose an element y_1 in $\mathfrak{q} \backslash \mathfrak{b}$. Clearly y_1 can be completed to a system of elements $y_\mathfrak{q}$ which form a regular system of parameters in $A_\mathfrak{q}$ and $(y_\mathfrak{q}) \not\subset \bigcup_{i=1}^{s} \mathfrak{p}_i$. Thus (1) holds.

It is enough to see that (1) implies

$$(2) \qquad \mathrm{height}(\mathfrak{b}_\mathfrak{q}, y_\mathfrak{q}) R_\mathfrak{q} > u$$

because then height$(\mathfrak{b}_\mathfrak{q}, y_\mathfrak{q})A > u$ (see, e.g., [20, (13.B)], the map $R_\mathfrak{q} \to A$ being flat), and so $\mathfrak{b}_\mathfrak{q} \not\subset \mathfrak{q}$ since $(y_\mathfrak{q}) \subset \mathfrak{q}$. Let $\mathfrak{p}'_1$ be a prime overideal of $(\mathfrak{b}_\mathfrak{q}, y_\mathfrak{q})R_\mathfrak{q}$ and $\mathfrak{p}'_2$ a minimal prime overideal of $\mathfrak{b}_\mathfrak{q}$ contained in $\mathfrak{p}'_1$. Choose a minimal prime overideal $\mathfrak{p}_2$ of $\mathfrak{a}$ lying over $\mathfrak{p}'_2$, the map $R_\mathfrak{q} \to A$ being finite. By the going-up theorem there exists a prime ideal $\mathfrak{p}_1 \supset \mathfrak{p}_2$ of A lying over $\mathfrak{p}'_1$. We have height $\mathfrak{p}_1 > u$ by (1) because $\mathfrak{p}_1 \supset (\mathfrak{a}, y_\mathfrak{q})$. Then height $\mathfrak{p}'_1 > u$ since $R_\mathfrak{q} \to A$ is flat and finite. Hence (2) holds. □

Proof of Theorem 3.11. Let $x = (x_1, \dots, x_d)$ be a system of parameters of A and $\delta = 1 + d$. By Cohen's Structure Theorem, A contains k, and is a finite extension of the regular local ring $R(x) = k[[x]]$. Consider the ideal

$$\mathfrak{c} = \sum_{\mathfrak{q} \in D(\mathfrak{a}\mathscr{I}_s(A))} (\mathfrak{b}_\mathfrak{q})^{t\delta} \mathfrak{N}_{A/R(x_\mathfrak{q})}$$

of A, where $x_\mathfrak{q}$ is given as in Lemma 3.12. By 3.10 we have

$$(1) \qquad \mathfrak{c}\,\mathrm{Ext}^1_A(M, N) = 0$$

for all $M \in \mathscr{C}_t(\mathfrak{a}, A)$ and all finite A-modules N. Using 3.12 we conclude that $\mathfrak{c}$ is not contained in any prime ideal $\mathfrak{q} \in D(\mathfrak{a}\mathscr{I}_s(A)) = \operatorname{Spec} A \setminus V(\mathfrak{a}\mathscr{I}_s(A))$. Then every prime ideal containing $\mathfrak{c}$ must contain also $\mathfrak{a}\mathscr{I}_s(A)$. Thus $\operatorname{Rad}\mathfrak{c} \supset \mathfrak{a}\mathscr{I}_s(A)$, and so $(\mathfrak{a}\mathscr{I}_s(A))^r \subset \mathfrak{c}$ for a certain $r \in \mathbb{N}$. Hence $(\Omega_\mathfrak{a})$ holds by (1). □

With the help of the following lemma and some techniques related to formally smooth algebras we can generalize 3.11.

