Optimal approximation of spherical squares by tensor product quadratic Bezier patches

2. Preliminaries

In[ ]:=
```
b0,0 = {-a, -a, Sqrt[1 - 2 a^2]};
b2,0 = {a, -a, Sqrt[1 - 2 a^2]};
b0,2 = {-a, a, Sqrt[1 - 2 a^2]};
b2,2 = {a, a, Sqrt[1 - 2 a^2]};
b1,0 = γ (b0,0 + b2,0); b0,1 = γ (b0,0 + b0,2); b2,1 = γ (b2,0 + b2,2); b1,2 = γ (b0,2 + b2,2);
b1,1 = δ {0, 0, 1};
B2,2[u_] = ((1 + u)/2)^2; B2,1[u_] = 2 ((1 + u)/2) ((1 - u)/2); B2,0[u_] = ((1 - u)/2)^2;
p[u_, v_, a_, γ_, δ_] = Sum[B2,i[u] B2,j[v] bi,j, {i, 0, 2}, {j, 0, 2}] // Simplify;
f[u_, v_, a_, γ_, δ_] = p[u, v, a, γ, δ].p[u, v, a, γ, δ] - 1 // Simplify;
fs[u_, a_, γ_] = f[u, 1, a, γ, 0] // Simplify;
fd[u_, a_, γ_, δ_] = f[u, u, a, γ, δ] // Simplify;
g[u_, v_, a_, γ_, δ_] = Sqrt[p[u, v, a, γ, δ].p[u, v, a, γ, δ]] - 1 // Simplify;
gs[u_, a_, γ_] = g[u, 1, a, γ, 0] // Simplify;
gd[u_, a_, γ_, δ_] = g[u, u, a, γ, δ] // Simplify;
```

Figure 1: The unit sphere S (light blue), boundary curves of a spherical square R (black) and a tensor product quadratic Bézier approximant P (orange) together with its control points (black).

In[ ]:=
```
geodesicArc[v1_List, v2_List, center_List, n_Integer] :=
  Module[{α0 = VectorAngle[v1 - center, v2 - center], β, f},
   f[β_] = (EuclideanDistance[v1, center] Sec[1/2 (α0 - 2 β)] Sin[β]) / EuclideanDistance[v1, v2];
   Line[Table[center + Norm[v1 - center] * Normalize[t * (v1 - center) + (1 - t) (v2 - center)],
     {t, Map[f, Subdivide[α0, n]]}]]];
```

In[ ]:=
```
aa = 1; γγ = 0.61;
δδ = 2.2;
Show[Graphics3D[{Gray, Opacity[0.3], Sphere[{0, 0, 0}, 1]}],
 Graphics3D[{Black, Thickness[0.005], geodesicArc[b0,0 /. a → 1/Sqrt[3], b0,2 /. a → 1/Sqrt[3], {0, 0, 0}, 10] // N}],
 Graphics3D[{Black, Thickness[0.005], geodesicArc[b0,2 /. a → 1/Sqrt[3], b2,2 /. a → 1/Sqrt[3], {0, 0, 0}, 10] // N}],
 Graphics3D[{Black, Thickness[0.0075], geodesicArc[b2,2 /. a → 1/Sqrt[3], b2,0 /. a → 1/Sqrt[3], {0, 0, 0}, 10] // N}],
 Graphics3D[{Black, Thickness[0.005], geodesicArc[b2,0 /. a → 1/Sqrt[3], b0,0 /. a → 1/Sqrt[3], {0, 0, 0}, 10] // N}],
 Graphics3D[{PointSize[0.025], Black,
   Point[{{1/Sqrt[3], 1/Sqrt[3], 1/Sqrt[3]}, {1/Sqrt[3], -1/Sqrt[3], 1/Sqrt[3]}, {-1/Sqrt[3], 1/Sqrt[3], 1/Sqrt[3]}, {-1/Sqrt[3], -1/Sqrt[3], 1/Sqrt[3]}}],
   Point[{b1,0, b0,1, b1,2, b2,1, {0, 0, 1.2}} /. {a → 1/Sqrt[3], γ → γγ, δ → δδ}]}],
 ParametricPlot3D[p[u, v, 1/Sqrt[3], γγ, δδ], {u, -1, 1}, {v, -1, 1}, Mesh → None,
  PlotStyle → {ColorData[97, "ColorList"][[2]]}, PlotPoints → 50],
```

```
Boxed → False]
```

Out[ ]=

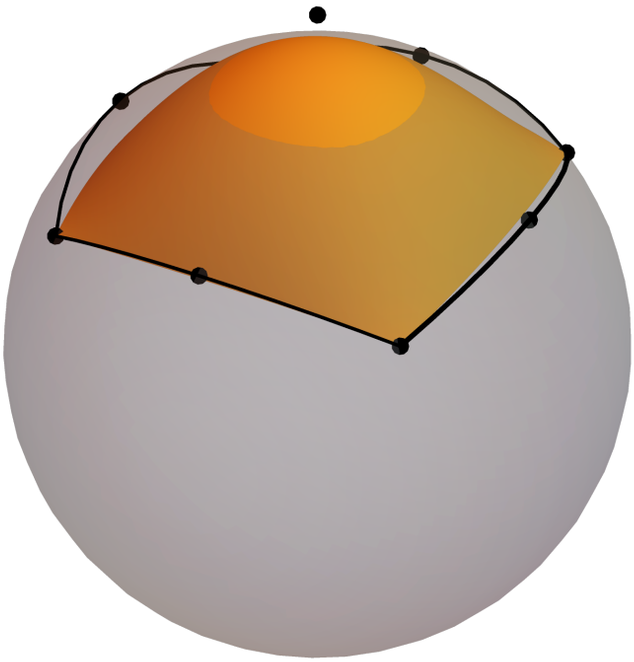


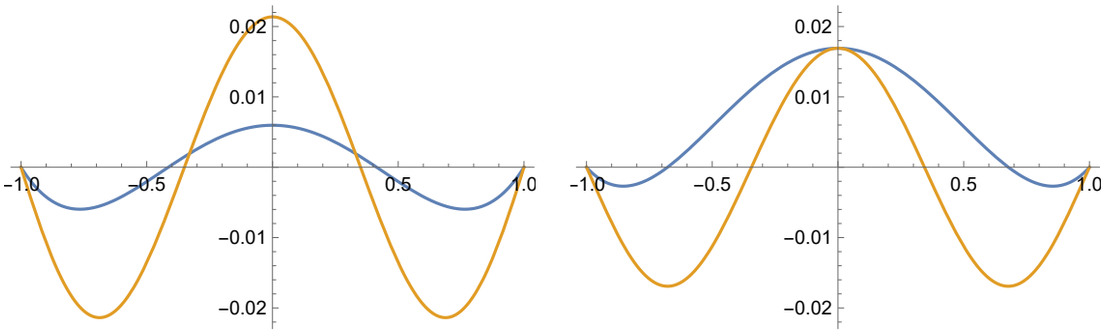
Figure 2: Graphs of simplified radial errors f_s (blue) and f_d (orange) for “the optimal" approximant from [1] (left) and for the optimal approximant constructed in this paper (right). In all cases $a = \frac{1}{\sqrt{3}}$. It is clearly seen that the one on the right has a smaller error on $\angle$.

