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FREE Z_8 ACTIONS ON S^3 (1)

BY

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ABSTRACT. This paper is devoted to the problem of classifying periodic homeomorphisms which act freely on the 3-sphere. The main result is the classification of free period eight actions and a generalization to free actions whose squares are topologically equivalent to orthogonal transformations. The result characterizes those 3-manifolds which have the 3-sphere as universal covering space and the cyclic group of order eight as fundamental group.

1. **Introduction.** This paper is devoted to the problem of classifying periodic homeomorphisms which act freely on the 3-sphere. Thus far, only free actions of period two and of period four have been classified. The main result of this paper is the classification of free period eight actions and a generalization to free actions whose squares are topologically equivalent to orthogonal transformations. The result characterizes those 3-manifolds which have the cyclic group of order eight as fundamental group and the 3-sphere as universal covering space.

It follows from the proofs of Corollaries 3.2 and 3.3 that the problem of showing a periodic homeomorphism acting freely on the 3-sphere, S^3, to be topologically an orthogonal transformation is equivalent to the problem of showing the existence of an unknotted simple closed curve which remains invariant under that homeomorphism. Roughly speaking, if Z_n acts freely on S^3, $h \in Z_n$ a generator, and J an unknotted simple closed curve which remains invariant under h, then there is a whole toroidal neighborhood N of J which remains invariant under h. The solid torus N is an n-fold cover for the orbit space N/Z_n and, similarily, the solid torus $\overline{S^3 - N}$ is an n-fold cover for $\overline{S^3 - N/Z_n}$. It will be shown that N/Z_n and $\overline{S^3 - N/Z_n}$ are solid tori. Since the orbit space S^3/Z_n is obtained from N/Z_n and $\overline{S^3 - N/Z_n}$ by identifying their boundaries via the projection map, S^3/Z_n is topologically the lens space $L(n, m)$. If p_1 denotes the covering projection of this construction and p_2 the covering projection for the standard construction of $L(n, m)$, then there is a homeomorphism g of S^3 onto itself such that $p_1 = p_2 g$.

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The conjugation of h with g will then be the required orthogonal map. More precisely, if T is the orthogonal transformation used for the construction of $L(n, m)$, then $h = g^{-1}Tg$. Our main theorem, therefore, shows the existence of an unknotted simple closed curve which is invariant under a given free Z_8 action.

2. Preliminaries.

Definitions and notation. If M is a topological manifold, then the interior of M will be denoted by int M and the set $M -$ int M is called the boundary of M and is denoted by ∂M.

Two homeomorphisms h_1 and h_2 of S^3 onto itself are said to be *topologically equivalent* if there is a homeomorphism g of S^3 onto itself such that $h_1 = g^{-1}h_2 g$. To say that a group G *acts freely on* S^3 means that each $g \in G$, $g \neq$ identity of G, is a fixed point free homeomorphism of S^3 onto itself. If G is cyclic of order p and $g \in G$ a generator, then g is called a *free action of* G *on* S^3 *of period* p.

We shall denote a cyclic group G of order p by Z_p. If Z_p acts freely on the closed combinatorial 3-manifold M, then the orbit space $M' = M/Z_p$ is also a closed 3-manifold. For let M' have the natural piecewise linear structure induced by the projection map $p: M \longrightarrow M'$. Let v' be a vertex of M' and v a vertex of M such that $v' = pv$. Since p is a local homeomorphism, the star of v in M, $\mathrm{st}(v, M)$, is homeomorphic to $\mathrm{st}(v', M')$. But $\mathrm{st}(v, M)$ is a 3-ball neighborhood of v in M, hence $\mathrm{st}(v', M')$ is a 3-ball neighborhood of v' in M'. According to R. H. Bing [1], M' can be triangulated and the triangulation can be lifted to M. The action h of Z_p on M is a deck-transformation on the covering space and hence simplicial. Thus, free actions on closed 3-manifolds are piecewise linear homeomorphisms. Henceforth, our objects (maps, embeddings, etc.) are always considered from the piecewise linear point of view.

In order to study free actions on S^3 it will be convenient to view S^3 as the join of the two circles $|z_0|^2 = 1$ and $|z_1|^2 = 1$, where z_0, z_1 are complex numbers with $S^3 = \{(z_0, z_1) \mid |z_0|^2 + |z_1|^2 = 1\}$, and investigate the behavior of the actions on these circles. The map $h: S^3 \longrightarrow S^3$ defined by $h: (z_0, z_1) \longrightarrow (e^{2\pi i/p} z_0, e^{2q\pi i/p} z_1)$, where p and q are relatively prime integers with $p > 0$, rotates the circles $|z_0|^2 = 1$ and $|z_1|^2 = 1$ through an angle of $2\pi/p$ and $2q\pi/p$, respectively. It follows that h is a fixed point free homeomorphism of period p and we call h the *standard* or *orthogonal* (p, q)*-action on* S^3.

The group of rotations on S^3 generated by the standard (p, q)-action is cyclic of order p and, hence, represented by Z_p. Furthermore, h is invariant on the two solid tori $|z_0|^2 \leq |z_1|^2$ and $|z_1|^2 \leq |z_0|^2$ with common boundary $|z_0|^2 = |z_1|^2$. Thus, the orbit space S^3/Z_p is the lens space $L(p, q)$ and we call this construction of $L(p, q)$ the *standard* construction.

Preliminary results. The following result of Livesay [6] classifies free Z_2 actions on S^3.

Theorem 2.1. *Every free action of Z_2 on S^3 is topologically equivalent to the antipodal map.*

Since results similar to the proposition below have appeared in numerous published works [3], [4], [5], [8], [10], we will omit its proof.

Proposition 2.2. *If Z_p acts freely on the closed combinatorial manifold M and P is a subpolyhedron of M invariant under a subgroup G of Z_p, then there is an arbitrarily small isotopy of M which takes P onto a polyhedron Q such that, for each $h \in Z_p/G$, Q is in general position with respect to hQ. Furthermore, Q is invariant under G.*

Rice [8], using Theorem 2.1 and Proposition 2.2, classified free Z_4 actions on S^3.

Theorem 2.3 (Rice). *Every free action of Z_4 on S^3 is topologically equivalent to the orthogonal action.*

If we consider $S^3 \subset E^4$ as the join of the two circles $x_1^2 + x_3^2 = 1$ and $x_2^2 + x_4^2 = 1$, then S^3 decomposes into two congruent solid tori having these circles as centerlines. The two congruent solid tori V^+ and V^- are defined by the equations, $x_1^2 + x_3^2 \geq x_2^2 + x_4^2$ and $x_1^2 + x_3^2 \leq x_2^2 + x_4^2$, respectively; their common boundary T is defined by the equation $x_1^2 + x_3^2 = x_2^2 + x_4^2$. If h denotes the standard orthogonal $(4, 1)$-action on S^3, then h maps the point $(x_1, x_2, x_3, x_4) \in S^3$ to the point $(-x_2, x_1, -x_4, x_3) \in S^3$. Here $z_0 = (x_1, x_2)$, $z_1 = (x_3, x_4)$ and $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$. It follows that $hT = T$ and h interchanges the closed complementary domains V^+ and V^- of T.

