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THE ELEMENTS OF EUCLID

FOR THE USE OF SCHOOLS AND COLLEGES

WITH NOTES, AN APPENDIX, AND EXERCISES BY

I. TODHUNTER, D.Sc., F.R.S.

NEW EDITION, REVISED AND ENLARGED, BY

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PREFACE.

In preparing a new edition of the late Dr. Todhunter's *Euclid*, the following are the principal alterations and additions that I have made:

(1) The text of the Propositions has been simplified and shortened. A sparing use has been made of symbols in the place of constantly recurring words. Some of the proofs (*e.g.* II. 13) have been altered in accordance with modern usage, but this has always been done subject to the regulations at present in force in the Universities of Oxford and Cambridge.

(2) Considerable trouble has been taken to arrange the book so that, with very few exceptions, each Proposition is commenced on a fresh page, and may be read by the student without his turning over a leaf.

(3) The proofs of Book V. have been much shortened. To them, and also to the proofs of Book II., the corresponding algebraic formulae have been added.

(4) The more important of Dr. Todhunter's Notes have been appended to the Propositions to which they refer.

(5) The total number of Exercises has been doubled.

(6) The easier of the Exercises in the previous edition, and a large number of additional ones, have been classified and follow, in the Text, the Propositions on which they depend. The more important of these have asterisks prefixed to them, and, as far as possible, their results should

be remembered by the student as part of his geometrical knowledge.

(7) All through the book, with the exception of the Exercises at the end, which are left to the student, hints have been annexed to the more difficult and more important Exercises.

(8) The Appendix has been more than doubled in quantity, and the theorems in it have been classified according to the Book to which they refer and on which they depend; a large number of Exercises has been incorporated in it.

(9) Sections have been added dealing with Poles and Polars, Orthogonal Circles, Pedal Triangles, The Pedal Line, The Nine-Point Circle, Co-axal Circles, Harmonic Ranges, Inversion, and the Properties of a Complete Quadrilateral.

Out of the large number of Propositions on these subjects it is clear that only a selection could be made, but I have endeavoured within the range chosen to omit no important Proposition. Without unduly increasing the size of the book it was impossible to prove all such important Propositions in the text. It is hoped, however, that the hints attached to those given as Exercises will be sufficient for any fairly intelligent student.

For any corrections of misprints or errors, or any suggestions for improvement, I shall be very grateful.

S. L. LONEY.

ROYAL HOLLOWAY COLLEGE, EGHAM,
SURREY, *April* 21*st*, 1899.

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EUCLID'S ELEMENTS.

BOOK I.

DEFINITIONS.

1. A **point** is that which has position but no magnitude.

2. A **line** is that which has length without breadth.

3. The extremities of a line are points, and the intersection of two lines is a point.

4. A **straight line** is one which lies evenly between its extreme points.

5. A superficies, or **surface**, is that which has only length and breadth.

6. The boundaries of a surface are lines.

7. A plane surface, or a **plane**, is that in which any two points being taken, the straight line between them lies wholly in that surface.

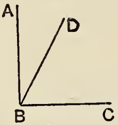
[Thus if we take a piece of wood, one of whose edges is straight, and apply it to the surface, and find that the edge fits closely to the surface everywhere, we know that the surface is a plane.]

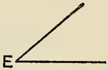
8. A **plane angle** is the inclination of two lines to one another in a plane, which meet together, but are not in the same direction.

9. A **plane rectilineal angle** is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.

The point where the two straight lines meet is called the **vertex** of the angle, and the straight lines themselves are sometimes called the **arms.**

Note. When several angles are at one point B, any one of them is expressed by three letters, of which the letter which is at the vertex of the angle is put between the other two letters, and one of these two letters is somewhere on one of those straight lines, and the other letter on the other straight line. Thus, the angle which is contained



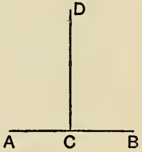


by the straight lines AB, CB is named the angle ABC, or CBA; the angle which is contained by the straight lines AB, DB is named the angle ABD, or DBA; and the angle which is contained by the straight lines DB, CB is named the angle DBC, or CBD; but if there be only one angle at a point, it may be expressed by a letter placed at that point, as the angle E.

Two such angles as ABD, DBC, which are on opposite sides of one common bounding line BD, are called **adjacent** angles.

The beginner must carefully observe that no change is made in an angle by prolonging the lines that form it, that is, by altering the length of its arms.

10. When a straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is called a **right angle**; and the straight line which stands on the other is called a **perpendicular** to it.



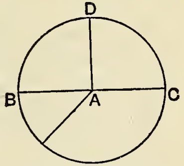
11. An **obtuse angle** is an angle which is greater than a right angle.

12. An **acute angle** is an angle which is less than a right angle.

13. A **plane figure** is one which is enclosed by one or more bounding lines, straight or curved; and the sum of these bounding lines is called a **perimeter**.

14. If the boundaries consist of straight lines only, it is called a **plane rectilineal figure**, and these straight lines are called its **sides**.

15. A **circle** is a plane figure contained by one line, which is called the **circumference**, and is such, that all straight lines drawn from a certain point within the figure to the circumference are equal to one another:



Also this point is called the **centre** of the circle.

16. A **radius** of a circle is a straight line drawn from the centre to the circumference.

17. A **diameter** of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

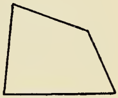
18. A **semicircle** is the figure contained by a diameter and the part of the circumference cut off by the diameter.

[Thus, in the above figure, A is the centre; AB, AD, AC are radii; BC is a diameter; and the figure bounded by the straight line BC and the curved line BDC is a semicircle.]

19. A **segment** of a circle is the figure contained by a straight line and the part of the circumference which it cuts off.

20. A **triangle** is a plane figure contained by three straight lines. [Any angular point may be called a **vertex** and the opposite side the **base**.]

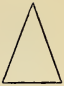
21. A **quadrilateral** is a plane figure contained by four straight lines.



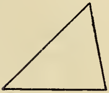
[The straight line joining two opposite corners of a quadrilateral is called a **diagonal**.]

22. A **polygon** is a plane figure contained by more than four straight lines.

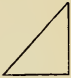
23. An **equilateral triangle** is one which has three equal sides.



24. An **isosceles triangle** is one which has two sides equal.



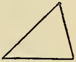
25. A **scalene triangle** is one which has three unequal sides.



26. A **right-angled triangle** is one which has a right angle.

[The side opposite to the right angle in a right-angled triangle is frequently called the **hypotenuse.**]

27. An **obtuse-angled triangle** is one which has an obtuse angle.



28. An **acute-angled triangle** is one which has three acute angles.

[It will be seen later that every triangle has at least two acute angles.]

29. **Parallel straight lines** are such as are in the same plane, and which being produced ever so far both ways do not meet.

30. A **parallelogram** is a four-sided figure which has its opposite sides parallel.

31. A **square** is a four-sided figure which has all its sides equal, and one of its angles a right angle.

32. A **rectangle** is a parallelogram which has one of its angles a right angle.

[It will be shewn (see Note, Prop. 46) that all the angles of a rectangle or a square are right angles.]

33. A **rhombus** is a four-sided figure which has all its sides equal, but its angles are not right angles.

34. A rhomboid is a four-sided figure which has its opposite sides equal to one another, but all its sides are not equal, nor its angles right angles.

35. A trapezium is a four-sided figure which has two of its sides parallel.

POSTULATES.

Let it be granted,

1. That a straight line may be drawn from any one point to any other point:

2. That a terminated straight line may be produced to any length in a straight line: and

3. That a circle may be described with any centre, at any distance from that centre, that is, with any given line drawn from the centre as radius.

Note. The postulates state what processes we assume that we can effect. It is sometimes stated that the postulates amount to requiring the use of a *ruler* and *compasses*. It must, however, be observed that the ruler is not supposed to be a *graduated* ruler, so that we cannot use it to measure off assigned lengths. Also, we are not supposed to use the compasses for the purpose of transferring any distance from one part of a figure to another; in other words, the compasses may be supposed to close of themselves, as soon as one of their points is removed from the paper. [After Prop. 3 it will be found that this restriction is no longer necessary.]

AXIOMS.

1. Things which are equal to the same thing are equal to one another.

2. If equals be added to equals the wholes are equal.

3. If equals be taken from equals the remainders are equal.

4. If equals be added to unequals the wholes are unequal.

5. If equals be taken from unequals the remainders are unequal.

6. Things which are double of the same thing are equal to one another.

7. Things which are halves of the same thing are equal to one another.

8. The whole is greater than its part.

9. Magnitudes which can be made to coincide with one another are equal to one another.

[This method of placing one geometrical magnitude upon a second is called the method of **superposition**, and the first magnitude is said to be **applied** to the other.]

10. Two straight lines cannot enclose a space.

This axiom should be extended thus :
If two straight lines coincide in two points, they must coincide both beyond and between these points.

11. All right angles are equal to one another.

[This axiom admits of proof ; see Note to I. 14.]

12. If a straight line meet two straight lines, so as to make the two interior angles on the same side of it together less than two right angles, these straight lines, being continually produced, shall at length meet on that side on which are the angles which are less than two right angles.

Note. An axiom is a truth which can be taken for granted, and requires no proof. The axioms are called in the original *Common Notions.*

The first eight are true of magnitudes of all kinds, and are sometimes called General Axioms; the remainder refer exclusively to geometrical magnitudes, and are sometimes called Geometrical Axioms.

The fourth axiom is sometimes referred to in editions of Euclid when in reality more is required than this axiom expresses. Euclid says that if A and B be unequal, and C and D equal, the sum of A and C is *unequal* to the sum of B and D. What Euclid often requires is something more, namely, that if A be greater than B, and C and D be equal, the sum of A and C is *greater* than the sum of B and D. Such an axiom as this is required, for example, in I. 17. A similar remark applies to the fifth axiom.

The eleventh axiom is not, strictly speaking, an axiom, for it can be proved; it is not required before I. 14, and the twelfth axiom is not required before I. 29; we shall not consider these axioms until we arrive at the propositions in which they are respectively required for the first time.

Note on Euclid's Propositions.

Euclid divides his different books into separate propositions, each proposition being derived from previous propositions. Propositions are of two kinds, **Problems** and **Theorems.**

In a problem Euclid states some definite construction which is to be made, such as to draw some particular figure. When this construction has been made the problem is *solved.*

In a theorem Euclid states some definite geometrical fact which he proceeds to prove. He names first the **Hypothesis**, or the conditions which he assumes, and then the **Conclusion** which he asserts will follow.

In a proposition we usually have:

(1) The *General Enunciation*, which is a preliminary statement.

(2) The *Particular Enunciation*, which applies the general enunciation to a particular diagram.

(3) The *Construction*, which shows what drawing of lines, etc., he wants.

(4) The *Demonstration*, or *Proof*, which shows that the problem has been solved, or that the theorem is true.

The letters Q.E.F. at the end of a problem stand for *Quod erat Faciendum*, that is, *which was to be done.*

The letters Q.E.D. at the end of a theorem stand for *Quod erat Demonstrandum*, that is, *which was to be proved.*

A **Corollary** is a statement which follows immediately from the proposition to which it is appended, and which thus requires no further proof.

Many of the corollaries are not in the original text but have been introduced by various editors.

SYMBOLS AND ABBREVIATIONS.

The following symbols are often used for words and expressions that continually occur in Euclid's Proofs. They may be used by the student in writing out propositions, and some of the more common will be gradually introduced in this Edition.

$\therefore$ *for* Therefore.

$=$ *for* Equals *or* Is equal to *or* Are equal to.

$>$ *for* Is greater than. $<$ *for* Is less than.

$\angle$ and $\angle^{s.}$ *for* Angle *and* Angles.

$\triangle$ and $\triangle^{s.}$ *for* Triangle *and* Triangles.

$\perp^{r.}$ *for* Perpendicular to.

$\|^{l.}$ *for* Parallel. $\|^{m.}$ and Par$^{m.}$ *for* Parallelogram.

Rt. $\angle$ *for* Right Angle. $\odot$ *for* Circle *or* Circumference.

The symbol $\because$ is sometimes used for Because. It is, however, liable to be confounded with the symbol for Therefore, and will not be used in this Edition.

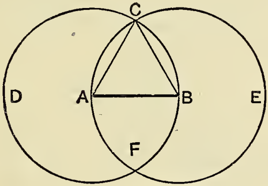
In addition any well understood abbreviations for words may be used; *e.g.*:

Def. *for* Definition; Ax. *for* Axiom; Post. *for* Postulate; Constr. *for* Construction; Hyp. *for* Hypothesis; Pt. *for* Point; Str. *for* Straight; Perp. *for* Perpendicular; Alt. *for* Altitude; Sq. *for* Square; Rect. *for* Rectangle; Quad$^{l.}$ *for* Quadrilateral; Adj. *for* Adjacent.

Proposition 1. Problem.

To describe an equilateral triangle on a given finite straight line.

Let AB be the given straight line:
it is required to describe an equilateral triangle on AB.



Construction. With centre A and radius AB, describe the circle BCD. [*Postulate* 3.

With centre B and radius BA, describe the circle ACE. [*Postulate* 3.

From the point C, at which the circles cut one another, draw the straight lines CA and CB. [*Postulate* 1.

ABC shall be an equilateral triangle.

Proof. Because A is the centre of the circle BCD,

AC is equal to AB. [*Definition* 15.

Also because B is the centre of the circle ACE,

BC is equal to BA. [*Definition* 15.

But it has been shewn that CA is equal to AB;
therefore CA and CB are each of them equal to AB.
But things which are equal to the same thing are equal to one another; [*Axiom* 1.

therefore CA is equal to CB.

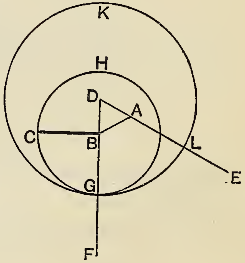
Therefore CA, AB, BC are equal to one another.

Wherefore *the triangle* ABC *is equilateral,* [*Definition* 23.
and it is described on the given straight line AB. [Q. E. F.

Proposition 2. Problem.

From a given point to draw a straight line equal to a given straight line.

Let A be the given point, and BC the given straight line: *it is required to draw from the point* A *a straight line equal to* BC.



Construction. Join AB, [*Post.* 1.
and on it describe the equilateral triangle DAB, [I. 1.
and produce the straight lines DA, DB to E and F. [*Postulate* 2.
With centre B and radius BC, describe the circle CGH,
meeting DF at G. [*Postulate* 3.
With centre D and radius DG, describe the circle GLK,
meeting DE at L. [*Postulate* 3.
AL shall be equal to BC.

Proof. Because the point B is the centre of the circle CGH,
BC is equal to BG. [*Definition* 15.
Also because D is the centre of the circle GLK,
DL is equal to DG; [*Definition* 15.
and DA, DB parts of them are equal; [*Definition* 23.
therefore the remainders AL, BG are equal. [*Axiom* 3.
But it has been shewn that BC is equal to BG;
therefore AL and BC are each of them equal to BG.
But things which are equal to the same thing are equal to one another. [*Axiom* 1.

Therefore AL is equal to BC.

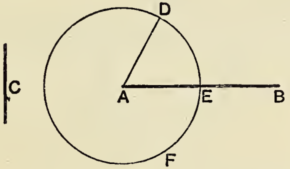
Wherefore *from the given point* A *a straight line* AL *has been drawn equal to the given straight line* BC. [Q.E.F.

PROPOSITION 3. PROBLEM.

From the greater of two given straight lines to cut off a part equal to the less.

Let AB and C be the two given straight lines, of which AB is the greater:

It is required to cut off from AB, *the greater, a part equal to* C, *the less.*



Construction. From the point A draw the straight line AD equal to C; [I. 2.
and with centre A and radius AD, describe the circle DEF, meeting AB at E. [*Postulate* 3.
Then AE shall be equal to C.

Proof. Because A is the centre of the circle DEF,
therefore AE is equal to AD. [*Def.* 15.
But C is equal to AD. [*Construction.*
Therefore AE and C are each of them equal to AD.
Therefore AE is equal to C. [*Axiom* 1.

Wherefore *from* AB, *the greater of two given straight lines, a part* AE *has been cut off equal to* C, *the less.*

EXERCISES.

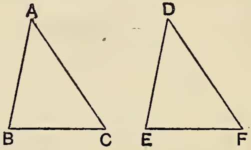
1. In the figure of I. 1, if the two circles meet again in F, prove that ACBF is a rhombus.

2. On a given straight line describe an isosceles triangle, having each of the sides equal to a given straight line.

3. On a given straight line describe an isosceles triangle, having each of the sides double the base.

PROPOSITION 4. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal to one another;
they shall also have their bases or third sides equal;
the two triangles shall be equal in area,
and their other angles shall be equal, each to each, namely, those to which the equal sides are opposite;
that is, the triangles shall be equal in all respects.



Let ABC, DEF be two triangles which have
the side AB equal to the side DE,
the side AC equal to the side DF,
and the angle BAC equal to the angle EDF;
then shall the base BC *be equal to the base* EF,
the triangle ABC *shall be equal in area to the triangle* DEF,
and the other angles shall be equal, each to each, to which the equal sides are opposite, namely, the angle ABC *to the angle* DEF, *and the angle* ACB *to the angle* DFE.

Proof. For if the triangle ABC be applied to the triangle DEF, so that the point A may be on the point D, and the straight line AB on the straight line DE,
the point B will coincide with the point E,
because AB is equal to DE. [*Hypothesis.*
And because AB coincides with DE,
and the angle BAC is equal to the angle EDF, [*Hypothesis.*
therefore AC will fall on DF.

Therefore also the point C will coincide with the point F,
because AC is equal to DF. [*Hypothesis.*

But the point B was shewn to coincide with the point E, therefore the base BC will coincide with the base EF; for if not, two straight lines will enclose a space, which is impossible. [*Axiom* 10.

Therefore the base BC coincides with the base EF, and is equal to it. [*Axiom* 9.

Therefore the whole triangle ABC coincides with the whole triangle DEF, and is equal to it. [*Axiom* 9.

And the other angles of the one coincide with the other angles of the other, and are equal to them, namely,

the angle ABC to the angle DEF,

and the angle ACB to the angle DFE.

Wherefore, *if two triangles, etc.* [Q.E.D.

Note. Triangles, such as ABC and DEF, which are equal in all respects, are said to be **Congruent.**

EXERCISES.

1. If two straight lines, AB and CD, bisect one another at right angles in O, any point P in either of them, AB, is equidistant from the ends, C and D, of the other.

****2.** If the straight line drawn from the vertex of a triangle to the middle point of the base cuts the base at right angles, the triangle is isosceles.

3. D and E are the middle points of the equal sides AB, AC of an isosceles triangle ABC; prove that CD and BE are equal.

4. ABCD is a quadrilateral, and its diagonals bisect one another at right angles; prove that ABCD is a rhombus.

5. The sides AB, AD of a quadrilateral ABCD are equal, and the diagonal AC bisects the angle BAD; prove that the sides CB, CD are equal, and that the diagonal AC bisects the angle BCD.

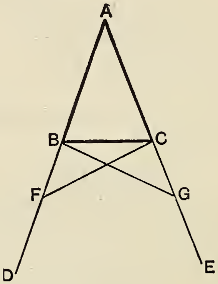
6. ABCD and EFGH are quadrilaterals, such that the sides AB, BC, CD are equal respectively to the sides EF, FG, GH, and the angles ABC, BCD are equal respectively to the angles EFG, FGH. Prove that the quadrilaterals are equal in all respects.

****7.** The straight line which bisects the vertical angle of an isosceles triangle bisects the base, and is also perpendicular to it.

PROPOSITION 5. THEOREM.

The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced, the angles on the other side of the base shall also be equal to one another.

Let ABC be an isosceles triangle, having the side AB equal to the side AC, and let the straight lines AB, AC be produced to D and E:
then the angle ABC *shall be equal to the angle* ACB,
and the angle CBD *to the angle* BCE.



Construction. In BD take any point F,
and from AE the greater cut off AG equal to AF the less. [I. 3.
Join FC and GB.

Proof. (1) In the two triangles AFC, AGB,

because { AF is equal to AG, [*Construction.*
and AC is equal to AB, [*Hypothesis.*
and the contained angle FAG is common to both;

therefore the triangles are equal in all respects, so that
the base FC is equal to the base GB,
the angle ACF is equal to the angle ABG,
and the angle AFC to the angle AGB. [I. 4.

(2) Because the whole AF is equal to the whole AG, of which the parts AB, AC are equal, [*Hypothesis.*
the remainder BF is equal to the remainder CG. [*Axiom* 3.

(3) Then in the triangles BFC, CGB,

because { BF is equal to CG, [*Proved in* (2).
and FC is equal to GB, [*Proved in* (1).
and the angle BFC is equal to the angle CGB; [*Proved in* (1).

therefore the triangles are equal in all respects, [I. 4.
so that the angle FBC is equal to the angle GCB,
and the angle BCF to the angle CBG.

(4) Since the whole angle ABG is equal to the whole angle ACF, [*Proved in* (1).
and that the parts of these, the angles CBG, BCF are also equal; [*Proved in* (3).
therefore the remaining angle ABC is equal to the remaining angle ACB, [*Axiom* 3.
and these are the angles at the base of the triangle ABC.

Also it has been shewn that the angle FBC is equal to the angle GCB, and these are the angles on the other side of the base.

Wherefore, *the angles, etc.* [Q.E.D.

Corollary. Every equilateral triangle is also equiangular.

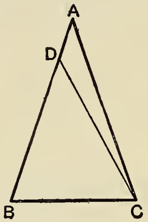
Note. This proposition is often found hard by beginners. This probably arises from the fact that the triangles ACF, ABG overlap one another; as also do the triangles BCF and CBG. For the part (1) of the proof the student is recommended to draw separately the two triangles ACF and ABG, and consider the figures thus obtained; for the part (3) of the proof he should similarly draw separately the two triangles BCF and CBG.

Proposition 6. Theorem.

If two angles of a triangle be equal to one another, the sides also which subtend, or are opposite to, the equal angles, shall be equal to one another.

Let ABC be a triangle, having the angle ABC equal to the angle ACB :

then the side AC *shall be equal to the side* AB.



Construction. For if AC be not equal to AB, one of them must be greater than the other.

Let AB be the greater, and from it cut off DB equal to AC the less, [I. 3.

and join DC.

Proof. In the triangles DBC, ACB,

because { DB is equal to AC, [*Construction.*
and BC is common to both triangles,
and the angle DBC is equal to the angle ACB, [*Hypothesis.* }

therefore the triangles are equal in all respects, [I. 4.

the less to the greater; which is absurd. [*Axiom* 8.

Therefore AB is not unequal to AC, that is, it is equal to it.

Wherefore, *if two angles, etc.* [Q. E. D.

Corollary. Every triangle, which has three equal angles, is also equilateral.

NOTE TO PROPOSITION 6.

Converse Theorem. One proposition is said to be the converse of another when the conclusion of each is the hypothesis of the other. I. 6 is the *converse* of part of I. 5. Thus in I. 5 the hypothesis is the equality of the sides, and one conclusion is the equality of the angles; in I. 6 the hypothesis is the equality of the angles and the conclusion is the equality of the sides.

The converse of a true proposition is not necessarily true; the student however will see, as he proceeds, that Euclid shews that the converses of many geometrical propositions are true.

I. 6 is an example of the **indirect** mode of demonstration, in which a result is established by shewing that some absurdity follows from supposing the required result to be untrue. Hence this mode of demonstration is called the **reductio ad absurdum.** Indirect demonstrations are often less esteemed than direct demonstrations; they are said to shew that a theorem *is* true rather than to shew *why* it is true. Euclid uses the *reductio ad absurdum* chiefly when he is demonstrating the converse of some former theorem; see I. 14, 19, 25, 40.

EXERCISES.

1. If the angles ABC, ACB at the base of an isosceles triangle ABC are bisected by the straight lines BD and CD, which meet at D, prove that BDC will be an isosceles triangle.

2. BAC is a triangle having the angle B double of the angle A. If BD bisects the angle B and meets AC at D, prove that BD and AD are equal.

In the figure of I. 5, if FC and BG meet at H, prove that

3. BH and CH are equal.

4. AH bisects each of the angles BAC, BHC.

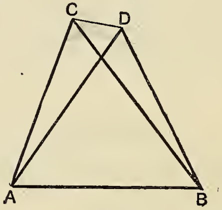
5. AH bisects BC at right angles.

6. If on the sides AB, BC, CA of an equilateral triangle ABC equal lengths AP, BQ, CR be measured, prove that PQR is also an equilateral triangle.

PROPOSITION 7. THEOREM.

On the same base, and on the same side of it, there cannot be two triangles having their sides which are terminated at one extremity of the base equal to one another, and likewise those which are terminated at the other extremity equal to one another.

If it be possible, on the same base AB, and on the same side of it, let there be two triangles ACB, ADB, having their sides CA, DA, which are terminated at the extremity A of the base, equal to one another,
and also their sides CB, DB, which are terminated at B, equal to one another.



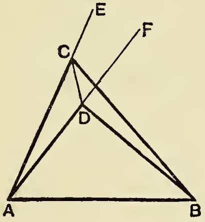
Construction. Join CD.

Proof. CASE I. Let the vertex of each triangle be without the other triangle.

Because AC is equal to AD, [*Hypothesis.*
the angle ACD is equal to the angle ADC. [I. 5.
But the angle ACD is greater than the angle BCD, [*Ax.* 8.
therefore the angle ADC is also greater than the angle BCD;
much more then is the angle BDC greater than the angle BCD.

Again, because BC is equal to BD, [*Hypothesis.*
the angle BDC is equal to the angle BCD. [I. 5.
But it has been shewn to be greater, which is impossible.

CASE II. Let one of the vertices as D be within the other triangle ACB, and produce AC, AD to E, F.



Then because AC is equal to AD, in the triangle ACD, [*Hyp.*
the angles ECD, FDC, on the other side of the base CD, are equal to one another. [I. 5.
But the angle ECD is greater than the angle BCD, [*Axiom* 8.
therefore the angle FDC is also greater than the angle BCD; much more then is the angle BDC greater than the angle BCD.
Again, because BC is equal to BD, [*Hypothesis.*
the angle BDC is equal to the angle BCD. [I. 5.
But it has been shewn to be greater, which is impossible.

CASE III. The case in which the vertex D of one triangle is on a side, BC, of the other needs no demonstration.

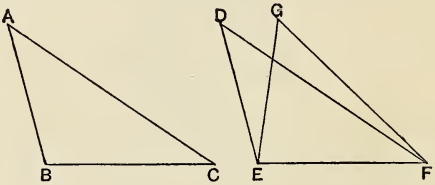
Wherefore, *on the same base, etc.* [Q.E.D.

Proposition 8. Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, the angle which is contained by the two sides of the one shall be equal to the angle which is contained by the two sides, equal to them, of the other.

Let ABC, DEF be two triangles, having the two sides AB, AC equal to the two sides DE, DF, each to each, namely, AB to DE, and AC to DF, and also the base BC equal to the base EF:

then shall the angle BAC *be equal to the angle* EDF.



Proof. For if the triangle ABC be applied to the triangle DEF, so that the point B may be on the point E,
and the straight line BC on the straight line EF,
the point C will also coincide with the point F,
because BC is equal to EF. [*Hypothesis.*

Therefore, BC coinciding with EF, BA and AC will coincide with ED and DF.

For if not, they must have a different situation as EG, GF; then on the same base and on the same side of it there would be two triangles having their sides DE, GE equal, and also their sides DF, GF equal.

But this is impossible. [I 7.

Therefore the sides BA, AC must coincide with the sides ED, DF.

Therefore also the angle BAC coincides with the angle EDF, and is equal to it. [*Axiom* 9.

Wherefore, *if two triangles, etc.* [Q.E.D.

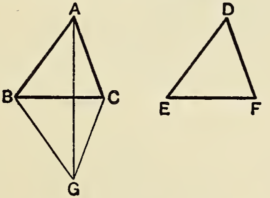
Cor. Since the triangle ABC would coincide with the triangle DEF, these two triangles *are equal in all respects.*

ALTERNATIVE PROOF OF PROPOSITION 8.

The following proof is independent of I. 7. It has been recommended by many writers, and is often known as Philo's proof.

Let ABC, DEF be two triangles, having the sides AB, AC equal to the sides DE, DF, each to each, and the base BC equal to the base EF:

the angle BAC *shall be equal to the angle* EDF.



For, let the triangle DEF be applied to the triangle ABC, so that the bases may coincide, the equal sides be conterminous, and the vertices fall on opposite sides of the base.

Let GBC represent the triangle DEF thus applied, so that G corresponds to D. Join AG.

Since, by hypothesis, BA is equal to BG, the angle BAG is equal to the angle BGA. [I. 5.

In the same manner the angle CAG is equal to the angle CGA.

Therefore the whole angle BAC is equal to the whole angle BGC, that is, to the angle EDF.

There are two other cases; for the straight line AG may pass through B or C, or it may fall outside BC: these cases may be treated in the same manner as that which we have considered.

EXERCISES.

**1. If D be the middle point of the base BC of an isosceles triangle ABC, prove that AD is perpendicular to BC.

**2. The opposite angles of a rhombus are equal.

**3. A diagonal of a rhombus bisects each of the angles through which it passes.

**4. The diagonals of a rhombus bisect one another at right angles.

5. ACB and ADB are two triangles on the same side of AB, such that AC is equal to BD, and AD is equal to BC; if AD and BC meet in O, prove that the triangle AOB is isosceles.

ABC is an isosceles triangle and D, E are points in the equal sides AB, AC, such that AD and AE are equal; if BE and CD meet in F, prove that

6. BCF is an isosceles triangle.

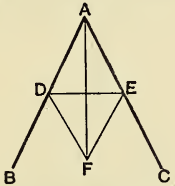
7. AF bisects the angle BAC.

8. AF produced bisects the base BC.

PROPOSITION 9. PROBLEM.

To bisect a given rectilineal angle, that is, to divide it into two equal angles.

Let BAC be the given rectilineal angle:
it is required to bisect it.



Construction. Take any point D in AB, and from AC cut off AE equal to AD; [I. 3.
join DE, and on DE, on the side remote from A, describe the equilateral triangle DEF. [I. 1.
Join AF.
AF shall bisect the angle BAC.

Proof. In the triangles DAF, EAF,

because { AD is equal to AE, [*Construction.*
and AF is common to both,
and the base DF is equal to the base EF, [*Def.* 23.

therefore the angle DAF is equal to the angle EAF. [I. 8.

Wherefore *the given rectilineal angle* BAC *is bisected by the straight line* AF. [Q.E.F.

Note. The equilateral triangle DEF is to be described on the side *remote* from A, because if it were described on the *same* side, its vertex F might coincide with A and then the construction would fail.

EXERCISES.

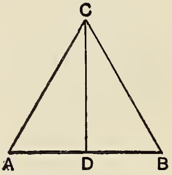
In the figure of I. 9 prove that

1. the lines AF and DE are at right angles.
2. any point P in AF is equally distant from the points D and E.
3. Divide any angle into four equal parts.

Proposition 10. Problem.

To bisect a given finite straight line, that is, to divide it into two equal parts.

Let AB be the given straight line:
it is required to divide it into two equal parts.



Construction. On AB describe an equilateral triangle ABC, [I. 1.
and bisect the angle ACB by the straight line CD, which meets AB at D. [I. 9.
AB shall be cut into two equal parts at the point D.

Proof. In the triangles ACD, BCD,

because { AC is equal to CB, [*Definition* 23.
and CD is common to both,
and the angle ACD is equal to the angle BCD, [*Construction.*

therefore the base AD is equal to the base DB. [I. 4.

Wherefore *the given straight line* AB *is divided into two equal parts at the point* D. [Q.E.F.

EXERCISE.

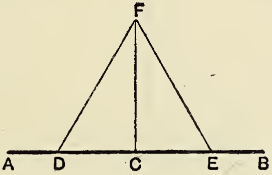
In the above figure prove that every point in the line CD, or CD produced, is equally distant from A and B.

PROPOSITION 11. PROBLEM.

To draw a straight line at right angles to a given straight line, from a given point in the same.

Let AB be the given straight line, and C the given point in it:

it is required to draw from the point C *a straight line at right angles to* AB.



Construction. Take any point D in AC, and make CE equal to CD. [I. 3.

On DE describe the equilateral triangle DFE, [I. 1.

and join CF.

The straight line CF shall be the line required

Proof. In the triangles DCF, ECF,

because { DC is equal to CE, [*Construction.*
and CF is common to both,
and the base DF is equal to the base EF, [*Def.* 23.

therefore the angle DCF is equal to the angle ECF, [I. 8.

and they are adjacent angles.

But when a straight line, standing on another straight line, makes the adjacent angles equal to one another, each of the angles is called a right angle; [*Definition* 10.

therefore each of the angles DCF, ECF is a right angle.

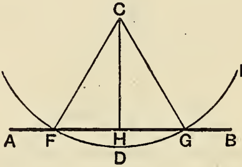
Wherefore, *from the given point* C *in the given straight line* AB, CF *has been drawn at right angles to* AB. [Q.E.F.

Proposition 12. Problem.

To draw a straight line perpendicular to a given straight line of an unlimited length from a given point without it.

Let AB be the given straight line, which may be produced to any length both ways, and let C be the given point without it:

it is required to draw from the point C *a straight line perpendicular to* AB.



Construction. Take any point D on the side of AB, remote from C, and with centre C and radius CD describe the circle EGF, meeting AB at F and G. [*Postulate* 3.

Bisect FG at H, [I. 10.

and join CH.

Then CH shall be the straight line required.

Join CF, CG.

Proof. In the triangles FHC, GHC,

because { FH is equal to HG, [*Construction.*
and HC is common to both,
and the base CF is equal to the base CG, [*Definition* 15.

therefore the angle CHF is equal to the angle CHG, [I. 8.

and they are adjacent angles.

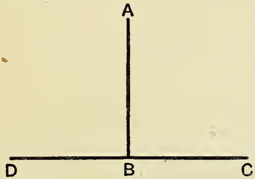
But when a straight line, standing on another straight line, makes the adjacent angles equal to one another, each of the angles is called a right angle, and the straight line which stands on the other is called a perpendicular to it. [*Def.* 10.

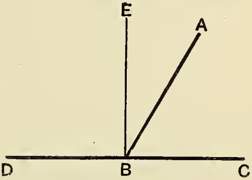
Wherefore *a perpendicular* CH *has been drawn to the given straight line* AB *from the given point* C *without it.* [Q.E.F.

PROPOSITION 13. THEOREM.

The angles which one straight line makes with another straight line on one side of it, are either two right angles, or are together equal to two right angles.

Let the straight line AB make with the straight line CD, on one side of it, the angles CBA, ABD:
these shall be either two right angles, or be together equal to two right angles.





CASE I. If the angle CBA is equal to the angle ABD, each of them is a right angle. [*Definition* 10.

CASE II. If not, from the point B draw BE at right angles to CD. [I. 11.

Proof. Since the angles DBE, EBC are equal to two right angles, [*Constr. and Def.* 10.
and the angle EBC is equal to the angles EBA, ABC,
therefore the angles DBE, EBA, ABC are equal to two right angles. [*Axiom* 2.
But the angles DBE, EBA are equal to the angle DBA;
therefore the angles DBA, ABC are equal to two right angles. [*Axiom* 2.

Wherefore, *the angles, etc.* [Q.E.D.

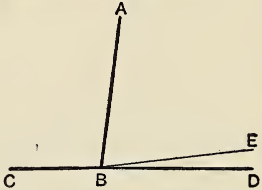
Note. Two angles, such as ABC and ABE, which together make up a right angle, EBC, are said to be **complementary**, and each is said to be the **complement** of the other.

Two angles, such as ABC and ABD, which together make up two right angles, are said to be **supplementary**, and each is said to be the **supplement** of the other.

PROPOSITION 14. THEOREM.

If, at a point in a straight line, two other straight lines, on the opposite sides of it, makes the adjacent angles together equal to two right angles, these two straight lines shall be in one and the same straight line.

At the point B in the straight line AB, let the two straight lines BC, BD, on the opposite sides of AB, make the adjacent angles ABC, ABD together equal to two right angles:
BD *shall be in the same straight line with* CB.



Proof. For if BD be not in the same straight line with CB, let BE be in the same straight line with it.
Then because the straight line AB meets the straight line CBE, the angles ABC, ABE are together equal to two right angles. [I. 13.
But the angles ABC, ABD are also together equal to two right angles. [*Hypothesis.*
Therefore the angles ABC, ABE are equal to the angles ABC, ABD. [*Axioms* 1 *and* 11.
From each of these equals take the common angle ABC, and the remaining angle ABE is equal to the remaining angle ABD, [*Axiom* 3.
the less to the greater, which is impossible.
Therefore BE is not in the same straight line with CB.

And in the same manner it may be shewn that no other can be in the same straight line with it but BD;
therefore BD is in the same straight line with CB.

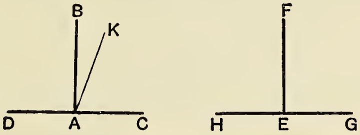
Wherefore, *if at a point, etc.* [Q.E.D.

NOTE TO I. 14.

Axiom 11, which is first used in this proposition, may be proved as follows:

Let AB be at right angles to DAC at the point A, and EF at right angles to HEG at the point E:

then shall the angles BAC *and* FEG *be equal.*



Take any length AC, and make AD, EH, EG all equal to AC. Now apply HEG to DAC, so that H may be on D, and HG on DC, and B and F on the same side of DC; then G will coincide with C, and E with A.

Also EF shall coincide with AB; for if not, suppose, if possible, that it takes a different position as AK.

Then the angle DAK is equal to the angle HEF, and the angle CAK to the angle GEF;

but the angles HEF and GEF are equal; [*Hyp. and Def.* 10.

therefore the angles DAK and CAK are equal.

But the angles DAB and CAB are also equal, [*Hyp. and Def.* 10.

and the angle CAB is greater than the angle CAK;

therefore the angle DAB is greater than the angle CAK.

Much more then is the angle DAK greater than the angle CAK.

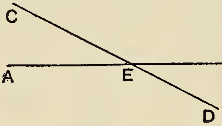
But the angle DAK was shewn to be equal to the angle CAK; which is absurd.

Therefore EF must coincide with AB; and therefore the angle FEG coincides with the angle BAC, and is equal to it.

Proposition 15. Theorem.

If two straight lines cut one another, the vertical, or opposite, angles shall be equal.

Let the two straight lines AB, CD cut one another at the point E:
the angle AEC *shall be equal to the angle* DEB,
and the angle CEB *to the angle* AED.



Proof. Because the straight line AE meets the straight line CD, the angles CEA, AED are together equal to two right angles. [I. 13.

Again, because DE meets AB, the angles AED, DEB are also together equal to two right angles. [I. 13.
Therefore the angles CEA, AED are equal to the angles AED, DEB. [*Axioms* 1 and 11.
From each of these equals take the common angle AED, and the remaining angle CEA is equal to the remaining angle DEB. [*Axiom* 3.

In the same manner it may be shewn that the angle CEB is equal to the angle AED.

Wherefore, *if two straight lines, etc.* [Q.E.D.

Corollary 1. From this it is clear that, if two straight lines cut one another, the angles which they make at the point where they cut are together equal to four right angles.

For the angles AEC, AED equal two right angles, and also the angles CEB, BED equal two right angles.

Corollary 2. It follows that all the angles made by any number of straight lines meeting at one point are together equal to four right angles.

EXERCISES ON PROPOSITION 12.

1. Prove that every point in CH, or CH produced, is equidistant from F and G.

2. Find a point in a given straight line such that its distances from two given points may be equal.

3. Find a point that shall be equidistant from three given points.

4. Through two given points on opposite sides of a given straight line draw two straight lines which shall meet in that given straight line, and include an angle bisected by that given straight line.

[Let A, B be the two points and KL the given straight line. Draw AM perpendicular to KL and produce to D, so that MD equals AM. Let DB meet KL in E; then AE, EB are the required lines.]

5. D is a given point in the base BC of a triangle ABC; find a straight line such that, if the triangle be folded along it, then the point A will coincide with D.

[The required straight line bisects AD at right angles.]

EXERCISES ON PROPOSITIONS 13-15.

1. If a right-angled triangle have one of its acute angles double the other, the hypotenuse is double the shorter side.

[If ABC be the right-angled triangle having the angle C double the angle B, produce CA to D, where CA=AD, and prove that CBD is equilateral, etc.]

2. If two isosceles triangles are on the same base, the straight line joining their vertices, or that straight line produced, will bisect the base at right angles.

3. A given angle BAC is bisected; if CA is produced to G and the adjacent angle BAG bisected, the two bisecting lines are at right angles. [These two bisecting lines are called the interior and the exterior bisectors of the angles BAC.]

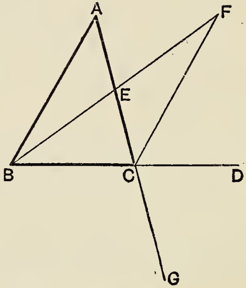
****4.** If four straight lines meet at a point so that the opposite angles are equal, these straight lines are two and two in the same straight line.

Proposition 16. Theorem.

If one side of a triangle be produced, the exterior angle shall be greater than either of the interior opposite angles.

Let ABC be a triangle, and let one side BC be produced to D:

the exterior angle ACD *shall be greater than either of the interior opposite angles* CBA, BAC.



Construction. Bisect AC at E, [I. 10.
join BE and produce it to F, making EF equal to EB, [I. 3.
and join FC.

Proof. In the triangles AEB, CEF,

because { AE = EC, [*Construction.*
and EB = EF, [*Construction.*
and the angle AEB = the angle CEF,

since they are opposite vertical angles; [I. 15.
therefore the angle BAE = the angle ECF. [I. 4.
But the angle ECD is greater than the angle ECF. [*Axiom* 8.
Therefore the angle ACD is greater than the angle BAE.

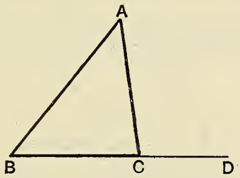
In the same manner, if BC be bisected, and the side AC be produced to G, it may be shewn that the angle BCG, that is, the angle ACD, is greater than the angle ABC. [I. 15.

Wherefore, *if one side, etc.* [Q.E.D.

PROPOSITION 17. THEOREM.

Any two angles of a triangle are together less than two right angles.

Let ABC be a triangle :
any two of its angles shall be together less than two right angles.



Construction. Produce BC to D.

Proof. Because ACD is the exterior angle of the triangle ABC, it is greater than the interior opposite angle ABC. [I. 16.
To each of these add the angle ACB.
Therefore the angles ACD, ACB are greater than the angles ABC, ACB.
But the angles ACD, ACB together = two right angles. [I. 13.
Therefore the angles ABC, ACB are together less than two right angles.

In the same manner it may be shewn that the angles BAC, ACB, as also the angles CAB, ABC, are together less than two right angles.

Wherefore, *any two angles, etc.* [Q.E.D.

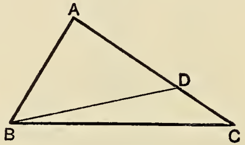
Note. Every triangle has at least two acute angles ; for if a triangle had two obtuse angles, their sum would not be less than two right angles, and this is impossible by the foregoing proposition.

PROPOSITION 18. THEOREM.

If one side of a triangle be greater than a second side, the angle opposite the first side shall be greater than the angle opposite the second.

Let ABC be a triangle, of which the side AC is greater than the side AB :

then shall the angle ABC *be also greater than the angle* ACB.



Construction. Because AC is greater than AB,
cut off AD equal to AB, [I. 3.
and join BD.

Proof. Because ADB is the exterior angle of the triangle BDC, it is greater than the interior opposite angle DCB. [I. 16.

But the angle ADB = the angle ABD, [I. 5.
because the side AD = the side AB. [*Construction.*

Therefore the angle ABD is also greater than the angle ACB.
Much more then is the angle ABC greater the angle ACB. [*Axiom* 8.

Wherefore, *the greater side, etc.* [Q.E.D.

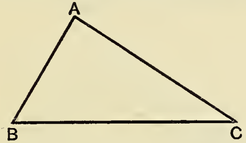
EXERCISE.

ABCD is a quadrilateral of which AD is the longest side and BC the shortest; shew that the angle ABC is greater than the angle ADC, and the angle BCD greater than the angle BAD.

PROPOSITION 19. THEOREM.

If one angle of a triangle be greater than a second angle, the side opposite the first angle shall be greater than the side opposite the second.

Let ABC be a triangle, of which the angle ABC is greater than the angle ACB:
then shall the side AC *be also greater than the side* AB.



Proof. For if not, AC must be either equal to AB or less than AB.

But AC is not equal to AB,
for then the angle ABC would be equal to the angle ACB; [I. 5.
but it is not; [*Hypothesis.*
therefore AC is not equal to AB.

Neither is AC less than AB,
for then the angle ABC would be less than the angle ACB; [I. 18.
but it is not; [*Hypothesis.*
therefore AC is not less than AB.
And it has been shewn that AC is not equal to AB.
Therefore AC is greater than AB.

Wherefore, *the greater angle, etc.* [Q. E. D.

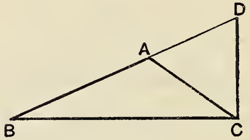
Note 1. In order to assist the student in remembering which of the two foregoing propositions is proved directly and which indirectly, it may be observed that the order is similar to that of Propositions 5 and 6.

Note 2. Simson's enunciations of Propositions 18 and 19 were: *the greater side of any triangle has the greater angle opposite it;* and *the greater angle of any triangle is subtended by the greater side, that is, has the greater side opposite to it.*

PROPOSITION 20. THEOREM.

Any two sides of a triangle are together greater than the third side.

Let ABC be a triangle:
any two sides of it are together greater than the third side,
namely, BA, AC *greater than* BC;
AB, BC *greater than* AC;
and BC, CA *greater than* AB.



Construction. Produce BA to D,
making AD equal to AC, [I. 3.
and join DC.

Proof. Because AD = AC, [*Construction.*
the angle ADC = the angle ACD. [I. 5.
But the angle BCD is greater than the angle ACD. [*Ax.* 8.
Therefore the angle BCD is greater than the angle BDC.
And because the angle BCD of the triangle BCD is greater than its angle BDC,
therefore the side BD is greater than the side BC. [I. 19.
But BD is equal to BA and AC.
Therefore BA and AC are together greater than BC.

In the same manner it may be shewn that
AB, BC are together greater than AC,
and BC, CA together greater than AB.

Wherefore, *any two sides, etc.* [Q.E.D.

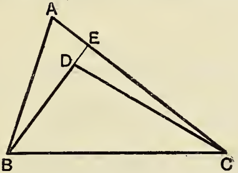
[For Exercises on Propositions 19 and 20 see page 39.]

PROPOSITION 21. THEOREM.

If from the ends of the side of a triangle there be drawn two straight lines to a point within the triangle, these shall be less than the other two sides of the triangle, but shall contain a greater angle.

Let ABC be a triangle, and from B and C, the ends of the side BC, let the two straight lines BD, CD be drawn to any point D within the triangle:

BD, DC *shall be less than the other two sides* BA, AC *of the triangle, but shall contain an angle* BDC *greater than the angle* BAC.



Construction. Produce BD to meet AC at E.

Proof. In the triangle ABE the two sides BA, AE are together greater than the side BE; [I. 20.

to each of these add EC;

∴ BA, AC are greater than BE, EC.

Again, the two sides DE, EC of the triangle DEC are greater than the third side DC; [I. 20.

to each of these add BD.

∴ BE, EC are greater than BD, DC.

But it has been shewn that BA, AC are greater than BE, EC; much more then are BA, AC greater than BD, DC.

Again, the exterior angle BDC of the triangle CDE is greater than the interior opposite angle CEB. [I. 16.

For the same reason, the exterior angle CEB of the triangle ABE is greater than the angle BAE;

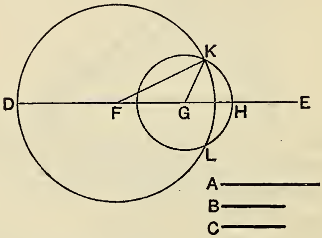
much more then is the angle BDC greater than the angle BAE, that is, BAC.

Wherefore, *if from the ends, etc.* [Q.E.D.

Proposition 22. Problem.

To make a triangle of which the sides shall be equal to three given straight lines, any two whatever of which are together greater than the third.

Let A, B, C be the three given straight lines, of which any two whatever are greater than the third; namely, A and B together greater than C; A and C together greater than B; and B and C together greater than A:
it is required to make a triangle of which the sides shall be equal to A, B, C, *each to each.*



Construction. Take a straight line DE terminated at the point D, but unlimited towards E, and make DF equal to A, FG equal to B, and GH equal to C. [I. 3.
With centre F, and radius FD, describe the circle DKL. [*Postulate* 3.
With centre G, and radius GH, describe the circle HKL, and let it cut the former circle at K.
Join KF, KG.
The triangle KFG shall be drawn as required.

Proof. Because the point F is the centre of the circle DKL,

FD = FK. [*Definition* 15.

But FD = A; [*Construction.*

∴ FK = A. [*Axiom* 1.

Again, because the point G is the centre of the circle HLK,

GH = GK. [*Definition* 15.

But GH = C; [*Construction.*

∴ GK = C. [*Axiom* 1.

Also FG = B. [*Construction.*

Therefore the three straight lines KF, FG, GK are equal to the three A, B, C.

Wherefore *the triangle* KFG *has its three sides* KF, FG, GK *equal to the three given straight lines* A, B, C. [Q.E.F.

EXERCISES ON PROPOSITION 19.

1. ABC is a triangle and the angle A is bisected by a straight line which meets BC at D; shew that BA is greater than BD, and CA greater than CD.

2. If a straight line be drawn through A one of the angular points of a square, cutting one of the opposite sides, and meeting the other produced at F, shew that AF is greater than the diagonal of the square.

3. Every straight line drawn from the vertex of a triangle to the base is less than the greater of the two sides, or than either of them if they be equal.

4. ABC is a triangle in which BA is greater than CA; the angle A is bisected by a straight line which meets BC at D; shew that BD is greater than CD.

[Take a point E on AB such that AE = AC; then the angle AED equals the angle ACB, and ED is equal to CD; ∴ the angle BED equals the exterior angle at C, and is thus greater than the angle ABC; ∴ BD is greater than DE, that is, than CD.]

****5.** The perpendicular is the shortest straight line that can be drawn from a given point to a given straight line; and, of others, that which is nearer to the perpendicular is less than the more remote; and two, and two only, straight lines, each equal to a given straight line, can be drawn from the given point to the given straight line, one on each side of the perpendicular.

EXERCISES ON PROPOSITION 20.

****1.** The difference between any two sides of a triangle is less than the third side.

****2.** The sum of the distances of any point from the three angles of a triangle is greater than half the sum of the sides of the triangle.

****3.** The two sides of a triangle are together greater than twice the straight line drawn from the vertex to the middle point of the base.

[*N.B.*—The straight line drawn from any angular point to the middle point of the opposite side is called a **Median.**]

[Let D be the middle point of the base BC of the triangle ABC. Produce AD to E where DE = AD, and join BE, EC; prove EC = AB, etc.]

4. The sum of the diagonals of a quadrilateral is less than the sum of the straight lines drawn to its angular points from any point except the intersection of its diagonals.

5. The four sides of any quadrilateral are together greater than the two diagonals together, but are less than twice the sum of the diagonals.

6. If through the ends of the base of a triangle with unequal sides lines be drawn to any point in the bisector of the vertical angle their difference is less than the difference of the sides.

[Let ABC be the △, AD the bisector of the angle A, and P any point on AD. Take a point E on AB such that AE = AC; then PC = PE. Now BP < BE, EP;

$\therefore$ BP - EP < BE, *i.e.* BP - CP < BA - AE, *i.e.* < BA - AC.]

7. If lines be drawn to any point in the bisector of the exterior angle of a triangle from the ends of its base, their sum is greater than the sum of the sides.

[Take a point E on BA produced such that AE equals AC, and proceed as in the last Example.]

EXERCISE ON PROPOSITION 21.

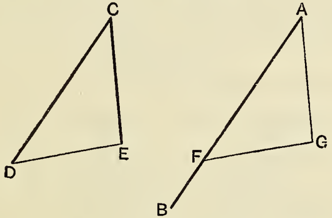
ABC is a triangle, and P is any point within it: shew that the sum of PA, PB, PC is less than the sum of the sides of the triangle.

Proposition 23. Problem.

At a given point in a given straight line, to make a rectilineal angle equal to a given rectilineal angle.

Let AB be the given straight line, and A the given point in it, and DCE the given rectilineal angle:
it is required to make at the given point A, *in the given straight line* AB, *an angle equal to the given rectilineal angle* DCE.

Construction. In CD, CE take any points D, E, and join DE.
From AB cut off AF equal to CD, and construct the triangle AFG so that the sides AF, FG, GA may be respectively equal to the sides CD, DE, EC. [I. 22.



The angle FAG shall be equal to the angle DCE.

Proof. In the triangles DCE, FAG,

because FA = DC,
and AG = CE,
and the base FG = the base DE; [Construction.

∴ the angle FAG = the angle DCE. [I. 8.

Wherefore, *at the given point* A, *in the given straight line* AB, *the angle* FAG *has been made equal to the given rectilineal angle* DCE. [Q.E.F.

[For Exercises see page 43.]

PROPOSITION 24. THEOREM.

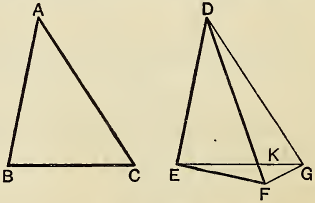
If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides, equal to them, of the other, the base of that which has the greater angle shall be greater than the base of the other.

Let ABC, DEF be two triangles, which have the side AB equal to the side DE, the side AC equal to the side DF, but the angle BAC greater than the angle EDF :
the base BC *shall be greater than the base* EF.

Of the two sides DE, DF, let DE be the side which is not greater than the other.

Construction. At the point D in the straight line DE, make the angle EDG equal to the angle BAC, [I. 23.
and make DG equal to AC or DF; [I. 3.
join EG, GF.

Let EG and DF, produced if necessary, meet in K.



Proof. Since DE is not greater than DF, that is, DG, that is, since DE is equal to, or less than, DG,
therefore the angle DGE is equal to, or less than, the angle DEG. [I. 5, 18.
But the angle DKG is greater than the angle DEG, [I. 16.
therefore the angle DKG is greater than the angle DGE;
therefore the side DG is greater than DK,
that is, DF is greater than DK.

In the triangles ABC, DEG,

because $\begin{cases} AB = DE, \\ \text{and } AC = DG, \\ \text{and the angle } BAC = \text{the angle } EDG, \end{cases}$ [*Hypothesis.* / [*Construction.* / [*Construction.*

∴ the base BC = the base EG. [I. 4.

And because DG = DF, [*Construction.*

the angle DGF = the angle DFG. [I. 5.

But the angle DGF is greater than the angle EGF; [*Axiom* 8.

∴ the angle DFG is greater than the angle EGF.

Much more then is the angle EFG greater than the angle EGF. [*Axiom* 8.

And because the angle EFG of the triangle EFG is greater than its angle EGF,

∴ the side EG is greater than the side EF. [I. 19.

But EG was shown to be equal to BC;

∴ BC is greater than EF.

Wherefore, *if two triangles, etc.* [Q.E.D.

EXERCISES ON PROPOSITION 23.

1. If one angle of a triangle is equal to the sum of the other two, the triangle can be divided into two isosceles triangles.

2. If the angle C of a triangle is equal to the sum of the angles A and B, the side AB is equal to twice the straight line joining C to the middle point of AB.

3. Construct a triangle, having given the base, one of the angles at the base, and the sum of the sides.

[If AB be the given base, make the angle BAD equal to the given ∠ and AD equal to the given sum of the sides; at B, on same side of BD as A, make ∠DBK = ∠ADB; let BK meet AD in K. Then KAB is the required △.]

4. Construct a triangle having given the base, one of the base angles, and the difference of the sides.

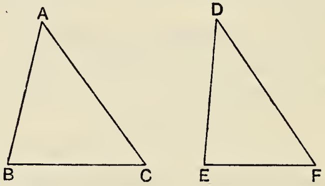
[If AB be the given base, make the angle BAD equal to the given ∠ and AD equal to the given difference; at B, on the other side of BD from A, make the ∠DBK = exterior ∠ of ADB; if BK meet AD produced in K, then KAB is the required △.]

5. A is a given point, and B is a given point in a given straight line: it is required to draw from A to the given straight line, a straight line AP, such that the sum of AP and PB may be equal to a given length.

Proposition 25. Theorem.

If two triangles have two sides of the one respectively equal to two sides of the other, but the base of the one greater than the base of the other, the angle contained by the sides of that which has the greater base shall be greater than the angle contained by the corresponding sides of the other.

Let ABC, DEF be two triangles, which have the side AB equal to the side DE, the side AC equal to the side DF, but the base BC greater than the base EF:
the angle BAC *shall be greater than the angle* EDF.



Proof. For if not, the angle BAC must be either equal to the angle EDF, or less than, the angle EDF.
Now the angle BAC does not = the angle EDF, for then
the base BC would = the base EF; [I. 4.
but it does not; [*Hypothesis.*
∴ the angle BAC does not = the angle EDF.
Neither is the angle BAC less than the angle EDF,
for then the base BC would be less than the base EF; [I. 24.
but it is not; [*Hypothesis.*
∴ the angle BAC is not less than the angle EDF.
∴ the angle BAC is not equal, or less than, the angle EDF.
∴ the angle BAC must be greater than the angle EDF.
Wherefore, *if two triangles, etc.* [Q.E.D.

Note. Proposition 25, as well as Proposition 19, are proved by the method of "Exhaustion," that is, by shewing that no other conclusion, *except the given one*, can follow.

EXERCISES.

1. ABCD is a quadrilateral having AB, CD equal, but the diagonal BD greater than the diagonal AC; prove that the angle DCB is greater than the angle ABC.

2. ABC is a triangle having AB less than AC, and G is any point in the straight line joining A to the middle point of BC; prove that G is nearer to B than to C.

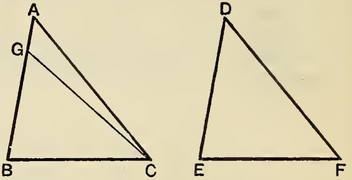
PROPOSITION 26. THEOREM.

If two triangles have two angles of the one respectively equal to two angles of the other, and one side equal to one side, namely, either the sides adjacent to the equal angles, or sides which are opposite to equal angles in each, then shall the triangles be equal in all respects, these sides being equal which are opposite to equal angles.

CASE I. When the equal sides are adjacent to the equal angles.

Let ABC, DEF be two triangles. which have the angle ABC equal to DEF, and the angle BCA equal to EFD, and the side BC equal to the side EF,

then shall the triangles be equal in all respects, so that AB = DE, AC = DF, *and the angle* BAC = *the angle* EDF.



For if AB be not equal to DE, one of them must be greater than the other. Let AB be the greater, and make BG equal to ED, [I. 3.
and join GC.

Then, in the triangles GBC, DEF,

because { GB = DE, [*Construction.*
and BC = EF,
and the angle GBC = the angle DEF, [*Hypothesis.*

∴ the triangles are equal in all respects,

and the angle GCB = the angle DFE. [I. 4.

But the angle DFE = the angle ACB. [*Hypothesis.*

∴ the angle GCB = the angle ACB, [*Axiom* 1.

the less to the greater, which is impossible.

Therefore AB is not unequal to DE, that is, it is equal to it.

Then, in the triangles ABC, DEF,

because { AB = DE, [*Proved.*
and BC = EF, [*Hypothesis.*
and the angle ABC = the angle DEF; [*Hypothesis.*

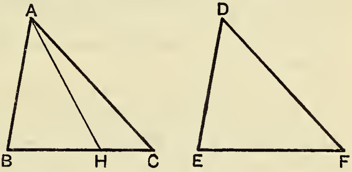
∴ the base AC = the base DF,

and the third angle BAC = the third angle EDF. [I. 4.

CASE II. When the equal sides are opposite to equal angles in each triangle.

Let ABC, DEF be two triangles having the two angles ABC, ACB respectively equal to the angles DEF, DFE, and the side AB equal to DE,

then the triangles shall be equal in all respects, so that BC = EF, AC = DF, *and also the third angle* BAC = *the third angle* EDF.



For if BC be not equal to EF, one of them must be greater than the other.

Let BC be the greater, and make BH equal to EF, [I. 3.

and join AH.

Then, in the triangles ABC, DEF,

because { AB = DE, [*Hypothesis.*
and BH = EF, [*Construction.*
and the angle ABH = the angle DEF ; [*Hypothesis.*

∴ the triangle ABH = the triangle DEF in all respects,

and therefore the angle BHA = the angle EFD. [I. 4.

But the angle EFD = the angle BCA ; [*Hypothesis.*

∴ the angle BHA = the angle BCA, [*Axiom* 1.

that is, the exterior angle BHA of the triangle AHC is equal to its interior opposite angle BCA, which is impossible. [I. 16.

Therefore BC is not unequal to EF, that is, it is equal to it.

Hence in the triangles ABC, DEF,

because { AB = DE, [*Hypothesis.*
and BC = EF, [*Proved.*
and the angle ABC = the angle DEF,

∴ the base AC = the base DF,

and the third angle BAC = the third angle EDF. [I. 4.

Wherefore, *if two triangles, etc.* [Q.E.D.

EXERCISES ON PROPOSITION 26.

**1. The perpendiculars from the ends of the base of an isosceles triangle upon the opposite sides are equal.

**2. The perpendicular from the vertex on the base of an isosceles triangle bisects both the base and the vertical angle.

**3. The perpendiculars let fall on two given straight lines AB, AC from any point in the straight line bisecting the angle between them are equal.

4. Find a point whose distances from the sides of a given triangle are equal.

5. In a given straight line find a point such that the perpendiculars drawn from it to two given straight lines which intersect shall be equal.

6. Through a given point draw a straight line such that the perpendiculars on it from two given points may be on opposite sides of it and equal to each other.

7. A straight line bisects the angle A of a triangle ABC; from B a perpendicular is drawn to this bisecting straight line, meeting it at D, and BD is produced to meet AC or AC produced at E: shew that BD is equal to DE.

8. AB, AC are any two straight lines meeting at A: through any point P draw a straight line meeting them at E and F, such that AE may be equal to AF.

9. If the diagonal AC of a quadrilateral ABCD bisect the angles at A and C, prove that it is at right angles to the other diagonal BD.

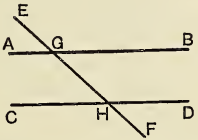
10. If the sides AB, AC of a triangle ABC be produced to D and E and the bisectors of the angles BCE, CBD meet in O, prove that the perpendiculars from O upon AD, AE, and BC are equal.

Note to I. 26. The first twenty-six Propositions constitute a distinct section of the first Book of the *Elements*. The principal results are those contained in Propositions 4, 8, 26; in each of these Propositions it is shewn that two triangles which agree in three respects agree entirely. For another case in which two triangles are equal in all respects (in addition to the cases considered in I. 4, 8, 26) the student may refer to Page 322.

The Propositions from I. 27 to I. 34 inclusive may be said to constitute the second section of the first Book. They relate to the theory of parallel straight lines.

On the Angles made by one straight Line with two other straight Lines.

When two straight lines AB and CD are met by a third straight line EF in two points G, H, names for the sake of distinction are given to the angles at G and H.



Thus EGB, AGE, CHF and FHD are called **exterior** angles;
BGH, AGH, CHG, GHD are called **interior** angles;
BGH and GHC are called **alternate** angles,
and so are also AGH and GHD;
also GHD is said to be *the interior and opposite angle of the exterior angle* EGB.

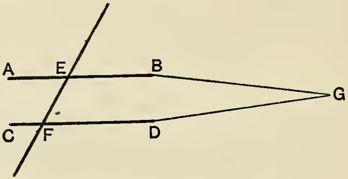
The words *interior*, *exterior*, and *alternate* are often abbreviated into *int.*, *ext.*, and *alt.*

Proposition 27. Theorem.

If a straight line, meeting two other straight lines, make the alternate angles equal to one another, the two straight lines shall be parallel to one another.

Let the straight line EF, which meets the two straight lines AB, CD, make the alternate angles AEF, EFD equal to one another:

AB *shall be parallel to* CD.



Proof. For if not, AB and CD, being produced, will meet either towards B, D or towards A, C. Let them be produced and meet towards B, D at the point G.

Therefore GEF is a triangle, and its exterior angle AEF is greater than the interior opposite angle EFG. [I. 16.
But the angle AEF also = the angle EFG; [*Hypothesis.*
which is impossible.

∴ AB and CD, being produced, do not meet towards B, D.

In the same manner it may be shewn that they do not meet towards A, C.

∴ AB is parallel to CD. [*Definition* 29.

Wherefore, *if a straight line, etc.* [Q.E.D.

Proposition 28. Theorem.

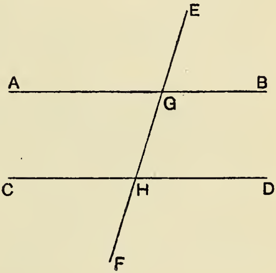
If a straight line, meeting two other straight lines, make the exterior angle equal to the interior and opposite angle on the same side of the line, or if it make the interior angles on the same side together equal to two right angles, the two straight lines shall be parallel to one another.

Let the straight line EF, which meets the two straight lines AB, CD,

(1) make the exterior angle EGB equal to the interior and opposite angle GHD on the same side;

or (2) make the interior angles on the same side BGH, GHD together equal to two right angles:

AB *shall be parallel to* CD.



Proof. (1) Because the angle EGB = the angle GHD, [*Hyp.*
and the angle EGB = the angle AGH, [I. 15.
∴ the angle AGH = the angle GHD, [*Axiom* 1.
and they are alternate;
∴ AB is parallel to CD. [I. 27.

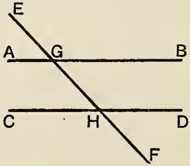
(2) Again, because the angles BGH, GHD are together equal to two right angles, [*Hypothesis.*
and the angles AGH, BGH together = two right angles, [I. 13.
∴ the angles AGH, BGH = the angles BGH, GHD. [*Axs.* 1, 11.
Take away the common angle BGH;
∴ the angle AGH = the angle GHD, [*Axiom* 3.
and they are alternate;
∴ AB is parallel to CD. [I. 27.

Wherefore, *if a straight line, etc.* [Q.E.D.

PROPOSITION 29. THEOREM.

If a straight line meet two parallel straight lines, it makes
(1) *the alternate angles equal to one another ;*
(2) *the exterior angle equal to the interior and opposite angle on the same side ;*
and also (3) *the two interior angles on the same side together equal to two right angles.*

Let the straight line EF meet the two parallel straight lines AB, CD:
(1) *then the alternate angles* AGH, GHD *shall be equal to one another*;
(2) *the exterior angle* EGB *shall be equal to the interior and opposite angle on the same side* GHD; *and*
(3) *the two interior angles on the same side* BGH, GHD *shall be together equal to two right angles.*



Proof. (1) If the angle AGH be not equal to the angle GHD, one of them must be greater than the other; let the angle AGH be the greater.
Then the angle AGH is greater than the angle GHD;
to each of them add the angle BGH;
∴ the angles AGH, BGH are greater than the angles BGH, GHD.
But the angles AGH, BGH together = two right angles; [I. 13.
∴ the angles BGH, GHD are together less than two right angles;
∴ AB, CD, if continually produced, will meet on the side of GH towards B and D. [*Axiom* 12.
But they never meet, since they are parallel by hypothesis.

∴ the angle AGH is not unequal to the angle GHD, that is, it is equal to it.

(2) Again, because the angle AGH = the angle EGB; [I. 15.
therefore the angle EGB = the angle GHD. [*Axiom* 1.

(3) Add to each of these the angle BGH.
∴ the angles EGB, BGH = the angles BGH, GHD. [*Axiom* 2.
But the angles EGB, BGH together = two right angles. [I. 13.
∴ the angles BGH, GHD together = two right angles. [*Ax.* 1.

Wherefore, *if a straight line, etc.* [Q.E.D.

[For Exercises see page 55.]

NOTE ON EUCLID'S TWELFTH AXIOM.

In I. 29 Euclid uses for the first time his twelfth axiom. The theory of parallel lines has always been considered the great difficulty of elementary geometry, and many attempts have been made to overcome this difficulty in a better way than Euclid has done. We shall not give an account of these attempts.

Speaking generally, it may be said that the methods which differ substantially from Euclid's involve, in the first place, an axiom as difficult as his, and then an intricate series of propositions; while in Euclid's method, after the axiom is once admitted, the remaining process is simple and clear.

One modification of Euclid's axiom has been proposed, which appears to diminish the difficulty of the subject. This consists in assuming, instead of Euclid's axiom, the following:

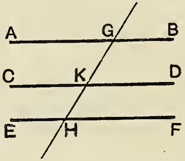
Two intersecting straight lines cannot be both parallel to a third straight line. The propositions in the *Elements* are then demonstrated as in Euclid up to I. 28, inclusive. Then, in I. 29, we proceed with Euclid up to the words, "therefore the angles BGH, GHD are together less than two right angles." We then infer that BGA and CHD must meet: because if a straight line be drawn through G so as to make the interior angles together equal to two right angles, this straight line will be parallel to CD, by I. 28; and, by our axiom, there cannot be two parallels to CD, both passing through G.

This form of making the necessary assumption has been recommended by various eminent mathematicians, among whom may be mentioned Playfair and De Morgan. It is hence known as **Playfair's Axiom.** By postponing the consideration of the axiom until it is wanted, that is, until after I. 28, and then presenting it in the form here given, the theory of parallel straight lines appears to be treated in the easiest manner that has hitherto been proposed.

PROPOSITION 30. THEOREM.

Straight lines which are parallel to the same straight lines are parallel to each other.

Let AB, CD be each of them parallel to EF:
then shall AB *be parallel to* CD.



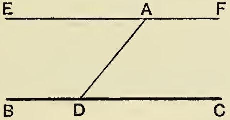
Construction. Let the straight line GKH cut AB, EF, CD.

Proof. Because GKH cuts the parallel lines AB, EF, the angle AGH = the alternate angle GHF. [I. 29.
Again, because GK cuts the parallel lines EF, CD, the exterior angle GKD = the interior opposite angle GHF. [I. 29.
∴ the angle AGK = the angle GKD, [*Axiom* 1.
and they are alternate angles;
∴ AB is parallel to CD. [I. 27.
Wherefore, *straight lines, etc.* [Q.E.D.

PROPOSITION 31. PROBLEM.

To draw a straight line through a given point parallel to a given straight line.

Let A be the given point, and BC the given straight line: *it is required to draw a straight line through the point* A *parallel to the straight line* BC.



Construction. In BC take any point D, and join AD; at the point A in the straight line AD make the angle DAE equal to the alternate angle ADC, [I. 23.

and produce EA to F.

EF shall be parallel to BC.

Proof. Because AD, which meets the two straight lines BC, EF, makes the alternate angles EAD, ADC equal. [*Constr.*

∴ EF is parallel to BC. [I. 27.

Wherefore *the straight line* EAF *is drawn through the given point* A *parallel to the given straight line* BC. [Q.E.F.

EXERCISES ON PROPOSITIONS 27–29.

1. In the figure of I. 16 prove that AB and FC are parallel.

****2.** Straight lines which are perpendicular to the same straight line are parallel.

3. Any straight line parallel to the base of an isosceles triangle makes equal angles with the sides.

4. If two straight lines A and B are respectively parallel to two others C and D, shew that the inclination of A to B is equal to that of C to D.

5. A straight line is drawn terminated by two parallel straight lines; through its middle point any straight line is drawn and terminated by the parallel straight lines. Shew that the second straight line is bisected at the middle point of the first.

****6.** If through any point equidistant from two parallel straight lines, two straight lines be drawn cutting the parallel straight lines, they will intercept equal portions of these parallel straight lines.

7. If the straight line bisecting the exterior angle of a triangle be parallel to the base, shew that the triangle is isosceles.

EXERCISES ON PROPOSITION 31.

1. Place between two parallel straight lines, and terminated by them, a straight line of given length.

2. Find a point B in a given straight line CD, such that if AB be drawn to B from a given point A, the angle ABC will be equal to a given angle.

[Draw AE parallel to CD and make the angle EAK equal to the given angle, AK being on the same side of AE as CD; AK meets CD in the required point B.]

3. Find a point such that the perpendiculars from it upon two given straight lines may be given. How many such points are there?

4. From a point D in the base BC of an isosceles triangle ABC a straight line DEF is drawn perpendicular to BC to meet the sides in E and F; prove that AEF is an isosceles triangle.

5. Through the middle point M of the base BC of a triangle a straight line DME is drawn, so as to cut off equal parts from the sides AB, AC, produced if necessary; show that BD is equal to CE.

[Through C draw CF parallel to AB to meet DE in F; prove that CF=BD, and also that CF=CE.]

6. Construct a triangle, having given the base, the altitude, and the length of the line joining the vertex to the middle point of the base.

7. Find points D, E in the sides AB, AC of a triangle ABC, such that DE is parallel to BC and equal to BD.

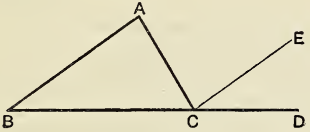
[BE bisects the angle ABC.]

8. Given the altitude and the base angles of a triangle, construct it.

Proposition 32. Theorem.

If a side of any triangle be produced,
(1) *the exterior angle is equal to the two interior and opposite angles;* and (2) *the three interior angles of the triangle are together equal to two right angles.*

Let ABC be a triangle, and let one of its sides BC be produced to D:
then shall (1) *the exterior angle* ACD *be equal to the two interior and opposite angles* CAB, ABC;
and (2) *the three interior angles* ABC, BCA, CAB *shall be together equal to two right angles.*



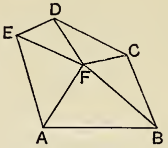
Construction. Through the point C draw CE parallel to AB. [I. 31.

Proof. (1) Because AB is parallel to CE, and AC meets them, the alternate angles BAC, ACE are equal. [I. 29.
Again, because AB is parallel to CE, and BD meets them, the exterior angle ECD is equal to the interior and opposite angle ABC. [I. 29.
∴ the whole exterior angle ACD = the two interior and opposite angles CAB, ABC. [*Axiom* 2.

(2) To each of these equals add the angle ACB;
∴ the angles ACD, ACB = the three angles CBA, BAC, ACB. [*Axiom* 2.
But the angles ACD, ACB together = two right angles; [I. 13.
∴ also the angles CBA, BAC, ACB together = two right angles. [*Axiom* 1.

Wherefore, *if a side of any triangle, etc.* [Q.E.D.

Corollary 1. *All the interior angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has sides.*



Take any rectilineal figure, and join any point F within it to its angular points.

We thus have as many triangles as there are sides to the figure.

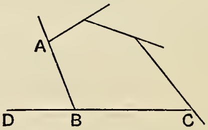
Also, by the preceding propositions, the three angles of each triangle make up two right angles.

∴ all the angles of these triangles = twice as many right angles as the figure has sides.

But all the angles of the triangles = the interior angles of the figure, together with the angles at the point F, which are equal to four right angles. [I. 15. *Cor.* 2.

∴ all the interior angles of the figure, together with four right angles, = twice as many right angles as the figure has sides.

Corollary 2. *All the exterior angles of any rectilineal figure are together equal to four right angles.*



Because every interior angle ABC, with its adjacent exterior angle ABD, is equal to two right angles; [I. 13.

∴ all the interior angles of the figure, together with all its exterior angles, = twice as many right angles as the figure has sides.

But all the interior angles of the figure, together with four right angles, = twice as many right angles as the figure has sides. [*Corollary* 1.

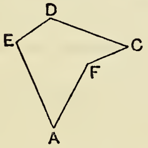
∴ all the interior angles of the figure, together with all its exterior angles, = all the interior angles of the figure, together with four right angles.

∴ all the exterior angles = four right angles.

Note on the Corollaries to I. 32.

I. 32. The corollaries to I. 32 were added by Simson. In the second corollary it ought to be stated what is meant by an *exterior* angle of a rectilineal figure. At each angular point let *one* of the sides meeting at that point be produced; then the exterior angle at that point is the angle contained between this produced part and the side which is not produced. *Either* of the sides may be produced, for the two angles which can thus be obtained are equal, by I. 15.

The rectilineal figures to which Euclid confines himself are those in which the angles all face inwards; we may here however notice another class of figures. In the accompanying diagram the angle AFC faces



outwards, and it is an angle less than two right angles; this angle however is not one of the interior angles of the figure AEDCF. We may consider the corresponding interior angle to be the excess of four right angles above the angle AFC.

An angle such as AFC, greater than two right angles, is called a **re-entrant angle.**

The *first* of the corollaries to I. 32 is true for a figure which has a re-entrant angle or re-entrant angles; but the *second* is not.

EXERCISES.

****1.** If two triangles have two angles of the one equal to two angles of the other, the third angle of the one is equal to the third angle of the other.

****2.** Each angle of an equilateral triangle is equal to two-thirds of a right angle. Hence trisect a right angle.

****3.** The sum of the angles of any quadrilateral figure is equal to four right angles.

****4.** If one angle of a triangle be equal to the sum of the other two, the triangle is right-angled; if it be less than the sum of the other two it is acute; if greater, then obtuse.

5. In an obtuse-angled triangle the side opposite the obtuse angle is the greatest side.

6. What is the magnitude of an angle of a regular (1) pentagon, (2) hexagon, and (3) octagon?

7. The bisectors of the exterior angles of a quadrilateral form a quadrilateral, the sum of whose two opposite angles is two right angles.

****8.** The straight line joining the middle point of the hypotenuse of a right-angled triangle to the right angle is equal to half the hypotenuse.

[A being the right $\angle$ of the $\triangle ABC$, make $\angle BAD = \angle ABC$, and let AD meet BC in D; then $\angle DAC = \angle DCA$, etc.]

9. From the extremities of the base of an isosceles triangle straight lines are drawn perpendicular to the sides; shew that the angles made by them with the base are each equal to half the vertical angle.

10. If the straight lines bisecting the angles at the base of an isosceles triangle be produced to meet, they will contain an angle equal to an exterior angle of the triangle.

11. ABC is a triangle, and the exterior angles at B and C are bisected by the straight lines BD, CD respectively, meeting at D; shew that the angle BDC together with half the angle BAC make up a right angle.

12. The angle between the internal bisector of one base angle and the external bisector of the other is equal to one-half the vertical angle.

13. The angle included between the bisector of the angle A of a triangle and the perpendicular from A upon the opposite side is half the difference of the base angles of the triangle.

14. The bisector of the exterior vertical angle of a triangle makes with the base an angle equal to half the difference of the base angles, and with either side an angle equal to half the sum of the base angles.

15. On the sides of any triangle ABC equilateral triangles BCD, CAE, ABF are described, all external; shew that the straight lines AD, BE, CF are all equal.

16. Through two given points draw two straight lines forming with a straight line given in position an equilateral triangle.

[Through the points draw straight lines making with the given line angles equal to two-thirds of a right angle.]

17. A is the vertex of an isosceles triangle ABC, and BA is produced to D, so that AD is equal to BA, and DC is drawn; shew that BCD is a right angle.

18. The median passing through the vertex of a triangle is equal to, greater than, or less than half of the base according as the vertical angle is a right, an acute, or an obtuse angle. [Use Ex. 4.]

19. If one angle of a triangle be triple another the triangle may be divided into two isosceles triangles.

[Let ABC be the △ where ∠BCA = 3∠ABC; at C make ∠BCD = ∠ABC, and let CD meet AB in D. Then DBC, ADC can be proved to be isosceles triangles.]

20. If one angle of a triangle be double another, an isosceles triangle may be added to it so as to form together with it a single isosceles triangle.

[Let ABC be the △ where ∠BCA = 2∠ABC; with centre A and radius AB describe a circle meeting BC produced in D; then ACD will be the △ to be added.]

21. Given two angles of a triangle and the side opposite one of them, construct the triangle.

22. Within a triangle ABC straight lines AD, BE, and CF are drawn, making with AB, BC, and CA respectively the angles DAB, EBC, FCA equal to each other. If AD, BE, and CF do not meet in one point they will form by their intersections a triangle whose angles are equal to those of the triangle ABC.

23. Through the vertex A of a triangle ABC a straight line is drawn parallel to BC. If, on either side of A, lengths AD, AE be measured off from it, each equal to AC, and CD, CE be joined, prove that DCE is a right angle.

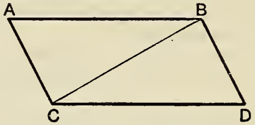
****24.** Each of the angles of a polygon of n sides, whose angles are all equal, is equal to $\frac{2n-4}{n}$ right angles.

Proposition 33. Theorem.

The straight lines which join the extremities of two equal and parallel straight lines towards the same parts are also themselves equal and parallel.

Let AB and CD be equal and parallel straight lines, and let them be joined towards the same parts by the straight lines AC and BD:

then shall AC *and* BD *be equal and parallel.*



Construction. Join BC.

Proof. Because AB is parallel to CD, [*Hypothesis.*
and BC meets them,
the alternate angles ABC, BCD are equal. [I. 29.

Then, in the triangles ABC, BCD,

because { AB = CD, [*Hypothesis.*
and BC is common,
and the angle ABC = the angle BCD; [*Proved.*

∴ the triangles are equal in all respects, so that
the base AC = the base BD,
and the angle ACB = the angle CBD. [I. 4.

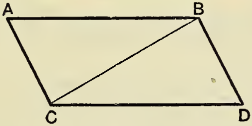
Also these are alternate angles;
∴ AC is parallel to BD, [I. 27.
and it was shewn to be equal to it.

Wherefore, *the straight lines, etc.* [Q.E.D.

PROPOSITION 34. THEOREM.

The opposite sides and angles of a parallelogram are equal to one another, and the diagonal bisects the parallelogram, that is, divides it into two equal parts.

Let ACDB be a parallelogram, of which BC is a diagonal: *then* (1) *the opposite sides and angles of the figure shall be equal to one another, and* (2) *the diagonal* BC *shall bisect it.*



Proof. (1) Because AB is parallel to CD, and BC meets them, the alternate angles ABC, BCD are equal. [I. 29.

And because AC is parallel to BD, and BC meets them, the alternate angles ACB, CBD are equal. [I. 29.

Then, in the triangles ABC, DCB,

because { the angle ABC = angle DCB,
and the angle BCA = angle CBD,
and the side BC adjacent to the equal angles in each is common to both;

∴ the triangles are equal in all respects, so that

{ AB = CD,
AC = BD,
and the angle BAC = the angle CDB. [I. 26.

And because the angle ABC = the angle BCD,
and the angle CBD = the angle ACB,
∴ the whole angle ABD = the whole angle ACD. [*Ax.* 2.

And the angle BAC has been proved equal to the angle CDB.

∴ the opposite sides and angles are equal.

(2) Also it has been proved that the triangles ABC, DCB are equal in all respects;

∴ the diagonal BC bisects the parallelogram ACDB.

Wherefore, *the opposite sides, etc.* [Q.E.D.

EXERCISES.

1. If a quadrilateral have two of its opposite sides parallel, and the two others equal but not parallel, any two of its opposite angles are together equal to two right angles.

****2.** If the opposite angles of a quadrilateral are equal it is a parallelogram. [$A+B=C+D=$ two rt. $\angle^s$, by I. 32, Cor. 2; $\therefore$ etc.]

****3.** If the opposite sides of a quadrilateral are equal it is a parallelogram.

****4.** The diagonals of a parallelogram bisect each other.

****5.** If the diagonals of a quadrilateral bisect each other it is a parallelogram.

****6.** If the straight line joining two opposite angles of a parallelogram bisect the angles the parallelogram is a rhombus.

7. Draw a straight line through a given point such that the part of it intercepted between two given parallel straight lines may be of given length. [Use Ex. 1, page 56.]

****8.** Straight lines bisecting two adjacent angles of a parallelogram intersect at right angles.

9. Straight lines bisecting two opposite angles of a parallelogram are either parallel or coincident.

****10.** If the diagonals of a parallelogram are equal all its angles are equal, and it is a rectangle.

11. Shew that any straight line passing through the middle point of the diagonal of a parallelogram and terminated by two opposite sides, bisects the parallelogram.

12. Bisect a parallelogram by a straight line drawn through any given point. [Use Ex. 11.]

****13.** The diagonals of a rhombus are at right angles.

14. A, B, C are three points in a straight line, such that AB is equal to BC; shew that the sum of the perpendiculars from A and C on any straight line which does not pass between A and C is double the perpendicular from B on the same straight line.

15. If straight lines be drawn from the angles of any parallelogram perpendicular to any straight line which is outside the parallelogram, the sum of those from one pair of opposite angles is equal to the sum of those from the other pair of opposite angles. [Use Ex. 14.]

****16.** The parallel to any side of a triangle through the middle point of another bisects the third side. [See Appendix, Art. 1.]

17. On the sides of a parallelogram ABCD four points E, F, G and H are taken, and parallels to the sides are drawn through these points to form a parallelogram PQRS. Prove that the sum of the areas ABCD and PQRS is equal to twice that of EFGH.

[Twice EFGH = twice PQRS + ||gms PB, QC, RD, SA = etc.]

18. In the base BC of an isosceles triangle ABC any point D is taken, and perpendiculars are drawn from D upon the two sides AB, AC. Prove that the sum of these two perpendiculars is equal to the perpendicular from B upon AC.

[Let DE, DF be the perpendiculars, and draw BK perpr to AC and DL perpr to BK; prove the two △s BED, DLB equal in all respects and ∴ BL = DE, etc.]

19. From a given point O draw to a given straight line AB a straight line that shall be bisected by another given straight line AC.

[Draw OP ||l to AB to meet AC in P, and PQ ||l to OA to meet AB in Q; then OQ is the required straight line, by Ex. 4.]

20. It is required to draw a straight line which shall be equal to one straight line and parallel to another, and be terminated by two given straight lines.

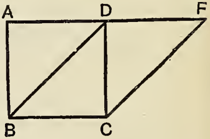
****21.** AB, AC are two given straight lines; through a given point E between them it is required to draw a straight line GEH such that the intercepted portion GH shall be bisected at the point E. [See Appendix, Art. 18.]

Note.—The Propositions from I. 35 to I. 48 may be said to constitute the third section of the first Book of the *Elements*. They relate to equality of area in figures which are not necessarily identical in form.

Proposition 35. Theorem.

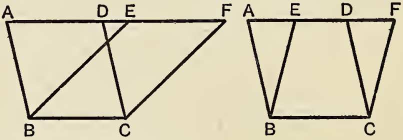
Parallelograms on the same base, and between the same parallels, are equal to one another.

Let the parallelograms ABCD, EBCF be on the same base BC, and between the same parallels AF, BC:
then shall the parallelogram ABCD *be equal to the parallelogram* EBCF.



Proof. Case I. If the sides AD, DF opposite to the base BC be terminated at the same point D, it is plain that each of the parallelograms is double of the triangle BDC; [I. 34.
and they are therefore equal. [*Ax.* 6.

Case II. But if the sides AD, EF, opposite to the base BC be not terminated at the same point, then
because ABCD is a parallelogram, AD = BC; [I. 34.



for the same reason EF = BC; ∴ AD = EF; [*Axiom* 1.
∴ the whole, or the remainder, AE = the whole, or the remainder, DF. [*Axioms* 2, 3.

Then, in the triangles EAB, FDC,

because { AB = DC,
and AE = DF,
and the exterior angle FDC = the interior opposite angle EAB; [I. 29.

∴ the △EAB = the △FDC. [I. 4.

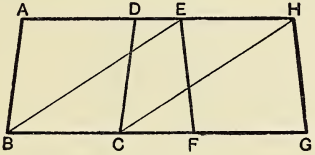
Take the △FDC from the figure ABCF, and from the same, or a similar, figure take the △EAB, and the remainders are equal; [*Axiom* 3.
that is, the parallelogram ABCD = the parallelogram EBCF.

Wherefore, *parallelograms on the same base, etc.* [Q.E.D.

Proposition 36. Theorem.

Parallelograms on equal bases, and between the same parallels, are equal to one another,

Let ABCD, EFGH be parallelograms on equal bases BC, FG, and between the same parallels AH, BG:
then shall ABCD *be equal to* EFGH.



Construction. Join BE, CH.

Proof. Because BC = FG, [*Hypothesis.*
and FG = EH, [I. 34.
∴ BC = EH, [*Axiom* 1.
and they are parallels; [*Hypothesis.*
∴ BE and CH, which join their extremities toward the same parts, are both equal and parallel. [I. 33.
∴ EBCH is a parallelogram, [*Definition.*
and it is equal to ABCD, because they are on the same base BC, and between the same parallels BC, AH. [I. 35.