In[ ]:=
```
γ1 = γ /. Solve[
    D[f_s[u, 1/√3, γ], u] == 0 &&
     f_s[0, 1/√3, γ] + f_s[u, 1/√3, γ] == 0 &&
     0 < u < 1 && γ > 0, {u, γ}][[1]];
δ1 = δ /. Solve[
    D[f_d[u, 1/√3, γ1, δ], u] == 0 &&
     f_d[0, 1/√3, γ1, δ] + f_d[u, 1/√3, γ1, δ] == 0 &&
     0 < u < 1 && δ > 0, {u, δ}][[1]];
```

In[ ]:=
```
{γ2, δ2} = {γ, δ} /. Solve[
    D[f_d[u, 1/√3, γ, δ], u] == 0 &&
     f_d[0, 1/√3, γ, δ] + f_d[u, 1/√3, γ, δ] == 0 &&
     f_s[0, 1/√3, γ] == f_d[0, 1/√3, γ, δ] &&
     0 < u < 1 && δ > 0, {u, γ, δ}][[1]];
```

In[ ]:=
```
GraphicsRow[{Plot[{f_s[u, 1/√3, γ1], f_d[u, 1/√3, γ1, δ1]}, {u, -1, 1}, PlotRange → {-0.023, 0.023}],
  Plot[{f_s[u, 1/√3, γ2], f_d[u, 1/√3, γ2, δ2]}, {u, -1, 1}, PlotRange → {-0.023, 0.023}]}]
```

Out[•]=



3. Main results

Lemma 3.1. For every $u \in [0, 1)$ and $\gamma, \delta > 0$ functions $f_s(u, \cdot)$, $f_d(u, \cdot, \delta)$, and $f_d(u, \gamma, \cdot)$ are strictly increasing.

In[•]:= $D[f_s[u, a, \gamma], \gamma] == (1 - a^2)(1 - u^2)(1 + u^2 + 2\gamma(1 - u^2))$ // Simplify

Out[•]= True

In[•]:= $D[f_d[u, a, \gamma, \delta], \gamma] == \frac{1}{2}\sqrt{1 - 2a^2}\,\delta(1 - u^4)(1 - u^2)^2 + 2\gamma(1 - 2a^2)(1 - u^4)^2 + 4a^2\gamma u^2(1 - u^2)^2 + \frac{1}{2}(1 - 2a^2)(1 - u^8) + u^2(1 - u^4)$ // Simplify

Out[•]= True

In[•]:= $D[f_d[u, a, \gamma, \delta], \delta] == \frac{1}{8}(1 - u^2)^2\left(\delta(1 - u^2)^2 + \sqrt{1 - 2a^2}(1 + u^2)^2 + 4\sqrt{1 - 2a^2}\,\gamma(1 - u^4)\right)$ // Simplify

Out[•]= True

Lemma 3.2. Let $\gamma \in \left[\frac{1}{2}, \frac{1}{2 - 2a^2}\right]$ and let $f_d(0, \gamma, \delta) \le f_s(0, \gamma)$. Then $f_d(u, \gamma, \delta) < f_s(u, \gamma)$ for all $u \in (0, 1)$.

In[•]:= Solve[$f_s[0, a, \gamma] == f_d[0, a, \gamma, \delta], \delta$] // Simplify

Out[•]= $\left\{\left\{\delta \to -\sqrt{1 - 2a^2} - 4\sqrt{1 - 2a^2}\,\gamma - 2\sqrt{-((-1 + a^2)(1 + 2\gamma)^2)}\right\}, \left\{\delta \to -\sqrt{1 - 2a^2} - 4\sqrt{1 - 2a^2}\,\gamma + 2\sqrt{-((-1 + a^2)(1 + 2\gamma)^2)}\right\}\right\}$

$$\left\{\left\{\delta \to -\sqrt{1 - 2a^2} - 4\sqrt{1 - 2a^2}\,\gamma - 2\sqrt{-((-1 + a^2)(1 + 2\gamma)^2)}\right\}, \left\{\delta \to -\sqrt{1 - 2a^2} - 4\sqrt{1 - 2a^2}\,\gamma + 2\sqrt{-((-1 + a^2)(1 + 2\gamma)^2)}\right\}\right\}$$

In[•]:= Assuming$\left[\gamma > 0 \,\&\&\, 0 < a < \frac{1}{\sqrt{2}},\right.$
$\left.-\sqrt{1 - 2a^2} - 4\sqrt{1 - 2a^2}\,\gamma + 2\sqrt{-((-1 + a^2)(1 + 2\gamma)^2)} == 2(1 + 2\gamma)\sqrt{1 - a^2} - (1 + 4\gamma)\sqrt{1 - 2a^2}\right.$ // Simplify$\left.\right]$

Out[•]= True

In[•]:= $\gamma t = \frac{a^2}{2(1 - a^2)}t + \frac{1}{2};$

$\delta\gamma t = 2(1 + 2\gamma)\sqrt{1 - a^2} - (1 + 4\gamma)\sqrt{1 - 2a^2}$ /. $\gamma \to \gamma t;$

In[•]:= $f_s[u, a, \gamma t] - f_d[u, a, \gamma t, \delta\gamma t] == u^2(1 - u^2)\left(2\left(1 - \sqrt{1 - 2a^2}\sqrt{1 - a^2} - 3\frac{a^2}{2}\right)(1 - u^2)(2 - u^2) + a^2\right)(1 - t) + \frac{u^2(1 - u^2)^2}{1 - a^2}\left(\left(1 - \sqrt{1 - 2a^2}\sqrt{1 - a^2} - 3\frac{a^2}{2}\right)(4 - 2u^2 - a^2u^2) + a^4\frac{u^2}{2}\right)t(1 - t) + \frac{u^2(1 - u^2)^2}{4(1 - a^2)^2}\left(\left(1 - \sqrt{1 - 2a^2}\sqrt{1 - a^2} - 3\frac{a^2}{2} - \frac{a^4}{8}\right)(8 - 4a^2)(2 - a^2 - u^2) + \frac{1}{2}a^6(a^2 + 3u^2)\right)t^2$ // Simplify

Out[•]= True

Equations (4)

In[ ]:= $f_s\left[u, a, \frac{1}{2}\right] == -a^2 (1 - u^2)$ // Simplify

Out[ ]= True

In[ ]:= $f_s\left[u, a, \frac{1}{2 - 2 a^2}\right] == \frac{a^4 (1 - u^2)^2}{4 (1 - a^2)}$ // Simplify

Out[ ]= True

Lemma 3.5. If $f_s(0, \gamma) = f_d(0, \gamma, \delta) = 0$, then $f_s(u, \gamma) \le 0$ and $f_d(u, \gamma, \delta) \le 0$ for all $u \in [-1, 1]$.