Theorem 2.4. *If Z_p acts freely on S^3 and there is a torus $T \subset S^3$ whose complementary domains interchange under a generator h of Z_p, then $p = 4k$ and there is a $(1, 1)$-curve on T which remains invariant under h. In particular, h is equivalent to the standard $(4k, 2k - 1)$-action. Conversely, if h is equivalent to the standard $(4k, 2k - 1)$-action, then there is a torus $T \subset S^3$ whose complementary domains interchange under h.*

Proof. Let V^+ and V^- denote the closed complementary domains of T in S^3. Since h interchanges V^+ and V^-, p must be even and T unknotted. The action h^2 generates $Z_{p/2}$ and, since $h^2V^+ = V^+$, it follows from our introductory remarks that h^2 is topologically an orthogonal transformation.

We suppose that h^2 is topologically equivalent to the standard $(p/2, r)$-action, where r is some positive integer relatively prime to $p/2$, that is $(p/2, r) = 1$. Any meridian simple closed curve m on the torus $\partial(V^+/Z_{p/2})$ lifts to $p/2$ disjoint meridian simple closed curves on ∂V^+ which permute under h^2. We let $m_1, m_2, \cdots, m_{p/2}$ denote this collection of meridian curves and assume that the subscripts are arranged in an order such that $h^2 m_{p/2} = m_1$ and $h^2 m_i = m_{i+1}$ for $i = 1, \cdots, p/2 - 1$. Denoting the image of m_i under h by l_i, then, since $hV^+ = V^-$, l_i is a meridian simple closed curve on ∂V^- and, hence, a longitudinal simple closed curve for V^+.

We set $m_1, \cdots, m_{p/2}$ into general position with respect to the collection $l_1, \cdots, l_{p/2}$ by adjusting the m_i's as follows. Let X be a component of $m_1 \cap l_1$ such that X is not a crossing point of $m_1 \cup l_1$. Let D be a sufficiently small polyhedral disc on T such that the following four properties are satisfied:

(i) $X \subset \operatorname{int} D$ and $h^i D \cap D = \emptyset$ for $i = 1, \cdots, p/2 - 1$.

(ii) $D \cap m_1$ is an arc α such that $\alpha \cap \partial D = \partial\alpha$ and $D \cap l_1$ is an arc β such that $\beta \cap \partial D = \partial\beta$.

(iii) $D \cap m_i = D \cap l_i = \emptyset$ for $i = 2, \cdots, p/2$.

(iv) If Y is a component of $m_1 \cap l_1$ and $Y \neq X$, then $Y \cap D = \emptyset$.

Let a_1, a_2 and b_1, b_2 be the end points of α and β, respectively. The set $a_1 \cup a_2$ divides ∂D into two arcs w_1 and w_2 and, similarly, $b_1 \cup b_2$ divides ∂D into two arcs v_1 and v_2. Either v_1 is a subset of w_1 or w_2, or v_2 is a subset of w_1 or w_2, or int v_1 contains exactly one end point a_i, $i = 1$ or 2. If w_1 contains v_1, we adjust m_1 (Figure 2.1) by replacing the arc α by the arc w_2. Similarly, if $v_1 \subset w_2$, we replace α by w_1. We then copy this replacement in the images of h^2. If int v_1 contains

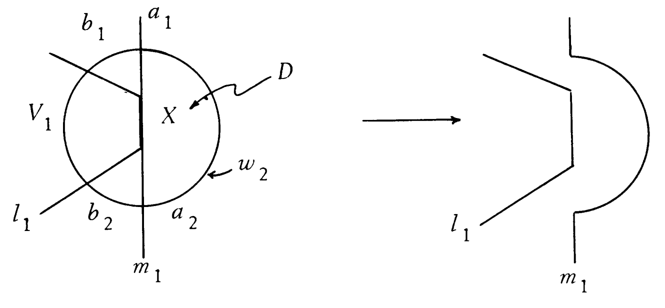

Figure 2.1

exactly one end point, we adjust m_1 by replacing α by w_1 (Figure 2.2) and, again, copy this adjustment in the images of h^2. We repeat this process a finite number of times until $m_1 \cap l_1$ consists of only a finite number of crossing points. In an analogous fashion, we set m_1 in general position with respect to l_i for $i = 2, 3, \cdots, p/2$. It follows that $m_j \cap l_i$ consists of only a finite number of crossing points as i and j range over the set $\{1, 2, \cdots, p/2\}$.

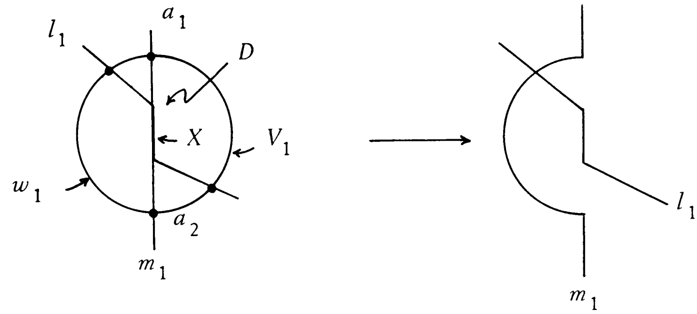

Figure 2.2

Since l_i is a longitudinal simple closed curve on ∂V^+, the number of crossing points in $m_j \cap l_i$ must be odd. In particular, if $p/2$ is odd, then the number of crossing points in $\bigcup_{i=1}^{p/2} [(\bigcup_{j=1}^{p/2} m_j) \cap l_i]$ must be odd. But this is impossible since h is fixed point free, of even period, and invariant on $\bigcup_{i=1}^{p/2} [(\bigcup_{j=1}^{p/2} m_j) \cap l_i]$. Thus, $p/2$ is even and $p = 4k$ for some positive integer k.

Since h^2 is topologically the $(2k, r)$-action, m_i and $h^{2r}m_i$ are adjacent; that is, there is an annulus $A \subset T$ with boundary components m_i and $h^{2r}m_i$ such that $m_j \cap \operatorname{int} A = \emptyset$ for $j = 1, \cdots, p/2$. Furthermore, since $(r, 2k) = 1$, h^{2r} generates $Z_{p/2}$. On the other hand, $(r, 2k) = 1$ implies that $(r, p) = (r, 2(2k)) = (r, 2) = 1$ and, hence, h^r also generates Z_p and interchanges V^+ with V^-. In the remaining part of this proof we let $h_1 = h^r$ and suppose that the subscripts of the meridians and longitudes have been relabeled so that $h_1^2 m_i = m_{i+1}$ and $h_1^2 l_i = l_{i+1}$.