Also the parallelogram EFGH is equal to the same EBCH, since they are on the same base EH, and between the same parallels EH, BG. [I. 35.
∴ the parallelogram ABCD = the parallelogram EFGH. [*Ax.* 1.

Wherefore, *parallelograms, etc.* [Q.E.D.

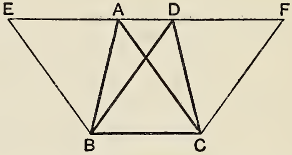
EXERCISES.

1. Construct a rectangle equal to a given parallelogram.
2. Construct a rhombus equal to a given parallelogram.
3. Divide a parallelogram into four equal parallelograms.
4. If two adjacent sides of a parallelogram be given, its area is greatest when the sides are perpendicular.

Proposition 37. Theorem.

Triangles on the same base, and between the same parallels, are equal.

Let the triangles ABC, DBC be on the same base BC, and between the same parallels AD, BC:
then shall the triangle ABC *be equal to the triangle* DBC.



Construction. Produce AD both ways to the points E, F; through B draw BE parallel to CA, and through C draw CF parallel to BD. [I. 31.

Proof. The parallelograms EBCA and DBCF are equal, because they are on the same base BC, and between the same parallels BC, EF. [I. 35.
Also the △ ABC is half of the parallelogram EBCA, because the diagonal AB bisects it, [I. 34.
and the △ DBC is half of the parallelogram DBCF, because the diagonal DC bisects it; [I. 34.
∴ the △ ABC = the △ DBC. [*Axiom* 7.
Wherefore, *triangles, etc.* [Q.E.D.

EXERCISES.

1. PQR is a straight line parallel and equal to the base BC of a triangle ABC and meets the sides in P and Q. Prove that the triangles BPQ, AQR are equal.

[Draw QS parallel to AB to meet BC in S; then QR = SC; ∴ RC is parallel to QS or AB; ∴ △ˢ AQR, RQC = △ ARC = $\frac{1}{2}$BR = $\frac{1}{2}$BQ + $\frac{1}{2}$SR = △ˢ BPQ, RQC, etc.

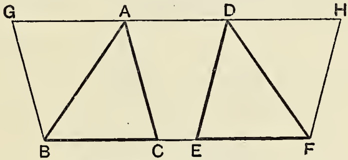
2. Describe a triangle equal to a given quadrilateral ABCD.

[Draw AE parallel to DB to meet BC produced in E; then DEC is the △ required.]

PROPOSITION 38. THEOREM.

Triangles on equal bases, and between the same parallels, are equal to one another.

Let the triangles ABC, DEF be on equal bases BC, EF, and between the same parallels BF, AD:
then shall the triangle ABC *be equal to the triangle* DEF.



Construction. Produce AD both ways to the points G, H; through B draw BG parallel to CA, and through F draw FH parallel to ED. [I. 31.

Proof. Each of the figures GBCA, DEFH is a parallelogram, [*Definition.*
and they are equal because they are on equal bases BC, EF, and between the same parallels BF, GH. [I. 36.
Also the △ABC is half of the parallelogram GBCA, because the diagonal AB bisects it; [I. 34.
and the △DEF is half of the parallelogram DEFH, because the diagonal DF bisects it;
∴ the △ABC = the △DEF. [*Axiom* 7.

Wherefore, *triangles, etc.* [Q.E.D.

EXERCISES.

**1. A triangle is bisected by either of its medians.

2. ABC is a triangle and E any point in the median AD; prove that the triangles ABE, ACE are equal.

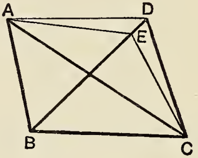
**3. If two triangles have two sides of the one equal to two sides of the other, and the contained angles supplementary, the triangles are equal in area.
[For they can be placed so that they have one equal side of each coincident, and the other two equal sides in the same straight line. Then apply Ex. 1.]

**4. Prove that the four triangles into which a parallelogram is divided by its diagonals are equal in area. [Use Ex. 4, page 64.]

Proposition 39. Theorem.

Equal triangles on the same base, and on the same side of it, are between the same parallels.

Let the equal triangles ABC, DBC be on the same base BC, and on the same side of it:
they shall be between the same parallels.



Construction. Join AD.
AD shall be parallel to BC.

For if it is not, let AE be parallel to BC, and let it meet BD at E, [I. 31.
and join EC.

Proof. The △ABC = the △EBC, because they are on the same base BC, and between the same parallels BC, AE. [I. 37.

But the △ABC = the △DBC; [*Hypothesis.*
∴ also the △DBC = the △EBC, [*Axiom* 1.
the greater to the less, which is impossible;
∴ AE is not parallel to BC.
In the same manner it can be shewn that no other straight line through A except AD is parallel to BC;
∴ AD is parallel to BC.

Wherefore, *equal triangles, etc.* [Q.E.D.

[For Exercises see page 72.]

PROPOSITION 40. THEOREM.

Equal triangles, on equal bases, in the same straight line, and on the same side of it, are between the same parallels.

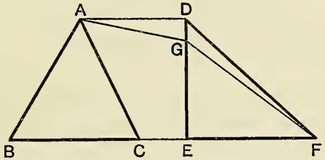
Let the equal triangles ABC, DEF be on equal bases BC, EF, in the same straight line BF, and on the same side of it: *they shall be between the same parallels.*

Construction. Join AD.

AD shall be parallel to BF.

For if it is not, let AG be drawn parallel to BF, and let it meet ED at G, [I. 31.

and join GF.



Proof. The △ABC = the △GEF, because they are on equal bases BC, EF, and between the same parallels. [I. 38.

But the △ABC = the △DEF; [*Hypothesis.*

∴ also the △DEF = the △GEF, [*Axiom* 1.

the greater to the less, which is impossible.

∴ AG is not parallel to BF.

In the same manner it can be shewn that no other straight line through A except AD is parallel to BF;

∴ AD is parallel to BF.

Wherefore, *equal triangles, etc.* [Q.E.D.

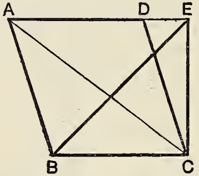
[For Exercises see page 73.]

Proposition 41. Theorem.

If a parallelogram and a triangle be on the same base and between the same parallels, the parallelogram shall be double of the triangle.

Let the parallelogram ABCD and the triangle EBC be on the same base BC, and between the same parallels BC, AE: *the parallelogram* ABCD *shall be double of the triangle* EBC.

Construction. Join AC.



Proof. The △ABC = the △EBC, because they are on the same base BC, and between the same parallels BC, AE. [I. 37.
But the parallelogram ABCD is double of the △ABC, because the diagonal AC bisects the parallelogram. [I. 34.
∴ the parallelogram ABCD is also double of the △EBC.

Wherefore, *if a parallelogram, etc.* [Q.E.D.

EXERCISES ON PROPOSITION 39.

**1. Two straight lines AB and CD intersect at E, and the triangle AEC is equal to the triangle BED: shew that BC is parallel to AD.

**2. The straight line which joins the middle points of two sides of any triangle is parallel to the base and is equal to half the base. [See Appendix, Art. 1.]

**3. Straight lines joining the middle points of adjacent sides of a quadrilateral form a parallelogram. [Use Ex. 2.]

4. In the base AC of a triangle take any point D; bisect AD, DC, AB, BC at the points E, F, G, H respectively: shew that EG is equal and parallel to FH. [Use Ex. 2.]

5. Two triangles of equal area stand on the same base and on opposite sides: shew that the straight line joining their vertices is bisected by the base or the base produced. [See App., Art. 2.]

6. If a quadrilateral figure be bisected by one diagonal the second diagonal is bisected by the first. [Use Ex. 5.]

7. Any quadrilateral figure which is bisected by both of its diagonals is a parallelogram.

EXERCISES ON PROPOSITION 40.

1. A quadrilateral is divided into four triangles by its diagonals; prove that if two adjacent triangles be equal the other two triangles will also be equal.

2. The straight lines AD, BE bisecting the sides BC, AC of a triangle intersect at G: shew that AG is double of GD. [See App., Art. 7.]

EXERCISES ON PROPOSITION 41.

1. ABCD is a quadrilateral having BC parallel to AD; shew that its area is the same as that of the parallelogram which can be formed by drawing through the middle point of DC a straight line parallel to AB to meet AD and BC.

2. ABCD is a quadrilateral having BC parallel to AD, E is the middle point of DC; shew that the triangle AEB is half the quadrilateral.

3. If any point be taken within a parallelogram the sum of the triangles formed by joining the point with the extremities of a pair of opposite sides is half the parallelogram.

[Through the point draw a straight line parallel to the two opposite sides to meet the other sides.]

4. ABCD is a parallelogram; from any point P in the diagonal BD the straight lines PA, PC are drawn. Shew that the triangles PAB and PCB are equal in area.

[By Ex. 3, $\triangle BPC + \triangle APD = \frac{1}{2}ABCD = \triangle ABD = \triangle APB + \triangle APD$.]

****5.** If the middle points of any two sides of a triangle be joined, the triangle so cut off is one quarter of the whole.

6. On the same side of the straight line ABC equal rectangles ABDE, ACFG are described. Prove that BG and DF are parallel.

[$\triangle GBF = \frac{1}{2}$ rect. $AF = \frac{1}{2}$ rect. $AD = \triangle GDB$; $\therefore$ etc.]

****7.** If two sides of a triangle be given the area of the triangle is greatest when the angle between them is a right angle.

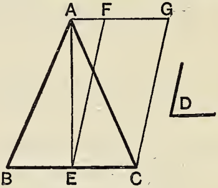
8. Through the vertices of a quadrilateral straight lines are drawn parallel to the diagonals; prove that the figure thus obtained is a parallelogram whose area is twice that of the quadrilateral.

Proposition 42. Problem.

To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let ABC be the given triangle, and D the given rectilineal angle:

it is required to describe a parallelogram that shall be equal to the given triangle ABC, *and have one of its angles equal to* D.



Construction. Bisect BC at E; [I. 10.

join AE, and at the point E, in the straight line EC, make the angle CEF equal to D; [I. 23.

through A draw AFG parallel to EC, and through C draw CG parallel to EF. [I. 31.

Then FECG is the parallelogram constructed as required.

Proof. The △ ABE = the △ AEC, because they are on equal bases BE, EC, and between the same parallels BC, AG. [I. 38.

∴ the △ ABC is double of the △ AEC.

But the parallelogram FECG is also double of the △ AEC, because they are on the same base EC, and between the same parallels EC, AG. [I. 41.

∴ the parallelogram FECG = the △ ABC, [*Axiom* 6.

and its angle CEF is equal to the given angle D. [*Construction.*

Wherefore *a parallelogram* FECG *has been described equal to the given triangle* ABC, *and having one of its angles* CEF *equal to the given angle* D. [Q.E.F.

EXERCISE.

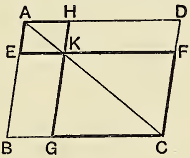
Construct a triangle equal to a given parallelogram, and having an angle equal to a given angle.

PROPOSITION 43. THEOREM.

The complements of the parallelograms which are about the diagonal of any parallelogram are equal.

Let ABCD be a parallelogram, of which the diagonal is AC; and EH, GF parallelograms about AC, that is, through which AC passes; and BK, KD the other parallelograms which make up the whole figure ABCD, and which are therefore called the complements:

the complement BK *shall be equal to the complement* KD.



Proof. Because AEKH is a parallelogram, and AK its diagonal, the △ AEK = the △ AHK. [I. 34.

For the same reason the △ KGC = the △ KFC;

∴ the two triangles AEK, KGC together = the two triangles AHK, KFC. [*Axiom* 2.

But the whole △ ABC = the whole △ ADC, because the diagonal AC bisects the parallelogram ABCD; [I. 34.

∴ the remainder, the complement BK, is equal to the remainder, the complement KD. [*Axiom* 3.

Wherefore, *the complements, etc.* [Q.E.D.

EXERCISES.

1. Each of the parallelograms about the diagonal of a rhombus is a rhombus.

In the figure of I. 43 prove that

2. EH, DB, GF are parallel.

[△ BED = △ BCE = $\frac{1}{2}$BF = $\frac{1}{2}$CH = △ HCD = △ BHD, etc.]

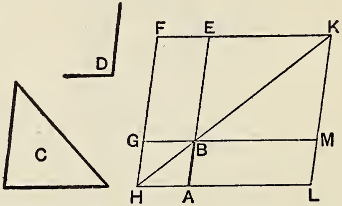
3. The triangle EKC is half of the complement BK.

4. The triangles EKC, HKC are equal.

PROPOSITION 44. PROBLEM.

To a given straight line to apply a parallelogram, which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let AB be the given straight line, C the given triangle, and D the given rectilineal angle:
it is required to apply to AB *a parallelogram equal to the triangle* C, *and having an angle equal to* D.



Construction. Make the parallelogram BEFG equal to the △ C, and having the angle EBG equal to the angle D, so that BE may be in the same straight line with AB; [I. 42.
produce FG to H;
through A draw AH parallel to BG or EF, [I. 31.
and join HB.

Because the straight line HF meets the parallels AH, EF, the angles AHF, HFE are together equal to two right angles; [I. 29.

Therefore the angles BHF, HFE are together less than two right angles;
∴ HB and FE will meet towards B and E, if produced far enough; [*Axiom* 12.
let them meet at K.

Through K draw KL parallel to EA or FH; [I. 31.
and produce HA, GB to meet KL in the points L, M.

Proof. HLKF is a parallelogram, of which the diagonal is HK; and AG, ME are parallelograms about HK; and LB, BF are the complements;

∴ LB = BF; [I. 43.

But BF = the △ C; [*Construction.*

∴ LB = the △ C. [*Axiom* 1.

And because the angle GBE = the angle ABM, [I. 15.

and likewise = the angle D; [*Construction.*

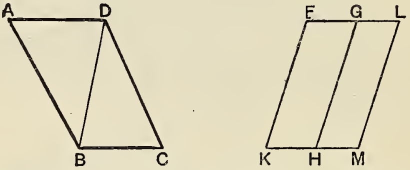
∴ the angle ABM = the angle D. [*Axiom* 1.

Wherefore *to the given straight line* AB *the parallelogram* LB *is applied, equal to the triangle* C, *and having the angle* ABM *equal to the angle* D. [Q. E. F.

Proposition 45. Problem.

To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.

Let ABCD be the given rectilineal figure, and E the given rectilineal angle:
it is required to describe a parallelogram equal to ABCD, *and having an angle equal to* E.



Construction. Join DB, and describe the parallelogram FH equal to the △ADB, and having the angle FKH equal to the angle E; [I. 42.
and to the straight line GH apply the parallelogram GM equal to the △DBC, and having the angle GHM equal to the angle E. [I. 44.
The figure FKML shall be the parallelogram required.

Proof. Because the angle E = each of the angles FKH, GHM, *Construction.*
the angle FKH is equal to the angle GHM. [*Axiom* 1.
Add to each of these equals the angle KHG;
∴ the angles FKH, KHG are equal to the angles KHG, GHM, [*Axiom* 2.
But FKH, KHG together = two right angles; [I. 29.
∴ KHG, GHM together = two right angles.
∴ KH is the same straight line with HM. [I. 14.

And because HG meets the parallels KM, FG, the alternate angles MHG, HGF are equal. [I. 29.
Add to each of these the angle HGL;

∴ the angles MHG, HGL together = the angles HGF, HGL. [*Axiom* 2.

But MHG, HGL together = two right angles; [I. 29.

∴ HGF, HGL together = two right angles;

∴ FG, GL are in the same straight line. [I. 14.

And because KF is parallel to HG, and HG to ML, [*Constr.*

∴ KF is parallel to ML, [I. 30.

and KM, FL are parallels; [*Construction.*

∴ KFLM is a parallelogram. [*Definition.*

And because the △ ABD = the parallelogram HF, [*Constr.*

and the △ DBC = the parallelogram GM, [*Constr.*

∴ the whole rectilineal figure ABCD = the whole parallelogram KFLM. [*Axiom* 2.

Wherefore *the parallelogram* KFLM *has been described equal to the given rectilineal figure* ABCD, *and having the angle* FKM *equal to the given angle* E. [Q.E.F.

Corollary. From this it is clear how, to a given straight line, to apply a parallelogram, which shall have an angle equal to a given rectilineal angle, and shall be equal to a given rectilineal figure, namely, by applying to the given straight line a parallelogram equal to the first triangle ABD, and having an angle equal to the given angle; and so on. [I. 44.

EXERCISE.

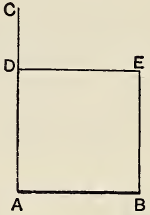
Describe a triangle equal to a given rectilineal figure.

[Let ABCDE be the given figure; join BD, AD and draw CP, EQ parallel to DB, DA respectively to meet AB produced if necessary in P and Q; then △ DCB = △ DPB and △ DEA = △ DQA [I. 37], and ∴ whole figure = △ DQP. Similarly for a figure with a larger number of sides.]

Proposition 46. Problem.

To describe a square on a given straight line.

Let AB be the given straight line:
it is required to describe a square on AB.



Construction. From the point A draw AC at right angles to AB, [I. 11.
and make AD equal to AB; [I. 3.
through D draw DE parallel to AB, and through B draw BE parallel to AD. [I. 31.
ADEB shall be a square.

Proof. ADEB is by construction a parallelogram;

AB = DE, and AD = BE. [I. 34.

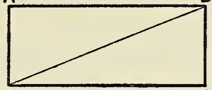
But AB = AD. [*Construction.*

∴ BA, AD, DE, EB are all equal, and the parallelogram ADEB is therefore equilateral. [*Axiom* 1.
Also the angle BAD is a right angle; [*Construction.*
∴ the figure ADEB is equilateral, and it has one angle a right angle.

Therefore *it is a square,* [*Definition* 30.
and it is described on the given straight line AB. [Q.E.F.

Note to Proposition 46.

It can be proved that squares and also rectangles have all their angles right angles.



Let ABCD be a rectangle having A a right angle. Join BD.
Then in the triangles ABD, CBD we have AB=CD, AD=BC, and the base BD common;
∴ the angle BCD=the angle BAD=a right angle,
and the angle ABD=the angle BDC, so that AB, CD are parallel,
and the angle ADB=the angle DBC, so that AD, BC are parallel.

Since AD, BC are parallel, the angles DAB, ABC are equal to two right angles, of which DAB is a right angle;
∴ the angle ABC is a right angle.

Similarly ADC is a right angle.

EXERCISES.

1. Prove that the sides of two equal squares must be equal.

2. The square on a given straight line is four times the square on half the line.

3. If, in the sides of AB, BC, CD, DA of a square ABCD points E, F, G, H be taken so that AE, BF, CG, DH are equal, then EFGH is a square.

4. If the diagonals of a quadrilateral are equal and bisect each other at right angles, the quadrilateral is a square.

5. On the sides AC, BC of a triangle ABC, squares ACDE, BCFH are described: shew that the straight lines AF and BD are equal.

6. Squares are described on the three sides of any triangle, the squares being all outside the triangle, and adjacent corners of the squares are joined; shew that the triangles so formed are each equal in area to the original triangle.

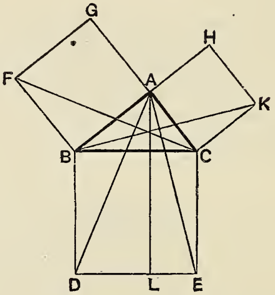
[Let ABC be the △, and BDEC, CFGA, AHKB the squares on its sides; since BCE, ACF are right ∠s, ∴ ECF, ACB are supplementary angles [I. 15, Cor. 2]; ∴ ACB, ECF are two △s with two sides equal and the included ∠s supplementary. Now use Ex. 3 on Prop. 38.]

PROPOSITION 47. THEOREM.

In any right-angled triangle the square which is described on the side subtending the right angle is equal to the squares described on the sides which contain the right angle.

Let ABC be a right-angled triangle, having the right angle BAC;

the square described on the side BC *shall be equal to the squares described on the sides* BA, AC.



Construction. On BC describe the square BDEC, and on BA, AC describe the squares BFGA, AHKC; [I. 46.
through A draw AL parallel to BD or CE, [I. 31.
and join AD, FC.

Proof. Because the angle BAC is a right angle, [*Hypothesis.*
and that the angle BAG is also a right angle; [*Definition* 31.
∴ CA is in the same straight line with AG. [I. 14.
Similarly, AB and AH are in the same straight line.
Now the angle DBC = the angle FBA,
for each of them is a right angle; [*Axiom* 11.
add to each the angle ABC.
∴ the whole angle DBA = the whole angle FBC [*Axiom* 2.

Then, in the triangles DBA, FBC,

because { AB = FB, [*Constr.*
and BD = BC, [*Constr.*
and the angle ABD = the angle FBC, [*Proved.*

∴ the △ ABD = the △ FBC. [I. 4.

Now the parallelogram BL is double of the △ ABD, because they are on the same base BD, and between the same parallels BD, AL. [I. 41.

And the square GB is double of the △ FBC, because they are on the same base FB, and between the same parallels FB, GC. [I. 41.

But the doubles of equals are equal. [*Axiom* 6.

∴ the parallelogram BL = the square GB.

Similarly, by joining AE, BK, it can be shewn that the parallelogram CL = the square CH.

∴ the whole square BDEC = the two squares GB, HC. [*Ax.* 2.

And the square BDEC is described on BC, and the squares GB, HC on BA, AC.

∴ the square described on the side BC

= the squares described on the sides BA, AC.

Wherefore, *in any right-angled triangle, etc.* [Q.E.D.

NOTE ON I. 47.

Tradition ascribed the discovery of I. 47 to Pythagoras, who flourished about 570 to 500 B.C. Many demonstrations have been given of this celebrated proposition; the following is one of the most interesting:

Let ABCD, AEFG be any two squares, placed so that their bases may join and form one straight line. Take GH and EK each equal to AB, and join HC, CK, KF, FH.

Since GH = AB, ∴ HB = GA = FE = FG.

Since EK = AD, ∴ DK = AE = FG = HB.

∴ the △s FGH, FEK, HBC, KDC are all equal in all respects.

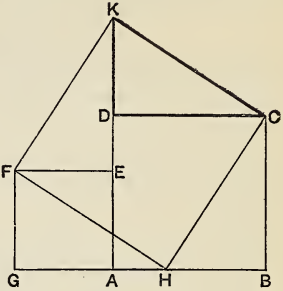
∴ the two figures AEFG, ADCB together = the fig. FHCK.

Also, since the above four △s are equal,

∴ CH, FH, FK, KC are all equal, and ∠KCD = ∠HCB and ∴ ∠HCK = ∠BCD = a rt. ∠.

∴ HCKF is a square.

Also the side CH is the hypotenuse of a right-angled triangle of which the sides CB, BH are equal to the sides of the two given squares.



This demonstration requires no proposition of Euclid after I. 32, and it shews how two given squares may be cut into pieces which will fit together so as to form a third square. *Quarterly Journal of Mathematics*, Vol. I.

It will be noted that if the △ CHB be conceived as turning round the point C as a pivot it may be rotated into the position CKD. Similarly, the △ FGH may be rotated round F into the position FED.

EXERCISES.

****1.** The square described on the diagonal of a given square is twice the given square.

2. Construct a square equal to half a given square.

[Let sq. be on given str. line AB. Make ∠ABC = ∠BAC = half a rt. ∠, so that ACB = a rt. ∠ and AC = BC.

∴ sq. on AB = twice sq. on AC, etc.] [I. 48.

3. Construct a line the square on which shall be equal to the sum of three given squares.

****4.** ABC is an equilateral triangle and AD is drawn perpendicular to BC; prove that the square on AD is three times the square on BD.

5. If two opposite sides of a quadrilateral be at right angles, the sum of the squares on the diagonals is equal to the sum of the squares on the other two sides.

6. The sum of the squares on the sides of a rhombus is equal to the sum of the squares on its diagonals.

7. If ABC be a triangle whose angle A is a right angle, and BE, CF be drawn bisecting the opposite sides respectively, shew that four times the sum of the squares on BE and CF is equal to five times the square on BC.

****8.** The square on the side subtending an acute angle of a triangle is less than the squares on the sides containing the acute angle.

[Let ABC be the △, C being acute; make BCD a rt. ∠ and CD = CA. Then, by I. 24, $BA < BD$ ∴ $BA^2 < BD^2$, *i.e.* $< BC^2 + CD^2$, *i.e.* $< BC^2 + CA^2$.]

****9.** The square on the side subtending an obtuse angle of a triangle is greater than the squares on the sides containing the obtuse angle.

****10.** If the square on one side of a triangle be less than the squares on the other two sides, the angle contained by these sides is an acute angle; if greater, an obtuse angle.

In the figure of I. 47, prove that

11. AD and FC are at right angles.

[Let AD meet FC in O and BC in V.
Then ∠AOC = ∠OVC + ∠OCV = ∠BVD + ∠BDV = a rt. ∠.]

12. F, A, K are in a straight line.

13. FG, HK, and AL meet in a point.

[Let FG meet KH in U; then △ˢ AHU, CAB are equal in all respects, so that ∠HAU = ∠ACB = complement of ∠ABC = ∠BAL, etc.]

14. BG, CH are parallel.

15. If DM, EN are drawn perpendicular to AC, AB, then AM equals AB, and AN equals AC.

[Draw DT perpʳ to AB produced; then DTAM is a rectangle and ∴ AM = DT. It can then be proved that △ˢ DTB, BAC are equal in all respects, and ∴ DT = AB.]

16. Divide a straight line into two parts the sum of the squares on which is equal to a given square. What limit is there to the size of this given square?

[AB being given, make ∠ABC = half a rt. ∠. With centre A and radius = side of given square describe a circle to meet BC in P. Draw PM perpʳ to AB. Then AB is divided as required in M.]

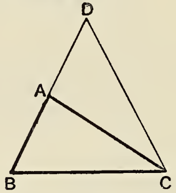
17. In a straight line AB, produced if necessary, find a point D such that the difference of the squares on AD, BD may be equal to a given square. [See App. Art. 17.]

18. Shew that a right-angled triangle can be made whose sides are proportional to the numbers 3, 4, and 5.

PROPOSITION 48. THEOREM.

If the square described on one of the sides of a triangle be equal to the squares described on the other two sides of it, the angle contained by these two sides is a right angle.

Let the square described on BC, one of the sides of the triangle ABC, be equal to the squares described on the other sides BA, AC:
the angle BAC *shall be a right angle.*



Construction. From the point A draw AD at right angles to AC; [I. 11.
make AD equal to BA, and join DC.

Proof. Because DA = BA, the square on DA = the square on BA. To each of these add the square on AC.

∴ the squares on DA, AC = the squares on BA, AC. [*Axiom* 2.
But because the angle DAC is a right angle, [*Construction.*
the square on DC = the squares on DA, AC; [I. 47.
and the square on BC = the squares on BA, AC; [*Hypothesis.*
∴ the square on DC = the square on BC; [*Axiom* 1.
∴ DC = BC.

Then, in the triangles BAC, DAC,

because { BA = AD, [*Construction.*
and AC is common,
and the base BC = the base CD;

∴ the angle DAC = the angle BAC. [I. 8.
But DAC is a right angle; [*Construction.*
∴ also BAC is a right angle. [*Axiom* 1.

Wherefore, *if the square, etc.* [Q.E.D.

BOOK II.

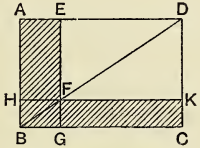
DEFINITIONS.

1. Every right-angled parallelogram, or rectangle, is said to be contained by any two of the straight lines which contain one of the right angles.

Thus the rectangle ABCD is said to be *contained* by the straight lines AB and AD, or by the straight lines BA, BC, etc.

We shall use the abbreviations "the rectangle AB, BC," or "the rect. AB, BC" for the expression "the rectangle contained by AB, BC."

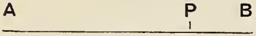
2. In every parallelogram, any of the parallelograms about a diameter, together with the two complements, is called a Gnomon.



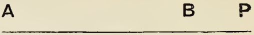
Thus the parallelogram HG, together with the complements AF, FC, is the gnomon, which is more briefly expressed by the letters AGK, or EHC, which are at the opposite angles of the parallelograms which make the gnomon.

Similarly, the figure consisting of EK together with the complements AF, FC (that is, all the figure except HG) is called the gnomon AKG.

3. When a straight line is divided into two parts, each part is called a segment by Euclid. It is found convenient to extend the meaning of the word *segment*, and to lay down the following definition: When a point P is taken in a straight line AB, or in the straight line AB produced, its distances from the ends of the straight line are called segments of the straight line. When it is necessary to distinguish them, such segments are called **internal** or **external**, according as the point is in the straight line, or in the straight line produced.



Case I.



Case II.

Note.

There is an analogy between the first ten propositions of this book and some elementary facts in Arithmetic and Algebra.

It is shewn in Arithmetic that if one side of a rectangle contains a unit of length an exact number of times, and if an adjacent side also contains the same unit an exact number of times, the product of these units will be the number of square units in the area of the rectangle.

Thus, if the sides be respectively m inches and n inches, the area is mn square inches.

Similarly, if a square have each of its sides equal to m units, its area is m^2 square units.

We thus see that the area of a rectangle in geometry corresponds to a product of two numbers in Arithmetic or Algebra; whilst the area of a square corresponds to a square of a number.

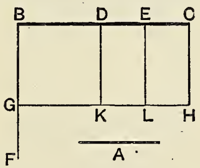
We shall add to these ten propositions the corresponding algebraic formulae. By means of the latter the student is enabled to more easily keep in his memory the results of the Propositions. The proofs in the Propositions are, however, more general than those of the formulae, since in Geometry it is not assumed that the lines spoken of are *commensurable*. We do not enter on this subject as it would lead us too far from Euclid's *Elements of Geometry* with which we are here occupied.

Owing to the above analogy the expression "the square on AB" is often abbreviated into "AB^2," and "the rectangle AB, BC" into "AB . BC." We shall sometimes use these abbreviations in the course of this book, and the student may use them in Exercises. We shall not use them in the text of the Propositions, nor should the student do so in writing out the Proposition in an Examination. The signs $+$ and $-$ may be used in Deductions. But the student must always carefully note that such an expression as "$AB^2 + BC . CD$" is only an *abbreviation* for "the square on AB together with the rectangle BC, CD."

PROPOSITION 1. THEOREM.

If there be two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the rectangles contained by the undivided line, and the several parts of the divided line.

Let A and BC be two straight lines; and let BC be divided into any number of parts at the points D, E:
the rectangle contained by A, BC *shall be equal to the sum of the rectangles contained by* A, BD, *by* A, DE, *and by* A, EC.



Construction. From B draw BF at right angles to BC, [I. 11.
and make BG equal to A; [I. 3.
through G draw GH parallel to BC; and through D, E, C draw DK, EL, CH, parallel to BG. [I. 31.

Proof. The rectangle BH = the sum of the rectangles BK, DL, EH.
But BH is contained by A, BC, for it is contained by GB, BC, and GB = A. [*Construction.*
And BK is contained by A, BD, for it is contained by GB, BD, and GB = A;
and DL is contained by A, DE, because DK is equal to BG, which is equal to A; [I. 34.
and in like manner EH is contained by A, EC;
∴ the rectangle contained by A, BC = the sum of the rectangles contained by A, BD, and by A, DE, and by A, EC.

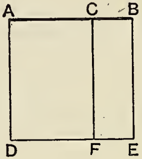
Wherefore, *if there be two straight lines, etc.* [Q.E.D.

Algebraic Formula. $a(b+c+d) = ab + ac + ad.$

Proposition 2. Theorem.

If a straight line be divided into any two parts, the rectangles contained by the whole and each of the parts are together equal to the square on the whole line.

Let the straight line AB be divided into any two parts at the point C:
the rectangle AB, BC, *together with the rectangle* AB, AC, *shall be equal to the square on* AB.



Construction On AB describe the square ADEB; [I. 46.
and through C draw CF parallel to AD or BE. [I. 31.

Proof. AE is equal to the rectangles AF, CE.
But AE is the square on AB.
Also AF is the rectangle contained by BA, AC;
for it is contained by DA, AC, of which DA = BA;
and CE is contained by AB, BC, for BE = AB;
∴ the rectangle AB, AC, together with the rectangle AB, BC, = the square on AB.

Wherefore, *if a straight line, etc.* [Q.E.D.

Algebraic Formula. Let AC be a units, and CB be b units; then

$$a(a+b)+b(a+b)=(a+b)^2$$

EXERCISE.

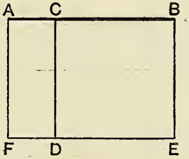
If a straight line be divided internally into any number of segments, the square on the straight line is equal to the sum of the rectangles contained by the straight line and the several segments.

PROPOSITION 3. THEOREM.

If a straight line be divided into any two parts, the rectangle contained by the whole and one of the parts is equal to the square on that part, together with the rectangle contained by the two parts.

Let the straight line AB be divided into any two parts at the point C:

the rectangle AB, BC *shall be equal to the square on* BC, *together with the rectangle* AC, CB.



Construction. On BC describe the square CDEB; [I. 46.
produce ED to F, and through A draw AF parallel to CD or BE. [I. 31.

Proof. The rectangle AE = the rectangles AD, CE.
But AE is the rectangle contained by AB, BC; for it is contained by AB, BE, of which BE = BC;
and AD is contained by AC, CB, since CD = CB;
and CE is the square on BC;

the rectangle AB, BC = the square on BC, together with the rectangle AC, CB.

Wherefore, *if a straight line, etc.* [Q. E. D.

Algebraic Formula. Let AC be a units, and CB be b units; then

$$(a+b)b = ab + b^2.$$

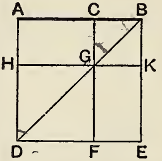
EXERCISE.

A, B, C, D are four points in a straight line taken in order; prove that the rectangle AC, BD is equal to the sum of the rectangles AB, CD and AD, BC.

Proposition 4. Theorem.

If a straight line be divided into any two parts, the square on the whole line is equal to the squares on the two parts, together with twice the rectangle contained by the two parts.

Let the straight line AB be divided into any two parts at the point C:
the square on AB *shall be equal to the squares on* AC, CB, *together with twice the rectangle* AC, CB.



Construction. On AB describe the square ADEB; [I. 46.
join BD; through C draw CGF parallel to AD or BE, and through G draw HGK parallel to AB or DE. [I. 31.

Proof. Because CF is parallel to AD, and BD meets them, the exterior angle CGB = the interior and opposite angle ADB; [I. 29.

but the angle ADB = the angle ABD, [I. 5.

because BA = AD, being sides of a square,

∴ the angle CGB = the angle CBG; [*Axiom* 1.

∴ CG = CB. [I. 6.

But CB = GK, and CG = BK; [I. 34.

∴ CK is equilateral.

Also, CBK is a right angle; [I. 46.

∴ CK is a parallelogram with all its sides equal, and one angle a right angle;

∴ it is square, and it is on the side CB. [*Definition* 31.

Similarly, HF is the square on HG which = AC. [I. 34.

∴ HF, CK are the squares on AC, CB.

Now, the complement AG = the complement GE; [I. 43.
and AG is the rectangle AC, CB, since CG = CB;
∴ GE also = the rectangle AC, CB; [*Ax.* 1.
∴ AG, GE = twice the rectangle AC, CB.

Finally, the square on AB = the figure AE,
that is, = the four figures HF, CK, AG, GE,
that is, = the squares on AC, CB, together with twice the rectangle AC, CB.

Wherefore, *if a straight line, etc.* [Q.E.D.

Corollary 1. Parallelograms about the diameter of a square are likewise squares.

Corollary 2. If, in the previous proposition, C bisect AB, the rectangle AC, CB is equal to the square on AC, so that the proposition states that, in this case, the square on AB is four times the square on AC, that is,
the square on twice a given line is four times the square on the line.

Algebraic Formula. Let AC be a units, and CB be b units; then $$(a+b)^2 = a^2 + b^2 + 2ab.$$

Alternative Proof.

Because AB is divided into two parts at C;
∴ square on AB = sum of rectangles AB, AC and AB, CB. [II. 2.
Also, by II. 3,
the rectangle AB, AC = square on AC, together with rect. AC, CB;
and the rectangle AB, CB = square on CB, together with rect. AC, CB;
∴ the square on AB = the squares on AC, CB, together with twice the rectangle AC, CB.

EXERCISES.

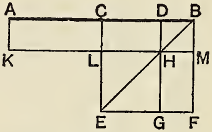
1. Prove the truth of II. 4 as follows:—On the sides AD, DE, EB of the square ADEB take points P, Q, R such that AP, DQ, and ER are each equal to BC; shew that the four triangles APC, DQP, ERQ and BCR are each one-half the rect. AC, CB, and that PQRC is the square on CP, which equals the squares on AP, AC, that is, on BC, AC.

2. In the figure of I. 1 if the circles meet again at F prove that the square on CF is three times the square on AB.

Proposition 5. Theorem.

If a straight line be divided into two equal parts and also into two unequal parts, the rectangle contained by the unequal parts, together with the square on the line between the points of section, is equal to the square on half the line.

Let the straight line AB be divided into two equal parts at the point C, and into two unequal parts at the point D :
the rectangle AD, DB, *together with the square on* CD, *shall be equal to the square on* CB.



Construction. On CB describe the square CEFB ; [I. 46.
join BE ; through D draw DHG parallel to CE or BF ; through H draw KLM parallel to CB or EF ; and through A draw AK parallel to CL or BM. [I. 31.

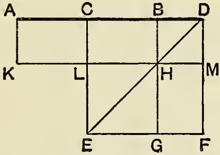
Proof. The complement CH = the complement HF ; [I. 43.
to each of these add DM ;
∴ the whole CM = the whole DF. [*Axiom* 2.
But CM = AL, because AC = CB ; [I. 36.
∴ also AL = DF ; to each add CH ;
∴ the whole AH = DF and CH. [*Axiom* 2.
But AH is the rectangle AD, DB, for DH = DB ; [II. 4, *Corollary* 1.
and DF together with CH is the gnomon CMG ;
therefore the rectangle AD, DB = the gnomon CMG.
To each add the square on CD, which = LG ; [II. 4, *Cor.* 1, and I. 34.
∴ the rectangle AD, DB, together with the square on CD,
= the gnomon CMG, together with LG,
that is, = the figure CEFB, which is the square on CB.

Wherefore, *if a straight line, etc.* [Q.E.D.

PROPOSITION 6. THEOREM.

If a straight line be bisected, and produced to any point, the rectangle contained by the whole line thus produced, and the part of it produced, together with the square on half the line bisected, is equal to the square on the straight line which is made up of the half and the part produced.

Let the straight line AB be bisected at the point C, and produced to the point D:
the rectangle AD, DB, *together with the square on* CB, *shall be equal to the square on* CD.



Construction. On CD describe the square CEFD; [I. 46.
join DE; through B draw BHG parallel to CE or DF; through H draw KLM parallel to AD or EF; and through A draw AK parallel to CL or DM. [I. 31.

Proof. Because AC = CB, [*Hypothesis.*
the rectangle AL = the rectangle CH; [I. 36.
but the complement CH = the complement HF; [I. 43.
∴ also AL = HF. [*Axiom* 1.
To each add CM;
∴ the whole AM = the gnomon CMG. [*Axiom* 2.
But AM is the rectangle contained by AD, DB, since DM = DB; [II. 4, *Corollary* 1.
∴ the rectangle AD, DB = the gnomon CMG. [*Axiom* 1.
To each add the square on CB, which = LG; [II. 4, *Cor.* 1, and I. 34.
∴ the rectangle AD, DB, together with the square on CB, = the gnomon CMG and the figure LG,
that is, = the figure CEFD, which is the square on CD.

Wherefore, *if a straight line, etc.* [Q.E.D.

Algebraic Formula for Prop. 5. Let AC or CB be a units and CD be b units, so that

$$AD=(a+b) \text{ units and } DB=(a-b) \text{ units};$$

then $$(a+b)(a-b)+b^2=a^2.$$

Algebraic Formula for Prop. 6. Let $AC=CB=a$ units and $CD=b$ units; then

$$(a+b)(b-a)+a^2=b^2.$$

Alternative Proof of II. 5.

The sq. on CB = sqs. on CD, DB and twice the rect. CD, DB [II. 4.
= sq. on CD with rect. CD, DB and rect. DB, CB [II. 3.
= sq. on CD with rect. CD, DB and rect. AC, DB
= sq. on CD and rect. AD, DB. [II. 1.

Alternative Proof of II. 6.

The sq. on CD = sqs. on CB, BD with twice the rect. CB, BD [II. 4.
= sq. on CB with rect. CB, BD and rect. CD, BD [II. 3.
= sq. on CB with rect. AC, BD and rect. CD, BD
= sq. on CB with rect. AD, DB. [II. 1.

Note on II. 5.

From this proposition it is clear that *the difference of the squares on two unequal straight lines* AC, CD *is equal to the rectangle contained by their sum and difference.*

For the proposition states that the difference between the squares on AC, CD = the rect. AD, DB.

Also AD = the sum of AC and CD, and DB = difference between CB and CD = the diff. between AC and CD.

Again, the proposition says that *the rectangle contained by two straight lines is equal to the difference between the squares on their semi-sum and their semi-difference.*

For the rect. AD, DB = difference of sqs. on CB, CD.

Also CB = half of AB = half the sum of AD, DB,

and CD = half the difference of AD and DB,

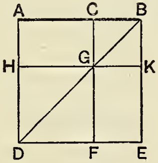
since $AD-DB=AC+CD-DB=CB+CD-DB=2CD.$

With a slight alteration of lettering the above is also true for the figure of II. 6.

PROPOSITION 7. THEOREM.

If a straight line be divided into any two parts, the squares on the whole line, and on one of the parts, are equal to twice the rectangle contained by the whole and that part, together with the square on the other part.

Let the straight line AB be divided into any two parts at the point C:
the squares on AB, BC *shall be equal to twice the rectangle* AB, BC, *together with the square on* AC.



Construction. On AB describe the square ADEB, and construct the figure as in the preceding propositions.

Proof. The complement AG = the complement GE; [I. 43.
to each of these add CK;
∴ the whole AK = the whole CE;
∴ AK, CE are double of AK.
But twice the rect. AB, BC is double of AK,
since BK = BC. [II. 4, *Cor.* 1.
∴ twice the rectangle AB, BC = the figures AK, CE,
that is, = gnomon AKF, together with the square CK.
To each add the square on AC, which = HF; [II. 4, *Cor.* 1 and I. 34.
∴ twice the rectangle AB, BC, together with the square on AC,
= the gnomon AKF, together with the squares CK, HF,
that is, = the whole figure ADEB, together with CK,
that is, = the squares on AB, BC.

Wherefore, *if a straight line, etc.* [Q.E.D.

Algebraic Formula. Let AB be a units and CB be b units; then $a^2 + b^2 = 2ab + (a - b)^2$.

Alternative Proof.

Since AB is divided into two parts at C,
∴ sq. on AB = sqs. on AC, CB, and twice the rectangle AC, CB. [II. 4.
To each add the square on CB;
∴ the squares on AB, CB = the square on AC, together with twice the square on CB, and twice the rectangle AC, CB.

But the rect. AB, BC = sq. on CB, and the rect. AC, CB; [II. 3.
∴ twice the rect. AB, BC = twice the square on CB and twice the rect. AC, CB; [*Axiom* 6.
∴ the squares on AB, CB = the square on AC, together with twice the rectangle AB, BC.

Note to Proposition 7.

This proposition may be enunciated thus:

The square described on a straight line which is equal to the difference of two straight lines is less than the sum of the squares on the two straight lines by twice the rectangle contained by them.

EXERCISES.

****1.** Divide a given straight line into two parts such that the rectangle contained by them shall be the greatest possible.

[In the figure to II. 5 the rect. AD, DB is greatest when CD vanishes, *i.e.* when D is at C.]

2. The least square that can be inscribed in a given square is that which is one half the given square.

[Let ABCD be the given square, and E, F, G, H points in the sides such that AE = BF = CG = DH; then the △s AEH, EBF, FCG, GDH are all equal; ∴ the sq. EFGH is least when the △ AEH is greatest, *i.e.* when the rect. AE, AH is greatest (I. 41), *i.e.* when AE, EB is greatest, *i.e.* by Ex. 1, when E is the middle point of AB; ∴ etc.]

****3.** Of all rectangles with the same perimeter, the square has the greatest area. [Use Ex. 1.]

****4.** If ABC is an isosceles triangle and D any point on the base BC, prove that the rectangle BD, DC is equal to the difference of the squares on AC, AD.

5. The square on either of the sides about the right angle of a right-angled triangle is equal to the rectangle contained by the sum and the difference of the hypotenuse and the other side.

[Use I. 47 and the Note to II. 5.]

6. Construct a rectangle equal to the difference of two given squares. [Use II. 5 or 6.]

****7.** ABC is a triangle having a right angle at A, and AD is the perpendicular on BC; prove that

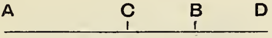
(1) AD^2 = rect. BD, DC, and (2) AC^2 = rect. BC, CD.

PROPOSITION 8. THEOREM.

If a straight line be divided into any two parts, four times the rectangle contained by the whole and one of the parts, together with the square on the other part, is equal to the square on the straight line which is made up of the whole and that part.

[Euclid's proof of this proposition is cumbrous; it is seldom read or required. The following proof is easy.]

Let the straight line AB be divided into two parts at B: *then four times the rectangle* AB, BC, *together with the square on* AC, *shall be equal to the square on the straight line made up of* AB *and* BC *together.*



Construction. Produce AB to D, making BD equal to CB.

Proof. Since AD is divided into two parts at B;

∴ the square on AD = the squares on AB, BD, together with twice the rect. AB, BD. [II. 4.

But sq. on BD = sq. on BC, and rect. AB, BD = rect. AB, BC, since BD = CB;

∴ sq. on AD = sqs. on AB, BC and twice the rect. AB, BC

But by II. 7,

the sqs. on AB, BC = sq. on AC and twice the rect. AB, BC;

∴ sq. on AD = sq. on AC, and four times the rect. AB, BC.

Also AD is made up of AB and CB together, since BD = CB.

Wherefore, *etc.*

Algebraic Formula. Let AB = a units, and CB = BD = b units; then $4ab + (a - b)^2 = (a + b)^2$.

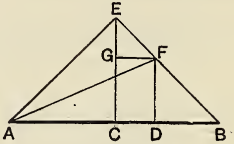
NOTE TO PROPOSITION 8.

Since AD is the sum of the two lines AB, BC, and AC is the difference of the same two lines, this proposition may be enunciated thus: *The square on the sum of two straight lines = the square on the difference of the two lines, together with four times the rect. contained by the two lines.*

PROPOSITION 9. THEOREM.

If a straight line be divided into two equal, and also into two unequal parts, the squares on the two unequal parts are together double of the square on half the line and of the square on the line between the points of section.

Let the straight line AB be divided into two equal parts at the point C, and into two unequal parts at the point D: *the squares on* AD, DB *shall be together double of the squares on* AC, CD.



Construction. From C draw CE at right angles to AB, [I. 11.
and make it equal to AC or CB, [I. 3.
and join EA, EB;
draw DF parallel to CE to meet BE in F,
and FG parallel to BA to meet CE in G, [I. 31.
and join AF.

Proof. (1) Because ACE is a right angle, [*Construction.*
the two other angles AEC, EAC are together equal to one right angle; [I. 32.
and they are equal, since CE was made equal to AC; [I. 5.
∴ each of them is half a right angle.
Similarly, each of the angles CEB, EBC is half a right angle;
∴ the whole angle AEB is a right angle.

(2) Because the angle GEF is half a right angle, and the angle EGF a right angle,
for it = the interior and opposite angle ECB; [I. 29.
∴ the remaining angle EFG is half a right angle;
∴ the angle GEF = the angle EFG, and EG = GF. [I. 6.

(3) Again, because the angle at B is half a right angle, and FDB a right angle,
for it = the interior and opposite angle ECB; [I. 29.
∴ the remaining angle BFD is half a right angle; [I. 32.
∴ the angle at B = the angle BFD, and DF = DB. [I. 6.

(4) The square on AE = the squares on AC, CE, [I. 47.
that is, = twice the square on AC, since CE = AC. [*Constr.*
Again, the square on EF = the squares on EG, GF, [I. 47.
that is, = twice the square on GF, since EG = GF; [*Proved.*
that is, = twice the square on CD, since GF = CD. [I. 34.
∴ twice the squares on AC, CD = the sqs. on AE, EF,
that is, = the square on AF, since AEF is a right angle, [I. 47.
that is, = the squares on AD, DF, since ADF is a rt. angle, [I. 47.
that is, = the sqs. on AD, DB, since DB = DF.
Wherefore, *if a straight line, etc.* [Q.E.D.

Algebraic Formula. Let AC = CB = a units and CD = b units; then

$$(a+b)^2+(a-b)^2=2a^2+2b^2.$$

Alternative Proof of II. 9.

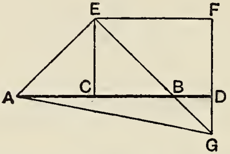
Since AD is divided at C,
∴ sq. on AC, CD + twice the rect. AC, CD = sq. on AD. [II. 4.
Again, since BC is divided internally at D,
∴ sqs. on BC, CD = twice the rect. BC, CD + sq. on BD; [II. 7.
that is, sqs. on AC, CD = twice the rect. AC, CD + sq. on BD,
since AC = BC; add these results;
∴ twice the squares on AC, CD, together with twice the rect. AC, CD = the squares on AD, BD, together with twice the rect. AC, CD;
∴ twice the squares on AC, CD alone = the squares on AD, BD.

[If in the third line of this proof we read "externally" instead of "internally," it will apply also to Prop. 10.]

PROPOSITION 10. THEOREM.

If a straight line be bisected, and produced to any point, the square on the whole line thus produced, and the square on the part of it produced, are together double of the square on half the line bisected and of the square on the line made up of the half and the part produced.

Let the straight line AB be bisected at C, and produced to D: *the squares on* AD, DB *shall be together double of the squares on* AC, CD.



Construction. From C draw CE at right angles to AB, [I. 11.
and make it equal to AC or CB; [I. 3.
and join AE, EB;
draw EF parallel to AB, and DF parallel to CE. [I. 31.

Proof. (1) Because EF meets the parallels EC, FD, the angles CEF, EFD are together equal to two right angles; [I. 29.
∴ BEF, EFD are together less than two right angles;
∴ EB, FD will meet, if produced, towards B, D. [*Axiom* 12.
Let them meet at G, and join AG.

Then, because ACE is a right angle; [*Construction.*
the angles AEC, EAC together = a rt. angle, [I. 32.
and they are equal, since AC = CE; [*Construction.*
∴ each of the angles CEA, EAC is half a right angle. [I. 32.
Similarly, each of the angles CEB, EBC is half a right angle;
∴ AEB is a right angle.

(2) And because EBC is half a right angle, the vertically opposite angle DBG is also half a right angle; [I. 15.

but BDG is a right angle, because it is equal to the alternate angle DCE; [I. 29.
∴ the remaining angle DGB is half a right angle, [I. 32.
and is therefore equal to the angle DBG;
∴ also the side BD = the side DG. [I. 6.

(3) Again, because EGF is half a right angle, and the angle at F a right angle, for it = the angle ECD; [I. 34.
∴ the remaining angle FEG is half a right angle, [I. 32.
and is therefore equal to the angle EGF;
∴ also GF = FE. [I. 6.

(4) The square on AE = the squares on EC, CA, [I. 47.
that is, = twice the square on AC,
since EC = AC. [*Constr.*
Also the square on EG = the squares on GF, FE, [I. 47.
that is, = twice the square on FE,
since GF = FE, [*Proved in* (3).
that is, = twice the square on CD. [I. 34.
Hence, twice the sqs. on AC, CD = the sqs. on AE, EG,
that is, = the sq. on AG, since AEG is a rt. angle, [I.47.
that is, = the sqs. on AD, DG, since ADG is a rt. angle, [I. 47.
that is, = the sqs. on AD, DB, since DB = DG. [*Proved in* (2).
Wherefore, *if a straight line, etc.* [Q. E. D.

Algebraic Formula. Let AC = CB = a units and CD = b units; then $(a+b)^2 + (a-b)^2 = 2a^2 + 2b^2$.

[For Exercises, see Page 105.]

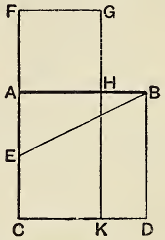
Note on Propositions 9 and 10.

These two propositions may be enunciated in one thus:
The sum of the squares on the sum and difference of two lines = twice the sum of the squares on the two lines. For in each case AD is the sum, and DB the difference of the two lines AC, CD.

Proposition 11. Problem.

To divide a given straight line into two parts, so that the rectangle contained by the whole and one of the parts may be equal to the square on the other part.

Let AB be the given straight line:
it is required to divide it into two parts, so that the rectangle contained by the whole and one of the parts may be equal to the square on the other part.



Construction. On AB describe the square ABDC; [I. 46.
bisect AC at E; [I. 10.
join BE; produce CA to F, and make EF equal to EB; [I. 3.
on AF describe the square AFGH. [I. 46.
AB shall be divided at H so that the rectangle AB, BH is equal to the square on AH.

Produce GH to meet CD in K.

Proof. Because AC is bisected at E, and produced to F, the rectangle CF, FA, together with the square on AE,
= the square on EF; [II. 6.
that is, = the square on EB, since EF = EB. [*Construction.*
that is, = the sqs. on AE, AB, since EAB is a rt. angle; [I. 47.
From each of these equals take the square on AE
∴ the rect. CF, FA = the square on AB. [*Axiom* 3.
But FK is the rectangle CF, FA; for FG = FA, [*Constr.*
and AD is the square on AB;

∴ FK = AD.

Take away the common part AK,
and the remainder FH = the remainder HD. [*Axiom* 3.
But HD is the rectangle AB, BH; for AB = BD; [*Constr.*
and FH is the square on AH; [*Constr.*
∴ the rectangle AB, BH = the square on AH.

Wherefore *the straight line* AB *is divided at* H, *so that the rectangle* AB, BH *is equal to the square on* AH. [Q.E.F.

Note. When a straight line AB is divided as in the above proposition, it is said to be divided in **medial section**

EXERCISES ON PROPOSITIONS 9 and 10.

1. Divide a given straight line into two parts such that the sum of the squares on the two parts may be the least possible.

[In the figure of II. 9, $AD^2 + DB^2$ is least when CD vanishes, that is, when D is the middle point of AB.]

2. The sum of the squares on two straight lines is never less than twice the rectangle contained by them and is never less than half the square on their sum. [Use Propositions 7 and 9.]

3. If AB be bisected in C and divided unequally at D, then the sum of the squares on AD, DB is equal to twice the rectangle AD, DB together with four times the square on CD. [Use Propositions 9 and 5.]

4. C is the middle point of AB and D any point on AB produced; prove that the square on AD is equal to the square on BD, together with four times the rectangle AC, CD.

EXERCISES ON PROPOSITION 11.

****1.** *Divide a given straight line externally in medial section.*

Let AB be the given straight line. On it describe the square ACDB. Bisect AC in E and produce EC to F, making EF equal to EB. On AF, on the side remote from BD, describe the square AFGH. H shall be the point required, that is, the rectangle AB, BH shall equal the square on AH. On completing the parallelogram CFGL the proof is similar to that of II. 11.

In the figure of II. 11 prove that

2. If CH be produced to meet BF at L, CL is at right angles to BF.

[The $\triangle^s$ HCA, FBA are equal in all respects. $\therefore$ etc.]

3. If BE and CH meet at O, AO is at right angles to CH.

[$\angle EFB = \angle EBF$. $\therefore$, by Ex. 2, $\angle ECO = \angle HOB = \angle EOC$; $\therefore EO = EC = EA$, etc.]

4. GB, FD, and AK are all parallel.

[$2\triangle GFB = FH = HD = 2\triangle DGB$; $\therefore$ etc.,

$2\triangle GAB = 2\triangle GHB + FH = 2\triangle GHB + HD$

$= 2\triangle GHB + 2\triangle HKB = 2\triangle GBK$; $\therefore$ etc.]

5. If FG meet DB in M, then CHM is a straight line.

[Use the converse of I. 43.]

6. The rectangles GD, FB, AK are all equal.

7. KF and HD are parallel.

[$2\triangle KHF = AK = DG$ [by Ex. 6] $= \triangle KDF$; $\therefore$ etc.]

8. The rectangle AH, HB is equal to the difference of the squares on AH, HB.

9. $AB^2 + BH^2 = 3AH^2$.

10. The square on the sum of AB, BH = five times the square on AH.

11. CF is divided at A in medial section.

12. The square on EF = five times the square on EA.

13. If in HA a point P be taken so that HP = HB, then AH is divided in medial section at P.

14. Shew that in a straight line, divided as in II. 11, the rectangle contained by the sum and difference of the parts is equal to the rectangle contained by the parts.

15. Produce a given straight line so that the rectangle contained by the whole straight line thus produced and the part produced may be equal to the square on the given straight line.

[In the figure of II. 11, if CA be the given straight line, then F is the required point; hence the required construction.]

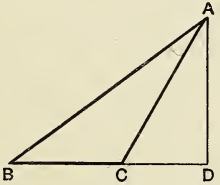
Note. In the figures of the two following propositions the line BD is often called the **projection** of the side AB upon the base BC.

PROPOSITION 12. THEOREM.

In an obtuse-angled triangle, if a perpendicular be drawn from either of the acute angles to the opposite side produced, the square on the side subtending the obtuse angle is greater than the squares on the sides containing the obtuse angle, by twice the rectangle contained by the side on which, when produced, the perpendicular falls, and the straight line intercepted without the triangle, between the perpendicular and the obtuse angle.

Let ABC be an obtuse-angled triangle, having the obtuse angle ACB, and from the point A let AD be drawn perpendicular to BC produced:

The square on AB *shall be greater than the squares on* AC, CB, by *twice the rectangle* BC, CD.



Proof. Because BD is divided at C, the sq. on BD = sqs. on BC, CD, and twice the rect. BC, CD. [II. 4.

To each add the square on DA;

∴ the squares on BD, DA = the squares on BC, CD, DA, and twice the rectangle BC, CD. [*Axiom* 2.

But the square on BA = the squares on BD, DA, } [I. 47.
and the square on CA = the squares on CD, DA, }
because the angle D is a right angle;

∴ the square on BA = the squares on BC, CA, and twice the rectangle BC, CD;

that is, the square on BA is greater than the squares on BC, CA by twice the rectangle BC, CD.

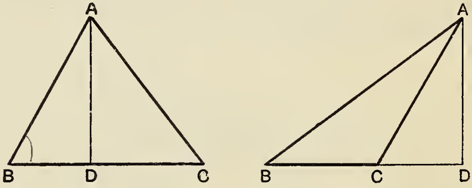
Wherefore, *in obtuse-angled triangles, etc.* [Q.E.D.

Proposition 13. Theorem.

In every triangle, the square on the side subtending an acute angle is less than the squares on the sides containing that angle, by twice the rectangle contained by either of these sides, and the straight line intercepted between the perpendicular let fall on it from the opposite angle and the acute angle.

Let ABC be any triangle, and the angle at B an acute angle; on BC let fall the perpendicular AD from the opposite angle:

the square on AC *shall be less than the squares on* CB, BA, *by twice the rectangle* CB, BD.



Proof. First, let AC be not perpendicular to BC.

In Fig. 1 BC is divided at D, and in Fig. 2 BD is divided at C; therefore, in both cases,

the squares on CB, BD = twice the rectangle CB, BD, together with the square on CD. [II. 7.

To each add the square on DA;

∴ the squares on CB, BD, DA = twice the rectangle CB, BD, together with the squares on CD, DA.

But the square on BA = the squares on BD, DA,

and the square on CA = the squares on CD, DA; [I. 47.

∴ the squares on CB, BA = twice the rectangle CB, BD, together with the square on AC.

that is,

the square on AC alone is less than the squares on AB, BC by twice the rectangle CB, BD.

Secondly, let AC be perpendicular to BC.

Then BC is the straight line between the perpendicular and the acute angle at B;

and it is clear that the squares on AB, BC

= the sq. on AC, and twice the square on BC. [I. 47 and *Ax*. 2.

Wherefore, *in every triangle, etc.* [Q.E.D.

EXERCISES ON PROPOSITIONS 12 AND 13.

****1.** *In any triangle the sum of the squares on the sides is equal to twice the square on half the base, together with twice the square on the line joining the vertex to the middle point of the base (that is, together with twice the square on the median through the vertex).*

Let D be the middle point of the base BC of the triangle ABC.

If ADB is a right angle the theorem is clear by I. 47. If not, of the two angles ADB, ADC, one is obtuse and the other acute.

Let ADB be the obtuse angle.

Draw AE perpendicular to BC.

Then by Euc. II. 12, 13,

square on AB = squares on AD, DB + twice rect. BD, DE.

and square on AC = squares on AD, DC − twice rect. DC, DE

= squares on AD, DB − twice rect. BD, DE,

since BD and DC are equal;

∴ by addition,

the squares on AB, AC = twice the squares on AD, DB.

The student will have no difficulty in drawing the figure (or see App., Art. 32). There will be two cases according as the angle ACB is acute or obtuse.

The above is a very important proposition. Many of the following exercises depend on it.

****2.** ABCD is a rectangle and P any point; prove that the squares on PA, PC are together equal to the squares on PB, PD.

**3. Four times the sum of the squares on the medians of a triangle is equal to three times the sum of the squares on the sides of the triangle.

**4. The sum of the squares on the sides of a parallelogram is equal to the sum of the squares on the diagonals.

5. The base of a triangle is given and is bisected by the centre of a given circle; if the vertex be at any point of the circumference, shew that the sum of the squares on the two sides of the triangle is invariable.

6. In any quadrilateral the squares on the diagonals are together equal to twice the sum of the squares on the straight lines joining the middle points of opposite sides. [Use Ex. 4, and Page 72, Ex. 2.]

7. If a circle be described round the point of intersection of the diameters of a parallelogram as a centre, shew that the sum of the squares on the straight lines drawn from any point in its circumference to the four angular points of the parallelogram is constant.

8. The sides of a triangle are 8, 12, and 13 inches; prove that it is acute-angled.

9. The sides of a triangle are 8, 12, and 15 inches; prove that it is obtuse-angled.

10. If the angle ACB in II. 12 become more and more obtuse, until finally A falls on BC produced, what does the proposition become?

11. The square on the base of an isosceles triangle is equal to twice the rectangle contained by either side and by the straight line intercepted between the perpendicular let fall on the side from the opposite angle and the extremity of the base.

12. If one angle of a triangle be equal to two-thirds of a right angle the square on the opposite side is less than the sum of the squares containing that angle by the rectangle contained by these two sides.

13. Find the obtuse angle of a triangle, when the square on the side opposite to the obtuse angle is greater than the sum of the squares on the sides containing it by the rectangle of the sides.

14. ABC is a triangle having the sides AB and AC equal; if AB is produced beyond the base to D so that BD is equal to AB, shew that the square on CD is equal to the square on AB, together with twice the square on BC.

15. In AB the diameter of a circle take two points C and D equally distant from the centre, and from any point E in the circumference draw EC, ED; shew that the squares on EC and ED are together equal to the squares on AC and AD. [Use Ex. 1 and II. 10.]

16. In BC the base of a triangle take D such that the squares on AB and BD are together equal to the squares on AC and CD; then the middle point of AD will be equally distant from B and C.

17. A square BDEC is described on the hypotenuse BC of a right-angled triangle ABC: shew that the squares on DA and AC are together equal to the squares on EA and AB. [Use Ex. 1.]

18. ABC is an acute-angled triangle, and BE, CF are the perpendiculars on CA, AB; prove that the rectangles AB, AF and AC, AE are equal.

[By II. 13, $AB^2 + AC^2 - 2BA.AF = BC^2 = AB^2 + AC^2 - 2AC.AE$; ∴ etc.]

19. In a triangle ABC the angles B and C are acute: if E and F be the points where perpendiculars from the opposite angles meet the sides AC, AB, shew that the square on BC is equal to the rectangle AB, BF, together with the rectangle AC, CE.

20. Describe an isosceles obtuse-angled triangle such that the square on the largest side may be equal to three times the square on either of the equal sides. [The obtuse ∠ must = four-thirds of a right ∠.]

21. ABC is an equilateral triangle, and AB is produced to D so that BD is twice AB. Prove that the square on CD is seven times the square on AB.

22. Squares are described on the sides of any triangle; the sum of the squares on the straight lines joining adjacent corners of the squares are equal to three times the sum of the squares on the sides.

[The letters being as in I. 47, produce BC to X, so that BC = CX. The △ˢ KCE, ACX are equal in all respects, so that KE = AX;

$$\therefore\ KE^2 + AB^2 = AX^2 + AB^2 = 2AC^2 + 2BC^2, \text{ by Ex. 1.}$$

Similarly for HG^2, FD^2; ∴ etc.]

23. The squares on the two equal sides of an isosceles triangle are together less than the squares on the two sides of any other triangle on the same base and between the same parallels. [Use Ex. 1.]

****24.** The squares on the sides of a quadrilateral are together greater than the squares on its diagonals by four times the square on the straight line joining the middle points of its diagonals.

[Let E, F be the middle points of the diagonals AC, BD of the quad[1] ABCD. Then, by Ex. 1

$$AB^2 + BC^2 = 2BE^2 + 2EC^2, \text{ and } AD^2 + DC^2 = 2DE^2 + 2EC^2;$$

∴ sum of the squares on the sides

$$= 4EC^2 + 2BE^2 + 2DE^2 = 4EC^2 + 4EF^2 + 4DF^2 = AC^2 + BD^2 + 4EF^2.]$$

25. If the squares on the sides of a quadrilateral are together equal to the squares on its diagonals it must be a parallelogram.

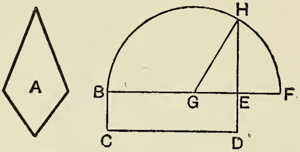
26. Construct a triangle having given its area, its base, and the sum of the squares on its sides.

[By I. 39 the vertex lies on a line parallel to the base, and by Ex. 1 its distance from the middle point of the base is given.]

Proposition 14. Problem.

To describe a square that shall be equal to a given rectilineal figure

Let A be the given rectilineal figure:
it is required to describe a square that shall be equal to A.



Construction. Describe the rectangular parallelogram BCDE equal to the figure A. [I. 45.

Then, if its sides BE, ED are equal, it is a square, and what was required is now done.

But if they are not equal, produce one of them BE to F,

make EF equal to ED, [I. 3.

and bisect BF at G; [I. 10.

with centre G, and radius GB, or GF, describe the semi-circle BHF, and produce DE to meet it in H.

The square on EH shall = the figure A. Join GH.

Proof. Because BF is divided into two equal parts at G, and into two unequal parts at E,

the rectangle BE, EF, together with the square on GE

= the square on GF, [II. 5.

that is, = the square on GH, since GF = GH, [*Constr.*

that is, = the squares on GE, EH; [I. 47.

Take away the square on GE, which is common to both;

∴ the rectangle BE, EF = the square on EH. [*Axiom* 3.

But BD is the rectangle BE, EF, since EF = ED; [*Constr.*

∴ BD = the square on EH.

But BD = the rectilineal figure A; [*Construction.*

∴ the square on EH = the rectilineal figure A.

Wherefore *a square has been made equal to the given rectilineal figure* A, *namely, the square described on* EH. [Q.E.F.

BOOK III.

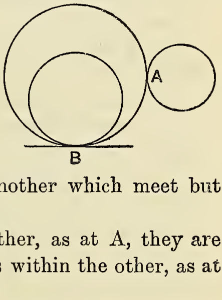
DEFINITIONS.

1. Equal circles are those of which the diameters are equal, or from the centres of which the straight lines to the circumferences are equal.

[This is not a definition, but a theorem, the truth of which is evident; for, if the circles be applied to one another, so that their centres coincide, the circles must likewise coincide, since the straight lines from the centres are equal.]

2. A straight line is said to **touch** a circle when it meets the circle, and, being produced, does not cut it.

Such a straight line is called a **tangent**, and the point at which it touches the circle is called its **point of contact.**



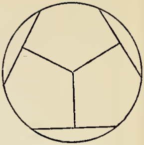
3. Circles are said to touch one another which meet but do not cut one another.

When each circle is without the other, as at A, they are said to touch **externally**; when one is within the other, as at B, they are said to touch **internally.**

4. A **secant** of a circle is a straight line which cuts the circumference in two points.

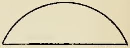
5. A **chord** of a circle is a straight line which joins any two points on the circle.

6. Chords are said to be **equally distant** from the centre of a circle when the perpendiculars drawn to them from the centre are equal.



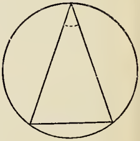
Also the chord on which the greater perpendicular falls is said to be farther from the centre.

7. A **segment** of a circle is the figure contained by a chord and the circumference it cuts off.



The chord of the segment is sometimes called its **base.**

8. An **angle in a segment** is the angle contained by two straight lines drawn from any point in the circumference of the segment to the extremities of the straight line which is the base of the segment.



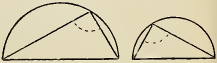
Also an angle is said to insist or stand on the circumference intercepted between the straight lines which contain the angle.

9. Any part of the circumference of a circle is called an **arc.**

10. A **sector** of a circle is the figure contained by two straight lines drawn from the centre, and the circumference between them.

The angle between the straight lines is called the angle of the sector.

11. Similar segments of circles are those in which the angles are equal, or which contain equal angles.



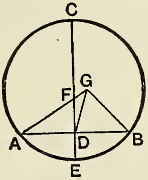
12. Two circles are said to be **concentric** when they have the same centre.

[*Note.* In the following propositions, whenever the expression "straight lines from the centre," or "drawn from the centre," occurs, it is to be understood that the lines are drawn to the circumference.]

PROPOSITION 1. PROBLEM.

To find the centre of a given circle.

Let ABC be the given circle:
it is required to find its centre.



Construction. Draw any chord AB, and bisect it at D; [I. 10.
from D draw DC at right angles to AB; [I. 11.
produce CD to meet the circumference at E, and bisect CE at F. [I. 10.

Then F shall be the centre of the circle ABC.

For if F be not the centre, if possible, let G be the centre; and join GA, GD, GB.

Proof. In the triangles GDA, GDB,

because { DA = DB, [*Construction.*
and DG is common to both,
and GA = GB,

because they are drawn from the centre G; [I. *Definition* 15.

∴ the ∠ADG = the ∠BDG. [I. 8.

these angles, being adjacent, are right angles; [I. *Def.* 10.

∴ the ∠BDG is a right ∠.

But the ∠BDF is also a right ∠; [*Construction.*

∴ the ∠BDG = the ∠BDF, [*Axiom* 11.

the less to the greater; which is impossible;

∴ G is not the centre of the circle ABC.

In the same manner it may be shewn that no other point out of the line CE is the centre;

and since CE is bisected at F, any other point in CE divides it into unequal parts, and cannot be the centre.

∴ no point but F is the centre;

that is, F is the centre of the circle ABC:

which was to be found.

Corollary. If in a circle a straight line bisect another at right angles, the centre of the circle is in the straight line which bisects the other.

EXERCISES.

****1.** The line joining the centres of two circles which meet in two points is perpendicular to the line joining the two points, and bisects it.

2. Shew that the straight lines drawn at right angles to the sides of a quadrilateral inscribed in a circle from their middle points intersect at a fixed point.

3. Find the shortest distance between two circles which do not meet.

4. Given two chords of a circle in magnitude and position, find its centre.

5. Describe a circle with a given centre cutting a given circle at the extremities of a diameter.

6. Describe a circle which shall have its centre in a given straight line and pass through two given points.

7. Describe a circle which shall pass through two given points and have a given radius.

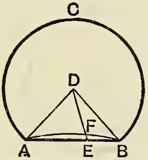
Proposition 2. Theorem.

If any two points be taken in the circumference of a circle, the straight line which joins them shall fall within the circle.

Let ABC be a circle, and A and B any two points in the circumference:

the straight line drawn from A *to* B *shall fall within the circle.*

For if it do not, let it fall, if possible, without, as AEB.



Construction. Find D, the centre of the circle ABC; [III. 1. join DA, BD; in the arc AB take any point F, join DF, and produce it to meet AB at E.

Proof. Because DA = DB, [I. *Definition* 15.
the ∠DAB = the ∠DBA. [I. 5.

But the exterior ∠DEB of the △DAE is greater than the interior opposite ∠DAE, [I. 16.

and the ∠DAE was shewn to be equal to the ∠DBE;

∴ the ∠DEB is greater than the ∠DBE;

∴ DB is greater than DE. [I. 19.

But DB = DF; [I. *Definition* 15.

∴ DF is greater than DE, the less than the greater; which is impossible;

∴ the straight line AB does not fall without the circle.

In the same manner it may be shewn that it does not fall on the circumference.

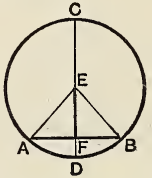
Therefore it falls within the circle.

Wherefore, *if any two points, etc.* [Q. E. D.

PROPOSITION 3. THEOREM.

If a straight line drawn through the centre of a circle bisect a straight line in it which does not pass through the centre, it shall cut it at right angles; and if it cut it at right angles, it shall bisect it.

Let ABC be a circle; and let CD, a straight line drawn through the centre, bisect any straight line AB, which does not pass through the centre, at the point F:
CD *shall cut* AB *at right angles.*



Construction. Take E, the centre of the circle; and join EA, EB. [III. 1.

Proof. In the triangles AFE, BFE,

because { AF = FB, [*Hypothesis.*
and FE is common,
and the base EA = the base EB; [I. *Definition* 15.

∴ the ∠AFE = the ∠BFE; [I. 8.

∴ these angles being adjacent, are right angles. [I. *Def.* 10.

∴ CD cuts AB at right angles.

Conversely: Let CD cut AB at right angles;
CD *shall also bisect* AB, *that is,* AF *shall be equal to* FB.

The same construction being made, because EA = EB,

the ∠EAF = the ∠EBF. [I. 5.

Then in the triangles AFE, BFE,

because { the ∠EAF = the ∠EBF,
and the right ∠AFE = the right ∠BFE,
and the side EF is common;

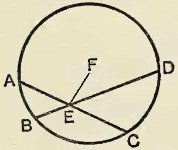
∴ AF = FB. [I. 26.

Wherefore, *if a straight line, etc.* [Q.E.D.

PROPOSITION 4. THEOREM.

If in a circle chords cut one another which do not both pass through the centre, they do not bisect one another.

Let ABCD be a circle, and AC, BD two chords in it, which cut one another at the point E, and do not both pass through the centre:
then AC, BD *shall not bisect one another.*



CASE I. If one of the chords pass through the centre, it is plain that it cannot be bisected by the other which does not pass through the centre.

CASE II. If neither of them pass through the centre, if possible, let AE = EC and BE = ED.

Construction. Find F, the centre of the circle, [III. 1.
and join EF.

Proof. Because FE, passing through the centre, bisects AC which does not pass through the centre; [*Hypothesis.*
∴ the ∠FEA is a right ∠. [III. 3.
Again, because FE bisects BD, [*Hypothesis.*
∴ the ∠FEB is a right ∠. [III. 3.
But the ∠FEA was shewn to be a right ∠;
∴ the ∠FEA = the ∠FEB, [*Axiom* 11.
the less to the greater; which is impossible.
∴ AC, BD do not bisect each other.

Wherefore, *if in a circle, etc.* [Q.E.D.

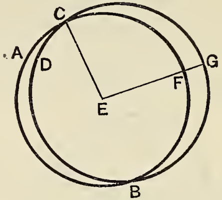
[For Exercises see next page.]

PROPOSITION 5. THEOREM.

If two circles cut one another, they shall not have the same centre.

Let the two circles ABC, CDG cut one another at the points B, C:

they shall not have the same centre.



Construction. If it be possible, let E be their centre; join EC, and draw any straight line EFG, meeting the circumferences at F and G.

Proof. Because E is the centre of the circle ABC,

∴ EC = EF. [I. *Definition* 15.

Again, because E is the centre of the circle CDG,

∴ EC = EG. [I. *Definition* 15.

∴ EF = EG, [*Axiom* 1.

the less to the greater, which is impossible;

∴ E is not the centre of the circles ABC, CDG.

Wherefore, *if two circles, etc.* [Q.E.D.

EXERCISES ON PROPOSITIONS 3 AND 4.

****1.** Through a given point within a given circle draw the chord which is bisected at the point.

****2.** The middle points of all parallel chords of a circle lie on a straight line which passes through the centre and is perpendicular to them.

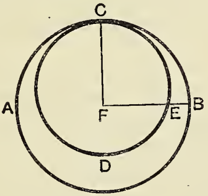
****3.** The middle points of all chords of a circle, which are of the same length, all lie on a concentric circle.

4. If two circles ABCD, ABEF cut each other in points A, B, any two parallel straight lines DAF, CBE drawn through the points of section to cut the circles are equal. [Use Ex. 2.]

Proposition 6. Theorem.

If two circles touch one another internally, they shall not have the same centre.

Let the two circles ABC, CDE touch one another internally at the point C:
they shall not have the same centre.



Construction. If it be possible, let F be their centre; join FC, and draw any straight line FEB, meeting the circumferences at E and B.

Proof. Because F is the centre of the circle ABC,

FC = FB. [I. *Definition* 15.

Again, because F is the centre of the circle CDE,

FC = FE; [I. *Definition* 15.

∴ FE = FB,

the less to the greater, which is impossible;
∴ F is not the centre of the circles ABC, CDE.

Wherefore, *if two circles, etc.* [Q.E.D.

Note. Propositions 5 and 6 amount to this proposition: *If two circles meet in a point they cannot have the same centre;* so that circles which have the same centre and one point common must coincide altogether.

PROPOSITION 7. THEOREM.

If any point be taken in the diameter of a circle which is not the centre, of all the straight lines which can be drawn from this point to the circumference, then

(1) *the greatest is that in which the centre is, and*

(2) *the other part of the diameter is the least;*

(3) *of any others, that which is nearer to the straight line which passes through the centre is always greater than one more remote; and*

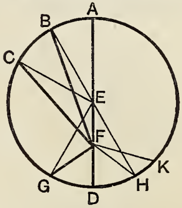
(4) *from the same point there can be drawn to the circumference two straight lines, and only two, which are equal to one another, one on each side of the shortest line.*

Let ABCD be a circle and AD its diameter, in which let any point F be taken which is not the centre; let E be the centre: of all the straight lines FB, FC, FG, etc., that can be drawn from F to the circumference,

(1) FA, *which passes through* E, *shall be the greatest, and*

(2) FD, *the other part of the diameter* AD, *shall be the least; and*

(3) *of the others* FB *shall be greater than* FC, *and* FC *than* FG.



Construction. Join BE, CE, GE.

Proof. (1) Because any two sides of a triangle are together greater than the third side, [I. 20.

therefore BE, EF are together greater than BF.

But BE = AE; [I. *Definition* 15.

∴ AE, EF are together greater than BF,

that is, AF is greater than BF.

Similarly it can be shewn that FA is greater than any other line drawn from F to the circumference;

∴ FA is the greatest of all such lines.

(2) Because GF, FE are together greater than EG, [I. 20.
and that EG = ED ; [I. *Definition* 15.
∴ GF, FE are together greater than ED.
Take away the common part FE, and the remainder GF is greater than the remainder FD.

Similarly any other straight line drawn from F can be shewn to be greater than FD ;
∴ FD is the least of all such lines.

(3) In the triangles BEF, CEF,

because { BE = CE, [I. *Definition* 15.
and EF is common,
but the ∠BEF is greater than the ∠CEF ;

∴ the base BF is greater than the base CF. [I. 24.

Similarly it may be shewn that CF is greater than GF.

(4) Also, there can be drawn two equal straight lines from the point F to the circumference, one on each side of the shortest line FD.

At the point E make the ∠FEH equal to the ∠FEG, [I. 23.
and join FH.

Then in the triangles GEF, HEF,

because { EG = EH, [I. *Definition* 15.
and EF is common,
and the ∠GEF = the ∠HEF ; [*Construction.*

∴ the base FG = the base FH. [I. 4.

But, besides FH, no other straight line can be drawn from F to the circumference equal to FG.

For, if it be possible, let FK be equal to FG.
Then, because FK = FG, and FH = FG ; [*Hypothesis.*
∴ FH = FK ; [*Axiom* 1.
that is, a line nearer to that which passes through the centre is equal to a line which is more remote,
which is impossible by what has been already shewn.

Wherefore, *if any point be taken, etc.* [Q.E.D.

Proposition 8. Theorem.

If any point be taken without a circle, and straight lines be drawn from it to the circumference, one of which passes through the centre ;

(1) *of those which fall on the concave circumference, the greatest is that which passes through the centre; and of the rest, that which is nearer to the one passing through the centre is always greater than one more remote ; but*

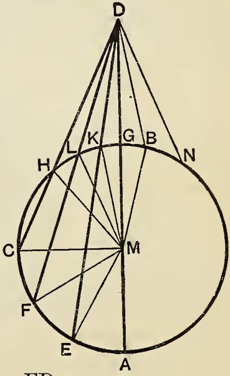
(2) *of those which fall on the convex circumference, the least is that between the point without the circle and the diameter ; and of the rest, that which is nearer to the least is always less than one more remote ; and*

(3) *from the same point there can be drawn to the circumference two straight lines, and only two, which are equal to one another, one on each side of the shortest line.*

Let ABC be a circle, and D any point without it, and from D let the straight lines DA, DE, DF, DC be drawn to the circumference, of which DA passes through the centre ;

(1) *of those which fall on the concave circumference* AEFC, *the greatest shall be* DA, *and* DE *shall be greater than* DF, *and* DF *than* DC ; *but*

(2) *of those which fall on the convex circumference* GKLH, *the least shall be* DG, *and* DK *less than* DL, *and* DL *than* DH.



Construction. Take M, the centre of the circle ABC, [III. 1.
and join ME, MF, MC, MH, ML, MK.

Proof. (1) In the △EMD the two sides EM, MD are together greater than ED. [I. 20.
But EM = AM ; [I. *Definition* 15.
∴ AM, MD are together greater than ED,
that is, AD is greater than ED.

Again, in the triangles EMD, FMD,

because { EM = FM,
and MD is common,
but the ∠EMD is greater than the ∠FMD;

$\therefore$ the base ED is greater than the base FD. [I. 24.

Similarly, it may be shewn that FD is greater than CD. Therefore DA is the greatest, and DE greater than DF, and DF greater than DC.

(2) Again, because MK, KD are greater than MD, [I. 20.

and MK = MG, [I. *Definition* 15.

the remainder KD is greater than the remainder GD,

that is, GD is less than KD.

Also, in the △MLD, because straight lines MK, KD are drawn to a point K within the triangle from the ends of its base;

$\therefore$ MK, KD are less than ML, LD; [I. 21.

but MK = ML; [I. *Definition* 15.

$\therefore$ the remainder KD is less than the remainder LD.

Similarly, it may be shewn than LD is less than HD;

$\therefore$ DG is the least, and DK less than DL, and DL than DH.

(3) Also, there can be drawn two equal straight lines from D to the circumference, one on each side of the least line.

Construction. At the point M, in the straight line MD, make the angle DMB equal to the angle DMK, [I. 23.

and join DB.

Proof. In the triangles KMD, BMD,

because { MK = MB, [I. *Def.* 15.
and MD is common,
and the ∠DMK = the ∠DMB; [*Constr.*

$\therefore$ the base DK = the base DB. [I. 4.

But, besides DB, no other straight line can be drawn from D to the circumference equal to DK.

For, if it be possible, let DN = DK.

Then, because DB = DK, $\therefore$ DB = DN; [*Axiom* 1.

that is, a line nearer to the least = one which is more remote, which is impossible by what has been already shewn.

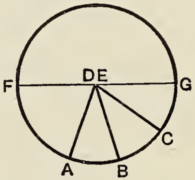
Wherefore, *if any point be taken, etc.* [Q.E.D.

Proposition 9. Theorem.

If a point be taken within a circle, from which there can be drawn more than two equal straight lines to the circumference, that point is the centre of the circle.

Let the point D be taken within the circle ABC, from which, to the circumference, there are drawn more than two equal straight lines, namely, DA, DB, DC:
the point D *shall be the centre of the circle.*

For, if not, let E be the centre.



Construction. Join DE, and produce it both ways to meet the circumference at F and G.

Proof. D is a point within the circle which is not its centre, and from it more than two equal straight lines, namely, the three straight lines DA, DB, DC, have been drawn to the circumference;

but this is impossible; [III. 7.

∴ E is not the centre of the circle ABC.

In the same manner it may be shewn that any other point than D is not the centre;
therefore D is the centre of the circle ABC.

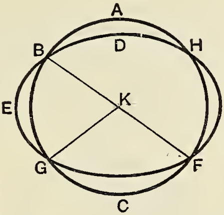
Wherefore, *if a point be taken, etc.* [Q.E.D.

Note. For other demonstrations of Propositions 9 and 10 see Notes, Pages 325 and 326.

Proposition 10. Theorem.

One circumference of a circle cannot cut another at more than two points.

If it be possible, let the circumference ABC cut the circumference DEF at more than two points, namely, at the points B, G, F.



Construction. Take K, the centre of the circle ABC, [III. 1.
and join KB, KG, KF.

Proof. Because K is the centre of the circle ABC,
∴ KB, KG, KF are all equal. [I. *Definition* 15.
And because within the circle DEF, the point K is taken, from which, to the circumference DEF, are drawn more than two equal straight lines KB, KG, KF,
∴ K is the centre of the circle DEF. [III. 9.
But K is also the centre of the circle ABC; [*Construction.*
∴ the same point is the centre of two circles which cut one another, which is impossible. [III. 5.

Wherefore, *one circumference, etc.* [Q.E.D.

EXERCISES.

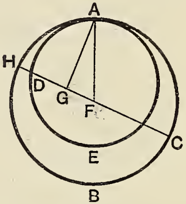
1. Two circles cannot meet in three points without coinciding entirely.

2. Through three points, which are not in the same straight line, only one circle can be drawn.

PROPOSITION 11. THEOREM.

If two circles touch one another internally, the straight line which joins their centres, being produced, shall pass through the point of contact.

Let the two circles ABC, ADE touch one another internally at the point A; let F be the centre of the circle ABC, and G the centre of the circle ADE:
the straight line FG, *being produced, shall pass through the point* A.



Proof. For, if not, let it pass otherwise, if possible, as FGDH, and join AF, AG.

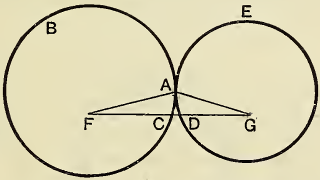
Then, because AG, GF are together greater than AF, [I. 20.
and AF = HF; [I. *Definition* 15.
∴ AG, GF are together greater than HF.
Take away the common part GF;
∴ the remainder AG is greater than the remainder HG.
But AG = DG; [I. *Definition* 15.
∴ DG is greater than HG, the less than the greater,
which is impossible.
∴ FG, being produced, cannot pass otherwise than through the point A,
that is, it must pass through A.

Wherefore, *if two circles, etc.* [Q.E.D.

Proposition 12. Theorem.

If two circles touch one another externally, the straight line which joins their centres shall pass through the point of contact.

Let the two circles ABC, ADE touch one another externally at the point A; and let F be the centre of the circle ABC, and G the centre of the circle ADE:
the straight line FG *shall pass through the point* A.



Proof. For, if not, let it pass otherwise, if possible, as FCDG, and join FA, AG.

Then, because F is the centre of the circle ABC,

FA = FC; [I. *Definition* 15.

and because G is the centre of the circle ADE, GA = GD;

∴ FA, AG are equal to FC, DG; [*Axiom* 2.

∴ the whole FG is greater than FA, AG.

But FG is also less than FA, AG, [I. 20.

which is impossible;

∴ the straight line FG cannot pass otherwise than through the point A,

that is, it must pass through A.

Wherefore, *if two circles, etc.* [Q.E.D.

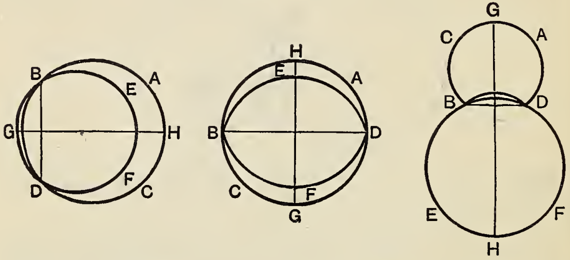
Note. Propositions 11 and 12 may be enunciated in one, thus: *If two circles touch one another, their two centres and the point of contact are in a straight line.*

PROPOSITION 13. THEOREM.