In[ ]:= $\delta_0[\gamma_] = 2 (1 + 2 \gamma) \sqrt{1 - a^2} - (1 + 4 \gamma) \sqrt{1 - 2 a^2}$;

In[ ]:= $f_s\left[u, a, \frac{1}{\sqrt{1 - a^2}} - \frac{1}{2}\right] == -\left(1 - \sqrt{1 - a^2}\right)^2 u^2 (1 - u^2)$ // Simplify

Out[ ]= True

In[ ]:= $f_d\left[u, a, \frac{1}{\sqrt{1 - a^2}} - \frac{1}{2}, \delta_0\left[\frac{1}{\sqrt{1 - a^2}} - \frac{1}{2}\right]\right] ==$

$$-\frac{2 u^2 (1 - u^2)}{1 - a^2} \left(2 \left(1 - \sqrt{1 - a^2} - \frac{a^2}{2}\right) u^2 \left(u^2 (1 - a^2) + a^2 (1 - u^2)\right) + (1 - u^2)^2 \left(\sqrt{1 - 2 a^2} - \sqrt{1 - a^2}\right)^2 + \left(1 - \sqrt{1 - 2 a^2} - a^2\right) u^2 (1 - u^2) (1 - a^2)\right)$$ // Simplify

Out[ ]= True

Lemma 3.6. Let $f_d(0, \gamma, \delta) = f_s(0, \gamma) \ge 0$. Then
$\frac{d f_d}{d \gamma}(0, \gamma, \delta_0(\gamma)) \ge \frac{d f_d}{d \gamma}(u, \gamma, \delta_0(\gamma))$
for all $u \in [-1, 1]$.

In[ ]:= $k = \left(\sqrt{1 - 2 a^2} - \sqrt{1 - a^2}\right)^2 (1 - u^2) \left(2 (1 - u^2)^2 + u^2 + u^4\right) + 2 \left(\sqrt{1 - 2 a^2} \sqrt{1 - a^2}\right) u^2 (1 - u^4) + u^6 (1 - a^2) + a^2 u^2 (1 - u^2) (3 - u^2)$;

$n = \left(\sqrt{1 - 2 a^2} - \sqrt{1 - a^2}\right)^2 \left(2 - 5 u^2 + 5 u^4 - \frac{3}{2} u^6\right) + \sqrt{1 - 2 a^2} \sqrt{1 - a^2} u^2 + a^2 \left(u^2 - u^4 + \frac{u^6}{2}\right)$;

In[ ]:= $D[f_d[0, a, \gamma, \delta_0[\gamma]], \gamma] - D[f_d[u, a, \gamma, \delta_0[\gamma]], \gamma] == 2 u^2 \left(k \left(\gamma - \frac{1}{2}\right) + n\right)$ // Simplify

Out[ ]= True

Corollary 3.7. If $f_s(0\, \gamma) = f_d(0, \gamma, \delta) \ge 0$ then $\max f_s(\cdot, \gamma) = \max f_d(\cdot, \gamma, \delta) = f_s(0, \gamma) = f_d(0, \gamma, \delta)$.

In[ ]:= $f_s\left[u, a, \frac{1}{\sqrt{1 - a^2}} - \frac{1}{2}\right] == -\left(1 - \sqrt{1 - a^2}\right)^2 u^2 (1 - u^2)$ // Simplify

Out[ ]= True

Lemma 3.8. TFor the optimal parameters γ^* and δ^* we have $f_d(0\, \gamma^*, \delta^*) \ge f_s(0, \gamma^*)$.

Lemma 3.9. Let $f_d(0, \gamma, \delta) > f_s(0, \gamma) \ge 0$. Let α' be such that $f_s(0, \gamma') = f_d(0, \gamma, \delta) = f_s(0, \gamma', \delta')$.
Then $f_d(u, \gamma, \delta) \le f_d(u, \gamma', \delta') \le f_d(0, \gamma, \delta)$ for all $u \in [-1, 1]$. In particular, $f_d(0, \gamma^*, \delta^*) \le f_s(0, \gamma^*)$.

In[ ]:= $\gamma z = \frac{1 - a^2 - 2 \sqrt{1 + z} \sqrt{1 - a^2}}{2 (-1 + a^2)}$;

$$\delta z = \frac{-\sqrt{1-2a^2} - 4\sqrt{1+z} + 4\sqrt{1+z}\,\sqrt{1-2a^2}\,\sqrt{1-a^2} + a^2\left(\sqrt{1-2a^2} + 4\sqrt{1+z}\right)}{-1+a^2};$$

$$\delta\gamma z = -\sqrt{1-2a^2} + 4\sqrt{1+z} - 4\sqrt{1-2a^2}\,\gamma;$$

$$A_1 = 2\sqrt{1-2a^2}\,(1-a^2)\,(1-u^2)^2;$$

$$A_2 = 2\,(1-u^2)\left(2\,(1-u^2)\,\sqrt{1-2a^2}\,\sqrt{1-a^2} + 2\,u^2\,(1-2a^2) + a^2\right);$$

$$A_3 = 2\left(\sqrt{1-2a^2}\,\sqrt{1-a^2}\,2\,(1-u^2)^2\left(1-\sqrt{1-a^2}\right) + \sqrt{1-a^2}\,\left(u^4\,(1-2a^2) + a^2u^2\right) + (1-u^2)\left((1-2a^2)\,2u^2 + a^2\right)\right);$$

In[ ]:= $D[f_d[u, a, \gamma z, \delta z], z] - D[f_d[u, a, \gamma, \delta\gamma z], z] == \frac{u^2\,(1-u^2)\left(A_1\left(\frac{3}{2}-\gamma\right) + A_2\left(\sqrt{1+z}-1\right) + A_3\right)}{(1-a^2)\,\sqrt{1+z}}$ // Simplify

Out[ ]= True

New parametrization with symmetric polynomials

In[ ]:= $g[e1_, e2_, \gamma_] = -1 + \frac{1}{4}(1-a^2)\,(1+2\gamma)^2 + \frac{1}{4}\left(\sqrt{1-a^2} - \sqrt{1-2a^2}\right)^2 (1+2\gamma)^2\,(e1^2 - e1 - e1\,e2) + \frac{1}{2}(1+2\gamma)\left(\left(\sqrt{1-a^2} - \sqrt{1-2a^2}\right)^2 2\gamma + 1 + a^2 - 2\gamma\right) e2 - \left(\left(\sqrt{1-a^2} - \sqrt{1-2a^2}\right)^2 (1+2\gamma)\,\gamma + a^2\gamma\,(1-2\gamma)\right) e1\,e2 + \frac{1}{4}\left(\sqrt{1-a^2}\,(1+2\gamma) - 4\sqrt{1-2a^2}\,\gamma\right)^2 e2^2;$