Lemma 3.13. *Let $\widehat{A}$ be the completion of A with respect to $\mathfrak{m}$ and suppose that the canonical map $A \to \widehat{A}$ is regular (i.e., its fibers are geometrically regular). If $\widehat{A}$ satisfies $(\Omega_{\mathfrak{a}\widehat{A}})$ then A satisfies $(\Omega_\mathfrak{a})$.*

Proof. Since $A \to \widehat{A}$ is regular we get $\mathscr{I}_s(\widehat{A}) = \operatorname{Rad}(\mathscr{I}_s(A)\widehat{A})$ (cf., e.g., [20, (33.B)]). If $\widehat{A}$ satisfies $(\Omega_{\mathfrak{a}\widehat{A}})$ then there is a positive integer r such that

$$(1) \qquad (\mathfrak{a}\mathscr{I}_s(A))^r\,\mathrm{Ext}^1_{\widehat{A}}(M', N') \subset \mathfrak{a}^r\mathscr{I}_s(\widehat{A})^r\,\mathrm{Ext}^1_{\widehat{A}}(M', N') = 0$$

for all modules $M' \in \mathscr{C}_t(\mathfrak{a}\widehat{A}, \widehat{A})$ and all finite $\widehat{A}$-modules N'. We claim that r works also for A. Let $M \in \mathscr{C}_t(\mathfrak{a}, A)$, N be a finite A-module and $\widehat{M} = \widehat{A} \otimes_A M$, $\widehat{N} = \widehat{A} \otimes_A N$. Then $\mathfrak{a}^t H^i_{\mathfrak{m}\widehat{A}}(\widehat{M}) = \mathfrak{a}^t H^i_\mathfrak{m}(M) = 0$ for $i \neq d$, and so $\widehat{M}$ belongs to $\mathscr{C}_t(\mathfrak{a}\widehat{A}, \widehat{A})$. Since the completion map is faithfully flat, (1) implies that

$$(\mathfrak{a}\mathscr{I}_s(A))^r\,\mathrm{Ext}^1_A(M, N) = 0. \quad \square$$

Theorem 3.14. *Suppose that A is a reduced excellent ring containing a field and its residue field k is perfect or finite over k^p if* $\operatorname{char} k = p > 0$. *Then A satisfies $(\Omega_\mathfrak{a})$.*

Proof. By 3.13 it is enough to treat the case where A is complete. If k is perfect then 3.11 gives the result. Suppose now that k is not perfect, but $[k : k^p] < \infty$. Thus $p > 0$ and $K = k^{1/p^\infty}$ is a perfect field with

$$e = \operatorname{rank}_K \Gamma_{K/k} = \operatorname{rank}_k \Omega_{k/k^p} = [k : k^p]$$

(see, e.g., [24, (4.4)]), $\Gamma_{K/k}$ being the imperfection module of K over k and Ω_{k/k^p} being the module of differentials of k over k^p. The complete ring A is a factor of a regular local ring $(R, \mathfrak{n})$, let us say $A \cong R/\mathfrak{b}$, $u = \dim R$. By [16, (22.2.6)] there exists a formally smooth Noetherian (it follows regular) complete local R-algebra $(R', \mathfrak{n}')$ such that $R'/\mathfrak{n}' \cong K$ and $\dim R' = u + e$. Then $A' = R'/\mathfrak{b}R'$ is a formally smooth Noetherian complete local A-algebra, and so $(A', \mathfrak{m}')$, $\mathfrak{m}' = \mathfrak{n}'/\mathfrak{b}$, is a reduced Cohen-Macaulay local ring and $\mathcal{J}_s(A') \supset \mathcal{J}_s(A)A'$ (the structural morphism $A \to A'$ is regular, A being excellent (see, e.g., [1], or [5] and [6])). As A' satisfies $(\Omega_{\mathfrak{a}A'})$ there exists $r \in \mathbb{N}$ such that

$$(1) \qquad (\mathfrak{a}\mathcal{J}_s(A))^r \operatorname{Ext}^1_{A'}(\widetilde{M}, \widetilde{N}) \subset (\mathfrak{a}\mathcal{J}_s(A'))^r \operatorname{Ext}^1_{A'}(\widetilde{M}, \widetilde{N}) = 0$$