If n denotes the number of crossing points in $m_1 \cap l_i$, then n is odd, and if $n > 1$, then there are $(n-1)$-discs on ∂V^+ such that the boundary of each disc is the union of two arcs meeting only in their end points, one of which is a subset of m_1 and the other of l_i. Suppose D is a disc with $\partial D = \alpha \cup \beta$, where $\alpha \subset m_1$ and $\beta \subset l_i$ are arcs for some $i = 1, 2, \cdots, p/2$, such that $\operatorname{int} D \cap m_j = \operatorname{int} D \cap l_j = \emptyset$ for $j = 1, 2, \cdots, p/2$. We call such a disc innermost with respect to $m_1 \cup l_i$. Let p_1 and p_2 be the common end points of α and β. Let p_1' and p_2' be points on $m_1 - \alpha$ near p_1 and p_2, respectively, such that the arc α' on m_1 containing α and having end points p_1', p_2' has the property that $\alpha' \cap l_i = p_1 \cup p_2$ and $\alpha' \cap l_j = \emptyset$ for $j \neq i$. Let β' be an arc near β satisfying the following three properties:

(i) $\beta' \cap m_1 = p_1' \cup p_2' = \partial\beta'$.

(ii) $\beta' \cap l_i = \beta' \cap m_j = \emptyset$ for $i = 1, \cdots, p/2$, $j = 2, \cdots, p/2$.

(iii) $h_1^i \beta' \cap \beta' = \emptyset$ for $i = 1, \cdots, p/2 - 1$.

Property (iii) is easily satisfied since int β intersects no other meridian or longitudinal simple closed curves and h is a free action. We now adjust m_1 by replacing α' by β' (Figure 2.3). If we again denote

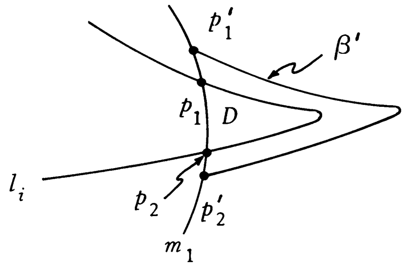

Figure 2.3

by m_1 the adjustment of m_1, then $m_1 \cap l_i$ contains two fewer crossing points. We copy this adjustment in all the images of h. Repeating this process, we eliminate all innermost discs with respect to $m_1 \cup l_i$ for all i. It now follows that each $m_i \cap l_j$, $i, j = 1, 2, \cdots, p/2$, consists of exactly one crossing point. For if some $m_r \cap l_s$ contained more than one crossing point, then there is an innermost disc D with respect to some $m_i \cup l_j$. But, then, if $i + m = p/2$, $h_1^{2(m+1)}D$ is a disc innermost with respect to $m_1 \cup h_1^{2(m+1)}l_j$, contrary to our assumption that all discs innermost with respect to $m_1 \cup l_i$, $i = 1, \cdots, p/2$, have been eliminated.

We now assume that each $m_i \cap l_j$, $i, j = 1, 2, \cdots, p/2$, consists of exactly one point. Let $x \in m_1 \cap l_1$, then $h_1 x \in h_1 m_1 \cap h_1 l_1 = l_1 \cap m_2$. The points x and $h_1 x$ divide l_1 into two arcs. We let $\alpha \subset l_1$ be the arc with the properties that $\alpha \cap m_1 = x$, $\alpha \cap m_2 = h_1 x$, and $\alpha \cap m_i \neq 0$, $i = 3, \cdots, p/2$. The simple closed curve $J = \bigcup_{i=1}^{p/2} h_1^i \alpha$ (Figure 2.4) is clearly invariant under h_1 and, therefore, nontrivial on T. Thus J is a (p, q)-curve on T with respect to V^+ and at least one of p or q is not 0.

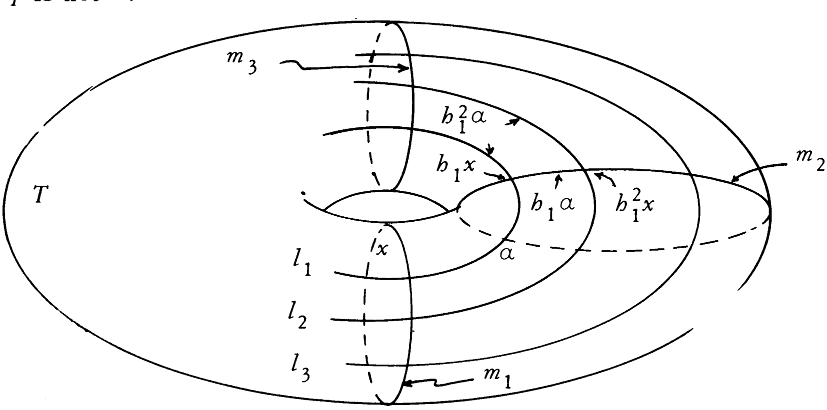

Figure 2.4

Since $h_1 J$ is a (p, q)-curve on T with respect to $h_1 V^+ = V^-$, $h_1 J$ must be a (q, p)-curve on T with respect to V^+. But $h_1 J = J$ and, hence, $p = q = 1$. We note that since J meets every meridian m_i and every longitude l_i exactly once in an arc, it also follows from the construction of J that J must be a $(1, 1)$-curve on T.

Finally, since J is invariant under h_1, J is invariant under every power of h_1 and, in particular, $hJ = J$. Again, by our introductory remarks, h is equivalent to a standard $(4k, q)$-action and $S^3/h = L(4k, q)$. If $e\colon S^3 \to S^3/h$ denotes the natural projection, then, since $hV^+ = V^-$, eT is one-sided and, therefore, a Klein bottle. By [2] this is possible only if $q \equiv \pm(2k-1) \bmod 4k$.

In order to prove the converse, let f be a homeomorphism of S^3 onto itself defined by $f\colon (z_0, z_1) \to (z_1, e^{\pi i/k}z_0)$. It follows that f is a free action of period $4k$ which interchanges the two solid tori V^+ and V^- defined by $|z_1|^2 \le |z_0|^2$ and $|z_0|^2 \le |z_1|^2$, respectively, and satisfying $|z_0|^2 + |z_1|^2 = 1$. By the first part of the theorem, f is equivalent to h. Hence there is a homeomorphism g of S^3 onto itself such that $f = g^{-1}hg$. We now let $T = g(\partial V^+)$. This verifies Theorem 2.4.

3. **The main theorem.** We will divide the proof of the main theorem into four steps, commencing each step with a statement of what is to be shown.

Theorem 3.1. *If h is a homeomorphism of period eight acting freely on S^3, then there is a polyhedral unknotted simple closed curve J in S^3 such that $hJ = J$.*

Proof. Since h acts freely on S^3, h^2 is a free action of Z_4 on S^3 and, by Theorem 2.3, we may assume that h^2 is equivalent to the orthogonal $(4, 1)$-action. Hence, there is a polyhedral torus T in S^3 with closed complementary domains V^+ and V^- such that T is invariant under h^2 and h^2 interchanges V^+ and V^-. Furthermore, by Proposition 2.2, we may assume that T is in general position with respect to hT.