One circle cannot touch another at more points than one, whether it touches it on the inside or outside.

For, if it be possible, let the circle EBF touch the circle ABC at more points than one, namely, at B and D, internally as in Figs. 1 and 2, or externally as in Fig. 3.

Join BD, and draw GH bisecting BD at right angles. [I. 10, 11.



Proof. Because, in all three figures, BD is a chord of each circle, and GH bisects it at right angles;

∴ the centre of each circle is in GH. [III. 1, *Corollary.*

∴ GH passes through the point of contact. [III. 11, 12.

But GH does not pass through the point of contact, because the points B, D are out of the line GH,

which is absurd.

Wherefore, *one circle cannot, etc.* [Q.E.D.

EXERCISES.

****1.** If the distance between the centres of two circles be equal to the sum of their radii, the circles touch externally.

****2.** If it be equal to their difference, the circles touch internally.

3. Describe a circle with a given centre which shall touch a given circle. How many such circles can be drawn?

4. Describe a circle to pass through a given point and touch a given circle at a given point.

[Let B be the given point on the given circle whose centre is O and A the other given point. Join AB; produce OB to C, and at A make the $\angle BAD$ equal to the $\angle ABC$; let AD meet OC in D; then D is the centre of the required circle, etc.]

5. Describe a circle with a given radius to pass through a given point and touch a given circle.

[The centre of the required circle is at a distance from the centre of the given circle equal to the sum of the given radius and the radius of the given circle; it is also at a distance from the given point equal to the given radius; $\therefore$ etc.]

6. Describe a circle with a given radius to touch two given circles.

7. Three circles touch one another externally at the points A, B, C; from A the straight lines AB, AC are produced to cut the circle BC at D and E. Show that DE is a diameter of BC, and is parallel to the straight line joining the centres of the other circles.

8. If in any two given circles which touch one another there be drawn two parallel diameters, an extremity of each diameter and the point of contact shall lie in the same straight line.

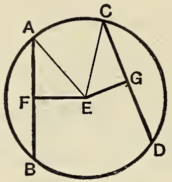
9. If circles be described on the two sides of a right-angled triangle as diameters, they will be touched by a circle whose centre is the middle point of the hypotenuse, and whose diameter is equal to the sum of the sides.

10. If a variable circle touch two fixed circles the sum, or the difference, of the distances of its centre from those of the fixed circles is equal to the sum, or the difference, of their radii.

Proposition 14. Theorem.

Equal chords in a circle are equally distant from the centre; and those which are equally distant from the centre are equal to one another.

Let the chords AB, CD in the circle ABCD be equal to one another:
they shall be equally distant from the centre.



Construction. Find E, the centre of the circle ABDC; [III. 1.
from E draw EF, EG perpendiculars to AB, CD, [I. 12.
and join EA, EC.

Proof. Because EF, passing through the centre, cuts AB, which does not pass through the centre, at right angles, it also bisects it; [III. 3.
∴ AF = FB, and AB is double of AF.
For the like reason CD is double of CG.
But AB = CD; [*Hypothesis.*
∴ AF = CG. [*Axiom* 7.
Also because AE = CE, [I. *Definition* 15.
the square on AE = the square on CE.
But the square on AE = the squares on AF, FE, because the angle AFE is a right angle; [I. 47.
similarly the square on CE = the squares on CG, GE;
∴ the squares on AF, FE = the squares on CG, GE. [*Axiom* 1.

But the square on AF = the square on CG, since AF = CG;

∴ the remaining sq. on FE = the remaining sq. on GE; [*Ax.* 3.

∴ EF = EG;

∴ AB, CD are equally distant from the centre. [III. *Def.* 6.

Conversely: Let AB, CD be chords equally distant from the centre, that is, let EF = EG;

then shall AB = CD.

For, the same construction being made, it may be shewn, as before, that AB is double of AF, and CD double of CG, and that the squares on EF, FA = the squares on EG, GC; but the square on EF = the square on EG, because EF = EG; [*Hypothesis.*

∴ the remaining sq. on FA = the remaining sq. on GC; [*Ax.* 3.

∴ AF = CG.

But AB was shewn to be double of AF, and CD double of CG;

∴ AB = CD. [*Axiom* 6.

Wherefore, *equal straight lines, etc.*

EXERCISE.

In a given circle draw a chord of given length, not greater than the diameter, so that its centre may be on a given chord.

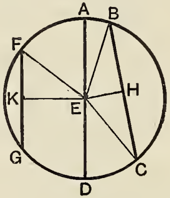
PROPOSITION 15. THEOREM.

The diameter is the greatest chord in a circle;

and, of all others, that which is nearer to the centre is always greater than one more remote;

and the greater is nearer to the centre than the less.

Let ABCD be a circle, of which AD is a diameter, and E the centre; and let BC be nearer to the centre than FG: AD *shall be greater than any chord* BC *which is not a diameter, and* BC *shall be greater than* FG.



Construction. From the centre E draw EH, EK perpendiculars to BC, FG, [I. 12.

and join EB, EC, EF.

Proof. Because AE = BE, and ED = EC, [I. *Definition* 15.

∴ AD = BE, EC; [*Axiom* 2.

but BE, EC are together greater than BC; [I. 20.

∴ also AD is greater than BC.

Also, because BC is nearer to the centre than FG, [*Hypothesis*.

EH is less than EK. [III. *Definition* 6.

Now it may be shewn, as in the preceding proposition, that BC is double of BH, and FG double of FK, and that the squares on EH, HB = the squares on EK, KF.

But the square on EH is less than the square on EK,
because EH is less than EK;
∴ the square on HB is greater than the square on KF;
and therefore BH is greater than FK;
∴ BC is greater than FG.

Conversely: Let BC be greater than FG: then, the same construction being made, EH shall be less than EK.

For, because BC is greater than FG, BH is greater than FK.
But the squares on BH, HE = the squares on FK, KE;
and the square on BH is greater than the square on FK,
because BH is greater than FK;
∴ the square on HE is less than the square on KE,
and therefore EH is less than EK.

Wherefore, *the diameter, etc.* [Q.E.D.

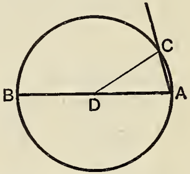
EXERCISE.

**Through a given point within a circle, draw the shortest chord.

Proposition 16. Theorem.

(1) *The straight line drawn at right angles to the diameter of a circle from the extremity of it falls without the circle;*
and (2) *any other straight line drawn from the extremity must cut the circle.*

Let ABC be a circle, of which D is the centre and AB a diameter:
the straight line drawn at right angles to AB, *from its extremity* A, *shall fall without the circle.*



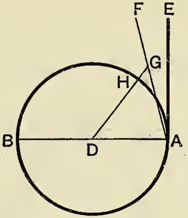
For, if not, let it fall, if possible, within the circle, as AC, and draw DC to the point C, where it meets the circumference.

(1) **Proof.** Because DA = DC, [I. *Definition* 15.
the ∠DAC = the ∠DCA. [I. 5.
But the ∠DAC is a right ∠; [*Hypothesis.*
∴ the ∠DCA is a right ∠;
∴ the angles DAC, DCA = two right angles,
which is impossible. [I. 17.
∴ the straight line drawn from A at right angles to AB does not fall within the circle.

And in the same manner it may be shewn that it does not fall on the circumference;
∴ it must fall without the circle, as AE.

(2) If possible, let AF be another straight line drawn through A which does not cut the circle; and from the centre D draw DG perpendicular to AF; [I. 12.
let DG meet the circumference at H.

Then, because the ∠DGA is a right ∠, [*Construction.*
the ∠DAG is less than a right ∠; [I. 17.
∴ DA is greater than DG. [I. 19.



But DA = DH; [I. *Definition* 15.
∴ DH is greater than DG, the less than the greater, which is impossible;
∴ no straight line, except AE, can be drawn from the point A which does not cut the circle.

Wherefore, *the straight line, etc.* [Q.E.D.

Corollaries. (1) The straight line which is drawn at right angles to the diameter of a circle from the extremity of it touches the circle. [III. *Definition* 2.

(2) A tangent touches the circle at one point only, because if it did meet the circle at two points it would fall within it. [III. 2.

(3) There can be but one tangent at the same point.

EXERCISES.

1. Draw parallel to a given straight line a straight line to touch a given circle.

2. Draw perpendicular to a given straight line a straight line to touch a given circle.

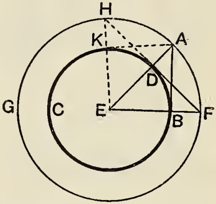
3. Two circles have the same centre. Show that all chords of the outer circle which touch the inner circle are equal, and are bisected at the point of contact.

Proposition 17. Problem.

From a given point, either without or on the circumference, to draw a tangent to a given circle.

Case I. Let the given point A be without the given circle BCD:

it is required to draw from A *a tangent to the given circle.*



Construction. Find E, the centre of the circle, [III. 1.
and join AE, cutting the given circle at D; with centre E and distance EA describe the circle AFG; from D draw HDF at right angles to EA to meet the circle AFG in F and H, [I. 11.
and join EF and EH, cutting the given circle at B and K; join AB and AK. AB and AK shall touch the circle BCD.

Proof. In the triangles AEB, FED,

because { AE = FE, being radii of the outer circle,
and EB = ED, being radii of the given circle,
and the angle AEB is common;

∴ the triangles are equal in all respects; [I. 4.

∴ the ∠ABE = the ∠FDE.

But the ∠FDE is a right ∠; [*Construction.*

∴ the ∠ABE is a right ∠. [*Axiom* 1.

∴ BA, being drawn at right angles to a diameter from one of its extremities B, touches the circle. [III. 16, *Corollary.*

Similarly it can be shewn that AK touches the circle at K.

Also AB and AK are drawn from the given point A. [Q.E.F.

Case II. If the given point be in the circumference of the circle, as the point D, draw DE to the centre E, and DF at right angles to DE; then DF touches the circle. [III. 16, *Cor.*

Corollaries. (1) Two, and only two, tangents can be drawn to a circle from an external point.

(2) These two tangents are equal, subtend equal angles at the centre, and make equal angles with the line joining the centre to the given external point.

For each of the angles ABE, AKE is a right angle;
∴ the squares on AB, BE = the square on AE,
and the squares on AK, KE = the square on AE.
But the square on BE = the square on KE;
∴ the square on AB = the square on AK; ∴ AK = AB,
and the triangles AKE, ABE are equal in all respects;
∴ the ∠KAE = the ∠BAE, and
the ∠KEA = the ∠BEA.

EXERCISES.

**1. Prove that the following construction also gives the tangents from A. *Join* EA *and bisect it in* O; *with centre* O *and radius* OA *or* OE, *describe a circle and let it meet the given circle in* P *and* Q; *join* AP *and* AQ: *these are the required tangents.*

[Join OP; since OP = OE, ∴ ∠OEP = ∠OPE; since OP = OA, ∴ ∠OAP = ∠OPA; ∴ ∠s OEP, OAP = ∠s OPE, OPA = ∠APE. Hence, by I. 32, ∠APE is a right angle, and AP is a tangent at P; so AQ is a tangent at Q.]

**2. The centre of any circle which touches two given straight lines lies on the bisector of the angle between them.

**3. A quadrilateral is described so that its sides touch a circle. Shew that two of its opposite sides are together equal to the other two sides, and conversely. [Use Cor. 2.]

4. If a hexagon, or any polygon having an even number of sides, circumscribe a circle, the sums of its alternate sides are equal.

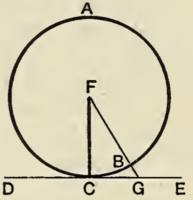
**5. CN is drawn from the centre C of a circle perpendicular to the chord AB. Prove that the tangents at A and B meet at a point T on CN produced.

Prove also that the rectangle CN, CT equals the square on CA.

Proposition 18. Theorem.

If a straight line touch a circle, the radius drawn from the centre of the point of contact shall be perpendicular to the line touching the circle.

Let the straight line DE touch the circle ABC at the point C;
the radius FC, *drawn from the centre* F, *shall be perpendicular to* DE.



Construction. For if not, let FG be drawn from F perpendicular to DE, meeting the circumference at B.

Proof. In the △FCG, because FGC is a right ∠, [*Hypoth.*
the ∠FCG is less than a right ∠; [I. 17.
∴ FC is greater than FG. [I. 19.
But FC = FB; [I. *Definition* 15.
∴ FB is greater than FG, the less than the greater, which is impossible.
∴ FG is not perpendicular to DE.

Similarly, it may be shewn that no other straight line from F is perpendicular to DE except FC;
∴ FC is perpendicular to DE.

Wherefore, *if a straight line, etc.* [Q.E.D.

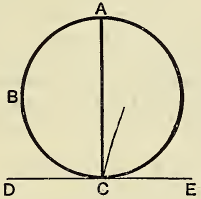
PROPOSITION 19. THEOREM.

If a straight line touch a circle, and from the point of contact a straight line be drawn at right angles to the touching line, the centre of the circle shall be in that line.

Let the straight line DE touch the circle ABC at C, and from C let CA be drawn at right angles to DE:

the centre of the circle shall be in CA.

For, if not, if possible, let F be the centre, and join CF.



Proof. Because DE touches the circle ABC, and FC is drawn from the centre to the point of contact, FC is perpendicular to DE; [III. 18.

∴ the ∠FCE is a right ∠.

But the ∠ACE is also a right ∠; [*Construction.*

∴ the ∠FCE = the ∠ACE, [*Axiom* 11.

that is, the less to the greater, which is impossible.

∴ F is not the centre of the circle ABC.

Similarly, it may be shewn that no other point out of CA is the centre; therefore the centre is in CA.

Wherefore, *if a straight line, etc.* [Q.E.D.

EXERCISES.

****1.** If two tangents to a circle be parallel, the points of contact are at the extremities of a diameter.

****2.** Shew that no parallelogram can be described about a circle except a rhombus.

[Let ABCD be a $\|^{gm}$ described about a circle, centre O, and let E, F, G, H be the points of contact of the sides AB, BC, CD, DA.

By Ex. 1, EOG, FOH are straight lines;

$\therefore$ by III. 16 Cor., OA, OC bisect the $\angle^s$ EOH, FOG, and are $\therefore$ in a straight line;

$\therefore$ $\triangle^s$ CGO, AOE are equal in all respects, so that $CG = AE = AH$;

$\therefore$ $DC = DG + GC = DH + HA = DA$, etc.]

****3.** If a quadrilateral be described about a circle, the angles subtended at the centre of the circle by any two opposite sides of the figure are together equal to two right angles.

4. Two parallel tangents to a circle intercept on a third tangent a distance which subtends a right angle at the centre.

5. The tangent at any point A of a circle meets two fixed tangents in P and Q. Prove that the points P and Q subtend a constant angle at the centre of the circle.

6. A series of circles touch a given straight line at a given point. Where will their centres all lie?

7. Describe a circle of given radius to pass through a given point and touch a given straight line.

8. A straight line is drawn touching two circles. Shew that the chords are parallel which join the points of contact and the points where the straight line through the centres meets the circumferences.

9. If two tangents TP, TQ be drawn to a circle, prove that the angle between them is double the angle between the line PQ joining their points of contact, and the diameter through either of them.

10. In the diameter of a circle produced, determine a point so that the tangent drawn from it to the circumference shall be of given length.

11. Two circles touch externally in A and the straight line BC touches them in B and C. Prove that BAC is a right angle.

12. Through a given point within or without a given circle draw a chord of given length not greater than the diameter.

[Use Page 137, Ex 3.]

****13.** Draw a straight line to touch each of two given circles.

[See App., Art. 39.]

14. Construct a triangle, given the vertical angle, one of the sides containing it, and the perpendicular altitude.

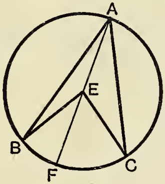
Proposition 20. Theorem.

The angle at the centre of a circle is double of the angle at the circumference on the same arc.

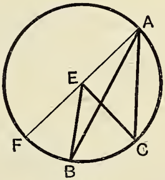
Let ABC be a circle, and BEC an angle at the centre, and BAC an angle at the circumference, which have the same arc BC for their base:

the angle BEC *shall be double of the angle* BAC.

Construction. Join AE, and produce it to F.



Case I.



Case II.

Proof. Case I. Let the centre of the circle be within the angle BAC.

Then, because EA = EB, the ∠EAB = the ∠EBA; [I. 5.

∴ the angles EAB, EBA are double of the angle EAB.

But the angle BEF = the angles EAB, EBA; [I. 32.

∴ the ∠BEF is double of the ∠EAB.

Similarly, the ∠FEC is double of the ∠EAC.

∴ the whole ∠BEC is double of the whole ∠BAC.

Case II. Let the centre E be without the ∠BAC.

As in Case I., the ∠FEC is double of the ∠FAC, and the ∠FEB, a part of the first, is double of the ∠FAB, a part of the other;

∴ the remaining ∠BEC is double of the remaining ∠BAC.

Case III. If the centre E lie on the straight line AB it is clear, if the corresponding figure be drawn, that, as in Case I., the ∠BEC is double of the ∠BAC.

Wherefore, *the angle at the centre, etc.* [Q.E.D.

[For an important note on III. 20, see Page 326.]

Proposition 21. Theorem.

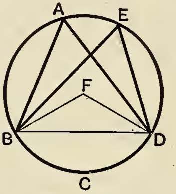
The angles in the same segment of a circle are equal to one another.

Let ABCD be a circle, and BAD, BED angles in the same segment BAED:
the angles BAD, BED *shall be equal to one another.*

Construction. Find F the centre of the circle ABCD. [III. 1.

Case I. Let the segment BAED be greater than a semicircle.

Join BF, DF.



Proof. Because the angles BFD and BAD are angles at the centre and circumference standing on the same arc BD.

∴ the ∠BFD is double the ∠BAD. [III. 20.

Similarly the ∠BFD is double the ∠BED.

∴ the ∠BAD = the ∠BED. [*Axiom* 7.

Case II. Let the segment BAED be not greater than a semicircle.

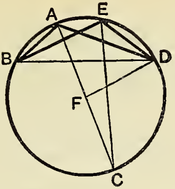
Construction. Draw AF to the centre, and produce it to meet the circumference at C, and join CE.

Proof. Since CFD and CED are angles at the centre and circumference standing on the same arc CD;

∴ the ∠CFD is double the ∠CED; [III. 20.

Similarly, the ∠CFD is double the ∠CAD;

∴ the ∠CAD = the ∠CED.



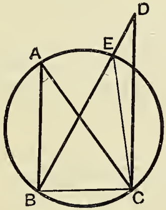
But the segment BAEC is greater than a semicircle, and therefore the ∠BAC = the ∠BEC, by Case I.

∴ the whole ∠BAD = the whole ∠BED. [*Axiom* 2.

Wherefore, *the angles in the same segment, etc.* [Q.E.D.

Converse of Proposition 21.

The vertices of equal angles, standing on the same base and on the same side of it, lie on a segment of a circle passing through the extremities of the base.



Let two of the angles be ~~ABC~~ and ~~DBC~~.

Describe a circle about ABC.

[This may be done by Prop. 5 of Book IV., which the student is now recommended to read.]

If this circle do not pass through D, let it cut BD in E, and join EC.

Then ∠BEC = ∠BAC, by the preceding proposition,

= ∠BDC, by hypothesis.

But this is impossible by I. 16,

∴ D must lie on the circle through B, A, C.

Similarly the vertex of any other such equal angle, whose arms pass through B and C, must lie on this same circle.

EXERCISES.

1. Two circles cut at A and B, and any chord PAQ is drawn terminated by the two circumferences; prove that the angle PBQ is constant.

2. If two straight lines AEB, CED in a circle intersect at E, the angles subtended by AC and BD at the centre are together double of the angle AEC.

3. Two tangents AB, AC are drawn to a circle; D is any point on the circumference outside of the triangle ABC; shew that the sum of the angles ABD and ACD is constant.

4. P, Q are any points in the circumferences of two segments described on the same straight line AB, and on the same side of it; the angles PAQ, PBQ are bisected by the straight lines AR, BR meeting at R; shew that the angle ARB is constant.

[$\angle PAR = \angle RAQ$, *i.e.* $\angle PAB - \angle RAB = \angle RAB - \angle QAB$;
$\therefore\ 2\angle RAB = \angle PAB + \angle QAB$.
So $2\angle RBA = \angle PBA + \angle QBA$;
$\therefore\ 2\angle^s RAB, RBA = \angle^s PAB, PBA, QAB, QBA$,
i.e. 4 rt. $\angle^s - 2\angle ARB = 2$ rt. $\angle^s - \angle APB + 2$ rt. $\angle^s - \angle AQB$;
$\therefore\ 2\angle ARB = \angle APB + \angle AQB =$ const.]

5. Two segments of a circle are on the same base AB, and P is any point in the circumference of one of the segments; the straight lines APD, BPC are drawn meeting the circumference of the other segment at D and C; AC and BD are drawn intersecting at Q; shew that the angle AQB is constant.

[$\angle AQB = \angle ACB - \angle CBQ$, and $\angle APB = \angle ADB + \angle CBQ$
$\therefore\ \angle^s AQB, APB = \angle^s ACB, ADB = 2\angle ACB$.
But the $\angle^s APB$, ACB are constant; $\therefore$ etc.]

6. APB is a fixed chord passing through P a point of intersection of two circles AQP, PBR, and QPR is any other chord of the circles passing through P; shew that AQ and RB when produced meet at a constant angle.

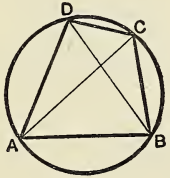
7. Two circles meet in A and B; through A two chords ACD, AC′D′ are drawn cutting the circles in C, D and C′, D′; prove that the $\triangle^s$BCD, BC′D′ are equiangular.

Proposition 22. Theorem.

The opposite angles of any quadrilateral figure inscribed in a circle are together equal to two right angles.

Let ABCD be a quadrilateral figure inscribed in the circle ABCD:

any two of its opposite angles shall be together equal to two right angles.



Construction. Join AC, BD.

Proof. The ∠CAB = the ∠CDB, because they are in the same segment CDAB; [III. 21.

and the ∠ACB = the ∠ADB, because they are in the same segment ADCB;

∴ the two angles CAB, ACB together = the whole ∠ADC. [*Axiom* 2.

To each of these equals add the ∠ABC;

∴ the three angles CAB, ACB, ABC = the two angles ABC, ADC.

But the three angles CAB, ACB, ABC, being the angles of the △ACB, are together equal to two right angles; [I. 32.

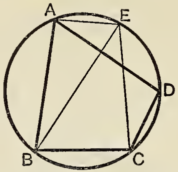
∴ also the angles ABC, ADC together = two right angles.

In the same manner it may be shewn that the angles BAD, BCD are together equal to two right angles.

Wherefore, *the opposite angles, etc.* [Q.E.D.

CONVERSE OF III. 22.

This is true and very important; namely,
If two opposite angles of a quadrilateral be together equal to two right angles, a circle may be circumscribed about the quadrilateral.



For, let ABCD denote the quadrilateral. Describe a circle round the triangle ABC, by IV. 5. Take any point E on the arc cut off by AC and on the same side of AC that D is.
Then the angles at B and E together = two right angles, by III. 22,
and the angles at B and D together = two right angles, by hypothesis;
∴ the angle at E = the angle at D;
∴ by the converse of III. 21, D is on the same segment as E.

Note. A quadrilateral, such as ABCD in the above figure, which can be inscribed in a circle is called a **Cyclic Quadrilateral**, and the four points A, B, C, D are said to be **Concyclic**.

EXERCISES.

****1.** If one side of a quadrilateral inscribed in a circle be produced, the exterior angle so formed is equal to the interior opposite angle of the quadrilateral.

2. The straight lines which bisect any angle of a quadrilateral inscribed in a circle and the opposite exterior angle meet on the circumference of the circle. [Use Ex. 1.]

3. A triangle is inscribed in a circle; shew that the sum of the angles in the three segments exterior to the triangle is equal to four right angles.

4. A quadrilateral is inscribed in a circle; shew that the sum of the angles in the four segments of the circle exterior to the quadrilateral is equal to six right angles.

5. If a polygon of an even number of sides be inscribed in a circle, the sum of the alternate angles together with two right angles is equal to as many right angles as the figure has sides.

[Let ABCDEF... be the polygon and O the centre of the circle. The $\angle ABC = 2$ rt. $\angle^s - \frac{1}{2}\angle AOC$; $\angle CDE = 2$ rt. $\angle^s - \frac{1}{2}\angle COE$, etc.;
$\therefore$ on addition, the sum of the alternate angles

$$=2 \text{ rt. } \angle^s \times \tfrac{1}{2} \text{ no. of the sides} - \tfrac{1}{2} \text{ all the } \angle^s \text{ at } O = \text{etc.}]$$

6. Shew that the four straight lines bisecting the interior (or the exterior) angles of any quadrilateral form a quadrilateral which can be inscribed in a circle.

[Let the bisectors of the interior angles A and B, B and C, C and D, D and A of a quadrilateral meet in E, F, G, H. Then

$$\angle FEH = 2 \text{ rt. } \angle^s - \tfrac{1}{2}\angle A - \tfrac{1}{2}\angle B,$$

$$\text{and } \angle FGH = 2 \text{ rt. } \angle^s - \tfrac{1}{2}\angle C - \tfrac{1}{2}\angle D;$$

$\therefore$ $\angle^s$FEH, FGH $=4$ rt. $\angle^s$ − half the sum of the interior angles of ABCD

$$=4 \text{ rt. } \angle^s - 2 \text{ rt. } \angle^s = 2 \text{ rt. } \angle^s; \ \therefore \text{ etc.}$$

Similarly for the bisectors of the exterior angles.]

****7.** Shew that no parallelogram except a rectangle can be inscribed in a circle.

8. D is any point on the arc BC of a circle whose centre is A; CD is produced to E; prove that the angle BDE is half the angle BAC.

9. AOB is a triangle; C and D are points in BO and AO respectively, such that the angle ODC is equal to the angle OBA; shew that a circle may be described round the quadrilateral ABCD.

10. ABCD is a quadrilateral inscribed in a circle, and the sides AB, DC when produced meet at O; shew that the triangle AOC is equiangular to the triangle BOD, and the triangle AOD to the triangle BOC.

****11.** If any two consecutive sides of a hexagon inscribed in a circle be respectively parallel to their opposite sides, the remaining sides are parallel to each other.

[Let ABCDEF be the hexagon having FA, AB parallel to CD, DE. Then $\angle FEB = 2$ rt. $\angle^s - \angle FAB$ [III. 22] $=2$ rt. $\angle^s$ − DEC [I. 29] $=\angle EBC$.

12. ABCD is a quadrilateral inscribed in a circle; AB and CD meet in P; AD and BC meet in Q; prove that the bisectors of the angles P and Q are at right angles.

[If O be the intersection of the bisectors, and OQ meet CD in L, then

$$\angle OPC = \tfrac{1}{2}D - \tfrac{1}{2}PAD \text{ and } \angle OQD = \tfrac{1}{2}D - \tfrac{1}{2}DCQ;$$

$$\therefore \angle POQ = \angle PLQ - \angle OPC = \angle D - \angle OQD - \angle OPC$$

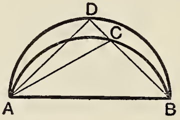
$$= \angle D - \tfrac{1}{2}(2\angle D - \angle PAD - \angle PCQ)$$

$$= \tfrac{1}{2}\angle PAD + \tfrac{1}{2}\angle PCQ = \text{a rt. } \angle.]$$

Proposition 23. Theorem.

On the same chord, and on the same side of it, there cannot be two similar segments of circles which do not coincide with one another.

If it be possible, on the same chord AB, and on the same side of it, let there be two similar segments of circles ACB, ADB not coinciding with one another.



Construction. Because the circle ACB cuts the circle ADB at the two points A, B, they cannot cut one another at any other point; [III. 10.

therefore one of the segments must fall within the other;

let ACB fall within ADB;

draw the straight line BCD, and join AC, AD.

Proof. Because ACB, ADB are, by hypothesis, similar segments of circles, they contain equal angles; [III. *Definition* 11.

∴ the ∠ACB = the ∠ADB,

that is, the exterior ∠ of the △ACD is equal to the interior and opposite ∠, which is impossible. [I. 16.

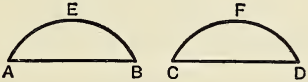
Wherefore, *on the same straight line, etc.* [Q.E.D.

PROPOSITION 24. THEOREM.

Similar segments of circles on equal chords are equal to one another.

Let AEB, CFD be similar segments of circles on the equal chords AB, CD:

the segment AEB *shall be equal to the segment* CFD.



Proof. If the segment AEB be applied to the segment CFD, so that the point A may be on the point C, and AB on CD,

then B will coincide with D, because AB = CD; [*Hyp.*

∴ AB coinciding with CD, the segment AEB must coincide with the segment CFD, [III. 23.

and is therefore equal to it.

Wherefore, *similar segments, etc.* [Q.E.D.

Proposition 25. Problem.

A segment of a circle being given, to describe the circle of which it is a segment.

Let ABC be the given segment of a circle:
it is required to describe the circle of which it is a segment.

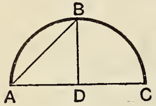


Fig. 1.

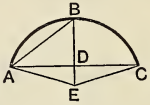


Fig. 2.

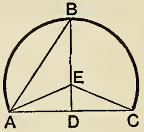


Fig. 3.

Construction. Bisect AC at D; [I. 10.
from D draw DB at right angles to AC to meet the circumference in B, [I. 11.
and join AB.

Case I. Let the angles ABD, BAD be equal [Fig 1].

Then DB = DA; [I. 6.
but DA = DC; [*Construction.*
therefore DB = DC. [*Axiom* 1.

∴ DA, DB, DC are all equal,
and therefore D is the centre of the circle. [III. 9.

With D, and radius equal to any of the three DA, DB, DC, describe a circle; this will pass through the other points, and the circle of which ABC is a segment is described.

Also because the centre D is in AC, the segment ABC is a semicircle.

Case II. Let the angles ABD, BAD be not equal [Figs. 2, 3].

Construction. At the point A make the angle BAE equal to the angle ABD; [I. 23.
produce BD, if necessary, to meet AE at E, and join EC.

Proof. Because the $\angle$BAE = the $\angle$ABE, [*Construction.*

EA = EB. [I. 6.

Then, in the triangles ADE, CDE,

because { AD = CD, [*Construction.*
and DE is common,
and the right $\angle$ADE = the right $\angle$CDE;

$\therefore$ the base EA = the base EC. [I. 4.

But EA was shewn to be equal to EB;

therefore EB = EC. [*Axiom* 1.

$\therefore$ the three straight lines EA, EB, EC are all equal, and therefore a circle, whose centre is E and whose radius is any of the three EA, EB, EC, will pass through the other points, and the circle of which ABC is a segment is described. [III. 9.

And it is evident that, if the angle ABD be greater than the angle BAD, the centre E falls without the segment ABC, which is therefore less than a semicircle;
but if the angle ABD be less than the angle BAD, the centre E falls within the segment ABC, which is greater than a semicircle.

Wherefore *a segment of a circle being given, the circle has been described of which it is a segment.* [Q.E.F.

Alternative Method for Prop. 25.

The following is an alternative method: *Let* A, B, C *be three points on the arc*; *join* AB *and* BC; *bisect* AB *and* BC *in* D *and* E *respectively*; *draw* DF *perpendicular to* AB *and* EF *perpendicular to* BC, *and let them meet in* F. Then F shall be the centre of the required circle.

For by III. 1 Cor. the centre is in the straight line DF, which bisects AB at right angles.

Similarly, it is in EF.

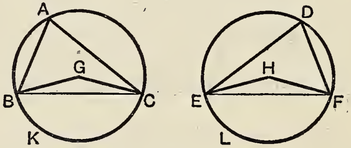
$\therefore$ it is at F, and a circle whose centre is at F and radius equal to either FA, FB, or FC will be the circle required; for as in III. 1,

FA = FB and FB = FC.

PROPOSITION 26. THEOREM.

In equal circles the arcs which are subtended by equal angles, whether they be at the centres or circumferences, are equal.

Let ABC, DEF be equal circles; and let BGC, EHF be equal angles in them at their centres, and BAC, EDF equal angles at their circumferences:
the arc BKC *shall be equal to the arc* ELF.



Construction. Join BC, EF.

Proof. Because the circles ABC, DEF are equal, [*Hyp.*
their radii are equal; [III. *Definition* 1.
Then, in the triangles BGC, EHF,

because { BG = EH,
and GC = HF,
and the ∠BGC = the ∠EHF; [*Hypothesis.*

∴ the base BC = the base EF. [I. 4.

And because the ∠BAC = the ∠EDF, [*Hypothesis.*
the segment BAC is similar to the segment EDF, [III. *Def.* 11.
and they are on equal chords BC, EF.

∴ the segment BAC = the segment EDF. [III. 24.

But the whole circle ABC = the whole circle DEF; [*Hypothesis.*
∴ the remaining segment BKC = the remaining segment ELF; [*Axiom* 3.

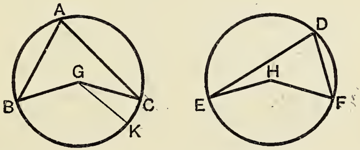
∴ the arc BKC = the arc ELF.

Wherefore, *in equal circles, etc.* [Q.E.D.

Proposition 27. Theorem.

In equal circles, the angles which stand on equal arcs are equal to one another, whether they be at the centres or circumferences.

Let ABC, DEF be equal circles, and let the angles BGC, EHF at their centres, and the angles BAC, EDF at their circumferences, stand on equal arcs BC, EF:
the angle BGC *shall be equal to the angle* EHF, *and the angle* BAC *equal to the angle* EDF.



If the ∠BGC be equal to the ∠EHF, it is manifest that the ∠BAC is also equal to the ∠EDF. [III. 20, *Axiom* 7.

But, if not, one of them must be the greater.

Let BGC be the greater, and at G make the ∠BGK equal to the ∠EHF. [I. 23.

Proof. Because the ∠BGK = the ∠EHF;

∴ the arc BK = the arc EF. [III. 26.

But the arc EF = the arc BC; [*Hypothesis.*

∴ the arc BK = the arc BC, [*Axiom* 1.

the less to the greater, which is impossible.

∴ the ∠BGC is not unequal to the ∠EHF,

that is, it is equal to it.

And the ∠ at A is half of the ∠BGC, and the ∠ at D is half of the ∠EHF; [III. 20.

∴ the ∠ at A = the ∠ at D. [*Axiom* 7.

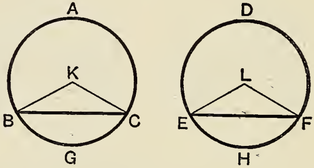
Wherefore, *in equal circles, etc.* [Q.E.D.

Note. Prop. 27 is the converse of Prop. 26.

PROPOSITION 28. THEOREM.

In equal circles, the arcs which are cut off by equal chords are equal, the greater equal to the greater, and the less equal to the less.

Let ABC, DEF be equal circles, and BC, EF equal chords in them, which cut off the two greater arcs BAC, EDF, and the two less arcs BGC, EHF:
the greater arc BAC *shall be equal to the greater arc* EDF, *and the less arc* BGC *equal to the less arc* EHF.



Construction. Find K, L, the centres of the circles, [III. 1.
and join BK, KC, EL, LF.

Proof. Because the circles are equal, [*Hypothesis.*
their radii are equal; [III. *Definition* 1.
Then, in the triangles BKC, ELF,

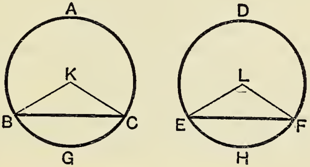
because { BK = EL,
and KC = LF,
and the base BC = the base EF, [*Hypothesis.*

∴ the ∠BKC = the ∠ELF; [I. 8.
∴ the arc BGC = the arc EHF. [III. 26.
But the circumference ABGC = the circumference DEHF; [*Hypothesis.*
∴ the remaining arc BAC = the remaining arc EDF. [*Ax.* 3.
Wherefore, *in equal circles, etc.* [Q.E.D.

PROPOSITION 29. THEOREM.

In equal circles, the chords which cut off equal arcs are equal.

Let ABC, DEF be equal circles, and let BGC, EHF be equal arcs in them, and join BC, EF:
the chord BC *shall be equal to the chord* EF.



Construction. Find K, L, the centres of the circles, [III. 1.
and join BK, KC, EL, LF.

Proof. Because the arc BGC = the arc EHF, [*Hypothesis.*
the angle BKC = the angle ELF. [III. 27.
And because the circles ABC, DEF are equal, [*Hypothesis.*
their radii are equal; [III. *Definition* 1.
∴ the two sides BK, KC = the two sides EL, LF,
and they contain equal angles; [*Proved.*
∴ the base BC = the base EF. [I. 4.
Wherefore, *in equal circles, etc.* [Q.E.D.

Note 1. Prop. 29 is the converse of Prop. 28.

Note 2. The Propositions 26–29 tell us that in equal circles
If two angles are equal, so also are the chords and arcs subtended;
If two chords are equal, so also are the angles subtended and the arcs cut off;
If two arcs are equal, so also are the angles and chords.

Note 3. It is clear that Props. 26–29 could easily be proved by *Superposition.*

EXERCISES.

1. The straight lines joining the extremities of the chords of two equal arcs of a circle, towards the same parts, are parallel to each other.

2. The straight lines in a circle which join the extremities of two parallel chords are equal to each other.

****3.** The straight lines which bisect the vertical angles of all triangles on the same base and on the same side of it, and having equal vertical angles, all intersect at the same point.

4. The internal and external bisectors of the vertical angle of a triangle inscribed in a circle meet the circumference again in points which are equidistant from the ends of the base, and which lie on the straight line bisecting the base at right angles.

5. If A, B be two fixed points on a circle, and CD a chord of constant length ; prove that the intersections of AD, BC and of AC, BD lie on two fixed circles.

[Since CD is constant in length,

$\therefore$ the $\angle^s$CAD, CBD are constant [Props. 28, 27].

Hence if AD, BC meet in O, then

$\angle AOB = \angle ACB + \angle DAC =$ const. ; $\therefore$ etc.]

6. AB is a common chord of two circles ; through C any point of one circumference straight lines CAD, CBE are drawn terminated by the other circumference ; shew that the arc DE is invariable.

7. If two equal circles cut each other, and if through one of the points of intersection a straight line be drawn terminated by the circles, the straight lines joining its extremities with the other point of intersection are equal.

8. The straight line joining the feet of the perpendiculars from any point of a circle upon two diameters given in position is constant.

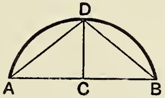
Conversely, if a straight line of given length move so that its ends slide along two given straight lines, the two perpendiculars at its ends to the given lines meet on a fixed circle.

PROPOSITION 30. PROBLEM.

To bisect a given arc, that is, to divide it into two equal parts.

Let ADB be the given arc :

it is required to bisect it.



Construction. Join AB, and bisect it at C ; [I. 10.
from the point C draw CD at right angles to AB meeting the arc at D. [I. 11.

The arc ADB shall be bisected at the point D.

Join AD, DB.

Proof. In the triangles ACD, BCD,

because { AC = CB, [Construction.
and CD is common to both,
and the right ∠ACD = the right ∠BCD ; [Const.

∴ the base AD = the base BD. [I. 4.

But equal straight lines cut off equal arcs, the greater equal to the greater, and the less equal to the less, [III. 28.
and each of the arcs AD, DB is less than a semicircle, because DC, if produced, is a diameter ; [III. 1, *Cor.*

∴ the arc AD = the arc DB.

Wherefore, *the given arc is bisected at D.* [Q.E.F.

PROPOSITION 31. THEOREM.

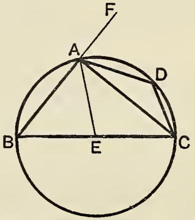
An angle in a semicircle is a right angle,
but an angle in a segment greater than a semicircle is less than a right angle,
and an angle in a segment less than a semicircle is greater than a right angle.

Let ABCD be a circle, of which BC is a diameter and E the centre; and draw CA, dividing the circle into the segments ABC, ADC:

(1) *the angle in the semicircle* BAC *shall be a right angle;*

(2) *the angle in the segment* ABC, *which is greater than a semicircle, shall be less than a right angle;*

(3) *the angle in the segment* ADC, *which is less than a semicircle, shall be greater than a right angle.*



Construction. Join AE, and produce BA to F.
Take any point D on the arc ADC, and join AD, DC.

Proof. (1) Because EA = EB, [I. *Definition* 15.
the ∠EAB = the ∠EBA; [I. 5.
and, because EA = EC,
the ∠EAC = the ∠ECA;
∴ the whole angle BAC = the two angles ABC, ACB. [*Ax.* 2.
But FAC, the exterior angle of the triangle ABC,
= the two angles ABC, ACB; [I. 32.
∴ the ∠BAC = the ∠FAC, [*Axiom* 1.
and therefore each of them is a right ∠; [I. *Definition* 10.
∴ the ∠ in a semicircle BAC is a right ∠.

(2) Because the two angles ABC, BAC of the triangle ABC are together less than two right angles, [I. 17.
and that BAC has been shewn to be a right ∠,
∴ the ∠ABC is less than a right ∠;
∴ the ∠ in a segment ABC, greater than a semicircle, is less than a right ∠.

(3) Because ABCD is a quadrilateral figure in a circle, any two of its opposite angles together = two right angles; [III. 22.
∴ the angles ABC, ADC together = two right angles.
But the ∠ABC has been shewn to be less than a right ∠.
∴ the ∠ADC is greater than a right ∠;
∴ the ∠ in a segment ADC, less than a semicircle, is greater than a right ∠.

Wherefore, *the angle, etc.* [Q.E.D.

Corollary. *If one angle of a triangle be equal to the other two, it is a right angle.*

For the angle adjacent to it is equal to the same two angles; [I. 32.
and when the adjacent angles are equal, they are right angles.
[I. *Definition* 10.

EXERCISES.

****1.** Right-angled triangles are described on the same hypotenuse; shew that the angular points opposite the hypotenuse all lie on a circle described on the hypotenuse as diameter.

2. The circles described on the sides of any triangle as diameters, will intersect on the base.

3. AOB and COD are two perpendicular diameters of a circle. If P be any point on the circumference, prove that CP and DP are the internal and external bisectors of the angle APB.

4. Chords AB, CD of a circle cut one another at right angles; the sum of the opposite arcs AC and BD is a semicircle.

5. AB is the diameter of a circle whose centre is C, and DCE is a sector having the arc DE constant; join AE, BD intersecting at P; shew that the angle APB is constant.

6. On the side AB of any triangle ABC as diameter a circle is described; EF is a diameter parallel to BC; shew that the straight lines EB and FB bisect the interior and exterior angles at B.

7. If AD, CE be drawn perpendicular to the sides BC, AB of a triangle ABC, and DE be joined, shew that the angles ADE and ACE are equal. [A, E, D, C lie on a circle.]

8. If two circles ABC, ABD intersect at A and B, and AC, AD be two diameters, shew that the straight line CD will pass through B.

9. If O be the centre of a circle and OA a radius, and a circle be described on OA as diameter, the circumference of this circle will bisect any chord of the exterior circle drawn from A.

10. If from the angles at the base of any triangle perpendiculars are drawn to the opposite sides, produced if necessary, the straight line joining the points of intersection will be bisected by a perpendicular drawn to it from the centre of the base.

11. If two chords of a circle meet at right angles within or without a circle, the squares on their segments are together equal to the squares on the diameter.

[Let the chords be AB, CD meeting within the circle at O and let AA′ be the diameter through A. Then

$$\angle CAB = \text{rt.} \angle - ACD = \text{rt.} \angle - AA'D = \angle DAA';$$

$$\therefore\ DA' = BC;$$

$$\therefore\ AO^2 + OD^2 + BO^2 + OC^2 = AD^2 + BC^2 = AD^2 + DA'^2 = AA'^2.$$

Similarly if the chords cut without the circle.]

12. AB is a diameter of a circle and C a given point in AB; find a point in the circumference at which both AC and CB will subtend half a right angle.

[Join C to the middle point D of the arc AB; the required point is the point where CD meets the circle again.]

13. Circles are described on the sides of a quadrilateral as diameters; prove that the common chord of two of these circles which are adjacent is parallel to the common chord of the other two.

14. ABCD is a quadrilateral figure inscribed in a circle; if the bisectors of two of its opposite angles meet the circle in E and F, then EF is a diameter.

15. Divide a straight line into two parts such that the rectangle contained by the two parts may be equal to a given square. When is this impossible? [Use II. 5.]

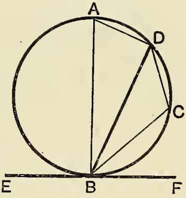
PROPOSITION 32. THEOREM.

If a straight line touch a circle, and from the point of contact a chord be drawn, the angles which this chord makes with the tangent shall be equal to the angles which are in the alternate segments of the circle.

Let the straight line EF touch the circle ABCD at the point B, and from B let the chord BD be drawn, cutting the circle: then

(1) *the angle* DBF *shall be equal to the angle in the segment* BAD, *and*

(2) *the angle* DBE *shall be equal to the angle in the segment* BCD.



Construction. From B draw BA at right angles to EF, [I. 11.
and take any point C in the arc BD.

Join AD, DC, CB.

Proof. (1) Because BA is drawn at right angles to the tangent EF at its point of contact B, [*Constr.*

∴ the centre of the circle is in BA; [III. 19.

∴ the ∠ADB, being in a semicircle, is a right ∠; [III. 31.

∴ the other two angles BAD, ABD together = a right ∠. [I. 32.

But ABF is also a right angle; [*Constr.*

∴ the ∠ABF = the angles BAD, ABD.

From each of these equals take the common ∠ABD;

∴ the remaining ∠DBF = the remaining ∠BAD, [*Ax.* 3.
which is in the alternate segment of the circle.

(2) Because ABCD is a quadrilateral in a circle, the angles BAD, BCD together = two right angles. [III. 22.

But the angles DBF, DBE together = two right angles; [I. 13.

∴ the angles DBF, DBE together = the angles BAD, BCD.

And the ∠DBF has been shewn equal to the ∠BAD;

∴ the remaining ∠DBE = the remaining ∠BCD, [*Ax.* 3.
which is in the alternate segment of the circle.

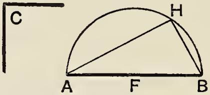
Wherefore, *if a straight line, etc.* [Q.E.D.

PROPOSITION 33. PROBLEM.

On a given straight line to describe a segment of a circle, containing an angle equal to a given rectilineal angle.

Let AB be the given straight line, and C the given rectilineal angle:

it is required to describe on AB *a segment of a circle containing an angle equal to* C.



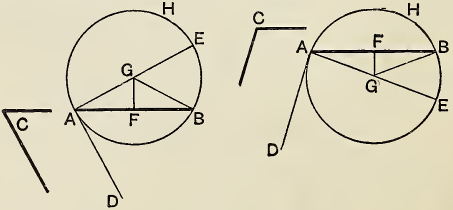
CASE I. First, let the ∠C be a right ∠.

Bisect AB at F, [I. 10.

and with centre F and radius FB describe the semicircle AHB.

Then the ∠AHB in a semicircle is equal to the right ∠C. [III. 31.

CASE II. Let the ∠C be not a right ∠.



Construction. At A make the ∠BAD equal to the ∠C; [I. 23.

from A draw AE at right angles to AD; [I. 11.

bisect AB at F; [I. 10.

from F draw FG at right angles to AB to meet AE in G. [I. 11.

Join GB.

Proof. In the triangles AFG, BFG,

because { AF = BF,
and FG is common,
and the right ∠AFG = the right ∠BFG; [I. *Ax.* 11.

∴ the base AG = the base BG, [I. 4.

and therefore the circle described from the centre G, with radius GA, will pass through the point B.

Let this circle be described; and let it be AHB.

The segment AHB shall contain an angle equal to the given rectilineal angle C.

Because AD is drawn at right angles to the diameter AE,

∴ AD touches the circle. [III. 16. *Corollary.*

Also, because AB is drawn from the point of contact A,

the ∠DAB = the ∠ in the alternate segment AHB. [III. 32.

But the ∠DAB = the ∠C; [*Construction.*

∴ the ∠ in the segment AHB = the ∠C. [*Axiom* 1.

Wherefore, *on the given straight line* AB, *the segment* AHB *of a circle has been described, containing an angle equal to the given angle* C. [Q.E.F.

EXERCISES ON PROPOSITION 32.

1. Prove the converse of III. 32. [See Notes, page 327.]

****2.** If a tangent to a circle is parallel to a chord, the point of contact of the tangent will be the middle point of the arc cut off by the chord.

3. Two circles touch each other externally. Prove that any straight line drawn through the point of contact cuts off similar segments from the two circles.

4. If two circles intersect one another, prove that each common tangent subtends at either common point angles that are equal or supplementary.

5. B is a point in the circumference of a circle whose centre is C; PA, a tangent at any point P, meets CB produced at A, and PD is drawn perpendicular to CB; shew that the straight line PB bisects the angle APD.

6. Two circles intersect at A and B, and through P any point in the circumference of one of them the chords PA and PB are drawn to cut the other circle at C and D; shew that CD is parallel to the tangent at P.

7. If from any point in the circumference of a circle a chord and tangent be drawn, the perpendiculars dropped on them from the middle point of the intercepted arc are equal.

8. AB is a chord and AD a tangent to a circle at A. DPQ is a straight line parallel to AB, meeting the circle in P and Q. Prove that the triangles PAD, QAB are equiangular.

9. On a straight line AB as base, and on the same side of it are described two segments of circles ; P is any point in the circumference of one of the segments, and the straight line BP cuts the circumference of the other segment at Q ; shew that the angle PAQ is equal to the angle between the tangents at A.

10. C is the centre of a circle ; CA, CB are two radii at right angles ; from B any chord BP is drawn cutting CA at N : a circle being described round ANP, shew that it will be touched by BA.

11. AB and CD are parallel straight lines, and the straight lines which join their extremities intersect at E : shew that the circles described round the triangles ABE, CDE touch one another.

12. If the centres of two circles which touch each other externally be fixed, the common tangent of those circles will touch another circle of which the straight line joining the fixed centres is the diameter.

EXERCISES ON PROPOSITION 33.

1. Construct a triangle, having given the base, the vertical angle, and the point in the base on which the perpendicular falls.

[On the given base AB describe a segment of a circle containing the given vertical angle ; by the converse of III. 22 the vertex lies somewhere on this segment. Through the given point in the base draw a straight line perpendicular to the base ; this will meet the segment in the required vertex. There are two solutions, since two segments can be drawn, one on each side of AB.]

2. Construct a triangle, having given the base, the vertical angle, and the altitude.

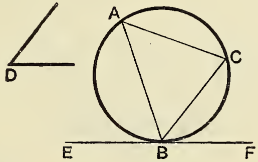
3. Construct a triangle, having given the base, the vertical angle, and the length of the straight line drawn from the vertex to the middle point of the base.

4. Having given the base and the vertical angle of a triangle, shew that the triangle will be greatest when it is isosceles.

Proposition 34. Problem.

From a given circle to cut off a segment containing an angle equal to a given rectilineal angle.

Let ABC be the given circle, and D the given angle: *it is required to cut off from the circle* ABC *a segment containing an angle equal to* D.



Construction. Draw the straight line EF touching the circle ABC at B; [III. 17.

and at B make the ∠FBC equal to the ∠D. [I. 23.

The segment BAC shall contain an ∠ equal to D.

Proof. Because EF touches the circle ABC, and BC is drawn from the point of contact B; [*Construction.*

∴ the ∠FBC = the ∠ in the alternate segment BAC. [III. 32.

But the ∠FBC = the ∠D; [*Construction.*

∴ the ∠ in the segment BAC = the ∠D. [*Axiom* 1.

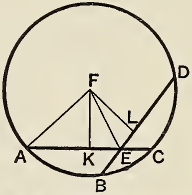
Wherefore, *from the given circle* ABC, *the segment* BAC *has been cut off, containing an angle equal to the given angle* D. [Q.E.F.

Proposition 35. Theorem.

If two chords of a circle cut one another within the circle, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.

Let the two chords AC, BD cut one another at the point E within the circle;

the rectangle AE, EC *shall be equal to the rectangle* BE, ED.



Construction. Case I. Find the centre F, and, if it be not in AC or BD, join FE.

Draw FK perpendicular to AC and FL perpendicular to BD. Join FA.

Proof. Since AC is bisected at K and divided unequally at E,

∴ the rect. AE, EC together with the square on KE
= the square on AK. [II. 5.

To each add the square on FK,

∴ the rect. AE, EC, together with the squares on FK, KE,
= the squares on AK, KF.

But, since the angles at K are right angles, [*Construction.*

the squares on FK, KE = the square on FE, [I. 47.

and the squares on AK, KF = the square on FA; [I. 47.

∴ the rect. AE, EC together with the square on FE
= the square on FA.

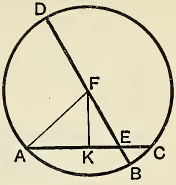
Similarly, the rect. BE, ED together with the square on FE = the square on FD, that is, the square on FA;

∴ the rect. AE, EC with the square on FE
= the rect. BE, ED with the square on FE.

Take away the common square on FE;

∴ the rect. AE, EC = the rect. BE, ED.

CASE II. Next, let the centre F be on one of the chords BD, and let BD be not perpendicular to AC.



Then since BD is bisected at F and divided unequally at E,

∴ the rect. BE, ED with the square on EF = the square on FB, [II. 5.

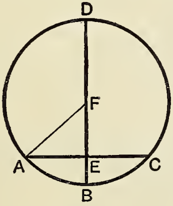
that is, the square on FA.

Also, as in Case I.,

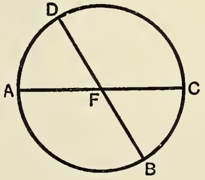
the rect. AE, EC with the square on FE = the square on FA;

∴ as before, the rect. AE, EC = the rect. BE, ED.

CASE III. Next, let F be on BD, and let BD be perpendicular to AC. Then BD will bisect AC;



CASE III.



CASE IV.

∴ the rect. AE, EC = the square on AE,

and ∴ the rect. AE, EC, together with the square on EF,
= the squares on AE, EF,

that is, = the square on FA, that is, as in the second case,
= the rect. BE, ED with the square on EF,

and thus the rect. AE, EC = the rect. BE, ED

CASE IV. Lastly, let both chords pass through the centre F. In this case the lines AF, FC, BF, FD are all equal;

∴ the rect. AF, FC = the rect. BF, FD;

for each is equal to the square on the radius of the circle.

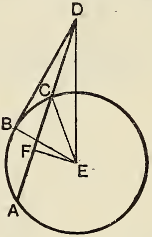
Wherefore, *if two chords, etc.*

PROPOSITION 36. THEOREM.

If from any point without a circle two straight lines be drawn, one of which cuts the circle, and the other touches it; the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, shall be equal to the square on the line which touches it.

From any point D without the circle ABC let there be drawn two straight lines DCA, DB, of which DCA cuts the circle and DB touches it:

the rectangle AD, DC *shall be equal to the square on* DB.

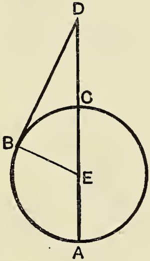


Construction. CASE I. Let DCA not pass through the centre of the circle ABC; find the centre E; [III. 1.
from E draw EF perpendicular to AC; [I. 12.
and join EB, EC, ED.

Proof. Because EF drawn from the centre cuts the chord AC at right angles at F, ∴ AF = FC. [III. 3.
And because AC is bisected at F, and produced to D, the rectangle AD, DC, together with the square on FC,
= the square on FD. [II. 6.
To each of these equals add the square on FE;
∴ the rect. AD, DC, with the squares on CF, FE,
= the squares on DF, FE. [*Axiom* 2.
But, because CFE is a right angle, the squares on CF, FE
= the square on CE, [I. 47.
that is, = the square on BE;

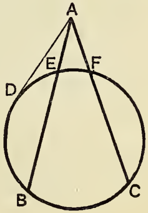
and the squares on DF, FE = the square on DE;
∴ the rect. AD, DC, together with the square on BE,
= the square on DE.
that is, = the squares on DB, BE,
because EBD is a right angle. [I. 47.
Take away the common square on BE;
then the remaining rect. AD, DC = the square on DB. [*Axiom* 3.

CASE II. If DCA passes through E, join EB.



Since AC is bisected at E and produced to D,
the rect. AD, DC, with the square on EC,
= the square on ED, [II. 6.
that is, = the sqs. on EB, BD, since EBD is a right ∠. [I. 47.
But the square on EC = the square on EB;
∴ the rect. AD, DC = the square on BD.

Wherefore, *if from any point, etc.* [Q.E.D.



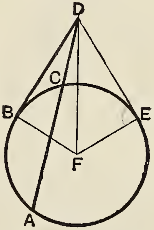
Corollary. If from any point without a circle there be drawn secants AEB, AFC, the rectangles contained by the whole secants and the parts of them without the circles are equal, that is, the rectangle BA, AE = the rectangle CA, AF; for each of them is equal to the square on the tangent AD.

Proposition 37. Theorem.

If from any point without a circle there be drawn two straight lines, one of which cuts the circle, and the other meets it, and if the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, be equal to the square on the line which meets the circle, the line which meets the circle shall be a tangent.

Let any point D be taken without the circle ABC, and from it let two straight lines DCA, DB be drawn, of which DCA cuts the circle, and DB meets it; and let the rectangle AD, DC be equal to the square on DB:

DB *shall touch the circle.*



Construction. Draw the straight line DE, touching the circle ABC; [III. 17.

find F the centre, [III. 1.

and join FB, FD, FE.

Proof. Because DE touches the circle ABC, and DCA cuts it,

the rect. AD, DC = the square on DE. [III. 36.

But the rect. AD, DC = the square on DB; [*Hypothesis.*

∴ the square on DE = the square on DB. [*Axiom* 1.

Hence, in the triangles DBF, DEF,

because { DE = DB,
and EF = BF, I. *Definition* 15.
and the base DF is common,

∴ the ∠DEF = the ∠DBF. [I. 8.

But DEF is a right ∠, since DE is a tangent ; [*Constr.* and III. 18.

∴ also DBF is a right ∠.

And BF, if produced, is a diameter;

∴ DB touches the circle ABC. [III. 16, *Corollary.*

Wherefore, *if from a point, etc.* [Q.E.D.

EXERCISES.

****1.** Prove the following converse of III. 35 and III. 36, Cor. : *If two straight lines* AB, CD *intersect in* O *and the rectangle* AO, OB = *the rectangle* CO, OD, *the four points* A, B, C, D *lie on a circle.*

[For if the circle passing through A, B, C do not pass through D, let it meet CD in E; then, by III. 35 or III. 36 Cor. (according as O is within or without CD) the rectangle CO, OE = the rectangle AO, OB = the rectangle CO, OD (Hyp.), ∴ OE = OD. Hence E coincides with D, and thus the circle through A, B, C passes through D.]

****2.** If two circles cut one another, the tangents drawn to the two circles from any point in the common chord produced are equal.

****3.** Two circles intersect. Shew that their common chord produced bisects their common tangent.

4. If AD, CE are drawn perpendicular to the sides BC, AB of a triangle ABC, shew that the rectangle contained by BC and BD is equal to the rectangle contained by BA and BE.

5. If through any point in the common chord of two circles which intersect one another, there be drawn any two other chords, one in each circle, their four extremities all lie on the circumference of a circle.

6. From a given point as centre, describe a circle cutting a given straight line in two points, so that the rectangle contained by their distances from a fixed point in the straight line may be equal to a given square.

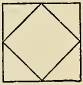
[The radius of the circle is such that its square is equal to the difference of the square on the line joining the two given points and the given square.]

7. A series of circles intersect each other, and are such that the tangents to them from a fixed point are equal. Shew that the straight lines joining the two points of intersection of each pair will pass through this point.

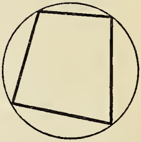
BOOK IV.

DEFINITIONS.

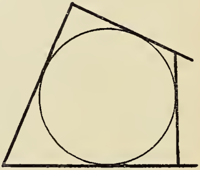
1. A rectilineal figure is said to be **inscribed** in another rectilineal figure when all the angles of the inscribed figure are on the sides of the figure in which it is inscribed, each on each.



2. In like manner a figure is said to be **described about**, or **circumscribed about**, another figure when all the sides of the circumscribed figure pass through the angular points of the figure about which it is described, each through each.



3. A rectilineal figure is said to be **inscribed in a circle** when all the angles of the inscribed figure are on the circumference of the circle.



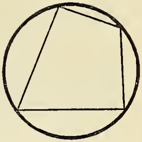
4. A rectilineal figure is said to be described about a circle when each side of the circumscribed figure touches the circumference of the circle.

5. In like manner a circle is said to be inscribed in a rectilineal figure when the circumference of the circle touches each side of the figure.

Such a circle is often called the **in-circle** and its centre the **in-centre**.

[A circle is said to be **escribed** to a triangle when it touches one side of the triangle and the other two sides produced. For a figure, see page 181.]

6. A circle is said to be described about a rectilineal figure when the circumference of the circle passes through all the angular points of the figure about which it is described.



Such a circle is often called the **circum-circle** and its centre the **circum centre.**

7. A straight line is said to be placed in a circle when the extremities of it are on the circumference of the circle.

8. A polygon is a rectilineal figure contained by more than four straight lines. [I. *Def.* 22.

A polygon of five sides is called a **pentagon** (Gk. πέντε).

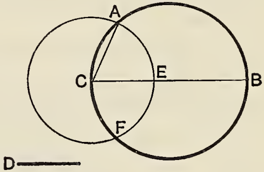
,,	six	,,	,,	**hexagon** (Gk. ἕξ).
,,	seven	,,	,,	**heptagon** (Gk. ἑπτά).
,,	eight	,,	,,	an **octagon** (Gk. ὀκτώ).
,,	ten	,,	,,	a **decagon** (Gk. δέκα).
,,	twelve	,,	,,	**dodecagon** (Gk. δώδεκα).
,,	fifteen	,,	,,	**quindecagon** (Lat. *quinque*; Gk. δέκα).

9. A **regular polygon** is one which has all its sides equal and also all its angles equal, *i.e.* it is equilateral and equi-angular.

PROPOSITION 1. THEOREM.

In a given circle, to draw a chord equal to a given straight line, which is not greater than the diameter of the circle.

Let ABC be the given circle, and D the given straight line, not greater than the diameter of the circle:
it is required to draw a chord of the circle ABC *equal to* D.



Construction. Draw BC, a diameter of the circle ABC. Then, if BC = D, the thing required is done; for, in the circle ABC, a straight line is placed equal to D.
But, if it is not, BC is greater than D. [*Hypothesis.*
Make CE equal to D, [I. 3.
and with centre C and radius CE describe the circle AEF and join CA.

Proof. Because C is the centre of the circle AEF,

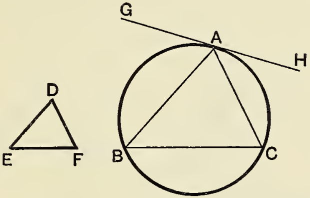
CA = CE; [I. *Definition* 15.
but CE = D; [*Construction.*
∴ CA = D. [*Axiom* 1.

Wherefore, *in the circle* ABC, *a straight line* CA *is placed equal to the given straight line* D, *which is not greater than the diameter of the circle.* [Q.E.F.

PROPOSITION 2. PROBLEM.

In a given circle, to inscribe a triangle equiangular to a given triangle.

Let ABC be the given circle, and DEF the given triangle: *it is required to inscribe in the circle* ABC *a triangle equiangular to* DEF.



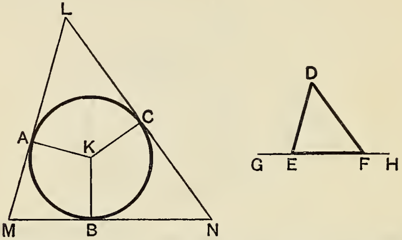
Construction. At any point A on the circumference draw the tangent GAH; [III. 17.
at A make the ∠HAC equal to the ∠DEF, [I. 23.
and also make the ∠GAB equal to the ∠DFE;
join BC. ABC shall be the triangle required.

Proof. Because GAH touches the circle ABC, and AC is drawn from the point of contact A; [*Construction.*
∴ the ∠HAC = the ∠ABC in the alternate segment. [III. 32.
But the ∠HAC = the ∠DEF; [*Construction.*
∴ the ∠ABC = the ∠DEF. [*Axiom* 1.
Similarly, the ∠ACB = the ∠DFE;
∴ the third ∠BAC = the third ∠EDF. [I. 32, *Axioms* 11 and 3.
Wherefore *the triangle* ABC *is equiangular to the triangle* DEF, *and it is inscribed in the circle* ABC. [Q.E.F.

Proposition 3. Problem.

About a given circle, to describe a traingle equiangular to a given triangle.

Let ABC be the given circle, and DEF the given triangle: *it is required to describe a triangle about the circle* ABC *equiangular to the* △DEF.



Construction. Produce EF both ways to G and H;
find K, the centre of the circle ABC; [III. 1.
from K draw any radius KB;
at K make the ∠BKA equal to the ∠DEG,
and the ∠BKC equal to the ∠DFH; [I. 23.
and through A, B, C draw the straight lines LAM, MBN, NCL at right angles to KA, KB, and KC respectively. [I. 11.
LMN shall be the triangle required.

Proof. Because LM, MN, NL are drawn perpendicular to the radii KA, KB, KC through their extremities, [*Construction.*
therefore LM, MN, NL all touch the circle,
and LMN is a △ described about the circle.
Also because the four angles of the figure AMBK are together equal to four right angles,
for it can be divided into two triangles,
and two of them KAM, KBM are right angles,
∴ the other two AKB, AMB are together equal to two right angles. [*Axiom* 3.

But the angles DEG, DEF together = two right angles; [I. 13.
∴ the angles AKB, AMB = the angles DEG, DEF;
of which the ∠AKB = the ∠DEG; [*Construction.*
∴ the remaining ∠AMB = the remaining ∠DEF. [*Ax.* 3.
Similarly, the angles LNM and DFE may be shewn to be equal;
∴ the third ∠MLN = the third ∠EDF. [I. 32, *Axioms* 11 and 3.

Wherefore *the triangle* LMN *is equiangular to the triangle* DEF, *and it is described about the circle* ABC. [Q.E.F.

EXERCISES.

1. Place a chord in a given circle equal to a given straight line, so that it shall be parallel to another given straight line.

2. Place a chord of given length in a given circle, so that it may pass through a given point within or without the circle. When is this impossible?

3. Inscribe in a circle a triangle MNP whose sides are parallel to three given straight lines.

4. Two triangles are circumscribed to a given circle, each of them being equiangular to a given triangle; prove that the triangles are equal in all respects.

5. Any rectilinear figure ABCDE is inscribed in a circle; the arcs AB, BC, CD, DE, EA are bisected, and tangents drawn at the points of bisection; shew that the resulting figure is equiangular to ABCDE.

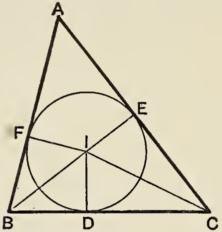
6. Prove that the area of an equilateral triangle inscribed in a circle is one-quarter that of the equilateral triangle circumscribed to the circle.

[Let ABC be an equilateral triangle inscribed in a given circle; at A, B, C draw tangents. Let the tangents at B, C meet in A′; those at C, A in B′; and those at A, B in C′. Then A′B′C′ is an equilateral △ described about the circle, and it is easy to prove that the △s A′CB, B′AC, C′BA and ABC are all equal.]

Proposition 4. Problem.

To inscribe a circle in a given triangle.

Let ABC be the given triangle:
it is required to inscribe a circle in the triangle ABC.



Construction. Bisect the angles ABC, ACB by the straight lines BI, CI, meeting one another at the point I; [I. 9.
and from I draw ID, IE, IF perpendiculars to BC, CA, AB. [I. 12.

Proof. In the triangles DBI, FBI,

because { the ∠DBI = the ∠FBI, [*Construction.*
and the right ∠BDI = the right ∠BFI,
and the side BI is common;

∴ ID = IF. [I. 26.

For the same reason ID = IE;

∴ IE = IF; [*Axiom* 1.

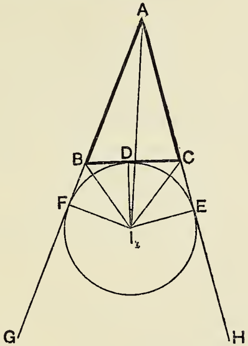
∴ the three straight lines ID, IE, IF are equal and the circle described with centre I, and radius equal to any one of them, will pass through the extremities of the other two;
and it will touch the straight lines BC, CA, AB, because the angles at the points D, E, F are right angles; [III. 16, *Cor.* 1.
∴ the circle DEF is inscribed in the triangle ABC.

Wherefore *a circle has been inscribed in the given triangle.* [Q.E.F.

Additional Proposition.

To draw an escribed circle of a given triangle.

Let ABC be the given triangle:
it is required to draw the escribed circle which is opposite to the angle A, *that is, which touches* BC *externally.*



Construction. Produce AB and AC to G and H;
bisect the exterior angles GBC, HCB by the straight lines BI_1, CI_1, which meet at I_1;
draw I_1D, I_1E, and I_1F perpendicular to BC, CA, and AB respectively.

Proof. In the triangles I_1BD, I_1BF,

because { the $\angle I_1BD$ = the $\angle I_1BF$, [*Construction.*
and the right $\angle I_1DB$ = the right $\angle I_1FB$,
and the side I_1B is common;

∴ the base I_1D = the base I_1F. [I. 26.
Similarly, it may be shewn that $I_1D = I_1E$;
therefore I_1D, I_1E, and I_1F are equal, and a circle described with centre I_1, and radius equal to either of these three, will pass through the extremities of the other two.
Also, since the angles at D, E, F are right angles, this circle will touch BC and AB, AC produced, and will therefore be the circle required.

Corollary. Since this circle touches AE, AF, the angles I_1AE and I_1AF will be equal, [III. 17, *Corollary.*
and I_1A will therefore bisect the angle BAC.

EXERCISES.

**1. In the figure of IV. 4, prove that AI bisects the angle BAC, and hence that the bisectors of the angles of a triangle meet in a point.

**2. In the figure of IV. 4, prove that BD + CA = CE + AB = AF + BC = the semi-perimeter of the triangle ABC.

[BD = BF, CD = CE, AE = AF (III. 17, *Cor.*);

∴ sum of twice BD, CE, AE = sum of BD, BF, CE, CD, AE, AF
= sum of BC, CA, AB;

∴ twice BD and twice AC = perimeter, etc.]

3. The circle inscribed in a triangle ABC touches the sides in the points D, E, and F. Prove that the angles of the triangle DEF are equal to the complements of half the angles of the triangle ABC, and hence that the triangle DEF is always acute-angled.

4. Without producing two straight lines to meet, find that straight line which would bisect the angle between them.

[Suppose BK, CL to be the two lines; draw any straight line BC to meet them; bisect ∠s at B, C by straight lines BI, CI meeting in I; draw IF, IE perpendicular to BK, CL and bisect FIE; this bisecting line will be the required line.]

5. With the vertices A, B, C of a triangle as centres draw three circles, each of which touches the other two.

6. A circle is inscribed in a triangle ABC, and a triangle is cut off at each angle by a tangent to the circle. Shew that the sides of the three triangles so cut off are together equal to the sides of ABC.

7. If the circle inscribed in a triangle ABC touch the sides AB, AC at the points D, E, prove that the middle point of the arc DE is the centre of the circle inscribed in the triangle ADE.

8. Find the centre of a circle cutting off three equal chords from the sides of a triangle.

The escribed circle opposite the angle A of a triangle touches the sides in the points D_1, E_1, F_1 and the inscribed circle touches them in D, E, and F; prove that

**9. $AE_1 = AF_1$ = the semi-perimeter of the △ ABC.

10. $BD = CD_1$.

11. DD_1 = the difference between the sides AB and AC.

12. Two sides of a triangle whose perimeter is constant are given in position; prove that the third side always touches a certain circle.

I is the in-centre and I_1, I_2, I_3 the centres of the escribed circles of the triangles ABC; prove that

**13. AII_1, BII_2, and CII_3 are straight lines. [See App. Art. 48.]

**14. I_2AI_3, I_3BI_1, and I_1CI_2 are straight lines.

****15.** AII_1 is perpendicular to I_2I_3, etc.

16. I, B, C, I_1 lie on a circle.

17. I_2, I_3, B, C lie on a circle.

18. the triangles BCI_1, CAI_2, and ABI_3 are equiangular.

19. If the escribed circles opposite to the angles A, B, C of a triangle touch the sides BC, CA, AB in D_1, D_2, D_3, prove that $AD_2 = BD_1$, $BD_3 = CD_2$, and $CD_1 = AD_3$.

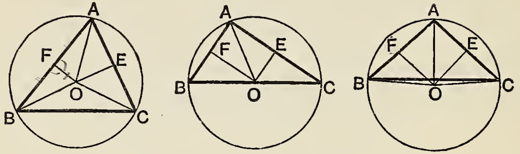
20. The triangle formed by joining the centres of the escribed circles of the triangle and that formed by joining the points of contact of the inscribed circle are equiangular.

21. How many circles can, in general, be drawn to touch three straight lines? What are the exceptional cases?

PROPOSITION 5. PROBLEM

To describe a circle about a given triangle.

Let ABC be the given triangle:
it is required to describe a circle about ABC.



Construction. Bisect AB, AC at the points F, E; [I. 10.
from these points draw FO, EO at right angles to AB, AC; [I. 11.

FO, EO, produced, will meet one another;
for if they do not meet they are parallel;
therefore AB, AC, which are at right angles to them, are parallel, which is absurd:
let them meet at O, and join OA;
also, if O be not in BC, join BO, CO.

Proof. In the triangles AFO, BFO,

because { AF = BF, [*Construction.*
and FO is common,
and the right ∠AFO = the right ∠BFO;

∴ the base OA = the base OB. [I. 4.

Similarly, it may be shewn that OC = OA;

∴ OB = OC, [*Axiom* 1.

and OA, OB, OC are all equal.

Therefore the circle described with centre O, and radius equal to any one of them, will pass through the extremities of the other two, and will be described about the triangle ABC.

Wherefore *a circle has been described about the given triangle.*

[Q.E.F.

Corollary. It is clear that

(i.) when the centre of the circle falls within the triangle, each of its angles is less than a right angle, each of them being in a segment greater than a semicircle;

(ii.) when the centre is in one of the sides of a triangle, the ∠ opposite to this side, being in a semicircle, is a right ∠; and

(iii.) when the centre falls without the triangle, the ∠ opposite to the side beyond which it is, being in a segment less than a semicircle, is greater than a right ∠. [III. 31.

Therefore, conversely, *if the given triangle be acute-angled, the centre of the circle falls within it; if it be a right-angled triangle, the centre is in the side opposite to the right angle; and if it be an obtuse-angled triangle, the centre falls without the triangle, beyond the side opposite to the obtuse angle.*

EXERCISES.

****1.** In IV. 5, shew that the perpendicular from O on BC will bisect BC, and hence that the straight lines which bisect at right angles the sides of a triangle all meet in a point.

****2.** If the inscribed and circumscribed circles of a triangle be concentric, shew that the triangle must be equilateral.

The perpendicular from O on BC meets the circum-circle in K and L (K being on the opposite side of BC from A). Prove that

3. ∠BOK = ∠COK = A.

4. AK and AL bisect the interior and exterior angles at A.

****5.** K is the centre of the circum-circle of BIC, where I is the in-centre.

6. The angle between the radius of the circum-circle passing through the vertex A of a triangle ABC, and the perpendicular from A upon BC is equal to the difference of the base angles of the triangle.

7. O is the centre of the circle circumscribing a triangle ABC; D, E, F the feet of the perpendiculars from A, B, C on the opposite sides. Shew that OA, OB, OC are respectively perpendicular to EF, FD, DE.

8. In an equilateral triangle, prove that the radius of the circum-circle is twice that of the in-circle, and the radius of the escribed circle is three times that of the in-circle.

[Let O be the in- and circum-centre of the equilateral △ ABC and O_1 the e-centre opposite A; let AD be ⊥ʳ to BC and P the middle point of OO_1. Then OCO_1 is a rt. ∠ and ∴ P is the circum-centre of △ OCO_1; ∴ $PO_1 = PC = PO$; ∴ ∠PCO = ∠POC = 2 ∠ACO = 2 ∠OCD = 2 complement of ∠DCO_1 = 2 ∠DO_1C = ∠DPC, and ∴ POC is equilateral; ∴ OC = OP = 2 OD, and $O_1D = O_1P + PD = 2\,OD + OD = 3\,OD$.]

9. The side BC of a triangle ABC is produced to D so that the triangles ABD, ACD are equiangular. Prove that AD touches the circum-circle of the triangle ABC.

10. A quadrilateral ABCD is inscribed in a circle, and AD, BC are produced to meet at E. Shew that the circle described about the triangle ECD will have the tangent at E parallel to AB.

11. If DE be drawn parallel to the base BC of a triangle ABC to meet the sides in D and E, shew that the circles described about the triangles ABC and ADE have a common tangent.

12. The diagonals of a given quadrilateral ABCD intersect at O. Shew that the centres of the circles described about the triangles OAB, OBC, OCD, ODA are at the angular points of a parallelogram.

[Each of its sides is perpendicular to one of the diagonals of ABCD.]

13. The opposite sides of a quadrilateral inscribed in a circle are produced to meet in P and Q, and about the triangles so formed without the quadrilateral circles are described, which meet in R. Prove that PRQ is a straight line.

14. Three circles whose centres are A, B, and C touch one another externally in D, E, and F. Prove that the in-circle of the triangle ABC is the circum-circle of the triangle DEF.

[Draw the common tangents at D, E to meet in O. Then shew that OFA is a right ∠ so that the common tangent at F goes through O. Then prove that OA = OB = OC and OD = OE = OF.]

**15. The four circles each of which passes through the centres of three of the four circles touching the sides of a triangle are equal to one another.

16. If L, M, N be any three points on the sides BC, CA, AB of a triangle, prove that the circles which circumscribe the triangles MAN, NBL, LCM meet in a point.

Proposition 6. Problem.

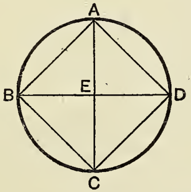
To inscribe a square in a given circle.

Let ABCD be the given circle:
is required to inscribe a square in ABCD.

Construction. Find the centre E of the circle, and draw wo diameters AC, BD at right angles to one another; [III. 1, I. 11.

and join AB, BC, CD, DA.

he figure ABCD shall be the square required.



Proof. In the two triangles BEA, DEA,

ecause { BE = DE, being radii,
and AE is common,
and the right ∠BEA = the right ∠DEA,

∴ the base BA = the base DA. [I. 4.

Similarly, BC, DC each = BA, or DA;

∴ the figure ABCD is equilateral.

Also, BD being a diameter of the circle ABCD,

BAD is a semicircle; [*Constr.*

∴ the ∠BAD is a right ∠. [III. 31.

. the figure ABCD is equilateral, and has one angle a right ngle;

∴ it is a square.

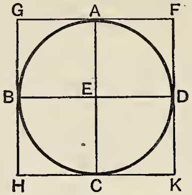
Wherefore *a square has been inscribed in the given circle.*

[Q.E.F.

PROPOSITION 7. PROBLEM.

To describe a square about a given circle.

Let ABCD be the given circle:
it is required to describe a square about it.



Construction. Find the centre E, and draw two diameters AC, BD at right angles to one another; [III. 1, I. 11.
and through A, B, C, D draw FG, GH, HK, KF perpendicular to EA, EB, EC, ED.

The figure GHKF shall be the square required.

Proof. (1) Because FG, GH, HK, KF are drawn at right angles to the radii EA, EB, EC, ED at their extremities,

therefore they touch the circle, [III. 16, *Cor.* 1.

and the circle is inscribed in the figure GHKF.

Also because AEB and EBG are both right angles, [*Constr.*

∴ GH is parallel to AC. [I. 28.

Similarly, AC is parallel to FK,

and GF, HK are each parallel to BD;

∴ the figures GK, GC, CF, FB, BK are parallelograms;

∴ GH and FK each = AC,

and GF and HK each = BD. [I. 34.

But AC = BD, both being diameters;

∴ GH, HK, KF, FG are all equal, and FGHK is equilateral.

(2) Again, since AEBG is a parallelogram, and AEB a right ∠,

∴ the opposite ∠AGB is also a right ∠. [I. 34.

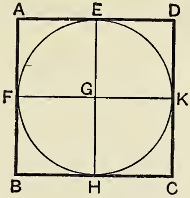
∴ the figure FGHK is equilateral, and has one of its angles a right angle; therefore it is a square.

Wherefore *a square has been described about the given circle.* [Q.E.F.

PROPOSITION 8. PROBLEM.

To inscribe a circle in a given square.

Let ABCD be the given square:
it is required to inscribe a circle in ABCD.



Construction. Bisect each of the sides AB, AD at the points F, E; [I. 10.
through E draw EH parallel to AB or DC, and through F draw FK parallel to AD or BC. [I. 31.

Proof. Let EH and FK meet in G.
AB and AD are equal, being sides of a square;
∴ their halves, AF and AE, are equal; [*Axiom* 7.
therefore, since AEGF is a parallelogram by construction, the opposite sides GF, GE are equal. [I. 34.

Similarly, it may be shewn that GK = GE, and GH = GF.
∴ GE, GF, GH, GK are all equal, and the circle described with centre G, and radius equal to any one of them, will pass through the extremities of the other three;
and it will touch AB, BC, CD, DA, because these are straight lines drawn through E, F, H, K perpendicular to the radii. [III. 16, *Cor.* 1.

Wherefore *a circle has been inscribed in the given square.* [Q.E.F.

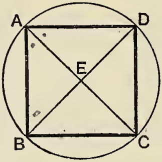
EXERCISE.

Prove the following alternative construction: *Draw the diagonals* AC *and* BD, *and let them meet in* O; *draw* OP, OQ, OR, OS *perpendicular to the sides; the required circle has its centre at* O *and its radius equal to either of the four* OP, OQ, OR, OS.

PROPOSITION 9. PROBLEM.

To describe a circle about a given square.

Let ABCD be the given square:
it is required to describe a circle about ABCD.



Construction. Join AC, BD, cutting one another at E.

Proof. In the triangles BAC, DAC,

because $\begin{cases} AB = AD, \\ \text{and } AC \text{ is common,} \\ \text{and the base } BC = \text{the base } DC; \end{cases}$

∴ the ∠BAC = the ∠DAC, [I. 8.

that is, the ∠BAD is bisected by AC.

Similarly, the other angles of the square are bisected by BD or AC.

Then, because the angles DAB, ABC are equal,
∴ their halves, the angles EAB, EBA, are equal,
and therefore the sides EB and EA are equal. [I. 6.

Similarly, it may be shewn that EC = EB, and ED = EA.
∴ EA, EB, EC, ED are all equal, and the circle described with centre E, and radius equal to any one of them, will pass through the ends of the other three, and will be described about the square ABCD.

Wherefore *a circle has been described about the given square.*

[Q.E.F.

EXERCISES.

1. Describe a circle about a given rectangle.

****2.** The square inscribed in a circle is double of the square on the radius.

****3.** The square circumscribed about a circle is double of the square inscribed in the same circle.

****4.** Shew that no rectangle except a square can be described about a circle.

5. Describe a square about a given rectangle.

[Let ABCD be the rectangle; at A draw a straight line outside the square making an ∠ equal to half a right ∠ with AD; similarly at B, C, D; the figure obtained is the required square.]

6. Inscribe a regular octagon in a given circle.

7. The area of a regular octagon inscribed in a circle is equal to the rectangle contained by the sides of the squares inscribed in and circumscribed about the circle.

[Let ABCD be the inscribed square and E, F, G, H the middle points of the arcs AB, BC, CD, DA. Let the tangents at A, B meet in K and AB meet OK in L. The rect. contained by the sides of the inscribed and circumscribed squares = rect. by 2 AK and AB = four times rect. AK, AL = four times rect. AL, OB = four times rect. AL, OE = eight times △ AOE (I. 41) = area of octagon AEBFCGDK.]

8. If from any point in the circumference of a given circle straight lines be drawn to the four angular points of an inscribed square, the sum of the squares on the four straight lines is double the square on the diameter. [Use Ex. 1, p. 109.]

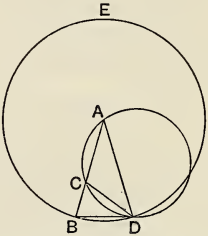
PROPOSITION 10. PROBLEM

To describe an isosceles triangle, having each of the angles at the base double of the third angle.

Construction. Take any straight line AB, and divide it at C, so that the rect. AB, BC = the square on AC; [II. 11.
with centre A and radius AB describe the circle BDE, and in it draw the chord BD equal to AC; [IV. 1.
join DA.

The triangle ABD shall be such as is required.

Join DC; and about the triangle ACD describe the circle ACD. [IV. 5.



Proof. Because the rect. AB, BC = the square on AC, [*Const.*
that is, = the square on BD, [*Construction.*
∴ BD touches the circle ACD. [III. 37.

Also, DC is drawn from the point of contact D;
∴ the ∠BDC = the ∠DAC in the alternate segment. [III. 32.

To each of these add the ∠CDA;
∴ the whole angle BDA = the two angles CDA, DAC. [*Ax.* 2.

But the exterior ∠BCD = the angles CDA, DAC; [I. 32.
∴ the ∠BDA = the ∠BCD. [*Axiom* 1.

But the ∠BDA = the ∠DBA, since AD = AB; [I. 5.
∴ the ∠DBA also = the ∠BCD; [*Axiom* 1.
∴ DC = DB; [I. 6.
but DB was made equal to CA; ∴ CA = CD, [*Axiom* 1.
and ∴ the ∠CAD = the ∠CDA. [I. 5.

But the angle BCD = the sum of the angles CAD, CDA; [I. 32.
∴ the ∠BCD is double of the ∠CAD.
And the angle BCD has been shewn to be equal to each of the angles BDA, DBA;
∴ each of the angles BDA, DBA is double of the ∠BAD.

Wherefore *an isosceles triangle has been described, having each of the angles at the base double of the third angle.* [Q.E.F.

EXERCISES.

****1.** Prove that the angle BAD is one-fifth part of two right angles.

****2.** Divide a right angle into 5 equal parts.

3. Divide a circle into two parts so that the angle in one segment may be four times that in the other.

4. Shew that the angle ACD in the figure of IV. 10 is equal to three times the angle at the vertex of the triangle.

5. Shew that the smaller of the two circles employed in the figure of IV. 10 is equal to the circle described round the required triangle.

[If two △s have equal bases and equal vertical angles then, as in the converse of III. 21, their circum-circles are equal. Also, in the above figure, we have BD = AC and ∠BAD = ∠ADC; ∴ etc.]

6. Shew that in the figure of IV. 10 there are two triangles which possess the required property. Shew that there is also an isosceles triangle whose equal angles are each one-third part of the third angle.

[These two △s are BCD and ACD.]

In the figure of IV. 10, if the two circles meet again in F, prove that

****7.** AC is the side of a regular decagon inscribed in the larger circle.

****8.** BD ,, ,, pentagon ,, smaller circle.

9. DF = BD.

[∠AFD = 2 rt. ∠s − ∠ACD = ∠BCD = ∠ABD, etc.]

10. BF is the side of a regular pentagon inscribed in the larger circle.

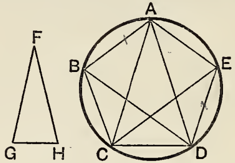
[∠FAD = ∠BAD; ∴ ∠FAB = 2∠BAD = one fifth of four rt. ∠s; ∴ etc.]

11. If AF meet BD in K, then CDKF is a parallelogram.

PROPOSITION 11. PROBLEM.

To inscribe a regular pentagon in a given circle.

Let ABCDE be the given circle:
it is required to inscribe a regular pentagon in the circle ABCDE.