for all $\widetilde{M} \in \mathcal{C}_t(\mathfrak{a}A', A')$ and any finite A'-module $\widetilde{N}$. We claim that the same r works also for A. Let $M \in \mathcal{C}_t(\mathfrak{a}, A)$, N be a finite A-module and set $M' = A' \otimes_A M$, $N' = A' \otimes_A N$. We have $\mathfrak{a}^t H^i_{\mathfrak{m}}(M)$ for $i \neq d$, and by local duality we get $\mathfrak{a}^t \operatorname{Ext}^{u-i}_R(M, R) = 0$ for $i \neq d$. Since R' is flat over R it follows that $\mathfrak{a}^t \operatorname{Ext}^{u-i}_{R'}(M', R') = 0$ for $i \neq d$, and using again local duality we get $\mathfrak{a}^t H^j_{\mathfrak{m}'}(M') = 0$ for $j \neq e + u$, that is $M' \in \mathcal{C}_t(\mathfrak{a}A', A')$. Now (1) and faithful flatness implies that

$$(\mathfrak{a}\mathcal{J}_s(A))^r \operatorname{Ext}^1_A(M, N) = 0. \quad \square$$

Theorem 3.15. *Suppose that* $(A, \mathfrak{m}, k)$ *is either* (a) *a Henselian regular local ring, or* (b) *a reduced excellent Henselian ring containing a field, and that* k *is perfect or finite over* k^p *if* $\operatorname{char} k = p > 0$. *Then a power of* $\mathfrak{a}\mathcal{J}_s(A)$ *is a* $\mathcal{C}_t(\mathfrak{a}, A)$*-reduction ideal.*

For the proof apply 3.8, 3.9, and 3.14.

As a first application of our reduction theory we present

Corollary 3.16. *Suppose that* A *is a reduced excellent Henselian ring containing a field, and that* k *is perfect or finite over* k^p *of* $\operatorname{char} k = p > 0$. *Let* A' *be the completion of* A *with respect to* $\mathfrak{a}\mathcal{J}_s(A)$, *and let* $I\mathcal{C}(\mathfrak{a}, A)$ (*resp.* $I\mathcal{C}(\mathfrak{a}A', A')$) *be the set of isomorphism classes of indecomposable modules of* $\mathcal{C}(\mathfrak{a}, A)$ (*resp.* $\mathcal{C}(\mathfrak{a}A', A')$). *Then the base change functor* $A' \otimes_A -$ *induces a bijection* $I\mathcal{C}(\mathfrak{a}, A) \to I\mathcal{C}(\mathfrak{a}A', A')$.

Proof. Since A is excellent the canonical map $A \to A'$ is regular, cf. [1] or [5] and [6]. Therefore A' is a reduced Cohen-Macaulay ring and $\mathcal{J}_s(A') = \operatorname{Rad}(\mathcal{J}_s(A)A')$ (see, e.g., [20, (33.B)]).

Let M be an indecomposable module $\mathcal{C}(\mathfrak{a}, A)$ which belongs to $\mathcal{C}_u(\mathfrak{a}, A)$ for some $u \in \mathbb{N}$. According to the above theorem, the ideal $\mathfrak{b} = (\mathfrak{a}\mathcal{J}_s(A))^r$ is a $\mathcal{C}_u(\mathfrak{a}, A)$-reduction ideal for a certain $r \in \mathbb{N}$, and so $A/\mathfrak{b} \otimes_A M$ is indecomposable as an $A/\mathfrak{b} \cong A'/\mathfrak{b}A'$-module. This means that $A' \otimes_A M/\mathfrak{b}(A' \otimes_A M)$ is indecomposable, whence $A' \otimes_A M$ is indecomposable by Nakayama's Lemma. Thus the base change functor $A' \otimes_A -$ defines a map $\varphi\colon I\mathcal{C}(\mathfrak{a}, A) \to I\mathcal{C}(\mathfrak{a}A', A')$.

Now we show that φ is one-to-one. Let M, N be two indecomposable modules of $\mathcal{C}(\mathfrak{a}, A)$ such that $A' \otimes_A M \cong A' \otimes_A N$. We can choose an integer $u' \in \mathbb{N}$ such that M, N both belong to $\mathcal{C}_{u'}(\mathfrak{a}, A)$. As above, a certain power $\mathfrak{b}'$ of $\mathfrak{a}\mathcal{J}_s(A)$ is a $\mathcal{C}_{u'}(\mathfrak{a}, A)$-reduction ideal. Then from

$$M/\mathfrak{b}'M \cong A' \otimes_A M/\mathfrak{b}'(A' \otimes_A M) \cong A' \otimes_A N/\mathfrak{b}'(A' \otimes_A N) \cong N/\mathfrak{b}'N$$

we get $M \cong N$ because $\mathfrak{b}'$ is a reduction ideal.