Step 1. $T \cap hT \neq \emptyset$. We suppose that $T \cap hT = \emptyset$. Then either $hT \subset \operatorname{int} V^+$ or $hT \subset \operatorname{int} V^-$. If $hT \subset \operatorname{int} V^+$, then either (i) $hV^+ \subset \operatorname{int} V^+$ or (ii) $hV^- \subset \operatorname{int} V^+$.

If we apply h to (i), we obtain $V^- = h^2V^+ \subset h(\operatorname{int} V^+) \subset \operatorname{int} V^+$ which is a contradiction.

If we apply h to (ii), we obtain $V^+ = h^2V^- \subset h(\operatorname{int} V^+)$. A second application of h yields $hV^+ \subset \operatorname{int} V^-$. Therefore, $V^+ \subset h(\operatorname{int} V^+) \subset hV^+ \subset \operatorname{int} V^-$ which is a contradiction.

Similarly, hT cannot be contained in $\operatorname{int} V^-$.

Step 2. T can be adjusted so that every simple closed curve in $T \cap hT$ is a $(1, 1)$-curve on T and h^2 interchanges the closed complementary domains of T.

Since T is in general position with respect to hT and $T \cap hT \neq \emptyset$, $T \cap hT$ consists of a finite collection of simple closed curves. If J is a simple closed curve in $T \cap hT$, then J satisfies one and only one of the following three properties:

(i) J is trivial on both T and hT.

(ii) J is trivial on one of T or hT but not both.

(iii) J is nontrivial on both T and hT.

If J is a (p, q)-curve on ∂V^+, then h^2J is a (p, q)-curve on $h^2(\partial V^+) = \partial V^-$

and, hence, a (q, p)-curve on ∂V^+. Since $\partial V^+ = \partial V^- = T$ and either $h^2 J = J$ or $h^2 J \cap J = \varnothing$, $h^2 J$ must be parallel to J on T. Therefore, $p = q = 1$ and all simple closed curves in $T \cap hT$ which are nontrivial on T are $(1, 1)$-curves on T.

If J is a simple closed curve of type (i) or (ii) in $T \cap hT$ such that J bounds a disc D on T or hT with the property that $\operatorname{int} D \cap (T \cap hT) = \varnothing$, then D is called an innermost disc with respect to $T \cap hT$. Our next step is to eliminate all simple closed curves of type (i) in $T \cap hT$—if there are such curves—which bound innermost discs on hT. We may then assume that there are no simple closed curves of type (i) in $T \cap hT$ bounding innermost discs on hT and it will then follow that there are no simple closed curves of type (ii) and, hence, no simple closed curves of type (i) in $T \cap hT$.

We suppose that J is a simple closed curve of type (i) in $T \cap hT$ bounding an innermost disc D on hT. Since $D \subset hT$ and is innermost, either $D \subset V^+$ or $D \subset V^-$ and we suppose, without loss of generality, that $D \subset V^+$. We denote by E the disc on T bounded by J and let J' be a simple closed curve in $T - E$ sufficiently close to J such that the annulus $A \subset T$ bounded by $J \cup J'$ has the property that $A \cap hT = J$. Next, we choose a disc $D' \subset V^+$ so close to D that D' satisfies $D' \cap T = \partial D' = J'$, $D' \cap hT = \varnothing$, and $h^4 D' \cap D' = \varnothing$. This choice of D' is possible since h and h^4 are fixed point free and since there are no intersections of T with hT on int D. Finally, we set $E' = E \cup A$ and replace the disc E' by D' (Figure 3.1) and copy this replacement in the images of h. This adjustment of T results in the torus T_1

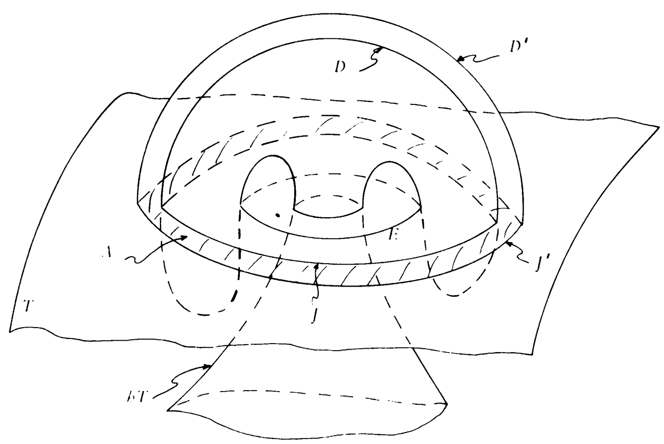

Figure 3.1

defined by

$$T_1 = [T - (E' \cup h^2 E' \cup h^4 E' \cup h^6 E')] \cup (D' \cup h^2 D' \cup h^4 D' \cup h^6 D')$$

with closed complementary domains given by

$$V_1^+ = \overline{V^+ - (B \cup h^4 B)} \cup (h^2 B \cup h^6 B)$$

and

$$V_1^- = \overline{V^- - (h^2 B \cup h^6 B)} \cup (B \cup h^4 B),$$

where B is the ball in V^+ bounded by $E' \cup D'$. Since V_1^+, V_1^- interchange under the action of h^2, it follows that T_1 is unknotted and that $T_1 \cap hT_1$ contains four fewer intersection curves of type (i) which bound innermost discs on hT_1. Repeating this process at most a finite number of times will then result in an unknotted torus T whose complementary domains interchange under h^2 and such that $T \cap hT$ contains no curves of type (i) which bound innermost discs on hT.

We suppose that $J \subset T \cap hT$ is a simple closed curve of type (ii) and assume that J is nontrivial on T. The argument is analogous if J is nontrivial on hT. Let D denote the disc on hT bounded by J. Since J is nontrivial on T, J is a $(1, 1)$-curve on T and, hence, D is not an innermost disc on hT. Thus, there is a finite number of simple closed curves of $T \cap hT$ on D and, hence, one of these, J', must bound an innermost disc D' on D (Figure 3.2) and hence on hT. Since J' bounds a disc on hT, J' is trivial on hT and since all type (i) simple closed curves bounding innermost discs on hT have been removed,

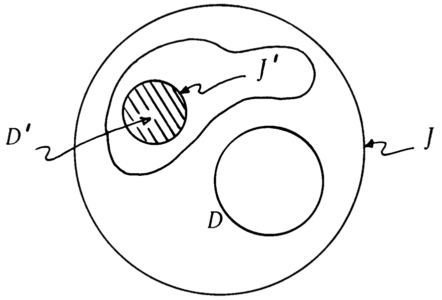

Figure 3.2

J' is also nontrivial on T. Therefore, J' is a $(1, 1)$-curve on T which bounds a disc in one of the complementary domains of T. But this is impossible and, therefore, no type (ii) curves exist. Furthermore, since all type (i) curves bounding innermost discs on hT have been removed, all simple closed curves in $T \cap hT$ are nontrivial on T and hT.