Construction. Describe an isosceles triangle FGH, having each of the angles at G, H double of the angle at F; [IV. 10.
in the circle ABCDE inscribe the triangle ACD equiangular to the triangle FGH, [IV. 2.
so that each of the angles ACD, ADC is double of the angle CAD;
bisect the angles ACD, ADC by the straight lines CE, DB, [I. 9.
and join AB, BC, AE, ED;
then ABCDE shall be the pentagon required.

Proof. (1) Because each of the angles ACD, ADC is double of the angle CAD,
and that they are bisected by the straight lines CE, DB;
∴ the five angles ADB, BDC, CAD, DCE, ECA are equal.
But equal angles stand on equal arcs;
∴ the five arcs AB, BC, CD, DE, EA are equal. [III. 26.
Also equal arcs are subtended by equal chords; [III. 29.
∴ the five chords AB, BC, CD, DE, EA are equal;
and therefore the pentagon ABCDE is equilateral.

(2) It is also equiangular.
For the arc AB = the arc DE; to each add the arc BCD;
∴ the whole arc ABCD = the whole arc BCDE. [*Axiom* 2.

Also the $\angle AED$ stands on the arc ABCD, and the $\angle BAE$ on the arc BCDE.

$\therefore$ the $\angle AED =$ the $\angle BAE$. [III. 27.

Similarly, each of the angles ABC, BCD, CDE is equal to the angle AED or BAE;

$\therefore$ the pentagon ABCDE is equiangular.

Also it has been shewn to be equilateral.

Wherefore *a regular pentagon has been inscribed in the given circle.* [Q.E.F.

EXERCISES.

****1.** What is the magnitude of the angle of a regular pentagon?

2. If the alternate sides of a regular pentagon be produced to meet, the five points of intersection form another regular pentagon.

ABCDE is a regular pentagon; prove that

****3.** Any angle of it is trisected by the straight lines joining it to the opposite angular points.

4. CD and AE meet at an angle equal to CAD.

[If CD, AE meet in X, the $\angle XDE = \angle CAE$ (III. 22) $= 2\,\angle CAD = \angle ACD$ and so $\angle XED = \angle ADC$; $\therefore$ $\angle EXC = \angle CAD$.]

****5.** The diagonals AC and AD are parallel to the sides ED and BC respectively.

****6.** All the diagonals intersect so as to form another regular pentagon.

7. If AC and BE meet in F, then AB, CF, and EF are equal, and hence that any two diagonals divide one another in medial section.

[$\angle FEC = \angle FCE = \angle$ subtended by a side; $\therefore$ $FE = FC$; $\angle CFB = \angle FBA + \angle FAB =$ twice $\angle$ subtended by a side $= \angle CBE$; $\therefore$ $CF = CB = AB$. Also $\triangle ACE$ is a $\triangle$ equiangular with $\triangle ABD$ in IV. 10 and since $\angle AEB = \angle BEC$, EF bisects $\angle E$ at its base; $\therefore$ as in IV. 10, rect. CA . AF $= CF^2$.]

8. CFED is a rhombus.

9. AB is a tangent to the circum-circle of the triangle BFC.

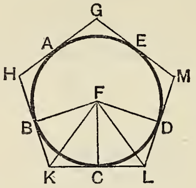
10. AE and BC are tangents to the circum-circle of the triangle CFE.

11. ABCDE is a regular pentagon inscribed in a circle; the arcs AB, BC, CD, DE, and EA are bisected, and lines drawn parallel to the sides of the inscribed pentagon. Prove that these lines form a circumscribing pentagon.

PROPOSITION 12. PROBLEM.

To describe a regular pentagon about a given circle.

Let ABCDE be the given circle :
it is required to describe a regular pentagon about it.



Construction. Let the angular points of a regular pentagon, inscribed in the circle, be A, B, C, D, E, so that the arcs AB, BC, CD, DE, EA are equal;
and through the points A, B, C, D, E draw GH, HK, KL, LM, MG touching the circle. [III. 17.
The figure GHKLM shall be the pentagon required.

Proof. (1) In the triangles FBK, FCK,

because { FB = FC, both being radii,
and FK is common,
and the base BK = the base KC, both being tangents. } [III. 17, *Cor.* 2.

∴ the $\angle$BFK = the $\angle$CFK, [I. 8.
and the $\angle$BKF = the $\angle$CKF ; [I. 4.
∴ the $\angle$BFC = twice the $\angle$CFK,
and the $\angle$BKC = twice the $\angle$CKF.
Similarly, the $\angle$CFD = twice the $\angle$CFL,
and the $\angle$CLD = twice the $\angle$CLF.

And because the arc BC = the arc CD,
the $\angle$BFC = the $\angle$CFD ; [III. 27.
and the $\angle$BFC is double of the $\angle$CFK, and the $\angle$CFD is double of the $\angle$CFL ;
∴ the $\angle$CFK = the $\angle$CFL. [*Axiom* 7.

Then, in the triangles FCK, FCL,

because $\begin{cases} \text{the } \angle CFK = \text{the } \angle CFL, \\ \text{and the right } \angle FCK = \text{the right } \angle FCL, \\ \text{and the side FC is common;} \end{cases}$ [*Proved.*

∴ CK = CL, and the ∠FKC = the ∠FLC. [I. 26.

Also because CK is equal to CL, LK is double of CK.

Similarly, it may be shewn that HK is double of BK.

And because BK is equal to CK, as was shewn, and that HK is double of BK, and LK double of CK;

∴ HK = LK. [*Axiom* 6.

Similarly, it may be shewn that any two consecutive sides of the pentagon are equal, and it is therefore equilateral.

(2) It is also equiangular.

For since the ∠FKC = the ∠FLC, and that the ∠HKL is double of the ∠FKC, and the ∠KLM double of the ∠FLC, as was shewn;

∴ the ∠HKL = the ∠KLM. [*Axiom* 6.

In the same manner it may be shewn that any two consecutive angles of the pentagon are equal, and it is therefore equiangular.

Also it has been shewn to be equilateral.

Wherefore *a regular pentagon has been described about the given circle.* [Q.E.F.

EXERCISES.

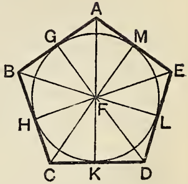
1. Prove the following alternative construction: *In the circle inscribe, by* IV. 11, *a regular polygon* ABCDE; *draw the tangents to the circle at* A, B, C, D, E, *and let them meet in* P, Q, R, S, T; *then* PQRST *is the required figure.*

2. The bisectors of all the angles of a regular polygon meet in a point.

PROPOSITION 13. PROBLEM.

To inscribe a circle in a given regular pentagon.

Let ABCDE be the given regular pentagon: *it is required to inscribe a circle in it.*



Construction. Bisect the angles BCD, CDE by CF, DF; [I. 9.
and from the point F, at which they meet, draw the straight lines FB, FA, FE.

Draw FG, FH, FK, FL, FM perpendiculars to AB, BC, CD, DE, EA. [I. 12.

Proof. In the triangles BCF, DCF,

because { BC = CD, [*Hypothesis.*
and CF is common,
and the ∠BCF = the ∠DCF; [*Construction.*

∴ the base BF = the base DF,
and the ∠CBF = the ∠CDF, [I. 4.
∴ twice the ∠CBF = twice the ∠CDF.
that is, = the ∠CDE, which = the ∠CBA.
that is, the ∠ABC is bisected by BF.

In the same manner it may be shewn that the angles BAE, AED are bisected by AF, EF.

Again, in the triangles FHC, FKC,

because { the ∠FCH = the ∠FCK,
and the right ∠FHC = the right ∠FKC,
and the side FC is common;

∴ FH = FK. [I. 26.

Similarly, it may be shewn that FL, FM, FG are each equal to FH or FK;

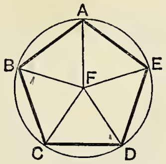
∴ FG, FH, FK, FL, FM are all equal, and the circle described, with centre F and radius equal to any one of them, will pass through the extremities of the other four;
and it will touch AB, BC, CD, DE, EA, because the angles at G, H, K, L, M are right angles. [III. 16.

Wherefore *a circle* GHKLM *has been inscribed in the given regular pentagon* ABCDE. [Q.E.F.

Proposition 14. Problem.

To describe a circle about a given regular pentagon.

Let ABCDE be the given regular pentagon: *it is required to describe a circle about it.*



Construction. Bisect the angles BCD, CDE by the straight lines CF, DF, which meet in F. [I. 9.
Join FB, FA, FE.

Proof. It may be shewn, as in the preceding proposition, that the angles CBA, BAE, AED are bisected by the straight lines BF, AF, EF.

Also because the $\angle$BCD = the $\angle$CDE,
and that the $\angle$FCD is half of the $\angle$BCD,
and the $\angle$FDC is half of the $\angle$CDE,
$\therefore$ the $\angle$FCD = the $\angle$FDC; [*Axiom* 7.
$\therefore$ FC = FD. [I. 6.

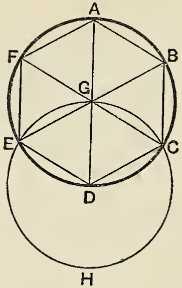
In the same manner it may be shewn that FB, FA, FE are each equal to FC or FD;
therefore FA, FB, FC, FD, FE are all equal, and the circle described, with centre F and radius equal to any one of them, will pass through the extremities of the other four, and will be described about the pentagon ABCDE.

Wherefore *a circle has been described about the given regular pentagon.* [Q.E.F.

Proposition 15. Problem.

To inscribe a regular hexagon in a given circle.

Let ABCDEF be the given circle: it is required to inscribe a regular hexagon in it.



Construction. Find the centre G of the circle, [III. 1.
and draw the diameter AGD;
with centre D and radius DG describe the circle EGCH,
join EG, CG, and produce them to meet the circumference of the circle ABCDEF in B and F;
join AB, BC, CD, DE, EF, FA.
Then ABCDEF is the hexagon required.

Proof. (1) Because
GE = GD, being radii of the circle ABCDEF,
and DE = DG, being radii of the circle EGCH,
∴ the triangle EGD is equilateral; [*Axiom* 1.
∴ the angles EGD, GDE, DEG are all equal. [I. 5, *Corollary*.
∴ the angle EGD = one-third of two right angles. [I. 32.

Similarly, it may be shewn that the angle DGC = one-third of two right angles.
∴ the angle EGC = two-thirds of two right angles;
but the angles EGC, CGB together = two right angles; [I. 13.

∴ the angle CGB = one-third of two right angles;
∴ the angles EGD, DGC, CGB are equal.

Also to these are equal the vertical opposite angles BGA, AGF, FGE; [I. 15.

∴ the six angles EGD, DGC, CGB, BGA, AGF, FGE are all equal;

∴ the six arcs AB, BC, CD, DE, EF, FA are all equal. [III. 26.

∴ the six chords AB, BC, CD, DE, EF, FA are all equal, and the hexagon is equilateral. [III. 29.

(2) It is also equiangular.

For the arc AF = the arc ED; to each add the arc ABCD;
∴ the whole arc FABCD = the whole arc ABCDE;
therefore the angles FED, AFE, which stand on these equal arcs, are equal. [III. 27.

Similarly, it may be shewn that any other two angles of the hexagon ABCDEF are equal;

∴ the hexagon is equiangular, and it has been shewn to be equilateral, and it is inscribed in the circle ABCDEF.

Wherefore *a regular hexagon has been inscribed in the given circle.* [Q.E.F.

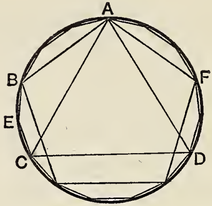
Corollaries. (1) Since EDG is an equilateral triangle, therefore DE = DG,
that is, the side of the hexagon = the radius of the circle.

(2) If through the points A, B, C, D, E, F there be drawn tangents to the circle, a regular hexagon will be described about the circle, as may be shewn from what was said of the pentagon; and a circle may be inscribed in a given regular hexagon, and circumscribed about it, by a method like that used for the pentagon.

Proposition 16. Problem.

To inscribe a regular quindecagon in a given circle.

Let ABCD be the given circle:
it is required to inscribe a regular quindecagon in it.



Let AC be the side of an equilateral △ inscribed in the circle, [IV. 2.
and AB the side of a regular pentagon inscribed in it. [IV. 11.

Then, of such equal parts, as the whole circumference ABCDF contains fifteen,
the arc ABC, which is the third part of the whole, contains five,
and the arc AB, which is the fifth part, contains three,
and therefore their difference, the arc BC, contains two.
Bisect the arc BC at E; [III. 30.
therefore each of the arcs BE, EC is the fifteenth part of the whole circumference.
∴ if BE, EC be drawn, and chords equal to them be placed round in the whole circle, [IV. 1.
a regular quindecagon will be inscribed in it. [Q.E.F.

Corollary. As in the case of the pentagon, if through the points of division made by inscribing the quindecagon tangents be drawn to the circle, a regular quindecagon will be described about it; and also, as for the pentagon, a circle may be inscribed in a given regular quindecagon, and circumscribed about it.

EXERCISES.

1. AB, BC are sides of a regular hexagon, and therefore AC is the side of an equilateral triangle inscribed in the same circle. Prove that the square on AC is three times the square on AB.

2. In the figure of IV. 15, prove that A, C, E are the angular points of an equilateral triangle.

****3.** The area of a regular hexagon is twice that of an equilateral triangle inscribed in the same circle.

[In the figure of IV. 15, AGBC is a $\|^{gm}$, and $\therefore$ = twice $\triangle$ AGC. But the hexagon = three times AGCB and $\triangle$ AEC = three times AGC, etc.]

4. Construct on a given straight line as one side a regular—

(1) pentagon ; (2) hexagon ; (3) octagon.

****5.** Inscribe a figure of 30 sides in a given circle.

****6.** Any equilateral figure which is inscribed in a circle is also equiangular.

7. If the alternate sides of a regular polygon be produced to meet, the points of intersection form another regular polygon of the same number of sides.

****8.** Prove that the centre of the inscribed circle and of the circumscribing circle of any regular polygon is found by bisecting any two consecutive angles.

****9.** If we have any regular polygon in a circle and draw tangents to the circle at its vertices, the latter form a regular circumscribed polygon having the same number of the sides as the first.

10. Verify that, by the constructions of this book, we can inscribe in a circle polygons of respectively 2×2^n, 3×2^n, 5×2^n, and 15×2^n sides, where n is any positive integer.

[See also Notes, page 328]

BOOK V.

DEFINITIONS.

1. A less magnitude is said to be an aliquot part, or a **measure**, or **submultiple**, of a greater magnitude, when the less is contained a certain number of times exactly in the greater.

2. A greater magnitude is said to be a **multiple** of a less, when the greater contains the less a certain number of times exactly.

3. Ratio is a mutual relation of two magnitudes of the same kind to one another in respect of quantity

The two magnitudes are called the terms of the ratio. The first term is called the **antecedent**, and the second term the **consequent.**

4. Magnitudes are said to have a ratio to one another, when the less can be multiplied so as to exceed the other.

[This definition amounts to saying that the quantities must be of the *same kind.*]

5. The first of four magnitudes is said to have the same **ratio** to the second that the third has to the fourth, when any equimultiples whatever of the first and the third being taken, and any equimultiples whatever of the second and the fourth; if according as the multiple of the first is greater than, equal to, or less than the second, the multiple of the third is greater than, equal to, or less than the fourth.

[Thus, to use the language of Algebra, four magnitudes, A, B, C, D, are proportionals, if $mC \gtreqqless nD$ according as $mA \gtreqqless mB$, where m and n are any whole numbers whatever.]

[*Note.* The fifth definition is the foundation of Euclid's doctrine of proportion. The student will find in works on Algebra (for example, in the present Editors' *Algebra for Beginners*, Arts. 373-377) a comparison of Euclid's definition of proportion with the simpler definitions which are employed in Arithmetic and Algebra. Euclid's definition is applicable to *incommensurable* quantities, as well as to *commensurable* quantities.

We should recommend the student to read the first proposition of the sixth Book immediately after the fifth definition of the fifth Book; he will there see how Euclid applies his definition, and will thus obtain a better notion of its meaning and importance.]

6. Magnitudes which have the same ratio are called **proportionals.**

When four magnitudes are proportionals, it is usually expressed by saying, the first is to the second as the third is to the fourth. The first and fourth magnitudes are sometimes called the **extremes**, and the second and third the **means.**

7. When of the equimultiples of four magnitudes, taken as in the fifth definition, the multiple of the first is greater than the multiple of the second, but the multiple of the third is not greater than the multiple of the fourth,

then the first is said to have to the second a greater ratio than the third has to the fourth;

and the third is said to have to the fourth a less ratio than the first has to the second.

8. Analogy, or **proportion**, is the similitude of ratios.

9. Proportion consists in three terms at least.

10. Three, or more, magnitudes are said to be in **continued proportion** when the ratio of the first to the second is the same as that of the second to the third, and the ratio of the second to the third is that of the third to the fourth, and so on.

When three magnitudes are continued proportionals, the first is said to have to the third the **duplicate ratio** of that which it has to the second.

Thus, if the ratio of A to B equals that of B to C, then A has to C the duplicate ratio of that which it has to B.

[The second magnitude is said to be a **mean proportional** between the first and the third.]

[It will be shewn in the corollary to VI. 20 that the duplicate ratio of two straight lines is the same as the ratio of the squares on the lines.]

When four magnitudes are continued proportionals, the first is said to have to the fourth the triplicate ratio of that which it has to the second.

11. *Definition of compound ratio.* When there are any number of magnitudes of the same kind, the first is said to have to the last of them the ratio which is compounded of the ratio which the first has to the second, and of the ratio which the second has to the third, and of the ratio which the third has to the fourth, and so on unto the last magnitude.

For example, if A, B, C, D be four magnitudes of the same kind, the first A is said to have to the last D the ratio compounded of the ratio of A to B, and of the ratio of B to C, and of the ratio of C to D; or, the ratio of A to D is said to be compounded of the ratios of A to B, B to C, and C to D.

And if A has to B the same ratio that E has to F; and B to C the same ratio that G has to H; and C to D the same ratio that K has to L; then, by this definition, A is said to have to D the ratio compounded of the ratios of E to F, G to H, and K to L.

[Since the ratio of A to C is compounded of that of A to B and of B to C, it follows from the preceding definition that the ratio compounded of two equal ratios is the duplicate of either of them.]

12. In proportionals, the antecedent terms are said to be **homologous** to one another; as also the consequents to one another.

Geometers make use of the following technical words to signify certain ways of changing either the order or the magnitude of four proportionals, so that they continue still to be proportionals.

13. *Permutando*, or *alternando*, by permutation or alternately; when it is inferred that the first is to the third as the second is to the fourth. V. 16.

14. *Invertendo*, by inversion; when it is inferred that the second is to the first as the fourth is to the third. V. *B*.

15. *Componendo*, by composition; when it is inferred that the first, together with the second, is to the second as the third, together with the fourth, is to the fourth. V. 18.

16. *Dividendo*, by division; when it is inferred that the excess of the first above the second is to the second as the excess of the third above the fourth is to the fourth. V. 17.

17. *Convertendo*, by conversion; when it is inferred that the first is to its excess above the second as the third is to its excess above the fourth. V. *E*.

18. *Ex æquali distantia*, or *ex æquo*, by equality; when there is any number of magnitudes more than two, and as many others, such that they are proportionals when taken two and two of each set, and it is inferred that the first is to the last of the first set of magnitudes as the first is to the last of the others.

Of this there are the two following kinds, which arise from the different order in which the magnitudes are taken.

19. *Ex æquali*, by direct equality. This term is used simply by itself, when the first magnitude is to the second of the first set as the first is to the second of the other set; and the second is to the third of the first set as the second is to the third of the other; and so on in order; and the inference is that mentioned in the preceding definition. V. 22.

Thus, suppose the magnitudes of the first set to be A, B, C, ..., and those of the second set to be P, Q, R, ..., and suppose we know

that A is to B as P to Q,

and B to C as Q to R;

then we infer that A is to C as P to R.

20. *Ex æquali in proportione perturbatâ seu inordinatâ*, from equality in perturbate or disorderly proportion, that is, by transverse equality. This term is used when the first magnitude is to the second of the first set as the last but one is to the last of the second set; and the second is to the third of the first set as the last but two is to the last but one of the second set; and the third is to the fourth of the first set as the last but three is to the last but two of the second set; and so on in a cross order; and the inference is that mentioned in the eighteenth definition. V. 23.

Thus, if A is to B as Q to R,
and B is to C as P to Q,
we infer that A is to C as P to R.

AXIOMS.

1. Equimultiples of the same, or of equal magnitudes, are equal to one another.

2. Those magnitudes, of which the same or equal magnitudes are equimultiples, are equal to one another.

3. A multiple of a greater magnitude is greater than the same multiple of a less.

4. That magnitude, of which a multiple is greater than the same multiple of another, is greater than that other magnitude.

Note 1. In the proofs of the propositions of this Book, capital letters A, B, C, ... will be used to denote the magnitudes themselves and small letters m, n, p, ... to denote whole numbers.

Thus, mA will mean the magnitude obtained by adding together m magnitudes each equal to A.

The symbol $>$ will sometimes be used for *greater than*, and the symbol $<$ for *less than*.

The notation A : B is often used as an abbreviation for "the ratio of A to B." Also the notation A : B : : C : D, or A : B = C : D, is often used as an abbreviation for "the ratio of A to B equals that of C to D," which, again, is generally abbreviated into "A is to B as C is to D."

Note 2. The propositions of the Fifth Book might be divided into four sections. Propositions 1 to 6 relate to the properties of equimultiples. Propositions 7 to 10 and 13 and 14 connect the notion of the *ratio* of magnitudes with the ordinary notions of *greater*, *equal*, and *less*. Propositions 11, 12, 15, and 16 may be considered as introduced to shew that, *if four quantities of the same kind be proportionals they will also be proportionals when taken alternately*. The remaining propositions shew that magnitudes are proportional by *composition*, by *division*, and *ex æquo*.

In this division of the Fifth Book propositions 13 and 14 are supposed to be placed immediately after proposition 10; and they might be taken in this order without any change in Euclid's demonstrations.

The propositions headed *A*, *B*, *C*, *D*, *E*, and also the preceding axioms, were supplied by Simson.

Note 3. At the end of each proposition has been appended the corresponding algebraic theorem; many of these theorems are so simple that they are self-evident. The others are proved in the *Algebra for Beginners*, new edition, Arts. 357-364 and 371.

Proposition 1. Theorem.

If any number of magnitudes be equimultiples of as many, each of each; whatever multiple any one of them is of its part, the same multiple shall all the first magnitudes be of all the other.

Let any number of magnitudes A, B, C be equimultiples of as many others E, F, G, each of each:

whatever multiple A *is of* E, *the same multiple shall* A, B, *and* C *together be of* E, F, *and* G *together.*

For let A contain E, B contain F, and C contain G any number of times, say m times.

Then $\mathrm{A} = m\mathrm{E}$, $\mathrm{B} = m\mathrm{F}$, and $\mathrm{C} = m\mathrm{G}$;

$\therefore$ A, B, and C together $= m$E, mF, and mG together

$= m$ times E, F, and G together.

Wherefore, *if any number, etc.* [Q.E.D.

Algebraically. If $a = me$, $b = mf$, and $c = mg$,

then $$a + b + c = m(e + f + g).$$

Proposition 2. Theorem.

If the first be the same multiple of the second that the third is of the fourth, and the fifth the same multiple of the second that the sixth is of the fourth; the first, together with the fifth, shall be the *same multiple of the second that the third, together with the sixth, is of the fourth.*

Let the first magnitude A be the same multiple of B the second that C the third is of D the fourth, and let E the fifth be the same multiple of B the second that F the sixth is of D the fourth:

then A, *together with* E, *shall be the same multiple of* B *that* C, *together with* F, *is of* D.

For let m be the first common multiple, so that

$$A = mB, \quad C = mD.$$

Also, let n be the second common multiple, so that

$$E = nB, \quad F = nD;$$

$\therefore$ A, together with E, $= mB$, together with nB, $= (m+n)B$, and C, together with F, $= mD$, together with nD, $= (m+n)D$.

Corollary. If $A = mB$, $C = mD$,

$E = nB$, $F = nD$,

and $G = pB$, $H = pD$,

then A, E, G together are the same multiple of B that C, F, H together are of D.

Wherefore, *if the first, etc.* [Q.E.D.

Algebraically. If $a = mb$, $c = md$, $e = nb$, and $f = nd$,

then $$\frac{a+e}{b} = \frac{c+f}{d}.$$

Proposition 3. Theorem.

If the first be the same multiple of the second that the third is of the fourth, and if of the first and the third there be taken equimultiples, these shall be equimultiples, the one of the second, and the other of the fourth.

Let A the first be the same multiple of B the second that C the third is of D the fourth; and of A and C let the equimultiples nA, nC be taken:

nA *shall be the same multiple of* B *that* nC *is of* D.

For let A be m times B, and C m times D, so that

$$A = mB, \text{ and } C = mD.$$

Then n times $A = n$ times $mB = mn$ times B,
and n times $C = n$ times $mD = mn$ times D;
$\therefore$ nA is the same multiple of B that nC is of D.

Wherefore, *if the first, etc.* [Q.E.D.

Algebraically. If $a = mb$ and $c = md$,

then $$\frac{na}{b} = \frac{nc}{d}.$$

Proposition 4. Theorem.

If the first have the same ratio to the second that the third has to the fourth, and if there be taken any equimultiples whatever of the first and the third, and also any equimultiples whatever of the second and the fourth, then the multiple of the first shall have the same ratio to the multiple of the second that the multiple of the third has to the multiple of the fourth.

Let A the first have to B the second the same ratio that C the third has to D the fourth; and of A and C let there be taken any equimultiples whatever mA and mC, and of B and D any equimultiples whatever nB and nD:
mA *shall have the same ratio to* nB *that* mC *has to* nD.

For of mA and mC take any equimultiple p,
and of nB and nD take any equimultiple q;
then mA has to nB the same ratio that mC has to nD,
if pmA is greater, equal to, or less than qnB,
according as pmC is greater, equal to, or less than qnD,
that is, if A has to B the same ratio that C has to D,
[V. *Definition* 5.
since pmA, pmC are equimultiples of A and C,
and qnB, qnD are equimultiples of B and D.

Wherefore, *if the first, etc.* [Q.E.D.

Algebraically. If $a : b :: c : d$, then $ma : nb :: mc : nd$.

Proposition 5. Theorem.

If one magnitude be the same multiple of another that a magnitude taken from the first is of a magnitude taken from the other, the remainder shall be the same multiple of the remainder that the whole is of the whole.

Let A be the same multiple of B that C, taken from the first, is of D taken from the other:

the remainder E *shall be the same multiple of the remainder* F *that the whole* A *is of the whole* B.

For let $A = mB$ and $C = mD$.

Then the difference between A and C

= the difference between mB and mD,

that is, E = the difference between mB and mD,

that is, = m times the difference between B and D,

that is, $E = m$ times F.

Wherefore, *if one magnitude, etc.* [Q.E.D.

Algebraically. If $a = mb$ and $c = md$, then $a - c = m(b - d)$.

Proposition 6. Theorem.

If two magnitudes be equimultiples of two others, and if equimultiples of these be taken from the first two, the remainders shall be either equal to these others, or equimultiples of them.

Let the two magnitudes A, B be equimultiples of the two C, D; and let E, F, taken from the first two, be equimultiples of the same C, D:

the remainders G, H *shall be equimultiples of* C *and* D.

For let $A = mC$ and $B = mD$,

and let $E = nC$ and $F = nD$.

Then the difference between A and E

= the difference between mC and nC,

that is, $G = (m - n)$ times C;

so $H = (m - n)$ times D.

Wherefore, *if two magnitudes, etc.* [Q.E.D.

Algebraically. If $a = mc$, $b = md$, $e = nc$, and $f = nd$,

then $$\frac{a-e}{c} = \frac{b-f}{d}.$$

PROPOSITION *A*. THEOREM.

If the first of four magnitudes have the same ratio to the second that the third has to the fourth, then, if the first be greater than the second, the third shall also be greater than the fourth; and if equal, equal; and if less, less.

The four magnitudes being A, B, C, and D, we know that if we take any equimultiples whatever mA, mC, nB, and nD, then as mA is greater, equal to, or less than nB,

so mC is greater, equal to, or less than nD; [V. *Def.* 5.

∴, if we make m and n each equal to unity, it follows that

as A is greater, equal to, or less than B,

so C is greater, equal to, or less than D.

Wherefore, *if the first, etc.* [Q.E.D.

Algebraically. If $a : b :: c : d$, then $a \gtreqless b$ according as $c \gtreqless d$.

PROPOSITION *B*. THEOREM.

If four magnitudes be proportionals, they shall also be proportionals when taken inversely.

Let A be to B as C is to D:

then also, inversely, B *shall be to* A *as* D *is to* C.

Take mA, mC, any equimultiples of A and C, and nB, nD, any equimultiples of B and D.

Then we are given that if mA is greater than nB,

then mC is greater than nD,

that is, if nB is less than mA, then nD is less than mC.

Similarly, if nB is equal to, or greater than mA,

then nD is equal to, or greater than mC;

∴ nB is less than, equal to, or greater than mA, according as nD is less than, equal to, or greater than mC;

∴ B is to A as D to C. [V. *Definition* 5.

Wherefore, *if four magnitudes, etc.* [Q.E.D.

Algebraically. If $a : b :: c : d$, then $b : a :: d : c$.

PROPOSITION *C*. THEOREM.

If the first be the same multiple of the second, or the same part of it, that the third is of the fourth, the first shall be to the second as the third is to the fourth.

Let A, B, C, D be the four magnitudes,
and, firstly, let A and C be equimultiples of B and D,
so that A = mB and C = mD,
then shall A *be to* B *as* C *to* D.

Take any multiples nA, nC of A and C,
and any multiples pB, pD of B and D.

Then A is to B as C to D,
if nC is greater than, equal to, or less than pD,
according as nA is greater than, equal to, or less than pB;
that is, if nmD is greater than, equal to, or less than pD,
according as nmB is greater than, equal to, or less than pB;
that is, if nm is greater than, equal to, or less than p,
according as nm is greater than, equal to, or less than p,
which is true.

Next, let A be the same part, or submultiple, of B that C is of D.

Then B is the same multiple of A that D is of C;

∴ by the first part, B is to A as D to C;

∴ inversely, A is to B as C to D. [V. *B*.

Wherefore, *if the first, etc.* [Q.E.D.

Algebraically. If $a = mb$ and $c = md$, then $a : b :: c : d$.

PROPOSITION *D*. THEOREM.

If the first be to the second as the third is to the fourth, and if the first be a multiple, or a part, of the second, the third shall be the same multiple, or the same part, of the fourth.

First, let $A = mB$, and take the same equimultiple mD of D; then shall $C = mD$.

Since A, B, C, D are proportionals,

∴ according as A is greater, equal to, or less than mB,

so C is greater, equal to, or less than mD.

But $A = mB$; ∴ $C = mD$.

Next, let A be a part of B.

Because A is to B as C to D,

∴ inversely, B is to A as D to C;

∴ B is the same multiple of A that D is of C, by the first case;

that is, A is the same part of B that C is of D.

Wherefore, *if the first be, etc.* [Q.E.D.

Algebraically. If $a : b :: c : d$, and if $a = mb$, then $c = md$.

PROPOSITION 7. THEOREM.

Equal magnitudes have the same ratio to the same magnitude; and the same has the same ratio to equal magnitudes.

Let A and B be equal magnitudes, and C any other magnitude:

each of the magnitudes A *and* B *shall have the same ratio to* C; *and* C *shall have the same ratio to each of the magnitudes* A *and* B.

For it is clear, since $A = B$,

that mA is greater than, equal to, or less than nC,

according as mB is greater than, equal to, or less than nC;

∴ A is to C as B to C;

∴ inversely, C is to A as C to B. [V. *B.*

Wherefore, *equal magnitudes, etc.* [Q.E.D.

Algebraically. If $a = b$, then $a : c :: b : c$ and $c : a :: c : b$.

PROPOSITION 8. THEOREM.

Of unequal magnitudes, the greater has a greater ratio to the same than the less has; and the same magnitude has a greater ratio to the less than it has to the greater.

Let A and B be unequal magnitudes, of which A is the greater; and let C be any other magnitude whatever:
A *shall have a greater ratio to* C *than* B *has to* C; *and* C *shall have a greater ratio to* B *than it has to* A.

Let m be such a number that mB and m times A – B are both greater than C, and let nC be the least multiple of C which is just greater than mA;
∴ nC – C is less than mA; that is, mA is $>(n-1)$C.

Again, since nC is greater than mA,
but C is less than mA – mB,
∴ $(n-1)$C is greater than mB;
∴ mA is greater, but mB is less, than $(n-1)$ times C;
∴ the ratio of A to C is greater than that of B to C. [V. *Def.* 7.

Again, since we have proved that $(n-1)$ times C is less than mA, but greater than mB;
∴ the ratio of C to A is less than that of C to B. [V. *Def.* 7.

Wherefore, *of unequal magnitudes, etc.* [Q.E.D.

Algebraically. If $a > b$, then $a : c > b : c$, and $c : b < c : a$.

PROPOSITION 9. THEOREM.

Magnitudes which have the same ratio to the same magnitude are equal to one another; and those to which the same magnitude has the same ratio are equal to one another.

First, let A and B have the same ratio to C:
A *shall be equal to* B.

For if not, A must be greater than B, or else B greater than A.
A is not greater than B; for then the ratio of A to C would be greater than that of B to C. [V. 8.

Neither is B greater than A; for then the ratio of B to C would be greater than that of A to C; [V. 8.
∴ A must be equal to C.

In a similar manner it may be shewn that, if C have the same ratio to A or B, then A and B are equal.

Wherefore, *magnitudes which have, etc.* [Q.E.D.

Algebraically. If $a : c :: b : c$, then $a = b$.
Also if $a = b$, then $a : c :: b : c$.

Proposition 10. Theorem.

That magnitude which has a greater ratio than another has to the same magnitude is the greater of the two; and that magnitude to which the same has a greater ratio than it has to another magnitude is the less of the two.

First, let A have to C a greater ratio than B has to C: A *shall be greater than* B.

Since the ratio of A to C is greater than that of B to C, two numbers m and n can be found, such that mA is greater than nC, and mB is less than nC; [V. *Definition* 7.
∴ mA is greater than mB; ∴ A is greater than B.

Next, if the ratio of C to B is greater than that of C to A, two numbers p and q can be found, such that pC is greater than qB, and pC is less than qA;
∴ qA is greater than qB; ∴ A is greater than B.

Wherefore, *that magnitude, etc.* [Q.E.D.

Algebraically. If $a : c > b : c$, then $a > b$;
and if $c : b > c : a$, then $b < a$.

Proposition 11. Theorem.

Ratios that are equal to the same ratio are equal to one another.

Let A be to B as C is to D, and let C be to D as E is to F: A *shall be to* B *as* E *is to* F.

Take any equimultiples mA, mC, mE, and any equimultiples nB, nD, nF.

Then, by hypothesis,

mA is greater than, equal to, or less than nB,
as mC is greater than, equal to, or less than nD,
that is, as mE is greater than, equal to, or less than nF; [*Hyp.*
∴ A is to B as E to F.

Wherefore, *ratios that are the same, etc.* [Q.E.D.

Algebraically. If $a:b::c:d$ and $c:d::e:f$, then $a:b::e:f$.

Proposition 12. Theorem.

If any number of magnitudes be proportionals, as one of the antecedents is to its consequent, so shall all the antecedents be to all the consequents.

Let any number of magnitudes A, B, C, D, E, F be proportionals; namely, as A is to B, so let C be to D, and E to F: *as* A *is to* B, *so shall* A, C, E *together be to* B, D, F *together.*

Take any equimultiples mA, mC, mE,
and any equimultiples nB, nD, nF.

Then, by hypothesis, if mA is greater than nB,
mC is greater than nD, and mE is greater than nF;
∴ mA, mC, mE are together greater than the sum of nB, nD, nF.

Similarly, if mA is equal to, or less than nB,
mA, mC, mE together are equal to, or less than the sum of nB, nD, nF;
∴ m times the sum of A, C, E is greater than, equal to, or less than n times the sum of B, D, F,
according as mA is greater, equal to, or less than nB;
∴ A, C, E together are to B, D, F together as A to B.

Wherefore, *if any number, etc.* [Q.E.D.

Algebraically. If $a:b::c:d::e:f$,
then $a:b::a+c+e:b+d+f$.

Proposition 13. Theorem.

If the first have the same ratio to the second which the third has to the fourth, but the third to the fourth a greater ratio than the fifth to the sixth, the first shall have to the second a greater ratio than the fifth has to the sixth.

Let A have the same ratio to B that C has to D, but C a greater ratio to D than E to F:
A *shall have to* B *a greater ratio than* E *has to* F.

Because the ratio of C to D is greater than that of E to F, there are two numbers m, n, such that $m\text{C} > n\text{D}$, but $m\text{E} < n\text{F}$. [V. *Definition* 7.
But if $m\text{C} > n\text{D}$, then $m\text{A} > n\text{B}$;
$\therefore$ $m\text{A} > n\text{B}$, but $m\text{E} < n\text{F}$;
$\therefore$ the ratio of A to B is greater than that of E to F.

Wherefore, *if the first, etc.* [Q.E.D.

Algebraically. If $a : b :: c : d$ and $c : d > e : f$, then $a : b > e : f$.

Proposition 14. Theorem.

If the first have the same ratio to the second that the third has to the fourth, then if the first be greater than the third the second shall be greater than the fourth; and if equal, equal; and if less, less.

Let A have the same ratio to B that C has to D:
if A *be greater than* C, B *shall be greater than* D; *if equal, equal; and if less, less.*

If $\text{A} > \text{C}$, then the ratio of A to B is greater than that of C to B, [V. 8.
that is, the ratio of C to D is greater than that of C to B; [*Hypothesis.*
$\therefore$ $\text{B} > \text{D}$. [V. 10.

Similarly, it may be shewn that if $\text{A} = \text{C}$, then $\text{B} = \text{D}$, and if $\text{A} < \text{C}$, then $\text{B} < \text{D}$.

Wherefore, *if the first, etc.* [Q.E.D.

Algebraically. If $a : b :: c : d$, then $a \gtreqless c$ according as $b \gtreqless d$.

PROPOSITION 15. THEOREM.

Magnitudes have the same ratio to one another that their equimultiples have.

Let A and B be any two magnitudes, and let any two equimultiples mA and mB be taken:
the ratio of A *to* B *shall be the same as that of* mA *to* mB.

Since A is to B as A to B;
∴ A is to B as A + A to B + B,
that is, as 2A to 2B. [V. 12.

Again, since A is to B as 2A to 2B;
∴ A is to B as A + 2A to B + 2B,
that is, as 3A to 3B. [V. 12.

Similarly, A is to B as 4A to 4B, [V. 12.
and so on whatever be the equimultiple m.

Wherefore, *magnitudes, etc.* [Q.E.D.

Algebraically. $a : b :: ma : mb$.

PROPOSITION 16. THEOREM.

If four magnitudes of the same kind be proportionals, they shall also be proportionals when taken alternately.

Let A, B, C, D be four magnitudes of the same kind which are proportionals; namely, as A is to B so let C be to D:
they shall also be proportionals when taken alternately, that is, A *shall be to* C *as* B *is to* D.

Let any equimultiples mA, mB, and any equimultiples nC, nD be taken.

Then the ratio of mA to mB = the ratio of A to B [V. 15.
= the ratio of C to D [*Hypothesis.*
= the ratio of nC to nD; [V. 15.
∴ mA, mB, nC, nD are proportionals;
∴ mA is greater than, equal to, or less than nC, according as mB is greater than, equal to, or less than nD; [V. 14.
∴ A, C, B, D are proportionals. [V. *Definition* 5.

Wherefore, *if four magnitudes, etc.* [Q.E.D.

Algebraically. If $a : b :: c : d$, then $a : c :: b : d$.

Proposition 17. Theorem.

If four magnitudes be proportional, the difference between the first and second shall be to the second as the difference between the third and fourth is to the fourth.

Let A, B, C, D be four magnitudes, such that A is to B as C to D:

the difference between A, B *shall be to* B *as the difference between* C, D *is to* D.

Take any equimultiples mA, mC, and any equimultiples nB, nD, and let p denote the difference between n and m.

First, let $mA > nB$; then, by hypothesis, $mC > nD$.

Since $mA > nB$, $\therefore\ mA - mB > nB - mB$,

that is, $m(A - B) > pB$.

Similarly $m(C - D) > pD$;

$\therefore\ m(A - B) > pB$, when $m(C - D) > pD$;

so $m(A - B) \leqq pB$, when $n(C - D) \leqq pD$;

$\therefore$ A − B, B, C − D, and D are proportionals. [V. *Definition* 5.

Wherefore, *if four magnitudes, etc.* [Q.E.D.

Algebraically. If $a : b :: c : d$, then $a - b : b :: c - d : d$.

Proposition 18. Theorem.

If four magnitudes be proportionals, the sum of the first and second is to the second as the sum of the third and fourth is to the fourth.

Let A, B, C, D be four magnitudes, such that A is to B as C to D:

then shall the sum of A *and* B *be to* B *as the sum of* C *and* D *to* D.

Take any equimultiples mA, mC, and any equimultiples nB, nD, and let p denote the sum of m and n.

First, let $mA > nB$; then, by hypothesis, $mC > nD$.

Since $mA > nB$; $\therefore\ mA + mB > mB + nB$,

that is, $m(A + B) > pB$.

So since $mC > nD$, $\therefore$ $m(C+D) > pD$;
$\therefore$ if $m(A+B) > pB$, then $m(C+D) > pD$.

Similarly, it can be shewn that
if $m(A+B) \leqq pB$, then $m(C+D) \leqq pD$;
$\therefore$ A + B, B, C + D, D are proportionals.

Wherefore, *if four magnitudes, etc.* [Q.E.D.

Algebraically. If $a : b :: c : d$, then $a + b : b :: c + d : d$.

Proposition 19. Theorem.

If a whole magnitude be to a whole as a magnitude taken from the first is to a magnitude taken from the other, the remainder shall be to the remainder as the whole is to the whole.

Let A be to B as C to D:
then shall A – C *be to* B – D *as* A *to* B.

For since A is to B as C to D;
$\therefore$ A is to C as B to D; [V. 16.
$\therefore$ A – C is to C as B – D to D; [V. 17.
$\therefore$ A – C is to B – D as C to D, [V. 16.
that is, as A to B.

Wherefore, *if a whole, etc.* [Q.E.D.

Algebraically. If $a : b :: c : d$, then $a - c : b - d :: a : b$.

Proposition *E*. Theorem.

If four magnitudes be proportionals, they shall also be proportionals by conversion; that is, the first shall be to its excess above the second as the third is to its excess above the fourth.

Let A be to B as C is to D:
then shall A *be to* A – B *as* C *to* C – D.

Since A is to B as C to D;
$\therefore$ A is to C as B to D; [V. 16.
$\therefore$ A – B is to C – D as A to C; [V. 19.
$\therefore$ A – B is to A as C – D to C; [V. 16.

∴ A is to A − B as C to C − D. [V. *B*.

Wherefore, *if four magnitudes, etc.* [Q.E.D.

Algebraically. If $a : b :: c : d$, then $a : a - b :: c : c - d$.

PROPOSITION 20. THEOREM.

If there be three magnitudes, and other three, which have the same ratio, taken two and two, then, if the first be greater than the third, the fourth shall be greater than the sixth; and if equal, equal; and if less, less.

Let A, B, C be three magnitudes, and D, E, F other three, such that A is to B as D is to E, and B to C as E to F: *if* A *be greater than* C, D *shall be greater than* F; *and if equal, equal; and if less, less.*

First, let A be greater than C,
then A has to B a greater ratio than C has to B. [V. 8.
But A is to B as D is to E; [*Hypothesis.*
∴ D has to E a greater ratio than C has to B. [V. 13.
Also because B is to C as E is to F, [*Hypothesis.*
∴ by inversion, C is to B as F is to E; [V. *B*.
∴ D has to E a greater ratio than F has to E;
∴ D is greater than F. [V. 10.

Secondly, let A = C; ∴ A is to B as C is to B. [V. 7.
But A is to B as D is to E, [*Hypothesis.*
and C is to B as F is to E; [*Hypothesis*, V. *B*.
∴ D is to E as F is to E; [V. 11.
∴ D = F. [V. 9.

Lastly, let A < C, then C > A;
∴ as in the first case, C is to B as F is to E,
and B is to A as E is to D;
therefore, by the first case, F > D, that is, D < F.

Wherefore, *if there be three, etc.* [Q.E.D.

Algebraically. If $a : b :: d : e$ and $b : c :: e : f$, then $a \gtreqless c$ according as $d \gtreqless f$.

PROPOSITION 21. THEOREM.

If there be three magnitudes, and other three, which have the same ratio, taken two and two, but in a cross order; then, if the first be greater than the third, the fourth shall be greater than the sixth; and if equal, equal; and if less, less.

Let A, B, C be three magnitudes, and D, E, F other three, such that A is to B as E is to F, and B to C as D to E: *if* A *be greater than* C, D *shall be greater than* F; *and if equal, equal; and if less, less.*

First, let $A > C$;

$\therefore$ A has to B a greater ratio than C has to B. [V. 8.

But A is to B as E to F; [*Hypothesis.*

$\therefore$ E has to F a greater ratio than C has to B. [V. 13.

Also because B is to C as D to E, [*Hypothesis.*

$\therefore$ by inversion, C is to B as E to D; [V. *B.*

$\therefore$ E has to F a greater ratio than E has to D;

$\therefore$ $F < D$, that is, $D > F$. [V. 10.

Secondly, let $A = C$; $\therefore$ A is to B as C to B. [V. 7.

But A is to B as E to F, [*Hypothesis.*

and C is to B as E to D; [*Hypothesis*, V. *B.*

$\therefore$ E is to F as E to D, [V. 11.

and therefore $D = F$. [V. 9.

Lastly, let $A < C$, then $C > A$;

$\therefore$ as in the first case, C is to B as E to D,

and B is to A as F to E;

$\therefore$ by the first case, $F > D$, that is, $D < F$.

Wherefore, *if there be three, etc.* [Q.E.D.

Algebraically. If $a : b :: e : f$ and $b : c :: d : e$, then $a \gtreqless c$ according as $d \gtreqless f$.

PROPOSITION 22. THEOREM.

If there be any number of magnitudes, and as many others, which have the same ratio, taken two and two in order, the first shall have *to the last of the first magnitudes the same ratio which the first of the others has to the last.*

[This proposition is usually cited by the words *ex æquali.*]

First, let there be three magnitudes A, B, C, and other three D, E, F, such that A is to B as D is to E, and B to C as E to F:
A *shall be to* C *as* D *is to* F.

Of A and D take any equimultiples whatever mA, mD; of B and E any equimultiples nB, nE; and of C and F any equimultiples pC, pF.

Because A is to B as D to E; [*Hypothesis.*
∴ mA is to nB as mD to nE. [V. 4.
Similarly nB is to pC as nE to pF; [V. 4.
∴ according as mA is greater, equal to, or less than pC, so is mD greater, equal to, or less than pF; [V. 20.
∴ A is to C as D to F. [V. *Definition* 5.

Next, let there be four magnitudes A, B, C, D, and other four E, F, G, H, such that A is to B as E to F, B to C as F to G, and C to D as G to H
then shall A *be to* D *as* E *to* H.
For, by the first case, A is to C as E to G.
But C is to D as G to H; [*Hypothesis.*
∴ by the first case, A is to D as E to H.

A similar demonstration applies whatever be the number of magnitudes.

Wherefore, *if there be any number, etc.* [Q.E.D.

Algebraically. If $a:b::d:e$ and $b:c::e:f$, then $a:c::d:f$.

PROPOSITION 23. THEOREM.

If there be any number of magnitudes, and as many others, which have the same ratio, taken two and two in a cross order, the first shall have to the last of the first magnitudes the same ratio which the first of the others has to the last.

First, let there be three magnitudes, A, B, C, and other three E, F, G, such that A is to B as F to G, and B to C as E to F:

A *shall be to* C *as* E *to* G.

Of A and E take any equimultiples whatever mA, mE;
of B and F any equimultiples nB, nF;
and of C and G any equimultiples nC, nG.

Because A is to B as F to G; [*Hypothesis.*

∴ mA is to mB as nF to nG. [V. 15 and 11.

Also since B is to C as E to F; [*Hypothesis.*

∴ mB is to nC as mE to nF; [V. 16, 15, 11.

∴ according as mA is greater, equal to, or less than nC, so is mE greater, equal to, or less than nG; [V. 21.

∴ A is to C as E to G. [V. *Definition* 5.

Next, let there be four magnitudes, A, B, C, D, and other four E, F, G, H, such that A is to B as G to H, B to C as F to G, and C to D as E to F:

then A *is to* D *as* E *to* H.

For, by the first case, A is to C as F to H.

But C is to D as E to F; [*Hypothesis.*

∴ A is to D as E to H, by the first case.

Similarly, whatever be the number of the magnitudes.

Wherefore, *if there be any number, etc.* [Q.E.D.

Algebraically. If $a : b :: f : g$ and $b : c :: e : f$,

then $a : c :: e : g$.

Note. Propositions 22 and 23 may be included in one enunciation, thus: *Ratios which are compounded of equal ratios are equal.*

Proposition 24. Theorem.

If the first have to the second the same ratio which the third has to the fourth, and the fifth have to the second the same ratio which the sixth has to the fourth, then the first and fifth together shall have to the second the same ratio which the third and sixth together have to the fourth.

Let A be to B as C to D, and E to B as F to D: *then shall* A *and* E *together be to* B *as* C *and* F *together to* D.

Because E is to B as F to D;

∴ B is to E as D to F, by inversion. [V. *B*.

But A is to B as C to D; [*Hypothesis*.

∴ *ex æquali*, A is to E as C to F; [V. 22.

∴ A + E is to E as C + F to F. [V. 18.

But E is to B as F to D; [*Hypothesis*.

∴ *ex æquali*, A + E is to B as C + F to D. [V. 22.

Wherefore, *if the first have, etc.* [Q.E.D.

Algebraically. If $a : b :: c : d$ and $e : b :: f : d$, then $a + e : b :: c + f : d$.

Proposition 25. Theorem.

If four magnitudes of the same kind be proportionals, the greatest and least of them together shall be greater than the other two together.

Let the four magnitudes A, B, C, D be proportionals; namely, let A be to B as C to D; and let A be the greatest of them, and consequently D the least: [V. *A*, V. 14.

A *and* D *together shall be greater than* B *and* C *together.*

Since A is to B as C to D;

∴ A is to C as B to D, *alternando*; [V. 16.

∴ A is to A − C as B is to B − D. [V. *E*.

But A is greater than B; [*Hypothesis*.

∴ A − C is greater than B − D; [V. 14.

∴ A + D is greater than B + C.

Wherefore, *if four magnitudes, etc.* [Q.E.D.

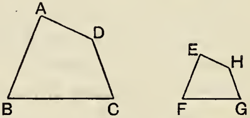
Algebraically. If $a : b :: c : d$, and a be the greatest of the four quantities, then $a + d > b + c$.

BOOK VI.

DEFINITIONS.

1. Two rectilineal figures are said to be **equiangular** when the angles of the first, taken in order, are respectively equal to the angles of the second, taken in order.

2. Similar rectilineal figures are those which have their several angles equal, each to each, and the sides about the equal angles proportionals. Also the corresponding sides of the two figures are said to be **homologous.**



Thus, in the case of the two figures ABCD, EFGH, they will be similar if the angles at A, B, C, D are respectively equal to the angles E, F, G, H; and if

AB is to BC as EF to FG,
BC is to CD as FG to GH,
and CD is to DA as GH to HE;
and ∴ by Book V., DA to AB as HE to EF.

Also AB and EF are homologous; also BC and FG, CD and GH, etc.

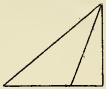
3. Reciprocal figures, namely, triangles and parallelograms, have their sides about two of their angles proportionals in such a manner, that a side of the first figure is to a side of the other, as the remaining side of this other is to the remaining side of the first.

Thus, if ABC, DEF be two triangles which have their sides about the angles A and D such that AB is to DE as DF to AC, they are reciprocal figures.

4. A straight line is said to be cut in **extreme and mean ratio** when the whole is to the greater segment as the greater segment is to the less.

Thus, in II. 11, the straight line AB is divided at H in extreme and mean ratio.

5. The altitude of any figure is the straight line drawn from its vertex perpendicular to the base.



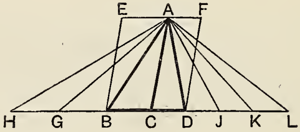
[This definition is strictly applicable only to a triangle because no other figure has a point that can exclusively be called its vertex. The altitude of a parallelogram is the perpendicular drawn to its base from any point in the opposite side.]

Proposition 1. Theorem.

Triangles and parallelograms of the same altitude are to one another as their bases.

Let the triangles ABC, ACD, and the parallelograms EC, CF have the same altitude, namely, the perpendicular drawn from A to BD:

as the base BC *is to the base* CD, *so shall the triangle* ABC *be to the triangle* ACD, *and the parallelogram* EC *to the parallelogram* CF.



Construction. Produce BD both ways;
take any number of straight lines BG, GH, each equal to BC, and any number DJ, JK, KL, each equal to CD, [I. 3.
and join AG, AH, AJ, AK, AL.

Proof. (1) Because CB, BG, GH are all equal, [*Construction.*
the triangles ABC, AGB, AHG are all equal; [I. 38.
∴ the △AHC is the same multiple of the △ABC that HC is of BC.

Similarly, the △ACL is the same multiple of the △ACD that CL is of CD.
And if HC = CL, the △AHC = the △ACL;
and if HC be greater than CL, the △AHC is greater than the △ACL; and if less, then less. [I. 38.

Then, since there are four magnitudes, namely, the two triangles ABC, ACD, and the two bases BC, CD;
and of the first and the third any equimultiples have been taken, namely, the △AHC and the base HC;
and of the second and fourth any equimultiples whatever have been taken, namely, the △ACL and the base CL;

and since it has been shewn that the $\triangle AHC$ is greater, equal to, or less than the $\triangle ACL$, according as HC is greater, equal to, or less than CL;

$\therefore$ the $\triangle ABC$: the $\triangle ACD$ :: the base BC : the base CD.

[V. *Definition* 5.

(2) Because the parallelogram CE is double of the $\triangle ABC$, and the parallelogram CF is double of the $\triangle ACD$; [I. 41.

and that magnitudes have the same ratio which their equimultiples have; [V. 15.

$\therefore$ the parallelogram EC : the parallelogram CF

:: the $\triangle ABC$: the $\triangle ACD$,

that is, :: the base BC : the base CD. [V. 11.

Wherefore, *triangles, etc.* [Q.E.D.

Corollary. From this it is plain that triangles and parallelograms which have equal altitudes, are to one another as their bases.

For, let the figures be placed so as to have their bases in the same straight line, and to be on the same side of it, and let there be drawn perpendiculars from the vertices of the triangles to the bases; then the straight line which joins the vertices is parallel to that in which their bases are; [I. 33.

because the perpendiculars are both equal and parallel to one another. [I. 28.

For, if the same construction be made as in the proposition, the demonstration will be the same.

EXERCISES.

1. The four triangles into which any quadrilateral is divided by its two diagonals are proportional.

2. Perpendiculars are drawn from any point within an equilateral triangle on the three sides: shew that their sum is always the same.

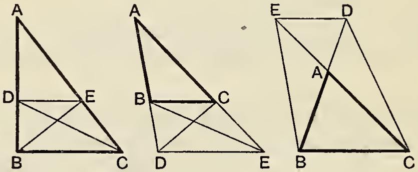
[Let O be any point within the equilateral $\triangle$ ABC; OP, OQ, OR the $\perp^{rs}$ upon the sides; AD, BE, CF the $\perp^{rs}$ from the angular points. Then we have $\triangle$ OBC : $\triangle$ ABC :: OP : AD; $\triangle$ OCA : $\triangle$ ABC :: OQ : BE, that is, as OQ : AD, etc.; $\therefore$ sum of $\triangle^{s}$ OBC, OCA, OAB : $\triangle$ ABC; $\therefore$ sum of OP, OQ, OR : AD; $\therefore$ OP, OQ, OR together = AD.]

Proposition 2. Theorem.

If a straight line be drawn parallel to one of the sides of a triangle, it shall cut the other sides, or those sides produced, proportionally.

Conversely, *if the sides, or the sides produced, be cut proportionally, the straight line which joins the points of section shall be parallel to the remaining side of the triangle.*

Let DE be drawn parallel to BC, one of the sides of the triangle ABC:
BD *shall be to* DA *as* CE *to* EA.



Construction. Join BE, CD.

Proof. The △BDE = the △CDE, because they are on the same base DE and between the same parallels DE, BC. [I. 37.
Also ADE is another △, and equal magnitudes have the same ratio to the same magnitude; [V. 7.

∴ the △BDE : the △ADE :: the △CDE : the △ADE.

But the △BDE : the △ADE :: BD : DA,
because they have the same altitude, namely, the perpendicular drawn from E to AB. [VI. 1.
Similarly, the △CDE : the △ADE :: CE : EA;

∴ BD : DA :: CE : EA. [V. 11.

Conversely, let BD be to DA as CE to EA, and join DE: DE *shall be parallel to* BC.

For, the same construction being made,
because BD : DA :: CE : EA, [*Hypothesis.*

and BD : DA :: the $\triangle$BDE : the $\triangle$ADE, [VI. 1.

and CE : EA :: the $\triangle$CDE : the $\triangle$ADE; [VI. 1.

$\therefore$ the $\triangle$BDE : the $\triangle$ADE :: the $\triangle$CDE : the $\triangle$ADE; [V. 11.

$\therefore$ the $\triangle$BDE = the $\triangle$CDE. [V. 9.

Also these triangles are on the same base DE and on the same side of it; therefore DE is parallel to BC. [I. 39.

Wherefore, *if a straight line, etc.* [Q.E.D.

EXERCISES.

1. The straight line joining the middle points of the sides of a triangle is parallel to the base and is one half the base.

****2.** A fixed point O is joined to all the points in a given straight line RS; prove that all the points which divide the joining lines in the same ratio lie on a straight line which is parallel to RS.

3. D is any point in the side BC of a triangle ABC; if BD, DC, BA, AC are bisected in E, F, G, H respectively, prove that EG is equal and parallel to FH.

****4.** If any two straight lines be cut by three parallel straight lines, the intercepts on the one are proportional to the corresponding intercepts on the other.

****5.** The diagonals of a trapezium cut one another proportionally, and any straight line drawn parallel to either of its parallel sides will cut the other sides in the same ratio.

6. Shew that the diagonals of a quadrilateral, two of whose sides are parallel and one of them double of the other, cut one another at a point of trisection.

7. From a point E in the common base of two triangles ACB, ADB, straight lines are drawn parallel to AC, AD, meeting BC, BD at F, G: shew that FG is parallel to CD.

8. From any point in the base of a triangle straight lines are drawn parallel to the sides: shew that the intersection of the diagonals of every parallelogram so formed lies in a certain straight line.

9. In a triangle ABC a straight line AD is drawn perpendicular to the straight line BD which bisects the angle B; shew that a straight line drawn from D parallel to BC will bisect AC.

10. ABC is a triangle; it is required to draw from a given point P, in the side AB, or AB produced, a straight line to AC, or AC produced, so that it may be bisected by BC. [Bisect PA in Q; draw QR ||[l] to AC to meet BC in R; PR produced is the required line.]

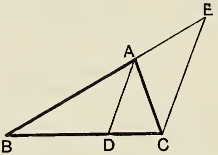
PROPOSITION 3. THEOREM.

If the vertical angle of a triangle be bisected by a straight line which also cuts the base, the segments of the base shall have the same ratio which the other sides of the triangle have to one another.

Conversely, *if the segments of the base have the same ratio which the other sides of the triangle have to one another, the straight line drawn from the vertex to the point of section shall bisect the vertical angle.*

Let ABC be a triangle, and let the angle BAC be bisected by AD, which meets the base at D:
BD *shall be to* DC *as* BA *is to* AC.

Construction. Through C draw CE parallel to DA, [I. 31.
and let BA produced meet CE at E.



Proof. (i.) Because AC meets the parallels AD, EC,
the ∠CAD = the ∠ACE, and the ∠BAD = the ∠AEC; [I. 29.
But the ∠CAD = the ∠BAD; [*Hypothesis.*
∴ the ∠ACE = the ∠AEC; [*Axiom* 1.
∴ AE = AC. [I. 6.
Also, because AD is parallel to EC, [*Construction.*
one of the sides of the △BCE;
∴ BD : DC :: BA : AE; [VI. 2.
∴ BD : DC :: BA : AC.

(ii.) *Conversely*, let BD : DC :: BA : AC, and join AD : *the angle* BAC *shall be bisected by* AD.

Let the same construction be made.

Then BD : DC :: BA : AC, [*Hypothesis.*

and BD : DC :: BA : AE, [VI. 2.

because AD is parallel to EC ; [*Construction.*

$\therefore$ BA : AC :: BA : AE ; [V. 11.

$\therefore$ AC = AE ; [V. 9.

$\therefore$ the $\angle$AEC = the $\angle$ACE. [I. 5.

But the $\angle$AEC = the exterior $\angle$BAD, [I. 29.

and the $\angle$ACE = the alternate $\angle$CAD ; [I. 29.

$\therefore$ the $\angle$BAD = the $\angle$CAD, [*Axiom* 1.

that is, the $\angle$BAC is bisected by AD.

Wherefore, *if the vertical angle, etc.* [Q.E.D.

EXERCISES

1. The side BC of a triangle ABC is bisected at D, and the angles ADB, ADC are bisected by the straight lines DE, DF, meeting AB, AC at E, F respeetively: shew that EF is parallel to BC.

2. If the bisectors of the angles A, C of a quadrilateral ABCD meet on the diagonal BD, prove that the bisectors of the angles B, D meet on the diagonal AC.

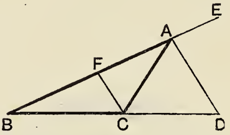
Proposition *A*. Theorem.

If the exterior angle of a triangle, made by producing one of its sides, be bisected by a straight line which also cuts the base produced, the segments between the bisector and the extremities of the base shall have the same ratio which the other sides of the triangle have to one another.

Conversely, *if the segments of the base produced have the same ratio which the other sides of the triangle have to one another, the straight line drawn from the vertex to the point of section shall bisect the exterior angle of the triangle.*

In the triangle ABC let one of its sides BA be produced to E; and let the exterior angle CAE be bisected by AD which meets the base produced at D:

BD *shall be to* DC *as* BA *is to* AC.



Construction. Through C draw CF parallel to AD, [I. 31. meeting AB at F.

Proof. (i.) Because AC meets the parallels CF, AD,

the ∠CAD = the alternate ∠ACF,

and the exterior ∠DAE = the interior opposite ∠AFC; [I. 29.

But the ∠CAD = the ∠DAE; [*Construction.*

∴ the ∠ACF = the ∠AFC, and AF = AC. [I. 6.

And because AD is parallel to FC, [*Construction.*

one of the sides of the △BCF;

∴ BD : DC :: BA :: AF; [VI. 2.

but AF = AC;

∴ BD : DC :: BA : AC.

(ii.) *Conversely*, let BD be to DC as BA to AC; and join AD:
the exterior ∠CAE *shall be bisected by* AD.

Let the same construction be made.

Then BD : DC :: BA : AC, [*Hypothesis*.
and BD : DC :: BA : AF; [VI. 2.
∴ BA : AC :: BA : AF; [V. 11.
∴ AC = AF, [V. 9.
and ∴ the ∠ACF = the ∠AFC. [I. 5.
But the ∠AFC = the exterior ∠DAE; [I. 29.
and the ∠ACF = the alternate ∠CAD; [I. 29.
∴ the ∠CAD = the ∠DAE, [*Axiom* 1.
that is, the ∠CAE is bisected by the straight line AD.

Wherefore, *if the exterior angle, etc.* [Q.E.D.

Note. Propositions 3 and *A* may be enunciated in one statement, thus: *If the interior, or exterior, vertical angle of a triangle be bisected by a straight line which meets the base, the segments of the base* [II. Def. 3] *are to one another as the sides of the triangle.*

EXERCISES.

1. In the circumference of the circle of which AB is a diameter, take any point P; and draw PC, PD on opposite sides of AP, and equally inclined to it, meeting AB at C and D: shew that AC is to BC as AD is to BD.

2. AB is a straight line and C a point in it; find a point D in AB produced such that AD is to DB as AC to CB. [Use Ex. 1.]

3. From the same point A straight lines are drawn making the angles BAC, CAD, DAE each equal to half a right angle, and they are cut by a straight line BCDE, which makes BAE an isosceles triangle: shew that BC or DE is a mean proportional between BE and CD.

[By VI. 3, BD : DC :: BA : AC;
∴ BD − DC : BD + DC :: BA − AC : BA + AC (V. 18, 17).
But BD − DC = BD + OD − OC = BO + 2 OD − OC = twice OD,
and BD + DC = BC = twice OB; ∴ etc.]

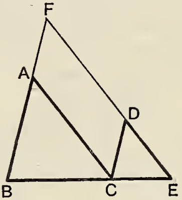
4. The angle A of a triangle ABC is bisected by AD which cuts the base at D, and O is the middle point of BC: shew that OD bears the same ratio to OB that the difference of the sides bears to their sum.

PROPOSITION 4. THEOREM.

The sides about the equal angles of triangles which are equiangular to one another are proportionals; and those which are opposite to the equal angles are homologous sides, that is, are the antecedents or the consequents of the ratios.

Let the triangle ABC be equiangular to the triangle DCE, having the angle ABC equal to the angle DCE, and the angle ACB equal to the angle DEC, and consequently the angle BAC equal to the angle CDE:
the sides about the equal angles of the triangles ABC, DCE *shall be proportionals, viz.,*

AB *to* BC *as* DC *to* CE,
BC *to* CA *as* CE *to* ED,
and CA *to* AB *as* ED *to* DC.



Proof. (1) Let the △DCE be placed so that CE may be contiguous to BC, and in the same straight line with it, and D and A on the same side of BCE. [I. 22.

Because the ∠s ABC, ACB are together < two right ∠s. [I. 32.
and that the ∠ACB = the ∠CED; [*Hypothesis.*
∴ the angles ABC, CED are together < two right angles;
∴ BA and ED, if produced, will meet. [*Axiom* 12.
Let them be produced and meet at the point F.

Then, because the ∠ABC = the ∠DCE, [*Hypothesis.*
BF is parallel to CD; [I. 28.
and because the ∠ACB = the ∠DEC, [*Hypothesis.*
AC is parallel to FE. [I. 28.
∴ FACD is a ||m; ∴ AF = CD, and AC = FD. [I. 34.

(2) Because AC is parallel to FE, a side of the △FBE,

∴ BA : AF :: BC : CE ; [VI. 2.

but AF = CD ;

∴ BA : CD :: BC : CE ; [V. 7.

and, alternately, AB : BC :: DC : CE. [V. 16.

(3) Again, because CD is parallel to BF,

∴ BC : CE :: FD : DE ; [VI. 2.

but FD = AC ;

∴ BC : CE :: AC : DE ;

and, alternately, BC : AC :: CE : DE. V. 16.

Also it has been shewn that AB : BC :: DC : CE ;

therefore, *ex æquali*, AB : AC :: DC : DE. [V. 22.

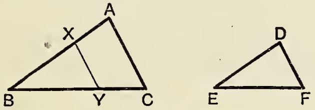
Wherefore, *the sides, etc.* [Q.E.D.

Corollary. Hence *Triangles which are equiangular are also similar.* This is not necessarily true of other rectilineal figures.

Alternative Proof of Proposition 4.

Let the side AB be greater than the corresponding side DE.

Superpose the triangle DEF upon the triangle ABC, so that the angle E coincides with the angle B, and let X, Y be the points in BA, BC with which D and F coincide.



Then ED = BX, EF = BY, and ∠BXY = ∠EDF ;

∴ ∠BXY = ∠BAC ; [*Hypothesis.*

∴ XY and AC are parallel ;

∴ AX is to XB as CY to YB ; [VI. 2.

∴ *componendo*, AB : BX :: CB : BY ; [V. 18.

∴ alternately, AB : BC :: BX : BY,

that is, :: DE : EF.

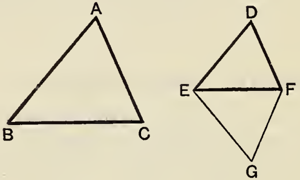
Similarly, by applying the △DEF to the △ABC so that the angle F coincides with C, we can shew that BC : CA :: EF : FD ;

and therefore, *ex æquali*, AB : AC :: DE : DF.

Proposition 5. Theorem.

If the sides of two triangles taken in order, about each of their angles, be proportionals, the triangles shall be equiangular to one another, and shall have those angles equal which are opposite to the homologous sides.

Let the triangles ABC, DEF have their sides proportional, so that AB is to BC as DE to EF; and BC to CA as EF to FD; and, consequently, *ex æquali*, BA to AC as ED to DF: *the two triangles shall be equiangular, and shall have the* ∠ABC *equal to the* ∠DEF, *the* ∠BCA *equal to the* ∠EFD, *and the* ∠BAC *equal to the* ∠EDF.



Construction. At E make the ∠FEG equal to the ∠ABC, the points D and G being on opposite sides of EF; and at F make the ∠EFG equal to the ∠BCA. [I. 23.

Proof. The remaining ∠EGF = the remaining ∠BAC;

∴ the △ABC is equiangular to the △GEF;

∴ AB : BC :: GE : EF. [VI. 4.

But AB : BC :: DE : EF; [*Hypothesis.*

∴ DE : EF :: GE : EF; [V. 11.

∴ DE = GE.

Similarly, DF = GF.

Then, in the triangles DEF, GEF,

because { DE = GE,
and EF is common,
and the base DF = the base GF ;

∴ the ∠DEF = the ∠GEF,
and the ∠DFE = the ∠GFE. [I. 8.
But the ∠GEF = the ∠ABC ; [*Construction.*
∴ the ∠ABC = the ∠DEF. [*Axiom* 1.

For the same reason, the ∠ACB = the ∠DFE,
and the ∠ at A = the ∠ at D ; [I. 32.

∴ the △ABC is equiangular to the △DEF.

Wherefore, *if the sides, etc.* [Q.E.D.

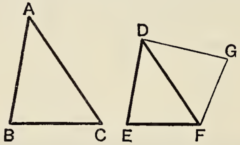
Note. Each of the two propositions VI. 4 and VI. 5 is the converse of the other. They shew that if two triangles have either of the two properties involved in the definition of similar figures they will have the other also, that is, *if two triangles are equiangular they have their sides proportional ; and if they have their sides proportional they are equiangular.* This is a special property of triangles. In other figures either of the properties may exist alone. For example, any rectangle and a square have their angles equal, but not their sides proportional ; while a square and any rhombus have their sides proportional, but not their angles equal.

[For Exercises on Propositions 4 and 5, see Pages 243 and 244.]

PROPOSITION 6. THEOREM.

If two triangles have one angle of the one equal to one angle of the other, and the sides about the equal angles proportionals, the triangles shall be equiangular to one another, and shall have those angles equal which are opposite to the homologous sides.

Let the triangles ABC, DEF have the angle BAC equal to the angle EDF, and the sides about those angles proportionals, namely, BA to AC as ED is to DF:
the △ABC *shall be equiangular to the* ∠DEF, *and shall have the* ∠ABC *equal to the* ∠DEF, *and the* ∠ACB *equal to the* ∠DFE.



Construction. At D make the ∠FDG equal to either of the angles BAC, EDF, the points G and E being on opposite sides of DF; and at F make the ∠DFG equal to the ∠ACB. [I. 23.

Proof. The remaining ∠ at G = the remaining ∠ at B·
∴ the △DGF is equiangular to the △ABC;
∴ GD : DF :: BA : AC, [VI. 4.
that is, :: ED : DF; [*Hyp.* and V. 11.
∴ ED = GD [V. 9.

Then, in the triangles EDF, GDF,

because { ED = GD,
and DF is common,
and the included ∠EDF = the included ∠GDF;

∴ the base EF = the base GF, the ∠DFG = the ∠DFE,
and the ∠ at G = the ∠ at E. [I. 4.
But the ∠DFG = the ∠ACB; [*Construction.*
∴ the ∠ACB = the ∠DFE. [*Axiom* 1.

Also the $\angle BAC =$ the $\angle EDF$; [*Hypothesis.*

$\therefore$ the remaining $\angle$ at B = the remaining $\angle$ at E;

$\therefore$ the $\triangle ABC$ is equiangular to the $\triangle DEF$.

Wherefore, *if two triangles, etc.* [Q.E.D.

EXERCISES ON PROPOSITION 4.

****1.** Every straight line drawn parallel to the base of a triangle cuts off a similar triangle.

2. If from a point A are drawn two straight lines, one to touch a circle in B, and the other to cut it in C and D, the triangles ABC, ADB are similar.

3. The side BC of a triangle ABC is produced to D so that the triangles ABD, ACD are similar. Prove that AD touches the circum-circle of the triangle ABC.

4. Two chords AB, CD of a circle are produced towards B and D to meet at E; CB is produced to meet at F a line through E parallel to AD; prove that BF, EF and CF are proportional.

5. Two circles intersect in A and the tangents at A to them meet the circles again in B and C; prove that AB is to AC in the ratio of the two radii.

6. Two chords AB, CD of a circle intersect at a point E, either within or without a circle; prove that AEC, BED are similar triangles and also AED, BEC.

7. ABC is a triangle having the angle C double of the angle B; if the bisector of C meet AB in D, prove that AC is a mean proportional between AD and AB.

[The two triangles ACD, ABC are similar.]

8. ABCD is a parallelogram; through D a straight line is drawn to cut BA and BC produced in E and F. Show that EAD and DCF are similar triangles.

9. D is a point in the base BC of a triangle ABC and EFG is a straight line parallel to BC which cuts AB, AD, AC in E, F, G respectively; prove that EF is to FG as BD to DC.

****10.** The diameters of the circles circumscribing the triangles formed by joining the vertex with any point in the base of a given triangle are proportional to the sides of that triangle.

11. ABCD is a parallelogram, and through D is drawn a straight line to meet AB in E and BC in F; prove that EA, AB, AD, and CF are proportionals.

EXERCISES ON PROPOSITIONS 5, 6.

1. AB and CD are two parallel straight lines; E is the middle point of CD; AC and BE meet at F, and AE and BD meet at G: shew that FG is parallel to AB.

2. A, B, C are three fixed points in a straight line; any straight line is drawn through C; shew that the perpendiculars on it from A and B are in a constant ratio.

3. If the perpendiculars from two fixed points on a straight line passing between them be in a given ratio, the straight line must pass through a third fixed point.

[This fixed point divides the line joining the two given points in the given ratio.]

4. Find a straight line such that the perpendiculars on it from three given points shall be in given ratios to each other.

[Use Ex. 3, and assume the construction of Prop. 10.]

5. C is the centre of a circle, and A any point within it; CA is produced through A to a point B such that the radius is a mean proportional between CA and CB: shew that if P be any point on the circumference, the angles CPA and CBP are equal.

6. In the figure of I. 43 shew that if EG and FH be produced they will meet on AC produced.

7. Two fixed circles touch at A and P and P′ are points, one on each circle, such that PAP′ is a right angle; prove that PP′ always meets the line joining the centres of the circles in a fixed point.

[Let PP′ meet the straight line joining the centres O and O′ in Q; then $\angle$QO′P′ = twice $\angle$QAP′ = twice complement of $\angle$PAO = $\angle$AOP; $\therefore$ $\triangle^s$ QO′P′, QOP are similar; $\therefore$ OQ : QO′ :: OP : O′P′, etc.]

8. ABCD is a parallelogram and DEF a straight line cutting AB in E and CB produced in F; prove that the triangle AEF is a mean proportional to the triangles AED, BEF.

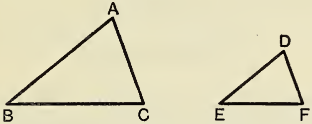
[The $\triangle^s$ AED, FEB are similar; $\therefore$ ED : EF :: AE : EB, etc.]

9. Through D, any point in the base of a triangle ABC, straight lines DE, DF are drawn parallel to the sides AB, AC, and meeting the sides at E, F: shew that the triangle AEF is a mean proportional between the triangles FBD, EDC.

PROPOSITION 7. THEOREM.

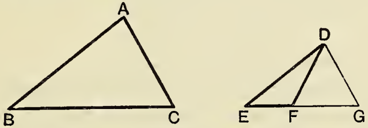
If two triangles have one angle of the one equal to one angle of the other, and the sides about another angle of each proportional, the sides opposite the equal angles being homologous, the third angles of the triangles shall be either equal or supplementary.

Let the triangles ABC, DEF have the angles ABC, DEF equal, and the sides about the angles A and D proportional, so that BA is to AC as ED to DF:
the angles ACB, DFE *shall be equal or supplementary.*



First, let the angles BAC, EDF be equal.
Then the third ∠ACB = the third ∠DFE. [I. 32.

Next, let BAC be the greater of the two angles BAC, EDF,
At the point D make the ∠EDG equal to the ∠BAC,
and produce EF to meet DG in G;

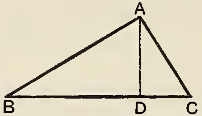


∴ the third ∠DGE of the △DGE = the third ∠ACB,
and the triangles ABC, DEG are equiangular;
∴ BA : AC :: ED : DG. [VI. 4.
But BA : AC :: ED : DF; [*Hypothesis.*
∴ DG = DF, and the ∠DGF = the ∠DFG.
But the ∠DGF = the ∠ACB, since ABC, DEG are equiangular;
and DFG, DFE are supplementary angles;
∴ ACB and DFE are supplementary angles.
Wherefore, *if two triangles, etc.* [Q.E.D.

PROPOSITION 8. THEOREM.

In a right-angled triangle, if a perpendicular be drawn from the right angle to the base, the triangles on each side of it are similar to the whole triangle, and to one another.

Let ABC be a triangle, having a right angle BAC, and let AD be drawn perpendicular to the base BC:
the triangles DBA, DAC *shall be similar to the triangle* ABC, *and to one another.*



Proof. The right ∠BAC = the right ∠BDA, [*Axiom* 11.
and the ∠ at B is common to the two triangles ABC, DBA;
∴ the remaining ∠ACB = the remaining ∠DAB.
∴ the triangles ABC, DBA are equiangular;
∴ they are similar. [VI. 4, *Corollary.*

Similarly, the △DAC is similar to the △ABC.
∴ the triangles DBA, DAC being both equiangular to the △ABC, are equiangular to each other.
∴ they are similar.

Wherefore, *in a right-angled triangle, etc.* [Q.E.D.

Corollary. From this it is clear that the perpendicular drawn from the right angle of a right-angled triangle to the base is a mean proportional between the segments of the base, and that each of the sides is a mean proportional between the base and the segment of the base adjacent to that side.

For, in the triangles DBA, DAC,
BD : DA :: DA : DC; [VI. 4.
and in the triangles ABC, DBA,
BC : BA :: BA : BD; [VI. 4.
and in the triangles ABC, DAC,
BC : CA :: CA : CD. [VI. 4.

EXERCISES.

****1.** The three sides of a right-angled triangle and the perpendicular from the right angle on the hypotenuse are proportionals

2. If one side of a right-angled triangle be double the other, prove that the perpendicular from the vertex on the hypotenuse divides it in the ratio 1 to 4.

3. ABC is a triangle, and a perpendicular is drawn from A to the opposite side, meeting it at D between B and C: shew that if AD is a mean proportional between BD and CD, the angle BAC is a right angle.

4. ABC is a triangle, and a perpendicular is drawn from A on the opposite side, meeting it at D between B and C: shew that if BA is a mean proportional between BD and BC, the angle BAC is a right angle.

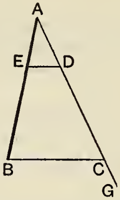
5. Two straight lines are drawn from a point A to touch a circle of which the centre is E; the points of contact are joined by a straight line which cuts EA at H; and on HA as diameter a circle is described: shew that the straight lines drawn through E to touch this circle will meet it on the circumference of the given circle.

6. ABC is a triangle, right-angled at A, and BP is drawn perpendicular to the tangent at A to the circumcircle of ABC; prove that BP, BA, and BC are proportionals.

Proposition 9. Problem.

From a given straight line to cut off any assigned sub-multiple.

Let AB be the given straight line:
it is required to cut off any assigned sub-multiple.



Construction. From the point A draw a straight line AG, making any angle with AB;
in AG take any point D, and take AC, the same multiple of AD, that AB is of the part which is to be cut off from it; join BC, and draw DE parallel to it. AE shall be the part required to be cut off.

Proof. Because ED is parallel to the side BC of the △ABC; [*Construction.*

∴ BE : EA :: CD : DA, [VI. 2.

and, by composition, BA : AE :: CA : AD; [V. 18.

∴ BA is the same multiple of AE that CA is of AD, [V. *D.*

that is, AE is the required sub-multiple.

Wherefore, *from the given straight line* AB *the assigned sub-multiple has been cut off.* [Q.E.F.

EXERCISES.

Trisect a given straight line.

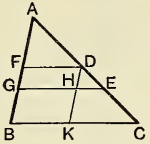
Cut off one-fifth of a given straight line.

Cut off four-sevenths of a given straight line.

Proposition 10. Problem.

To divide a given straight line similarly to a given divided straight line.

Let AB be the straight line given to be divided, and AC the given divided straight line:
it is required to divide AB *similarly to* AC.



Construction. Let AC be divided at the points D, E; and let AB, AC be placed so as to contain any angle, and join BC; through D draw DF parallel to BC, and through E draw EG parallel to BC. [I. 31.
AB shall be divided at F and G similarly to AC.

Through D draw DHK parallel to AB. [I. 31.

Proof. Each of the figures FH, HB is a parallelogram;
∴ DH = FG, and HK = GB. [I. 34.
Then, because HE is parallel to KC, [*Construction.*
∴ KH : HD :: CE : ED. [VI. 2.
But KH = BG, and HD = GF;
∴ BG : GF :: CE : ED. [V. 7.

Again, because FD is parallel to GE, [*Construction.*
∴ GF : FA :: ED : DA. [VI. 2.
And it has been shewn that BG : GF :: CE : ED.

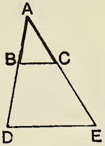
Wherefore *the given straight line* AB *is divided similarly to the given divided straight line* AC. [Q.E.F.

[For a note to this Proposition, see Page 329.]

PROPOSITION 11. PROBLEM.

To find a third proportional to two given straight lines.

Let AB, AC be the two given straight lines:
it is required to find a third proportional to AB, AC.



Construction. Let AB, AC be placed so as to contain any angle; produce AB, AC to D, E; and make BD equal to AC; [I. 3.
join BC, and through D draw DE parallel to BC. [I. 31.
CE shall be a third proportional to AB, AC.

Proof. Because BC is parallel to DE, a side of the △ADE; [*Construction.*

∴ AB : BD :: AC : CE; [VI. 2.

but BD = AC; [*Construction.*

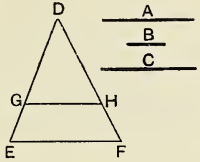
∴ AB : AC :: AC : CE.

Wherefore, *to the given straight lines* AB, AC, *a third proportional* CE *is found.* [Q.E.F.

PROPOSITION 12. PROBLEM.

To find a fourth proportional to three given straight lines.

Let A, B, C be the three given straight lines: *it is required to find a fourth proportional to* A, B, C.



Construction. Take two straight lines, DE, DF, containing any angle EDF; and in these make DG equal to A, GE equal to B, and DH equal to C; [I. 3.

join GH, and through E draw EF parallel to GH to meet DF in F. [I. 31.

HF shall be a fourth proportional to A, B, C.

Proof. Because GH is parallel to EF, a side of the △DEF, [*Construction.*

∴ DG : GE :: DH : HF. [VI. 2.

But DG = A, GE = B, and DH = C; [*Construction.*

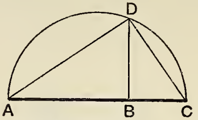
∴ A : B :: C : HF.

Wherefore, *to the three given straight lines* A, B, C, *a fourth proportional* HF *is found.* [Q.E.F.

Proposition 13. Problem.

To find a mean proportional between two given straight lines.

Let AB, BC be the two given straight lines:
it is required to find a mean proportional between them.



Construction. Place AB, BC in a straight line, and on AC describe the semicircle ADC;
from B draw BD at right angles to AC. [I. 11.
BD shall be a mean proportional between AB and BC.
Join AD, DC.

Proof. The ∠ADC, being in a semicircle, is a right ∠, [III. 31.
and DB is drawn from the right ∠ perpendicular to the base;
∴ DB is a mean proportional between AB, BC, the segments of the base. [VI. 8, *Corollary.*

Wherefore, *between the two given straight lines* AB, BC, *a mean proportional* DB *is found.* [Q.E.F.

EXERCISES.

****1.** If two circles touch each other, and also touch a given straight line, the part of the straight line between the points of contact is a mean proportional between the diameters of the circles.

[Let the circles, centres O and O′, touch one another in A and the straight line in P, P′; draw the tangent at A to meet PP′ in T. Then TP = TA = TP′. Also, OTO′ is a right ∠, since OT, O′T bisect ∠s PTA, P′TA. ∴ TA is a mean proportional between AO, AO′, etc.]

2. The bisector of the angle A of a triangle meets the base BC in D and the straight line bisecting BC at right angles in E; prove that EB is a mean proportional between EA and ED.

3. AB, AC are equal chords of a circle, and AED is any chord meeting BC in E; prove that AB is a mean proportional between AD and AE.

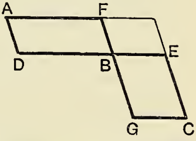
4. Two circles intersect in A and B, and the tangents at A to the two circles meet the circumferences in C and D; prove that AB is a mean proportional between DB and BC.

PROPOSITION 14. THEOREM.

Parallelograms of equal area, which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional;

Conversely, *parallelograms which have one angle of the one equal to one angle of the other, and their sides about the equal angles reciprocally proportional, are equal in area.*

Let AB, BC be parallelograms of equal area, which have the angle FBD equal to the angle EBG:



the sides of the parallelograms about the equal angles shall be reciprocally proportional, that is, DB *shall be to* BE *as* GB *to* BF.

Construction. Let the parallelograms be placed, so that the sides DB, BE may be in the same straight line;

∴ also FB, BG are in one straight line. [I. 14.

Complete the parallelogram FE.

Proof. Because the ||ms AB and BC are equal, [*Hypoth.*

and that FE is another parallelogram;

∴ AB : FE :: BC : FE.

But AB : FE :: the base DB : the base BE, [VI. 1.

and BC : FE :: the base GB : the base BF; [VI. 1.

∴ DB : BE :: GB : BF. [V. 11.

Conversely, let the angle FBD = the angle EBG, and let DB be to BE as GB to BF:

the parallelograms AB *and* BC *shall be equal in area.*

For, the same construction being made,

because DB : BE :: GB : BF, [*Hypothesis.*

and that DB : BE :: the ||m AB : the ||m FE, [VI. 1.

and that GB : BF :: the ||m BC : the ||m FE; [VI. 1.

∴ AB : FE :: BC : FE; [V. 11.

∴ the parallelogram AB = the parallelogram BC.

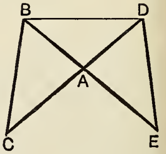
Wherefore, *equal parallelograms, etc.* [Q.E.D.

PROPOSITION 15. THEOREM.

Triangles of equal area, which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional.

Conversely, *triangles which have one angle of the one equal to one angle of the other, and their sides about the equal angles reciprocally proportional, are equal in area.*

Let ABC, ADE be triangles of equal area which have the angle BAC equal to the angle DAE:
the sides of the triangles about the equal angles shall be reciprocally proportional, that is, CA *shall be to* AD *as* EA *to* AB.



Construction. Let the triangles be placed so that the sides CA, AD may be in the same straight line;
therefore also EA, AB are in one straight line; [I. 14.

join BD.

Proof. Because the △ABC = the △ADE, [*Hypothesis.*
and that ABD is another △;
∴ △ABC : △ABD :: △ADE : △ABD. [V. 7.
But △ABC : △ABD :: base CA : base AD, [VI. 1.
and △ADE : △ADB :: base EA : base AB; [VI. 1.
∴ CA : AD :: EA : AB. [V. 11.