In[ ]:= $f[u, v, a, \gamma, \delta_0[\gamma]] == g\left[u^2 + v^2,\ u^2 v^2,\ \gamma\right]$ // Simplify

Out[ ]= True

$g(\cdot, \cdot, \gamma^*)$ has no local extrema on T

$g(\cdot, \cdot, \gamma^*)$ has one local extrema on R^2

In[ ]:= {x, y} = {e1, e2} /. Solve[D[g[e1, e2, γ], e1] == 0 && D[g[e1, e2, γ], e2] == 0, {e1, e2}][[1]] // Simplify;

In[ ]:= $x == \frac{4\,(1+2\gamma)\left(\gamma\,(1-2\gamma)\,\left(2-4\gamma+a^2\,(3+2\gamma)\right) + \left(1+7\gamma+8\gamma^2-20\gamma^3\right)\left(\sqrt{1-2a^2} - \sqrt{1-a^2}\right)^2\right)}{(1-2\gamma)^2\,(1+6\gamma)\left(a^2\,(2\gamma-1) - 2\,(1+2\gamma)\left(\sqrt{1-2a^2} - \sqrt{1-a^2}\right)^2\right)}$ // Simplify

Out[ ]= True

In[ ]:= $y == \frac{(1+2\gamma)^2\left(a^2\left(1-4\gamma^2\right) + 2\left(5+2\gamma-8\gamma^2\right)\left(\sqrt{1-2a^2} - \sqrt{1-a^2}\right)^2\right)}{(1-2\gamma)^2\,(1+6\gamma)\left(a^2\,(2\gamma-1) - 2\,(1+2\gamma)\left(\sqrt{1-2a^2} - \sqrt{1-a^2}\right)^2\right)}$ // Simplify

Out[ ]= True

In[ ]:= $\gamma_l = \frac{1 - \frac{1}{5}a^4}{2\,(1-a^2)};$

$\gamma_r = \frac{1 + \frac{1}{5}a^4}{2\,(1-a^2)};$

$\gamma^* > \gamma_l$

In[ ]:= $f_s[0, a, \gamma_l] == \frac{a^4\,(5 + 10a^2 + a^4)}{100\,(1-a^2)}$ // Simplify

Out[ ]= True

In[ ]:= For[i = 0, i ≤ 8, i++,
η_i = {1 033 449 984 000, 9 650 307 744 000, 37 616 758 528 225, 83 055 891 992 150, 127 733 097 043 935, 164 350 922 247 220, 170 358 020 376 975, 109 671 637 130 390, 29 988 229 774 801}[[i + 1]];
For[i = 0, i ≤ 4, i++, ξ_i = {2 028 600, 13 803 300, 34 512 660, 37 529 100 , 14 988 708}[[i + 1]];
For[i = 0, i ≤ 5, i++, ξ_i = {2 028 600, 14 817 600, 41 415 455, 53 704 805, 30 526 277, 5 476 151}[[i + 1]];

```
η = a^4 Sum[η_i a^(2 i) (1 - 2 a^2)^(8-i), {i, 0, 8}];
ϱ = Sum[ϱ_i a^(2 i) (1 - 2 a^2)^(4-i), {i, 0, 4}];
ξ = Sum[ξ_i a^(2 i) (1 - 2 a^2)^(5-i), {i, 0, 5}];
```

In[ ]:= $f_d\left[\frac{3}{4}, a, \gamma_1, \delta_0[\gamma_1]\right] + f_d[0, a, \gamma_1, \delta_0[\gamma_1]] == -\frac{\eta}{6553600\,(1-a^2)^2\left(\varrho\sqrt{1-2a^2}\sqrt{1-a^2}+\xi\right)}$ // Simplify

Out[ ]= True

$\gamma^* < \gamma_r$

In[ ]:= $\varrho = 4\,(10 - 5a^2 + a^4)\,u^2\,(1-u^2)^2\,(5\,(2 - a^2 - u^2) + a^4\,(1-u^2))$;

$\xi = (200\,u^2\,(-2+u^2)\,(-1+u^2)^2 - 200\,a^2\,u^2\,(-1+u^2)^2\,(-5+2u^2) + a^{10}\,(-2+6u^2-18u^4+22u^6-9u^8) + a^8\,(22-64u^2+150u^4-112u^6+15u^8) - 5a^6\,(22-62u^2+134u^4-110u^6+27u^8) + 5a^4\,(18-156u^2+330u^4-220u^6+37u^8))$;

In[ ]:= $f_d[0, a, \gamma_r, \delta_0[\gamma_r]] + f_d[u, a, \gamma_r, \delta_0[\gamma_r]] == \frac{\varrho\sqrt{1-2a^2}\sqrt{1-a^2}+\xi}{100\,(1-a^2)^2}$ // Simplify

Out[ ]= True

In[ ]:=
```
For[i = 0, i ≤ 4, i++,
 For[j = 0, j ≤ 9 - i, j++,
  k_{i,j} = {{644, 1952, 1236, 888, 9462, 21754, 22112, 11596, 3058, 322},
     {1646, 4488, 4940, 11572, 29861, 37620, 23474, 7064, 823},
     {1522, 3274, 4170, 11650, 20601, 16455, 5847, 761}, {610, 860, 1350, 4020, 4685, 2070, 305},
     {90, 50, 180 , 440, 265, 45}}[[i + 1, j + 1]]]]
```

In[ ]:= $\varrho\left(1-\frac{3}{2}a^2-a^4\right) + \xi == \text{Sum}\left[\text{Sum}\left[k_{i,j}\,u^{2j}\,(1-u^2)^{9-i-j}, \{j, 0, 9-i\}\right]\,a^{12-2i}\,(1-2a^2)^i, \{i, 0, 4\}\right]$ // Simplify

Out[ ]= True

$y_d < 0$

In[ ]:= $y_n = (1+2\gamma)^2\left(a^2\,(1-4\gamma^2) + 2\,(5+2\gamma-8\gamma^2)\left(\sqrt{1-2a^2}-\sqrt{1-a^2}\right)^2\right)$;

$y_d = (1-2\gamma)^2\,(1+6\gamma)\left(a^2\,(2\gamma-1) - 2\,(1+2\gamma)\left(\sqrt{1-2a^2}-\sqrt{1-a^2}\right)^2\right)$;

$y == \frac{y_n}{y_d}$ // Simplify

Out[ ]= True

In[ ]:= $\varrho = 4\,(-10 + 5a^2 + a^4\,(1-2t))$;

$\xi = 40 - 80a^2 + 7a^6\,(1-2t) + a^4\,(21+8t)$;

In[ ]:= $\left(-\frac{5\,(1-a^2)}{(1-2\gamma)^2\,(1+6\gamma)}\,y_d \;/.\; \gamma \to (1-t)\,\gamma_1 + t\,\gamma_r\right) == \xi + \varrho\sqrt{1-2a^2}\sqrt{1-a^2}$ // Simplify