Step 3. There is an unknotted torus $T \subset S^3$ such that $T \cap hT$ contains at least one pair of simple closed curves which remain invariant as a pair under the action of h. Furthermore, h^2 interchanges the closed complementary domains of T.

We let T denote the torus obtained in Step 2 and let n be the number of simple closed curves in $T \cap hT$. We may assume that n is even and set $n = 2r$ for some positive integer r, for otherwise there is an arc $\alpha \subset hT$ with both end points in int V^+ and piercing ∂V^+ in only an odd number of points. Furthermore, by Step 1, $r \neq 0$. The simple closed curves in $T \cap hT$ divide T into n annuli and, hence, divide $T \cup hT$ into $2n$ annuli. Furthermore, if A and A' are any two of these annuli with $A \neq A'$, then int $A \cap A' = \emptyset$. The $2n$ annuli divide S^3 into k closed 3-dimensional regions and any one such region is entirely contained in one of the following four sets: $V^+ \cap hV^+$, $V^- \cap hV^+$, $V^- \cap hV^-$, and $V^+ \cap hV^-$. Under the action of h on S^3, these sets permute as follows:

$$V^+ \cap hV^+ \longrightarrow V^- \cap hV^+ \longrightarrow V^- \cap hV^- \longrightarrow V^+ \cap hV^- \longrightarrow V^+ \cap hV^+$$

Therefore, $k \equiv 0 \bmod 4$.

There are $2r$ annuli on hT and if A is any one of these annuli, then either $A \subset V^+$ or $A \subset V^-$. Since $h^2V^+ = V^-$, there are exactly r annuli on hT which must be contained in V^+ and r annuli on hT which must be contained in V^-. If $A \subset V^+$ is an annulus on hT, then A spans V^+ and, therefore, divides V^+ into two 3-dimensional regions. Hence, the $2r$ annuli on hT divide $V^+ \cup V^- = S^3$ into $2(r+1)$ 3-dimensional regions. Therefore, $2(r+1) = k \equiv 0 \bmod 4$ and r is odd.

We let $J_1, \cdots, J_n$, where $n = 2r$, denote the components of $T \cap hT$ and define a permutation $\sigma \in \Sigma n$ by $hJ_i = J_{\sigma(i)}$. Writing σ as a product of disjoint cycles, $\sigma = \sigma_1, \cdots, \sigma_k$, we have $n = \Sigma \operatorname{length}(\sigma_i) = \Sigma \operatorname{order}(\sigma_i)$ and the order of σ is the least common multiple of the orders of the σ_i's. Since $\sigma^8 = 1$, each σ_i must have order 1, 2, 4 or 8. If some σ_i has order 1, then there is a component J in $T \cap hT$ with $hJ = J$. But then there remains an odd number of components in $T \cap hT$ whose union is invariant under h. Thus, since h has even period, there is another simple closed curve J' in $T \cap hT$ with $hJ' = J'$. If some σ_i has order 2, then the conclusion of Step 3 follows trivially. If each σ_i has order ≥ 4, then n is divisible by 4, contradicting the fact that r is odd.

Before continuing our proof of Theorem 3.1, we note that if P and Q denote the simple closed curves in $T \cap hT$ such that $P \cup Q$ is invariant under h, then it is possible that each curve, P and Q, remains invariant under h, thus establishing the theorem.

Step 4. If P and Q are simple closed curves in $T \cap hT$ pairwise invariant under h with $hP \neq P$, then there is a torus $T' \subset S^3$ such that $hT' = T'$ and such that h interchanges the closed complementary domains of T'.

The set $P \cup Q$ divides T into two annuli, A and B, and either $h^2A = A$ or $h^2A = B$. If $h^2A = A$, we let $A' \subset hA$ be an innermost annulus on hT with respect to $T \cap hT$ having P for one of its boundary curves. Either $A' \subset V^+$ or $A' \subset V^-$, and we suppose that $A' \subset V^+$ and, hence $h^2A' \subset V^-$. But then, since $h^2P = P$ and $h^2(hA) = hA$, $h^2A' = A' \subset V^+$. Similarly, A' is not contained in V^- and, therefore, only $h^2A = B$ is possible.

In order to obtain T' we shall consider the two cases $P \cup Q = T \cap hT$ and $P \cup Q \subsetneq T \cap hT$ separately.

Case (a). $T \cap hT = P \cup Q$. Since P and Q are disjoint, we may choose a sufficiently small regular neighborhood N of P such that $N \cap hN = \emptyset$, $h^2N = N$ and $T \cap hT$ divides ∂N into four annuli. Such a neighborhood N may be obtained by choosing any regular neighborhood N_p of P and then taking a second derived neighborhood of P in a second derived subdivision of N_p on which h is simplicial. We let a_1, a_2, a_3, and a_4 denote the four annuli on ∂N so that

$$a_1 \subset V^+ \cap hV^+, \qquad a_2 = h^2 a_1 \subset V^- \cap hV^-,$$

$$a_3 \subset V^+ \cap hV^-, \qquad a_4 = h^2 a_3 \subset V^- \cap hV^+,$$

and set $a_i \cap hT = J_i$ for $a = 1, 2$, and $a_i \cap T = J_i$ for $i = 3, 4$ (Figure 3.3). We define A_p to be the

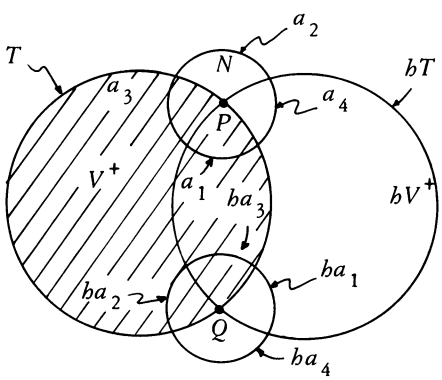

Figure 3.3

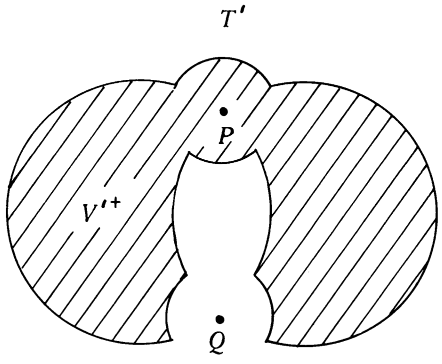

Figure 3.4

annulus on T containing P and such that $\partial A_p = J_3 \cup J_4$. Similarly, we let B_p be the annulus on hT containing P and such that $\partial B_p = J_1 \cup J_2$. The torus T' defined by

$$T' = ((T \cup hT) - (A_p \cup B_p \cup hA_p \cup hB_p)) \cup (a_1 \cup a_2 \cup ha_1 \cup ha_2)$$

is clearly invariant under h. Furthermore, since

$$\overline{(V^+ - hV^+) \cup (hV^+ - V^+)} = \overline{(V^- - hV^-) \cup (hV^- - V^-)}$$

and

$$\overline{(V^- - hV^+) \cup (hV^+ - V^-)} = \overline{(V^+ - hV^-) \cup (hV^- - V^+)},$$

the two closed complementary domains of T' in S^3 given by

$$V'^{+} = \overline{((V^{+} - hV^{+}) \cup (hV^{+} - V^{+}) \cup N) - hN}$$

and

$$V'^{-} = \overline{((V^{-} - hV^{+}) \cup (hV^{+} - V^{-}) \cup hN) - N}$$

(Figure 3.4) interchange under the action of h.