Finally, let $M' \in \mathscr{C}(\mathfrak{a}A', A')$. Then M' is locally free on $\operatorname{Spec} A' \setminus V(\mathfrak{a}\mathscr{I}_s(A'))$ and there exists a finite A-module M which is locally free on $\operatorname{Spec} A \setminus V(\mathfrak{a}\mathscr{I}_s(A))$ such that $A' \otimes_A M \cong M'$ (see [10, Theorem 3]). It is clear that $M \in \mathscr{C}(\mathfrak{a}A', A')$, and if M' is indecomposable then M is too by faithful flatness, i.e., φ is surjective. □

Remark 3.17. Using Artin approximation theory (e.g., applying [25, (3.6)]) we immediately get the above result for all excellent Henselian Cohen-Macaulay local rings, i.e., under weaker conditions as in 3.15(b).

Corollary 3.18. *Suppose that A is either* (a) *Henselian regular, or* (b) *reduced excellent Henselian with isolated singularity, containing a field, and that k is perfect or finite over k^p if* $\operatorname{char} k = p > 0$. *Then a power of $\mathfrak{m}$ is a $\mathscr{C}_t(A)$-reduction ideal. Moreover, if $\widehat{A}$ is the completion of A with respect to $\mathfrak{m}$, then the base change functor $\widehat{A} \otimes_A -$ induces a bijection $I\mathscr{C}(A) \to I\mathscr{C}(\widehat{A})$, where $I\mathscr{C}(A)$ (resp. $I\mathscr{C}(\widehat{A})$) denotes the set of isomorphism classes of indecomposable modules of $\mathscr{C}(A)$ (resp. $\mathscr{C}(\widehat{A})$).*

Let $\mathfrak{b} \subset A$ be a parameter ideal, M an A-module and $e(\mathfrak{b}; M)$ the multiplicity of M relative to $\mathfrak{b}$. By [8, (3.7)] (see also [31]) the difference

$$I(\mathfrak{b}; M) = l(M/\mathfrak{b}M) - e(\mathfrak{b}; M)$$

is bounded by the integer

$$I(M) = \sum_{i=0}^{d-1} \binom{d-1}{i} l(H^i_{\mathfrak{m}}(M))$$

for all parameter ideals $\mathfrak{b}$ of A. Let $h = (h_0, \dots, h_{d-1})$ be a sequence of nonnegative integers and $\mathscr{C}(h, A)$ the class of modules $M \in \mathscr{C}(A)$ with $l(H^i_{\mathfrak{m}}(M)) \le h_i$ for $0 \le i < d$.

Lemma 3.19. *Under the hypotheses of the above corollary there exists a parameter ideal $\mathfrak{b}$ of A such that*

(a) *$M/\mathfrak{b}M$ is indecomposable over $A/\mathfrak{b}$ for every indecomposable module of $\mathscr{C}(h, A)$,*
(b) *two indecomposable modules $M, N \in \mathscr{C}(h, A)$ are isomorphic if $M/\mathfrak{b}M$ and $N/\mathfrak{b}N$ are isomorphic,*
(c) *$l(M/\mathfrak{b}M) \le e(\mathfrak{b}; M) + u_h$ for all module $M \in \mathscr{C}(h, A)$, where*

$$u_h = \sum_{i=0}^{d-1} \binom{d-1}{i} h_i.$$

Proof. There is a positive integer s such that $\mathscr{C}(h, A) \subset \mathscr{C}_s(A)$. By 3.18 a power $\mathfrak{m}^r$ of $\mathfrak{m}$ is a $\mathscr{C}_s(A)$-reduction ideal. Choose a parameter ideal $\mathfrak{b} \subset \mathfrak{m}^r$. Then $\mathfrak{b}$ is still a $\mathscr{C}_s(A)$-reduction ideal, and so (a), (b) hold. Certainly (c) follows from the above formula for $I(M)$, because $I(\mathfrak{b}; M) \le I(M) \le u_h$ for all $M \in \mathscr{C}(h, A)$. □

The following corollary is a form of the Harada-Sai Lemma [17] for the category $\mathscr{C}(h, A)$ (see also [32, (3.1)], [33, (6.20)], [24, (5.2)], [26, (1.9)]).