Out[ ]= True

In[ ]:= $\xi == \left(40\,(1-2a^2)^3 + 160\,a^2\,(1-2a^2)^2 + 181\,a^4\,(1-2a^2) + 49\,a^6 + 8\,a^4\,(1-2a^2)\,t + 2\,a^6\,t\right)$ // Simplify

Out[ ]= True

In[ ]:= $\xi^2 - \left(\varrho\sqrt{1-2a^2}\sqrt{1-a^2}\right)^2 == a^6\left(17\,a^6\,(1-2t)^2 + 480\,(1-a^2)\,(1-t) + 320\,t + 12\,a^4\,t + 25\,a^2\,(1-2a^2)\,t + 105\,a^2\,(1-2a^2)\,(1-t) + 232\,a^4\,(1-t) + 32\,t\,a^4\,(1-t)\right)$ // Simplify

Out[ ]= True

$y > 1$

In[ ]:= Remove[ϱ];

```
For[i = 0, i ≤ 15, i++,
 For[j = 0, j ≤ 4, j++,
  ξ_{i,j} = {{40 000, 40 000, 40 000, 40 000, 40 000},
     {1 050 000, 1 050 000, 1 050 000, 1 050 000, 1 050 000},
     {12 788 000, 12 792 000, 12 792 000, 12 792 000, 12 792 000},
     {95 809 500, 95 885 500, 95 885 500, 95 885 500, 95 885 500},
     {493 565 700, 494 185 900, 494 182 700, 494 182 700, 494 182 700},
     {1 850 729 200, 1 853 443 400, 1 853 378 200, 1 853 378 200, 1 853 378 200},
     {5 214 530 120, 5 220 369 600, 5 219 770 240, 5 219 770 080, 5 219 770 080},
     {11 232 362 060, 11 230 397 300, 11 227 119 620, 11 227 117 540, 11 227 117 540},
     {18 631 647 684, 18 577 150 372, 18 565 319 428, 18 565 309 060, 18 565 309 124},
     {23 771 986 296, 23 586 788 408, 23 557 248 632, 23 557 228 920, 23 557 229 816},
     {23 108 818 256, 22 747 536 848, 22 695 612 752, 22 695 637 840, 22 695 643 216},
     {16 781 812 960, 16 318 939 360, 16 254 922 720, 16 255 137 760, 16 255 155 680},
     {8 796 756 160, 8 400 009 920, 8 346 057 920, 8 346 559 680, 8 346 595 520},
     {3 135 352 448, 2 915 641 984, 2 886 297 216, 2 886 907 520, 2 886 950 528},
     {677 756 672, 606 707 456, 597 575 424, 597 968 640, 597 997 312},
     {66 801 152, 56 649 216, 55 453 184, 55 559 680, 55 567 872}}[[i + 1, j + 1]]]
```

$$\xi = \frac{1}{625\,(1-a^2)^4}\left(\text{Sum}\left[\xi_{i,j}\,a^{2i}\,(1-2a^2)^{15-i}\,t^j\,(1-t),\ \{i,0,15\},\ \{j,0,3\}\right] + \text{Sum}\left[\xi_{i,4}\,a^{2i}\,(1-2a^2)^{15-i}\,t^4,\ \{i,0,15\}\right]\right);$$

In[ ]:= $k = \frac{103\,901}{256};$

```
For[i = 0, i ≤ 15, i++,
 For[j = 0, j ≤ 4, j++,
  ϱ_{i,j} = {{10 447 802, 10 447 802, 10 447 802, 10 447 802, 10 447 802},
     {259 674 060, 259 674 060, 259 674 060, 259 674 060, 259 674 060},
     {2 987 244 840, 2 988 268 840, 2 988 268 840, 2 988 268 840, 2 988 268 840},
     {21 086 895 280, 21 104 303 280, 21 104 303 280, 21 104 303 280, 21 104 303 280},
     {102 082 894 880, 102 203 266 080, 102 202 446 880, 102 202 446 880, 102 202 446 880},
     {358 756 614 592, 359 130 272 192, 359 114 502 592, 359 114 502 592, 359 114 502 592},
     {944 866 245 760, 944 788 135 040, 944 650 980 480, 944 650 939 520, 944 650 939 520},
     {1 897 672 426 240, 1 892 199 535 360, 1 891 487 384 320, 1 891 486 810 880, 1 891 486 810 880},
     {2 928 265 946 624, 2 903 827 287 552, 2 901 370 265 088, 2 901 366 586 880, 2 901 366 603 264},
     {3 469 548 612 096, 3 407 439 129 088, 3 401 508 608 512, 3 401 493 895 680, 3 401 494 084 096},
     {3 129 407 856 640, 3 023 970 844 672, 3 013 710 462 976, 3 013 668 552 704, 3 013 669 437 440},
     {2 109 908 940 800, 1 985 424 128 000, 1 972 627 650 560, 1 972 538 521 600, 1 972 540 651 520},
     {1 029 768 847 360, 927 851 028 480, 916 493 967 360, 916 355 686 400, 916 358 307 840},
     {343 888 683 008, 288 232 628 224, 281 344 598 016, 281 199 370 240, 281 200 549 888},
     {70 420 529 152, 52 041 220 096, 49 451 761 664, 49 361 059 840, 49 360 535 552},
     {6 687 981 568, 3 889 856 512, 3 430 318 080, 3 405 152 256, 3 404 627 968}}[[i + 1, j + 1]]]
```

$$\varrho = \frac{1}{625\,(1-a^2)^4}\left(\frac{1}{256}\left(\text{Sum}\left[\varrho_{i,j}\,a^{2i}\,(1-2a^2)^{15-i}\,t^j\,(1-t),\ \{i,0,15\},\ \{j,0,3\}\right] + \text{Sum}\left[\varrho_{i,4}\,a^{2i}\,(1-2a^2)^{15-i}\,t^4,\ \{i,0,15\}\right]\right) - 2k\right);$$

In[ ]:= $(y_d - y_n\ /.\ \gamma \to (1-t)\,\gamma_l + t\,\gamma_r) == \xi\,\sqrt{1-a^2}\,\sqrt{1-2a^2} - \varrho$ // Simplify

Out[ ]= True

4. Optimal G^1 approximation

In[ ]:= $\text{Cross}\left[D\left[p\left[u, v, \frac{\sqrt{3}}{3}, \gamma, \delta\right], u\right], D\left[p\left[u, v, \frac{\sqrt{3}}{3}, \gamma, \delta\right], v\right]\right]\ /.\ \{u \to -1, v \to -1\}$ // Simplify

Out[ ]= $\left\{\frac{2}{3}(1-2\gamma)\gamma,\ \frac{2}{3}(1-2\gamma)\gamma,\ -\frac{4}{3}(-1+\gamma)\gamma\right\}$

$\left\{\frac{2}{3}(1-2\gamma)\gamma,\ \frac{2}{3}(1-2\gamma)\gamma,\ -\frac{4}{3}(-1+\gamma)\gamma\right\}$