Case (b). $P \cup Q \subsetneq T \cap hT$. If n is the number of simple closed curves in $T \cap hT$, then $n = 2r$ for some positive integer $r > 1$, and each $A \cap hT$ and $B \cap hT$ contains $r + 1$ simple closed curves. We order the simple closed curves in $A \cap hT$ sequentially, letting $P = J_0, J_1, \cdots, J_r = Q$, so that J_i precedes J_{i+1} as we traverse A from P to Q. In a similar fashion we order the simple closed curves in $B \cap hT$ denoting them by $P = K_0, K_1, \cdots, K_r = Q$.

Since $r > 1$ and, by Step 3, r is odd, hT contains at least six innermost annuli. Hence, there is an innermost annulus A_1 on hT such that $P \cap \partial A_1 = Q \cap \partial A_1 = \varnothing$. The boundary of A_1 must be of one of the following three types:

Type 1. $J_i \cup J_j$; $i, j = 1, 2, \cdots, r-1$; $i \neq j$.

Type 2. $K_k \cup K_l$; $k, l = 1, 2, \cdots, r-1$; $k \neq l$.

Type 3. $J_m \cup K_n$; $m, n = 1, 2, \cdots, r-1$.

Our next step is to eliminate all innermost annuli on hT with boundaries of Type 1 or Type 2. It will then follow that we have either again Case (a), that is $T \cap hT$ consists of exactly two simple closed curves, or all innermost annuli on hT have boundaries of Type 3.

We suppose that $A_1 \subset V^{+}$ and has Type 1 boundary. If $\partial A_1 = J_i \cup J_j$, then $J_i \cup J_j$ divides T into two annuli which we denote by A' and B' with $P \cup Q \subset \operatorname{int} B'$. We choose two simple closed curves J_i' and J_j' on B', parallel and sufficiently close to J_i and J_j, respectively, satisfying the following property: If R_i and R_j denote the two annuli on B' bounded by $J_i \cup J_i'$ and $J_j \cup J_j'$, respectively, then $R_i \cap hT = J_i$, $R_j \cap hT = J_j$, $h^4 R_i = R_i$, and $h^4 R_j = R_j$. These last conditions are easily satisfied by observing that, since $h^4 A = A$, $h^4 P = P$ and $h^4 Q = Q$, we must have $h^4 J_i = J_i$ and $h^4 J_j = J_j$. Since A_1 is innermost, $hA_1 \cap \operatorname{int} A_1 = \varnothing$. If $hA_1 \cap \partial A_1 \neq \varnothing$, then since hA_1 is innermost on T, $hA_1 \subset A$ and also $h^2 A_1 \cap \partial hA_1 \neq \varnothing$. But this is impossible since $\partial h^2 A_1 \subset h^2 A = B$. Thus, $hA_1 \cap A_1 = \varnothing$, and we may choose an annulus $A_1' \subset V^{+}$ close to A_1 so that $A_1' \cap hT = \varnothing$, $A_1' \cap T = \partial A_1' = J_i' \cup J_j'$ and $A_1' \cap hA_1' = \varnothing$ (Figure 3.5). Replacing the annulus $A' \cup R_i \cup R_j$ by A_1' and copying this replacement in the image of h^2 results in the torus

$$T_1 = (T - (A' \cup R_i \cup R_j) \cup h^2(A' \cup R_i \cup R_j)) \cup (A_1' \cup h^2 A_1').$$

Since each of the annuli A', A_1', R_i, and R_j remain invariant under h^4, the torus T_1 remains invariant under h^2 and $T_1 \cap hT_1$ contains at least four fewer simple

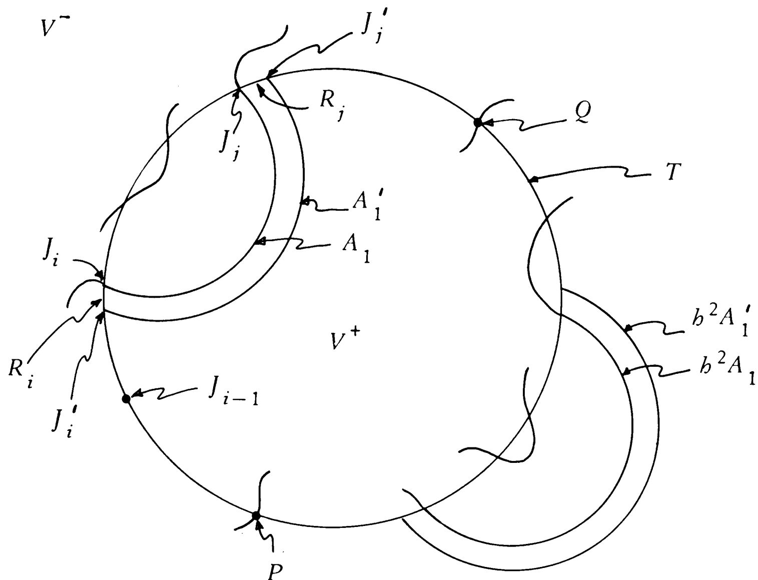

Figure 3.5

closed curves than $T \cap hT$. Furthermore, if V_1 denotes the closed 3-dimensional region in V^+ bounded by $A_1 \cup A \cup R_i \cup R_j$, then the two closed complementary domains of T_1 in S^3, given by

$$V_1^+ = (V^+ - V_1) \cup h^2 V_1 \quad \text{and} \quad V_1^- = (V^- - h^2 V_1) \cup V_1,$$

interchange under the action of h^2. We reason analogously if $A_1 \subset V^-$ and/or A_1 has Type 2 boundary.

If $T_1 \cap hT_1$ contains exactly two simple closed curves, we obtain the torus T' by Case (a). Otherwise we replace T by T_1 in Case (b) and repeat the entire argument. Since we started with only $2r$ intersection curves and the preceding argument reduces the number of innermost annuli having Type 1 or Type 2 boundaries by at least two, a finite number of at most $(r-1)/2$ repetitions of the above argument must eventually yield a torus T such that either $T \cap hT$ consists of exactly two simple closed curves or all innermost annuli on hT have Type 3 boundary.