Conversely, let the ∠BAC = the ∠DAE,
and let CA be to AD as EA to AB:
the triangles ABC, ADE *shall be equal in area.*

For, the same construction being made,
because CA : AD :: EA : AB, [*Hypothesis.*
and that CA : AD :: △ABC : △ABD, [VI. 1.
and that EA : AB :: △ADE : △ABD; [VI. 1.
∴ △ABC : △ABD :: △ADE : △ABD; [V. 11.
∴ △ABC = △ADE. [V. 9.

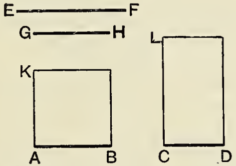
Wherefore, *equal triangles, etc.* [Q.E.D.

PROPOSITION 16. THEOREM.

If four straight lines be proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means.

Conversely, *if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four straight lines are proportionals.*

Let the four straight lines AB, CD, EF, GH be proportionals, namely, AB to CD as EF to GH:
the rectangle contained by AB *and* GH *shall be equal to the rectangle contained by* CD *and* EF.



Construction. From the points A, C draw AK, CL at right angles to AB, CD; [I. 11.
make AK equal to GH and CL to EF, [I. 3.
and complete the parallelograms BK, DL. [I. 31.

Proof. Because AB : CD :: EF : GH, [*Hypothesis.*
and that EF = CL, and GH = AK; [*Construction.*
∴ AB : CD :: CL : AK,
that is, the sides of the parallelograms BK, DL about the equal angles are reciprocally proportional;

∴ BK = DL. [VI. 14.

But BK = the rect. AB, GH, since AK = GH,
and DL = the rect. CD, EF, since CL = EF;
∴ the rect. AB, GH = the rect. CD, EF.

Conversely, let the rectangle AB, GH = the rect. CD, EF: *then shall* AB *be to* CD *as* EF *to* GH.

For, the same construction being made,
because the rect. AB, GH = the rect. CD, EF, [*Hypothesis.*

and that BK = the rect. AB, GH, since AK = GH, [*Const.*
and that DL = the rect. CD, EF, since CL = EF; [*Const.*

∴ BK = DL. [*Axiom* 1.

And these parallelograms are equiangular;
∴ the sides about their equal angles are reciprocally proportional, that is, AB : CD :: CL : AK. [VI. 14.

But CL = EF, and AK = GH;

∴ AB : CD :: EF : GH.

Wherefore, *if four straight lines, etc.* [Q.E.D.

EXERCISES.

1. Shew that the diagonals of any quadrilateral figure inscribed in a circle divide the quadrilateral into four triangles which are similar two and two; and deduce the theorem of III. 35.

2. The exterior angle A of a triangle ABC is bisected by a straight line which meets the base in D and the circumcircle of the triangle in E; prove that the rectangle BA, AC = the rectangle EA, AD.

3. ABC is a triangle. In AB, AC are taken points D, E so that AB is to AC as AE to AD. Prove that the diameter of the circumscribing circle of the triangle ABC which passes through A is perpendicular to DE.

[AE . AC = AB . AD. ∴ BDEC is a cyclic quad[l].

∴ ∠AED = ∠ABC = rt. ∠ − ∠EAO,

where O is the circumcentre of △ABC.]

4. A straight line meets two intersecting circles in P, Q, R, S and their common chord in O; prove that OP, OQ, OR, OS, taken in a suitable order, are proportionals.

5. If two lines, AD and AE, be drawn through the vertex equally inclined to the bisector of the vertical angle of a triangle ABC to meet the base and the circumcircle of the triangle in D and E, the rectangle contained by them is equal to the rectangle contained by the sides.

[The △s ABD, AEC are similar.]

6. I is the incentre of the triangle ABC and I_1 the centre of the escribed circle opposite the angle A; prove that the rectangles AI, AI_1 and AB, AC are equal.

[ACI_1, AIB are similar △s.]

****7.** ABCD is a straight line; find a point P in A such that PA, PB, PC, PD are proportionals.

[On AC, BD describe semi-circles; they meet in a point such that the foot of the perpendicular from it upon AD is the required point P.]

Proposition 17. Theorem.

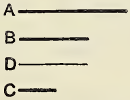
If three straight lines be proportionals, the rectangle contained by the extremes is equal to the square on the mean.

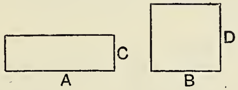
Conversely, *if the rectangle contained by the extremes be equal to the square on the mean, the three straight lines are proportionals.*

Let the three straight lines A, B, C be proportionals, namely, let A be to B as B is to C:

the rectangle contained by A *and* C *shall be equal to the square on* B.

Take D equal to B.





Proof. Then, because A : B :: B : C, [*Hypothesis.*

and that B = D;

∴ A : B :: D : C; [V. 7.

∴ the rect. A, C = the rect. B, D. [VI. 16.

But the rect. B, D = the square on B, since D = B; [*Const.*

∴ the rect. A, C = the square on B.

Conversely, let the rect. A, C = the square on B:

A *shall be to* B *as* B *to* C.

For, the same construction being made,

because the rect. A, C = the square on B, [*Hypothesis.*

and that the square on B = the rect. B, D, since D = B; [*Const.*

∴ the rect. A, C = the rect. B, D;

∴ A : B :: D : C; [VI. 16.

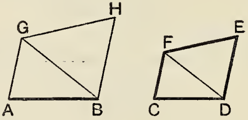
that is, A : B :: B : C.

Wherefore, *if three straight lines, etc.* [Q.E.D.

Proposition 18. Problem.

On a given straight line to describe a rectilineal figure similar and similarly situated to a given rectilineal figure.

Let AB be the given straight line, and CDEF the given rectilineal figure of four sides:
it is required to describe on AB *a rectilineal figure similar and similarly situated to* CDEF.



Construction. Join DF; at the point A make the ∠BAG equal to the ∠DCF; and at the point B make the ∠ABG equal to the ∠CDF; [I. 23.
∴ the remaining ∠AGB = the remaining ∠CFD,
and the △AGB is equiangular to the △CFD.
Again, at B make the ∠GBH equal to the ∠FDE,
and at G make the ∠BGH equal to the ∠DFE; [I. 23.
∴ the remaining ∠BHG = the remaining ∠DEF,
and the △BHG is equiangular to the △DEF.
Then ABHG shall be the figure required.

Proof. Because the ∠AGB = the ∠CFD,
and the ∠BGH = the ∠DFE, [*Construction.*
∴ the whole ∠AGH = the whole ∠CFE.
For the same reason the ∠ABH = the ∠CDE.
Also the ∠BAG = the ∠DCF, and the ∠BHG = the ∠DEF.
∴ the figure ABHG is equiangular to the figure CDEF.

Also these figures have their sides about the equal angles proportionals.
For, because the △BAG is equiangular to the △DCF,
∴ BA : AG :: DC : CF. [VI. 4.

Also, for the same reason, AG : GB :: CF : FD, and
BG : GH :: DF : FE;
∴ *ex æquali*, AG : GH :: CF : FE. [V. 22.
Similarly, AB : BH :: CD : DE,
and GH : HB :: FE : ED. [VI. 4.
∴ ABHG and CDEF are equiangular, and have their sides about the equal angles proportionals;
∴ they are similar. [VI. *Definition* 1.

The construction and proof would be similar to the above whatever be the number of sides that the given figure has.

EXERCISES ON PROPOSITION 17.

****1.** If the tangents to a circle at P and Q meet in T, and if C be the centre and CT meet PQ in N and the circle in A, prove that the rect. CN, CT = the square on CA.

****2.** If there be drawn two parallel tangents to a circle, and any variable tangent touch the circle at B and meet the parallel tangents in A and C, the rectangle AB, BC is equal to the square on the radius of the circle.

3. If AD be the tangent at a point D of a circle, and DE be the diameter through D, and AE be joined to cut the circle in F, prove that the rectangle AE, EF is the same for all positions of A.

4. The straight line bisecting at right angles AC, one of the equal sides of an isosceles triangle, meets the base BC produced in D. Prove that AC is a mean proportional between CB and CD.

****5.** The rectangle contained by the perpendiculars from any point P of a circle on any two tangents is equal to the square of the perpendicular from the same point upon their chord of contact.

[Let B, C be the points of contact; PK, PL, PM the $\perp^{rs}$ from P upon the tangents at B, C, and on BC. Then PKBM and PLCM are cyclic quadrilaterals;

$$\therefore \angle KMP = \angle KBP = \angle PCM = \angle PLM,$$
$$\text{and } \angle PKM = \angle PBM = \angle PCL = \angle PML;$$

∴ the $\triangle^s$ PKM, PLM are similar; ∴ etc.]

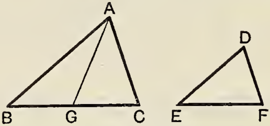
****6.** ABC is a straight line; find a point P in it such that PB may be a mean proportional between PA and PC.

[On AC describe a semi-circle; at B draw a str. line BQ such that ABQ = half a right ∠; let BQ meet the circle in Q; draw QP perpendicular to AB, etc.]

Proposition 19. Theorem.

Similar triangles are to one another in the duplicate ratio of their homologous sides.

Let ABC and DEF be similar triangles, having the angle B equal to the angle E, and let AB be to BC as DE is to EF, so that the side BC is homologous to the side EF:
the △ABC *shall be to the* △DEF *in the duplicate ratio of* BC *to* EF



Construction. Take BG a third proportional to BC and EF, so that BC may be to EF as EF to BG, [VI. 11.
and join AG.

Proof. Because AB : BC :: DE : EF; [*Hypothesis.*
∴ alternately, AB : DE :: BC : EF; [V. 16.
but BC : EF :: EF : BG; [*Construction.*
∴ AB : DE :: EF : BG, [V. 11.
that is, the sides of the triangles ABG and DEF, about their equal angles, are reciprocally proportional;
∴ the △ABG is equal in area to the △DEF. [VI. 15.

Also, because BC : EF :: EF : BG;
∴ BC is to BG in the duplicate ratio of BC to EF. [V. *Def.* 10.
But the △ABC : the △ABG :: BC : BG; [VI. 1.
∴ the △ABC is to the △ABG in the duplicate ratio of BC to EF.
But the △ABG was shewn equal to the △DEF;
∴ the △ABC is to the △DEF in the duplicate ratio of BC to EF.

Wherefore, *similar triangles, etc.* [Q.E.D.

Corollary. From this it is manifest, that if three straight lines be proportionals, as the first is to the third, so is any △ described on the first to a similar and similarly described △ on the second.

EXERCISES.

[*In the following examples the student may assume the result of Cor.* 3 *to Prop.* 20.]

1. Make a triangle four times as large as a given triangle and similar to it.

[If ABC be the given △, produce AB, AC to P, Q so that AP = twice AB and AQ = twice AC. Then APQ is the required △ by VI. 19 and 20, Cor. 2.]

2. Bisect the area of a triangle by a straight line drawn parallel to one of the sides.

[Let the required line meet the sides AB, AC in P, Q; then by VI. 19 and 20, Cor. 2, $AP^2 : AB^2 :: 1 : 2$. $\therefore AB^2 = 2AP^2$. To construct AP, on AB as diameter, describe a semicircle, and let R be its middle point; then $AB^2 = AR^2 + RB^2 = 2AR^2$. $\therefore AP = AR$. Hence the construction.]

3. Construct a triangle twice as large as a given triangle and similar to it. [Here $AP^2 = 2AB^2$. Use I. 47.]

4. Construct a triangle five times as large as a given triangle and similar to it. [Here $AP^2 = 5 . AB^2$.]

5. ABC is a triangle and BE, CF are perpendiculars upon the sides AC, AB. Prove that the triangle ABE is to the triangle ACF as the square on AB to the square on AC.

6. AB and CD are two chords of a circle intersecting in an external point E; prove that the triangle EAC is to the triangle EBD as AC^2 to BD^2.

7. O is a point without a circle, centre C, OT is a tangent and OPCQ a straight line cutting the circle; PN is another tangent cutting OT in N; shew that the triangle OPN, OTC are to one another in the ratio OP to OQ.

8. ABC is a triangle, and AD, BE, CF the perpendiculars on the opposite sides. Prove that the areas of the △s DBF, ABC are to one another as the squares on BD, AB.

9. Prove that the area of the regular hexagon inscribed in a circle is three-quarters of the regular hexagon described about the same circle.

10. Prove that similar triangles are to one another in the duplicate ratio of corresponding medians, and also in the duplicate ratio of the radii of their circumcircles.

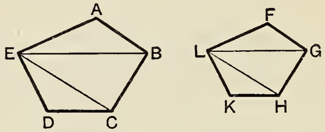
PROPOSITION 20. THEOREM.

Similar polygons may be divided into the same number of similar triangles, having the same ratio to one another that the polygons have; and the polygons are to one another in the duplicate ratio of their homologous sides.

Let ABCDE, FGHKL be similar polygons, and let AB be the side homologous to the side FG :

(1) *the polygons* ABCDE, FGHKL *may be divided into the same number of similar triangles, of which each shall have to each the same ratio which the polygons have; and*

(2) *the polygon* ABCDE *shall be to the polygon* FGHKL *in the duplicate ratio of* AB *to* FG.



Construction. Join BE, EC, GL, LH.

Proof. (1) Because the polygon ABCDE is similar to the polygon FGHKL, [*Hypothesis.*

the ∠BAE = the ∠GFL, and BA : AE :: GF : FL; [VI. *Definition* 2.

∴ the △ABE is equiangular to the △FGL, [VI. 6.

and therefore these triangles are similar; [VI. 4.

∴ the ∠ABE = the ∠FGL.

But, because the polygons are similar, [*Hypothesis.*

∴ the whole ∠ABC = the whole ∠FGH; [VI. *Definition* 2.

∴ the remaining ∠EBC = the remaining ∠LGH. [*Axiom* 3.

Also, because the triangles ABE and FGL are similar,

∴ EB : BA :: LG : GF;

and also, because the polygons are similar, [*Hypothesis.*

∴ AB : BC :: FG : GH; [VI. *Definition* 2.

∴ *ex æquali*, EB : BC :: LG : GH, [V. 22.

that is, the sides about the equal angles EBC and LGH are proportionals;

∴ the △EBC is equiangular to the △LGH; [VI. 6.

∴ these triangles are similar. [VI. 4.

For the same reason the △ECD is similar to the △LHK. Therefore the similar polygons ABCDE, FGHKL may be divided into the same number of similar triangles.

(2) Again, because the △ABE is similar to the △FGL,

∴ ABE is to FGL in the duplicate ratio of EB to LG. [VI. 19.

For the same reason the △EBC is to the △LGH in the duplicate ratio of EB to LG;

∴ △ABE : △FGL :: △EBC : △LGH. [V. 11.

Again, because the △EBC is similar to the △LGH,

∴ EBC is to LGH in the duplicate ratio of EC to LH. [VI. 19.

For the same reason the △ECD is to the △LHK in the duplicate ratio of EC to LH;

∴ △EBC : △LGH :: △ECD : △LHK. [V. 11.

But it has been shewn that the △EBC : the △LGH

:: the △ABE : the △FGL;

∴ △ABE : △FGL :: △EBC : △LGH,

and :: △ECD : △LHK; [V. 11.

and therefore as one of the antecedents is to its consequent, so are all the antecedents to all the consequents; [V. 12.

that is, as △ABE : △FGL,

:: the polygon ABCDE : the polygon FGHKL.

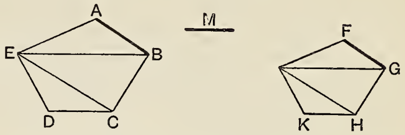
But the △ABE is to the △FGL in the duplicate ratio of AB to FG; [VI. 19.

∴ the polygon ABCDE is to the polygon FGHKL in the duplicate ratio of AB to FG.

Wherefore, *similar polygons, etc.* [Q.E.D.

Corollary 1. In like manner it may be shewn that similar four-sided figures, or figures of any number of sides, are to one another in the duplicate ratio of their homologous sides; and it has already been shewn for triangles; therefore, universally, *Similar rectilinear figures are to one another in the duplicate ratio of their homologous sides.*

Corollary 2. If to AB and FG, two of the homologous sides, a third proportional M be taken, [VI. 11.



then AB has to M the duplicate ratio of AB to FG. [V. *Def.* 10.
But any rectilineal figure described on AB is to the similar and similarly described rectilineal figure on FG in the duplicate ratio of AB to FG. [*Corollary* 1.
Therefore as AB : M :: the figure on AB : the figure on FG; [V. 11.
and this was shewn before for triangles. [VI. 19, *Corollary.*
Wherefore, universally, *If three straight lines be proportionals, as the first is to the third, so is any rectilineal figure described on the first to a similar and similarly described rectilineal figure on the second.*

Corollary 3. If these similarly described figures be squares, it follows that the first : the third :: the square on the first : the square on the second.
But the first is to the third in the duplicate ratio of the first to the second. [V. *Definition* 10.
Hence, *The duplicate ratio of two straight lines is equal to the ratio of the squares on the two straight lines.*

Propositions 19, 20 may therefore be enunciated thus:

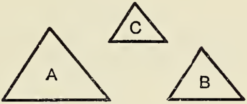
Similar triangles, and polygons, are to one another as the squares on their homologous sides.

PROPOSITION 21. THEOREM.

Rectilineal figures which are similar to the same rectilineal figure are also similar to each other.

Let each of the rectilineal figures A and B be similar to the rectilineal figure C:

the figure A *shall be similar to the figure* B.



Proof. Because A is similar to C, [*Hypothesis.*

A is equiangular to C, and they have their sides about the equal angles proportionals. [VI. *Definition* 2.

Again, because B is similar to C, [*Hypothesis.*

B is equiangular to C, and they have their sides about the equal angles proportionals; [VI. *Definition* 2.

∴ the figures A and B are each of them equiangular to C, and have the sides about the equal angles of each of them and of C proportionals;

∴ A is equiangular to B, [*Axiom* 1.

and they have their sides about the equal angles proportionals. [V. 11.

∴ the figure A is similar to the figure B. [VI. *Definition* 2.

Wherefore, *rectilineal figures, etc.* [Q.E.D.

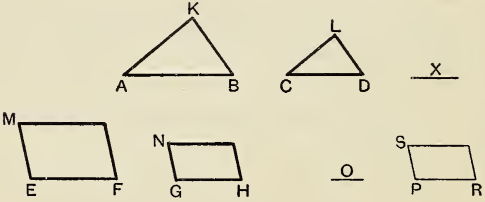
PROPOSITION 22. THEOREM.

If four straight lines be proportionals, and a pair of similar rectilineal figures be similarly described on the first and second, and also a pair on the third and fourth, these figures shall be proportionals.

Conversely, *if a rectilineal figure on the first of four straight lines be to a similar and similarly described figure on the second as a rectilineal figure on the third is to a similar and similarly described figure on the fourth, these four straight lines shall be proportionals.*

Let the four straight lines AB, CD, EF, GH be proportionals, namely, AB to CD as EF is to GH; and on AB, CD let the similar rectilineal figures KAB, LCD be similarly described; and on EF, GH let the similar rectilineal figures MF, NH be similarly described:

the figure KAB *shall be to the figure* LCD *as the figure* MF *is to the figure* NH.



Construction. To AB and CD take a third proportional X, and to EF and GH take a third proportional O.

Proof. Because AB : CD :: EF : GH, [*Hypothesis.*
and AB : CD :: CD : X, [*Construction.*
and EF : GH :: GH : O; [*Construction.*
∴ CD : X :: GH : O. [V. 11.

Also AB : CD :: EF : GH;

∴ *ex æquali*, AB : X :: EF : O. [V. 22.

But AB : X :: KAB : LCD;

and EF : O :: MF : NH; [VI. 20, *Cor.* 2.

∴ KAB : LCD :: MF : NH. [V. 11.

Conversely, let KAB be to the similar figure LCD as MF to the similar figure NH:

AB *shall be to* CD *as* EF *to* GH.

Construction. Make EF to PR as AB to CD; [VI. 12. and on PR describe SR, similar and similarly situated to either of the figures MF, NH. [VI. 18.

Proof. Because AB is to CD as EF to PR, and that on AB, CD are described the similar figures KAB, LCD, and on EF, PR the similar figures MF, SR;

therefore, by the former part of this proposition,

KAB : LCD :: MF : SR.

But, by hypothesis, KAB : LCD :: MF : NH;

∴ MF : SR :: MF : NH; [V. 11.

∴ SR = NH.

But SR and NH are similar and similarly situated; [*Constr.*

∴ PR = GH.

And because AB : CD :: EF : PR, and that PR = GH;

∴ AB : CD :: EF : GH.

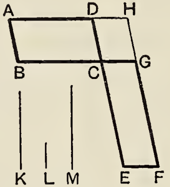
Wherefore, *if four straight lines, etc.* [Q.E.D.

[A note on this proposition will be found on Page 330.]

PROPOSITION 23. THEOREM.

Parallelograms which are equiangular to one another have to one another the ratio which is compounded of the ratios of their sides.

Let the parallelogram AC be equiangular to the parallelogram CF, having the ∠BCD equal to the ∠ECG:
AC *shall have to* CF *the ratio which is compounded of the ratios of their sides.*



Construction. Let BC and CG be placed in a straight line;
∴ DC and CE are also in a straight line; [I. 14.
complete the parallelogram DG;
take any straight line K, and make K to L as BC to CG, and L to M as DC to CE. [VI. 12.

Proof. The ratio of K to M is that which is compounded of the ratios of K to L and of L to M; [V. *Def.* 12.
that is, of BC to CG and DC to CE.

Now, the ||m AC : the ||m CH :: BC : CG, [VI. 1.
that is, :: K : L. [*Constr.*
Again, the ||m CH : the ||m CF :: DC : CE, [VI. 1.
that is, :: L : M; [*Constr.*
∴ *ex æquali*, the ||m AC : the ||m CF :: K : M. [V. 22.
But K has to M the ratio compounded of the ratios of the sides;
∴ also the parallelogram AC has to the parallelogram CF the ratio compounded of the ratios of the sides.

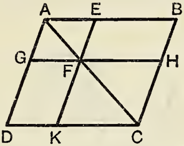
Wherefore, *parallelograms, etc.* [Q.E.D.

[An important note to this proposition will be found on Page 331.]

Proposition 24. Theorem.

Parallelograms about the diameter of any parallelogram are similar to the whole parallelogram, and to one another.

Let ABCD be a parallelogram, of which AC is a diameter; and let EG and HK be parallelograms about the diameter: EG *and* HK *shall be similar both to the whole parallelogram and to one another.*



Proof. Because DC and GF are parallel,

the ∠ADC = the ∠AGF. [I. 29.

Because BC and EF are parallel,

∴ the ∠ABC = the ∠AEF.

Also each of the angles BCD and EFG is equal to the opposite angle BAD, [I. 34.

and therefore they are equal;

∴ the parallelograms ABCD and AEFG are equiangular.

Again, because the ∠ABC = the ∠AEF, and the ∠BAC is common to the two triangles BAC and EAF;

∴ these triangles are equiangular;

∴ AB : BC :: AE : EF. [VI. 4.

Also the opposite sides of parallelograms are equal; [I. 34.

∴ AB : AD :: AE : AG,

and DC : CB :: GF : FE,

and CD : DA :: FG : GA. [V. 7.

∴ the sides of ABCD and AEFG about their equal angles are proportional, and they are therefore similar. [VI. *Def.* 1.

So also the parallelograms ABCD and FHCK are similar;

∴ each of the parallelograms EG and HK is similar to BD;

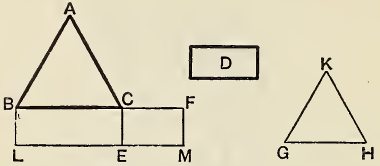
EG is similar to HK. [VI. 21.

Wherefore, *parallelograms, etc.* [Q.E.D.

PROPOSITION 25. PROBLEM.

To describe a rectilineal figure which shall be similar to one given rectilineal figure and equal to another given rectilineal figure.

Let ABC and D be the two given rectilineal figures: *it is required to describe a rectilineal figure similar to* ABC *and equal to* D.



Construction. On BC describe the parallelogram BE equal to the figure ABC.
On CE describe the parallelogram CM equal to D, and having the ∠FCE equal to the ∠CBL; [I. 45, *Corollary.*
therefore BC and CF will be in one straight line, and LE and EM will be in one straight line.
Between BC and CF find a mean proportional GH, [VI. 13.
and on GH describe the figure KGH, similar and similarly situated to the figure ABC. [VI. 18.
KGH shall be the figure required.

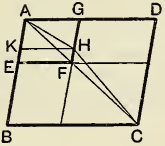
Proof. Because BC : GH :: GH : CF, [*Construction.*
∴ BC : CF :: ABC : KGH. [VI. 20, *Corollary* 2.
But BC : CF :: the ∥m BE : the ∥m CM; [VI. 1.
∴ ABC : KGH :: BE : CM. [V. 11.
Also, the figure ABC = the ∥m BE; [*Construction.*
∴ the figure KGH = the ∥m CM, [V. 14.
that is, = D, [*Construction.*
and it is similar to the figure ABC. [*Construction.*

Wherefore *the rectilineal figure* KGH *has been described similar to the figure* ABC, *and equal to* D. [Q.E.F.

Proposition 26. Theorem.

If two similar parallelograms have a common angle, and be similarly situated, they are about the same diagonal.

Let the parallelograms ABCD, AEFG be similar and similarly situated, and having the common angle BAD :
ABCD *and* AEFG *shall be about the same diagonal.*



Construction. For, if not, let, if possible, the parallelogram BD have its diagonal AHC in a different straight line from AF, the diagonal of the parallelogram EG;
let GF meet AHC at H, and draw HK parallel to AD. [I. 31.

Proof. The parallelograms ABCD and AKHG are about the same diagonal, and are therefore similar; [VI. 24.

∴ DA : AB :: GA : AK.

But because ABCD and AEFG are similar, [*Hypothesis.*

∴ DA : AB :: GA : AE; [VI. *Definition* 2.

∴ GA : AK :: GA : AE; [V. 11.

∴ AK = AE,

the less to the greater, which is impossible;

∴ ABCD and AEFG must have their diagonals in the same straight line, that is, they are about the same diagonal.

Wherefore, *if two similar parallelograms, etc.* [Q.E.D.

Note. Prop. 26 is the converse of Prop. 24.

EXERCISES.

1. In the figure of VI. 24 shew that EG and KH are parallel.

2. Prove also that the parallelograms EG, BF, FD, and HK are proportionals.

3. Construct a square equal to a given equilateral triangle.

4. Construct an equilateral triangle equal to a given square.

5. Construct an equilateral triangle equal to a given hexagon.

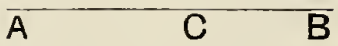
6. Through a given point draw a chord in a given circle so that it shall be divided at the point in a given ratio.

[The rect. contained by the segments is known; for it is equal to the rect. contained by the segments of any chord through the given point. Thus, the ratio of the segments being given, we have to construct a rectangle of given area, whose sides are in a given ratio, so that the required rectangle is to be similar to a given one.]

Proposition 30. Problem.

To cut a given straight line in extreme and mean ratio.

Let AB be the given straight line:
it is required to cut it in extreme and mean ratio.



Construction. Divide AB at the point C, so that the rectangle AB, BC may be equal to the square on AC. [II. 11.

Proof. Because the rectangle AB, BC

= the square on AC, [*Construction.*

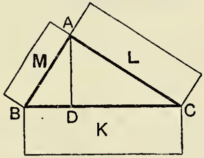
∴ AB : AC :: AC : CB. [VI. 17.

Wherefore AB *is cut in extreme and mean ratio at the point* C.
[Q.E.F. VI. *Definition* 4.

Proposition 31. Theorem.

In any right-angled triangle, any rectilineal figure described on the side subtending the right angle is equal to the similar and similarly described figures on the sides containing the right angle.

Let ABC be a triangle, right-angled at A:
the rectilineal figure described on BC *shall be equal to the similar and similarly described figures on* BA *and* CA.



Construction. Draw the perpendicular AD. [I. 12.
Let K, L, M be the similar and similarly described figures on BC, CA, and AB.

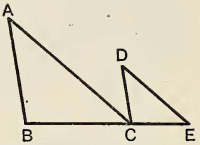
Proof. Because AD is drawn from the right angle A perpendicular to BC,
the △CBA is similar to the △ABD; [VI. 8.
∴ BC : BA :: BA : BD; [VI. *Def.* 2.
∴ BC : BD :: K : M; [VI. 20, *Cor.* 2.
and inversely, BD : BC :: M : K. [V. *B.*
Similarly, CD : BC :: L : K;
∴ as BD and CD together are to BC so are L and M together to K. [V. 24.
But BD and CD together = BC;
∴ the figure K = the figures L and M. [V. *A.*
Wherefore, *in any right-angled triangle, etc.* [Q.E.D.

Note. Of this general proposition I. 47 is a particular case.

PROPOSITION 32. THEOREM.

If two triangles, which have two sides of one proportional to two sides of the other, be joined at one angle so as to have their homologous sides parallel, the remaining sides shall be in a straight line.

Let ABC and DCE be two triangles which have BA to AC as CD is to DE, and let AB be parallel to DC and AC parallel to DE:
BC *and* CE *shall be in one straight line.*



Proof. Because AC meets the parallels AB, DC, [*Hyp.*
the alternate angles BAC, ACD are equal; [I. 29.
Similarly, the angles ACD, CDE are equal;
∴ the ∠BAC = the ∠CDE. [*Axiom* 1.
Again, because the ∠ at A = the ∠ at D,
and that BA : AC :: CD : DE, [*Hypothesis.*
∴ the △ABC is equiangular to the △DCE; [VI. 6.
∴ the ∠ABC = ∠DCE.
Also the ∠BAC = ∠ACD; [*Shewn.*
∴ the whole ∠ACE = the two angles ABC and BAC. [*Ax.* 2.
Add the ∠ACB to each of these equals;
∴ ∠ˢACE, ACB together = ∠ˢABC, BAC, ACB,
that is, = two right angles; [I. 32.
∴ BC and CE are in one straight line. [I. 14.
Wherefore, *if two triangles, etc.* [Q.E.D.

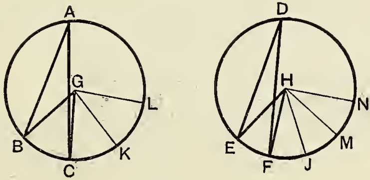
[For a note on this proposition, see Page 331.]

PROPOSITION 33. THEOREM.

In equal circles, angles, whether at the centres or at the circumferences, have the same ratio which the arcs on which they stand have to one another; so also have the sectors.

Let ABC and DEF be equal circles, and let BGC and EHF be angles at their centres, and BAC and EDF angles at their circumferences: then shall

(i.) *the* ∠BGC *be to the* ∠EHF *as the arc* BC *to the arc* EF;
(ii.) *the* ∠BAC *be to the* ∠EDF *as the arc* BC *to the arc* EF;
and (iii.) *the sector* BGC *be to the sector* EHF *as the arc* BC *to the arc* EF.

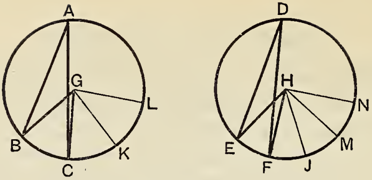


Construction. Take any number of arcs CK, KL, each equal to BC, and also any number of arcs FJ, JM, MN each equal to EF; and join GK, GL, HJ, HM, HN.

Proof. (i.) Because the arcs BC, CK, KL are all equal, the angles BGC, CGK, KGL are also all equal; [III. 27.
∴ the arc BL is the same multiple of the arc BC that the ∠BGL is of the ∠BGC.

Similarly, the arc EN is the same multiple of the arc EF that the ∠EHN is of the ∠EHF.
And if the arc BL = the arc EN, the ∠BGL = the ∠EHN; [III. 27.
and if the arc BL be greater than the arc EN, the ∠BGL is greater then the ∠EHN; and if less, then less.

Therefore, since there are four magnitudes, the two arcs BC, EF, and the two angles BGC, EHF;

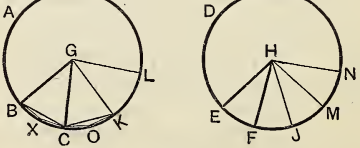


and of the first and third have been taken any equimultiples whatever, namely, the arc BL and the ∠BGL;
and of the second and fourth have been taken any equimultiples whatever, namely, the arc EN and the ∠EHN;

and since the ∠BGL is >, =, or < the ∠EHN, according as the arc BL is >, =, or < the are EN;
∴ arc BC : arc EF :: ∠BGC : ∠EHF. [V. *Definition* 5.

(ii.) But ∠BGC : ∠EHF :: ∠BAC : ∠EDF, [V. 15.
for each is double of each; [III. 20.
∴ the arc BC : the arc EF :: the ∠BGC : the ∠EHF,
and :: the ∠BAC : the ∠EDF.

(iii.) Join BC, CK, and in the arcs BC, CK take any points X, O, and join BX, XC, CO, OK.



Then, in the triangles BGC, CGK,

because { BG = GC,
GC = CK,
and ∠BGC = ∠CGK;

∴ the base BC = the base CK, and △BGC = △CGK. [I. 4.

Also because the arc BC = the arc CK, [*Construction.*
∴ the remaining arc BAC = the remaining arc CAK;
∴ the ∠BXC = the ∠COK; [III. 27.
∴ the segments BXC, COK are similar, [III. *Definition* 11.
and they are on equal chords BC, CK;
∴ the segment BXC = the segment COK. [III. 24.

Also the △BGC was shewn to be equal to the △CGK;
∴ the whole sector BGC = the whole sector CGK. [*Axiom* 2.
Similarly, the sector KGL = each of the sectors BGC, CGK.
In the same manner the sectors EHF, FHJ, JHM, MHN may be shewn to be equal,
∴ whatever multiple the arc BL is of the arc BC, the same multiple is the sector BGL of the sector BGC;
and, similarly, whatever multiple the arc EN is of the arc EF, the same multiple is the sector EHN of the sector EHF.
Also if arc BL = the arc EN, the sector BGL = the sector EHN; and if the arc BL be greater than the arc EN, the sector BGL is greater than the sector EHN; and if less, then less.

Therefore, since there are four magnitudes, the two arcs BC, EF, and the two sectors BGC, EHF;
and that of the first and third have been taken any equimultiples whatever, namely, the arc BL and the sector BGL;
and of the second and fourth have been taken any equimultiples whatever, namely, the arc EN and the sector EHN;
and since the sector BGL is >, =, or < the sector EHN, according as the arc BL is >, =, or < the arc EN;
∴ the arc BC : the arc EF
:: the sector BGC : sector EHF. [V. *Definition* 5.

Wherefore, *if equal circles, etc.* [Q.E.D.

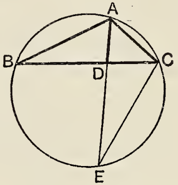
Note. In VI. 33 Euclid implicitly gives up the restriction, which he seems to have adopted hitherto, that no angle is to be considered greater than two right angles. For in the demonstration the angle BGL may be any multiple whatever of the angle BGC, and so may be greater than any number of right angles.

PROPOSITION *B*. THEOREM.

If the vertical angle of a triangle be bisected by a straight line which also cuts the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base, together with the square on the straight line which bisects the angle.

Let ABC be a triangle, and let the angle BAC be bisected by AD;

the rectangle BA, AC *shall be equal to the rectangle* BD, DC, *together with the square on* AD.



Construction. Describe the circle ACB about the triangle; [IV. 5.

produce AD to meet the circumference at E, and join EC.

Proof. Because the ∠BAD = the ∠EAC, [*Hypothesis.*

and the ∠ABD = the ∠AEC, in the same segment. [III. 21.

∴ the △BAD is equiangular to the △EAC;

∴ BA : AD :: EA : AC; [VI. 4.

∴ the rect. BA, AC = the rect. EA, AD, [VI. 16.

that is, = rect. ED, DA, and the square on AD [II. 3.

But the rect. ED, DA = the rect. BD, DC; [III. 35.

∴ the rect. BA, AC = the rect. BD, DC, together with the square on AD.

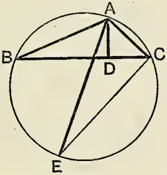
Wherefore, *if the vertical angle, etc.* [Q.E.D.

Proposition C. Theorem.

If from the vertical angle of a triangle a straight line be drawn perpendicular to the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circle described about the triangle.

Let ABC be a triangle, and let AD be the perpendicular from A to the base BC:

the rect. BA, AC *shall be equal to the rect. contained by* AD *and the diameter of the circle described about the triangle.*



Construction. Describe the circle ACB about the triangle; [IV. 5.

draw the diameter AE, and join EC.

Proof. Because the right angle BDA

= the ∠ECA in a semi-circle; [III. 31.

and the ∠ABD = the ∠AEC in the same segment; [III. 21.

∴ the △ABD is equiangular to the △AEC;

∴ BA : AD :: EA : AC; [VI. 4.

∴ the rect. BA, AC = the rect. EA, AD. [VI. 16.

Wherefore, *if from the vertical angle, etc.* [Q.E.D.

EXERCISES ON PROPOSITIONS B, C.

1. If the exterior angle of BAC of a triangle ABC be bisected by a straight line AE which cuts the base in F, and the circle in E, prove that the rectangle BF, FC = the rectangle BA, AC + the square on AF.

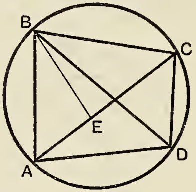
[The △s BAF, EAC are similar.]

2. Construct a triangle, having given the base, the vertical angle, and the rectangle contained by the sides.

PROPOSITION *D*. THEOREM.

The rectangle contained by the diagonals of a quadrilateral inscribed in a circle is equal to the sum of the two rectangles contained by its opposite sides.

Let ABCD be any quadrilateral inscribed in a circle, and join AC, BD:
the rect. AC, BD *shall be equal to the two rectangles contained by* AB, CD *and by* AD, BC.



Construction. Make the ∠ABE equal to the ∠DBC. [I. 23.

Proof. Add to each of these equals the ∠EBD;
then the ∠ABD = the ∠EBC. [*Axiom* 2.
Also the ∠BDA = the ∠BCE, in the same segment; [III. 21.
∴ the △ABD is equiangular to the △EBC;
∴ AD : DB :: EC : CB; [VI. 4.
∴ the rect. AD, CB = the rect. DB, EC. [VI. 16.

Again, because the ∠ABE = the ∠DBC, [*Construction.*
and the ∠BAE = the ∠BDC in the same segment; [III. 21.
∴ the △ABE is equiangular to the △DBC;
∴ BA : AE :: BD : DC; [VI. 4.
∴ the rect. BA, DC = the rect. AE, BD. [VI. 16.

But the rect. AD, CB = the rect. DB, EC; [*Shewn.*
∴ the rectangles AD, CB and BA, DC together
= the rectangles BD, EC and BD, AE;
that is, = the rectangle BD, AC. [II. 1.

Wherefore, *the rectangle contained, etc.* [Q.E.D.

EXERCISES.

1. *The sum of the rectangles contained by opposite sides of a quadrilateral is greater than that contained by the diagonals except when the quadrilateral can be inscribed in a circle, and then it is equal to the rectangle contained by the diagonals.*

Let ABCD be a quadrilateral not inscribable in a circle.

Make the $\angle ABE = \angle DBC$, and the $\angle BAE = \angle BDC$.

[Since ABCD is not inscribable in a circle, the $\angle BAC$ does not $= \angle BDC$, and $\therefore$ AE does not fall on AC as it does in Proposition D.]

The $\triangle^s$ABE, DBC are thus equiangular; $\therefore AB : AE :: DB : DC$.

$$\therefore AB . DC = AE . DB.$$

Again, since $\angle ABD = \angle EBC$, and that

$AB : BE :: BD : BC$, since $\triangle^s$ABE, DBC are similar,

$$i.e. \ AB : BD :: BE : BC.$$

$\therefore$ $\triangle^s$ABD, EBC are similar, and

$\therefore AD : DB :: CE : CB$, *i.e.* $AD . CB = CE . DB$

$\therefore AB . DC + AD . CB = AE . DB + CE . DB$

= rectangle contained by DB and the sum of AE, EC,

and $\therefore >$ rectangle contained by DB, AC. [I. 20.

2. A circle is described round an equilateral triangle, and from any point in the circumference straight lines are drawn to the angular points of the triangle; shew that one of these straight lines is equal to the sum of the other two.

3. From the extremities B, C of the base of an isosceles triangle ABC, straight lines are drawn at right angles to AB, AC respectively, and intersecting at D; shew that the rectangle BC, AD is double of the rectangle AB, DB.

4. If the diagonals of a quadrilateral inscribed in a circle be at right angles, the sum of the rectangles contained by opposite sides is equal to twice the area of the quadrilateral.

5. Prove that the product of the perpendiculars drawn from any point on a circle upon a pair of opposite sides of an inscribed quadrilateral is equal to the product of the perpendiculars from the same point on the other pair of sides, and also to the product of the perpendiculars upon the diagonals of the quadrilateral.

[If O be the point, and OF, OG, OH, OK, OL, OM the perpendiculars on the sides AB, BC, CD, DA and the diagonals AC, BD of the quadrilateral ABCD, then, by Prop. *C*,

$$OF \times diam^r = OA . OB; \ OH \times diam^r = OC . OD,$$

$$OL \times diam^r = OA . OC; \ OM \times diam^r = OB . OD;$$

$$\therefore OF . OH = OL . OM. \quad \text{So } OG . OK = OL . OM.]$$

BOOK XI.

DEFINITIONS.

1. A **solid** is that which has length, breadth, and thickness.

2. That which bounds a solid is a superficies.

3. A straight line is **perpendicular, or at right angles, to a plane**, when it is at right angles to every straight line meeting it in that plane. Such a straight line is often called a **normal** to the plane.

4. **A plane is perpendicular to a plane** when the straight lines drawn in one of the planes perpendicular to the common section of the two planes are perpendicular to the other plane.

5. **The inclination of a straight line to a plane** is the acute angle contained by that straight line, and another drawn from the point at which the first line meets the plane to the point at which a perpendicular to the plane drawn from any point of the first line above the plane meets the same plane.

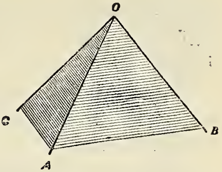
6. The **inclination of a plane to a plane** is the acute angle contained by two straight lines drawn from any the same point of their common section at right angles to it, one in one plane, and the other in the other plane.

7. Two planes are said to have the same or a like inclination to one another which two other planes have, when the said angles of inclination are equal to one another.

8. **Parallel planes** are such as do not meet one another though produced.

A straight line is said to be parallel to a plane when they do not meet if produced.

9. A **solid angle** is that which is made by more than two plane angles, which are not in the same plane, meeting at one point.



A solid angle contained by three plane angles is called a **trihedral** angle; one contained by more than three plane angles is called a **polyhedral** angle.

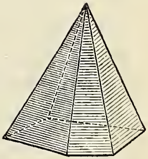
10. The angle made by two straight lines, which do not meet, is the angle contained by two straight lines parallel to them and drawn through any point.

11. Similar solid figures are such as have all their solid angles equal, each to each, and are contained by the same number of similar planes.

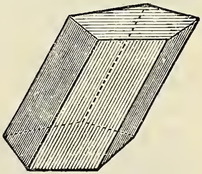
12. A **polyhedron** is a solid figure all of whose bounding surfaces are planes.

It is said to be **regular** when its bounding surfaces are equal and regular polygons.

13. A **pyramid** is a polyhedron which has for one face a triangle or polygon, and for the other faces triangles whose bases are the sides of the polygon, and a point which does not lie in the plane of the polygon as the common vertex.



14. A **prism** is a polyhedron contained by plane figures, of which two that are opposite are equal, similar and parallel to one another, and the others are parallelograms.



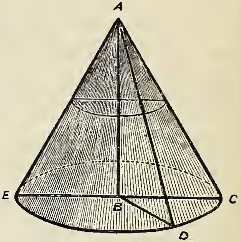
15. A **sphere** is a **solid** figure described by the revolution of a semicircle about its diameter, which remains fixed.

16. The axis of a sphere is the fixed straight line about which the semicircle revolves.

17. The centre of a sphere is the same as that of the semicircle.

The diameter of a sphere is any straight line which passes through the centre, and is terminated both ways by the superficies of the sphere.

18. A **right circular cone** is a solid figure described by the revolution of a right-angled triangle about one of the sides containing the right angle, which side remains fixed.

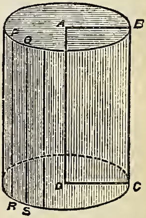


If the fixed side be equal to the other side containing the right angle, the cone is called a right-angled cone; if it be less than the other side, an obtuse-angled cone; and if greater, an acute-angled cone.

19. The axis of a cone is the fixed straight line about which the triangle revolves.

20. The base of a cone is the circle described by that side containing the right angle which revolves.

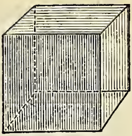
21. A **right circular cylinder** is a solid figure described by the revolution of a rectangle about one of its sides which remains fixed.



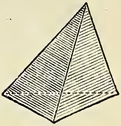
22. The axis of a cylinder is the fixed straight line about which the rectangle revolves.

23. The bases of a cylinder are the circles described by the two revolving opposite sides of the rectangle.

24. Similar cones and cylinders are those which have their axes and the diameters of their bases proportionals.



25. A **cube** is a solid figure contained by six equal squares.

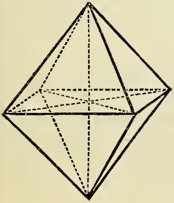


26. A **tetrahedron** is a solid figure contained by four triangles; when these bounding triangles are all equal and equilateral, the tetrahedron is said to be **regular.**

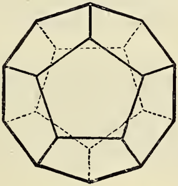
27. A regular **octahedron** is a solid figure contained by eight equal and equilateral triangles.

28. A regular **dodecahedron** is a solid figure contained by twelve equal pentagons which are equilateral and equiangular.

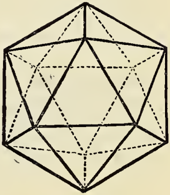
29. A regular **icosahedron** is a solid figure contained by twenty equal and equilateral triangles.



OCTAHEDRON.

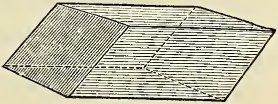


DODECAHEDRON.

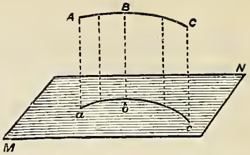


ICOSAHEDRON.

30. A **parallelepiped** is a solid figure contained by six quadrilateral figures, of which every opposite two are parallel.



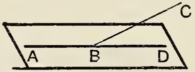
31. The projection of a line on a plane is the line joining the feet of the perpendiculars let fall from the points in that line upon the plane. Thus *abc* is the projection of ABC upon the plane MN.



PROPOSITION 1 THEOREM.

One part of a straight line cannot be in a plane, and another part without it.

If it be possible, let AB, part of the straight line ABC, be in a plane, and the part BC without it.



Construction. Since the straight line AB is in the plane, it can be produced in that plane; let it be produced to D; and let any plane pass through the straight line AD, and be turned about until it pass through the point C.

Proof. Because the points B and C are in this plane, the straight line BC is in it. [I. *Definition* 7.

Therefore there are two straight lines ABC, ABD in the same plane that have a common segment AB;

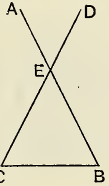
but this is impossible. [I. *Axiom* 10.

Wherefore, *one part of a straight line, etc.* [Q. E. D.

PROPOSITION 2. THEOREM.

Two straight lines which cut one another are in one plane; and three straight lines which meet one another are in one plane.

Let the two straight lines AB, CD cut one another at E: AB *and* CD *shall be in one plane; and the three straight lines* EC, CB, BE, *which meet one another, shall be in one plane.*



Construction. Let any plane pass through the straight line EB, and let the plane be turned about EB, produced if necessary, until it pass through the point C.

Proof. Because the points E and C are in this plane, the straight line EC is in it; [I. *Definition* 7.
for the same reason, the straight line BC is in the same plane; and, by hypothesis, EB is in it;

∴ EC, CB, BE are in one plane.

But AB, CD are in the plane in which EB, EC are; [XI. 1.

∴ AB and CD are in one plane.

Wherefore, *two straight lines, etc.* [Q.E.D.

Note. A plane is determined when it is given that it passes through

(1) two straight lines, intersecting or parallel; [Cf. XI. 7.

or (2) three points which are not in a straight line;

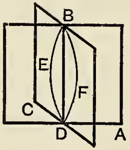
or (3) a given straight line and a given point not lying on this straight line.

PROPOSITION 3. THEOREM.

If two planes cut one another, their common section is a straight line.

Let two planes AB, BC cut one another, and let BD be their common section:

BD *shall be a straight line.*



Construction. If it be not, from B to D, draw in the plane AB the straight line BED, and in the plane BC the straight line BFD. [*Postulate* 1.

Proof. The two straight lines BED, BFD have the same extremities, and therefore include a space between them;

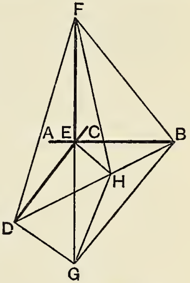
but this is impossible. [*Axiom* 10.

Therefore BD, the common section of the planes AB and BC, cannot but be a straight line.

Wherefore, *if two planes, etc.* [Q.E.D.

PROPOSITION 4. THEOREM.

If a straight line stand at right angles to each of two straight lines at the point of their intersection, it shall also be at right angles to the plane which passes through them, that is, to the plane in which they are.



Let the straight line EF stand at right angles to each of the straight lines AB, CD at E, the point of their intersection:

EF *shall also be at right angles to the plane passing through* AB, CD.

Construction. Produce FE to G, making EG equal to FE; through E draw any straight line EH, meeting BD in H, and join FD, FH, FB, GD, GH, GB.

Proof. (1) In the triangles BEF, BEG,

because { FE = EG, [*Construction.*
and EB is common,
and the ∠BEF = the ∠BEG, both being right angles;

∴ BF = BG. [I. 4.

Similarly, DF = DG.

The triangles FDB, GDB are therefore equal in all respects, and therefore the ∠FBD = the ∠GBD.

(2) In the triangles FBH, GBH,

because { FB = BG, [*Proved.*
and BH is common,
and the ∠FBD = the ∠GBD; [*Proved.*

∴ the base FH = the base GH.

(3) Lastly, in the triangles FEH, GEH,

because { FE = GE, [*Construction.*
and EH is common,
and the base FH = the base GH; [*Proved in* (2).

∴ the ∠GEH = the ∠FEH; ∴ each is a right ∠; [I. *Def.* 10.

∴ EF is perpendicular to EH.

Similarly, it may be shewn to be $\perp^r$ to every other straight line which meets it in the plane through AB and CD;

∴ EF is $\perp^r$ to the plane in which are AB and CD. [XI. *Def.* 3.

Wherefore, *if a straight line, etc.* [Q.E.D.

Note. This is not the proof as given by Euclid, but is much easier.

EXERCISES.

**1. The angle between two planes is equal to the angle between two perpendiculars, one to one plane and one to the other, which intersect.

2. Prove that three planes, in general, meet in a point. What are the exceptions?

**3. Of all the straight lines that can be drawn from a given point to a given plane, the least is that which is perpendicular to the given plane; and of other such lines that one, whose intersection with the plane is nearer to the foot of the perpendicular, is less than one whose intersection is farther from this foot.

**4. The sum of the squares on any two opposite edges of a tetrahedron is less than the sum of the squares on the other four edges.

[Let ABCD be the tetrahedron, E and F the middle points of AB and CD. Then by Ex. 1, page 109, we have

$AC^2 + AD^2 = 2AF^2 + 2FD^2$ and $BC^2 + BD^2 = 2BF^2 + 2FD^2$.

$$\therefore\ AC^2 + AD^2 + BC^2 + BD^2 = 2AF^2 + 2BF^2 + 4FD^2$$
$$= 4FE^2 + 4BE^2 + 4FD^2, \text{ by the same theorem,}$$
$$= 4FE^2 + AB^2 + CD^2,$$

that is, the sum of the squares on AB, CD is less than the sum of the squares on the other four edges by four times the square on the line oining the middle points of AB and CD.]

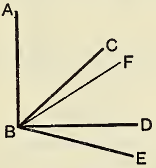
**5. In a tetrahedron the sum of the squares on the six edges is always equal to four times the sum of the squares on the straight lines joining the middle points of opposite edges. [Use Ex. 4.]

Proposition 5. Theorem.

If three straight lines meet all at one point, and a straight line be perpendicular to each of them at that point, the three straight lines shall be in one and the same plane.

Let the straight line AB stand at right angles to each of the straight lines BC, BD, BE, at B the point where they meet:

BC, BD, BE *shall be in one and the same plane.*



Construction. For, if not, let, if possible, BD and BE be in one plane, and BC without it; let a plane pass through AB and BC; the common section of this plane with the plane in which are BD and BE is a straight line; [XI. 3.
let this straight line be BF.

Proof. The three straight lines AB, BC, BF are all in one plane, namely, the plane which passes through AB and BC.

Because AB is perpendicular to both BD and BE, [*Hyp.*
∴ it is perpendicular to the plane through them; [XI. 4.
∴ it is perpendicular to the straight line BF which meets it, and is in that plane; [XI. *Definition* 3.

∴ the ∠ABF is a right ∠.

But the ∠ABC is, by hypothesis, a right ∠;

∴ the ∠ABC = the ∠ABF, [*Axiom* 11.

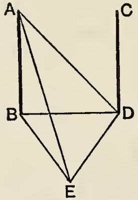
and they are in one plane, which is impossible; [*Axiom* 8.
∴ the straight line BC is not without the plane in which are BD and BE,
that is, BC, BD, BE are in one and the same plane.

Wherefore, *if three straight lines, etc.* [Q.E.D.

Proposition 6. Theorem.

If two straight lines be at right angles to the same plane, they shall be parallel to one another.

Let the straight lines AB, CD be at right angles to the same plane:
AB *shall be parallel to* CD.



Construction. Let them meet the plane at B, D; join BD; and in the plane draw DE at right angles to BD; [I. 11.
make DE equal to AB; and join BE, AE, AD.

Proof. Because AB is perpendicular to the plane, [*Hyp.*
it is at right angles to BD and BE, which meet it and are in that plane; [XI. *Definition* 3.
∴ each of the angles ABD, ABE is a right angle.
Similarly, each of the angles CDB, CDE is a right angle.

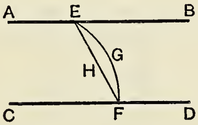
Also, because ABE is a right ∠,
∴ sq. on AE = sum of the sqs. on AB, BE [I. 47.
= sum of the sqs. on AB, BD, DE, since BDE is a right ∠,
= sum of the squares on AD, DE, since ABD is a rt. ∠;
∴ the ∠ADE is a right angle. [I. 48.
But the angles EDB, EDC are also right angles;
∴ DB, DA, DC are all in the same plane. [XI. 5.
But AB is in the plane in which are BD, DA; [XI. 2.
∴ AB, BD, CD are in one plane.
Also each of the angles ABD, CDB is a right ∠;
∴ AB is parallel to CD. [I. 28.

Wherefore, *if two straight lines, etc.* [Q.E.D.

PROPOSITION 7. THEOREM.

If two straight lines be parallel, the straight line drawn from any point in one to any point in the other is in the same plane with the parallels.

Let AB, CD be parallel straight lines, and take any point E in one and any point F in the other: *the straight line which joins* E *and* F *shall be in the same plane with the parallels* AB *and* CD.



For, if not, let it be, if possible, without the plane, as EGF; and in the plane ABCD, in which the parallels are, draw the straight line EHF from E to F.

Then, since EGF is also a straight line, [*Hypothesis.*
the two straight lines EGF, EHF include a space between them, which is impossible; [*Axiom* 10.
∴ the straight line joining the points E and F is not without the plane in which the parallels AB, CD are;

∴ it is in that plane.

Wherefore, *if two straight lines, etc.* [Q.E.D.

EXERCISES.

**1. If a straight line be parallel to a second straight line it is parallel to any plane passing through the second straight line.

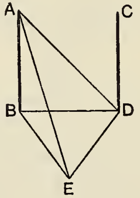
**2. If the perpendiculars drawn from two points to a plane be equal, the straight line joining the two points is parallel to the plane.

PROPOSITION 8. THEOREM.

If two straight lines be parallel, and one of them be at right angles to a plane, the other also shall be at right angles to the same plane.

Let AB, CD be two parallel straight lines; and let one of them AB be at right angles to a plane:

the other CD *shall be at right angles to the same plane.*



Construction. Let AB, CD meet the plane at the points B, D; join BD; then AB, CD, BD are in one plane. [XI. 7.

In the plane to which AB is at right angles, draw DE at right angles to BD; [I. 11.

make DE equal to AB; and join BE, AE, AD.

Proof. (1) Because AB is at right angles to the plane, [*Hyp.*

it is at right angles to every straight line meeting it in that plane; [XI. *Definition* 3.

∴ each of the angles ABD, ABE is a right angle.

And because BD meets the two parallels AB, CD,

the angles ABD, CDB together = two right angles. [I. 29.

But the ∠ABD is a right ∠; [*Hypothesis.*

∴ the ∠CDB is a right ∠,

that is, CD is at right angles to BD.

(2) Because ABE is a right ∠;

∴ the sq. on AE = the squares on AB, BE [I. 47.

= the squares on AB, BD, DE, since BDE is a right ∠,

= the squares on AD, DE, since ABD is a right ∠;

∴ the ∠ADE is a right angle, [I. 48.

that is, ED is at right angles to AD.

But ED is at right angles to BD; [*Construction.*

∴ ED is at right angles to the plane which passes through BD, DA. [XI. 4.

But CD is in the plane passing through BD, DA, because all three are in the plane in which are the parallels AB, CD;

∴ ED is at right angles to CD; [XI. *Def.* 3.

∴ CD is at right angles to ED.

But CD was shewn to be also at right angles to BD;

∴ CD is at right angles to the plane passing through BD, ED, [XI. 4.

that is, to the plane to which AB is at right angles.

Wherefore, *if two straight lines, etc.* [Q.E.D.

EXERCISE.

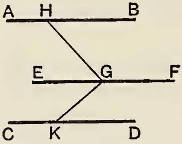
From a point A a perpendicular is drawn to a plane meeting it at B; from B a perpendicular is drawn to a straight line DE in the plane meeting it at C: shew that AC is perpendicular to the straight line DE.

[Through B draw BK ‖l to DE; then BK is in the plane and ABK is a right ∠ since AB is ⊥r to the plane; ∴ KBA, KBC are right ∠s; ∴ KB is ⊥r to the plane through BC, BA; ∴ the ‖l line DE is ⊥r to the plane through BC, CA; ∴ ∠DCA is a right ∠, etc.]

PROPOSITION 9. THEOREM.

Two straight lines which are each of them parallel to the same straight line, and are not in the same plane with it, are parallel to one another.

Let AB and CD be each of them parallel to EF, and not in the same plane with it:
AB *shall be parallel to* CD.



Construction. In EF take any point G;
in the plane passing through EF and AB, draw from G the straight line GH at right angles to EF;
and in the plane passing through EF and CD, draw from G the straight line GK at right angles to EF. [I. 11.

Proof. Because EF is at right angles to GH and GK, [*Constr.*
EF is at right angles to the plane HGK passing through them. [XI. 4.
Also EF is parallel to AB and also to CD; [*Hypothesis.*
∴ AB and CD are each of them at right angles to the plane HGK; [XI. 8.
∴ AB is parallel to CD. [XI. 6.
Wherefore, *if two straight lines, etc.* [Q.E.D.

EXERCISE.

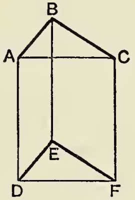
**The middle points of two pairs of opposite edges of a tetrahedron are in one plane and form a parallelogram.

[Let ABCD be the tetrahedron; P and Q the middle points of the opposite edges AB, CD, and R, S the middle points of the other edges AD, BC. Then QR is $\|^l$ to and half of AC; so PS is $\|^l$ to and half of AC; ∴ QR and PS are $\|^l$ and equal; ∴ etc.]

Proposition 10. Theorem.

If two straight lines meeting one another be parallel to two others that meet one another, and are not in the same plane with the first two, the first two and the other two shall contain equal angles.

Let the two straight lines AB, BC, which meet one another, be parallel to the two straight lines DE, EF, which meet one another, and are not in the same plane with AB, BC:
the angle ABC *shall be equal to the angle* DEF.



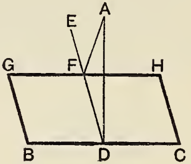
Construction. Take BA, BC, ED, EF all equal to one another, and join AD, BE, CF, AC, DF.

Proof. Because AB is equal and parallel to DE,
∴ AD is equal and parallel to BE. [I. 33.
Similarly, CF is equal and parallel to BE;
∴ AD and CF are each of them equal and parallel to BE;
∴ AD is equal and parallel to CF; [*Axiom* 1 *and* XI. 9.
∴ AC is equal and parallel to DF. [I. 33.
Also because AB, BC are equal to DE, EF, each to each,
and the base AC = the base DF;
∴ the ∠ABC = the ∠DEF. [I. 8.
Wherefore, *if two straight lines, etc.* [Q.E.D.

PROPOSITION 11. PROBLEM.

To draw a straight line perpendicular to a given plane from a given point without it.

Let A be the given point without the plane BH:
it is required to draw from A *a straight line perpendicular to the plane* BH.



Construction. Draw any straight line BC in the plane BH, and from A draw AD perpendicular to BC. [I. 12.
Then, if AD be also perpendicular to the plane BH, the thing required is done. But, if not, from D draw, in the plane BH, the straight line DE at right angles to BC, [I. 11.
and from A draw AF perpendicular to DE: [I. 12.
AF *shall be perpendicular to the plane* BH.

Through F draw GH parallel to BC. [I. 31.

Proof. Because BC is at right angles to ED and DA, [*Constr.*
BC is at right angles to the plane through ED, DA. [XI. 4.
Also GH is parallel to BC; [*Construction.*
∴ GH is at right angles to the plane passing through ED and DA. [XI. 8.
But AF meets GH, and is in this plane;

∴ GH is at right angles to AF; [XI. *Definition* 3.

∴ AF is at right angles to GH.

But AF is also at right angles to DE; [*Construction.*
∴ AF is perpendicular to the plane through GH and DE,

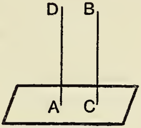
that is, to the plane BH. [XI. 4.

Wherefore, *from the given point* A, *without the plane* BH, *the straight line* AF *has been drawn perpendicular to the plane.* [Q.E.F.

Proposition 12. Problem.

To draw a straight line perpendicular to a given plane, from a given point in the plane.

Let A be the given point in the plane: *it is required to draw from* A *a straight line perpendicular to the plane.*



Construction. From any point B without the plane, draw BC perpendicular to the plane; [XI. 11.
and from A draw AD parallel to BC: [I. 31.
AD *shall be the straight line required.*

Proof. Because AD and BC are two parallel straight lines, [*Construction.*
and that one of them BC is perpendicular to the given plane, [*Construction.*
the other AD is also perpendicular to the given plane. [XI. 8.

Wherefore *a straight line has been drawn perpendicular to a given plane, from a given point in it.* [Q.E.F.

EXERCISES.

**1. Shew that equal straight lines drawn from a given point to a given plane are equally inclined to the plane.

2. If two straight lines in one plane be equally inclined to another plane, they will be equally inclined to the common section of these planes.

**3. P and Q are two points on the same side of a given plane; find a point R in the plane which is such that the sum of PR, RQ is the least possible.

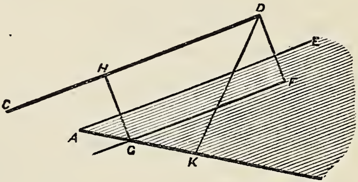
4. OA, OB, OC are three equal straight lines meeting in O which are not in the same plane; prove that the perpendicular from O upon the plane ABC meets it in the centre of the circumcircle of the triangle ABC. [Use Ex. 1.]

****5.** Give a geometrical construction for drawing a straight line which shall be equally inclined to three straight lines meeting at a point. [Use Exs. 1 and 4.]

6. Perpendiculars are drawn from a point to a plane, and to a straight line in that plane; shew that the straight line joining the feet of the perpendiculars is perpendicular to the former straight line.

7. If O is the centre of the circumcircle of a triangle ABC and OP be drawn perpendicular to the plane of the triangle, prove that any point in OP is equidistant from the vertices A, B, C.

****8.** *Show how to construct the straight line which is perpendicular to two straight lines which are not parallel and do not intersect; shew also that the common perpendicular is the shortest distance between the two given lines.*



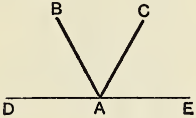
Let AB and CD be the two given straight lines; through AB draw the plane AEB, which contains AB and a straight line AE parallel to CD; this plane is therefore parallel to CD. Draw DF perp^r to this plane and let the plane CDF cut the plane AEB in the straight line FG. Let FG meet AB in G and draw GH perp^r to CD. Then GH is the line required, that is, it shall also be $\perp^r$ to AB. For CD, being parallel to the plane AEB, does not meet GF which is in this plane. Also CD and GF are in the same plane CDF; $\therefore$ CD and GF are parallel; $\therefore$ DF, which is $\perp^r$ to GF, is $\perp^r$ to CD, and is $\therefore \parallel^l$ to HG; $\therefore$ HG is $\perp^r$ to the plane AEB, and $\therefore$ is $\perp^r$ to the straight line AB.

Also HG is the shortest distance between the two given lines. For let DK be any other line joining two points on them. Since DF is $\perp^r$ to the plane EAB; $\therefore$ $\angle$DFK is a right $\angle$; $\therefore$ in the $\triangle$DKF, DK $>$ DF, *i.e.* $>$ HG.

PROPOSITION 13. THEOREM.

From the same point in a given plane there cannot be two straight lines at right angles to the plane on the same side of it; and there can be but one perpendicular to a plane from a point without the plane.

For, if it be possible, let the two straight lines AB, AC be at right angles to a given plane, from the same point A in the plane, and on the same side of it.



Let a plane pass through BA, AC;
the common section of this with the given plane is a straight line; [XI. 3.
let this straight line be DAE.

Proof. AB, AC, DAE are all in one plane. [*Const.*
Because CA is at right angles to the given plane, [*Hypothesis.*
it is at right angles to every straight line meeting it in the plane. [XI. *Definition* 3.
But DAE meets CA, and is in that plane;
∴ the ∠CAE is a right ∠.
For the same reason the ∠BAE is a right ∠;
∴ the ∠CAE = the ∠BAE,
and they are in one plane, which is impossible. [*Axiom* 8.

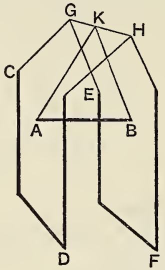
Also, from a point without the plane, there can be but one perpendicular to the plane.
For if there could be two, they would be parallel [XI. 6.
which is absurd.

Wherefore, *from the same point, etc.* [Q.E.D.

Proposition 14. Theorem.

Planes to which the same straight line is perpendicular are parallel to one another.

Let the straight line AB be perpendicular to each of the planes CD and EF:
these planes shall be parallel to one another.



Construction. For, if not, they will meet when produced; let them meet; their common section will be a straight line; let GH be this straight line; in it take any point K, and join AK, BK.

Proof. Because AB is perpendicular to the plane EF, [*Hyp.*
it is perpendicular to the straight line BK which is in that plane; [XI. *Definition* 3.

∴ the ∠ABK is a right ∠.

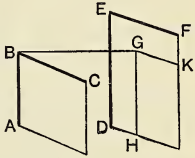
For the same reason the ∠BAK is a right ∠;
∴ the two angles ABK, BAK of the △ABK are equal to two right angles, which is impossible. [I. 17.
∴ the planes CD and EF, though produced, do not meet,
that is, they are parallel. [XI. *Definition* 8.

Wherefore, *planes, etc.* [Q.E.D.

PROPOSITION 15. THEOREM.

If two straight lines, which meet one another, be parallel to two other straight lines which meet one another, but are not in the same plane with the first two, the plane passing through these is parallel to the plane passing through the others.

Let AB, BC, two straight lines which meet one another, be parallel to two other straight lines DE, EF, which meet one another, but are not in the same plane with AB, BC:
the plane passing through AB, BC *shall be parallel to the plane passing through* DE, EF.



Construction. From B draw BG perpendicular to the plane passing through DE, EF, [XI. 11.
and let it meet that plane at G;
through G draw GH ||[l] to ED, and GK ||[l] to EF. [I. 31.

Proof. Because BG is perpendicular to the plane passing through DE, EF, [*Construction.*
it is perpendicular to each of the straight lines GH and GK, which meet it, and are in that plane; [XI. *Definition* 3.
∴ each of the angles BGH and BGK is a right ∠.

Now, because BA is parallel to ED, [*Hypothesis.*
and GH is parallel to ED, [*Construction.*
∴ BA is parallel to GH; [XI. 9.
∴ the angles ABG and BGH together = two right angles. [I. 29.
Also the angle BGH has been shewn to be a right angle;
∴ the ∠ABG is a right ∠.
For the same reason the ∠CBG is a right ∠;
∴ GB is perpendicular to the plane through AB, BC. [XI. 4.
And GB is also ⊥[r] to the plane through DE, EF; [*Constr.*
∴ the plane passing through AB, BC is parallel to the plane passing through DE, EF. [XI. 14.

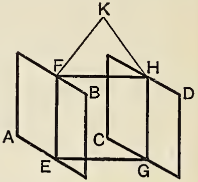
Wherefore, *if two straight lines, etc.* [Q.E.D.

PROPOSITION 16. THEOREM.

If two parallel planes be cut by another plane, their common sections with it are parallel.

Let the parallel planes AB, CD be cut by the plane EFHG, and let their common sections with it be EF, GH:

EF *shall be parallel to* GH.



Construction. For if not, EF and GH, being produced, will meet either towards F, H or towards E, G. Let them be produced and meet towards F, H at the point K.

Proof. Since EFK is in the plane AB, every point in EFK is in that plane; [XI. 1.

∴ K is in the plane AB.

For the same reason K is in the plane CD;

∴ the planes AB, CD, being produced, meet one another.

But they do not meet, since they are parallel by hypothesis;

∴ EF and GH, being produced, do not meet towards F, H.

In the same manner it may be shewn that they do not meet towards E, G;

∴ EF is parallel to GH. [I. *Definition* 29.

Wherefore, *if two parallel planes, etc.* [Q.E.D.

EXERCISES.

1. Draw two parallel planes, one through one straight line, and the other through another straight line which does not meet the former.

****2.** If two planes which are not parallel be cut by two parallel planes, the lines of section of the first two by the last two will contain equal angles.

3. Polygons formed by cutting a pyramid by parallel planes are similar.

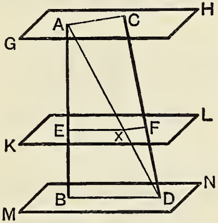
****4.** Polygons formed by cutting a prism by parallel planes are equal.

5. From the extremities of the two parallel straight lines AB, CD parallel straight lines A*a*, B*b*, C*c*, D*d* are drawn meeting a plane at *a*, *b*, *c*, *d*: shew that AB is to CD as *ab* is to *cd*.

PROPOSITION 17. THEOREM.

If two straight lines be cut by parallel planes, they shall be cut in the same ratio.

Let the straight lines AB and CD be cut by the parallel planes GH, KL, MN at the points A, E, B and C, F, D: AE *shall be to* EB *as* CF *to* FD.



Construction. Join AC, BD, AD; let AD meet the plane KL at the point X; and join EX, XF.

Proof. Because the parallel planes KL, MN are cut by the plane EBDX, the common sections EX, BD are parallel; [XI. 16.

and because the parallel planes GH, KL are cut by the plane AXFC, the common sections AC, XF are parallel. [XI. 16.

Also because EX is parallel to BD, a side of the △ABD,

∴ AE : EB :: AX : XD. [VI. 2.

Again, because XF is parallel to AC, a side of the △ADC,

∴ AX : XD :: CF : FD; [VI. 2.

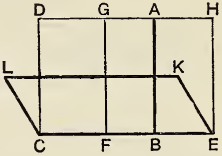
∴ AE : EB :: CF : FD. [V. 11.

Wherefore, *if two straight lines, etc.* [Q.E.D.

PROPOSITION 18. THEOREM.

If a straight line be perpendicular to a plane, every plane which passes through it shall be perpendicular to that plane.

Let the straight line AB be perpendicular to the plane CK: *every plane which passes through* AB *shall be perpendicular to the plane* CK.



Construction. Let any plane DE pass through AB, and let CE be the common section of the planes DE, CK; [XI. 3. take any point F in CE, from which draw FG, in the plane DE, at right angles to CE. [I. 11.

Proof. Because AB is perpendicular to the plane CK, [*Hypothesis.*

∴ it is perpendicular to CB which meets it, and is in that plane; [XI. *Definition* 3.

∴ the ∠ABF is a right ∠.

But the ∠GFB is also a right ∠; [*Construction.*

∴ FG is parallel to AB. [I. 28.

Also AB is at right angles to the plane CK; [*Hypothesis.*

∴ FG is also at right angles to the same plane, [XI. 8.

that is, any straight line FG drawn in the plane DE, at right angles to CE, the common section of the planes DE and CK, is at right angles to the plane CK;

∴ the plane DE is at right angles to the plane CK. [XI. *Definition* 4.

Similarly, any other plane which passes through AB is at right angles to the plane CK.

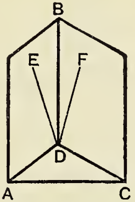
Wherefore, *if a straight line, etc.* [Q.E.D.

Proposition 19. Theorem.

If two planes which cut one another be each of them perpendicular to a third plane, their common section shall be perpendicular to the same plane.

Let the two planes BA, BC be each of them perpendicular to a third plane, and let BD be the common section of the planes BA, BC:

BD *shall be perpendicular to the third plane*



Construction. For, if not, from the point D, draw in the plane BA the straight line DE at right angles to AD, the common section of the plane BA with the third plane; [I. 11. and from D, draw in the plane BC the straight line DF at right angles to CD, the common section of the plane BC with the third plane. [I. 11.

Proof. Because the plane BA is perpendicular to the third plane, and DE is drawn in the plane BA at right angles to AD their common section; [*Construction.*

∴ DE is perpendicular to the third plane. [XI. *Definition* 4.

Similarly, DF is perpendicular to the third plane;

∴ from D two straight lines are drawn at right angles to the third plane, on the same side of it, which is impossible; [XI. 13.

∴ from D there cannot be drawn any straight line at right angles to the third plane, except BD the common section of the planes BA, BC;

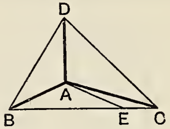
∴ BD is perpendicular to the third plane.

Wherefore, *if two planes, etc.* [Q.E.D.

PROPOSITION 20. THEOREM.

If a solid angle be contained by three plane angles, any two of them are together greater than the third.

Let the solid angle at A be contained by the three plane angles BAC, CAD, DAB:
any two of them shall be together greater than the third.



If the angles BAC, CAD, DAB be all equal, it is evident that any two of them are greater than the third.

If they are not all equal, let BAC be that angle which is not less than either of the other two, and is greater than one of them, BAD.

Construction. At the point A, in the straight line BA, make, in the plane which passes through BA, AC, the angle BAE equal to the angle BAD; [I. 23.

make AE equal to AD; [I. 3.

through E draw BEC, cutting AB, AC at the points B, C, and join DB, DC.

Proof. In the triangles BAD, BAE,

because { AD = AE, [*Construction.*
and AB is common,
and the ∠BAD = the ∠BAE, [*Construction.*

∴ the base BD = the base BE. [I. 4.

Also because BD, DC are together greater than BC, [I. 20.
and one of them BD has been shewn equal to BE, a part of BC;

∴ the other, DC, is greater than the remaining part EC.

Also because $AD = AE$, [*Construction.*

and AC is common to the two $\triangle^s$ DAC, EAC,

but the base DC > the base EC;

$\therefore$ the $\angle DAC >$ the $\angle EAC$. [I. 25.

Also, by construction, the $\angle BAD =$ the $\angle BAE$;

$\therefore$ the angles BAD, DAC are together greater than the angles BAE, EAC, that is, than the angle BAC.

Again, the $\angle BAC$ is not less than either of the angles BAD, DAC;

$\therefore$ the $\angle BAC$ together with either of the other angles is greater than the third.

Wherefore, *if a solid angle, etc.* [Q.E.D.

EXERCISES ON PROPOSITIONS 18 and 19.

1. From a point A in one of two planes are drawn AB at right angles to the first plane, and AC perpendicular to the second plane, and meeting the second plane at B, C; shew that BC is perpendicular to the line of intersection of the two planes.

[By Prop. 18 the plane ABC is perpendicular to each of the given planes, and $\therefore$ by Prop. 19 is at right angles to their intersection; $\therefore$ their intersection is $\perp^r$ to BC which is in the plane ABC.]

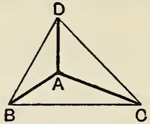
2. Perpendiculars AE, BF are drawn to a plane from two points A, B above it; a plane is drawn through A perpendicular to AB; shew that its line of intersection with the given plane is perpendicular to EF.

[Let EF meet the line of intersection OC in O. Then the plane AOE is $\perp^r$ to the plane AOC, since it goes through AB the perpendicular to AOC (XI. 18). Similarly it is $\perp^r$ to the plane EOC; $\therefore$ it is $\perp^r$ to their intersection OC; $\therefore$ OC is $\perp^r$ to the plane AOE, and $\therefore$ to OE.]

PROPOSITION 21. THEOREM.

Every solid angle is contained by plane angles, which are together less than four right angles.

(i.) Let the solid angle at A be contained by the three plane angles BAC, CAD, DAB:
these three shall be together less than four right angles.



Construction. In the straight lines AB, AC, AD take any points B, C, D, and join BC, CD, DB.

Proof. Because the solid angle at B is contained by the three plane angles CBA, ABD, DBC;

the angles CBA, ABD are together > the ∠DBC. [XI. 20.

Similarly, the ∠s BCA, ACD are together > the ∠DCB.

and the angles CDA, ADB are together > the ∠BDC;

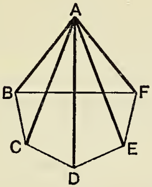
the six angles CBA, ABD, BCA, ACD, CDA, ADB are together > the three angles DBC, DCB, BDC,

that is, > two right angles. [I. 32.

Also because the three angles of each of the triangles ABC, ACD, ADB together = two right angles, [I. 32.
therefore the nine angles of these triangles, namely, the angles CBA, BCA, BAC, ACD, CDA, CAD, ADB, ABD, DAB = six right angles;
and of these, the six angles CBA, BCA, ACD, CDA, ADB, ABD are greater than two right angles;
∴ the remaining three angles BAC, CAD, DAB, which contain the solid angle at A, are together < four right ∠s.

(ii.) Let the solid angle at A be contained by any number of plane angles BAC, CAD, DAE, EAF, FAB:
these shall be together less than four right angles.

Construction. Let the planes, in which the angles are, be cut by a plane, and let the common sections of it with these planes be BC, CD, DE, EF, FB.



Proof. Because the solid angle at B is contained by the three plane angles CBA, ABF, FBC;

∴ the angles CBA, ABF are together > the ∠FBC. [XI. 20.

Similarly, at each of the points C, D, E, F, the two plane angles which are at the bases of the triangles having the common vertex A, are together > the third ∠ at the same point, which is one of the angles of the polygon BCDEF;

∴ all the angles at the bases of the triangles are together greater than all the angles of the polygon.

Now all the angles of the triangles together

= twice as many right angles as there are triangles,

that is, as there are sides in the polygon BCDEF; [I. 32.

and all the ∠s of the polygon, together with four right ∠s, also = twice as many right angles as there are sides in the polygon; [I. 32, *Corollary* 1.

∴ all the angles of the triangles = all the angles of the polygon, together with four right angles. [*Axiom* 1.

But it has been shewn that all the angles at the bases of the triangles are together greater than all the angles of the polygon;

∴ the remaining ∠s of the triangles, namely, those at the vertex, which contain the solid angle at A are < four right angles.

Wherefore, *every solid angle, etc.* [Q.E.D.

Note 1. In the second case of the foregoing proposition it is assumed that the polygon BCDEF has no *re-entrant* angle. [See note to I. 32.] Otherwise the proposition will not be true.

Note 2. There cannot be more than five *regular* polyhedra.

For we require not less than three plane angles to form a solid angle. Also by the preceding proposition the angles forming any solid angle are together less than four right angles.

Now three times the angle of a regular hexagon equals four right angles; and three times the angle of a polygon, with a number of sides larger than six, is greater than four right angles. The faces of a regular polyhedron must thus have less than six sides each, and must therefore be triangles, squares, or pentagons.

It is therefore only possible that a solid angle of a regular polyhedron should be formed by

(1) three equilateral triangles,

(2) four equilateral triangles,

(3) five equilateral triangles,

(4) three squares, or

(5) three pentagons.

For any higher number such as six equilateral triangles, or four squares, or four pentagons, would make the plane angles at each solid angle together not less than four right angles.

The solids corresponding to the above five cases are

(1) a tetrahedron formed by four equilateral triangles,

(2) an octahedron formed by eight equilateral triangles,

(3) an icosahedron formed by twenty equilateral triangles,

(4) a cube formed by six squares, and

(5) a dodecahedron formed by twelve pentagons.

There are no other *regular* polyhedra besides these five.

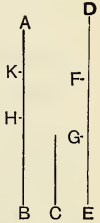
BOOK XII.

LEMMA.

[This lemma is the First Proposition of the Tenth Book.

If from the greater of two unequal magnitudes there be taken more than its half, and from the remainder more than its half, and so on, there shall at length remain a magnitude less than the smaller of the proposed magnitudes.

Let AB and C be two unequal magnitudes, of which AB is the greater:
if from AB *there be taken more than its half, and from the remainder more than its half, and so on, there shall at length remain a magnitude less than* C.



For C may be multiplied so as to become greater than AB.

Let it be so multiplied, and let DE its multiple be greater than AB, and be divided into DF, FG, GE, each equal to C.

From AB take BH greater than its half, and from the remainder AH take HK greater than its half, and so on, until there be as many divisions in AB as in DE; and let the divisions in AB be AK, KH, HB.

Then, because $DE > AB$,
and that EG taken from DE is not > its half,
but BH taken from $AB >$ its half;
$\therefore$ the remainder $DG >$ the remainder AH.
Again, because $DG > AH$, and that
GF is not > the half of DG, but $HK >$ the half of AH;
$\therefore$ the remainder $DF >$ the remainder AK.

But $DF = C$; $\therefore$ $C > AK$; that is, $AK < C$. [Q.E.D.

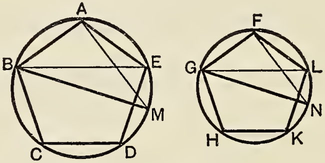
Also if only the halves be taken away, the same thing may in the same way be demonstrated.

PROPOSITION 1. THEOREM.

Similar polygons inscribed in circles are to one another as the squares on their diameters.

Let ABCDE, FGHKL be two circles, and in them the similar polygons ABCDE, FGHKL; and let BM, GN be the diameters of the circles:

the polygon ABCDE *shall be to the polygon* FGHKL *as the square on* BM *is to the square on* GN.



Construction. Join AM, BE, FN, GL.

Proof. Because the polygons are similar

∴ the ∠BAE = the ∠GFL,

and BA : AE :: GF : FL; [VI. *Definition* 2.

∴ the △BAE is equiangular to the △GFL; [VI. 6.

∴ the ∠AEB = the ∠FLG.

But ∠AEB = ∠AMB, and ∠FLG = ∠FNG; [III. 21.

∴ the ∠AMB = the ∠FNG.

Also, the ∠BAM = the ∠GFN; for each is a right ∠; [III. 31.

∴ the remaining angles in the triangles AMB, FNG are equal, and the triangles are equiangular;

∴ BA : BM :: GF : GN, [VI. 4.

and, alternately, BA : GF :: BM : GN; [V. 16.

But the polygon ABCDE is to the polygon FGHKL in the duplicate ratio of BA to GF, [VI. 20.

that is, in the duplicate ratio of BM to GN,

that is, in the ratio of the sq. on BM to the sq. on GN. [VI. 20.

Wherefore, *similar polygons, etc.* [Q.E.D.

PROPOSITION 2. THEOREM.

Circles are to one another as the squares on their diameters.

Let ABCD, EFGH be two circles, and BD, FH their diameters:

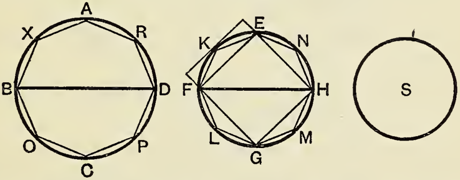
the circle ABCD *shall be to the circle* EFGH *as the square on* BD *is to the square on* FH.

For, if not, the square on BD must be to the square on FH as the circle ABCD is to some space either less than the circle EFGH, or greater than it.

First, if possible, let it be as the circle ABCD is to a space S less than the circle EFGH.

In the circle EFGH inscribe the square EFGH. [IV. 6.

This square shall be > half of the circle EFGH.



For the square EFGH is half of the square which can be formed by drawing straight lines to touch the circle at the points E, F, G, H;

and the square thus formed > the circle;

∴ the square EFGH > half of the circle.

Bisect the arcs EF, FG, GH, HE at the points K, L, M, N; and join EK, KF, FL, LG, GM, MH, HN, NE. Then each of the triangles EKF, FLG, GMH, HNE shall be > half of the segment of the circle in which it stands.

For the triangle EKF is half of the parallelogram which can be formed by drawing a straight line to touch the circle at K, and parallels through E and F,

and the parallelogram thus formed > the segment FEK;

∴ the $\triangle EKF >$ half of the segment.

Also similarly for the other triangles;

∴ the sum of all these triangles together > half of the sum of the segments of the circle in which they stand.

Again, bisect EK, KF, etc., and form triangles as before; then the sum of these triangles > half of the sum of the segments of the circle in which they stand.

If this process be continued, and the triangles be supposed to be taken away, there will at length remain segments of circles which are together < the excess of the circle EFGH above the space S, by the preceding Lemma.

Let then the segments EK, KF, FL, LG, GM, MH, HN, NE be those which remain, and which are together < the excess of the circle above S;

∴ the rest of the circle, namely, the polygon EKFLGMHN, > the space S.

In the circle ABCD describe the polygon AXBOCPDR similar to the polygon EKFLGMHN.

Then the polygon AXBOCPDR is to the polygon EKFLGMHN as the square on BD to the square on FH, [XII. 1.

that is, as the circle ABCD is to the space S. [*Hyp.*, V. 11.

But the polygon AXBOCPDR < the circle ABCD in which it is inscribed;

∴ the polygon EKFLGMHN < the space S; [V. 14.

but it is also greater, as has been shewn,

which is impossible;

∴ the square on BD is not to the square on FH as the circle ABCD is to any space less than the circle EFGH.

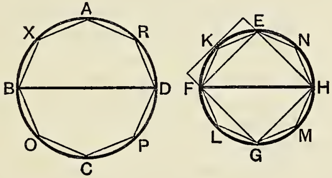
In the same way it may be shewn that the square on FH is not to the square on BD as the circle EFGH is to any space less than the circle ABCD.

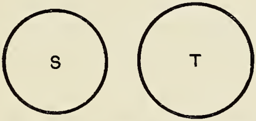
Next, if possible, let the square on BD be to the square on FH as the circle ABCD is to any space **greater than** the circle EFGH.

For, if possible, let it be as the circle ABCD is to a space T greater than the circle EFGH.

Then, inversely, the square on FH is to the square on BD as the space T is to the circle ABCD.

But as the space T is to the circle ABCD, so is the circle EFGH to some space, which must be less than the circle ABCD, because, by hypothesis, the space T is greater than the circle EFGH; [V. 14.





∴ the square on FH is to the square on BD as the circle EFGH is to some space less than the circle ABCD, which has been shewn to be impossible;

∴ the square on BD is not to the square on FH as the circle ABCD is to any space greater than the circle EFGH.

Also it has been shewn that the square on BD is not to the square on FH as the circle ABCD is to any space less than the circle EFGH;

∴ the square on BD is to the square on FH as the circle ABCD is to the circle EFGH.

Wherefore, *circles, etc.* [Q.E.D.

NOTES ON EUCLID'S ELEMENTS.

The article "Eucleides," in Dr. Smith's *Dictionary of Greek and Roman Biography*, was written by Professor De Morgan; it contains an account of the works of Euclid, and of the various editions of them which have been published. To that article we refer the student who desires full information on these subjects. Perhaps the only work of importance relating to Euclid which has been published since the date of that article is a work on the *Porisms of Euclid*, by Chasles. Paris, 1860.