In[ ]:= $\text{Solve}\left[\text{Cross}\left[\left\{\frac{2}{3}(1-2\gamma)\gamma, \frac{2}{3}(1-2\gamma)\gamma, -\frac{4}{3}(-1+\gamma)\gamma\right\}, \left\{-\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right\}\right] == \{0, 0, 0\}, \gamma\right]$

Out[ ]= $\left\{\{\gamma \to 0\}, \left\{\gamma \to \frac{3}{4}\right\}\right\}$

$\left\{\{\gamma \to 0\}, \left\{\gamma \to \frac{3}{4}\right\}\right\}$

In[ ]:= $\gamma_G = \frac{3}{4};$

In[ ]:= $\left(\text{Cross}\left[\left(D\left[p\left[u, v, \frac{\sqrt{3}}{3}, \gamma_G, \delta\right], u\right] /. v \to -1\right), \left(D\left[p\left[u, v, \frac{\sqrt{3}}{3}, \gamma_G, \delta\right], v\right] /. v \to -1\right)\right].\right.$
$\left.\text{Cross}\left[p\left[0-1, -1, \frac{\sqrt{3}}{3}, \gamma_G, \delta\right], p\left[1, -1, \frac{\sqrt{3}}{3}, \gamma_G, \delta\right]\right]\right) \text{ // Simplify}$

Out[ ]= $\frac{1}{18}\left(-1+u^2\right)\left(-7+2\sqrt{3}\,\delta\right)$

$\frac{1}{18}\left(-1+u^2\right)\left(-7+2\sqrt{3}\,\delta\right)$

In[ ]:= $\delta_G = \frac{7\sqrt{3}}{6};$

In[ ]:= $g\left[0, 0, \frac{\sqrt{3}}{3}, \gamma_G, \delta_G\right] \text{ // Simplify}$

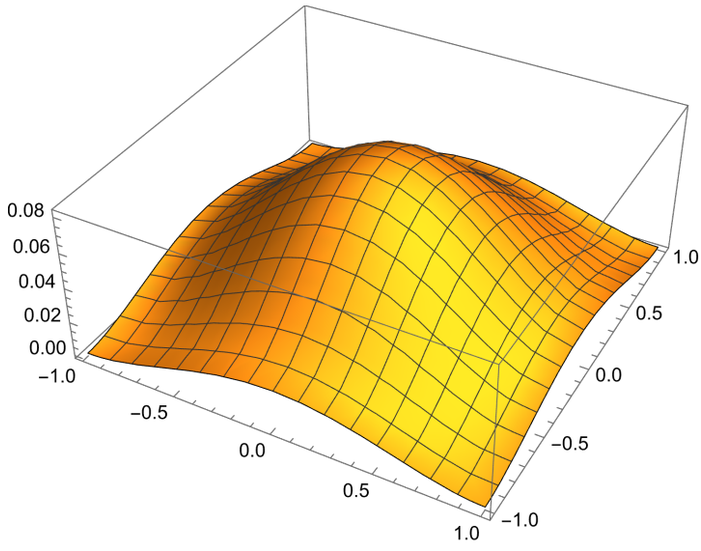
Out[ ]= $-1 + \frac{5\sqrt{3}}{8}$

In[ ]:= $N\left[g\left[0, 0, \frac{\sqrt{3}}{3}, \gamma_G, \delta_G\right]\right]$

Out[ ]= 0.0825318

In[ ]:= $\text{Plot3D}\left[g\left[u, v, \frac{\sqrt{3}}{3}, \gamma_G, \delta_G\right], \{u, -1, 1\}, \{v, -1, 1\}\right]$

Out[ ]=



5. Rectangular case

In[ ]:= $\text{Clear}[a, b, \gamma 1, \gamma 2, \delta];$

$b_{\{0,0\}} = \left\{-a, -b, \sqrt{1-a^2-b^2}\right\};$

$b_{\{2,0\}} = \left\{a, -b, \sqrt{1-a^2-b^2}\right\};$

$b_{\{0,2\}} = \left\{-a, b, \sqrt{1-a^2-b^2}\right\};$

$b_{\{2,2\}} = \left\{a, b, \sqrt{1-a^2-b^2}\right\};$

```
b{1,0} = γ1 (b{0,0} + b{2,0});
b{0,1} = γ2 (b{0,0} + b{0,2}); b{2,1} = γ2 (b{2,0} + b{2,2});
b{1,2} = γ1 (b{0,2} + b{2,2});
b{1,1} = δ {0, 0, 1};
B2₂[u_] = u^2; B2₁[u_] = 2 u (1 - u); B2₀[u_] = (1 - u)^2;
p[u_, v_, a_, b_, γ1_, γ2_, δ_] = Sum[B2i[(u + 1)/2] B2j[(v + 1)/2] b{i,j}, {i, 0, 2}, {j, 0, 2}] // Simplify;
f[u_, v_, a_, b_, γ1_, γ2_, δ_] = p[u, v, a, b, γ1, γ2, δ].p[u, v, a, b, γ1, γ2, δ] - 1 // Simplify;
fs1[u_, a_, γ1_] = f[u, 1, a, 0, γ1, 0, 0] // Simplify;
fs2[u_, b_, γ2_] = f[1, u, 0, b, 0, γ2, 0] // Simplify;
fd[u_, a_, b_, γ1_, γ2_, δ_] = f[u, u, a, b, γ1, γ2, δ] // Simplify;
g[u_, v_, a_, b_, γ1_, γ2_, δ_] = Sqrt[p[u, v, a, b, γ1, γ2, δ].p[u, v, a, b, γ1, γ2, δ]] - 1 // Simplify;
gs1[u_, a_, γ1_] = g[u, 1, a, 0, γ1, 0, 0] // Simplify;
gs2[u_, b_, γ2_] = g[1, u, 0, b, 0, γ2, 0] // Simplify;
gd[u_, a_, b_, γ1_, γ2_, δ_] = g[u, u, a, b, γ1, γ2, δ] // Simplify;
```

Find the minimal length of the shorter edged such that we have exactly one solution

```
In[ ]:= ShorterEdge[a_, h_] := Module[{γγ1, γ1, γ2, δ, u, uu, v, b},
  {γγ1, uu} = {γ1, u} /. FindInstance[
      D[h[u, 1, a, 0, γ1, 0, 0], u] == 0 &&
       h[u, 1, a, 0, γ1, 0, 0] + h[0, 1, a, 0, γ1, 0, 0] == 0 && 0 < u < 1 && γ1 > 0, {γ1, u}, WorkingPrecision → 20][[1]];
  b /. FindInstance[
     h[0, 1, a, 0, γγ1, 0, 0] == h[1, 0, 0, b, 0, γ2, 0] &&
      (D[h[u, v, a, b, γγ1, γ2, δ], v] /. {u → uu, v → 1}) == 0 &&
      h[0, 1, a, 0, γγ1, 0, 0] == h[0, 0, a, b, γγ1, γ2, δ] && 0 < b < a && γ2 > 0 && δ > 0, {b, γ2, δ},
     WorkingPrecision → 20][[1]]]
```