Corollary 3.20 (Harada-Sai Lemma for generalized Cohen-Macaulay modules). *With the hypotheses and notations of the above lemma there is a function* $\rho\colon \mathbb{N} \to \mathbb{N}\setminus\{0\}$ *with the following property: Let* $s > 0$ *be an integer and*

$$M_0 \overset{\varphi_1}{\to} M_1 \overset{\varphi_2}{\to} \cdots \overset{\varphi_{\rho(s)}}{\to} M_{\rho(s)}$$

a sequence of $\rho(s)$*-homomorphisms* φ_i *in* $\mathscr{C}(h, A)$ *which are not isomorphisms and such that for all* i *the module* M_i *is indecomposable and* $e(\mathfrak{m}; M_i) \le s$. *Then*

$$\operatorname{Im}(\varphi_{\rho(s)} \cdots \varphi_1) \subset \mathfrak{m} M_{\rho(s)}.$$

Proof. Let $\mathfrak{b}$ be the parameter ideal given by 3.19 and v a positive integer such that $\mathfrak{m}^v \subset \mathfrak{b}$. We claim that the function ρ given by $\rho(s) = 2^w$, $w = v^d s + u_h$ has the required property. Indeed, given M_i, φ_i as in our corollary we get

$$e(\mathfrak{b}; M_i) \le e(\mathfrak{m}^v; M_i) = v^d e(M_i) \le v^d s$$

by [21, (14.4), (14.3)], and so $l(M_i/\mathfrak{b}M_i) \le \rho(s)$ by 3.19(c). Note that $\overline{M}_i = A/\mathfrak{b} \otimes M_i$ are indecomposable and $\overline{\varphi}_i = A/\mathfrak{b}_i \otimes \varphi_i$ are not isomorphisms because of 3.19(a), (b). Then it is enough to apply the Harada-Sai Lemma [17] to $(\overline{M}_i)$, $(\overline{\varphi}_i)$. □

Note added in proof

Our Remark 2.20(b) says that the problem of finding pairs (α, β) such that there exists an indecomposable B-module M with $\mu(M) = \beta$, $r(M) = \alpha$ is equivalent with classifying $(e\beta - \alpha) \times \beta$-matrices $L = (l_{ij})$ of linear polynomials $l_{ij} = \sum_{t=1}^{e} a_{ij}^{(t)} X_t$ over $k := B/\mathfrak{n}$ in the indeterminates $X_1, \ldots, X_e$, $e = \operatorname{emb}\dim B$ up to conjugation by elements from $GL(k, e\beta - \alpha) \times GL(k, \beta)^0$ (i.e., the classification of e-tuples of linear maps given by $(a_{ij}^{(t)})$, $t = 1, \ldots, e$), and finding the sizes of the indecomposables under this classification.

As the second author learned from Bautista in a conference (Mexico, January 20–26, 1991) this was already done by V. Kac (Invent. Math. **56** (1980), 57–96, §2) or C. M. Ringel (J. Algebra **41** (1976), 269–302, Theorem 3), and so there exist indecomposable B-modules M with $\mu(M) = \beta$, $r(M) = \alpha$ if and only if $\alpha^2 + \beta^2 - e\alpha\beta \le 1$ with exactly one module M for each α, β satisfying the equality, and otherwise infinitely many M for each solution α, β, provided k is infinite. Thus our Theorem 2.13 together with 2.8 and 2.3(b) gives all admissible sequences $(h_0, \ldots, h_{d-1})$ with just two nonzero integers.

After the submission of this article, the authors became aware of two papers: M. Amasaki, *Maximal quasi Buchsbaum graded modules over polynomial rings with* $\#\{i\colon H_{\mathfrak{m}}^i(M) \ne 0,\ i < \dim R\} \le 2$ (abstract), and Y. Yoshino, *Maximal Buchsbaum modules of finite projective dimension* (preprint (1991). Using a completely different technique, Amasaki proves a result similar to our Theorem 2.3, whereas Yoshino extended it to the case when A is Gorenstein.

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