Euclid appears to have lived in the time of the first Ptolemy, B.C. 323–283, and to have been the founder of the Alexandrian mathematical school. The work on Geometry known as *The Elements of Euclid* consists of thirteen books; two other books have sometimes been added, of which it is supposed that Hypsicles was the author. Besides the *Elements*, Euclid was the author of other works, some of which have been preserved and some lost.

We will now mention the three editions which are the most valuable for those who wish to read the *Elements of Euclid* in the original Greek.

(1) The Oxford edition in folio, published in 1703, by David Gregory, under the title Εὐκλείδου τὰ σωζόμενα. "As an edition of the whole of Euclid's works, this stands alone, there being no other in Greek."—*De Morgan.*

(2) *Euclidis Elementorum Libri sex priores*, edidit Joannes Gulielmus Camerer. This edition was published at Berlin in two volumes octavo, the first volume in 1824 and the second in 1825. It contains the first six books of the *Elements* in Greek, with a Latin translation, and very good notes, which form a mathematical commentary on the subject.

(3) *Euclidis Elementa ex optimis libris in usum tironum Græce, edita ab Ernesto Ferdinando August.* This edition was published at Berlin in two volumes octavo, the first volume in 1826 and the second in 1829. It contains the thirteen books of the *Elements* in Greek, with a collection of various readings. A third volume, which was to have contained the remaining works of Euclid, never appeared. "To the scholar who wants one edition of the *Elements* we should decidedly recommend this, as bringing together all that has been done for the text of Euclid's greatest work."—*De Morgan.*

An edition, in five volumes, of the whole of Euclid's works in the original has been issued by Teubner, the well-known German publisher, as one of his series of compact editions of Greek and Latin authors.

Robert Simson's edition of the *Elements of Euclid*, which we have in substance adopted in the present work, differs considerably from the original. The English reader may ascertain the contents of the original by consulting the work entitled *The Elements of Euclid with dissertations*, by James Williamson. This work consists of two volumes quarto; the first volume was published at Oxford in 1781, and the second at London in 1788. Williamson gives a close translation of the thirteen books of the *Elements* into English, and he indicates by the use of italics the words which are not in the original, but which are required by our language.

For the history of Geometry the student is referred to Montucla's *Histoire des Mathématiques*, and to Chasles's *Aperçu historique sur l'origine et le devéloppement des Méthodes en Géométrie.*

BOOK I.

Definitions. The first seven definitions have given rise to considerable discussion, on which however we do not propose to enter. Such a discussion would consist mainly of two subjects, both of which are unsuitable to an elementary work, namely, an examination of the origin and nature of some of our elementary ideas, and a comparison of the original text of Euclid with the substitutions for it proposed by Simson and other editors. For the former subject the student may hereafter consult Whewell's *History of Scientific Ideas* and Mill's *Logic*, and for the latter the notes in Camerer's edition of the *Elements of Euclid.*

We will only observe that the ideas which correspond to the words *point*, *line*, and *surface*, do not admit of such definitions as will really supply the ideas to a person who is destitute of them. The so-called definitions may be regarded as cautions or restrictions.

Thus a *point* is not to be supposed to have any *size*, but only *position*;
a *line* is not to be supposed to have any *breadth* or *thickness*, but only *length*;
a *surface* is not to be supposed to have any *thickness*, but only *length* and *breadth*.

The eighth definition seems intended to include the cases in which an angle is formed by the meeting of two *curved* lines, or of a *straight* line and a *curved* line; this definition however is of no importance, as the only angles ever considered are such as are formed by straight lines.

Some writers object to such definitions as those of an equilateral triangle, or of a square, in which the existence of the object defined is *assumed* when it ought to be *demonstrated*. They would present them in such a form as the following: if there be a triangle having three equal sides, let it be called an equilateral triangle.

Moreover, some of the definitions are introduced prematurely. Thus, for example, take the definitions of a right-angled triangle and an obtuse-angled triangle; it is not shewn until I. 17 that a triangle cannot have both a right angle and an obtuse angle, and so cannot be at the same time right-angled and obtuse-angled. And before Axiom 11 has been given, it is conceivable that the same angle may be greater than one right angle, and less than another right angle, that is, obtuse and acute at the same time.

On the *method of superposition* we may refer to papers by Professor Kelland in the *Transactions of the Royal Society of Edinburgh*, Vols. XXI. and XXIII.

The first book is chiefly devoted to the properties of triangles and parallelograms.

We may observe that Euclid himself does not distinguish between problems and theorems except by using at the end of the investigation phrases which correspond to Q.E.F. and Q.E.D. respectively.

I. 2. This problem admits of *eight* cases in its figure. For it will be found that the given point may be found with *either* end of the given straight line; there the equilateral triangle may be described on *either* side of the straight line which is drawn, and the sides of the equilateral triangle which are produced may be produced through *either* extremity. These various cases may be left for the exercise of the student, as they present no difficulty.

There will not however always be eight different straight lines obtained which solve the problem. For example, if the point A falls on BC produced, some of the solutions obtained coincide; this depends on the fact which follows from I. 32, that the angles of all equilateral triangles are equal.

I. 5. "Join FC." Custom seems to allow this singular expression as an abbreviation for "draw the straight line FC," or for "join F to C by the straight line FC."

It has been suggested to demonstrate I. 5 by *superposition*. Conceive the isosceles triangle ABC to be taken up, and then replaced so that AB falls on the old position of AC, and AC falls on the old position of AB. Thus, in the manner of I. 4, we can shew that the angle ABC is equal to the angle ACB.

I. 6. This proposition is not required by Euclid before he reaches II. 4; so that I. 6 might be removed from its present place and demonstrated hereafter in other ways if we please. For example, I. 6 might be placed after I. 18, and demonstrated thus. Let the angle ABC be equal to the angle ACB; then the side AB shall be equal to the side AC. For if not, one of them must be greater than the other; suppose AB greater than AC. Then the angle ACB is greater than the angle ABC, by I. 18. But this is impossible, because the angle ACB is equal to the angle ABC, by hypothesis. Or I. 6 might be placed after I. 26, and demonstrated thus. Bisect the angle BAC by a straight line meeting the base at D. Then the triangles ABD and ACD are equal in all respects, by I. 26.

I. 12. Here the straight line is said to be of *unlimited* length, in order that we may ensure that it shall meet the circle.

Euclid distinguishes between the terms *at right angles* and *perpendicular*. He uses the term *at right angles* when the straight line is drawn from a point *in* another, as in I. 11; and he uses the term *perpendicular* when the straight line is drawn from a point *without* another, as in I. 12. This distinction, however, is often disregarded by modern writers.

I. 20. "Proclus, in his *Commentary*, relates, that the Epicureans derided Prop. 20, as being manifest even to asses, and needing no demonstration; and his answer is, that though the truth of it be manifest to our senses, yet it is science which must give the reason why two sides of a triangle are greater than the third: but the right answer to this objection, against this and the 21st, and some other plain propositions, is, that the number of axioms ought not to be increased without necessity, as it must be if these propositions be not demonstrated."—*Simson.*

I. 21. Here it must be carefully observed that the two straight lines are to be drawn *from the ends of the side* of the triangle. If this condition be omitted the two straight lines will not necessarily be less than two sides of the triangle.

I. 22. "Some authors blame Euclid because he does not demonstrate that the two circles made use of in the construction of this problem must cut one another: but this is very plain from the determination he has given, namely, that any two of the straight lines A, B, C, must be greater than the third.

The condition that B and C are greater than A, ensures that the circle described from the centre G shall not fall entirely within the circle described from the centre F; the condition that A and B are greater than C, ensures that the circle described from the centre F shall not fall entirely within the circle described from the centre G; the condition that A and C are greater than B, ensures that one of these circles shall not fall entirely without the other. Hence the circles must meet.

I. 26. It will appear after I. 32 that two triangles which have two angles of the one equal to two angles of the other, each to each, have also their third angles equal. Hence we are able to include the two cases of I. 26 in one enunciation thus; *if two triangles have all the angles of the one respectively equal to all the angles of the other, each to each, and have also a side of the one, opposite to any angle, equal to the side opposite to the equal angle in the other, the triangles shall be equal in all respects.*

Besides the cases of I. 4, 8, 26 there are two other cases which will naturally occur to a student to consider, namely,

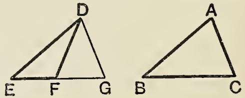
(1) when two triangles have the three angles of the one respectively equal to the three angles of the other;

(2) when two triangles have two sides of the one equal to two sides of the other, each to each, and an angle opposite to one side of one triangle equal to the angle opposite to the equal side of the other triangle.

In the first of these two cases the student will easily see, after reading I. 29, that the two triangles are not necessarily equal. In the second case also the triangles are not necessarily equal, as will be seen from a proposition which we shall now demonstrate.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles opposite to one pair of equal sides equal, the angle opposite the other pair of equal sides shall be either equal or supplementary.

Let ABC, DEF be two triangles having AB = DE, AC = DF, and the ∠ABC opposite AC = the ∠DEF opposite the equal side DF.



Let the △ABC be applied to the △DEF so that AB coincides with DE and the ∠ABC with the ∠DEF. Then the side BC falls on the side EF.

The point C then coincides with F, or falls in EF, or in EF produced.

If C coincides with F, the triangle ABC coincides with the triangle DEF, and they are equal in all respects.

If not, let C fall on the point G in EF, or EF produced.

The $\triangle^s ABC$, DEG then coincide and are equal in all respects, so that $\angle DGE = \angle ACB$, and the side $DG =$ the side AC.

But $AC = DF$; $\therefore$ $DG = DF$; $\therefore$ $\angle DGF = \angle DFG$. [I. 6.

But the $\angle^s DFG$, $DFE =$ two right angles.

$\therefore$ the $\angle^s DGE$, $DFE =$ two right angles;

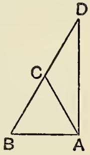
$\therefore$ the $\angle^s ACB$, $DFE =$ two right angles;

that is, the $\angle^s ACB$, DFE are supplementary.

The student who wishes to examine the attempts that have been made to avoid the assumption made in Euclid's twelfth Axiom may consult Camerer's *Euclid*, Gergonne's *Annales de Mathématiques*, Volumes XV. and XVI., the work by Colonel Perronet Thompson entitled *Geometry without Axioms*, the article "Parallels" in the *English Cyclopœdia*, a memoir by Professor Baden Powell in the second volume of the *Memoirs of the Ashmolean Society*, an article by M. Bouniakofsky in the *Bulletin de l'Académie Impériale*, Volume V., St. Petersburg, 1863, articles in the volumes of the *Philosophical Magazine* for 1856 and 1857, and a dissertation entitled *Sur un point de l'histoire de la Géométrie chez les Grecs*, par A. J. H. Vincent, Paris, 1857.

I. 32. If two triangles have two angles of the one equal to two angles of the other, each to each, they shall also have their third angles equal. This is a very important result, which is often required in the *Elements*. The student should notice how this result is established on Euclid's principles. By Axioms 11 and 2 one pair of right angles is equal to any other pair of right angles. Then, by I. 32, the three angles of one triangle are together equal to the three angles of any other triangle. Then, by Axiom 2, the sum of the two angles of one triangle is equal to the sum of the two equal angles of the other; and then, by Axiom 3, the third angles are equal.

After I. 32 we can draw a straight line at right angles to a given straight line from its extremity, without producing the given straight line.



Let AB be the given straight line. It is required to draw from A a straight line at right angles to AB.

On AB describe the equilateral triangle ABC. Produce BC to D, so that CD may be equal to CB. Join AD. Then AD shall be at right angles to AB. For the $\angle CAD =$ the $\angle CDA$, and the $\angle CAB =$ the $\angle CBA$, by I. 5. $\therefore$ the $\angle BAD =$ the two angles ABD, BDA, by Axiom 2. Therefore the $\angle BAD$ is a right $\angle$, by I. 32.

I. 35. The equality of the parallelograms in I. 35 is an equality of area, and not an identity of figure. Legendre proposed to use the word *equivalent* to express the equality of area, and to restrict the word *equal* to the case in which magnitudes admit of superposition and coincidence. This distinction, however, has not been generally adopted, probably because there are few cases in which any ambiguity can arise; in such cases we may say especially, *equal in area*, to prevent misconception.

Cresswell, in his *Treatise of Geometry*, has given a demonstration of I. 35, which shews that the parallelograms may be divided into pairs of pieces admitting of superposition and coincidence; see also his Preface, page x.

I. 38. An important case of I. 38 is that in which the triangles are on equal bases and have a *common* vertex.

I. 40. We may demonstrate I. 40 without adopting the indirect method. Join BD, CD. The triangles DBC and DEF are equal, by I. 38; the triangles ABC and DEF are equal, by hypothesis; therefore the triangles DBC and ABC are equal, by the first Axiom. Therefore AD is parallel to BC, by I. 39.

THE SECOND BOOK.

The second book is devoted to the investigation of relations between the rectangles contained by straight lines divided into segments in various ways.

II. 2 and II. 3 are particular cases of II. 1.

II. 11. The student should notice that II. 11 gives a geometrical construction for the solution of a particular quadratic equation.

For if $AB = a$, and AH, the part to be found, $= x$, then $HB = a - x$, and the relation given, namely, rect. AB, BH = sq. on AH, says that

$$a(a-x) = x^2; \qquad \therefore \; x^2 + ax = a^2,$$

that is,

$$x^2 + ax + \frac{a^2}{4} = a^2 + \frac{a^2}{4} = \frac{5a^2}{4}.$$

Taking the square root, we have

$$x + \frac{a}{2} = \pm\frac{\sqrt{5}}{2}a, \text{ that is, } x = \frac{\sqrt{5}-1}{2}a \text{ or } -\frac{\sqrt{5}+1}{2}a.$$

The first value of x corresponds to the position of H found in the proposition. The second value corresponds to the position found in the first exercise following the proposition.

II. 12, II. 13. These are interesting in connection with I. 47; and, as the student may see hereafter, they are of great importance in Trigonometry; they are however not required in any of the parts of Euclid's *Elements* which are usually read. The converse of I. 47 is proved in I. 48; and we can easily shew that converses of II. 12 and II. 13 are true.

Take the following, which is the converse of II. 12: *if the square described on one side of a triangle be greater than the sum of the squares described on the other two sides, the angle opposite to the first side is obtuse.*

For the angle cannot be a right angle, since the square described on the first side would then be equal to the sum of the squares described on the other two sides, by I. 47; and the angle cannot be acute, since the square described on the first side would then be less than the sum of the squares described on the other two sides, by II. 13; therefore the angle must be obtuse.

Similarly, we may demonstrate the following, which is the converse of II. 13: *if the square described on one side of a triangle be less than the sum of the squares described on the other two sides, the angle opposite to the first side is acute.*

II. 14. This is not required in any of the parts of Euclid's *Elements* which are usually read; it is included in VI. 22.

THE THIRD BOOK.

The third book of the *Elements* is devoted to properties of circles.

III. 1. In the construction, DC is said to be *produced* to E; this assumes that D is within the circle, which Euclid demonstrates in III. 2.

III. 3. This consists of two parts, each of which is the converse of the other; and the whole proposition is the converse of the corollary in III. 1.

The following proposition is analogous to III. 7 and III. 8.

If any point be taken on the circumference of a circle, of all the straight lines which can be drawn from it to the circumference, the greatest is that in which the centre is; and of any others, that which is nearer to the straight line which passes through the centre is always greater than one more remote; and from the same point there can be drawn to the circumference two straight lines, and only two, which are equal to one another, one on each side of the greatest line.

The first two parts of this proposition are contained in III. 15; all three parts might be demonstrated in the manner of III. 7.

III. 9. Euclid has given three demonstrations of III. 9, of which Simson has chosen the second. Euclid's other demonstration is as follows. Join D with the middle point of the straight line AB; then it may be shewn that this straight line is at right angles to AB; and therefore the centre of the circle must lie in this straight line, by III. 1, Corollary. In the same manner it may be shewn that the centre of the circle must lie in the straight line which joins D with the middle point of the straight line BC. The centre of the circle must therefore be at D, because two straight lines cannot have more than one common point.

III. 10. Euclid has given two demonstrations of III. 10, of which Simson has chosen the second. Euclid's first demonstration resembles his first demonstration of III. 9. He shews that the centre of each circle is on the straight line which joins K with the middle point of the straight line BG, and also on the straight line which joins K with the middle point of the straight line BH; therefore K must be the centre of each circle.

The demonstration which Simson has chosen requires some additions to make it complete. For the point K might be supposed to fall *without* the circle DEF, or *on* its circumference, or *within* it; and of these three suppositions Euclid only considers the last. If the point K be supposed to fall *without* the circle DEF we obtain a contradiction of III. 8; which is absurd. If the point K be supposed to fall *on* the circumference of the circle DEF we obtain a contradiction of the proposition which we have enunciated at the end of the note on III. 7 and III. 8; which is absurd.

What is demonstrated in III. 10 is that the circumference of two circles cannot have more than *two* common points; there is nothing in the demonstration which assumes that the circles *cut* one another, but the ennunciation refers to this case only because it is shewn in III. 13 that if two circles *touch* one another, their circumference cannot have more than *one* common point.

III. 11 may be deduced from III. 7. For GH is the least line that can be drawn from G to the circumference of the circle whose centre is F, by III. 7. Therefore GH is less than GA, that is, less than GD; which is absurd. Similarly, III. 12 may be deduced from III. 8.

III. 18. It does not appear that III. 18 adds anything to what we have already obtained in III. 16. For in III. 16 it is shewn that there is only one straight line which touches a given circle at a given point, and that the angle between this straight line and the radius drawn to the point of contact is a right angle.

III. 20. An important extension may be given to III. 20 by introducing angles greater than two right angles. For, in the first figure, suppose we draw the straight lines BF and CF. Then, the ∠BEA is double of the angle ∠BFA, and the ∠CEA is double of the ∠CFA;

∴ the sum of the angles BEA and CEA is double of the ∠BFC. The sum of the angles BEA and CEA is greater than two right angles; we will call the sum the *re-entrant* ∠BEC. Thus the re-entrant ∠BEC is double of the ∠BFC. (See note on I. 32.) If this extension be used some of the demonstrations in the third book may be abbreviated. Thus III. 21 may be demonstrated without making two cases; III. 22 will follow immediately from the fact that the sum of the angles at the centre is equal to four right angles; and III. 31 will follow immediately from III. 20.

III. 21. In III. 21 Euclid himself has given only the first case; the second case has been added by Simson and others. In either of the figures of III. 21 if a point be taken on the same side of BD as A, the angle contained by the straight lines which join this point to the extremities of BD is *greater* or *less* than the angle BAD, according as the point is *within* or *without* the circle BAD; this follows from I. 21.

III. 32. The converse of III. 32 is true and important; namely, *if a straight line meet a circle, and from the point of meeting a straight line be drawn cutting the circle, and the angle between the two straight lines be equal to the angle in the alternate segment of the circle, the straight line which meets the circle shall touch the circle.*

This may be demonstrated indirectly. For, if possible, suppose that the straight line which meets the circle does not touch it. Draw through the point of meeting a straight line to touch the circle. Then, by III. 32 and the hypothesis, it will follow that two different straight lines pass through the same point, and make the same angle, on the same side, with a third straight line which also passes through that point; but this is impossible.

III. 35, III. 36. The following proposition constitutes a large part of the demonstrations of III. 35 and III. 36. *If any point be taken in the base, or the base produced, of an isosceles triangle, the rectangle contained by the segments of the base is equal to the difference of the square on the straight line joining this point to the vertex and the square on the side of the triangle.*

This proposition is in fact demonstrated by Euclid, without using any property of the circle; if it were enunciated and demonstrated before III. 35 and III. 36 the demonstrations of these two propositions might be shortened and simplified.

THE FOURTH BOOK.

The fourth Book of the *Elements* consists entirely of problems. The first five propositions relate to triangles of any kind; the remaining propositions relate to polygons which have all their sides equal and all their angles equal.

IV. 3. We can also describe a triangle equiangular to a given triangle, and such that one of its sides and the other two sides produced shall touch a given circle. For, in the figure of IV. 3, suppose AK produced to meet the circle again; and at the point of intersection draw a straight line touching the circle; this straight line, with parts of NB and NC, will form a triangle, which will be equiangular to the triangle MLN, and therefore equiangular to the triangle EDF; and one of the sides of this triangle, and the other two sides produced, will touch the given circle.

It was first demonstrated by Gauss in 1801, in his *Disquisitiones Arithmeticæ*, that it is possible to describe geometrically a regular polygon of 2^n+1 sides, provided 2^n+1 be a prime number; the demonstration is not of an elementary character. As an example, it follows that a regular polygon of seventeen sides can be described geometrically; this example is discussed in Catalan's *Théorèmes et Problèmes de Géométrie Elémentaire.*

THE FIFTH BOOK.

The fifth Book of the *Elements* is on *Proportion.* Much has been written respecting Euclid's treatment of this subject; besides the Commentaries on the *Elements* to which we have already referred, the student may consult the articles "Ratio" and "Proportion" in the *English Cyclopædia*, and the tract on the *Connexion of Number and Magnitude* by Professor De Morgan.

The fifth Book relates not merely to length and space, but to any kind of magnitude of which we can form multiples.

V. Def. 3. Simson considers that the definitions 3 and 8 are "not Euclid's, but added by some unskilful editor." Other commentators also have rejected these definitions as useless. The last word of the third definition should be *quantuplicity*, not *quantity*; so that the definition indicates that ratio refers to the *number of times* which one magnitude contains another. See De Morgan's *Differential and Integral Calculus*, page 18.

Compound Ratio. The definition of compound ratio was supplied by Simson. The Greek text does not give any definition of compound ratio here, but gives one as the fifth definition of the sixth Book, which Simson rejects as absurd and useless.

V. Defs. 18, 19, 20. The definitions 18, 19, 20 are not presented by Simson precisely as they stand in the original. The last sentence in definition 18 was supplied by Simson. Euclid does not connect definitions 19 and 20 with definition 18. In 19 he defines *ordinate proportion*, and in 20 he defines *perturbate proportion*. Nothing would be lost if Euclid's definition 18 were entirely omitted, and the term *ex æquali* never employed. Euclid employs such a term in the enunciations of V. 20, 21, 22, 23; but it seems quite useless, and is accordingly neglected by Simson and others in their translations.

THE SIXTH BOOK.

The sixth Book of the *Elements* consists of the application of the theory of proportion to establish properties of geometrical figures.

VI. Def. 3. The third definition is useless, for Euclid makes no mention of reciprocal figures.

VI. 2. The enunciation of this important proposition is open to objection, for the manner in which the sides may be cut is not sufficiently limited. Suppose, for example, that AD is double of DB, and CE double of EA; the sides are then cut proportionally, for each side is divided into two parts, one of which is double of the other; but DE is not parallel to BC. It should therefore be stated in the enunciation that *the segments terminated at the vertex of the triangle are to be homologous terms in the ratios, that is, are to be the antecedents or the consequents of the ratios.*

VI. A. This proposition was supplied by Simson.

VI. 4. By superposition we might deduce VI. 4 immediately from VI. 2.

VI. 10. The most important case of this proposition is that in which a straight line is to be divided either *internally* or *externally* into two parts which shall be in a given ratio.

The case in which the straight line is to be divided *internally* is given in the text; suppose, for example, that the given ratio is that of AE to EC; then AB is divided at G in the given ratio.

Suppose, however, that AB is to be divided *externally* in a given ratio; that is, suppose that AB is to be produced so that the whole straight line made up of AB and the part produced may be to the part produced in a given ratio. Let the given ratio be that of AC to CE. Join EB; through C draw a straight line parallel to EB; then this straight line will meet AB, produced through B, at the required point.

VI. 11. This is a particular case of VI. 12.

VI. 14. The following is a full exhibition of the steps which lead to the result that FB and BG are in one straight line.

The $\angle$DBF = the $\angle$GBE ; add to each the $\angle$FBE ;

$\therefore$ the $\angle^s$ GBE, FBE = the $\angle^s$ DBF, FBE = two rt. $\angle^s$.

$\therefore$ FB and BG are in one straight line. [I. 14.

VI. 15. This may be inferred from VI. 14, since a triangle is half of a parallelogram with the same base and altitude.

It is not difficult to establish a third proposition conversely connected with the two involved in VI. 14, and a third proposition similarly conversely connected with the two involved in VI. 15. These propositions are the following.

Equal parallelograms which have their sides reciprocally proportional, have their angles equal, each to each.

Equal triangles which have the sides about a pair of angles reciprocally proportional, have also angles equal or together equal to two right angles.

We will take the latter proposition.

Let ABC, ADE be equal triangles ; and let CA be to AD as AE is to AB : either the angle BAC shall be equal to the angle DAE, or the angles BAC and DAE shall be together equal to two right angles.

[The student can construct the figure for himself.]

Place the triangles so that CA and AD may be in one straight line ; then if EA and AB are in one straight line the $\angle$BAC = $\angle$DAE, [I. 15. If EA and AB are not in one straight line, produce BA through A to F, so that AF may = AE ; join DF and EF.

Then because CA is to AD as AE to AB, [*Hypothesis.*
and AF = AE, [*Construction.*
$\therefore$ CA is to AD as AF to AB.
$\therefore$ the $\triangle$DAF = the $\triangle$BAC = $\triangle$DAE. [VI. 15 and *Ax.* 1.
$\therefore$ EF is parallel to AD. [I. 39.

Suppose now that the $\angle$DAE > the $\angle$DAF.

Then the $\angle$CAE = the $\angle$AEF = $\angle$AFE = $\angle$BAC. [I. 29.
$\therefore$ the angles BAC and DAE together = two right angles. [I. 29 and 5.

Similarly, if the $\angle$DAE is less than the $\angle$DAF.

VI. 16. This is a particular case of VI. 14.

VI. 17. This is a particular case of VI. 16.

VI. 22. There is a step in the second part of VI. 22 which requires examination. After it has been shewn that the figure SR is equal to the similar and similarly situated figure NH, it is added "therefore PR is equal to GH." In the Greek text reference is here made to a *lemma* which follows the proposition. The word *lemma* is occasionally used in mathematics to denote an auxiliary proposition. From the unusual circumstance of a reference to something following, Simson probably concluded that the lemma could not be Euclid's, and accordingly he takes no notice of it. The following is the substance of the lemma.

If PR be not equal to GH, one of them must be greater than the other; suppose PR greater than GH.

Then, because SR and NH are similar figures, PR is to PS as GH is to GN. [VI. *Definition* 1.

But PR > GH; ∴ PS > GN. [*Hypothesis.*

∴ the △ RPS > the △ HGN. [I. 4, *Axiom* 9.

But, because SR and NH are similar, the △ RPS = the △ HGN; [VI. 20.

which is impossible. ∴ PR = GH.

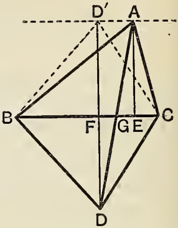
VI. 23. In the figure of VI. 23 suppose BD and GE drawn. Then the △ BCD is to the △ GCE as AC to the ||^m CF. Hence the result may be extended to triangles, and we have the following theorem; *Triangles which have one angle of the one equal to one angle of the other have to one another the ratio which is compounded of the ratios of their sides.*

Then VI. 19 is an immediate consequence of this theorem. For let ABC and DEF be similar triangles, so that AB is to BC as DE to EF; and therefore, alternately, AB is to DE as BC to EF. Then by the theorem, the △ ABC has to the △ DEF the ratio which is compounded of the ratios of AB to DE and of BC to EF, that is, the ratio which is compounded of the ratios of BC to EF and of BC to EF. And, from the definitions of duplicate ratio and of compound ratio (V. Def. 11), it follows that the ratio compounded of the ratios of BC to EF and of BC to EF is the duplicate ratio of BC to EF.

We have omitted in the sixth Book Propositions 27, 28, 29, and the first solution which Euclid gives of Proposition 30, as they appear now to be never required, and have been condemned as useless by various modern commentators. Some idea of the nature of these propositions may be obtained from the following statement of the problem proposed by Euclid in VI. 29. AB is a given straight line; it has to be produced through B to a point O, and a parallelogram described on AO subject to the following conditions; the parallelogram is to be equal to a given rectilineal figure, and the parallelogram on the base BO which can be cut off by a straight line through B is to be similar to a given parallelogram.

VI. 32. This proposition seems of no use. Moreover the enunciation is imperfect. For suppose ED to be produced through D to a point F, such that DF is equal to DE; and join CF. Then the triangle CDF will satisfy all the conditions in Euclid's enunciation, as well as the triangle CDE; but CF and CB are not in one straight line. It

2. *Two equal triangles are on the same base and on opposite sides of it; the straight line joining their vertices is bisected by the base, or the base produced.*



Let ABC, DBC be the equal triangles, and let AD meet BC in G.

Then shall AG = GD.

Draw AE, DF ⊥ʳ to BC and produce DF to D′ making FD′ = DF. Then from the △ˢ DCF, D′CF it is easily seen that D′C = CD; so D′B = BD.

∴ the △ˢ DBC, D′BC are equal in all respects.

∴ area △D′BC = area △DBC = area △ABC. [*Hypothesis.*

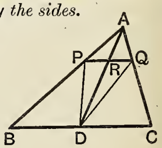
∴ AD′, BC are parallel (I. 39), and ∴ AEFD′ is a ‖ᵍᵐ. ∴ AE = D′F = FD. [I. 34.

∴ in the △ˢ AGE, DGF we have ∠AGE = ∠DGF, rt. ∠AEG = rt. ∠DFG, and side AE = FD. ∴ AG = GD. [Q.E.D.

The same proof holds if AD cuts BC produced.

Corollary. It follows that all triangles with equal areas and equal bases have equal heights.

3. *Any medium of a triangle bisects all lines which are parallel to the base to which it is drawn and which are intercepted by the sides.*



Let PQ, ‖ˡ to BC, meet the median AD through A in R.

Then shall PR = RQ.

Since BD = DC, ∴, by I. 38, △ABD = △ADC, and △PBD = △QDC.

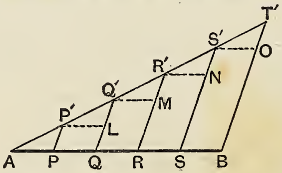
∴, by subtraction, △APD = △AQD.

i.e. the △ˢ APD, AQD on the same base AD are equal.

∴, by the previous theorem, PQ is bisected by AD.

Also PQ is any straight line ‖ˡ to the base. ∴ etc. [Q.E.D.

4. *To divide a given straight line into any number of given parts.*



Let AB be the given straight line, and let it be required to divide it into five given parts.

Through A draw a straight line AT′, making any angle with AB.

On it take any point P′ and cut off portions P′Q′, Q′R′, R′S′, S′T′ each equal to AP′.

Join T′B, and draw P′P, Q′Q, R′R, S′S all ∥ˡ to T′B to meet AB in P, Q, R, S respectively.

Then AB shall be divided at P, Q, R, S as required.

Draw P′L, Q′M, R′N, S′O ∥ˡ to AB to meet Q′Q, R′R, S′S, and T′B in L, M, N, O.

Then ∠Q′LP′ = ∠LQP = ∠P′PA,

and ∠Q′P′L = ∠P′AP. [I. 29.

∴ since AP′ = P′Q′, the △ˢ APP′, P′LQ are equal in all respects.

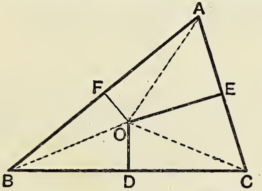
Similarly with the △ˢ Q′MR′, R′NS′, S′OT′.

∴ AP = P′L = Q′M = R′N = S′O.

∴ AP = PQ = QR = RS = SB. [I. 34.

The method is the same whatever be the number of parts into which AB is to be divided.

5. *The straight lines drawn at right angles to the sides of a triangle from the points of bisection of the sides meet in a point.*



Let ABC be a △; bisect BC, CA, and AB at D, E, F respectively. Draw DO perpʳ to BC and EO perpʳ to CA, and let them meet in O. Join OA, OB, OC, OF.

In the △ˢ BDO, CDO we have BD = CD, DO common, and the rt. ∠BDO = the rt. ∠CDO; [I. 4.

∴ BO = CO.

Similarly, from the △ˢ CEO, AEO we have CO = AO;

∴ AO = BO.

Thus the △ˢ AFO, BFO have their sides respectively equal;

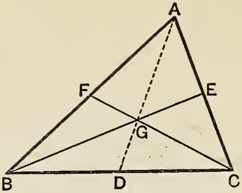
∴ the ∠ˢ AFO, BFO are equal, and ∴ each is a rt. ∠; [I. 8 and *Def.* 10.

∴ FO bisects AB at rt. ∠ˢ; ∴ etc.

Corollary. Since AO = BO = CO, the point O is equidistant from the angular points of the △.

When three straight lines, such as OA, OB, OC (or DO, EO, FO), meet in a point they are said to be **concurrent**.

6. *The straight lines drawn from the angles of a △ to bisect the opposite sides meet in a point.*



Let ABC be a △; D, E, F the middle points of its sides; let BE, CF meet in G.

Join AG, GD; *they shall be in the same straight line.*

The △BEA = △BEC, and △GEA = △GEC; [I. 38.

∴ by subtraction, △BGA = △BGC.

Similarly, △CGA = △BGC;

∴ △BGA = △CGA.

Also △BGD = △CGD; [I. 38.

∴ the △s BGA, BGD together = △s CGA, CGD;

∴ the △s BGA, BGD together = half the △ABC

= △BDA. [I. 38.

∴ G must fall on the straight line AD, otherwise the whole would be equal to its part;

that is, AG, GD are in one straight line. [I. 38.

7. *To prove* AG = 2 GD, BG = 2 GE, *and* CG = 2 GF.

By the previous article, △AGB = △BGC,

and △BGC = sum of △s BGD, CGD = twice △BGD. [I. 38.

∴ △AGB = twice △BGD.

If K be the middle point of AG, we then have, by I. 38,

△ABK = △KBG = △GBD;

∴ AK = KG = GD; ∴ AG = 2 GD.

Similarly BG = 2 GE, and CG = 2 GF.

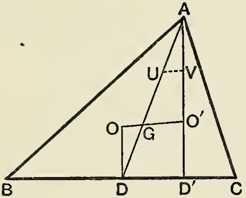
This point G is called the **Centroid** of the △ABC, and AD, BE, CF are the **Medians.**

8. *The perpendiculars drawn from the vertices of a triangle upon the opposite sides meet in a point.*

Let ABC be a △, O the point in which meet the perp^rs to the sides through the middle points, G the point in which meet the straight lines joining the vertices to opposite sides. [Arts. 5, 6.

Let D be the middle point of BC and AD′ the perp^r from A on BC.

Let OG meet AD′ in O′, and let U, V be the middle points of AG, AO′.



Then UV is parallel to, and one-half of, GO′. [Art. 1.

Because AD′, OD are parallel, both being perp^r to BC, [I. 28.

$\therefore\ \angle ADO = \angle UAV.$ [I. 29.

Also, since UV, GO′ are parallel, $\therefore\ \angle AUV = \angle AGO' = \angle OGD.$

Also, $AU = \frac{1}{2} AG = GD$; [Art. 7.

$\therefore$ the △^s AUV, DGO are equal in all respects; [I. 4.

$\therefore\ OG = UV = \frac{1}{2} GO'; \quad \therefore\ GO' = 2\,OG.$

The point in which OG meets the perp^r from A on BC is thus a fixed point on OG produced; similarly the perp^rs from B on CA, and from C on AB pass through this same fixed point O′;

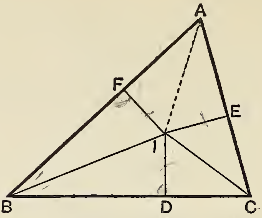
Hence the three perpendiculars from A, B, and C, upon the opposite sides meet in a point O′ which lies on OG produced, and is such that $GO' = 2\,OG$.

This point O′ is called the **Orthocentre.**

Corollary. Since the triangles AUV, DGO are equal,

$$\therefore\ OD = AV = \tfrac{1}{2} AO';$$
$$\therefore\ AO' = 2\,OD.$$

9. *The straight lines which bisect the angles of a triangle meet in a point.*



Let ABC be a triangle; bisect the angles at B and C by straight lines meeting at I; join AI:
then AI *shall bisect the angle at* A.

From I draw ID perpendicular to BC, IE perpendicular to CA, and IF perpendicular to AB.

In the $\triangle^s$ BIF, BID we have BI common, $\angle$DBI$=\angle$FBI, and the rt. $\angle$BDI$=$rt. $\angle$BFI; [*Construction.*

$\therefore$ IF$=$ID. [I. 26.

Similarly from the $\triangle^s$ CID, CIE we have ID$=$IE;

$\therefore$ IE$=$IF.

Then since E, F are right $\angle^s$, we have $AF^2+FI^2=AI^2=AE^2+IE^2$. [I. 47.

$\therefore$ AF$=$AE, and the $\triangle^s$ AEI, AFI are equal in all respects;

$\therefore$ $\angle$IAE$=\angle$IAF.

Corollary. Since ID$=$IE$=$IF, the point I is such that its perpendicular distances from the three sides of the $\triangle$ are equal.

10. *Let two sides of a triangle be produced through the base; then the straight lines which bisect the two exterior angles thus formed, and the straight line which bisects the vertical angle of the triangle, meet at the same point.*

This may be shewn in the same manner as the last theorem.

11. *If two medians of a triangle be equal, the angles through which they are drawn shall be equal.*

In the figure of Art. 6, let BE$=$CF. Then, by Art. 7, GB$=$GC. Hence, since BD$=$DC, the $\triangle^s$ GDB, GDC have their sides respectively equal.

Hence they are equal in all respects,

and $\therefore \angle GDB = \angle GDC$. [I. 8.

$\therefore$ GDB, GDC are both right $\angle^s$. [*Def.* 10.

Hence from the $\triangle^s$ ADB, ADC we have

AB = AC. [Ax. 11, and I. 4.

$\therefore \angle ABC = \angle ACB$. [I. 5.

EXERCISES.

1. The sides AB, AC of a triangle ABC are bisected at D, E; BE and CD are drawn and produced to F and G so that EF = BE and DG = DC; prove that AF and AG are in the same straight line.

[Use Art. 1.]

2. The middle points of the sides BC, CA, and AB of a triangle are D, E, F; FG is drawn parallel to BE to meet DE in G. Prove that the sides of the triangle CFG are equal to the medians of the triangle ABC, and hence that the sum of any two medians of a triangle is always greater than the third.

3. Prove that the sum of the medians of a triangle is less than the perimeter, but greater than three quarters of the perimeter, of the triangle.

[$GB + GC > BC$, *i.e.* $\frac{2}{3}BE + \frac{2}{3}CF > BC$, etc.

Also $AD < AE + ED$, *i.e.* $< \frac{1}{2}AC + \frac{1}{2}AB$, etc. (Fig. Art. 6)].

4. An angle of a triangle is right, acute, or obtuse according as the median drawn through it is =, >, or < half the side it bisects.

5. If one side of a triangle be longer than another, the corresponding median is shorter.

[Apply I. 25 to the $\triangle^s$ ADB, ADC (Fig. Art. 6), and then I. 24 to the $\triangle^s$ GDB, GDC.]

6. Of the two angles formed by a median with the two adjacent sides, that made with the shorter side is the greater.

7. If G be the centroid of a triangle ABC, the triangle BGC is equal in area to the quadrilateral AEGF. [Fig. Art. 6.]

ON GEOMETRICAL ANALYSIS.

12. The substantives *analysis* and *synthesis*, and the corresponding adjectives *analytical* and *synthetical*, are of frequent occurrence in mathematics. In general *analysis* means decomposition, or the separating a whole into its parts, and *synthesis* means composition, or making a whole out of its parts. In Geometry, however, these words are used in a more special sense. In **synthesis** we begin with results already established, and end with some new result; thus, by the aid of theorems already demonstrated, and problems already solved, we demonstrate some new theorem, or solve some new problem. In **analysis** we begin with assuming the truth of some theorem or the solution of some problem, and we deduce from the assumption consequences which we can compare with results already established, and thus test the validity of our assumption.

13. The propositions in Euclid's Elements are all exhibited synthetically; the student is only employed in examining the soundness of the reasoning by which each successive addition is made to the collection of geometrical truths already obtained; and there is no hint given as to the manner in which the propositions were originally discovered. Some of the constructions and demonstrations appear rather artificial, and we are thus naturally induced to enquire whether any rules can be discovered by which we may be guided easily and naturally to the investigation of new propositions.

14. Geometrical analysis has sometimes been described in language which might lead to the expectation that directions could be given which would enable a student to proceed to the demonstration of any proposed theorem, or the solution of any proposed problem, with confidence of success; but no such directions can be given. We will state the exact extent of these directions. Suppose that a new theorem is proposed for investigation, or a new problem for trial. Assume the truth of the theorem or the solution of the problem, and deduce consequences from this assumption combined with results which have been already established. If a consequence can be deduced which contradicts some result already established, this amounts to a demonstration that our assumption is inadmissible; that is, the theorem is not true, or the problem cannot be solved. If a consequence can be deduced which coincides with some result already established, we cannot say that the assumption is inadmissible; and it *may happen* that by starting from the consequence which we deduced, and retracing our steps, we can succeed in giving a synthetical demonstration of the theorem, or solution of the problem. These directions, however, are very vague, because no certain rule can be prescribed by which we are to combine our assumption with results already established; and moreover no test exists by which we can ascertain whether a valid consequence which we have drawn from an assumption will enable us to establish the assumption itself. That a proposition may be false and yet furnish consequences which are true, can be seen from a simple example. Suppose a theorem were proposed for investigation in the following words; *one angle of a triangle is to another as the side opposite to the first angle is to the side opposite to the other.* If this be assumed to be true we can immediately deduce Euclid's result in I. 19; but from Euclid's result in I. 19 we cannot retrace our steps and establish the proposed theorem, and in fact the proposed theorem is false.

Thus the only definite statement in the directions respecting geometrical analysis is, that if a consequence can be deduced from an assumed proposition which contradicts a result already established, that assumed proposition must be false.

15. We may mention, in particular, that a consequence would contradict results already established, if we could shew that it would lead to the solution of a problem already given up as impossible. There are three famous problems which are now admitted to be beyond the power of Geometry; namely:

To find a straight line equal in length to the circumference of a given circle,

To trisect any given angle, and

To find two mean proportionals between two given straight lines.

The grounds on which the geometrical solution of these problems is admitted to be impossible cannot be explained without a knowledge of the higher parts of mathematics; the student may, however, be content with the fact that innumerable attempts have been made to obtain solutions, and that these attempts have been made in vain.

The first of these problems is usually referred to as the *Quadrature of the Circle.* For the history of it the student should consult the article in the *English Cyclopædia* under that head, and also a series of papers in the *Athenæum* for 1863 and subsequent years, entitled a *Budget of Paradoxes*, by Professor De Morgan.

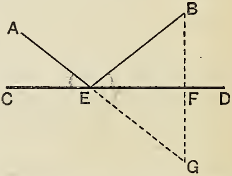
The third of the three problems is often referred to as the *Duplication of the Cube.*

We will now give some examples of geometrical analysis.

16. *From two given points it is required to draw to the same point in a given straight line two straight lines equally inclined to the given straight line.*

Let A and B be the given points, and CD the given straight line.

Suppose AE and EB to be the two straight lines equally inclined to CD. Draw BF perpendicular to CD, and produce AE and BF to meet at G.



Then the ∠BED = the ∠AEC, by hypothesis; and

the ∠AEC = the ∠DEG. [I. 15.

Hence the △s BEF and GEF are equal in all respects; [I. 26.

∴ FG = FB.

This result gives the required

Construction. Draw BF perpendicular to CD, and produce it to G, so that FG may be equal to FB; then join AG, and AG will intersect CD at the required point.

For the △s BEF, GEF are equal in all respects, and

∴ ∠BEF = ∠GEF = ∠AEC.

If A be on the opposite side of CD from B, join GA, and let it be produced to meet CD in K. Then AK, BK are equally inclined to BD.

17. *To divide a given straight line into two parts such that the difference of the squares on the parts may be equal to a given square.*

Let AB be the given straight line, and suppose C the required point.

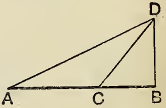
Then $AC^2 - CB^2$ = given sq.; ∴ $AC^2 = CB^2$ + given square.

Erect at B a perpendicular BD, so that

BD^2 = given sq.

∴ $AC^2 = CB^2 + BD^2 = CD^2$; [I. 47.

∴ AC = CD, and the △ACD is isosceles.



The ∠CAD therefore = the ∠ADC.

Hence we have the following

Construction. At B draw BD ⊥r to AB and equal to the side of the given sq. Join AD, and make the ∠ADC equal to the ∠BAD, and let DC meet AB in C. Then C is the required point.

For $AC^2 = CD^2 = CB^2 + BD^2$. [I. 47.

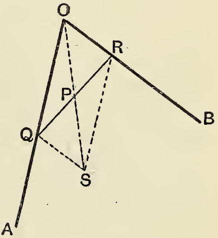
It is obvious that the given square must not exceed the square on AB, in order that the problem may be possible.

There are two positions of C, if it is not specified which of the two segments AC and CB is to be greater than the other; but only one position, if it is specified.

In like manner we may solve the problem, *to produce a given straight line so that the square on the whole straight line made up of the given straight line and the part produced, may exceed the square on the part produced by a given square, which is not less than the square on the given straight line.* For, in this case, BD > BA; ∴ ∠BDA < ∠BAD, and DC therefore meets AB produced.

The two problems may be combined in one enunciation thus, *to divide a given straight line internally or externally so that the difference of the squares on the segments may be equal to a given square.*

18. *Given two intersecting straight lines,* OA *and* OB, *and any point* P *between them; to draw through* P *a straight line such that the portion of it intersected between* OA *and* OB *is bisected at* P.



Let QPR be the required line. Join OP and produce to S so that OP = PS. Then QR and OS bisect one another at P, and this is a property of the diagonals of a ||m. Hence

Construction. Join OP and produce it to S so that OP = PS. Draw ||ls SQ, SR as in the figure. Then SQOR is a ||m;

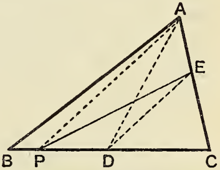
∴ SO, QR bisect one another.

But P is the middle point of OS.

∴ P is the middle point of QR.

∴ QPR is drawn as required.

19. *Bisect a triangle* ABC *by a straight line drawn through a given point* P *in the base* BC.



Let PE be the required straight line.

Then area ABPE = half the △ ABC.

But, if D be the middle point of BC,

△ ADB = half the △ ABC. [I. 38.

∴ ABPE = △ ADB.

∴ subtracting △ ABP from each, we have

△ AEP = △ ADP.

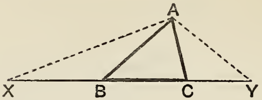
∴ AP, DE are parallel. [I. 39.

Hence the construction required: *Join* AP; *through* D, *the middle point of* BC, *draw* DE ||l *to* PA *to meet* AC *in* E; *then* PE *is the required straight line.*

For AEPB = △ AEP + △ ABP = △ ADP + △ ABP

= △ ADB = half of △ ABC. [I. 37.

20. *Construct a triangle, given its perimeter and its base angles.*



Suppose that ABC is the required △; produce CB to X and BC to Y, so that BX = BA and CY = CA. Then XY is equal to the given perimeter, and ∠BXA = ∠BAX, and ∴ ∠ABC = twice ∠AXY.

∴ ∠AXY = half ∠ABC, and similarly ∠AYX = half ∠ACB.

Hence the construction required; Let XY equal the given perimeter; at X make ∠YXA = half one given base ∠, and at Y make ∠XYA = half the other given base ∠. At A make ∠XAB = ∠AXB and ∠YAC = ∠AYC where AB, AC lie between AX, AY and meet XY in B, C. Clearly ABC is now the required △.

EXERCISES.

1. Construct an isosceles triangle having given the base and the perpendicular on it from the opposite angle.

Construct a right-angled triangle having given

2. the hypotenuse and an acute angle;

3. the hypotenuse and a side;

4. the hypotenuse and the sum of the remaining sides;

5. the hypotenuse and the difference of the other sides;

6. the hypotenuse and the perp^{r} on it from the right angle. [Use Page 60, Ex. 8.]

7. a side and the perpendicular on the hypotenuse from the right angle;

8. the perimeter and an acute angle. [Use Art. 20.]

9. Construct a triangle having given the middle points of its sides.

10. Construct a triangle, given its three medians.

[Let ABC be a △, G its centroid and AD the median through A; complete the ‖^{gm} BGCO; then GO is bisected at D and GCO is a △ having its sides respectively equal to two-thirds of the medians. Hence the construction.]

11. Construct a triangle, given the base, the difference of the angles at the base, and the difference of its sides.

[Let BC be the given base; make ∠BCD = $\frac{1}{2}$ the given diff. of the base ∠^{s}, and take D on CD such that BD = the given difference of the sides; produce BD to A, making ∠ACD = ∠ADC. Then ABC is the required △.]

12. Construct a triangle, given its base, one of its base angles, and (1) the sum, (2) the difference of its sides.

13. Bisect a parallelogram by a straight line drawn through a given point in its plane.

14. Trisect a right angle.

15. Bisect a quadrilateral by a straight line drawn through (1) an angular point, (2) a point in one of the sides.

[(1) Let ABCD be the quad[l], B being nearer to AC than D; through E, the middle point of BD draw a line ∥[l] to AC to meet DC in G; then AG bisects the quad[l]. (2) Let the given point be P in AB; draw DE ∥[l] to AC to meet BA in E; through F, the mid-point of BE, draw FQ ∥[l] to CP to meet CD in Q; then PQ bisects the quad[l].]

16. Construct a parallelogram, given one side and the two diagonals.

17. Trisect a given straight line. [On the given straight line AB describe an equilateral △ ABC, and through its centroid G draw GX, GY ∥[l] to AC, CB to meet AB in X, Y.]

18 Trisect a triangle by a straight line through an angular point.

[Use Ex. 17.]

19. Trisect a triangle by a straight line drawn through a point in one side. [Let P be the point in the side BC, and X, Y the points of trisection of BC (Ex 17); draw XR, YS ∥[l] to AP to meet the sides in R and S; then, as in Art. 19, PR and PS divide the △ as required.]

20. On the side AB of a parallelogram ABCD describe a triangle equal in area to ABCD and having the ∠ at A common.

21. Construct a triangle of given area with two sides of given length.

22. Inscribe a square of given magnitude in a given square.

23. Trisect a parallelogram by straight lines drawn through one angular point. [By Ex. 17 trisect the sides of the parallelogram opposite the given angular point.]

24. Describe a △ equal to a given quadrilateral. [See Page 68, Ex. 2.]

25. Describe a △ equal in area to a given rectilineal figure. [Use Ex. 24, or see page 79.]

ON LOCI.

21. A *locus* consists of all the points which satisfy certain conditions and of those points alone. Thus, for example, the locus of the points in a fixed plane which are at a given distance from a given point is the circumference of the circle described from the given point as centre, with the given distance as radius; for all the points on the circle, and no others, are at the given distance from the given point.

Again, the locus of all points which are at a given distance from a given straight line is one or other of the two straight lines parallel to the given straight line and at the given distance from the given straight line, one on one side and the other on the other side of it.

We shall restrict ourselves to loci which are situated in a fixed plane, and which are properly called *plane loci.*

Several of the propositions in Euclid furnish good examples of loci. Thus the locus of the vertices of all triangles which are on the same base and on the same side of it, and which have the same area, is a straight line parallel to the base; this is shewn in I. 37 and I. 39.

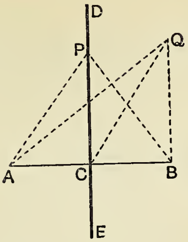
22. We will now give some examples. In each example we ought to shew not only that all the points which we indicate as the locus do fulfil the assigned conditions, but that no other points do. This second part however we shall generally leave to the student.

23. *Required the locus of points which are equidistant from two given points.*

Let A and B be the two given points. Join AB, and bisect it at C. Draw DCE at right angles to AB. Then DCE shall be the required locus. For take *any* point P on it and join PA and PB. Then in the $\triangle^s$ PCA, PCB we have PC common, AC=CB, and the right $\angle$PCA=the right $\angle$PCB;

$\therefore$ the base PA=the base PB, and $\therefore$ P is equidistant from A and B.

Also, no point out of DCE is equidistant from A and B. For, if



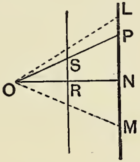
possible, let Q be such a point. Join QA, QC, QB. In the $\triangle^s$ QCA, QCB we then have AC = CB, QC common, and the base QA = the base QB;

$\therefore\ \angle QCA = \angle QCB$; [I. 8.

$\therefore$ each is a rt. $\angle$; $\therefore$ Q lies on the st. line DCE.

This line is thus the required locus.

24. *Find the locus of the middle points of the straight lines drawn from a fixed point* O *to all points on a given straight line.*



Let LM be the given straight line. Draw ON perpendicular to it, and bisect ON at R. Take *any* point P on LM; join OP, and bisect it in S. Join RS. Since R, S are the middle points of the sides ON, OP; $\therefore$ RS is parallel to NP; [Art. 1.

$\therefore\ \angle ORS = \angle ONP =$ a right $\angle$; [I. 29.

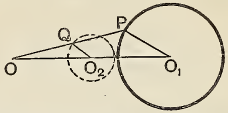
$\therefore$ S lies on the straight line bisecting ON at right angles.

Similarly, the middle point of the straight line joining O to any other point of LM lies on RS;

$\therefore$ the required locus is the straight line bisecting ON at right angles; it is therefore $\|^l$ to LM.

25. *The locus of the middle point of the straight lines drawn from a fixed point* O *to all points on a given circle is another circle.*

Let the given circle have O_1 as centre. Join O to any point P on the circle. Bisect OP in Q and OO_1 in O_2. Join O_2Q.



Since O_2, Q are the middle points of OO_1 and OP;

$\therefore$ O_2Q is $\|^l$ to and one half of O_1P;

$\therefore$ O_2Q is constant; also O_2 is a fixed point; [Constr.

$\therefore$ Q always lies on a fixed circle whose centre is O_2, and whose radius is one-half that of the given circle.

26. *Required the locus of the vertices of all triangles on a given base* AB, *such that the square on the side terminated at* A *may exceed the square on the side terminated at* B, *by a given square.*

Suppose C to denote a point on the required locus; from C draw CD perpendicular to AB, produced if necessary.

Then sq. on AC = sqs. on AD, DC,

and sq. on BC = sqs. on BD, DC; [I. 4.

$\therefore$ sq. on AC − sq. on BC = sq. on AD − sq. on BD;

that is, given sq. = sq. on AD − sq. on BD;

$\therefore$ D is a fixed point either in AB or in AB produced through B. [Art. 17.

Also CDB is a right $\angle$.

The required locus is the straight line drawn through the known point D, at right angles to AB.

EXERCISES ON LOCI (BOOK I.).

1. The locus of points equally distant from two given intersecting straight lines is one or other of two fixed straight lines.

2. What is the locus of points equidistant from two parallel straight lines?

3. Find the locus of the middle points of all straight lines parallel to the base of a triangle and terminated by the sides.

4. From any point on the base of a triangle straight lines are drawn parallel to the sides; the locus of the point of intersection of the diagonals of the $\|^{gms}$ so formed is a straight line parallel to the base.

5. The locus of all points, the sum of whose perpendicular distances from two given perpendicular straight lines is given, is a straight line which is equally inclined to the two given lines.

6. Find the locus of all points the difference of whose perpendicular distances from two given straight lines is given.

7. Find the locus of a point P such that the sum of the areas of the triangles PAB, PCD with given bases AB, CD is constant.

[Let AB, CD meet in O; take E, F on OA, OC so that OE=AB and OF=CD. Then $\triangle$PAB = $\triangle$POE and $\triangle$PCD = $\triangle$POF;

$\therefore$ area OEPF is const. But area OEF is the same for all positions of P:

$\therefore$ $\triangle$EPF is of constant area;

$\therefore$ locus of P is a straight line parallel to EF by I. 39.]

8. AB and CD are two straight lines given in magnitude and position, and a point P moves so that the $\triangle^s$ PAB, PCD are equal in area; the locus of P is a straight line passing through the intersection of AB and CD, produced if necessary.

[As in the previous solution we have $\triangle$OPE = $\triangle$OFP. Hence if OP meet EF in N, we have EN=FN (Art. 2); $\therefore$ P lies on the straight line joining O and the middle point of EF.]

9. Find the locus of the points of intersection of the diagonals of a parallelogram whose base and area are given.

10. Find the locus of the intersection of the medians of all triangles whose base and area are given.

11. The locus of the vertices of all right-angled triangles described upon a given straight line as hypotenuse is a circle.

12. A ladder is raised gradually, its ends being always in contact with a vertical wall and the ground respectively; the locus of its centre is a portion of a circle.

13. The locus of the vertex A of a triangle ABC, given the base BC and the length of the median through B, is a circle.

[If BD be the median through D the locus of D is a circle, by hypothesis, and CA always=twice CD; then proceed as in Art. 25.]

14. Find a point in a given straight line which is equidistant from (1) two given points, and (2) two given straight lines.

15. Find a point which is equidistant from three points which are not in the same straight line.

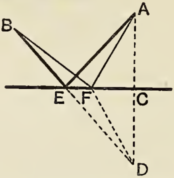
16. If two opposite ends of a square of given magnitude move along two straight lines which are at right angles, then the remaining angles move along two perpendicular straight lines.

17. Construct a triangle having given the base, the altitude, and the length of the median which bisects the base.

18. Construct an isosceles triangle which has the same base and the same area as a given triangle.

MAXIMA AND MINIMA.

27. *In a given indefinite straight line it is required to find a point such that the sum of its distances from two given points on the same side of the straight line shall be the least possible.*



Let A and B be the two given points. From A draw AC perpendicular to the given straight line, and produce AC to D so that CD = AC. Join DB meeting the given straight line at E.

Then E *shall be the required point.*

For let F be any other point in the given straight line. Then, because AC = DC, and EC is common to the two $\triangle$s ACE, DCE, and that the right $\angle$ACE = the right $\angle$DCE, $\therefore$ AE = DE. [I. 4.

Similarly, AF = DF. Also the sum of DF and FB > BD. [I. 20.

$\therefore$ the sum of AF and FB > BD;

that is, the sum of AF and FB > the sum of AE and EB.

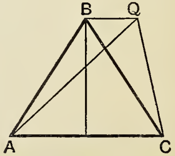
Cor. Since the $\triangle$s AEC, CED are equal in all respects, the $\angle$AEC = $\angle$DEC, and $\therefore$ the point E is such that the sum of its distances from A, B is a minimum when these distances are equally inclined to the given straight line.

28. *The perimeter of an isosceles triangle is less than that of any other triangle of equal area standing on the same base.*

Let ABC be an isosceles triangle; QAC any other triangle equal in area and standing on the same base AC.

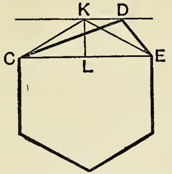
Join AQ; then BQ is parallel to AC. [I. 39.

Since AB, BC are equally inclined to AC, and $\therefore$ to BQ, it follows, from Art. 27 Cor., that the sum of AB, BC is always less than the sum of AQ, QC.



Cor. Of all $\triangle$s of the same perimeter that which has the greatest area is isosceles.

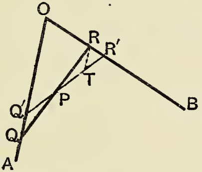
29. *If a polygon be not equilateral a polygon may be found of the same number of sides, and equal in area, but having a less perimeter.*



For, let CD, DE be two adjacent unequal sides of the polygon. Join CE. Through D draw a straight line parallel to CE. Bisect CE at L; from L draw a straight line at right angles to CE to meet the straight line drawn through D at K. Then by removing from the given polygon the triangle CDE and adding the triangle CKE, we obtain a polygon having the same number of sides as the given polygon, and equal to it in area, but having a less perimeter. [Art. 28.

30. *Through a given point* P *draw a straight line which shall cut off from two given straight lines* OA, OB *a triangle of minimum area, and shew that the required straight line is bisected at* P.

Draw through P a straight line QR which is bisected at P. [Art. 18.



Then QOR shall be the required minimum △.

For draw any other line Q′PR′ to cut off the △OQ′R′.

Draw RT parallel to OQ′ to meet Q′PR′ in T.

Then PR = PQ, ∠TPR = ∠Q′PQ, and ∠PRT = ∠PQQ′. [I. 29.

∴ △PQQ′ = △PTR.

∴ △PRR′ > △PQQ′.

Add to each the area ROQ′P. ∴ △Q′OR′ > △QOR.

∴ any other straight line through P cuts off a greater △ than QR cuts off.

∴ QOR is the required minimum △.

In this question it will be noticed that the position of QR, when it cuts off the minimum area, is a **symmetrical** one, in that it is bisected at P. We shall find that this characteristic of being a symmetrical position is a general property of lines, areas, and angles, which are either maxima or minima.

Thus in Art. 27, if F is to the right of E, the $\angle AFC > \angle BFE$; if it be to the left, then $\angle AFC <$ the corresponding angle; it is only when F is at the critical, or turning, point E that these angles are just equal.

Similarly in Art. 28, if Q be anywhere on AQ to the right or the left of A, the sides AQ, QC are not equal, they are only equal when Q is at the critical point B.

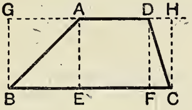
EXERCISES.

1. Find the triangle of greatest area, two of its sides being given.

2. Find the parallelogram of greatest area, two of its adjacent sides being given.

THEOREMS AND EXAMPLES ON BOOK II.

31. *The area of a trapezium equals half the rectangle contained by the sides and the perpendicular distance between them.*



Let ABCD be the trapezium with AD, BC the parallel sides. Draw perp^{rs} AE, DF, BG, CH on the opposite sides.

Then △ ABE = half the rect. EG,

and △ DFC = half the rect. FH. [I. 41.

∴ twice △ ABE + twice △ DFC + twice rect. AF

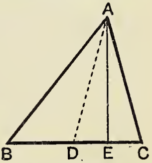
= rect. EG + rect. FH + twice rect. AF

= rect. BH + rect. AF = rect. AE, BC + rect. AD, AE.

that is, area ABCD

= half the rect. contained by AE and the sum of BC, AD.

32 *If* ABC *be a triangle,* D *the middle point of* BC *and* E *the foot of the perpendicular from* A *upon* BC, *the difference of the squares on* AB, AC = *twice the rect.* BC, DE.



We have $AB^2 = AE^2 + BE^2$,

and $AC^2 = AE^2 + CE^2$,

∴ the diff. of the sqs. on AB, AC

= the diff. of the sqs. on BE, EC

= the rect. contained by the sum and difference of BE, EC. [II. 5, Note.

Now the sum of BE, EC = BC, and

their difference = BE − EC = BE + DE − DC = BD + 2DE − DC

= 2DE, since D is the middle pt. of BC.

∴ diff. of sqs. on AB, AC = 2 rect. BC, DE.

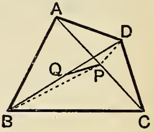
This proposition may be enunciated thus: *The difference of the squares on the sides of a* △ = *twice the rect. contained by the base and the projection on the base of the median through the vertex.*

It has already been shewn [II. 13, Exercise I.] that

$$AB^2 + AC^2 = 2AD^2 + 2DB^2.$$

33. ABCD *is any quadrilateral and* P, Q *the middle points of its diagonals* AC, BD ; *then the sum of the squares on its sides exceeds the squares on its diagonals by four times the square on the line joining the middle points of the diagonals, i.e.*

$$AB^2+BC^2+CD^2+DA^2=AC^2+BD^2+4PQ^2.$$



For [by the previous theorem]

$$AB^2+BC^2=2AP^2+2BP^2,$$

and $AD^2+DC^2=2AP^2+2DP^2.$

$$\therefore\ AB^2+BC^2+CD^2+DA^2=4AP^2+2BP^2+2DP^2$$
$$=AC^2+2BP^2+2PD^2.$$

Also $BP^2+PD^2=2PQ^2+2DQ^2$

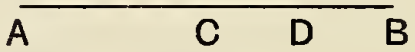
$$\therefore\ 2BP^2+2PD^2=4PQ^2+4DQ^2=4PQ^2+BD^2.$$
$$\therefore\ AB^2+BC^2+CD^2+DA^2=AC^2+BD^2+4PQ^2.$$

In the particular case when ABCD is a ||gm P and Q coincide, and ∴ PQ vanishes. The proposition then is ; *The sum of the sqs. on the sides of a parallelogram = sum of the sqs. on the diagonals.*

34. *If a straight line* AB *is divided into two parts at* D,

(1) *the sum of the squares on* AD, DB *is least,*

and (2) *the rect.* AD, DB *is greatest, when the straight line is bisected.*



Let C be the middle point of AB.

(1) By II. 9 we have $AD^2+DB^2=2AC^2+2CD^2$.

Now AC is of the same length wherever the point D is.

∴ $2AC^2+2CD^2$ is least when CD is least, and this is when CD is zero, *i.e.* when D coincides with C.

∴ AD^2+DB^2 is least when D is at C.

(2) By II. 5 we have rect. AD, $DB+CD^2=AC^2$ = a constant for all positions of D.

∴ rect. AD, DB is greatest when CD^2 is least, *i.e.* when CD is zero, *i.e.* when D coincides with C.

Corollary. Since, by (2), the rectangle contained by two straight lines, AD, DB, whose sum is given, is greatest when they are equal, it follows that *Of all rectangles with a given perimeter the greatest is a square.*

EXERCISES.

**1. The locus of the vertex of a triangle on a given base, when the sum of the squares on its sides is given, is a circle.

[For, in the figure of Art, 32, DA is constant and known.]

**2. When the difference of the squares is given, the locus is a straight line.

[For in this case DE is constant and known.]

**3. The base BC of a triangle ABC is divided at D, so that

$$m \,.\, BD = n \,.\, CD;$$

prove that

$$m \,.\, AB^2 + n \,.\, AC^2 = m \,.\, BD^2 + n \,.\, DC^2 + (m+n) AD^2.$$

If D be in BC produced, then

$$m \,.\, AB^2 - n \,.\, AC^2 = m \,.\, BD^2 - n \,.\, DC^2 + (m-n) \,.\, AD^2.$$

[Use II. 12 and 13.]

**4. Given the base of a △ in magnitude and position, and the sum, or difference, of m times the square on one side and n times that on the other, prove that the locus of the vertex is a circle.

[Use the previous Exercise.]

**5. If the medians of a triangle ABC meet in G, prove that

$$AB^2 + BC^2 + CA^2 = 3(GA^2 + GB^2 + GC^2).$$

**6. If G be the centroid of a △ABC and P any point in its plane, prove that $PA^2 + PB^2 + PC^2 = GA^2 + GB^2 + GC^2 + 3 \,.\, PG^2$. [Use Ex. 3.]

7. The locus of a point, which moves so that the sum of the squares of its distances from the angular points of a triangle is constant, is a circle whose centre is the centroid of the triangle. [Use Ex. 6.]

8. Find a point in the plane of a triangle such that the sum of the squares of its distances from the angular points is a minimum.

[Use Ex. 6.]

9. Find a point in a given straight line such that the sum of the squares of its distances from two given points is a minimum.

Divide a given straight line AB at C so that

10. $AB \,.\, BC =$ given square. [Use I. 44.]

11. $AC^2 = 2BC^2$.

12. $AB^2 + BC^2 = 2AC^2$.

Produce a given straight line AB to C, so that

13. $AB^2 + AC^2 = 2AC \,.\, BC$.

14. $AB^2 + BC^2 = 2AC \,.\, BC$.

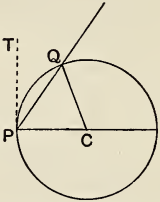
15. $AC \,.\, BC = AB^2$. [BC = AH, Fig. II., 11.]

16. $AC \,.\, AB =$ a given square.

THEOREMS AND EXAMPLES ON BOOK III.

DEFINITION OF A TANGENT AS A LIMIT.

35. Another definition of the tangent is often much more useful than Euclid's.



If PQ be a straight line through a point P on a circle, of centre C, which meets it again in Q, then PQ is called a **secant**.

If P be kept fixed and Q move along the circle until it coincides with P the limiting position of PQ, when Q becomes indefinitely close to P, is called the **tangent** at P.

[If the student conceive the circumference of the circle as made up of infinitely small dots, packed infinitely close together, then this method of looking upon the tangent at P conceives it as the line joining P and the very next dot to P.]

36. We can easily shew that the tangent PT is perpendicular to PC. For since $CQ = CP$, the $\angle CPQ = \angle CQP$.

$$\therefore \text{ twice } \angle CPQ = \angle CPQ + \angle CQP = 2 \text{ rt. } \angle^s - \angle PCQ. \qquad \text{[I. 32.}$$

This holds for all positions of Q. When Q is indefinitely close to P, the $\angle PCQ$ vanishes, and then in the limiting position twice the $\angle CPQ$ is two right $\angle^s$.

But in the limit the $\angle CPQ$ becomes the $\angle CPT$,

$$\therefore \text{ twice the } \angle CPT = \text{two right } \angle^s,$$

$$\textit{i.e. } \angle CPT = \text{a right } \angle.$$

We can also deduce III. 32. For let Q be a point close to B, in the figure of that proposition, and produce BQ to R. Then BQDA is a cyclic quadrilateral, and

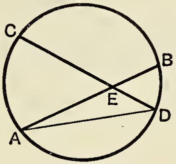
$$\therefore \angle BAD = \text{two rt. } \angle^s - BQD \qquad \text{[III. 22.}$$

$$= \angle DQR.$$

Now let Q become indefinitely close to B, then BQR ultimately coincides with BF, and

∴ ∠BAD = the ultimate position of ∠DQR = ∠DBF

37. *If two chords intersect within a circle, the angle which they include is measured by half the sum of the intercepted arcs.*



Let the chords AB and CD intersect at E; join AD.

The ∠AEC = the ∠s ADE, DAE, [I. 32.

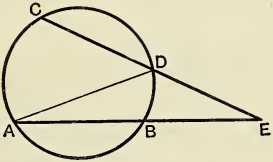
that is, = the ∠s standing on the arcs AC, BD,

that is, = the ∠ at the *circumference* standing on the sum of the arcs AC, BD,

that is, = the ∠ at the *centre* standing on half the sum of the arcs AC, BD. [III. 20.

Similarly the angle CEB is measured by half the sum of the arcs CB and AD.

38. *If two chords produced intersect without a circle, the angle which they include is measured by half the difference of the intercepted arcs.*



Let the chords AB and CD, produced, meet in E; join AD.

The ∠ADC = the ∠s EAD, AED, [I. 32.

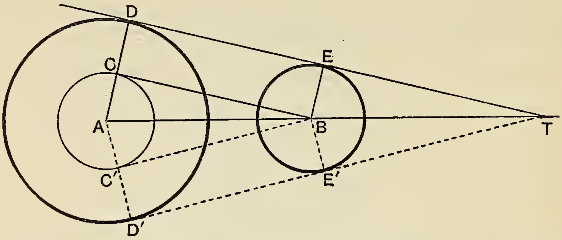
that is, ∠AEC = difference of the ∠s ADC, BAD,

that is, = ∠ at *circumference* standing on the difference of the arcs AC, BD,

and ∴ = ∠ at *centre* standing on half the difference of these arcs.

39. *To draw a common tangent to two given circles.*

Let A be the centre of the greater circle, and B the centre of the less circle. With centre A, and radius equal to the difference of the



radii of the given circles, describe a circle; from B draw a tangent BC to the circle so described.

Join AC and produce it to meet the circumference at D.

Draw the radius BE parallel to AD, and on the same side of AB; and join DE.

Then DE *shall touch both circles.*

For BE and CD are equal and parallel lines. [*Construction.*

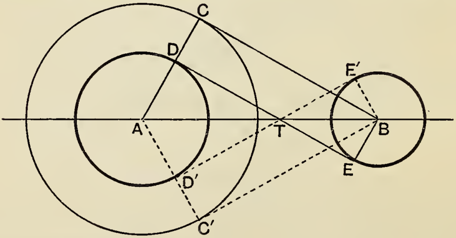
∴ BC and DE are equal and parallel. [I. 33.

∴ ∠ADE = ∠ACB [I. 29] = a right ∠.

∴ since BEDC is a ||^{m},

∠BED = ∠BCD = a right ∠.

∴ DE touches both circles at D, E. [III. 16, *Corollary.*



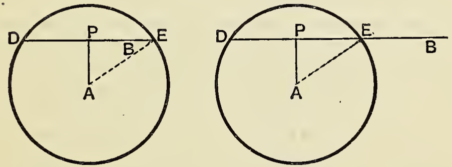
Since two tangents can be drawn from B to the described circle, two solutions can be obtained; and the two straight lines which are thus drawn to touch the two given circles can be shewn to meet AB, produced through B, at the same point T. The construction is applicable when each of the given circles is without the other, and also when they intersect.

When each of the given circles is without the other, we can obtain two other solutions. For, as in the second figure, describe a circle with A as centre and radius equal to the sum of the radii of the given circles; and continue as before, except that BE and AD will now be on *opposite* sides of AB. The two straight lines which are thus drawn to touch the two given circles can be shown to intersect AB at the same point. [See Art. 84.]

LOCI [BOOK III.].

40. *Required the locus of the middle points of all the chords of a circle which pass through a fixed point.*

Let A be the centre of the given circle; B the fixed point; let any



chord DE of the circle be drawn so that, produced if necessary, it may pass through B. Let P be the middle point of this chord, so that P is a point on the required locus.

Since P is the middle point of the chord DE,

∴ ∠APE, that is ∠APB, is a right ∠. [III. 3.

∴ P is on a circle of which AB is a diameter. Hence if B be within the given circle the locus is the circle described on AB as diameter; if B be without the given circle the locus is that part of the circle described on AB as diameter, which is within the given circle.

41. *A chord of given length slides round with its ends on the circumference of a given circle; the loci of*
(1) *its middle point,*
(2) *any fixed point on it, are circles.*

Let DE [Fig. Art. 40] be the chord of given length, P its middle point; A the centre.

Then $\angle$APE is a rt. $\angle$. [III. 3.

$\therefore$ $AP^2 = AE^2 - PE^2$ = a constant, since the radius of the circle and the length of the chord are given.

$\therefore$ AP is constant, and hence P moves on a circle whose centre is A.

(2) If B be the fixed point on the chord, then PB is constant.

$\therefore$ $AB^2 = AP^2 + PB^2$ = constant.

$\therefore$ AB is constant, and the locus of B is a circle whose centre is A.

EXERCISES.

1. Find the locus of the middle points of parallel chords of a circle.

2. Find the locus of the middle points of equal chords of a circle.

3. Find the locus of the point of intersection of tangents to a given circle which meet at a given angle.

4. Find the locus of all points from which tangents to a given circle are equal.

5. Find the locus of the points from which the tangents drawn to a given circle are of given length.

6. Find the locus of the point of contact of all tangents drawn from a given point to a system of concentric circles.

7. Find the locus of the vertex of a triangle whose base BC is given and of which the median through B is of given length.

8. The locus of the middle point of a straight line of constant length, whose ends move on two fixed perpendicular straight lines, OA and OB, is a circle whose centre is O.

9. Given the base and vertical angle of a triangle; find the locus of the middle point of the straight lines joining the vertices of all such triangles to the middle point of the base.

10. Given the base BC of a triangle and its vertical angle, prove that the loci of
(1) the intersection of the bisectors of the base $\angle^s$,
(2) the intersection of the bisectors of the exterior base $\angle^s$,
(3) the intersection of the bisector of one exterior base $\angle$ with the bisector of the other interior base $\angle$,
are all arcs of circles passing through B and C.

11. If two segments of circles have a common chord AB and any two points P and Q be taken, one on each segment, prove that the locus of O, the point of intersection of the bisectors of the angles PAQ, PBQ, is another segment of a circle passing through A and B.

[Twice $\angle AOB = 4$ rt. $\angle^s - 2\angle OAB - 2\angle OBA$

$= 4$ rt. $\angle^s - \angle PAB - \angle QAB - \angle PBA - \angle QBA$

$= \angle APB + \angle AQB =$ const. $\therefore$ etc.]

12. If A and B be two fixed points on a circle and C, D the extremities of a chord of constant length, then the intersection of AD, BC and also that of AC, BD lie on fixed circles.

13. O is a fixed point on the circumference of a circle and OP is any chord. On OP is described a circle containing a given angle; the locus of its centre is one or other of two fixed circles.

14. PQ is a straight line of given length which moves so that its ends, P and Q, slide on two given straight lines OA and OB; the locus of the intersection of perpendiculars to OA and OB, drawn through P and Q respectively, is a circle.

15. If a chord of a circle, centre O, subtend a right angle at a fixed point P, the locus of its middle point is a circle whose centre is the middle point of PO.

[If QR be the chord, and C, T be the middle points of OP, QR then by Page 109, Ex. 1, $4OC^2 + 4CT^2 = 2OT^2 + 2TP^2 = 2OQ^2 - 2TR^2 + QP^2 + PR^2 - 2TR^2 = 2OQ^2 - 4TR^2 + QR^2 = 2OQ^2$, etc.]

Find the locus of the centres of all circles which

16. pass through two given points.

17. touch two given intersecting straight lines.

18. touch a given straight line at a given point.

19. touch a given straight line and are of given radius.

20. touch a given circle at a given point.

21. touch a given circle and are of given radius.

22. A triangle is formed by a fixed tangent to a circle, a variable tangent, and the chord joining their points of contact. Find the locus of the centre of its circumscribing circle.

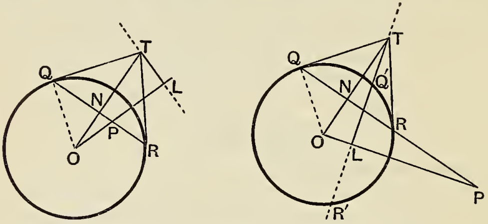
23. Tangents are drawn, one to each of two concentric circles, and the angle between them is constant. Prove that the locus of their point of intersection is one or other of two fixed circles concentric with the given one.

24. Two adjacent corners A, B of a sheet of paper are doubled down and meet at a point P in such a manner that the three parts of the edge AB form a triangle right-angled at P. Prove that the locus of P is an arc of a circle.

POLE AND POLAR.

42. The Polar of any point P with respect to a circle is the locus of the point of intersection of tangents drawn at the extremities of any chord of the circle which passes through P.

To find the polar of any given point P *and to construct it geometrically.*



Through P draw any straight line to cut the circle in Q and R. At Q and R draw tangents and let them meet in T. [III. 17.
Join OP and draw from T a perpendicular TL on OP or OP produced. Join OT and let it meet QR in N.
Then OQT is a △ having OQT a right ∠, and QN is perpendicular to the base OT;

$$\therefore\ ON \,.\, OT = OQ^2 = (\text{radius})^2.$$

[For, by III. 13, $TQ^2 = TO^2 + OQ^2 - 2\,ON \,.\, OT$;

$$\therefore\ 2\,ON \,.\, OT = TO^2 + OQ^2 - TQ^2 = 2\,OQ^2 \text{ (I. 47).]}$$

But since TNP, TLP are both right angles, a circle will go round T, N, P, L, and

$$\therefore\ OP \,.\, OL = ON \,.\, OT; \qquad \text{[III. 36.}$$

$$\therefore\ OP \,.\, OL = (\text{radius})^2;$$

∴ OL is constant, since P is a given point, and therefore OP constant;
∴ L is a fixed point and ∠OLT is a right angle;
∴ the locus of T (that is, the polar of P) is a straight line perpendicular to OL and passing through L.

The polar of P is thus a straight line, and may be constructed as follows:

On OP *take a point* L, *such that* OL . OP = (*radius*)^2, *and through* L *draw a straight line perpendicular to* OP.

The two points, L and P, are often called **inverse points.**

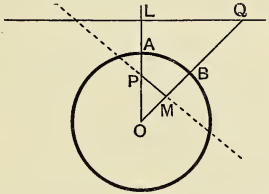
Corollary 1. If P be on the circle so also is L, and the polar of P in this case is the tangent at P.

Corollary 2. If, as in the right-hand figure, the line TL meet the circle in Q′ and R′ then the straight lines PQ′, PR′ would be tangents at Q′, R′.

For OL . OP = $(\text{radius})^2$, and Q′LR′ is perpendicular to OP.

Hence, if the point P lie outside the circle, its polar coincides with the line joining the points of contact of tangents drawn from it to the circle.

43. *If the polar of a point* P *passes through* Q *then the polar of* Q *passes through* P.



Let Q be any point on the polar LQ of a point P. Draw OPL perpendicular to LQ, O being the centre.

Then OP . OL = $(\text{radius})^2$. [Art. 42.

Join OQ and draw PM perpendicular to OQ.

Then since PLQ, PMQ are right angles, a circle will pass through P, L, Q, M;

∴ OM . OQ = OP . OL [III. 36.

= $(\text{radius})^2$.

Hence, since OMP is a right angle, it follows, by Art. 42, that PM is the polar of Q,

that is, *the polar of* Q *passes through* P.

The polars of P *and* Q *meet at an angle equal to that subtended by* P *and* Q *at the centre of the circle.*

For, if O be the centre, the polar of P is perpendicular to OP,

and the polar of Q is perpendicular to OQ.

Also, the ∠ between two straight lines is equal to the angle between their perpendiculars;

∴ the ∠ between the polars of P and Q = the ∠POQ.

EXERCISES.

****1.** The straight line joining two points, A and B, is the polar of the point of intersection of the polars of A and B.

[Let the latter intersect at T; then A lies on the polar of T, since T lies on the polar of A (43); so B lies on the polar of T; ∴ AB is the polar of T.]

****2.** The point of intersection of any two straight lines is the pole of the straight line joining their poles.

****3.** Find the locus of the poles of all straight lines which pass through a given point. [Use Art. 43.]

****4.** A and B are two points in a plane of a circle whose centre is C; AX and BY are the perpendiculars from A and B on the polars of B and A respectively; prove that the rectangles CA . BY and CB . AX are equal. [Salmon's Theorem.]

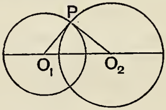
[Let the polar of A meet CA in M, and that of B meet CB in N; also draw AU, BV perp[r] to CB, CA respectively.

Then A, V, U, B lie on a circle; ∴ CU . CB = CA . CV. [III. 36.

But CN . CB = CM . CA = sq. on radius (Art. 42). Subtract;

∴ CB . NU = CA . MV, *i.e.* CB . AX = CA . BY.]

44. Orthogonal Circles. Def. Two circles are said to intersect orthogonally when the tangents at their points of intersection are at right angles.



If the two circles intersect at P, the radii O_1P and O_2P, which are perpendicular to the tangents at P, must also be at right angles.

Hence $$O_1O_2^2 = O_1P^2 + O_2P^2,$$

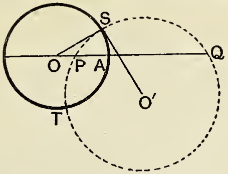
i.e. the square of the distance between the centres must be equal to the sum of the squares of the radii.

Also the tangent from O_2 to the other circle is equal to the radius a_2, *i.e. if two circles be orthogonal the length of the tangent drawn from the centre of one circle to the second circle is equal to the radius of the first.*

Either of these two conditions will determine whether the circles are orthogonal.

It follows that if we want the circle whose centre is O_2 which shall cut a given circle, centre O_1, orthogonally, we must take its radius equal to the tangent from O_2 to the given circle.

45. *Given a circle and two inverse points,* P *and* Q, *with respect to it; any circle which passes through* P *and* Q *cuts the given circle orthogonally.*



Let any such circle cut the given circle in S and T. Let O, O′ be the centres of the two circles and join OS, O′S. Then, by hypothesis, OP . OQ = $OA^2 = OS^2$.

∴ OS touches the second circle at S. [III. 37.

∴ OSO′ is a right angle.

∴ the two circles cut orthogonally at S and similarly at T.

EXERCISES.

****1**. What is the locus of the centres of all circles which cut a given circle orthogonally at a given point?

****2**. Through a given point draw a circle to cut a given circle orthogonally at a given point.

****3**. If two circles cut orthogonally the extremities, P and Q, of any diameter of either, are conjugate points with respect to the other, *i.e.* the polar of P passes through Q.

[Let the centres of the circles be O and O′, and let them cut in C; let PO′Q be any diameter of the second; join OP and draw QS perp[r] to OP; then S lies on the second circle. Since the circles cut othogonally the tangent at C to the circle O′ passes through O.

∴ OS . OP = OC^2 = sq. on radius of first circle;

∴ since PSQ is a right ∠, SQ is the polar of P with respect to the circle O (Art. 42); ∴ etc.]

4. P is any point in the plane of a circle C and Q any point on its polar with respect to C; the circle on PQ as diameter cuts C orthogonally.

5. All circles which pass through a given point and cut a given circle orthogonally pass through another given point.

[This is the converse of Art. 45.]

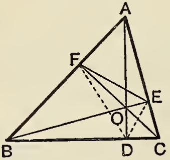
6. The chords joining two fixed points on a circle to the ends of any diameter intersect on a fixed circle which cuts the former orthogonally.

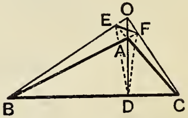
BOOK IV.

46. *The perpendiculars from the angles of a triangle on the opposite sides meet in a point.*

[A proof of this theorem has already been given in Art. 8].

Let ABC be a △ ; from B draw BE perpendicular to CA and CF perpendicular to AB ; let BE, CF meet in O ; join AO and produce it to meet BC in D ; *then* AD *shall be perpendicular to* BC.





[If the △ be obtuse-angled, as in the second figure, some of these perpendiculars will meet the opposite sides produced].

Since AEO, AFO are right ∠s a circle will pass through A, E, O, F. [III. 22, *Converse.*

∴ ∠FAO = ∠FEO. [III. 21.

Similarly, a circle will pass through B, C, E, F.

∴ ∠FEO = ∠FCB ;

∴ ∠FAO = ∠FCB, that is, ∠BAD = ∠FCB.

Also the ∠ at B is common to the △s BAD, BCF.

∴ the third ∠BDA = the third ∠BFC [I. 32.

= a right ∠. [*Construction.*

∴ AD is perpendicular to BC.

47. *When* ABC *is an acute-angled triangle the angles of the triangle* DEF *in the previous figure are equal to the supplements of twice the angles of the original triangle* ABC, *and its sides are equally inclined to the sides of the triangle* ABC.

Since O, F, A, E lie on a circle,

∴ ∠OFE = ∠OAE = a right ∠ − ∠ACD. [III. 21.

So, since O, F, B, D lie on a circle,

∴ ∠OFD = ∠OBC = a right ∠ − ∠ECB.

∴ OFE and OFD are equal angles, and their sum

= two right $\angle^s - 2\angle$ ACB.

that is, $\angle$DFE = the supplement of twice the $\angle$C.

So the angles FDE, DEF are the supplements of twice the angles A and B.

Also since $\angle$EFO = $\angle$DFO, ∴ $\angle$EFA = $\angle$DFB.

∴ FE, FD are equally inclined to AB.

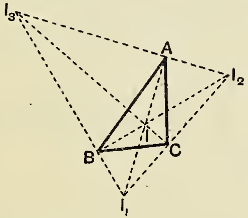
Similarly FE, ED are equally inclined to AC, and ED, DF are equally inclined to BC.

Definition. The triangle DEF is often called the **Pedal** Triangle of ABC.

48. *If* I *be the in-centre and* I_1, I_2, I_3 *the three e-centres of a triangle* ABC, *then*

(1) I *is the orthocentre of the triangle* $I_1I_2I_3$.

(2) ABC *is the pedal triangle of* $I_1I_2I_3$.



For AI bisects the $\angle$BAC, since AB, AC are tangents to a circle centre I. [III. 17, *Cor.* 2.

For a similar reason AI_1 bisects the same $\angle$.

∴ AII_1 is a straight line, and similarly BII_2, CII_3.

Also since BA produced and AC touch a circle centre I_2.

∴ AI_2 bisects the exterior angle of BAC. [III. 17, *Cor.* 2.

Also AI bisects $\angle$BAC.

∴ IAI_2 is a right angle. Similarly IAI_3 is a right $\angle$.

∴ I_2AI_3 is a straight line. Similarly I_3BI_1 and I_1CI_2 are straight lines.

∴ I_1IA, I_2IB, and I_3IC are the perpendiculars from I_1, I_2, I_3 on the sides of the $\triangle I_1I_2I_3$.