We now suppose that all innermost annuli on hT have Type 3 boundary. If $c_1 \subset V^+$ is an innermost annulus on hT with respect to $T \cap hT$ and $\partial c_1 = P \cup J_i$, then $i = 1$. For if $i > 1$, then there is an innermost annulus c on hT (Figure 3.6) with $\partial c = J_1 \cup J_k$, $k < i$, contrary to our assumption that all innermost annuli on hT have boundaries of Type 3. Similarly, if $c_2 \subset V^+$ is innermost on hT and $\partial c_2 = K_1 \cup J_i$ (Figure 3.7), then $i = 2$. Continuing this argument for $i = 3, \cdots, r$,

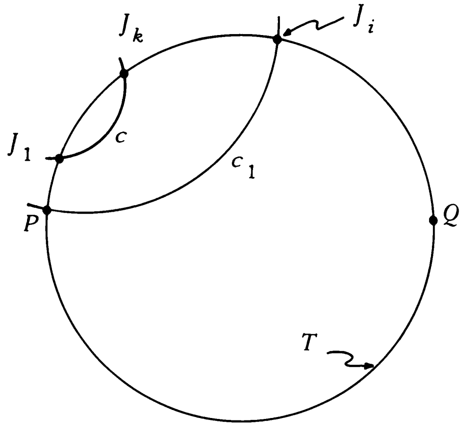

Figure 3.6

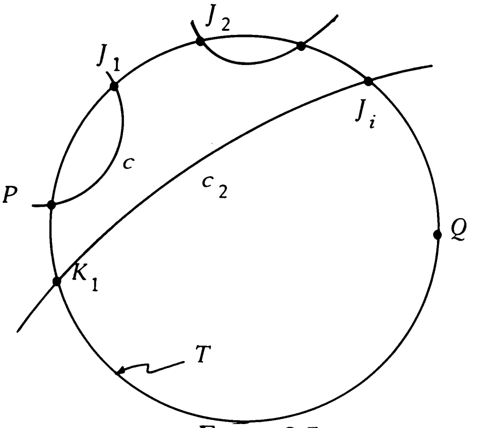

Figure 3.7

we see that if $c_i \subset V^+$ is innermost on hT, then $\partial c_i = K_{i-1} \cup J_i$.

If $c_1 \subset V^+$ were innermost on hT with $\partial c_1 = P \cup K_i$, then, using the same reasoning as above, the innermost annuli of hT in V^+ would have boundaries of form $J_{i-1} \cup K_i$, where $i = 1, \cdots, r$.

We shall assume that the innermost annuli of hT, contained in V^+, have boundaries of form $K_{i-1} \cup J_i$, $i = 1, \cdots, r$ (Figure 3.8) and argue analogously if, instead, the boundaries are of the form $J_{i-1} \cup K_i$, $i = 1, \cdots, r$.

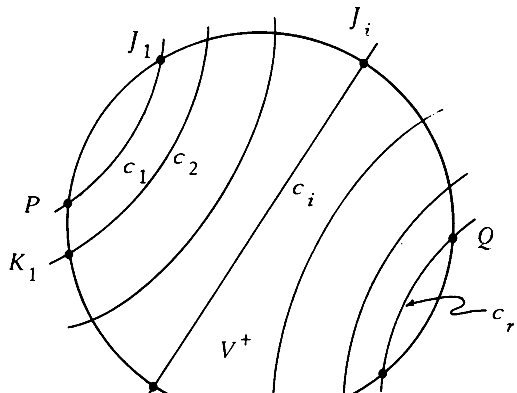

Figure 3.8

We denote the innermost annuli on T by a_i and b_i, where $a_i \subset A$, $\partial a_i = J_{i-1} \cup J_i$, $b_i \subset B$, $\partial b_i = K_{i-1} \cup K_i$, and $i = 1, \cdots, r$. We are interested in the images of the boundaries of these annuli under h. Since ha_1 is innermost on hT and $\partial ha_1 = Q \cup hJ_1$, we must have either $hJ_1 = J_{r-1}$ or $hJ_1 = K_{r-1}$ (Figure 3.9). There is no loss of generality if we assume that $hJ_1 = J_{r-1}$, since the case

$hJ_1 = K_{r-1}$ is argued in a likewise fashion. Thus, $\partial hb_1 = Q \cup hK_1 = Q \cup K_{r-1}$ and $\partial ha_2 = J_{r-1} \cup K_{r-2}$.

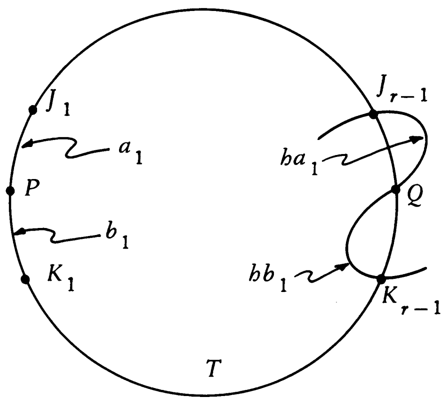

Figure 3.9

It follows that

$$\partial ha_1 = Q \cup J_{r-1}$$

$$\partial ha_2 = J_{r-1} \cup K_{r-2}$$

$$\partial ha_3 = K_{r-2} \cup J_{r-3}$$

$$\vdots \qquad \vdots$$

$$\partial ha_i = \begin{cases} J_{r-(i-1)} \cup K_{r-i}, & \text{if } i \text{ is even.} \\ K_{r-(i-1)} \cup J_{r-i}, & \text{if } i \text{ is odd.} \end{cases}$$

Similarly,

$$\partial hb_1 = Q \cup K_{r-1} \quad \text{and} \quad \partial hb_i = \begin{cases} K_{r-(i-1)} \cup J_{r-i}, & \text{if } i \text{ is even.} \\ J_{r-(i-1)} \cup K_{r-i}, & \text{if } i \text{ is odd.} \end{cases}$$

In particular,

$$hJ_i = \begin{cases} J_{r-i}, & \text{if } i \text{ is odd,} \\ K_{r-i}, & \text{if } i \text{ is even,} \end{cases} \quad \text{and} \quad hK_i = \begin{cases} K_{r-i}, & \text{if } i \text{ is odd.} \\ J_{r-i}, & \text{if } i \text{ is even.} \end{cases}$$

Thus, since r is odd, $h^2J_i = K_i$ if i is odd, and $h^2J_i = J_i$ if i is even. But $h^2A = B$ and $h^2P = P$ so $h^2J_i = K_i$ which is a contradiction if $r > 1$. This establishes Step 4 if Case (b) occurs.

We have shown that $T \cap hT$ contains a pair of unknotted simple closed curves, P and Q, remaining pairwise invariant under h and either $hP = P$ or there is a torus $T' \subset S^3$ invariant under h and such that h interchanges the closed complementary domains of T'. However, if the latter case occurs, then, by Theorem 2.4, there is an unknotted simple closed curve $J \subset T'$ with $hJ = J$. Thus, Theorem 3.1 has been verified.