Find the only solution

```
In[ ]:= OneApproximant[a_, b_, h_, P_] := Module[{γ1, γγ1, γ2, γγ2, δ, δδ, i, u, uu, v, vv, newvalue},
  {γγ1, γγ2, δδ} = {γ1, γ2, δ} /. FindInstance[
      h[0, 1, a, 0, γ1, 0, 0] == h[0, 0, a, b, γ1, γ2, δ] &&
       h[1, 0, 0, b, 0, γ2, 0] == h[0, 1, a, 0, γ1, 0, 0] &&
       D[h[v, v, a, b, γ1, γ2, δ], v] == 0 &&
       h[v, v, a, b, γ1, γ2, δ] + h[0, 1, a, 0, γ1, 0, 0] == 0 && 0 < v < 1 && γ1 > 0 && γ2 > 0 && δ > 0,
      {γ1, γ2, δ, v}, WorkingPrecision → 20][[1]];
  For[i = 1, i ≤ 10, i++,
   {uu, vv} =
    {u, v} /.
     FindInstance[D[h[u, v, a, b, γγ1, γγ2, δδ], u] == 0 && D[h[u, v, a, b, γγ1, γγ2, δδ], v] == 0 &&
        0.2 ≤ u ≤ 1 && 0.2 ≤ v ≤ 1, {u, v}, WorkingPrecision → 20][[1]];
   newvalue = (h[0, 1, a, 0, γγ1, 0, 0] - h[uu, vv, a, b, γγ1, γγ2, δδ]) / 2;
   γγ1 = γ1 /. FindInstance[h[0, 1, a, 0, γ1, 0, 0] == newvalue && γ1 > 0, γ1, WorkingPrecision → 20][[1]];
   γγ2 = γ2 /. FindInstance[h[1, 0, 0, b, 0, γ2, 0] == newvalue && γ2 > 0, γ2, WorkingPrecision → 20][[1]];
   δδ = δ /. FindInstance[h[uu, vv, a, b, γγ1, γγ2, δ] + newvalue == 0 && δ > 0, δ, WorkingPrecision → 20][[1]]
  ];
  If[P == 1,
   Print["Parameters of the best approximant are γ1=", γγ1, ", γ2=", γγ2, ", and δ=", δδ, "."];
   Print["A minimum of the error function is at a point (", uu, ",", vv, "). And the error is ",
    h[0, 0, a, b, γγ1, γγ2, δδ], "."];
   Plot3D[h[u, v, a, b, γγ1, γγ2, δδ], {u, -1, 1}, {v, -1, 1}],
   {γγ1, γγ2, δδ, uu, vv}]]
OneApproximantPlot[a_, b_, h_] := OneApproximant[a, b, h, 1]
OneApproximantParameters[a_, b_, h_] := OneApproximant[a, b, h, 0]
```

Find more solutions

```
In[ ]:= MoreApproximant[a_, b_, h_] := Module[{γ1, γγ1, γ2, γγ21, γγ22, γγ23, δ, δδ1, δδ2, δδ3, u, uu, v},
  {γγ1, uu} = {γ1, u} /. FindInstance[
      D[h[u, 1, a, 0, γ1, 0, 0], u] == 0 &&
```

```
      h[u, 1, a, 0, γ1, 0, 0] + h[0, 1, a, 0, γ1, 0, 0] == 0 && 0 < u < 1 && γ1 > 0, {γ1, u}, WorkingPrecision → 20][[1]];
  {γγ21, δδ1} = {γ2, δ} /. FindInstance[
      h[0, 1, a, 0, γγ1, 0, 0] == h[1, 0, 0, b, 0, γ2, 0] &&
       h[0, 1, a, 0, γγ1, 0, 0] == h[0, 0, a, b, γγ1, γ2, δ] && γ2 > 0 && δ > 0, {γ2, δ}, WorkingPrecision → 20][[1]];
  {γγ22, δδ2} = {γ2, δ} /. FindInstance[
      h[0, 1, a, 0, γγ1, 0, 0] == h[1, 0, 0, b, 0, γ2, 0] &&
       (D[h[u, v, a, b, γγ1, γ2, δ], v] == 0 /. {u → uu, v → 1}) && γ2 > 0 && δ > 0, {γ2, δ}, WorkingPrecision → 20][[1]];
  {γγ23, δδ3} = {γ2, δ} /. FindInstance[
      h[0, 1, a, 0, γγ1, 0, 0] == h[0, 0, a, b, γγ1, γ2, δ] &&
       (D[h[u, v, a, b, γγ1, γ2, δ], v] == 0 /. {u → uu, v → 1}) && γ2 > 0 && δ > 0, {γ2, δ}, WorkingPrecision → 20][[1]];
  Print["The parameter γ1=", γγ1, " and for the other two parameters we have more possibilities."];
  Print["Every pair (γ2,δ) in the triangle with vertices (", N[γγ21, 4], ",", N[δδ1, 4], "), (",
   N[γγ22, 4], ",", N[δδ2, 4], "), (", N[γγ23, 4], ",", N[δδ3, 4], ") induces a best approximant."];
  GraphicsRow[{
    Plot3D[h[u, v, a, b, γγ1, γγ21, δδ1], {u, -1, 1}, {v, -1, 1}],
    Plot3D[h[u, v, a, b, γγ1, γγ22, δδ2], {u, -1, 1}, {v, -1, 1}],
    Plot3D[h[u, v, a, b, γγ1, γγ23, δδ3], {u, -1, 1}, {v, -1, 1}]}]]
```

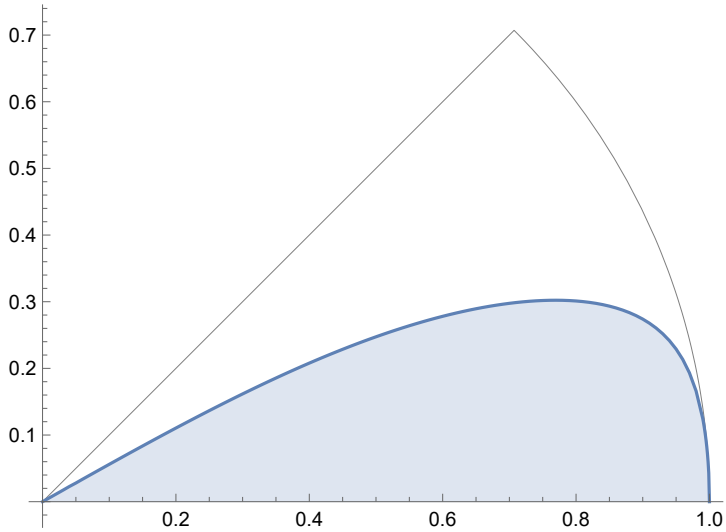
Figure 3 : The domain (light blue) in $[0, 1]^2$ for which pairs (a, b) imply several optimal approximants of the spherical rectangle . The grey dashed curve is the graph of the function $\min\left\{a, \sqrt{1-a^2}\right\}$ and together with the a-axis determine the admissible domain of pairs (a, b).