∴ I is the orthocentre and ABC the pedal $\triangle$ of the $\triangle I_1I_2I_3$.

EXERCISES.

1. The perpendicular from the middle point of the side of any triangle on the opposite side of its pedal triangle bisects the latter.

2. O is the orthocentre of a triangle ABC and the perpendicular AD on BC, when produced, meets the circumcircle in H; prove that DH = DO.

[For ∠DCH = ∠DAB (III. 21) = rt. ∠ − ABD = ∠BCF.

∴ △ˢ OCD, HCD are equal in all respects; ∴ etc.]

3. If O is the orthocentre of the triangle ABC, then either of the four points O, A, B, C is the orthocentre of the triangle formed by the other three.

4. If ABC be a triangle obtuse-angled at A, and AD, BE, CF be the perpendiculars on the sides, then BE, CF bisect the exterior angles of the △DEF, and AD bisects the interior angle.

5. The radii from the circumcentre of a triangle to the angular points are respectively perpendicular to the straight lines joining the feet of the perpendiculars on the sides of the triangle from the opposite vertices.

6. With the letters of Art. 46, prove that

AO . OD = BO . OE = CO . OF, and that DB . DC = DO . DA, etc.

[Taking the left-hand figure, we have AO . AD = AF . AB, since F, B, D, O lie on a circle.

$$\therefore\ AO \cdot OD = AF \cdot AB - AO^2 = AF \cdot FB + AF^2 - AO^2 = AF \cdot FB - OF^2$$
$$= AF \cdot FB + BF^2 - BO^2 = BA \cdot BF - BO^2$$
$$= BO \cdot BE - BO^2 \text{ (since O, F, A, E lie on a circle)}$$
$$= BO \cdot OE.$$

This is more easily proved by Book VI., since EOA, DOB are similar △ˢ].

7. In the second figure of Art. 46, if with centre O and radius, whose square is equal to either of the rectangles OA . OD, OB . OE, OC . OF, a circle be described, then each angular point of the triangle ABC is the polar with respect to this circle of the opposite side.

[This follows from the last Exercise and Art. 42. Such a circle is called the **Polar Circle** of the triangle; it is also called the **Self-Conjugate Circle**. If the △ be acute-angled, as in the first figure of Art. 46, there is no such circle. For its centre, if any, must, by Art. 42, lie on each of the three lines AD, BE, and CF, and must therefore be at O. But since in this case the point A and the line BC lie on **opposite** sides of O, it is impossible by Art. 42 that A should be the polar of BC.]

8. The circles described on the sides of a triangle as diameters cut the polar circle orthogonally.

[Taking the right-hand figure of Art. 46, the circle on AC as diameter passes through A and D, which are inverse points with respect to the polar circle, since OA . OD = sq. on radius of the polar circle. Then apply Art. 45.]

9. Find a point O within or without a triangle ABC such that the circumcircles of the triangles OAB, OBC, OCA are all equal.

10. Prove that the orthocentre and vertices of any triangle are the in- and escribed centres of the pedal triangle of the given triangle.

11. Given the pedal triangle DEF of a certain triangle ABC shew how to construct ABC.

12. If DEF is the pedal △ of the △ABC, the triangles DEC, EFA, FDB, ABC are all equiangular,

and their circumcircles are all of the same size.

13. In the figure of Art. 48 prove that

(1) the triangles BI_1C, CI_2A, AI_3B are equiangular.

(2) the four circles each of which passes through three of the four points I, I_1, I_2, I_3 are all equal.

14. The internal and external bisectors of the angle A of a triangle meet the base BC in E, E′ and the circumcircle in D and D′; prove that D is the orthocentre of the △EE′D′.

15. O is the orthocentre of a triangle ABC, and D, E, F the circumcentres of the triangles OAB, OBC, and OCA. Prove that ABC and DEF are equal in all respects:

16. ABCD is a quadrilateral inscribed in a circle; E and F are the orthocentres of the △s ABC and ABD respectively; prove that CDFE is a ||m.

Shew also that the orthocentres of the four △s ABC, BCD, CDA, DAB form a quadrilateral equal and similar to the given one.

[Use the Corollary to Art. 8 in this and the three following Exercises.]

17. Prove that the sum of the squares on the straight line joining the vertex of a triangle to the orthocentre, and on the opposite side = the sq. on the diameter of the circumcircle.

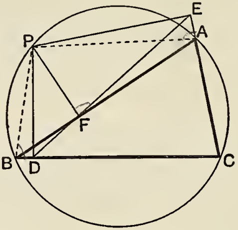
18. The orthocentre of a triangle is joined to the middle point of a side; prove that this joining line produced will meet the circumcircle in a point which is at the other end of the diameter through the angular point opposite to the bisected side.

19. A line MN of given length slides between two fixed lines OM, ON; prove that the orthocentre of the △OMN always lies on a certain circle.

20. A circular piece of paper is folded into any inscribed triangle; prove that the arcs of the folded segments all pass through a fixed point.

49. *If from any point in the circumference of the circle described round a triangle perpendiculars be drawn to the sides of the triangle, the three points of intersection are in the same straight line.*

Let ABC be a triangle, P any point on the circumcircle; from P draw PD, PE, PF perpendiculars to the sides BC, CA, AB respectively: D, E, F shall be in the same straight line.



[We will suppose that P is on the arc cut off by AB, on the opposite side from C, and that E is on CA produced through A; the demonstration will only have to be slightly modified for any other figure.]

A circle will go round PEAF; [III. 22 converse.

∴ ∠PFE = ∠PAE [III. 21.

= two rt. ∠s − ∠PAC [I. 13.

= ∠PBC. [III. 22.

Again, a circle will go round PFDB; [III. 21.

∴ ∠PFD = two rt. ∠s − ∠PBC. [III. 22 converse.

∴ ∠s PFD and PFE are together = two right ∠s.

∴ EF and FD are in the same straight line.

Conversely. If the feet of the perpendiculars PD, PE, PF drawn from any point P to the sides of a triangle ABC are collinear, then P lies on the circumcircle of ABC.

For ∠BPD = ∠BFD (III. 21) = ∠AFE = ∠APE (III. 21).

∴ ∠PBD = rt. ∠ − ∠BPD = rt. ∠ − APE = ∠PAE.

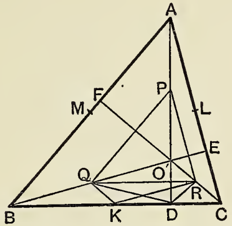
∴ ∠PBD + ∠PAC = 2 rt. ∠s.

∴ P, A, B, C lie on a circle.

Definition. The above straight line DFE is sometimes called the **Pedal Line** of P, and sometimes it is called the **Simson Line** of P from the name of its supposed discoverer.

50. ABC *is a triangle, and* O′ *is its orthocentre; the circle which passes through the middle points of* O′A, O′B, O′C *will pass through the feet of* *the perpendiculars and through the middle points of the sides of the triangle.*

Let P, Q, R be the middle points of OA, OB, OC respectively; let D be the foot of the perpendicular from A on BC, and K the middle point of BC.



Then O'BD is a right-angled triangle and Q is the middle point of the hypotenuse O'B;

$\therefore$ QD = QO';

$\therefore$ ∠QDO' = ∠QO'D.

Similarly, the ∠RDO' = RO'D;

$\therefore$ ∠RDQ = RO'Q.

But ∠s CO'B, BAC together = two rt. ∠s. [III. 22.

$\therefore$ ∠s RDQ, BAC together = two rt. ∠s.

Also ∠BAC = ∠QPR, [Art. 1.

since QP, PR are ‖l to BA, AC.

$\therefore$ ∠s RDQ, QPR = two rt. ∠s.

$\therefore$ D is on the circle through P, Q, R.

Again, RK is parallel to O'B, and QK parallel to O'C; [Art. 1.

$\therefore$ ∠QKR = ∠QO'C = ∠RDQ;

$\therefore$ K is also on the circumference of the circle.

Similarly, the two points in each of the other sides of the triangle ABC may be shewn to be on the circle.

51. The circle which is thus shewn to pass through these nine points is called the *Nine-point circle*: It has some curious properties, some of which we will now give.

52. *The radius of the nine-point circle is half of the radius of the circle described about the original triangle.*

For the △PQR has its sides respectively halves of the sides of the triangle ABC, so that the triangles are equiangular. Hence the radius of the circle described round PQR is half of the radius of the circle described round ABC.

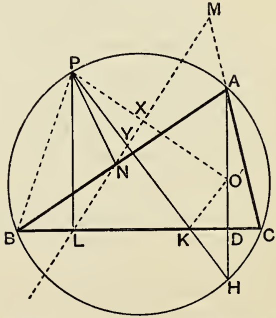
53. *If* O *be the centre of the circle described round the triangle* ABC, *the centre of the nine-point circle is the middle point of* OO′.

For OK is at right angles to BC, and therefore parallel to O′D. Hence the straight line which bisects KD at right angles must bisect OO′. And K and D are on the circumference of the nine-points circle, so that the straight line which bisects KD at right angles must pass through the centre of the nine-point circle. Similarly, from the other sides of the triangle ABC two other straight lines can be obtained, which pass through the centre of the nine-point circle and also bisect OO′. Hence the centre of the nine-point circle must coincide with the middle point of OO′.

Cor. The circumcentre O, the centroid G, the nine-point centre N, and the orthocentre O′ lie on a straight line; also G is a point of trisection [Art. 7], and N the point of bisection of OO′.

54. We may state that the nine-point circle of any triangle touches the inscribed circle and the escribed circles of the triangle: a demonstration of this theorem will be found in Dr. Todhunter's *Plane Trigonometry*, Chapter XXIV. For the history of the theorem see the *Nouvelles Annales de Mathématiques* for 1863, page 562.

55. *The line joining any point* P *on the circumcircle of any triangle* ABC *to the orthocentre* O′ *is bisected by the pedal line of* P *with respect to the triangle.*



Draw PL, PM, PN perpendicular to the sides of ABC; then LNM is the pedal line of P.

Draw AD perpendicular to BC and produce it to meet the circumcircle in H. Then, if O′ be the orthocentre, it lies on AD and O′D = DH. [Ex. 2, Art. 48.]

Let O′P meet LM in X. Then we have to prove that O′X = XP.

Join HP and let it meet BC in K. Join O′K.

Since PNB, PLB are both right angles, a circle goes through P, N, L, B;

$\therefore\ \angle PLN = \angle PBN = \angle PBA = \angle PHA$ [III. 21.

$= \angle HPL$, since AH, PL are parallel.

$\therefore\ \angle YLP = \angle YPL.$

$\therefore\ YP = YL,$

and thus Y is the centre of the circumcircle of the △PLK.

$\therefore\ PY = YK = YL.$

Again, since O′D = DH, and O′DK, HDK are right ∠s,

$\therefore\ \angle O'KD = \angle HKD$ [I. 4.

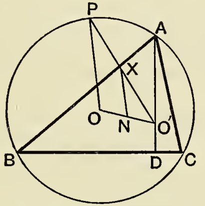
$= \angle YKL = \angle YLK$, since YK = YL. (*Proved.*)

∴ O′K and LNM are parallel.

∴ LNM is a straight line drawn through the middle point Y of the side PK of the △PKO′ parallel to its base KO′.

∴ LNM bisects PO′, *i.e.* PX = XO′. [Art. 1.

56. *The middle point of the straight line joining any point* P *on the circumcircle of any triangle to the orthocentre* O′ *lies on the nine-point circle of the triangle.*



Let O be the centre of the circumcircle. Bisect OO′ in N and join NX.

Then, by Art. 53, N is the centre of the nine-point circle.

Hence, since X and N are the middle points of O′P and O′O respectively, NX is parallel to, and is one-half of, OP.

∴ NX = one-half of the circum-radius

= the radius of the nine-point circle. [Art. 52.

∴ X lies on the nine-point circle.

Hence the intersections of O′P with the pedal line of P always lies on the nine-point circle.

EXERCISES.

1. In the figure of Art. 50 prove that PQKL, PMKR are rectangles, and hence that the circle on PK as diameter goes through Q, L, M, R and also through D, E, F. [This is an easy method of proof of Art. 50.] Prove that PK, QL, RM meet at the centre of the nine-point circle.

2. If the perpendicular from any point P on the side BC of a $\triangle$ ABC meet the circumcircle in Y, then AY is $\parallel^{l}$ to the pedal line of P.

3. If through any point O on a circle three chords be drawn, and on each a circle be described, shew that these three circles, besides intersecting in O, meet in three points lying on a straight line.

[The three points are the feet of the $\perp^{rs}$ from O upon the sides of the $\triangle$ whose angular points are the other ends of the chords through O.]

4. *The pedal lines of a triangle with respect to two points on the circumcircle is equal to half the angle the two points subtend at the circumcentre.*

For, in the figure of Art. 55,

$\angle NLC = \angle LNB + B = \angle LPB + B = \text{rt.} \angle - \angle PBL + B$

$= \text{rt.} \angle - PBN = \text{rt.} \angle - \frac{1}{2} \angle POA$, where O is the circumcentre,

that is, inclination of pedal line of P to BC

$= \text{rt.} \angle - \frac{1}{2} \angle POA.$

So the inclination of the pedal line of any other point P′ to BC

$= \text{rt.} \angle - \frac{1}{2} P'OA.$

$\therefore$ angle between these pedal lines

$=$ the difference of these inclinations $= \frac{1}{2} \angle P'OA - \frac{1}{2} \angle POA$

$= \frac{1}{2} \angle P'OP.$

If P, P′ are at the extremities of a diameter of the circumcircle, then $\frac{1}{2} \angle P'OP =$ a right $\angle$, and $\therefore$ the corresponding pedal lines are perpendicular.

5. *The pedal lines of the ends of a diameter of the circumcircle meet at right angles on the nine-point circle.*

For if the ends of the diameter be P, P′, and X, X′ be the middle points of O′P, O′P′, then XX′ bisects OO′ and thus passes through the nine-point centre. XX′ is thus a diameter of the nine-point circle. Also the pedal lines of P, P′ pass through X, X′, and they meet at a right $\angle$, by the last example; $\therefore$ they meet on the nine-point circle.

[III. 31.

6. P is any point on the circumcircle of the triangle ABC, and PL, drawn parallel to BC, meets the circle in Q; prove that AQ is perpendicular to the pedal line of P.

7. I is the in-centre and I_1, I_2, I_3 are the e-centres of a triangle ABC; prove that the straight lines II_1, II_2, II_3, I_2I_3, I_3I_1, I_1I_2 are all bisected by the circumcircle of the triangle ABC.

Prove also that the middle points of II_1 and I_2I_3 are at the ends of a diameter of this circle.

8. A, A′, B, C are four points on a circle. Prove that the pedal line of A′ with respect to the $\triangle$ ABC, of A with respect to the $\triangle$ A′BC, and the nine-point circles of these two $\triangle$s meet in a point.

[This point is at the intersection of the diagonals of the parallelogram formed by A, A′ and the orthocentres of the two triangles.]

9. The nine-point circles of the triangles formed by four points, taken three at a time, meet in a point.

[Let A, B, C, D be the points, and P, Q, R, S, T, U the middle points of BC, CA, AB, BD, CD, AD. If O be the intersection of the nine-point circles of ABC and BCD, then $\angle POQ = \angle PRQ = \angle ACB$, and $\angle POT = \angle PST = \angle BCD$; $\therefore$ $\angle QOT = \angle ACD = \angle QUT$; $\therefore$ etc.]

10. If A, B, C, D be four points on a circle, the pedal lines of the triangles ABC, BCD, CDA, DAB with respect to D, A, B, C respectively meet in a point. [Use the two previous Exercises.]

11. Prove also that the centroids of the four triangles are concyclic.

12. A triangle inscribed in a given circle has its orthocentre at a fixed point. Prove that the middle points of its sides lie on a fixed circle.

13. Having given an angular point A of a triangle, its circumcentre, and the length of the base BC, the loci of its orthocentre and its nine-point centre are both circles.

14. Four concyclic points, taken three by three, determine four triangles whose nine-point centres are concyclic.

15. If the nine-point circle and one of the angular points of a triangle be given, the locus of the orthocentre is a circle.

16. O is the orthocentre of a triangle ABC; the nine-point circles of the triangles OAB, OBC, OCA, ABC all coincide.

17. D and E are points on the circumcircle of the triangle ABC, and their pedal lines meet in P; prove that the locus of P is a circle when A moves on the circumcircle and B, C, D, E are fixed.

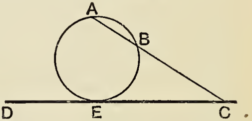
THE TANGENCIES OF CIRCLES.

57. A circle can be found to satisfy three conditions in general. Conversely, if three independent conditions be given, one circle, or at any rate only a finite number of circles can be found to satisfy them.

To describe a circle which shall pass through three given points not in the same straight line.

This is solved in Euclid IV. 5.

58. *To describe a circle which shall pass through two given points on the same side of a given straight line, and touch that straight line.*

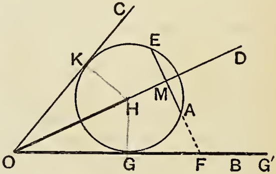


Let A and B be the given points; join AB and produce it to meet the given straight line at C. Make a square equal to the rectangle CA, CB (II. 14), and on the given straight line take CE equal to a side of this square. Describe a circle through A, B, E (IV. 5); this will be the circle required (III. 37).

Since E can be taken on either side of C, there are two solutions.

The construction fails if AB is parallel to the given straight line. In this case bisect AB at D, and draw DC at right angles to AB, meeting the given straight line at C. Then describe a circle through A, B, C.

59. *To describe a circle which shall pass through a given point and touch two given straight lines.*



Let A be the given point, and OB, OC the given straight lines. Bisect the angle BOC by the straight line OD. Draw AM perpendicular to OD and produce it to E, so that AM = ME.

Through A, E, draw a circle (by Art. 58) to touch the given straight line OB. This is done by producing EA to meet OB in F and taking $FG^2 = FA \cdot FE$.

Since the circle passes through A and E, its centre lies on MO which bisects AE at right $\angle$s. [III. 1.

Hence, if H be the centre, since it lies on the straight line OD which bisects the $\angle$BOC the perpendiculars HG, HK on OB, OC are equal, and therefore the circle just drawn also touches OC. It is therefore the required circle.

There are two such circles; for if we take G′ on the other side of F from G, such that FG′ = FG, a second circle can be drawn through A, E, and G′ which will also satisfy the required conditions.

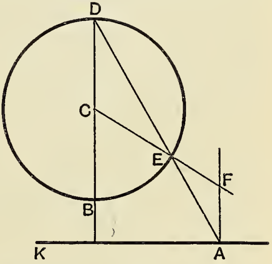
If A is on one of the given straight lines, draw from A a straight line at right angles to this given straight line; the point of intersection of this straight line with either of the two straight lines which bisect the angles made by the given straight lines may be taken for the centre of the required circle.

If the two given straight lines are parallel, instead of drawing a straight line BC to bisect the angle between them, we must draw it parallel to them, and equidistant from them.

60. *To describe a circle which shall touch three given straight lines, not more than two of which are parallel.*

Proceed as in Euclid IV. 4. If the given straight lines form a triangle, four circles can be described, namely, one as in Euclid, and three others, as on Page 181, each touching one side of the triangle and the other two sides produced. If two of the given straight lines are parallel, two circles can be described, namely, one on each side of the third given straight line.

61. *To describe a circle which shall touch a given circle, and touch a given straight line at a given point.*



Let A be the given point in the given straight line AK, and C be the centre of the given circle. Through C draw a straight line perpendicular to the given straight line to meet the circle in B and D, of which D is the more remote from the given straight line. Join AD, meeting the circle in E. From A draw a straight line at right angles to the given straight line, meeting CE produced at F.

Then F *shall be the centre of the required circle, and* FA *its radius.*

For the ∠AEF = the ∠CED; [I. 15.

and the ∠EAF = the ∠CDE; [I. 29.

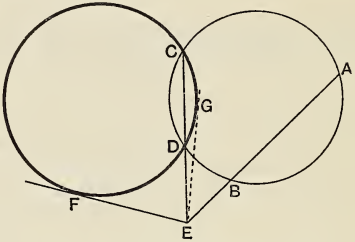
∴ the ∠AEF = the ∠EAF;

∴ AF = EF. [I. 6.

In a similar manner another solution may be obtained by joining AB. If the given straight line falls without the given circle, the circle obtained by the first solution touches the given circle externally, and the circle obtained by the second solution touches the given circle internally.

If the given straight line cuts the given circle, both the circles obtained touch the given circle externally.

✓ **62**. *To describe a circle which shall pass through two given points and touch a given circle.*



Let A and B be the given points. Take any point C on the circumference of the given circle, and describe a circle through A, B, C. If this described circle touches the given circle, it is the required circle. But if not, let D be the other point of intersection of the two circles. Let AB and CD be produced to meet at E; from E draw a tangent EF to the given circle.

Then the circle through A, B, F shall be the required circle. See III. 35 and III. 37.

There are two solutions, because two tangents EF, EG can be drawn from E to the given circle.

If the straight line which bisects AB at right angles passes through the centre of the given circle, the construction fails, for AB and CD are parallel. In this case F must be determined by drawing a straight line parallel to AB so as to touch the given circle.

63. *To describe a circle which shall touch two given straight lines and a given circle.*

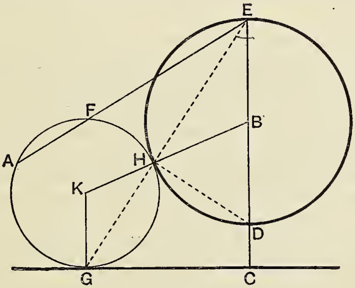
Draw two straight lines parallel to the given straight lines, at a distance from them equal to the radius of the given circle, and on the sides of them remote from the centre of the given circle. Describe a circle touching the straight lines thus drawn, and passing through the centre of the given circle (Art. 59). A circle having the same centre as the circle thus described, and a radius equal to the excess of its radius over that of the given circle, will be the required circle.

Two solutions will be obtained, because there are two solutions of the problem in Art. 59; the circles thus obtained touch the given circle externally.

We may obtain two circles which touch the given circle internally, by drawing the straight lines parallel to the given straight lines on the same sides of them as the centre of the given circle.

64. *To describe a circle which shall pass through a given point and touch a given straight line and a given circle.*

We will suppose the given point and the given straight line without the circle; other cases of the problem may be treated in a similar manner.



Let A be the given point, and B the centre of the given circle. From B draw a perpendicular to the given straight line, meeting it at C, and meeting the circumference of the given circle at D and E, so that D is between B and C.

Join EA and determine a point F in EA, produced if necessary, such that the rect. EA, EF = the rect. EC, ED; this can be done by describing a circle through A, C, D, which will meet EA at the required point. [III. 36, *Corollary.*

Describe a circle to pass through A and F and touch the given straight line (Art. 58); this shall be the required circle.

For, let the circle thus described touch the given straight line at G; join EG meeting the given circle at H, and join DH.

Since the ∠ˢ GHD, GCD are right ∠ˢ, [III. 31, and *Construction.*

∴ H, G, C, D lie on a circle.

∴ rect. EC, ED = rect. EG, EH. [III. 36.

∴ H is on the described circle. [III. 36, *Corollary.*

Take K the centre of the described circle; join KG, KH, and BH. Then ∠KHG = ∠KGH [I. 5] = ∠HEB [I. 29] = ∠EHB [I. 5].

∴ KHB is a straight line;

∴ the described circle touches the given circle.

Two solutions will be obtained, because there are two solutions of the problem in Art. 58; the circles thus described touch the given circle externally.

By joining DA instead of EA we can obtain two solutions in which the circles described touch the given circle internally.

65. *To describe a circle which shall touch a given straight line and two given circles.*

Let A be the centre of the larger circle and B the centre of the smaller circle. Draw a straight line parallel to the given straight line, at a distance from it equal to the radius of the smaller circle, and on the side of it remote from A. Describe a circle with A as centre, and radius equal to the difference of the radii of the given circles. Describe a circle which shall pass through B, touch externally the circle just described, and also touch the straight line which has been drawn parallel to the given straight line [Art. 64]. Then a circle having the same centre as the second described circle, and a radius equal to the difference between its radius and the radius of the smaller given circle, will be the required circle.

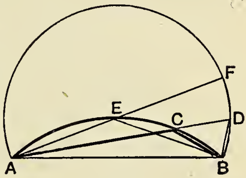
Two solutions will be obtained, because there are two solutions of the problem in Art. 64; the circles thus described touch the given circles externally.

We may obtain in a similar manner circles which touch the given circles internally, and also circles which touch one of the given circles internally and the other externally.

[For other cases of circles satisfying given conditions see Arts. 87, 88, 90, and 91.]

CONSTRUCTIONS.

66. *Construct a triangle given its base, its vertical angle, and the sum of its sides equal to a given straight line.*



Let AB be the base; on it describe a segment of a circle containing an ∠ACB equal to the given vertical angle. Let C be a point on such that the sum of AC, CD is equal to the given sum.

Produce AC to D so that CD=CB, and join DB.

Then AD is equal to the given straight line.

Also the ∠ACB=the sum of the ∠s CDB and CBD [I. 32], that is, =twice the ∠CDB [I. 5].

∴ the ∠ADB is half of the given vertical ∠.

Hence we have the following solution. Describe on AB a segment of a circle containing an ∠ equal to the given ∠ and a second segment containing an ∠ equal to half the given ∠.

With A as centre, and a radius equal to the given straight line, describe a circle.

Join A with a point of intersection D of this circle and the second segment; this joining straight line will cut the circumference of the first segment at a point C which solves the problem.

The given straight line must exceed AB, and it must not exceed a certain straight line which we will now determine. Suppose the circumference of the first segment bisected at E; join AE, and produce it to meet the circumference of the second segment at F.

Then AE=EB [III. 28], and EB=EF,

since ∠s EFB, EBF=∠AEB=∠ACB

=twice ∠ADB=twice ∠AFB,

and ∴ ∠EBF=∠EFB.

Thus EA, EB, EF are all equal;

∴ E is the centre of the circle of which ADB is a segment. [III. 9.

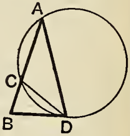
Hence AF is the longest straight line which can be drawn from A to the circumference of the described segment; so that the given straight line must not exceed twice AE.

67. *To describe an isosceles triangle having each of the angles at the base double of the third angle.*

This problem is solved in IV. 10; we may suppose the solution to have been discovered by such an analysis as the following.

Suppose the triangle ABD such a △ as is required, so that each of the ∠s at B and D is double of the ∠ at A.

Bisect the ∠ at D by the straight line DC.



Then the ∠ADC = the ∠ at A;

∴ CA = CD.

The ∠CBD = the ∠ADB, by hypothesis,

and the ∠CDB = the ∠ at A;

∴ the third ∠BCD = the third ∠ABD. [I. 32.

∴ BD = CD. [I. 6.

∴ BD = AC.

Since ∠BDC = the ∠ at A, BD is a tangent at D to the circle described round the △ACD (*Note on* III. 32).

∴ the rect. AB, BC = the square on BD. [III. 36.

∴ the rect. AB, BC = the square on AC.

∴ AB is divided at C in the manner required in II. 11.

Hence the synthetical solution of the problem is evident.

EXERCISES.

Describe a triangle having given its base, its vertical angle, and

1. another side;

2. its altitude;

3. the point where the perpendicular from the vertical angle meets the base;

4. the difference of its sides;

5. the sum of the squares on its sides;

[By Page 145 the vertex lies on a known arc of a circle going through the ends of the base; also, by Page 109, Ex. 1, the vertex lies on a circle whose centre is the middle point of the base; the vertex is thus at the intersection of these two circles.]

6. the difference of the squares on its sides;

7. the point where the bisector of the vertical angle meets the base;

8. the length of the median through the vertex;

9. the length of the median through one end of the base;

10. the difference of the angles at the ends of its base.

Describe a circle

11. with a given radius to touch two given straight lines;

12. with a given radius to touch two given circles;

13. with a given radius to touch a given circle and a given straight line;

[Since the required circle is to touch the given circle its centre is somewhere on a circle whose centre is that of the given circle and whose radius = sum of radius of given circle and given radius; since it touches given straight line the centre is on a straight line parallel to the given one; ∴ etc.]

14. with a given radius to pass through a given point and touch a given straight line;

15. with a given radius and centre in a given straight line to touch another given straight line;

16. through a given point to touch a given straight line at a given point;

17. to touch a given circle at a given point and also to touch a given straight line;

18. through a given point to touch a given circle at a given point;

19. to touch a given circle and also to touch two tangents to the circle;

20. to touch a given straight line at a given point and bisect the circumference of a given circle;

21. to pass through a given point and bisect the circumferences of two given circles;

22. Inscribe a circle in a given sector of a circle.

23. Find the centre of a circle cutting off equal chords from the sides.

24. With three given points as centre draw three circles which touch in pairs.

25. Find a point outside a circle such that the tangents from it to a circle meet at a given angle.

26. Draw a chord in a given circle equal to one given straight line and parallel to another.

Describe a triangle having given

27. the centres of its three escribed circles;

28. its in-centre and the centres of two escribed circles;

29. its pedal triangle;

30. its base, altitude, and circum-radius;

31. an ∠A, the perp^{r} from A upon the opposite side, and the in-radius;

32. the vertical ∠, the perimeter, and the in-radius;

33. the vertical ∠, the perimeter, and the altitude;

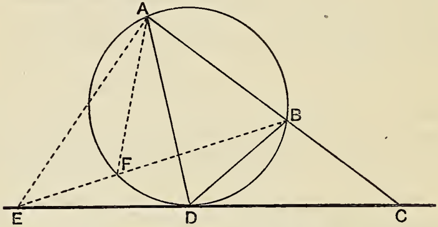
34. the base, one ∠ at the base, and the in-radius.

35. an angular point, the ortho-centre, and the circum-centre.

MAXIMA AND MINIMA [BOOKS III. AND IV.].

68. A *and* B *are two given points on the same side of a given straight line, and* AB *produced meets the given line at* C; *of all points in the given straight line on each side of* C, *it is required to determine that at which* AB *subtends the greatest angle.*

Describe a circle to pass through A and B, and to touch the given straight line on that side of C which is to be considered. Let D be the point of contact;
D shall be the required point.



For, take any other point E in CD on the same side of C as D; draw EA, EB; then one at least of these straight lines will cut the circumference ADB.

Suppose that BE cuts the circumference at F; join AF. Then the ∠AFB = the ∠ADB [III. 21]; and the ∠AFB > the ∠AEB [I. 16];

∴ the ∠ADB > the ∠AEB.

∴ the ∠ subtended by A, B at D is greater than the ∠ subtended at any other point of CD.

69. A *and* B *are two given points within a circle; and* AB *is drawn and produced both ways so as to divide the whole circumference into two arcs; it is required to determine the point in each of these arcs at which* AB *subtends the greatest angle.*

Describe a circle to pass through A and B and to touch the circumference considered [Art. 62]; the point of contact will be the required point. The demonstration is similar to that in the preceding proposition.

70. A *and* B *are two given points without a given circle; it is required to determine the points on the circumference of the given circle at which* AB *subtends the greatest and least angles.*

Suppose that neither AB nor AB produced cuts the given circle.

Describe two circles to pass through A and B, and to touch the given circle [Art. 62]; then G the point of contact of the circle which touches the given circle externally, will be the point where the angle is greatest, and F, the point of contact of the circle which touches the given circle internally, will be the point where the angle is least. The demonstration is similar to that in Art. 68.

If AB cuts the given circle, both the circles obtained by Art. 62 touch the given circle internally; in this case the angle subtended by AB at a point of contact is less than the angle subtended at any other point of the circumference of the given circle which is on the same side of AB. Here the angle is greatest at the points where AB cuts the circle, and is there equal to two right angles.

If AB *produced* cuts the given circle, both the circles obtained by Art. 62 touch the given circle externally; in this case the angle subtended by AB at a point of contact is greater than the angle subtended at any other point of the circumference of the given circle which is on the same side of AB. Here the angle is least at the points where AB produced cuts the circle, and is there zero.

EXERCISES.

****1.** Given the base and vertical angle, construct it when its area is greatest, and show that the tangent at its vertex is then parallel to its base.

****2.** Of all triangles inscribed in a circle, the equilateral one is the greatest.

[For if any triangle is such that the tangent at one vertex A is not $\|^l$ to the base BC, we can by Ex. 1 obtain a greater $\triangle$ by keeping BC fixed, and taking the vertex at a point where the tangent is $\|^l$ to the base; $\therefore$ the greatest $\triangle$ has the tangent at each angular point $\|^l$ to the opposite side; $\therefore$ etc.]

3. The pedal triangle of a triangle ABC is the triangle of least perimeter with its vertices on the sides of ABC. [Use Art. 27.]

4. Of all quadrilaterals inscribed in a circle, the inscribed square has the greatest area and the greatest perimeter.

5. Of all chords drawn through a given point within a circle, the least is the one that is bisected at the given point; prove also that it cuts off the least area from the circle.

6. Given a circle and two tangents, prove that the tangent which is such that the portion of it intercepted between the two tangents is a minimum, is bisected at the point of contact. Prove also that it cuts off a maximum or a minimum triangle.

BOOK VI.

71. Radical Axis. The radical axis of two circles is the locus of a point which moves so that the lengths of the tangents drawn from it to the two circles are equal.

72. A geometrical construction can be given for the radical axis of two circles.

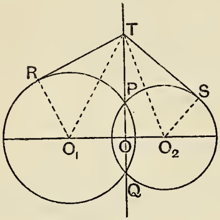


Fig. 1.

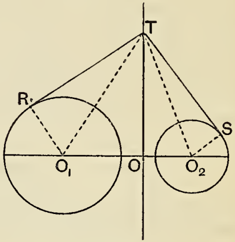


Fig. 2.

If the circles intersect in real points, P and Q, as in Fig. 1, the radical axis is clearly the straight line PQ. For if T be any point on PQ and TR and TS be the tangents from it to the circles, we have, by III. 36,

$$TR^2 = TP \cdot TQ = TS^2.$$

If they do not intersect in real points, as in the second figure, let T be a point such that the tangents TR and TS are equal in length.

Draw TO perpendicular to O_1O_2.

Since $$TR^2 = TS^2,$$

we have $$TO_1{}^2 - O_1R^2 = TO_2{}^2 - O_2S^2,$$

i.e. $$TO^2 + O_1O^2 - O_1R^2 = TO^2 + OO_2{}^2 - O_2S^2,$$

i.e. $$O_1O^2 - OO_2{}^2 = O_1R^2 - O_2S^2,$$

i.e. $$(O_1O - OO_2)(O_1O + OO_2) = O_1R^2 - O_2S^2,$$

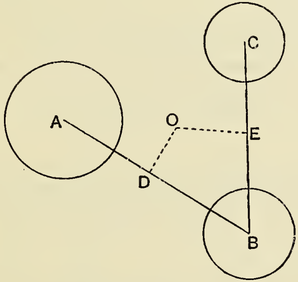
i.e. $$O_1O - OO_2 = \frac{O_1R^2 - O_2S^2.}{O_1O_2} = \text{a constant quantity.}$$

Hence O is a fixed point since it divides the fixed straight line O_1O_2 into parts whose difference is constant.

Therefore, since O_1OT is a right angle, the locus of T, *i.e.* the radical axis, is a straight line perpendicular to the line of centres.

73. If the radical axis be assumed to be a straight line it may also be constructed as follows: Draw any circle meeting one of the circles in E, F and the other in G, H; let EF and GH meet in P, then by III. 36 rect. PE, PF = rect. PG, PH. ∴ tangents from P to the two circles are equal and P is on the radical axis. The straight line through P perpendicular to the line of centres will be the radical axis.

74. *The radical axis of three circles, taken in pairs, meet in a point, which is such that the tangents from it to the three circles are equal.*



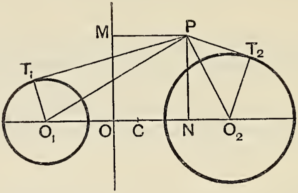
Let the three circles be called A, B, and C, and let the radical axis of A and B and that of B and C meet in a point O.

By the definition of the radical axis the tangent from O to the circle A = the tangent from O to the circle B, and the tangent from O to the circle B = tangent from it to the circle C.

Hence the tangent from O to the circle A = the tangent from it to the circle C, *i.e.* O is also a point on the radical axis of the circles A and C.

This point O is called the radical centre of the three circles.

75. *The difference in the squares on the tangents drawn from any point* P *to two circles is equal to twice the rectangle contained by the distance between their centres and the perpendicular from* P *upon their radical axis.*



Let PT_1, PT_2 be the tangents from a point P to two circles; O_1 and O_2 their centres and OM their radical axis; draw PM and PN perpendicular to the radical axis and the line of centres. Let C be the middle point of O_1O_2.

$$\text{Then } PO_1^2 - PO_2^2 = O_1N^2 - O_2N^2.$$

Also, since the tangents from O to the circles are equal,

$$\therefore\ OO_2^2 - O_2T_2^2 = OO_1^2 - O_1T_1^2,$$
$$O_2T_2^2 - O_1T_1^2 = OO_2^2 - OO_1^2.$$

$\therefore$ by addition,

$$PO_1^2 - O_1T_1^2 - (PO_2^2 - O_2T_2^2) = O_1N^2 - O_2N^2 + OO_2^2 - OO_1^2$$
$$= (O_1N + O_2N)(O_1N - O_2N) + (OO_2 + OO_1)(OO_2 - OO_1)$$
$$= O_1O_2(O_1N - OO_1 + OO_2 - O_2N) = 2\,O_1O_2 \,.\, ON$$
$$= 2\,O_1O_2 \,.\, PM.$$

Hence the proposition is true.

Corollary. If P lies on the circle of centre O_2 then PT_2 vanishes, and the theorem becomes: *The square of the tangent drawn from a point on one circle to another equals twice the rectangle contained by the perpendicular from the point on the radical axis of the circles and the distance between the centres of the circles.*

EXERCISES.

****1.** The radical axis of two circles bisects either of their common tangents.

****2.** If, of three circles, each touches the other two, the common tangents at the points of contact are concurrent.

**3. The locus of a point which moves so that the difference in the squares of the tangents drawn from it to two given circles is constant is a straight line parallel to the radical axis of the two circles.

4. Two circles meet in AB, and through A is drawn a chord PAQ cutting off from each circle segments containing equal angles. Prove that the tangents at P and Q meet in AB produced.

5. Prove the following construction for the radical axis of the circumcircle and the nine-point circle of a △ABC: Let D, E, F be the feet of the perpendiculars on the sides BC, CA, AB of the △ from the opposite vertices; let EF and BC meet in L, FD and CA in M, and DE and AB in N; then LMN is the required radical axis.

6. The radical centre of the three circles on the sides of a triangle as diameters is the orthocentre of the triangle. [Use Page xxxvi, Ex. 6.]

76. Coaxal Circles. A system of circles is said to be coaxal when they have a common radical axis, *i.e.* when the radical axis of each pair of circles of the system is the same.

Since the radical axis of any pair of the circles is perpendicular to the line joining their centres, it follows that the centres of all such circles of a coaxal system must lie on a straight line which is perpendicular to the radical axis.

If two of the circles intersect in two points P and Q, as in Fig. 1,

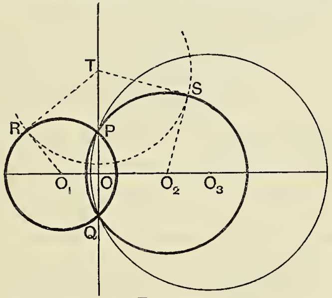


FIG. 1.

then all the circles of the system intersect in the same two points, and any circle of the system is then obtained by taking a point O_3 on the line of centres as centre, and the length of the straight line joining this point to the point P or Q as radius. The circles are then said to be of the Intersecting Species.

When two of the circles do not intersect as in Fig. 2 then any circle of the system can be constructed as follows:

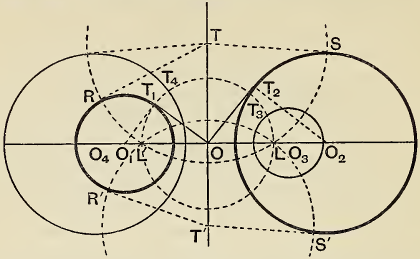


FIG. 2.

If O be the point of intersection of the radical axis and the line of centres, the length of the tangent drawn to every circle of the system from O is the same. If a circle be drawn with centre O, and radius equal to the tangent, OT_1 or OT_2, drawn to the two given circles as radius, the points of contact of tangents from O to any circle of the system lie on this circle. To construct the circle of the system whose centre is *any* point O_3, we draw a tangent O_3T_3 to the circle centre O; with centre O_3 and radius O_3T_3 describe a circle; this circle is one of the system. For OT_3 is the tangent to it from O and is of the required length. The circle (whose centre is either of the points of intersection, L and L′, of the dotted circle O with the line of centres) is of zero radius. For the length of the tangent from either L or L′ to the dotted circle is zero.

These two circles therefore reduce to points, and the points L and L′ are hence called the **Limiting Points**, since they are limiting positions of the circles of the system.

There are no circles of the system whose centres lie between O and L. For the tangent drawn from any such centre to the dotted circle, whose centre is O, is imaginary.

77. Let T be any point on the common radical axis of a system of coaxal circles and let TR be the tangent from it to any circle of the system.

Then a circle, whose centre is T and whose radius is TR (Fig. Art. 76) will cut each circle of the coaxal system orthogonally.

[For the radius TR of this circle is at right angles to the radius O_1R, and so for its intersection with any other circle of the system.]

Hence the limiting points (being point-*circles* of the system) are on this orthogonal circle.

The limiting points are therefore the intersections with the line of centres of *any* circle whose centre is on the common radical axis and whose radius is the tangent from it to any of the circles of the system.

Since, in Fig. 1, the limiting points are imaginary these orthogonal circles do not meet the line of centres in real points.

In Fig. 2 they pass through the limiting points L_1 and L_2.

These orthogonal circles (since they all pass through two points, real or imaginary) are therefore a coaxal system.

Also if the original circles, as in Fig. 1, intersect in real points, the orthogonal circles intersect in imaginary points; in Fig. 2 the original circles intersect in imaginary points, and the orthogonal circles in real points.

We therefore have the following theorem:

A set of coaxal circles can be cut orthogonally by another set of coaxal circles, the centres of each set lying on the radical axis of the other set; also one set is of the limiting-point species and the other set of the other species.

78. Without reference to the limiting points of the original system, it may be easily found whether or not the orthogonal circles meet the original line of centres.

For the circle, whose centre is T and whose radius is TR, meets or does not meet the line O_1O_2 according as TR^2 is $>$ or $<$ TO^2,

i.e. according as $TO_1^2 - O_1R^2$ is $\gtrless TO^2$,

i.e. according as $TO^2 + OO_1^2 - O_1R^2$ is $\gtrless TO^2$,

i.e. according as OO_1 is $\lessgtr O_1R$,

i.e. according as the radical axis is without, or within, each of the circles of the original system.

79. We can now deduce an easy construction for the circle that cuts any three circles orthogonally.

Consider the three circles in the figure of Art. 74.

By Art. 77 any circle cutting A and B orthogonally, has its centre on their common radical axis, *i.e.* on the straight line OD.

Similarly any circle cutting B and C orthogonally has its centre on the radical axis OE.

Any circle cutting all three circles orthogonally must therefore have its centre at the intersection of OD and OE, *i.e.* at the radical centre O. Also its radius must be the length of the tangent drawn from the radical centre to any one of the three circles.

EXERCISES.

**1. If there be three coaxal circles whose centres are O_1, O_2, O_3, the squares on the tangents drawn from any point of the third circle to the first two are to one another as O_1O_3 to O_2O_3. [Use Art. 75, Cor.]

**2. Conversely, if the squares on the tangents drawn from a point P to two circles (centres O_1 and O_2) be in a given ratio, the locus of P is a circle coaxal with the given ones, whose centre O_3 is such that $O_1O_3 : O_2O_3$ in the given ratio.

**3. Any circle passing through the limiting points of a set of coaxal circles cuts them all orthogonally.

4. The radical axis of a circle centre O_1 and a point O (considered as a point-circle) bisects at right angles the distance between O and the point in which its polar with respect to the circle meets OO_1.

5. Construct the circle which passes through a given point and cuts two circles orthogonally. [Treat the point as a point-circle, and apply Art. 79 and Ex. 4.]

**6. The polar of either limiting point with respect to any circle of the system passes through the other.

[The polar of L with respect to the circle O_1 goes through L′ if

$$O_1T_1{}^2 = O_1L' \cdot O_1L = O_1O^2 - OL^2$$

i.e. if $OO_1{}^2 = O_1T_1{}^2 + OL^2$

$$= O_1T_1{}^2 + OT_1{}^2, \text{ which is true.}]$$

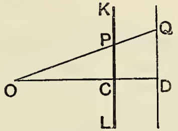
EXAMPLES OF LOCI. [BOOK VI.]

80. O *is a fixed point from which any straight line is drawn meeting a fixed straight line* KL *at* P; *in* OP *a point* Q *is taken such that* OQ *is to* OP *in a fixed ratio: determine the locus of* Q.

We shall shew that the locus of Q is a straight line.

Draw OC $\perp^r$ to KL and on it, produced if necessary, take a point D such that OD is to OC in the given ratio.

Join O to any point P on KL and on it take a point Q such that OQ is to OP in the fixed ratio; join QD.



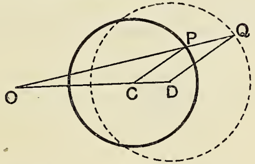
The $\triangle^s$ ODQ and OCP are similar (VI. 6);

$\therefore$ the $\angle$ODQ = the $\angle$OCP, and $\therefore$ = a right angle.

$\therefore$ Q lies in the straight line drawn through D at right $\angle^s$ to OD.

81. O *is a fixed point from which any straight line is drawn meeting a fixed circle at* P; *in* OP *a point* Q *is taken such that* OQ *is to* OP *in a fixed ratio: determine the locus of* Q.

We shall shew that the locus is a circle.



For let C be the centre of the fixed circle; in OC take a point D such that OD is to OC in the fixed ratio, and draw any radius CP of the fixed circle; draw DQ parallel to CP meeting OP, produced if necessary, at Q. Then the $\triangle^s$ OCP and ODQ are similar; [VI. 4.

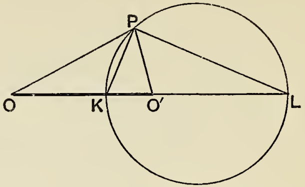
$\therefore$ OQ : OP :: OD : OC, that is, in the fixed ratio.

$\therefore$ Q is a point on the locus.

And DQ is to CP in the fixed ratio, so that DQ is of constant length. Hence the locus is a circle of which D is the centre.

82. *Find the locus of a point which moves so that its distances from two fixed points may be in a given ratio.*

Let O, O′ be the two fixed points, and let P be any point satisfying



the given condition, so that OP : PO′ in the given ratio. Bisect the interior angle OPO′ by a straight line PK meeting OO′ in K, and bisect the exterior angle at P by the straight line PL meeting OO′ in L.

Then, by VI. 3 and A,

OK : KO′ :: OP : PO′,

and OL : LO′ :: OP : PO′.

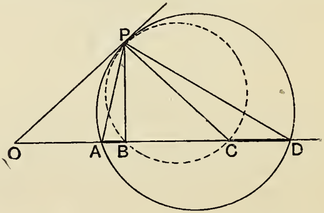
∴ the ratios OK : KO′ and OL : LO′ are each equal to the given ratio, and thus K, L are both fixed.

Since PK, PL bisect the interior and exterior angles at P,

∴ KPL is a right angle.

∴ the locus of P is a circle described on the straight line joining the two fixed points K, L as diameter.

83. *There are four given points* A, B, C, D *in a straight line; required the locus of a point at which* AB *and* CD *subtend equal angles.*



[We will take the case in which the points are in the following order, O, A, B, C, D.]

Let P be a point such that ∠APB = ∠CPD.

Describe a circle round the $\triangle APD$, and let PO be the tangent to it at P meeting the given line at O.

Then $\angle OPA = \angle ODP$; [III. 32.

$\therefore \angle OPB = \angle OPA + \angle APB = \angle ODP + \angle CPD = \angle OCP$;

$\therefore$ by the converse of III. 32, OP is the tangent at P to the circle circumscribing the $\triangle BPC$, and

$$\therefore OB \cdot OC = OP^2 = OA \cdot OD.$$

It follows, as will be shewn in Art. 89, that O is a determinate point and OB . OC is a known rect.;

$\therefore$ OP^2 is known, and thus OP is known, and the locus of P is a circle with O as centre.

EXERCISES.

**1. The locus of the centroid of a triangle, whose base and area are given, is a straight line.

**2. Given the base and vertical angle of a triangle, prove the locus of the intersection of its medians is an arc of a circle.

**3. Prove also that its in-centre, its orthocentre, and its e-centre opposite to the given vertical angle all move on arcs of circles which pass through the extremities of the base.

4. If a triangle of given species (*i.e.* one whose angles are given) has one angular point fixed, and if a second moves on a given straight line, the locus of the third angular point is also a straight line.

5. If the second angular point move on a circle, the locus of the third is also a circle.

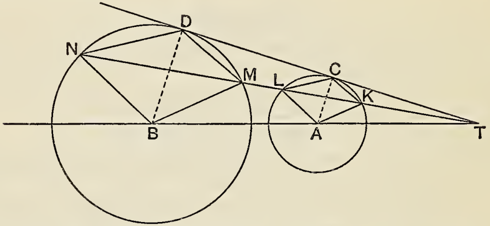
6. Given two points on a circle, find a third point on it whose distances from the two points are in a given ratio.

7. Find the locus of a point the perpendiculars from which on two given intersecting straight lines are in a given ratio.

8. Find a point the perpendiculars from which upon the sides of a triangle are in a given ratio

CENTRES OF SIMILITUDE.

84. *Let* A *be the centre of a circle, and* B *the centre of a larger circle; let a straight line be drawn touching the former circle at* C *and the latter circle at* D, *and meeting* BA *produced through* A *at* T. *From* T *draw any straight line meeting the smaller circle at* K *and* L, *and the larger circle at* M *and* N, *so that the five letters* T, K, L, M, N *are in this order. Then the straight lines* AK, KC, CL, LA *shall be respectively parallel to the straight lines* BM, MD, DN, NB; *and the rectangle* TK, TN *shall be equal to the rectangle* TL, TM, *and equal to the rectangle* TC, TD.



Join AC, BD. Then the △s TAC and TBD are equiangular;

∴ TA : TB :: AC : BD; [VI. 4.

:: AK : BM;

∴ the △s TAK, TMB are similar; [VI. 7.

∴ ∠TAK = ∠TBM;

∴ AK, BM are parallel.

Similarly AL, BN are parallel.

And because AK, BM are ∥l and AC, BD are ∥l;

∴ ∠CAK = ∠DBM;

∴ ∠CLK = ∠DNM; [III. 20.

∴ CL is parallel to DN.

Similarly CK is parallel to DM.

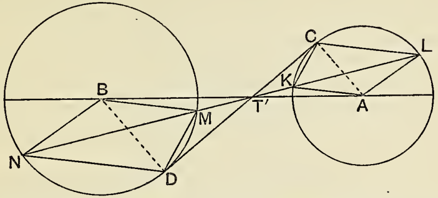
Now TD : TN :: TM : TD; [III. 37, VI. 16.

:: TK : TC; [VI. 4.

∴ rect. TC, TD = rect. TK, TN.

Similarly rect. TL, TM = rect. TC, TD.

If each of the given circles is without the other we may suppose the straight line which touches both circles to meet AB at T′ *between* A and B, and the above results will all hold, provided we interchange the letters K and L; so that the five letters are now to be in the following order, L, K, T′, M, N.



Since AK, BM are parallel, ∴ TK is to TM as AK to BM;
and similarly TL is to TN as AK to BN,
i.e. in the ratio of the radii;
∴ all straight lines drawn through T, or T′, are cut by the circles in the same constant ratio.

Hence either of the points, T or T′, is called a **Centre of Similitude**, one being the external centre and the other the internal centre.

Corollary. Since BDT, ACT are similar △s, we have
BT : AT = BD : AC = the ratio of the radii,
and similarly BT′ : AT′ = the ratio of the radii.
Hence the points, T and T′, in which the common tangents meet the line of centres divide the line joining the centres externally and internally in the ratio of the radii.

EXERCISES.

****1.** Two unequal similar figures of any number of sides ABCDE, A′B′C′D′E′ are placed so that the corresponding sides AB and A′B′, BC and B′C′, etc., are parallel; the lines AA′, BB′, CC′, etc., joining corresponding vertices all meet in the same point.

[This point is called the centre of similitude for the two figures.]

****2.** The circle of similitude of two circles is the locus of points at which the circles subtend the same angle; prove that it is the circle whose diameter is the straight line joining the centres of similitude.

[Let PQ, PQ′ be two tangents to the circles, centres A, B, such that ∠APQ = ∠BPQ′, and ∴ P a point on the required circle. Since AQP, BQ′P are similar △s, ∴ AP : BP = AQ : BQ′ = ratio of the radii;
∴ the locus of P is a circle on TT′ as diameter (Art. 82).]

**3. The orthocentre and the centroid of a triangle are the centres of similitude of the circumcircle and the nine-point circle.

4. The vertices of a triangle are the external centres of similitude of the in-circle and the three escribed circles, and are the internal centres of similitude for the three escribed circles taken in pairs.

5. The points in which the internal bisectors of the angles of a triangle meet the opposite sides are the internal centres of similitude of the in-circle and each of the escribed circles.

6. The points in which the exterior bisectors of the angles meet the opposite sides are the external centres of similitude of the escribed circles taken in pairs.

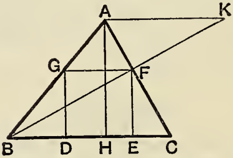
7. The circle of similitude of two circles is coaxal with them.

[Use Page lx, Ex. 2.]

8. The three circles of similitude of three circles, taken in pairs, are coaxal.

Constructions involving Book VI.

85. *To inscribe a square in a given triangle.*



Let ABC be the given △, and suppose DEFG the required square. Draw AH perpendicular to BC, and AK parallel to BC; and let BF produced meet AK at K.

Then BG : GF :: BA : AK,

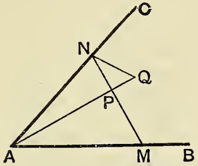
and BG : GD :: BA : AH; [VI. 4.

but GF = GD; [*Hypothesis.*

∴ AH = AK.

Hence we have the following synthetical solution. Draw AK parallel to BC, and equal to AH; and join BK. Then BK meets AC at one of the corners of the required square, and the solution can be completed.

86. *Through a given point between two given straight lines, it is required to draw a straight line, such that the rectangle contained by the parts between the given point and the given straight lines may be equal to a given rectangle.*



Let P be the given point, and AB and AC the given straight lines; suppose MPN the required straight line, so that the rectangle MP, PN is equal to a given rectangle.

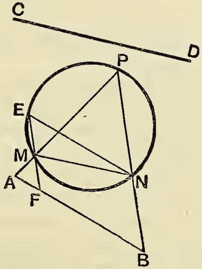
Produce AP to Q so that the rectangle AP, PQ may be equal to the given rectangle. Then the rectangle MP, PN is equal to the rectangle AP, PQ;

∴ a circle will go round AMQN; [Page 173, Ex. 1.

∴ ∠PNQ=∠PAM. [III. 21.

Hence we have the following synthetical solution. Produce AP to Q, so that the rectangle AP, PQ may=the given rectangle; describe on PQ a segment of a circle containing an ∠ equal to the ∠PAM, and let it meet AC in N; the straight line NPM solves the problem.

87. *In a given circle it is required to inscribe a triangle so that two sides may pass through two given points, and the third side be parallel to a given straight line.*



Let A and B be the given points, and CD the given straight line.

Suppose PMN to be the required △ inscribed in the given circle.

Draw NE parallel to AB; join EM, and produce it if necessary to meet AB at F.

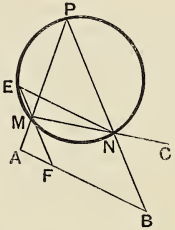
If the point F were known the problem might be considered solved. For ENM is a known ∠, and therefore the chord EM is known in magnitude. And then, since F is a known point, and EM is a known magnitude, the position of M becomes known. [Page 142, Ex. 12.]

We have then only to shew how F is to be determined.

The ∠MFA = the ∠MEN, [I. 29.
and ∴ = the ∠MPN; [III. 21.
∴ MAF, BAP are similar △s; [VI. 4.
∴ MA : AF :: BA : AP;
∴ rect. MA, AP = rect. AF, BA.

But since A is a *given* point the rectangle MA, AP is known; and AB is known; thus AF is determined.

88. *In a given circle it is required to inscribe a triangle so that the sides may pass through three given points.*

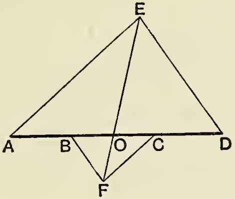


Let A, B, C be the three given points. Suppose PMN to be the required △ inscribed in the given circle.

Draw NE ||l to AB, and determine the point F as in the preceding problem. We shall then have to describe in the given circle a △EMN so that two of its sides may pass through given points, F and C, and the third side be ||l to a given straight line AB. This can be done by the preceding problem.

This example and the preceding are taken from the work of Catalan already cited. The present problem is sometimes called *Castillon's* and sometimes *Cramer's*; the history of the general researches to which it has given rise will be found in a series of papers in the *Mathematician*, Vol. III., by the late T. S. Davies.

89. *It is required to find a point in a given straight line, such that the rectangle contained by its distances from two given points in the straight line may be equal to the rectangle contained by its distances from two other given points in the straight line.*



Let A, B, C, D be four given points in the same straight line: it is required to find a point in AD such that the rectangle contained by its distances from A and B may = the rectangle contained by its distances from C and D.

On AD describe any triangle AED; and on CB describe a similar triangle CFB, so that CF is parallel to AE, and BF to DE; join EF, and let it meet the given straight line at O. Then O shall be the required point.

For, OE : OA :: OF : OC; [VI. 4.

∴ OE : OF :: OA : OC. [V. 16.

Similarly OE : OF :: OD : OB;

∴ OA : OC :: OD : OB; [V. 11.

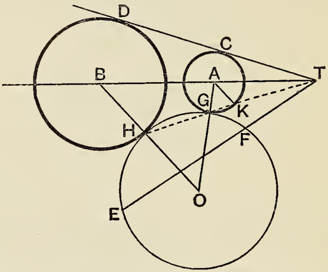
∴ rect. OA, OB = rect. OC, OD.

Aliter. Through A, B describe any circle and through C, D any other circle. Find the radical axis of these two circles. This radical axis meets the line ABCD in the required point O. For the rects. OA, OB and OC, OD are equal to the squares on the tangents from O to the circles, which are equal since O is on the radical axis.

The figure will vary slightly according to the situation of the four given points, but corresponding to an assigned situation there will be only *one* point such as is required.

90. *To describe a circle which shall pass through a given point and touch two given circles.*

Let A be the centre of the smaller circle and B the centre of the larger circle; and let E be the given point. Draw a straight line touching the former circle at C and the latter at D, and meeting BA, produced at T. Join TE and divide it at F so that the rect. TE, TF may = the rect. TC, TD.



Describe a circle to pass through E and F and touch either of the given circles (Art. 62); this shall be the required circle.

For suppose that the circle is described so as to touch the smaller given circle; let G be the point of contact; we have then to shew that the described circle will also touch the larger given circle. Join TG, and produce it to meet the larger given circle at H.

Then the rect. TG, TH = the rect. TC, TD [Art. 84.

that is, = the rect. TE, TF;

∴ the described circle passes through H.

Let O be its centre, so that OGA is a straight line; we have to shew that OHB is a straight line.

Let TG intersect the smaller circle again at K; then AK is parallel to BH (Art. 84); ∴ $\angle AKT = \angle BHG$,

and $\angle AKG = \angle AGK = \angle OGH = \angle OHG$;

∴ $\angle$s BHG, OHG = $\angle$s AKT, AKG = two right $\angle$s;

∴ OHB is a straight line.

Two solutions will be obtained, because there are two solutions of the problem in Art. 62. Also, if each of the given circles is without the other, two other solutions can be obtained by taking for T the point between A and B where a straight line touching the two given circles meets AB. The various solutions correspond to the cases when the contact of circles is external or internal.

91. *To describe a circle which shall touch three given circles.*

Let A be the centre of that circle which is not greater than either of the other circles; let B and C be the centres of the other circles. With centre B, and radius equal to the excess of the radius of the circle B over the radius of the circle A, describe a circle. Also, with centre C, and radius equal to the excess of the radius of the circle C over the radius of the circle A, describe a circle. Describe a circle to touch externally these two described circles and to pass through A (Art. 90). Then a circle having the same centre as the last described circle, and having a radius equal to the excess of its radius over the radius of the circle A, will touch externally the three given circles.

In a similar way we may describe a circle touching internally the three given circles, or touching one of them externally and the two others internally, or touching one of them internally and the two others externally.

EXERCISES.

****1.** Construct a triangle given the base, the vertical angle, and the ratio of the sides.

[By Page 145 its vertex lies on a known circle, and by Art. 82 its vertex also lies on a second known circle.]

****2.** Given the base and the ratio of the sides of a triangle, construct it when

(1) its area is given.

(2) the sum of the squares on its sides is given.

(3) the difference of the squares on its sides is given.

[Use Art. 82, I. 40, and Art. 32.]

Describe a triangle, having given

3. its vertical ∠, its circum-radius, and the ratio of its sides.

4. its vertical angle, its base, and the rectangle contained by its sides. [Use VI. C.]

5. the lengths of the perpendiculars from its vertices upon the opposite sides.

6. Inscribe a square in a given segment of a circle.

7. Describe an equilateral triangle to be equal in area to a given isosceles triangle.

8. Inscribe a regular octagon in a given square.

ARITHMETIC, GEOMETRIC, AND HARMONIC MEANS.

92. The Arithmetic mean between two straight lines is a straight line which is equal to half their sum.

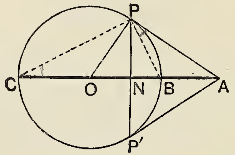
The Geometric mean is a straight line the square on which is equal to the rectangle contained by them.

The Harmonic mean is such that the first of the given straight lines is to the other as the excess of the first over the harmonic mean is to the excess of the harmonic mean over the second.

[These definitions are similar to those given in Algebra.]

93. *To construct these means.*

Let the two straight lines between which the means are to be found



be placed so as to have a common end, and be in the same direction as AB and AC.

Bisect BC in O. With centre O and radius OB, or OC, describe a circle, and draw tangents AP, AP′ to this circle. Let PP′ meet ABC in N.

Then AO *is the Arithmetic,* AP *the Geometric, and* AN *the Harmonic mean between* AB *and* AC.

For $AC - AO = OC = BO = AO - AB$;

$\therefore\ 2AO = AB + AC$, *i.e.* $AO = \dfrac{AB+AC}{2}$,

i.e. AO is the Arithmetic mean.

Also, by III. 36, $AP^2 = AB \cdot AC$,

i.e. AP is the Geometric mean.

Again, by III. 32, $\angle APB = \angle PCB$;

$\therefore$ the $\triangle^s$ APB, ACP are similar;

$\therefore\ AB : BP :: AP : PC$;

$\therefore\ AB^2 : AP^2 :: BP^2 : PC^2$;

i.e. $AB^2 : AB \cdot AC :: BN \cdot BC : CN \cdot CB$; [III. 36, and VI. 8.

$\therefore AB : AC :: BN : NC$;

$\therefore AC : AB :: NC : BN$

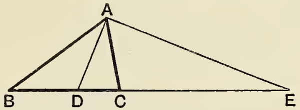
$:: AC - AN : AN - AB$;

$\therefore$ AN is the harmonic mean between AB and AC.

Corollary.—Since $AP^2 = AN \cdot AO$, AP is the geometric mean between AN and AO, that is,

The geometric mean between two given straight lines AB, AC *is also the geometric mean between the arithmetic and harmonic means between the same two straight lines.*

94. *Let* ABC *be a triangle, and let the interior and exterior angles at* A *be bisected by straight lines* AD, AE *which meet* BC *and* BC *produced in* D *and* E: *then* BD, BC, BE *shall be in harmonical progression.*



For $BD : DC :: BA : AC$, [VI. 3.

and $BE : EC :: BA : AC$; [VI. 4.

$\therefore BD : DC :: BE : EC$.

$\therefore BE : BD :: EC : DC$; [V. 16.

that is, as $BE - BC$ to $BC - BD$.

$\therefore$ BD, BC, BE are in harmonical progression.

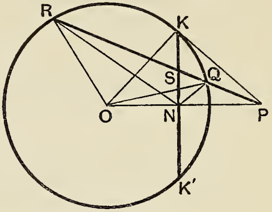
95. Def. When a straight line BC is divided internally and externally in the same ratio at D and E, the four points B, D, C, E are said to form a **Harmonic Range.** This result is sometimes expressed by saying that BC is divided harmonically at D and E.

The two points B and C are called **Harmonic Conjugates**, and so also the two points D and E.

If BDCE is a harmonic range and the points B, D, C, E be joined to a point A outside the range, the four straight lines AB, AD, AC, AE are called a **Harmonic Pencil,** and either of these four lines is called a **Ray** of the Pencil; also, the point A is called the **Vertex** of the pencil.

96. *Any straight line which passes through a fixed point is cut harmonically by the point, any circle, and the polar of the point.*

Let P be the fixed point and PQSR any straight line drawn through P to cut the circle in Q and R and the polar KNK′ of P in S. Then PQ, PS, PR shall be in harmonical progression.



Let KK′ meet OP in N. Join OK, OQ, and PK; then PK is the tangent at K and ∠PKO is a right ∠. [Art. 42, Cor. 2.

We have PN . PO = PK^2 [VI. 8.

= PQ . PR; [III. 36.

∴ Q, R, O, N lie on a circle;

∴ ∠ORQ = supplement of ∠ONQ = ∠PNQ.

But ∠ORQ = ∠OQR, since OQ = OR.

Also ∠OQR = ∠ONR in the same segment,

since O, N, Q, R lie on a circle;

∴ ∠ONR = ∠PNQ.

∴ ∠RNS = ∠QNS, since ONK, PNK are both right angles.

∴ NS is the bisector of the ∠RNQ, and since KNP is a right ∠, NP is the exterior bisector of the same ∠. ∴ (by Art. 94) PQ, PS, and PR are in harmonical progression.

EXERCISES.

1. If ACBD be a harmonic range and E be the middle point of CD, prove that EA . EB = EC^2, and conversely.

2. Prove also that AC . BD = AD . BC.

3. If in a harmonic pencil one ray bisect the angle between a pair of rays the fourth ray is at right angles to the first, and conversely.

[Let ACBD be a harmonic range, and O the vertex of the corresponding pencil, so that OC bisects the ∠AOB and

AC : CB :: AO : OB.

But, by the previous exercise,

AC : CB :: AD : DB ∴ AD : DB :: AO : OB.

∴ OD bisects the exterior ∠ of AOB, etc.]

4. Two circles cut at right angles, prove that any diameter of the one is divided harmonically by the other. [Use Ex. 3, Page xxxiii.]

5. The circum-centre, the centroid, the nine-point centre, and the orthocentre of a triangle form a harmonic range.

6. Any straight line parallel to one of the rays of a harmonic pencil is divided into equal parts by the other rays, and conversely.

[Let ACBD be the harmonic range, and OA, OC, OB, OD the rays of the pencil. Through C draw NCM ||[l] to OD to meet OA, OB in N, M.

Then we are given that AC : CB :: AD : DB,

i.e. AC : AD :: CB : DB,

i.e. CN : DO :: CM : DO;

∴ CN = CM.

Also, by Art. 3, the median OC of the △ NCM bisects all lines parallel to the base NM; ∴ etc.]

7. If the rays of any harmonic pencil be cut by *any* transversal, the points of intersection form a harmonic range. [Proceed as in Exercise 6.]

8. If four points form a harmonic range, their polars with respect to any circle give a harmonic pencil.

[For the straight lines joining the four points to the centre are, by Art. 43, inclined at the same angles as the polars of the points, etc.]

9. The polar circle of a triangle divides the sides harmonically.

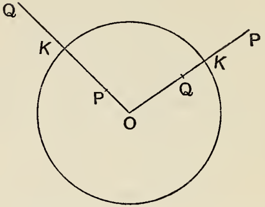
10. If two harmonic ranges are such that the straight lines joining three pairs of corresponding points meet in a point, the straight line joining the fourth pair passes through the same point.

[Let the two ranges be ACBD, A′C′B′D′ and let AA′, CC′, BB′ meet in O, and let OD meet A′C′B′ in D_1. Then, by Ex. 7, A′C′B′D_1 is a harmonic range; but, by hypothesis, so also is A′C′B′D′. ∴ D′ and D_1 coincide, etc.]

11. If two harmonic pencils are such that the intersections of three pairs of corresponding rays are on a straight line, the intersection of the fourth pair is on the same straight line.

INVERSION.

97. Let O be the centre of a fixed circle, and let P be any point in its plane, within or without the circle. Join OP; and let OP, pro-



duced if necessary, meet the circle in K. On OP take a point Q, such that

$$OP \cdot OQ = OK^2 = (\text{radius})^2.$$

Then Q is called the inverse of the point P with respect to the circle.

[By Art. 42 it follows that Q is the point where OP meets the polar of P.]

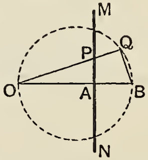
There is no real necessity for drawing the circle. If we put $OK = k$, then $OQ \cdot OP = k^2$, *i.e.* $OQ = \frac{k^2}{OP}$, and k^2 may be called the constant of inversion.

The inverse of a point P with respect to a fixed point O is thus a point Q lying on OP, or OP produced, such that $OQ = \frac{k^2}{OP}$.

If the point P move on a certain curve, the corresponding point Q will move on another curve, and the locus of Q is called the inverse of the locus of P.

We shall find the inverse of a straight line and also that of a circle.

98. *To find the inverse of a straight line.*



Let MN be the straight line whose inverse with respect to O is required.

Take any point P on MN ; join OP and on it, produced if necessary, take a point Q such that OP . OQ $=k^2$.

Draw OA perpendicular to MN and on it take a point B, such that OA . OB $=k^2$. Join BQ.

Since OA . OB = OP . OQ,

∴ A, P, Q, B lie on a circle ; [III. 36.

∴ ∠PAB + ∠PQB = two rt. ∠s. [III. 22.

But ∠PAB = a rt. ∠ ; [*Construction.*

∴ ∠PQB, that is ∠OQB, is a right angle.

∴ the locus of Q is a circle described on OB as diameter.

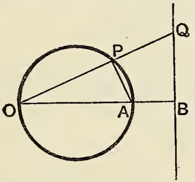
The inverse of a straight line with respect to a fixed point O *is therefore a circle, which passes through* O, *and whose diameter lies along the perpendicular drawn from* O *to the given straight line.*

Conversely, since if $OP=\frac{k^2}{OQ}$, then $OQ=\frac{k^2}{OP}$, the locus of P is obtained from that of Q by the same process by which the locus of Q is derived from that of P.

It follows that the inverse of the circle OQB with respect to O is the straight line MN.

This is more directly proved in the following theorem.

99. *To find the inverse of a circle with respect to a point* O, *which is* (1) *on the circumference,* (2) *not on the circumference.*



(1) Draw the diameter OA and on it take a point B such that OA . OB $=k^2$, the constant of inversion. B is then a fixed point.

Join O to any point P on the curve, and on it take Q such that OP . OQ $=k^2$. Then Q is the inverse of P.

Since OP . OQ = OA . OB,

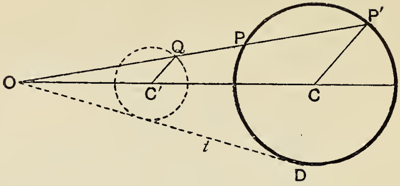
∴ P, Q, B, A lie on a circle ; [III. 36.

∴ ∠ABQ + ∠APQ = two right ∠s. [III. 22.

But ∠APQ is a right ∠, [III. 31.

∴ OBQ is a right angle, and thus Q lies on a straight line which is at right angles to OA and passes through the fixed point B.

(2) Let O not lie on the given circle.



Let C be the centre of the circle. Join O to any point P on the circle, and on it take Q such that OP . OQ $=k^2$, the constant of inversion.

Let OP produced meet the circle in P′. Draw OD the tangent from O to the circle, and for simplicity put OD $=t$.

Then OP . OP′ = OD$^2=t^2$; [III. 36.

$\therefore$ OP . OQ : OP . OP′ :: k^2 : t^2,

that is, OQ : OP′ :: k^2 : t^2.

Draw QC′ parallel to P′C to meet OC in C′.

Then OC′ : OC :: OQ : OP′ [VI. 2.

:: k^2 : t^2;

$\therefore$ OC′ is in a constant ratio to OC, and

$\therefore$ C′ is a fixed point.

Also C′Q : CP′ :: OQ : OP′

:: k^2 : t^2;

$\therefore$ C′Q is constant, since CP′ is constant;

$\therefore$ since Q is at a constant distance from a fixed point C′ its locus is a circle whose centre is C′.

The inverse of a circle with respect to a point outside it in its own plane is therefore a circle.

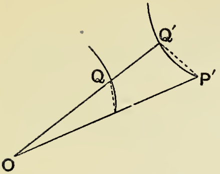
100. *If* P *be any point on a circle which is inverted with regard to a point* O, *and if* P′ *be the point corresponding to* P, *the tangent to the first circle at* P *makes the same angle with* OPP′ *that the tangent to the second circle at* P′ *makes with* OPP′.

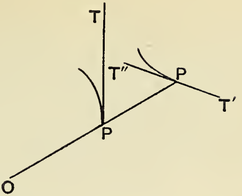
Let Q be a point on the first circle very close to P. Join OP, OQ, and on them, produced if necessary, take points P′, Q′ such that

OP . OP′ = OQ . OQ′ = constant of inversion.

Then P′, Q′ are points, very close together, on the inverse circle.

Also, since OP . OP′ = OQ . OQ′, the four points P, Q, Q′, P′ lie on a circle, and $\therefore$ $\angle$OPQ = supplement of $\angle$QPP′ = $\angle$OQ′P′. [III. 22.





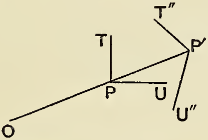
Now let Q move up indefinitely close to P and then PQ becomes the tangent at P, and the angle OPQ becomes the ∠ between OP and the tangent at P, viz. the ∠OPT. Similarly P′Q′ becomes the tangent at P′, and OQ′P′ the angle between OP′ and the tangent at P′, viz. the ∠OP′T′.

These two tangents are thus equally inclined to OP, but they are not parallel.

Corollary 1. The same is true if the circle PQ be replaced by a straight line or if the inverse curve P′Q′ be a straight line.

Corollary 2. The angles which the two tangents PT, P′T″ make on the same side of OPP′ [viz. ∠OPT and T″P′P] are supplementary.

101. *A circle and a straight line, or two circles, intersect at the same angle as their inverses.*



Let P be a point of intersection of the first two circles, and PT, PU their tangents at P.

Let P′ be the inverse point of P and P′T″, P′U″ the tangents there.

$$\left.\begin{array}{r}\text{Then } \angle TPP' = T''P'P,\\ \text{and } \angle UPP' = U''P'P\,;\end{array}\right\}$$

$$\therefore \text{ by addition, } \angle TPU = \angle T''P'U'';$$

that is, the tangents at P intersect at the same angle as the tangents at P′.

Corollary. If the two given circles touch at P, the angle between their tangents at P vanishes; ∴ the angle between the tangents to their inverses at P′ vanishes, *i.e.* their inverses touch.

EXERCISES.

1. Any two points and their inverses are concyclic.

2. The ratio of the squares on the distances of any point on a circle from two inverse points with respect to the circle is equal to the ratio of the distances of the centre of the circle from the same two points.

[Let P, Q be the inverse points, U the middle point of PQ, O the centre, T the point on the circle, TN the perp^r on OP.

$$\begin{aligned} \text{Then } TP^2 &= OT^2 + OP^2 - 2OP \cdot ON \\ &= OP \cdot OQ + OP^2 - 2OP \cdot ON \\ &= 2OU \cdot OP - 2OP \cdot ON \\ &= 2NU \cdot OP. \end{aligned}$$

$$\text{So } TQ^2 = 2NU \cdot OQ;$$

$$\therefore\ TP^2 : TQ^2 :: OP : OQ.]$$

3. The centre of inversion O is the centre of similitude of the original circle and its inverse.

[For CP′, C′Q are parallel radii. Art. 84.]

4. Any circle can be inverted into itself.

[If the constant of inversion be taken equal to the square on the tangent OD that can be drawn from O to the circle, every such point as P inverts into P′.]

5. Any two circles can be inverted into themselves.

[Take as centre of inversion any point O on their radical axis and as constant of inversion the square on the equal tangents from O to either circle; by Art. 99 each circle inverts into itself.]

6. Any three circles can be inverted into themselves.

[Take as centre of inversion the radical centre, and proceed as in Ex. 5.]

7. A circle and a diameter invert, in general, into two circles cutting orthogonally.

[For the circle and diameter cut orthogonally and the diameter inverts into a circle; then use Art. 101.]

8. Any three circles can be inverted into three circles whose centres lie on a straight line.

[Invert with respect to any point on their orthogonal circle (Art. 79); the latter becomes a straight line which cuts the inverses of the three circles orthogonally, and the only straight line cutting three circles orthogonally passes through their centres; these latter thus lie on a straight line.]

9. Any two circles can be inverted into equal circles.

[If O be the centre of inversion and t, t' the lengths of the tangents to the circles, of radii r and r', then by Art. 99 (2) we must have $t^2 : t'^2 :: r : r'$. Hence O is a point such that the ratio of the squares of the tangents from it to the two given circles is known. Hence by Art. 79, Ex. 2, O is any point on a known circle, viz. the circle coaxal with the given ones whose centre is the external centre of similitude.]

10. Any three circles can be inverted into three equal circles.

11. When two circles intersect the circle through their common points, and the centre of inversion inverts into the radical axis of the new circles.

12. If A, B, C, D be four points and A′, B′, C′, D′ their inverses,

$$\text{then } A'B' : AB :: k^2 : OA \,.\, OB,$$

$$\text{and } AC \,.\, BD : AB \,.\, CD = A'C' \,.\, B'D' : A'B' \,.\, C'D'.$$

[The $\triangle^s$ OB′A′, OAB are similar;

$$\therefore\ A'B' : AB :: OB' : OA$$
$$:: OB \,.\, OB' : OA \,.\, OB$$
$$:: k^2 : OA \,.\, OB \text{ etc.]}$$

13. Prove Euc. VI., D by inversion.

[Use Ex. 12 and the Exercise on page 91.]

14. P, Q are a pair of inverse points with respect to a circle C; any circle through them cuts any diameter of the circle in a pair of inverse points.

ON MODERN GEOMETRY.

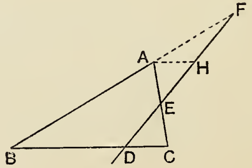
102. We have hitherto restricted ourselves to Euclid's Elements, and propositions which can be demonstrated by strict adherence to Euclid's methods. In modern times various other methods have been introduced, and have led to numerous and important results. These methods may be called semi-geometrical, as they are not confined within the limits of the ancient pure geometry; in fact the power of the modern methods is obtained chiefly by combining arithmetic and algebra with geometry. The student who desires to cultivate this part of mathematics may consult Townsend's *Chapters on the Modern Geometry of the Point, Line, and Circle.*

We will give as specimens some important theorems, taken from what is called the theory of transversals.

Any line, straight or curved, which cuts a system of other lines is called a **transversal**; in the examples which we shall give, the lines will be straight lines, and the system will consist of three straight lines forming a triangle.

We will give a brief enunciation of the theorem which we are about to prove, for the sake of assisting the memory in retaining the result; but the enunciation will not be fully comprehended until the demonstration is completed.

103. *If a straight line cut the sides, or the sides produced, of a triangle, the product of three segments in order is equal to the product of the other three segments.*



Let ABC be a triangle, and let a straight line be drawn cutting the side BC at D, the side CA at E, and the side AB produced through

B at F. Then BD and DC are called *segments* of the side BC, whilst CE and EA are called *segments* of the side CA, and AF and FB are called *segments* of the side AB.

Through A draw a straight line parallel to BC, meeting DF produced at H.

Then the $\triangle^s$ CED and EAH are equiangular;

∴ AH : CD :: AE : EC. [VI. 4.

∴ the rect. AH, EC = the rect. CD, AE. [VI. 16.

Again, the $\triangle^s$ FAH and FBD are equiangular;

∴ AH : BD :: FA : FB. [VI. 4.

∴ the rect. AH, FB = the rect. BD, FA. [VI. 16.

Now suppose the straight lines represented by numbers in the manner explained in the notes to the second Book of the Elements. We have then two results which we can express arithmetically: namely,

the *product* AH . EC = the *product* CD . AE;

and the *product* AH . FB = the *product* BD . FA.

Therefore, by the principles of Arithmetic, the product

AH . EC . BD . FA = the product AH . FB . CD . AE,

and therefore the product BD . CE . AF = the product DC . EA . FB.

This is the result intended by the enunciation given above. Each product is made by three segments, one from every side of the triangle: and the two segments which terminated at any angular point of the triangle are never in the same product.

Thus if we begin one product with the segment BD, the other segment of the side BC, namely, DC, occurs in the other product; then the segment CE occurs in the first product, so that the two segments CD and CE, which terminate at C, do not occur in the same product; and so on.

The student should for exercise draw another figure for the case in which the transversal meets *all* the sides *produced*, and obtain the same result.

The above theorem is generally known as **Menelaus' Theorem**, after Menelaus, an Alexandrian mathematician who lived in the 1st century after Christ.

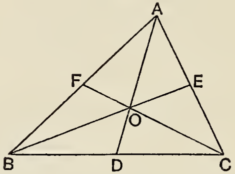
104. Conversely, it may be shewn by an indirect proof that if the product BD . CE . AF be equal to the product DC . EA . FB, the three points D, E, F lie in the same straight line.

105. *If three straight lines be drawn through the angular points of a triangle to the opposite sides, and meet at the same point, the product of three segments in order is equal to the product of the other three segments.*

Let ABC be a triangle. From the angular points to the opposite sides let the straight lines AOD, BOE, COF be drawn, which meet at the point O: *the product* AF . BD . CE = *the product* FB . DC . EA.

For the △ ABD is cut by the transversal FOC;

∴ AF . BC . DO = FB . CD . OA. [Art. 103.



Again the △ ACD is cut by the transversal EOB;

∴ AO . DB . CE = OD . BC . EA. [Art. 103.

∴ by the principles of arithmetic,

AF . BC . DO . AO . DB . CE = FB . CD . OA . OD . BC . EA.

∴ AF . BD . CE = FB . DC . EA.

We have supposed the point O to be within the triangle; if O be without the triangle two of the points D, E, F will fall on the sides produced.

Conversely, it may be shewn by an indirect proof that if the product AF . BD . CE be equal to the product FB . DC . EA, the three straight lines AD, BE, CF meet at the same point.

The theorem of the above article is generally known as **Ceva's Theorem**, after Ceva, in whose work, published A.D. 1678, it is first found.

EXERCISES.

****1.** The three medians of a triangle are concurrent.

****2.** The three internal bisectors of the angles of a triangle are concurrent.

****3.** Any two external bisectors and the third internal bisector of the ∠ˢ of a △ are concurrent.

****4.** The straight lines joining each angular point to the point of contact of the opposite side of the triangle with the in-circle are concurrent.

5. ABC is a triangle, and O any point in its plane, prove that the external bisectors of the angles BOC, COA, AOB meet BC, CA, AB respectively in points lying on a straight line.

6. The in-circle of a triangle ABC touches the sides BC, CA, AB in D, E, and F; EF, FD, DE produced meet BC, CA, AB respectively in L, M, and N; prove that L, M, N are collinear.

7. Two parallelograms ACBD, A′CB′D′ have a common angle at C. Prove that DD′, A′B, and AB′ are concurrent.

[Let DD′, A′B meet in E and A′D′, BD in G.

Then $BD \cdot GD' \cdot A'E = DG \cdot D'A' \cdot EB$, [Art. 103.

that is, $AC \cdot BB' \cdot A'E = AA' \cdot B'C \cdot EB$,

that is, $CA \cdot A'E \cdot BB' = AA' \cdot EB \cdot B'C$,

so that B′AE is a transversal of the △A′BC.]

8. The line joining the points of contact of the in-circle with two sides of a triangle meets the third side in a point which, with the ends of the third side and the point of contact with the third side, form a harmonic range. [Use Menelaus' Theorem.]

9. If the tangents to the circumcircle of the △ABC at its angular points meet the opposite sides in D, E, F, prove that DEF is a straight line.

[$BD : CD = BD^2 : BD \cdot CD = BD^2 : DA^2$.

$= BA^2 : AC$ by similar triangles; ∴ etc.]

10. ABC is a triangle inscribed in a circle, and D, E, F are the middle points of its sides BC, CA, AB; if A′, B′, C′ are the points of the circumcircle diametrically opposite to A, B, C, prove that A′D, B′E, C′F are concurrent. [They meet in the orthocentre.]

11. If X, Y, Z are points in the sides BC, CA, AB of a △ABC such that the perpendiculars to the sides at these points are concurrent, then $BX^2 + CY^2 + AZ^2 = CX^2 + AY^2 + BZ^2$, and conversely.

12. The perpendiculars from the e-centres upon the sides of a triangle are concurrent. [Use the previous Exercise.]

13. If three concurrent straight lines through the angular points of a triangle ABC meet the opposite sides in D, E, F, and if the circle through D, E, F meet these sides in D′, E′, F′, prove that AD′, BE′, CF′ are also concurrent.

14. The internal and external bisectors of the angles of a triangle are drawn; their six points of intersection with the opposite sides lie, three by three, on four straight lines.

15. The six centres of similitude of three non-intersecting circles lie, three by three, on four straight lines.

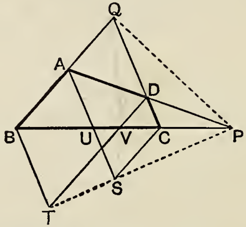
COMPLETE QUADRILATERALS.

106. Let ABCD be an ordinary Euclidean quadrilateral. Let AD and BC be produced to meet in P, and let AB and CD meet in Q. The figure is now said to be a complete quadrilateral, since it gives all the points of intersection of the four lines AB, BC, CD, DA produced indefinitely.

The straight line PQ is called the third diagonal, the other two being AC and BD.

Also, if AC, BD meet in R, then P, Q, and R are called the three vertices.

107. *The middle points of the diagonals of a complete quadrilateral lie on a straight line.*



Complete the parallelograms QASC, QBTD.

Let AS, DT meet BP in U and V.

Then PC : CU :: PD : DA, since CD, UA are ||[l],

i.e. .: PV : VB, since VD, BA are ||[l];

∴ PC : PV :: CU : VB,

:: CS : VT, since CUS, VBT are similar △[s];

∴ PST is a straight line;

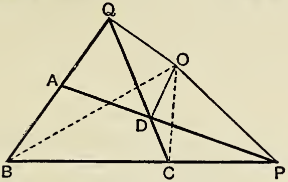
∴ the middle points of QP, QS, QT are in a straight line parallel to PST. [Art. 24.

But since QBTD, QASC are ||[gms], the middle points of QT, QS are the middle points of BD, AC;

∴ the middle points of the three diagonals BD, AC, QP lie on a straight line.

108. *The four circles circumscribing the four triangles that can be formed by four straight lines meet in a point.*

Let the four lines be BAQ, BCP, ADP, CDQ, as in the figure.



Let the circles circumscribing the two $\triangle^s$ QAD, DCP meet in the point O.

Then O *shall lie on the circumcircle of the* $\triangle$ BQC.

Because D, O, Q, A lie on a circle,

$\therefore$ $\angle$DOQ = the supplement of $\angle$QAD [III. 22] = $\angle$BAP.

Since D, O, P, C lie on a circle, $\therefore$ $\angle$DOC = $\angle$DPC, [III. 21.

$\therefore$ $\angle$QOC = $\angle$QOD + $\angle$DOC = $\angle$BAP + $\angle$BPA ;

$\therefore$ $\angle^s$ QOC, QBC together = the three $\angle^s$ of the $\triangle$ PAB = 2 rt. $\angle^s$;

$\therefore$ Q, B, C, O lie on a circle, that is, O is on the circumcircle of the $\triangle$ QBC.