We conclude this section by stating two easy corollaries of Theorem 3.1.

Corollary 3.2. *If Z_8 acts freely on S^3, then the orbit space S^3/Z_8 is either homeomorphic to the lens space $L(8, 1)$ or to the lens space $L(8, 3)$.*

Proof. If h is a generator of Z_8, then, by Theorem 3.1, there is an unknotted simple closed curve $J \subset S^3$ invariant under h. We denote by N_J a regular neighborhood of J in S^3. If $hN_J \neq N_J$, we take a second derived subdivision which is obtained from a lifting of a second derived subdivision of S^3/Z_8 and let N be the second derived neighborhood of J. Since N consists of all 3-simplexes which intersect J and since J is invariant under h, we must necessarily have N invariant under h. The neighborhood N is a solid torus with torus boundary ∂N. Furthermore, if $v_0, v_1, \cdots, v_k$ denote the vertices of J in this second derived subdivision, then $B_i = \operatorname{star}(v_i, N)$ is a 3-ball around v_i and $B_i \cap B_j \neq \emptyset$ if and only if either $i = j$ or $i - j = \pm 1$. Setting $D_i = B_i \cap B_{i+1}$, then D_i is a disc and $D_i \cap D_j = \emptyset$ for $i \neq j$. Thus, $N = \bigcup B_i$ and N/h is obtained by identifying two disjoint discs in the boundary of a 3-cell. By the Lefschetz fixed point theorem, h is orientation preserving and, therefore, N/h must be a solid torus. Similar considerations show that $\overline{S^3 - N/h}$ must also be a solid torus. Thus, the solid torus N is an 8-fold cover of the solid torus N/h and, similarly, the solid torus $\overline{S^3 - N}$ is an 8-fold cover of the solid torus $(\overline{S^3 - N})/h$. The orbit space S^3/Z_8 is obtained from N/h and $(\overline{S^3 - N})/h$ by identifying their boundaries via the projection map. Hence, S^3/Z_8 is homeomorphic to the lens space $L(8, q)$ and by [9] we may assume that $q \leq 4$ and relatively prime to 8. Thus, either $q = 1$ or $q = 3$. Furthermore, by [7], the lens spaces $L(8, 1)$ and $L(8, 3)$ are not homeomorphic. This proves Corollary 3.2.

Corollary 3.3. *Every free action of Z_8 on S^3 is topologically equivalent to either the orthogonal $(8, 1)$-action or the orthogonal $(8, 3)$-action.*

Proof. If h is a generator of Z_8, then, by Corollary 3.2, the orbit space S^3/Z_8 is either topologically the lens space $L(8, 1)$ or the lens space $L(8, 3)$. We denote by t_1 and t_2 the orthogonal $(8, 1)$- and $(8, 3)$-actions, respectively, and let p_1 and p_2 be the projections of S^3 onto the orbit spaces of t_1 and t_2, respectively. If the orbit space of h is the lens space $L(8, 1)$ and p the projection map of $S^3 \to L(8, 1)$ given by $p(x) = ph(x)$, $x \in S^3$, then, since S^3 is a universal covering space of $L(8, 1)$, there exists a homeomorphism g of S^3 onto itself such that the following diagram commutes:

$$\begin{tikzcd} S^3 \arrow[rr, "g"] \arrow[dr, "p"'] & & S^3 \arrow[dl, "p_1"] \\ & L(8, 1) & \end{tikzcd}$$

It follows that $h = g^{-1}t_1g$.

Similarly, if the orbit space of h is the lens space $L(8, 3)$, then there is a homeomorphism g such that $h = g^{-1}t_2g$. This proves Corollary 3.3.

4. **Free Z_n actions on S^3.** We shall view $S^3 \subset E^4$ as the join of the two circles $x_1^2 + x_3^2 = 1$ and $x_2^2 + x_4^2 = 1$ and let $z_0 = (x_1, x_3)$ and $z_1 = (x_2, x_4)$. The transformation $f\colon S^3 \to S^3$, defined by $f\colon (z_0, z_1) \to (z_1, e^{\pi i/p}z_0)$, interchanges the two solid tori $V^+\colon |z_1|^2 \leq |z_0|^2$ and $V^-\colon |z_0|^2 \leq |z_1|^2$, where $|z_0|^2 + |z_1|^2 = 1$. It follows from Theorem 2.4 that f is equivalent to an orthogonal $(4p, q)$-action, where $q \equiv \pm(2p - 1) \bmod 4p$. If $e\colon S^3 \to S^3/f = L(4p, q)$ is the natural projection, then $e(\partial V^+)$ is one-sided and, therefore, a Klein bottle. It is known [2] that a Klein bottle embeds in $L(4p, q)$ if and only if $q \equiv \pm(2p - 1) \bmod 4p$. In particular, the action which interchanged the closed complementary domains of the torus T' in the proof of Theorem 3.1 must be the action which is topologically equivalent to the $(8, 3)$-action. On the other hand, since there is no Klein bottle in $L(8, 1)$, there is no join construction of S^3 such that the join circles interchange under the orthogonal $(8, 1)$-action. It is this latter obstruction which prevents us from classifying all free Z_{2k} actions on S^3 using the methods developed in the preceding section. However, the method used for proving Theorem 3.1 does enable us to extend the results of the last section to those free actions on S^3 whose squares interchange the two circles $|z_0|^2 = 1$ and $|z_1|^2 = 1$.

Theorem 4.1. *Let Z_n act freely on S^3, $n = 4p$, p even and $h \in Z_n$ a generator. If h^2 is topologically equivalent to the orthogonal $(2p, q)$-action, where $q \equiv \pm(p - 1) \bmod 2p$, then h is topologically equivalent to either the orthogonal (n, q)-action or the orthogonal $(n, 2p - q)$-action.*

Proof. For $p = 2$, the result follows immediately from Corollary 3.3. For $p > 2$, there is a torus $T \subset S^3$ whose complementary domains in S^3 interchange under the action of h^2. Hence, by an exact duplicate of the argument given for the proof of Theorem 3.1, there is an unknotted simple closed curve in S^3 which remains invariant under the action of h. Furthermore, using the same reasoning as presented in the proofs of Corollaries 3.2 and 3.3, the orbit space S^3/Z_n, is homeomorphic to a lens space $L(n, q')$ and h is equivalent to an orthogonal (n, q')-action. By [7], [9], there is no loss of generality in assuming $q < p$ and $q' < 2p$. Thus, since $(e^{2q'\pi i/n})^2 = e^{q'\pi i/p}$, h^2 is topologically equivalent to the $(2p, q')$-action. Therefore, $q' \equiv \pm q^{\pm 1} \bmod 2p$ and, hence either $q' = q$ or $q' = 2p - q$. Finally, since

$2p - q \not\equiv \pm q \bmod 4p$ for $p \geq 2$, the two orthogonal actions are topologically distinct. This establishes Theorem 4.1.

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