Similarly it is on the circumcircle of the $\triangle$ PAB. Hence etc.

Corollary 1. The feet of the perpendiculars from O on the four straight lines are collinear.

For considering the $\triangle$QAD, since O is on its circumcircle the feet of the perpendiculars from O on the straight lines AB, AD, CD are collinear.

So considering the $\triangle$QBC, the feet of the perpendiculars on the straight line AB, BC, CD are collinear.

Hence the feet of the perpendiculars on the four straight lines AB, BC, CD, DA lie on a common line.

Corollary 2. The orthocentres of the four are collinear.

For, since O lies on each circumcircle, the middle points of the lines joining O to the orthocentres all lie on the common pedal line. [Art. 55.

$\therefore$ by Art. 24 the orthocentres all lie on a straight line parallel to the common pedal line.

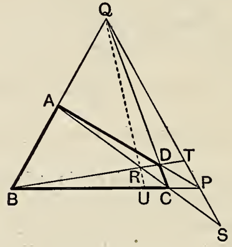
Corollary 3. If ABCD be a cyclic quad[l], then the point O lies on the third diagonal PQ. For $\angle$POD = $\angle$BCQ and $\angle$QOD = $\angle$BAD.

$\therefore$ $\angle$POD + $\angle$QOD = 2 rt. $\angle^s$, since ABCD is cyclic.

$\therefore$ POQ is a str. line.

109. *In a complete quadrilateral each diagonal is divided harmonically by the other two diagonals.*

Let the quadrilateral be formed by the straight lines AB, BC, C,D DA and let P, Q, R be the vertices of the complete quadrilateral; let AC, BD produced meet PQ in S and T. Join QR, and produce it to meet BP in U.



By Art. 105, since R is a point in the △ BCQ,

∴ BU . CD . QA = UC . DQ . AB. ....(1)

By Art. 103, since ADP is a transversal of the same △,

∴ BP . CD . QA = PC . DQ . AB. ....(2)

From (1), (2), BU : BP :: UC : PC, *i.e.* BU : UC :: BP : PC.

∴ B, U, C, P form a harmonic range; [Art. 95.

∴ QB, QU, QC, QP are a harmonic pencil; [Art. 95.

∴ B, R, D, T is a harmonic range, [Page lxxv, Ex. 7]

and similarly A, R, C, S is a harmonic range, so that the diagonal BD is divided harmonically at R, T and the diagonal AC at R, S.

Also, because A, R, C, S form a harmonic range;

∴ BA, BR, BC, BS form a harmonic pencil,

and ∴ Q, T, P, S form a harmonic range, so that the diagonal QP is divided harmonically at S, T.

Corollary. Since B, R, D, T and A, R, C, S and also Q, T, P, S are harmonic ranges;

∴ the pencils at P, Q, R, viz. PB, PR, PA, PQ and QA, QR, QD, QP and also RQ, RD, RP, RC are harmonic pencils.

110. *Given three points of a harmonic range, to find the fourth.*

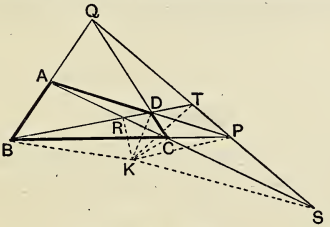
Let B, C, P be the three collinear points. Through P draw any line, and through B, C draw any two lines, BAQ and CDQ, meeting in Q, and cutting the line through P in A and D. Join AC and BD and let them meet in R. Join QR and produce it to meet BCP in U. Then by the last article U is the required fourth point of the range.

It will be noted that this construction requires the use of the ruler only.

111. *The circles on the three diagonals of a complete quadrilateral as diameters have a common radical axis, which is perp^r to the line through the middle points of the diagonals and contains the orthocentres of the four triangles formed by taking the sides of the quadrilateral, three by three. Prove also that these three circles cut orthogonally the self-conjugate circles of these four triangles.*

Draw a complete quadrilateral as in Art. 109. Let K be one point of intersection of the circles on AC, BD as diameters.

Then KB, KR, KD, KT form a harmonic pencil and BKD is a right ∠.



∴ KD bisects ∠ RKT; [Page lxxiv, Ex. 3]

∴ RD : DT :: RK : KT.

Similarly KC bisects ∠ RKS; [Page lxxiv, Ex. 3]

∴ RC : CS :: RK : KS.

∴ RD . KT = DT . RK and RC . KS = CS . RK;

∴ RD . KT . CS = DT . RC . KS.

But since CDQ is a transversal of △ RST,

∴ RD . TQ . SC = DT . QS . CR;

∴ KT : TQ :: KS : QS,

i.e. TK : KS :: TQ : QS.

∴ KQ bisects ∠ TKS externally.

Also KS, KP, KT, KQ is a harmonic pencil; [Art. 109.

∴ PKQ is a right angle. [Page lxxiv, Ex. 3]

Hence the circle on PQ as diameter goes through K, and similarly through the other point of intersection of the circles on AC, BD as diameter.

Again, the circle on PQ as diameter passes through P and the foot of the perpendicular from P on the side of BA of the △ ABP, *i.e.* it passes through two inverse points with respect to the self-conjugate circle of this △. [Page xxxvi, Ex. 7]

Hence, by Art. 45, it cuts this self-conjugate circle orthogonally. Similarly for the self-conjugate circles of the other three triangles.

Similarly the circles on AC, BD cut these same four self-conjugate circles orthogonally.

Also, since the three circles on the diameters are a coaxal system cut orthogonally by the four self-conjugate circles, the latter are a system whose centres are on the radical axis of the first three;
i.e. the four orthocentres lie on the radical axis of the circles on the diagonals as diameters, and therefore are on a straight line which is perpendicular to the line through their centres, and this is the line through the middle point of the diagonals of the quadrilateral.

EXERCISES.

1. The centres of the four circles circumscribing the four triangles formed by four co-planar lines lie on a circle which passes through the common point of the four circles.

2. The sides of a quadrilateral inscribed in a circle are produced to meet in P and Q; prove that (1) the bisectors of the angles at P and Q are perpendicular; (2) the square on PQ = the sum of the squares on the tangents from P, Q to the circle; and (3) the circle on PQ as diameter cuts the given circle orthogonally.

3. The circle described on any of the three diagonals of a complete quadrilateral as diameter cuts orthogonally the circle circumscribing the triangle formed by the three diagonals.
[Use Art. 109 and Page lxxv, Ex. 4.]

4. Draw tangents to a circle by means of a ruler only.
[Use Art. 96 and the harmonic properties of the complete quadrilateral.]

5. If a quadrilateral be inscribed in a circle and its sides be produced, and the angular points joined, so as to make a complete quadrilateral, the triangle formed by the three vertices of the latter is self-polar with respect to the circle. [Use Art. 96 and Cor. Art. 109.]

6. If ABCD (Fig. Art. 107) be a quadrilateral circumscribed about a circle, one pair of the straight lines joining the points of contact with the circle passes through the intersection of the diagonals AC, BD, and the other two pairs intersect on the third diagonal PQ.

[Let the circle touch the sides AB, BC, CD, DA in K, L, M, N. Let AC meet BD in R; let KL, MN meet in E; and KN, LM meet in F.

Then A is the pole of KN and C is the pole of LM;

∴ F is the pole of AC. [Page xxxii, Ex. 1.]

So E is the pole of BD;

∴ R (the intersection of AC, BD) is the pole of EF.

But the intersection of KM, LN is the pole of EF; [Ex. 5.

∴ KM, LN intersect in R.

Again, P is the pole of LN, and Q is the pole of KM;

∴ PQ is the pole of the intersection of LN and KM, *i.e.* of R;

∴ PQ and EF are each the polar of R;

∴ PQ and EF are the same str. line; ∴ E and F lie on PQ.]

7. If a quadrilateral be described about a circle, the triangle formed by the three diagonals is self-polar with respect to the circle.

[Take the figure of the previous Exercise.

Then F has been shown to be the pole of AC.

But, by Ex. 5, F is the pole of ER;

∴ AC, ER coincide, *i.e.* AC goes through E.

So BD goes through F, *i.e.* EFR is the △ formed by the three diagonals of ABCD, since E and F lie on PQ. [Ex. 6.

But KLMN is a quad[l] inscribed in the circle of which E, F, R are the vertices.

∴ EFR is a △ self-polar with respect to the circle. [Ex. 5.

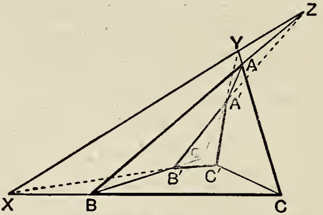
∴, etc.]

8. If a circle be inscribed in a quadrilateral, shew that its centre is at the orthocentre of the triangle formed by the diagonals of the complete quadrilateral.

[Use the previous Exercise and Page xxxvi, Ex. 7.]

9. If a circle be inscribed in a quadrilateral its centre lies on the straight line joining the middle points of the three diagonals.

112. *If two triangles* ABC, A′B′C′ *are so placed that the straight lines* AA′, BB′, CC′ *are concurrent, then the intersections of* BC *and* B′C′, CA *and* C′A′, AB *and* A′B′ *are collinear and conversely.*



Let AA′, BB′, CC′ meet in O, and let X, Y, Z be the intersections of BC and B′C′, of CA and C′A′, and of AB and A′B′.

Since ZAB cuts the sides of the △ A′B′O,

∴ A′Z . B′B . OA = ZB′ . BO . AA′. ....(1) [Art. 103.

Since XBC cuts the sides of the △ B′C′O,

∴ B′X . C′C . OB = XC′ . CO . BB′. ....(2) [Art. 103.

Since YAC cuts the sides of the △ C′A′O,

∴ C′Y . A′A . OC = YA′ . AO . CC′. ....(3) [Art. 103.

Multiplying together (1), (2), and (3), and cancelling like quantities on each side, we have

A′Z . B′X . C′Y = ZB′ . XC′ . YA′.

∴ X, Y, Z lie on a straight line. [Art. 104.

Conversely, if X, Y, Z are collinear, join CC′, BB′ and let them meet in O.

Then the triangles BB′Z, CC′Y, have the straight lines BC, B′C′, ZY, joining corresponding vertices, concurrent;

∴ by the previous part of this proposition, the intersections of BB′ and CC′, of B′Z and C′Y, and of ZB and YC are collinear, that is, O, A′, A are concurrent,

that is, AA′, BB′, CC′ meet in a point.

Two △s, which are such that the lines joining corresponding vertices meet, are called Co-polar triangles; when their corresponding sides meet in three points which are collinear the triangles are said to be Co-axial. The previous theorem therefore states that *Co-polar triangles are also Co-axial and conversely.*

EXERCISES IN GEOMETRY

EXERCISES.

I. 1–31.

1. In the figure of I. 2 if the diameter of the smaller circle is the radius of the larger, shew where the given point and the vertex of the constructed triangle will be situated.

2. In the figure of I. 17 shew that ABC and ACB are together less than two right angles, by joining A to any point in BC.

3. Two right-angled triangles have their hypotenuses equal, and a side of one equal to a side of the other: shew that they are equal in all respects.

4. If a straight line be drawn bisecting one of the angles of a triangle to meet the opposite side, the straight lines drawn from the point of section parallel to the other sides, and terminated by these sides, will be equal.

5. The side BC of a triangle ABC is produced to a point D; the angle ACB is bisected by the straight line CE which meets AB at E. A straight line is drawn through E parallel to BC, meeting AC at F, and the straight line bisecting the exterior angle ACD at G. Shew that EF is equal to FG.

6. AB is the hypotenuse of a right-angled triangle ABC: find a point D in AB such that DB may be equal to the perpendicular from D on AC.

7. ABC is an isosceles triangle: find points D, E in the equal sides AB, AC such that BD, DE, EC may all be equal.

8. A straight line drawn at right angles to BC the base of an isosceles triangle ABC cuts the side AB at D and CA produced at E: shew that AED is an isosceles triangle.

I. 32.

9. Shew that any angle of a triangle is obtuse, right, or acute, according as it is greater than, equal to, or less than the other two angles of the triangle taken together.

10. Construct an isosceles triangle having the vertical angle four times each of the angles at the base.

11. Construct an isosceles triangle which shall have one-third of each angle at the base equal to half the vertical angle.

12. AB, AC are two straight lines given in position: it is required to find in them two points P and Q, such that, PQ being joined, AP and PQ may together be equal to a given straight line, and may contain an angle equal to a given angle.

13. Straight lines are drawn through the extremities of the base of an isosceles triangle, making angles with it, on the side remote from the vertex, each equal to one-third of one of the equal angles of the triangle and meeting the sides produced: shew that three of the triangles thus formed are isosceles.

14. AEB, CED are two straight lines intersecting at E; straight lines AC, DB are drawn forming two triangles ACE, BED; the angles ACE, DBE are bisected by the straight lines CF, BF, meeting at F. Shew that the angle CFB is equal to half the sum of the angles EAC, EDB.

15. From the angle A of a triangle ABC a perpendicular is drawn to the opposite side, meeting it, produced if necessary, at D; from the angle B a perpendicular is drawn to the opposite side, meeting it, produced if necessary, at E: shew that the straight lines which join D and E to the middle point of AB are equal.

16. From the angles at the base of a triangle perpendiculars are drawn to the opposite sides, produced if necessary: shew that the straight line joining the points of intersection will be bisected by a perpendicular drawn to it from the middle point of the base.

17. In the figure I. 1, if C and F be the points of intersection of the circles, and AB be produced to meet one of the circles at K, shew that CFK is an equilateral triangle.

18. The straight lines bisecting the angles at the base of an isosceles triangle meet the sides at D and E: shew that DE is parallel to the base.

19. AB, AC are two given straight lines, and P is a given point in the former: it is required to draw through P a straight line to meet AC at Q, so that the angle APQ may be three times the angle AQP.

20. From a given point it is required to draw to two parallel straight lines, two equal straight lines at right angles to each other.

21. In a right-angled triangle if one of the acute angles be double the other, the hypotenuse is double the shorter side.

22. On the sides AB, BC of a parallelogram ABCD equilateral triangles ABP, BCQ are described having their vertices remote from the parallelogram. Prove that PQD is an equilateral triangle.

23. In the figure of I. 1, if CA and CB on being produced meet the circles again in G and H, and the circles meet again in F, then GCH is an equilateral triangle and GF and FH are in the same straight line.

24. In an equiangular polygon each exterior angle is one-sixth of a right angle. How many sides has it?

25. In an equiangular polygon each interior angle is five-thirds of a right angle. How many sides has it?

26. What is the least number of triangles into which a polygon of n sides can be divided?

I. 33–34.

27. If a straight line which joins the extremities of two equal straight lines, not parallel, make the angles on the same side of it equal to each other, the straight line which joins the other extremities will be parallel to the first.

28. No two straight lines drawn from the extremities of the base of a triangle to the opposite sides can possibly bisect each other.

29. On the sides AB, BC, and CD of a parallelogram ABCD three equilateral triangles are described, that on BC towards the same parts as the parallelogram, and those on AB, CD towards the opposite parts: shew that the distances of the vertices of the triangles on AB, CD from that on BC are respectively equal to the two diagonals of the parallelogram.

30. If a six-sided plane rectilineal figure have its opposite sides equal and parallel, the three straight lines joining the opposite angles will meet at a point.

31. Inscribe a rhombus within a given parallelogram, so that one of the angular points of the rhombus may be at a given point in a side of the parallelogram.

32. ABCD is a parallelogram, and E, F, the middle points of AD and BC respectively; shew that BE and DF will trisect the diagonal AC.

33. Find a point D on the base BC of a triangle ABC such that, if DE, DF be drawn parallel to AC, AB respectively, AE shall be equal to AF.

34. The feet of the perpendiculars drawn from A upon the internal and external bisectors of the angles B, C of the triangle ABC lie upon the straight line joining the middle points of AB and AC.

35. ABCD is a parallelogram, X any point on BC and Y any point on AX; prove that the triangles DXY and YBC are equal.

I. 35-45.

36. Construct a rhombus equal to a given parallelogram.

37. A straight line is drawn bisecting a parallelogram ABCD and meeting AD at E and BC at F: shew that the triangles EBF and CED are equal.

38. If a triangle is described having two of its sides equal to the diagonals of any quadrilateral, and the included angle equal to either of the angles between these diagonals, then the area of the triangle is equal to the area of the quadrilateral.

39. D, E are the middle points of the sides AB, AC of a triangle, and CD, BE intersect at F: shew that the triangle BFC is equal to the quadrilateral ADFE.

40. The sides AB, AC of a given triangle ABC are bisected at the points E, F; a perpendicular is drawn from A to the opposite side, meeting it at D. Shew that the angle FDE is equal to the angle BAC. Shew also that AFDE is half the triangle ABC.

41. Three parallelograms which are equal in all respects are placed with their equal bases in the same straight line and contiguous; the extremities of the base of the first are joined with the extremities of the side opposite to the base of the third, towards the same parts: shew that the portion of the new parallelogram cut off by the second is one half the area of any one of them.

42. ABCD is a parallelogram; from D draw any straight line DFG meeting BC at F and AB produced at G; draw AF and CG: shew that the triangles ABF, CFG are equal.

43. ABC is a given triangle: construct a triangle of equal area, having for its base a given straight line AD, coinciding in position with AB.

44. ABC is a given triangle: construct a triangle of equal area, having its vertex at a given point in BC and its base in the same straight line as AB.

45. ABCD is a given quadrilateral: construct another quadrilateral of equal area having AB for one side, and for another a straight line drawn through a given point in CD parallel to AB.

46. ABCD is a given quadrilateral: construct a triangle whose base shall be in the same straight line as AB, vertex at a given point P in CD, and area equal to that of the given quadrilateral.

47. ABC is a given triangle: construct a triangle of equal area, having its base in the same straight line as AB, and its vertex in a given straight line.

48. If through the point O within a parallelogram ABCD two straight lines are drawn parallel to the sides, and the parallelograms OB and OD are equal, the point O is in the diagonal AC.

I. 46–48.

49. A straight line is drawn intersecting the two sides of a right-angled triangle, and each of the acute angles is joined with the points where this straight line intersects the sides respectively opposite to them: shew that the squares on the joining straight lines are together equal to the square on the hypotenuse and the square on the straight line originally drawn.

50. If any point P be joined to A, B, C, D, the angular points of a rectangle, the squares on PA and PC are together equal to the squares on PB and PD.

51. In a right-angled triangle if the square on one of the sides containing the right angle be three times the square on the other, and from the right angle two straight lines be drawn, one to bisect the opposite side, and the other perpendicular to that side, these straight lines divide the right angle into three equal parts.

52. On the hypotenuse BC, and the sides CA, AB of a right-angled triangle ABC, squares BDEC, AF, and AG are described: shew that the squares on DG and EF are together equal to five times the square on BC.

53. ABC is a triangle, right-angled at A, and the square on AB is three times that on AC; prove that the angle ACB is twice the angle ABC.

54. In the figure to I. 47, prove that the triangles DBF, ECK, AHG, ABC are equal.

I. 1–48.

55. From the centres A and B of two circles parallel radii AP, BQ are drawn; the straight line PQ meets the circumferences again at R and S: shew that AR is parallel to BS.

56. In the figure of I. 5, if the equal sides of the triangle be produced upwards through the vertex, instead of downwards through the base, a demonstration of I. 15 may be obtained without assuming any proposition beyond I. 5.

57. Shew that by superposition the first case of I. 26 may be immediately demonstrated, and also the second case, with the aid of I. 16.

58. A straight line is drawn, terminated by one of the sides of an isosceles triangle and by the other side produced, and bisected by the

base: shew that the straight lines thus intercepted between the vertex of the isosceles triangle and this straight line are together equal to the two equal sides of the triangle.

59. Of all parallelograms which can be formed with diagonals of given lengths the rhombus is the greatest.

60. If two equal straight lines intersect each other anywhere at right angles, the quadrilateral formed by joining their extremities is equal to half the square on either straight line.

61. Inscribe a parallelogram in a given triangle, in such a manner that its diagonals shall intersect at a given point within the triangle.

62. AB, AC are two given straight lines: it is required to find in AB a point P, such that if PQ be drawn perpendicular to AC, the sum of AP and AQ may be equal to a given straight line.

63. On a given straight line as base, construct a triangle, having given the difference of the sides and a point through which one of the sides is to pass.

64. Let one of the equal sides of an isosceles triangle be bisected at D, and let it also be doubled by being produced through the extremity of the base to E, then the distance of the other extremity of the base from E is double its distance from D.

65. Determine the locus of a point whose distance from one given point is double its distance from another given point.

66. A straight line AB is bisected at C, and on AC and CB as diagonals any two parallelograms ADCE and CFBG are described; let the parallelogram whose adjacent sides are CD and CF be completed, and also that whose adjacent sides are CE and CG: shew that the diagonals of these latter parallelograms are in the same straight line.

67. ABCD is a rectangle of which A, C are opposite angles; E is any point in BC and F is any point in CD: shew that twice the area of the triangle AEF, together with the rectangle BE, DF, is equal to the rectangle ABCD.

68. ABC, DBC are two triangles on the same base, and ABC has the side AB equal to the side AC; a circle passing through C and D has its centre E on CA, produced if necessary; a circle passing through B and D has its centre F on BA, produced if necessary: shew that the quadrilateral AEDF has the sum of two of its sides equal to the sum of the other two.

69. Two straight lines AB, AC are given in position: it is required to find in AB a point P, such that a perpendicular being drawn from it to AC, the straight line AP may exceed this perpendicular by a proposed length.

70. Shew that the opposite sides of any equiangular hexagon are parallel, and that any two sides which are adjacent are together equal to the two to which they are parallel.

71. ABC is a triangle in which C is a right angle: shew how to draw a straight line parallel to a given straight line, so as to be terminated by CA and CB, and bisected by AB.

72. ABC is an isosceles triangle having the angle at B four times either of the other angles; AB is produced to D so that BD is equal to twice AB, and CD is joined: shew that the triangles ACD and ABC are equiangular to one another.

73. Through a point K within a parallelogram ABCD straight lines are drawn parallel to the sides: shew that the difference of the parallelograms of which KA and KC are diagonals is equal to twice the triangle BKD.

74. Construct a right-angled triangle, having given one side and the difference between the other side and the hypotenuse.

75. BAC is a right-angled triangle; one straight line is drawn bisecting the right angle A, and another bisecting the base BC at right angles; these straight lines intersect at E: if D be the middle point of BC, shew that DE is equal to DA.

76. On AC the diagonal of a square ABCD, a rhombus AEFC is described of the same area as the square, and having its acute angle at A: if AF be joined, shew that the angle BAC is divided into three equal angles.

77. AB, AC are two fixed straight lines at right angles; D is any point in AB, and E is any point in AC; on DE as diagonal a half square is described with its vertex at G: shew that the locus of G is the straight line which bisects the angle BAC.

78. Shew that a square is greater than any other parallelogram of the same perimeter.

79. ABC is a triangle; AD is a third of AB, and AE is a third of AC; CD and BE intersect at F: shew that the triangle BFC is half the triangle BAC, and that the quadrilateral ADFE is equal to either of the triangles CFE or BDF.

80. ABC is a triangle, having the angle C a right angle; the angle A is bisected by a straight line which meets BC at D, and the angle B is bisected by a straight line which meets AC at E; AD and BE intersect at O: shew that the triangle AOB is half the quadrilateral ABDE.

81. Shew that a scalene triangle cannot be divided by a straight line into two parts which will coincide.

82. ABCD, ACED are parallelograms on equal bases BC, CE, and between the same parallels AD, BE; the straight lines BD and AE intersect at F: shew that BF is equal to twice DF.

83. Parallelograms AFGC, CBKH are described on AC, BC outside the triangle ABC; FG and KH meet at Z; ZC is joined, and through A and B straight lines AD and BE are drawn, both parallel to ZC, and meeting FG and KH at D and E respectively: shew that the figure ADEB is a parallelogram, and that it is equal to the sum of the parallelograms FC, CK.

84. If a quadrilateral have two of its sides parallel shew that the straight line drawn parallel to these sides through the intersection of the diagonals is bisected at that point.

85. Two triangles are on equal bases and between the same parallels: shew that the sides of the triangles intercept equal lengths of any straight line which is parallel to their bases.

86. In a right-angled triangle, right-angled at A, if the side AC be double of the side AB, the angle B is more than double of the angle C.

87. AHK is an equilateral triangle; ABCD is a rhombus, a side of which is equal to a side of the triangle, and the sides BC and CD of which pass through H and K respectively: shew that the angle A of the rhombus is ten-ninths of a right angle.

88. In the figure of I. 35 if two diagonals be drawn to the two parallelograms respectively, one from each extremity of the base, and the intersection of the diagonals be joined with the intersection of the sides (or sides produced) in the figure, shew that the joining straight line will bisect the base.

89. In the figure of I. 47 prove that AL, BK, CF meet in a point.

II. 1–14.

90. A straight line is divided into two parts; shew that if twice the rectangle of the parts is equal to the sum of the squares described on the parts, the straight line is bisected.

91. Construct a line the square on which shall be equal to the difference of two given squares.

92. If a straight line AB be bisected in C and produced to D so that the square on AD is three times the square on CD, and if CB be bisected in E, prove that the square on ED is three times the square on EB.

93. Divide a straight line into two parts so that the square on one part may equal twice the rectangle contained by the whole and the other part.

94. ABCDE is a straight line such that AB, BC, CD, DE are equal and O is an external point; prove that the difference of the squares on OA, OE is twice the difference of the squares on OB, OD.

95. In the figure of II. 11, if AB is produced to Q so that

$$BQ = BH, \text{ then } AQ^2 = 5 . AH^2.$$

96. If a straight line be drawn through one of the angles of an equilateral triangle to meet the opposite side produced, so that the rectangle contained by the whole straight line thus produced and the part of it produced is equal to the square on the side of the triangle, shew that the square on the straight line so drawn will be double the square on a side of the triangle.

97. Divide a given straight line into two parts so that the rectangle contained by them may be equal to the square described on a given straight line which is less than half the straight line to be divided.

98. If the angle between two adjacent sides of a parallelogram increase, while their lengths do not alter, the diagonal through their point of intersection will diminish.

99. Produce one side of a given triangle so that the rectangle contained by this side and the produced part may be equal to the difference of the squares on the other two sides.

100. Construct a rectangle equal to a given square when the sum of two adjacent sides of the rectangle is equal to a given quantity.

101. Construct a rectangle equal to a given square when the difference of two adjacent sides of the rectangle is equal to a given quantity.

102. Two rectangles have equal areas and equal perimeters: shew that they are equal in all respects.

103. ABCD is a rectangle; P is a point such that the sum of PA and PC is equal to the sum of PB and PD: shew that the locus of P consists of the two straight lines through the centre of the rectangle parallel to its sides.

III. 1–15.

104. Two circles whose centres are A and B intersect at C; through C two chords DCE and FCG are drawn equally inclined to AB and terminated by the circles: shew that DE and FG are equal.

105. Through either of the points of intersection of two given circles draw the greatest possible straight line terminated both ways by the two circumferences.

106. If from any point in the diameter of a circle straight lines are drawn to the extremities of a parallel chord, the squares on these straight lines are together equal to the squares on the segments into which the diameter is divided.

107. A and B are two fixed points without a circle PQR; it is required to find a point P in the circumference, so that the sum of the squares described on AP and BP may be the least possible.

108. A circle is described on the radius of another circle as diameter, and two chords of the larger circle are drawn, one through the centre of the less at right angles to the common diameter, and the other at right angles to the first through the point where it cuts the less circle. Shew that these two chords have the segments of the one equal to the segments of the other, each to each.

109. O is the centre of a circle, P is any point in its circumference, PN a perpendicular on a fixed diameter: shew that the straight line which bisects the angle OPN always passes through one or the other of two fixed points.

110. Describe a circle which shall touch a given circle, have its centre in a given straight line, and pass through a given point in the given straight line.

III. 16–19.

111. A circle is drawn to touch a given circle and a given straight line. Shew that the points of contact are always in the same straight line with a fixed point in the circumference of the given circle.

112. Draw a straight line to touch one given circle so that the part of it contained by another given circle shall be equal to a given straight line not greater than the diameter of the latter circle.

113. Draw a straight line cutting two given circles so that the chords intercepted within the circles shall have given lengths.

114. ABD, ACE are two straight lines touching a circle at B and C, and if DE be joined DE is equal to BD and CE together: shew that DE touches the circle.

115. Two radii of a circle at right angles to each other when produced are cut by a straight line which touches the circle: shew that the tangents drawn from the points of section are parallel to each other.

116. If two circles can be described so that each touches the other and three of the sides of a quadrilateral figure, then the difference between the sums of the opposite sides is double the common tangent drawn across the quadrilateral.

117. AB is the diameter and C the centre of a semicircle: shew that O, the centre of any circle inscribed in the semicircle, is equidistant from C and from the tangent to the semicircle parallel to AB.

118. A quadrilateral is bounded by the diameter of a circle, the tangents at its extremities, and a third tangent: shew that its area is equal to half that of the rectangle contained by the diameter and the side opposite to it.

119. If a quadrilateral, having two of its sides parallel, be described about a circle, a straight line drawn through the centre of the circle, parallel to either of the two parallel sides, and terminated by the other two sides, shall be equal to a fourth part of the perimeter of the figure.

120. A series of circles touch a fixed straight line at a fixed point: shew that the tangents at the points where they cut a parallel fixed straight line all touch a fixed circle.

121. Of all straight lines which can be drawn from two given points to meet on the convex circumference of a given circle, the sum of the two is least which make equal angles with the tangent at the point of concourse.

122. C is the centre of a given circle, CA a radius, B a point on a radius at right angles to CA; join AB and produce it to meet the circle again at D, and let the tangent at D meet CB produced at E: shew that BDE is an isosceles triangle.

123. Let the diameter BA of a circle be produced to P, so that AP equals the radius; through A draw the tangent AED, and from P draw PEC touching the circle at C and meeting the former tangent at E; join BC and produce it to meet AED at D: then will the triangle DEC be equilateral.

124. If two circles touch one another externally the square on the common tangent is equal to the rectangle contained by the diameters.

125. ABCD is a straight line, and circles are described on AB and CD as diameters, and a common tangent to the circles is drawn meeting them in E and F. Prove that the triangles AEB and CFD are equiangular.

126. AB is a diameter and AP a chord of a circle; AQ is a chord bisecting the angle BAP: prove that the tangent at Q is perpendicular to AP.

127. The difference of the squares on the tangents drawn from any point to two concentric circles is equal to the square on the tangent drawn to the inner circle from any point on the outer circle.

III. 20–22.

128. Divide a circle into two parts so that the angle contained in one segment shall be equal to twice the angle contained in the other.

129. Divide a circle into two parts so that the angle contained in one segment shall be equal to five times the angle contained in the other.

130. If the angle contained by any side of a quadrilateral and the adjacent side produced be equal to the opposite angle of the quadrilateral, shew that any side of the quadrilateral will subtend equal angles at the opposite angles of the quadrilateral.

131. If a quadrilateral be inscribed in a circle, and a straight line be drawn making equal angles with one pair of opposite sides, it will make equal angles with the other pair.

132. A quadrilateral can have one circle inscribed in it and another circumscribed about it: shew that the straight lines joining the opposite points of contact of the inscribed circle are perpendicular to each other.

133. A and B are two points on a circle centre C. An arc of a circle is drawn through A, C, and B, and a straight line APQ is drawn to cut the two circles in P and Q. Prove that PB and PQ are equal.

134. On the sides BC, CA, AB of a triangle any points α, β, γ are taken. Prove that the centres of the circles circumscribing the triangles $\beta A\gamma$, $\gamma B\alpha$, $\alpha C\beta$ are the angular points of a triangle equiangular with ABC.

135. A and B are fixed points on a circle and N any other point on AB. Two circles are drawn, each passing through N and touching the first circle in A and B respectively. Prove that the point of intersection lies on a fixed circle.

136. A, B, C, D are four points on the circumference of a circle and the arcs AB, BC, CD, DA are bisected in E, F, G, H respectively; prove that EG, FH are perpendicular.

137. Circles are described upon the sides of any quadrilateral inscribed in a circle as diameters; prove that they intersect again in four points lying on a circle and that these four points form a quadrilateral equiangular with the given one.

138. Circles are described on the sides of any quadrilateral ABCD as diameters, and the four points of intersection of consecutive circles form a quadrilateral PQRS. Prove that the angles of PQRS and ABCD are equal or supplementary.

III. 23-30.

139. Through a point C in the circumference of a circle two straight lines ACB, DCE are drawn cutting the circle at B and E: shew that the straight line which bisects the angles ACE, DCB meets the circle at a point equidistant from B and E.

140. AB is a diameter of a circle, and D is a given point on the circumference, such that the arc DB is less than half the arc DA: draw a chord DE on one side of AB so that the arc EA may be three times the arc BD.

141. From A and B, two of the angular points of a triangle ABC, straight lines are drawn so as to meet the opposite sides at P and Q in given equal angles: shew that the straight line joining P and Q will be of the same length in all triangles on the same base AB, and having vertical angles equal to C.

142. OA, OB, OC are three chords of a circle; the angle AOB is equal to the angle BOC, and OA is nearer to the centre than OB. From B a perpendicular is drawn on OA, meeting it at P, and a perpendicular on OC produced, meeting it at Q: shew that AP is equal to CQ.

143. AB is a given finite straight line; through A two indefinite straight lines are drawn equally inclined to AB; any circle passing through A and B meets these straight lines at L and M. Shew that if AB be between AL and AM the sum of AL and AM is constant; if AB be not between AL and AM, the difference of AL and AM is constant.

144. AOB and COD are diameters of a circle at right angles to each other; E is a point in the arc AC, and EFG is a chord meeting COD at F, and drawn in such a direction that EF is equal to the radius. Shew that the arc BG is equal to three times the arc AE.

145. If two circles touch each other internally, any chord of the greater circle which touches the less shall be divided at the point of its contact into segments which subtend equal angles at the point of contact of the two circles.

146. ABCD is a semicircle whose diameter is AD; the chord BC produced meets AD produced in E; if CE is equal to the radius, prove that the arc AB is equal to three times the arc CD.

147. If two equal circles be drawn cutting each other in A and B, and if from A a chord is drawn cutting them in C and D, prove that the part CD between the circumferences is bisected by the circle on AB as diameter.

III. 31.

148. The greatest rectangle which can be inscribed in a circle is a square.

149. The hypotenuse AB of a right-angled triangle ABC is bisected at D, and EDF is drawn at right angles to AB, and DE and DF are cut off each equal to DA; CE and CF are joined: shew that the last two straight lines will bisect the angle C and its supplement respectively.

150. Describe a circle touching a given straight line at a given point, such that the tangents drawn to it from two given points in the straight line may be parallel.

151. Describe a circle with a given radius touching a given straight line, such that the tangents drawn to it from two given points in the straight line may be parallel.

152. AD is a diameter of a circle; B and C are points on the circumference on the same side of AD; a perpendicular from D on BC produced through C meets it at E: shew that the square on AD is greater than the sum of the squares on AB, BC, CD by twice the rectangle BC, CE.

153. AB is the diameter of a semicircle, P is a point on the circumference, PM is perpendicular to AB; on AM, BM as diameters two semicircles are described, and AP, BP meet these latter circumferences at Q, R: shew that QR will be a common tangent to them.

154. AB, AC are two straight lines, B and C are given points in the same; BD is drawn perpendicular to AC, and DE perpendicular to AB; in like manner CF is drawn perpendicular to AB, and FG to AC. Shew that EG is parallel to BC.

155. Two circles intersect at the points A and B, from which are drawn chords to a point C in one of the circumferences, and these chords, produced if necessary, cut the other circumference at D and E; shew that the straight line DE cuts at right angles that diameter of the circle ABC which passes through C.

156. If squares be described on the sides and hypotenuse of a right-angled triangle, the straight line joining the intersection of the diagonals of the latter square with the right angle is perpendicular to the straight line joining the intersections of the diagonals of the two former.

157. C is the centre of a given circle, CA a straight line less than the radius; find the point of the circumference at which CA subtends the greatest angle.

158. AB is the diameter of a semicircle, D and E are any two points in its circumference. Shew that if the chords joining A and B with D and E each way intersect at F and G, then FG produced is at right angles to AB.

159. Two equal circles touch one another externally, and through the point of contact chords are drawn, one to each circle, at right angles to each other: shew that the straight line joining the other extremities of these chords is equal and parallel to the straight line joining the centres of the circles.

160. A circle is described on the shorter diagonal of a rhombus as a diameter, and cuts the sides; and the points of intersection are joined crosswise with the extremities of that diagonal: shew that the parallelogram thus formed is a rhombus with angles equal to those of the first.

161. Through a fixed point O a straight line is drawn, cutting a fixed circle in P and Q, and upon OP, OQ as chords are described circles touching the fixed circles in P and Q. Prove that the two circles so described will intersect on another fixed circle.

162. Two segments of circles are on the same base AB, and P is any point on one of them; the straight lines APD, BPC are drawn meeting the circumferences of the other segment in D and C; AC and BD meet in Q. Prove that the angle AQB is constant.

163. Any point P is taken on the given segment of a circle described on a line AB, and AG and BH are let fall upon BP and AP respectively; prove that GH touches a fixed circle.

164. A straight line AD is trisected at BC. On AB, BD as diameters circles are described; shew how to draw a straight line through A so that the parts of it intercepted by the two circles may be equal in length.

165. Through a fixed point A on a given circle a line is drawn cutting the circle at P, and AP is produced to Q, so that PQ is constant. Prove that the straight line through Q, perpendicular to AQ, touches a fixed circle.

166. AB is the diameter of a circle, and AC, AD two chords which when produced meet the tangent at B in E and F. Prove that a circle will pass through C, D, E, F.

III. 32-34.

167. If two circles touch each other any straight line drawn through the point of contact will cut off similar segments.

168. AB is any chord, and AD is a tangent to a circle at A. DPQ is any straight line parallel to AB, meeting the circumference at P and Q. Shew that the triangle PAD is equiangular to the triangle QAB.

169. Two circles ABDH, ABG, intersect each other at the points A, B; from B a straight line BD is drawn in the one to touch the other; and from A any chord whatever is drawn cutting the circles at G and H: shew that BG is parallel to DH.

170. Two circles intersect at A and B. At A the tangents AC, AD are drawn to each circle and terminated by the circumference of the other. If CB, BD be joined, shew that AB or AB produced, if necessary, bisects the angle CBD.

171. AB is any chord of a circle, P any point on the circumference of the circle; PM is a perpendicular on AB and is produced to meet the circle at Q, and AN is drawn perpendicular to the tangent at P: shew that the triangle NAM is equiangular to the triangle PAQ.

172. Two diameters AOB, COD of a circle are at right angles to each other; P is a point in the circumference; the tangent at P meets COD produced at Q, and AP, BP meet the same line at R, S respectively: shew that RQ is equal to SQ.

173. From a given point A without a circle, whose centre is O, draw a straight line cutting the circle at the points B and C, so that the area BOC may be the greatest possible.

174. Find a point within a triangle at which the three sides subtend equal angles.

175. ABC is a triangle inscribed in a circle, and AP, BQ are drawn parallel to BC, CA respectively to meet the circle in P and Q; prove that the chord PQ is parallel to the tangent to the circle at C

176. Two circles PQA, PQB cut at P and Q. The tangent at P to the circle PQB meets the circle PQA at A, and the tangent at P to PQA meets PQB at B. Prove that the angles PQA, PQB are equal.

177. PMT is a tangent to the circle APC at the point P; CNAT is a diameter to which PN is drawn perpendicular and AM is perpendicular to PT; prove that AM, AN are equal.

178. In a right-angled triangle, if a semicircle be described on one of the sides, the tangent at the point where it cuts the hypotenuse bisects the other side.

179. The diagonals of a cyclic quadrilateral ABCD intersect in E; prove that the tangent at E to the circle about the triangle ABE is parallel to CD.

III. 35–37.

180. Two circles ABCD, EBCF, having the common tangents AE and DF, cut one another at B and C, and the chord BC is produced to cut the tangents at G and H: shew that the square on GH exceeds the square on AE or DF by the square on BC.

181. ABC is a right-angled triangle; from any point D in the hypotenuse BC a straight line is drawn at right angles to BC, meeting CA at E and BA produced at F: shew that the square on DE is equal to the difference of the rectangles BD, DC and AE, EC; and that the square on DF is equal to the sum of the rectangles BD, DC and AF, FB.

182. It is required to find a point in the straight line which touches a circle at the end of a given diameter, such that when a straight line is drawn from this point to the other extremity of the diameter, the rectangle contained by the part of it without the circle and the part within the circle may be equal to a given square not greater than that on the diameter.

III. 1-37.

183. AD, BE are perpendiculars from the angles A and B of a triangle on the opposite sides; BF is perpendicular to ED or DE produced: shew that the angle FBD is equal to the angle EBA.

184. If ABC be a triangle, and BE, CF the perpendiculars from the angles on the opposite sides, and K the middle point of the third side, shew that the angles FEK, EFK are each equal to A.

185. AB is a diameter of a circle; AC and AD are two chords meeting the tangent at B at E and F respectively: shew that the angles FCE and FDE are equal.

186. Two circles cut one another at a point A: it is required to draw through A a straight line so that the extreme length of it intercepted by the two circles may be equal to that of a given straight line.

187. Draw from a given point in the circumference of a circle, a chord which shall be bisected by its point of intersection with a given chord of the circle.

188. When an equilateral polygon is described about a circle it must necessarily be equiangular if the number of sides be odd, but not otherwise.

189. If any number of triangles on the same base BC, and on the same side of it have their vertical angles equal, and perpendiculars, intersecting at D, be drawn from B and C on the opposite sides, find the locus of D; and shew that all the straight lines which bisect the angle BDC pass through the same point.

190. Let O and C be any fixed points on the circumference of a circle, and OA any chord; then if AC be joined and produced to B, so that OB is equal to OA, the locus of B is an equal circle.

191. From any point P in the diagonal BD of a parallelogram ABCD, straight lines PE, PF, PG, PH are drawn perpendicular to the sides AB, BC, CD, DA: shew that EF is parallel to GH.

192. Through any fixed point of a chord of a circle other chords are drawn: shew that the straight lines from the middle point of the first chord to the middle points of the others will meet them all at the same angle.

193. ABC is a straight line, divided at any point B into two parts; ADB and CDB are similar segments of circles, having the common chord BD; CD and AD are produced to meet the circumferences at F and E respectively, and AF, CE, BF, BE are joined: shew that ABF and CBE are isosceles triangles, equiangular to one another.

194. A is a given point: it is required to draw from A two straight lines which shall contain a given angle and intercept on a given straight line a part of given length.

195. A and B are the centres of two circles which touch internally at C, and also touch a third circle, whose centre is D, externally and internally respectively at E and F: shew that the angle ADB is double of the angle ECF.

196. C is the centre of a circle, and CP is the perpendicular on a chord APB: shew that the sum of CP and AP is greatest when CP is equal to AP.

197. AB, BC, CD are three adjacent sides of any polygon inscribed in a circle; the arcs AB, BC, CD are bisected at L, M, N; and LM cuts BA, BC respectively at P and Q: shew that BPQ is an isosceles triangle, and that the angles ABC, BCD are together double of the angle LMN.

198. In the circumference of a given circle determine a point so situated that if chords be drawn to it from the extremities of a given chord of the circle their difference shall be equal to a given straight line less than the given chord.

199. Construct a triangle, having given the sum of the sides, the difference of the segments of the base made by the perpendicular from the vertex, and the difference of the base angles.

200. AKL is a fixed straight line cutting a given circle at K and L; APQ, ARS are two other straight lines making equal angles with AKL, and cutting the circle at P, Q and R, S: shew that whatever be the position of APQ and ARS, the straight line joining the middle points of PQ and RS always remains parallel to itself.

201. If about a quadrilateral another quadrilateral can be described such that every two of its adjacent sides are equally inclined to that side of the former quadrilateral which meets them both, then a circle may be described about the former quadrilateral.

202. Two circles touch one another internally at the point A : it is required to draw from A a straight line such that the part of it between the circles may be equal to a given straight line, which is not greater than the difference between the diameters of the circles.

203. ABCD is a parallelogram; AE is at right angles to AB, and CE is at right angles to CB: shew that ED, if produced, will cut AC at right angles.

204. The two angles at the base of a triangle are bisected by two straight lines on which perpendiculars are drawn from the vertex: shew that the straight line which passes through the feet of these perpendiculars will be parallel to the base and will bisect the sides.

205. In a given circle inscribe a rectangle equal to a given rectilineal figure.

206. In an acute-angled triangle ABC perpendiculars AD, BE are let fall on BC, CA respectively; circles described on AC, BC as diameters meet BE, AD respectively at F, G and H, K: shew that F, G, H, K lie on the circumference of a circle.

207. Two diameters in a circle are at right angles; from their extremities four parallel straight lines are drawn: shew that they divide the circumference into four equal parts.

208. E is the middle point of a semicircle arc AEB, and CDE is any chord cutting the diameter at D, and the circle at C: shew that the square on CE is twice the quadrilateral AEBC.

209. AB is a fixed chord of a circle, AC is a moveable chord of the same circle; a parallelogram is described of which AB and AC are adjacent sides: find the locus of the middle points of the diagonals of the parallelogram.

210. AB is a fixed chord of a circle, AC is a moveable chord of the same circle; a parallelogram is described of which AB and AC are adjacent sides: determine the greatest possible length of the diagonal drawn through A.

211. If two equal circles be placed at such a distance apart that the tangent drawn to either of them from the centre of the other is equal to a diameter, shew that they will have a common tangent equal to the radius.

212. Find a point in a given circle from which if two tangents be drawn to an equal circle, given in position, the chord joining the points of contact is equal to the chord of the first circle formed by joining the points of intersection of the two tangents produced; and determine the limit to the possibility of the problem.

213. AB is a diameter of a circle, and AF is any chord; C is any point in AB, and through C a straight line is drawn at right angles to AB, meeting AF, produced if necessary at G, and meeting the circumference at D: shew that the rectangle FA, AG, and the rectangle BA, AC, and the square on AD are all equal.

214. A, B, C are three given points in the circumference of a given circle: find a point P such that if AP, BP, CP meet the circumference at D, E, F respectively, the arcs DE, EF may be equal to given arcs.

215. Find the point in the circumference of a given circle, the sum of whose distances from two given straight lines at right angles to each other, which do not cut the circle, is the greatest or least possible.

216. On the sides of a triangle segments of a circle are described *internally*, each containing an angle equal to the excess of two right angles above the opposite angle of the triangle: shew that the radii of the circles are equal, that the circles all pass through one point, and that their chords of intersection are respectively perpendicular to the opposite sides of the triangle.

IV. 1–4.

217. In IV. 3 shew that the straight lines drawn through A and B to touch the circle will meet.

218. In IV. 4 shew that the straight lines which bisect the angles B and C will meet.

219. In IV. 4 shew that the straight line DA will bisect the angle at A.

220. Two opposite sides of a quadrilateral are together equal to the other two, and each of the angles is less than two right angles. Shew that a circle can be inscribed in the quadrilateral.

221. Two circles HPL, KPM, that touch each other externally, have the common tangents HK, LM; HL and KM being joined, shew that a circle may be inscribed in the quadrilateral HKML.

222. Given the base, the vertical angle, and the radius of the inscribed circle of a triangle, construct it.

IV. 5–9.

223. Shew that if the straight line joining the centres of the inscribed and circumscribed circles of a triangle passes through one of its angular points, the triangle is isosceles.

224. The common chord of two circles is produced to any point P; PA touches one of the circles at A, PBC is any chord of the other.

Shew that the circle which passes through A, B, and C touches the circle to which PA is a tangent.

225. Describe a circle which shall pass through two given points and cut off from a given straight linc a chord of given length.

226. Describe a circle which shall have its centre in a given straight line, and cut off from two given straight lines chords of equal given length.

227. Describe a circle which shall pass through two given points, so that the tangent drawn to it from another given point may be of a given length.

228. C is the centre of a circle; CA, CB are two radii at right angles; from B any chord BP is drawn cutting CA at N; prove that the circle described about the triangle ANP will be touched by BA.

229. The angle ACB of any triangle is bisected, and the base AB is bisected at right angles, by straight lines which intersect at D: shew that the angles ACB, ADB are together equal to two right angles.

230. ACDB is a semicircle, AB being the diameter, and the two chords AD, BC intersect at E: shew that if a circle be described round CDE it will cut the former at right angles.

231. A circle is described round the triangle ABC; the tangent at C meets AB produced at D; the circle whose centre is D and radius DC cuts AB at E: shew that EC bisects the angle ACB.

232. AB, AC are two straight lines given in position; BC is a straight line of given length; D, E are the middle points of AB, AC; DF, EF are drawn at right angles to AB, AC respectively. Shew that AF will be constant for all positions of BC.

233. A circle is described about an isosceles triangle ABC in which AB is equal to AC; from A a straight line is drawn meeting the base at D and the circle at E: shew that the circle which passes through B, D, and E, touches AB.

234. AC is a chord of a given circle; B and D are two given points in the chord, both within or both without the circle: if a circle be described to pass through B and D, and touch the given circle, shew that AB and CD subtend equal angles at the point of contact.

235. A and B are two points within a circle: find the point P in the circumference such that if PAH, PBK be drawn meeting the circle at H and K, the chord HK shall be the greatest possible.

236. The centre of a given circle is equidistant from two given straight lines: describe another circle which shall touch these two straight lines and shall cut off from the given circle a segment containing an angle equal to a given angle.

IV. 10–16.

237. On a given straight line as base describe an isosceles triangle having the third angle treble of each of the angles at the base.

238. If A be the vertex and BD the base of the constructed triangle in IV. 10, D being one of the two points of intersection of the two circles employed in the construction, and E the other, and AE be drawn meeting BD produced at G, shew that GAB is another isosceles triangle of the same kind.

239. In the figure of IV. 10 if the two equal chords of the smaller circle be produced to cut the larger, and these points of section be joined, another triangle will be formed having the property required by the proposition.

240. In the figure of IV. 10 if AF be the diameter of the smaller circle, DF is equal to a radius of the circle which circumscribes the triangle BCD.

241. Shew that each of the triangles made by joining the extremities of adjoining sides of a regular pentagon is less than a third and greater than a fourth of the whole area of the pentagon.

242. Shew how to derive a regular hexagon from an equilateral triangle inscribed in a circle, and from the construction shew that the side of the hexagon equals the radius of the circle, and that the hexagon is double of the triangle.

243. In a given circle inscribe a triangle whose angles are as the numbers 2, 5, 8.

244. If ABCDEF is a regular hexagon, and AC, BD, CE, DF, EA, FB be joined, another regular hexagon is formed whose area is one third of that of the former.

IV. 1–16.

245. The points of contact of the inscribed circle of a triangle are joined; and from the angular points of the triangle so formed perpendiculars are drawn to the opposite sides: shew that the triangle of which the feet of these perpendiculars are the angular points has its sides parallel to the sides of the original triangle.

246. Construct a triangle having given an angle and the radii of the inscribed and circumscribed circles.

247. ABCDE is a regular pentagon; join AC and BD intersecting at O: shew that AO is equal to DO, and that the rectangle AC, CO is equal to the square on BC.

248. A straight line PQ of given length moves so that its ends are always on two fixed straight lines CP, CQ; straight lines from P and Q at right angles to CP and CQ respectively intersect at R; perpendiculars from P and Q on CQ and CP respectively intersect at S: shew that the loci of R and S are circles having their common centre at C.

249. Right-angled triangles are described on the same hypotenuse: shew that the locus of the centres of the inscribed circles is a quarter of the circumference of a circle of which the common hypotenuse is a chord.

250. On a given straight line AB any triangle ACB is described; the sides AC, BC are bisected and straight lines drawn at right angles to them through the points of bisection to intersect at a point D; find the locus of D.

251. Construct a triangle, having given its base, one of the angles at the base, and the distance between the centre of the inscribed circle and the centre of the circle touching the base and the sides produced.

252. Within a given circle inscribe three equal circles, touching one another and the given circle.

253. If the radius of a circle be cut as in II. 11, the square on its greater segment, together with the square on the radius is equal to the square on the side of a regular pentagon inscribed in the circle.

254. From the vertex of a triangle draw a straight line to the base so that the square on the straight line may be equal to the rectangle contained by the segments of the base.

255. The perpendiculars from the angles A and B of a triangle ABC on the opposite sides meet at D; the circles described round ADC and DBC cut AB or AB produced at the points E and F: shew that AE is equal to BF.

256. Four circles are described so that each may touch internally three of the sides of a quadrilateral: shew that a circle may be described so as to pass through the centres of the four circles.

257. A circle is described round the triangle ABC, and from any point P of its circumference perpendiculars are drawn to BC, CA, AB, which meet the circle again at D, E, F: shew that the triangles ABC and DEF are equal in all respects, and that the straight lines AD, BE, CF are parallel.

258. With any point in the circumference of a given circle as centre, describe another circle, cutting the former at A and B; from B draw in the described circle a chord BD equal to its radius, and join AD, cutting the given circle at Q: shew that QD is equal to the radius of the given circle.

259. A point is taken without a square, such that straight lines being drawn to the angular points of the square, the angle contained by the two extreme straight lines is divided into three equal parts by the other two straight lines: shew that the locus of the point is the circumference of the circle circumscribing the square.

260. Circles are inscribed in the two triangles formed by drawing a perpendicular from an angle of a triangle on the opposite side; and analogous circles are described in relation to the two other like perpendiculars: shew that the sum of the diameters of the six circles together with the sum of the sides of the original triangle is equal to twice the sum of the three perpendiculars.

261. Three concentric circles are drawn in the same plane: draw a straight line, such that one of its segments between the inner and outer circumference may be bisected at one of the points at which the straight line meets the middle circumference.

VI. 1–A.

262. Shew that one of the triangles in the figure of IV. 10 is a mean proportional between the other two.

263. Find a point within a triangle such that if straight lines be drawn from it to the three angular points the triangle will be divided into three triangles equal in area.

264. ABC is a triangle; any straight line parallel to BC meets AB at D and AC at E; join BE and CD meeting at F: shew that the triangle ADF is equal to the triangle AEF.

265. ABC is a triangle; any straight line parallel to BC meets AB at D and AC at E; join BE and CD meeting at F: shew that if AF be produced it will bisect BC.

266. AB is a diameter of a circle, CD is a chord at right angles to it, and E is any point in CD; AE and BE are drawn and produced to cut the circle at F and G: shew that the quadrilateral CFDG has any two of its adjacent sides in the same ratio as the remaining two.

267. Apply VI. 3 to solve the problem of the trisection of a finite straight line.

268. Three points D, E, F in the sides of a triangle ABC being joined form a second triangle, such that any two sides make equal angles with the side of the former at which they meet: shew that AD, BE, CF are at right angles to BC, CA, AB respectively.

269. D, E, and F are the middle points of the sides BC, CA, AB of a triangle ABC. Through A a straight line is drawn cutting DF, DE in M and N respectively. Prove that BM and CN are parallel.

270. AB is a straight line given in magnitude and direction and D is any point in it. EF is another given straight line unlimited in length. Find a point P on it such that DP bisects the angle APB.

VI. 4–6.

271. If two triangles be on equal bases and between the same parallels, any straight line parallel to their bases will cut off equal areas from the two triangles.

272. A tangent to a circle at the point A intersects two parallel tangents at B, C, the points of contact of which with the circle are D, E respectively; and BE, CD intersect at F: shew that AF is parallel to the tangents BD, CE.

273. P and Q are fixed points; AB and CD are fixed parallel straight lines; any straight line is drawn from P to meet AB at M, and a straight line is drawn from Q parallel to PM meeting CD at N: shew that the ratio of PM to QN is constant, and thence shew that the straight line through M and N passes through a fixed point.

274. A and B are two points on the circumference of a circle of which C is the centre; draw tangents at A and B meeting at T; and from A draw AN perpendicular to CB: shew that BT is to BC as BN is to NA.

275. In the sides AB, AC of a triangle ABC are taken two points D, E, such that BD is equal to CE; DE, BC are produced to meet at F: shew that AB is to AC as EF is to DF.

276. Find a point the perpendiculars from which on the sides of a given triangle shall be in a given ratio.

277. On AB, AC, two adjacent sides of a rectangle, two similar triangles are constructed, and perpendiculars are drawn to AB, AC from the angles which they subtend, intersecting at the point P. If AB, AC be homologous sides, shew that P is in all cases in one of the diagonals of the rectangle.

278. APB and CQD are parallel straight lines directed towards the same parts; and AP is to PB as CQ is to QD: shew that the straight lines PQ, AC, BD when produced will meet at a point.

279. ACB is a triangle, and the side AC is produced to D so that CD is equal to AC, and BD is joined: if any straight line drawn parallel to AB cuts the sides AC, CB, and from the points of section straight lines be drawn parallel to DB, shew that these straight lines will meet AB at points equidistant from its extremities.

280. If a circle be described touching externally two given circles, the straight line passing through the points of contact will intersect the straight line passing through the centres of the given circles at a fixed point.

281. D is the middle point of the side BC of a triangle ABC, and P is any point in AD; through P the straight lines BPE, CPF are drawn meeting the other sides at E, F: shew that EF is parallel to BC.

282. AB is the diameter of a circle, E the middle point of the radius OB; on AE, EB as diameters circles are described; PQL is a common tangent meeting the circles at P and Q, and AB produced at L: shew that BL is equal to the radius of the smaller circle.

283. ABCD is a parallelogram; P and Q are points in a straight line parallel to AB; PA and QB meet at R, and PD and QC meet at S; shew that RS is parallel to AD.

284. A and B are two given points; AC and BD are perpendicular to a given straight line CD; AD and BC intersect at E, and EF is perpendicular to CD: shew that AF and BF make equal angles with CD.

285. From the angular points of a parallelogram ABCD perpendiculars are drawn on the diagonals meeting them at E, F, G, H respectively: shew that EFGH is a parallelogram similar to ABCD.

286. If at a given point two circles intersect, and their centres lie on two fixed straight lines which pass through that point, shew that whatever be the magnitude of the circles their common tangents will always meet in one of two fixed straight lines which pass through the given point.

287. A circle is described about an equilateral triangle ABC, and from A a straight line is drawn cutting BC and meeting the circle again in D; prove that BD, DC together are equal to AD.

VI. 7-18.

288. Divide a given arc of a circle into two parts, so that the chords of these parts shall be to each other in a given ratio.

289. In a given triangle draw a straight line parallel to one of the sides, so that it may be a mean proportional between the segments of the base.

290. O is a fixed point in a given straight line OA, and a circle of given radius moves so as always to be touched by OA; a tangent OP is drawn from O to the circle, and in OP produced PQ is taken a third proportional to OP and the radius: shew that as the circle moves along OA, the point Q will move in a straight line.

291. Find a point in a side of a triangle, from which two straight lines drawn, one to the opposite angle, and the other parallel to the base, shall cut off towards the vertex and towards the base, equal triangles.

292. ACB is a triangle having a right angle at C; from A a straight line is drawn at right angles to AB, cutting BC produced at E; from B a straight line is drawn at right angles to AB, cutting AC produced at D: shew that the triangle ECD is equal to the triangle ACB.

293. The straight line bisecting the angle ABC of the triangle ABC meets the straight lines drawn through A and C, parallel to BC and AB respectively, at E and F: shew that the triangles CBE, ABF are equal.

294. AB, CD are any two chords of a circle passing through a point O; EF is any chord parallel to OB; join CE, DF meeting AB at the points G and H, and DE, CF meeting AB at the points K and L: shew that the rectangle OG, OH is equal to the rectangle OK, OL.

295. ABCD is a quadrilateral in a circle; the straight lines CE, DE which bisect the angles ACB, ADB cut BD and AC at F and G respectively: shew that EF is to EG as ED is to EC.

296. From an angle of a triangle two straight lines are drawn, one to any point in the side opposite to the angle, and the other to the circumference of the circumscribing circle, so as to cut from it a segment containing an angle equal to the angle contained by the first drawn line and the side which it meets: shew that the rectangle contained by the sides of the triangle is equal to the rectangle contained by the straight lines thus drawn.

297. The vertical angle C of a triangle is bisected by a straight line which meets the base at D, and is produced to a point E, such that the rectangle contained by CD and CE is equal to the rectangle contained by AC and CB: shew that if the base and vertical angle be given, the position of E is invariable.

298. A square is inscribed in a right-angled triangle ABC, one side DE of the square coinciding with the hypotenuse AB of the triangle: shew that the area of the square is equal to the rectangle AD, BE.

299. ABCD is a parallelogram; from B a straight line is drawn cutting the diagonal AC at F, the side DC at G, and the side AD produced at E: shew that the rectangle EF, FG is equal to the square on BF.

300. If a straight line drawn from the vertex of an isosceles triangle to the base, be produced to meet the circumference of a circle described about the triangle, the rectangle contained by the whole line so produced, and the part of it between the vertex and the base shall be equal to the square on either of the equal sides of the triangle.

301. A triangle ABC is inscribed in a circle; a line BD drawn parallel to the tangent to the circle at A meets AC in D. Prove that the rectangles AB, BC and AC, BD are equal.

302. A square DEFG is inscribed in the right-angled triangle ABC so that the corners D, E lie on the hypotenuse AB and the other two corners on the sides BC, CA; prove that the square = the rect. AD, EB.

VI. 19-D.

303. An isosceles triangle is described having each of the angles at the base double of the third angle: if the angles at the base be bisected, and the points where the lines bisecting them meet the opposite sides be joined, the triangle will be divided into two parts in the proportion of the base to the side of the triangle.

304. Any regular polygon inscribed in a circle is a mean proportional between the inscribed and circumscribed regular polygons of half the number of sides.

305. Divide a triangle into two equal parts by a straight line at right angles to one of the sides.

306. If two isosceles triangles are to one another in the duplicate ratio of their bases, shew that the triangles are similar.

307. From a point without a circle draw a straight line cutting the circle, so that the two segments shall be equal to each other.

308. In the figure of II. 11 shew that four other straight lines, besides the given straight line are divided in the required manner.

309. ABC is an isosceles triangle, the side AB being equal to AC; F is the middle point of BC; on any straight line through A perpendiculars FG and CE are drawn: shew that the rectangle AC, EF is equal to the sum of the rectangles FC, EG and FA, FG.

310. A, B, C, and D are four points in a straight line and O is any point at which AB and CD subtend equal angles; prove that

$$BD \,.\, CD : BA \,.\, CA :: OD^2 : OA^2.$$

311. ABCD is a quadrilateral and AC, BD are the diagonals; prove that the centroids of the triangles ABD, CBD, ADC, ABC form a quadrilateral similar to ABCD and of one-ninth the area.

VI. 1-D.

312. AB is a diameter, and P any point in the circumference of a circle; AP and BP are joined and produced if necessary; from any point C in AB a straight line is drawn at right angles to AB meeting AP at D and BP at E, and the circumference of the circle at F: shew that CD is a third proportional to CE and CF.

313. A, B, C are three points in a straight line, and D a point at which AB and BC subtend equal angles: shew that the locus of D is the circumference of a circle.

314. If a straight line be drawn from one corner of a square cutting off one-fourth from the diagonal it will cut off one-third from a side. Also if straight lines be drawn similarly from the other corners so as to form a square, this square will be two-fifths of the original square.

315. The sides AB, AC of a given triangle ABC are produced to any points D, E, so that DE is parallel to BC. The straight line DE is divided at F so that DF is to FE as BD is to CE: shew that the locus of F is a straight line.

316. A, B are two fixed points on the circumference of a given circle, and P is a moveable point on the circumference; on PB is taken a point D such that PD is to PA in a constant ratio, and on PA is taken a point E such that PE is to PB in the same ratio: shew that DE always touches a fixed circle.

317. ABC is an isosceles triangle, the angle at A being four times either of the others: shew that if BC be trisected at D and E, the triangle ADE is equilateral.

318. Perpendiculars are let fall from two opposite angles of a rectangle on a diagonal: shew that they will divide the diagonal into equal parts, if the square on one side of the rectangle be double that on the other.

319. A straight line AB is divided into any two parts at C, and on the whole straight line and on the two parts of it equilateral triangles ADB, ACE, BCF are described, the two latter being on the same side of the straight line, and the former on the opposite side; G, H, K are the centres of the circles inscribed in these triangles: shew that the angles AGH, BGK are respectively equal to the angles ADC, BDC, and that GH is equal to GK.

320. On the two sides of a right-angled triangle squares are described: shew that the straight lines joining the acute angles of the triangle and the opposite angles of the squares cut off equal segments from the sides, and that each of these equal segments is a mean proportional between the remaining segments.

321. Two straight lines and a point between them are given in position: draw two straight lines from the given point to terminate in the given straight lines, so that they shall contain a given angle and have a given ratio.

322. With a point A in the circumference of a circle ABC as centre, a circle PBC is described cutting the former circle at the points B and C; any chord AD of the former meets the common chord BC at E, and the circumference of the other circle at O: shew that the angles EPO and DPO are equal for all positions of P.

323. ABC, ABF are triangles on the same base in the ratio of two to one; AF and BF produced meet the sides at D and E; in FB a part FG is cut off equal to FE, and BG is bisected at O: shew that BO is to BE as DF is to DA.

324. A is the centre of a circle, and another circle passes through A and cuts the former at B and C; AD is a chord of the latter circle meeting BC at E, and from D are drawn DF and DG tangents to the former circle: shew that G, E, F lie on one straight line.

325. In AB, AC, two sides of a triangle, are taken points D, E; AB, AC are produced to F, G such that BF is equal to AD, and CG equal to AE; BG, CF are joined meeting at H: shew that the triangle FHG is equal to the triangles BHC, ADE together.

326. In any triangle ABC if BD be taken equal to one-fourth of BC, and CE one-fourth of AC, the straight line drawn from C through the intersection of BE and AD will divide AB into two parts, which are in the ratio of nine to one.

327. Any rectilineal figure is inscribed in a circle: shew that by bisecting the arcs and drawing tangents to the points of bisection parallel to the sides of the rectilineal figure, we can form a rectilineal figure circumscribing the circle equiangular to the former.

328. Find a mean proportional between two similar right-angled triangles which have one of the sides containing the right angle common.

329. In the sides AC, BC of a triangle ABC points D and E are taken, such that CD and CE are respectively the third parts of AC and BC; BD and AE are drawn intersecting at O; shew that EO and DO are respectively the fourth parts of AE and BD.

330. CA, CB are diameters of two circles which touch each other externally at C; a chord AD of the former circle, when produced, touches the latter at E, while a chord BF of the latter, when produced, touches the former at G; shew that the rectangle contained by AD and BF is four times that contained by DE and FG.

331. Two circles intersect at A, and BAC is drawn meeting them at B and C; with B, C as centres are described two circles each of which intersects one of the former at right angles; shew that these circles and the circle whose diameter is BC meet at a point

332. ABCDEF is a regular hexagon; shew that BF divides AD in the ratio of one to three, and that EC is trisected by FD, BD.

333. ABC, DEF are triangles, having the angle A equal to the angle D, and AB is equal to DF; shew that the areas of the triangles are as AC to DE.

334. If M, N be the points at which the inscribed and an escribed circle touch the side AC of a triangle ABC; show that if BM be produced to cut the escribed circle again at P, then NP is a diameter.

335. The angle A of a triangle ABC is a right angle, and D is the foot of the perpendicular from A on BC; DM, DN are perpendiculars on AB, AC; shew that the angles BMC, BNC are equal.

336. If from the point of bisection of any given arc of a circle two straight lines be drawn, cutting the chord of the arc and the circumference, the four points of intersection shall also lie in the circumference of a circle.

337. The side AB of a triangle ABC is touched by the inscribed circle at D, and by the escribed circle at E: shew that the rectangle contained by the radii is equal to the rectangle AD, DB and also to the rectangle AE, EB.

338. A parallelogram is inscribed in a triangle, having one side on the base of the triangle, and the adjacent sides parallel to a fixed direction: shew that the locus of the intersection of the diagonals of the parallelogram is a straight line bisecting the base of the triangle.

339. From a given point outside two given circles which do not meet, draw a straight line such that the portions of it intercepted by each circle shall be respectively proportional to their radii.

340. In a given triangle inscribe a rhombus which shall have one of its angular points coincident with a point in the base, and a side on that base.

341. ABC is a triangle having a right angle at C; ABDE is the square described on the hypotenuse; F, G, H are the points of intersection of the diagonals of the squares on the hypotenuse and sides; shew that the angles DCE, GFH are together equal to a right angle.

XI. 1–21.

342. ABC is a triangle; the perpendiculars from A and B on the opposite sides meet at D; through D a straight line is drawn perpendicular to the plane of the triangle, and E is any point in this straight line; shew that the straight line joining E to any angular point of the triangle is at right angles to the straight line drawn through that angular point parallel to the opposite side of the triangle.

343. From a point E draw EC, ED perpendicular to two planes CAB, DAB which intersect in AB, and from D draw DF perpendicular to the plane CAB meeting it at F: show that the straight line CF, produced if necessary, is perpendicular to AB.

344. BCD is the common base of two pyramids, whose vertices A and E lie in a plane passing through BC; and AB, AC are respectively perpendicular to the faces BED, CED; shew that one of the angles at A together with the angles at E make up four right angles, supposing A and E to be on opposite sides of BC.

345. Within the area of a given triangle is inscribed another triangle: shew that the sum of the angles subtended by the sides of the interior triangle at any point not in the plane of the triangles is less than the sum of the angles subtended at the same point by the sides of the exterior angle.

346. Shew that the perpendicular drawn from the vertex of a regular tetrahedron on the opposite face is three times that drawn from its own foot on any of the other faces.

347. A triangular pyramid stands on an equilateral base and the angles at the vertex are right angles: shew that the sum of the perpendiculars on the faces from any point of the base is constant.

348. Three straight lines not in the same plane intersect at a point, and through their point of intersection another straight line is drawn within the solid angle formed by them; shew that the angles which this straight line makes with the first three are together less than the sum, but greater than half the sum, of the angles which the first three make with each other.

349. Three straight lines which do not all lie in one plane, are cut in the same ratio by three planes, two of which are parallel: shew that the third will be parallel to the other two, if its intersections with the three straight lines are not all in the same straight line.

350. The straight line PB*bp* cuts two parallel planes at B, *b*, and the points P, *p* are equidistant from the planes; PA*a*, *pc*C are other straight lines drawn from P, *p* to cut the planes: shew that the triangles ABC, *abc* are equal.

351. The locus of the points which are equidistant from two given points is a plane.

352. The locus of the points which are equidistant from three given points is a straight line which is perpendicular to the plane through the three points.

353. The locus of the straight lines which cut a given straight line perpendicularly at a given point in it is a plane.

354. The locus of the points which are equidistant from two given intersecting straight lines is a pair of planes.

355. The locus of the points which are equidistant from three straight lines, which meet in a point but are not in the same plane, is four straight lines.

356. P is any point in a plane and O a point outside the plane. On OP is taken a point Q such that the rectangle OP . OQ is constant; the locus of Q is a sphere passing through O.

357. Through a given point in space draw a straight line to meet each of two given straight lines which are not in the same plane.

358. From a point A two perpendiculars AP, AQ are drawn to each of two intersecting planes; prove that the common section of these planes is perpendicular to the plane through AP, AQ.

359. PA, PB, PC are three concurrent and mutually perpendicular straight lines, and PO is perpendicular to the plane ABC; shew that O is the orthocentre of the triangle ABC.

360. OA, OB, OC are three mutually perpendicular straight lines; from O perpendiculars OP, OQ, OR are let fall on BC, CA, AB. Prove that BC bisects the angle QPR externally.

361. The faces of a parallelopiped are parallelograms, and those which are opposite are identically equal.

362. The four diagonals of a parallelopiped have a common point and bisect one another.

363. Any section of a parallelopiped made by a plane cutting two opposite pairs of faces and not the remaining pair is a parallelogram.

364. The sum of the squares on the diagonals of a parallelopiped equals the sum of the squares on its edges.

365. Prove that four planes can be drawn through the centre of a cube, each of which cuts six edges of the cube in the corners of a regular hexagon and the other six sides produced in the corners of another regular hexagon.

366. If the sides of a skew quadrilateral ABCD (that is, a quadrilateral no three sides of which are in the same plane) be cut by a plane in the four points a, b, c, d, prove that

$$a\text{A} \,.\, b\text{B} \,.\, c\text{C} \,.\, d\text{D} = a\text{B} \,.\, b\text{C} \,.\, c\text{D} \,.\, d\text{A}.$$

367. The areas of the sections of a pyramid made by planes parallel to the base are in the duplicate ratio of the perpendicular distances of the sections from the vertex.

368. If the three plane angles at the vertex of a tetrahedron be bisected and the points in which the bisecting lines meet the sides of the base be joined with the opposite angles of the base, the three lines so formed will meet in a point.

369. The sum of the squares on the edges of a tetrahedron are equal to four times the squares on the straight lines joining the middle points of opposite edges.

370. The sum of the squares on any two opposite edges of a tetrahedron is less than the sum of the squares on the other four edges.

371. The shortest distance between two opposite edges of a regular tetrahedron equals half the diagonal of the square on an edge.

372. In any tetrahedron the straight lines joining each vertex to the centroid of the opposite face meet in a point and divide one another in the ratio 3 : 1.

373. If ABCD be a regular tetrahedron, and O its centre, prove that the radii of the spheres ABCD and OBCD are as 2 to 3.

374. A tetrahedron is cut by a plane so that the section is a rhombus; prove that the side of the rhombus is half the harmonic mean between a pair of opposite edges.

375. If the areas of all the faces of a tetrahedron are equal each edge is equal to the opposite edge.

376. If an equifacial tetrahedron be cut by a plane parallel to two edges which do not meet, the perimeter of the parallelogram so formed is double of either edge of the tetrahedron to which it is parallel.

377. If through the middle point of each edge of a tetrahedron a plane is drawn perpendicular to the opposite edge, prove that the six planes thus obtained meet in a point.

378. The six planes through the middle points of the edges of a tetrahedron and perpendicular to the edges meet in a point.

379. The straight lines which join the middle points of the opposite edges of a tetrahedron meet in a point.

380. In any tetrahedron the line joining the middle points of one pair of opposite edges is perpendicular to the shortest line between either of the other pair of opposite edges.

381. If the sum of the squares on opposite edges of a tetrahedron is the same for each pair, then the perpendiculars from the angular points on the opposite faces meet in a point.

382. If the perpendiculars from two of the angular points of a tetrahedron on the opposite faces meet in a point, the perpendiculars from the other two angular points meet in a point.

383. If the perpendiculars to two faces of a tetrahedron drawn from the opposite vertices meet, the edge in which the faces intersect is perpendicular to the opposite edge.

384. If the line joining any vertex to the orthocentre of the opposite face be perpendicular to that face, the same is true with regard to the other vertices.

If two pairs of opposite edges of a tetrahedron be at right angles to each other prove that

385. the other pair is at right angles also.

386. their six middle points lie on a sphere.

387. the sum of the squares on each pair of opposite edges is the same.

388. the four perpendiculars from the vertices upon the opposite faces, and the three shortest distances between opposite edges are concurrent.

389. If the circles inscribed in three of the faces of a tetrahedron touch one another, the circle inscribed in the fourth face will touch the other three.

MISCELLANEOUS.

390. Find the condition that must exist so that it may be possible to fold down the four corners of a quadrilateral flat down on the paper so that the four angular points may meet in a point and the paper be everywhere doubled.

391. The straight lines joining the middle points of opposite sides of a quadrilateral, and the straight line joining the middle points of its diagonals meet in a point.

392. The area of the greatest parallelogram that can be inscribed in a given $\triangle$ is one half of the area of the $\triangle$.

393. Prove that I. 47 can be proved by cutting off four right-angled $\triangle^s$ from each of two equal squares.

394. ABC is a given equilateral triangle, and in the sides BC, CA, AB the points X, Y, Z are taken so that BX, CY, AZ are equal. AX, BY, and CZ are drawn so as to form a triangle; prove that it is equilateral.

395. ABCD is a parallelogram, and on AB, BC, CD, DA are taken points P, Q, R, S, such that AP, BQ, CR, DS are all equal; shew that PQRS is a parallelogram.

396. The centre C of a circle BPQ lies on another circle APQ of which PBA is a diameter. Prove that PC and BQ are parallel.

397. The diagonals of a quadrilateral inscribed in a circle are at right angles; prove that the sum of the squares on either pair of opposite sides equals the square on the diameter.

398. A quadrilateral is inscribed in a circle; from its centre perpendiculars are drawn to the sides and a second quad^{l} formed by joining their feet. Prove that the area of the second quad^{l} is half that of the first.

399. Three circles touch one another externally at A, B, C; the chords AB, AC of two of them are produced to meet the third again in the points D, E; prove that DE is the diameter of the third circle and is parallel to the line joining the centres of the others.

400. Through P one of the points of intersection of two circles APB, APC a straight line BC is drawn perp^{r} to AP and BA, CA meet the circles in Q and R; prove that AP bisects the ∠QPR.

401. If two circles pass through the vertex and a point in the bisector of an angle they intercept equal lengths on the sides.

402. A′BC, B′CA, and C′AB are equilateral △^{s} described on the sides of a △ ABC with their vertices outside the △. Prove that AA′, BB′, CC′ are equal and meet in a point.

403. A, B, C are three points on a circle whose centre is O such that the arc AB is twice the arc BC. If AH, BK be drawn ⊥^{r} to OB, OC prove that HK and AB are parallel.

404. The arc BC of a circle ABC, of centre O, is double of the arc AB. If P be the middle point of the arc BC, and CP, AB be produced to meet at E, prove that a circle can be described about the quadrilateral BECO.

405. Two circles touch at A; through A are drawn two lines BAC B′AC′ to meet the circles in B, C and B′, C′; prove that BB′, and CC′, are parallel, and also that the tangents at B and C are parallel and also those at B′, C′.

406. Through one of the points of intersection of two circles draw a straight line that shall be bisected in that point and terminated by the circumference.

407. If a quad^{l} be inscribed in a circle and the middle points of the arcs subtended by its sides be joined to make another quadrilateral and so on; prove that these quadrilaterals tend to become squares.

408. Two circles meet in P and Q. A straight line MPN is drawn terminated by the circles in M and N. At M and N tangents are drawn to the circle which intersect in T. Prove that M, N, Q, and T all lie on a circle.

409. If the tangent PT at any point P of a circle meets a diameter AB produced at T and if P be joined to B, the extremity of the diameter nearer to T, prove that the ∠ATP is equal to the complement of twice the ∠BPT.

410. The sides AB, AC of a △ are bisected in E, F. EF cuts the circle on AB as diameter in H and K. Prove that BH and BK are the internal and external bisectors of the angle ABC.

411. ABCD are four collinear points and EF is a common tangent to the circles described on the lines AB, CD as diameters; prove that the triangles AEB, CFD are equiangular.

412. Three circles touch the sides of a △ ABC in the points D, E, F, in which the in-circle touches the sides and touch one another in G, H, K. Prove that AG, BH, CK meet in a point.

413. The difference in the squares on the direct and transverse common tangents to two circles is equal to the product of their diameters.

414. The centre A of a circle lies on another circle which cuts the former in B and C; AD is a chord of the latter circle cutting BC in E; from D are drawn tangents DF, DG to the former circle; prove that F, E, G are in one straight line.

415. If the sides of a cyclic quadrilateral touch a circle the lines joining the opposite points of contact are at right angles.

416. The circles which have for chords the sides of a cyclic quadrilateral intersect again in four concyclic points.

417. The sides AB, AC of a triangle are produced and the exterior angles are bisected by straight lines which meet in O; prove that the centre of the circumcircle of the triangle BOC lies on the circumcircle of ABC.

418. A straight line AB is divided into three equal parts at C and D and CPD is an equilateral triangle. Prove that AP is a tangent to the circle BPC.

419. AD bisects the angle A of a triangle and meets the base BC in D and a circle is described passing through A and touching BC at D. Prove that this circle will touch at A the circumcircle of ABC.

420. ABC is an equilateral triangle and A′, B′, C′ points on its sides such that BA′, CB′, AC′ are equal; prove that the circles about ABC, A′B′C′ are concentric and that the circles about the triangles AB′C′, BC′A′, CA′B′ intersect in the common centre of the first two.

421. The straight line joining the orthocentre of a triangle to the middle point of the base meets the circumcircle in K, and the perpendicular from the vertex upon the base meets the circumcircle in G; prove that KG is parallel to the base.

422. O being the orthocentre of a triangle ABC, prove that the circumcircles of the triangles BOC, COA, AOB are equal to that of ABC.

423. Three circles whose centres are A, B, and C touch one another externally at D, E, and F; prove that the in-circle of the triangle ABC is the circumcircle of the triangle DEF.

424. ABC is an isosceles triangle and on the base BC, or BC produced, a point D is taken; prove that the circumcircles of the triangles ABD, ACD are equal.

425. A circle passing through B, C and the in-centre of the △ ABC meets the sides AB, AC again in E and F; prove that EF touches the in-circle.

426. The in-circle of a triangle ABC touches AC, AB in E and F. Prove that the circle on BC as diameter meets the bisectors of the ∠s B, C in points lying on the straight line EF.

427. The point in which the external bisector of one angle of a triangle again cuts the circumcircle is equidistant from the other two angular points of the triangle and from the centres of two escribed circles.

428. If the bisector of the vertical angle A of a triangle ABC meets the circumcircle of the triangle in D and DE, DF be drawn ⊥r to the sides AB, AC, produced if necessary, prove that AE equals the semi-sum of the sides.

429. Compare the areas of a square and an equilateral △ inscribed in the same circle.

430. In a regular heptagon the ends, D and E, of a side are joined to the opposite vertex A; prove that in the △ ADE each angle at the base is three times the vertical angle.

431. If I be the in-centre of a △ ABC and D, E, F the circumcentres of the △s BIC, CIA, AIB respectively then the △ DEF is equiangular to $I_1I_2I_3$ and has the same circumcircle as ABC.

432. ABCD is a quadrilateral inscribed in a circle and BD bisects AC; prove that the rectangles AD, AB and DC, CB are equal.

433. AC and BD are the diagonals of a quadrilateral ABCD; prove that the sum of the perpendiculars from A and C on BD is to the sum of the perpendiculars from B and D on AC as AC to BD.

434. AB is any fixed straight line and CD a chord of a circle parallel to AB; AC being joined cuts the circle in E and BE meets it in F. Prove that DF will cut AB in a point G which is the same for all chords.

435. PHQ and POQ are two triangles on opposite sides of the same base PQ such that the angles QHP and HPQ are equal to the angles OQP and QPO respectively. Any parallel to OQ cuts PO and PQ in L and M. Prove that QL and HM meet on a fixed circle.

436. If ABC be a triangle, right-angled at A, and AD be the $\perp^r$ on BC, prove that $$\frac{1}{AD^2}=\frac{1}{AB^2}+\frac{1}{AC^2}.$$

437. Two points, P and Q, are taken on two given straight lines AB, AC such that the sum of AP, AQ is constant. Prove that the circle PAQ goes through a second fixed point.

438. From the vertex of a right-angled triangle a perpendicular is drawn on the hypotenuse, and from the foot of this perpendicular another is drawn on each side of the triangle; shew that the area of the triangle of which these two latter perpendiculars are two of the sides cannot be greater than one-fourth of the area of the original triangle.

439. If the extremities of two intersecting straight lines be joined so as to form two vertically opposite triangles, the figure made by connecting the points of bisection of the given straight lines, will be a parallelogram equal in area to half the difference of the triangles.

440. AB, AC are two tangents to a circle, touching it at B and C; R is any point in the straight line which joins the middle points of AB and AC; shew that AR is equal to the tangent drawn from R to the circle.

441. AB, AC are two tangents to a circle; PQ is a chord of the circle which, produced if necessary, meets the straight line joining the middle points of AB, AC at R; shew that the angles RAP, AQR are equal to one another.

442. Shew that the four circles each of which passes through the middle points of the sides of one of the four triangles formed by two adjacent sides and a diagonal of any quadrilateral all intersect at a point.

443. Perpendiculars are drawn from any point on the three straight lines which bisect the angles of an equilateral triangle: shew that one of them is equal to the sum of the other two.

444. Two circles intersect at A and B, and CBD is drawn through B perpendicular to AB to meet the circles; through A a straight line is drawn bisecting either the interior or exterior angle between AC and AD, and meeting the circumferences at E and F: shew that the tangents to the circumferences at E and F will intersect in AB produced.

445. Divide a triangle by two straight lines into three parts, which, when properly arranged, shall form a parallelogram whose angles are of given magnitude.

446. ABCD is a parallelogram, and P is any point: shew that the triangle PAC is equal to the difference of the triangles PAB and PAD, if P is within the angle BAD or that which is vertically opposite to it; and that the triangle PAC is equal to the sum of the triangles PAB and PAD, if P has any other position.

447. Two circles cut each other, and a straight line ABCDE is drawn, which meets one circle at A and D, the other at B and E, and their common chord at C; shew that the square on BD is to the square on AE as the rectangle BC, CD is to the rectangle AC, CE.

448. AD, BE, CF are the perpendiculars of a triangle ABC. If from D, E, F are drawn perpendiculars to the adjacent sides the feet of these six perpendiculars all lie on a circle.

[This circle is called TAYLOR'S CIRCLE.]

449. D is a point on the side BC of a triangle ABC, and EDF is a transversal through it; prove that the locus of the second point of intersection of the circumcircles of the triangles BDE, FDC is a circle.

450. If through any point on the radical axis of two circles a diameter be drawn to each circle, prove that the centre of the circle through the extremities of these diameters lies on a fixed line.

451. Circles are drawn with centres on a fixed line to cut a given circle orthogonally; prove that they all pass through two given points.

452. Any point E is taken outside a given circle and F is a point on its polar; prove that the circles whose centres are E, F and which cut the given circle orthogonally cut one another orthogonally.

453. If A′, B′, C′ be the midpoints of the sides of a triangle ABC, prove that the in-centre of A′B′C′ is collinear with the in-centre and centroid of the triangle ABC.

454. If from a point without a circle any straight lines APQ, ARS be drawn, making equal angles with the diameter through A and cutting the circle in P, Q, R, S respectively, prove that PS and QR meet in a fixed point.

455. If C be any point within a circle, centre O, and if CD, CE be drawn on the same side of OC, making equal angles with OC and cutting the circle in D and E, prove that the chord DE passes through a fixed point.

456. If G be the centroid of a triangle ABC, then tangents from A, B, C respectively to the circumcircles of GBC, GCA, GAB are all equal.

457. The locus of a point such that the rectangle contained by its distances from an opposite pair of sides of a cyclic quadrilateral is equal to the rectangle contained by its distances from the other pair is the circumcircle of the quadrilateral.

458. A and B are two fixed points on a fixed circle whose centre is C and QCR is any diameter; the circles described round ACQ and BCR will intersect on a circle cutting the fixed circle at right angles in A and B.

459. PAQ is a chord drawn through A, one of the points of intersection of two given circles, to meet them in P, Q; the locus of the middle point of PQ is a circle.

460. The locus of the foot of the perp^r from a fixed point upon a chord of a circle which subtends a right ∠ at it is another circle.

461. Straight lines are drawn through the middle points of the sides of a △ perpendicular to the bisectors of the opposite angle. Prove that the triangle formed by these straight lines has the same nine-point circle as the original △.

462. O is the orthocentre of a △ ABC and O′ is its circumcentre; A′, B′, C′ are the centres of the circumcentres of the △^{s}. BOC, COA, AOB respectively. Prove that O′ is the orthocentre of A′B′C′, O its circumcentre, A, B, C the centres of the circumcircles of B′O′C′, C′O′A′ and A′O′B′ respectively and that all the eight △^{s} mentioned above have the same nine-point circle.

463. AE, AD are the median line and perpendicular respectively from A to the base BC of a △ ABC and P, H, K are the orthocentres of the △^{s} ABC, AEC, ABE; prove that

(1) PHDK is a harmonic range;

(2) the nine-point circle of ABC and the circle EHK cut orthogonally.

464. Of the four triangles formed by the centres of the four circles touching the sides of any triangle, the centroid of any one is the orthocentre of the centroids of the other three.

465. P is a point on the circumcircle of the triangle ABC and PL, PM are the perpendiculars on the sides BC, CA and PY the perpendicular on the pedal line LM; prove that PY . PC = PL . PM.

466. The poles of the radical axis of two circles with respect to the circles are the harmonic conjugates with respect to the centres of similitude.

467. In a triangle if any two of the four points viz. the orthocentre, the centroid, the in-centre and the circumcentre coincide the triangle is equilateral.

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