In[ ]:=
```
points = {{0, 0}};
For[aa = 0.02, aa ≤ 0.99, aa += 0.01, AppendTo[points, {aa, ShorterEdge[aa, g]}]]
For[aa = 0.991, aa ≤ 0.999, aa += 0.001, AppendTo[points, {aa, ShorterEdge[aa, g]}]]
AppendTo[points, {1, 0}]
```

Out[ ]=
```
{{0, 0}, {0.02, 0.0112588799}, {0.03, 0.0168843}, {0.04, 0.0226792}, {0.05, 0.0280094}, {0.06, 0.0337178},
 {0.07, 0.0393225}, {0.08, 0.0449081}, {0.09, 0.050481}, {0.1, 0.056039511}, {0.11, 0.061582},
 {0.12, 0.0671069}, {0.13, 0.072617}, {0.14, 0.0780975}, {0.15, 0.0835598}, {0.16, 0.088998083},
 {0.17, 0.0944106}, {0.18, 0.099795644}, {0.19, 0.105152}, {0.2, 0.110477}, {0.21, 0.11577},
 {0.22, 0.121028580}, {0.23, 0.12625159784515857}, {0.24, 0.131437153}, {0.25, 0.136584}, {0.26, 0.141689},
 {0.27, 0.146752}, {0.28, 0.15177}, {0.29, 0.156743}, {0.3, 0.161668}, {0.31, 0.166543}, {0.32, 0.171367},
 {0.33, 0.176138}, {0.34, 0.18085386428567654234}, {0.35, 0.185513}, {0.36, 0.190114}, {0.37, 0.194655},
 {0.38, 0.199133530}, {0.39, 0.203548}, {0.4, 0.207897}, {0.41, 0.212179}, {0.42, 0.216391132},
 {0.43, 0.220532}, {0.44, 0.2246}, {0.45, 0.22859237906699595251}, {0.46, 0.232508}, {0.47, 0.236345},
 {0.48, 0.240101}, {0.49, 0.243774}, {0.5, 0.247362}, {0.51, 0.250863}, {0.52, 0.254275}, {0.53, 0.257596},
 {0.54, 0.26082289011990655533}, {0.55, 0.263955}, {0.56, 0.266988}, {0.57, 0.269922}, {0.58, 0.272752},
 {0.59, 0.275477}, {0.6, 0.27809345821681182032}, {0.61, 0.280599}, {0.62, 0.28299032798766346989},
 {0.63, 0.285264347}, {0.64, 0.28741760286277653655}, {0.65, 0.289446439}, {0.66, 0.291347},
 {0.67, 0.293114817}, {0.68, 0.294746}, {0.69, 0.296233969}, {0.7, 0.297575}, {0.71, 0.298762},
 {0.72, 0.299788}, {0.73, 0.300646751}, {0.74, 0.301329}, {0.75, 0.301826}, {0.76, 0.302127109},
 {0.77, 0.30222036170824285}, {0.78, 0.302092}, {0.79, 0.301728}, {0.8, 0.301109}, {0.81, 0.300215},
 {0.82, 0.299023}, {0.83, 0.297505}, {0.84, 0.295627}, {0.85, 0.293351112}, {0.86, 0.290629}, {0.87, 0.287405},
 {0.88, 0.283607}, {0.89, 0.279148}, {0.9, 0.273918}, {0.91, 0.267773340}, {0.92, 0.26052479477368250204},
 {0.93, 0.251917}, {0.94, 0.24159}, {0.95, 0.229019}, {0.96, 0.213385}, {0.97, 0.193295984}, {0.98, 0.166044},
 {0.99, 0.12471788187960836303}, {0.991, 0.119139}, {0.992, 0.113132}, {0.993, 0.106615}, {0.994, 0.0994762},
 {0.995, 0.0915552}, {0.996, 0.0826061}, {0.997, 0.072217479}, {0.998, 0.0595902}, {0.999, 0.0426712}, {1, 0}}
```

In[ ]:=
```
Show[Plot[Min[x, Sqrt[1 - x^2]], {x, 0, 1}, AspectRatio → Automatic, PlotStyle → {Gray, Thin}],
 ListLinePlot[points, Filling → Axis]]
```

Out[ ]=



In[ ]:=
```
BestApproximant[a_, b_, h_] := Module[{aa = Max[a, b], bb = Min[a, b], ba},
  If[aa^2 + bb^2 ≥ 1, Print["No approximant"],
   ba = ShorterEdge[aa, h];
   If[bb ≥ ba,
    OneApproximantPlot[aa, bb, h],
    MoreApproximant[aa, bb, h]]
  ]]
```

Figure 4: Graphs of f for three optimal approximants of the spherical rectangle given by parameters $a = 0.75$ and $b = 0.2$. Here $\gamma_1 = 1.0277$ and the left graph corresponds to the pair $(\gamma_2, \delta) = (0.5313, 1.4456)$, the middle one to $(\gamma_2, \delta) = (0.5313, 1.3881)$, and the right one to $(\gamma_2, \delta) = (0.5239, 1.4550)$.

In[ ]:=
```
BestApproximant[0.8, 0.6, g]
```

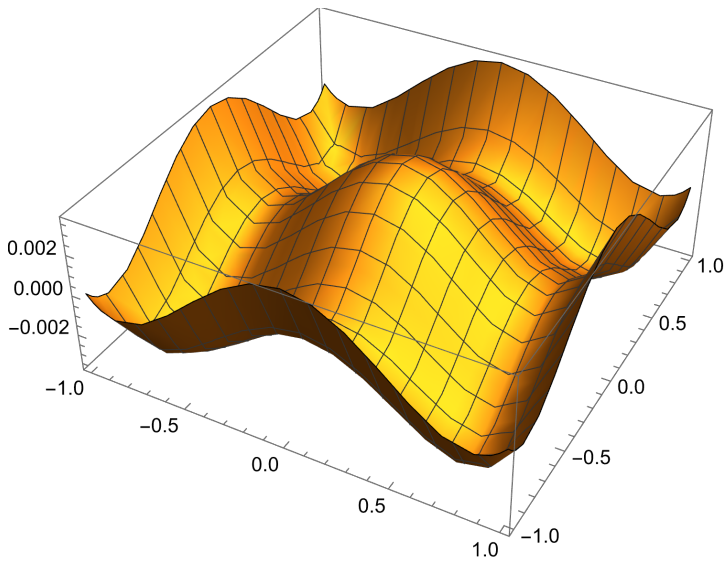
No approximant

In[ ]:=
```
BestApproximant[0.5, 0.5, g]
```

Parameters of the best approximant are γ_1=0.658956, γ_2=0.658956, and δ=1.44383.

A minimum of the error function is at a point (0.706274,0.706274). And the error is 0.00368559.

Out[ ]=

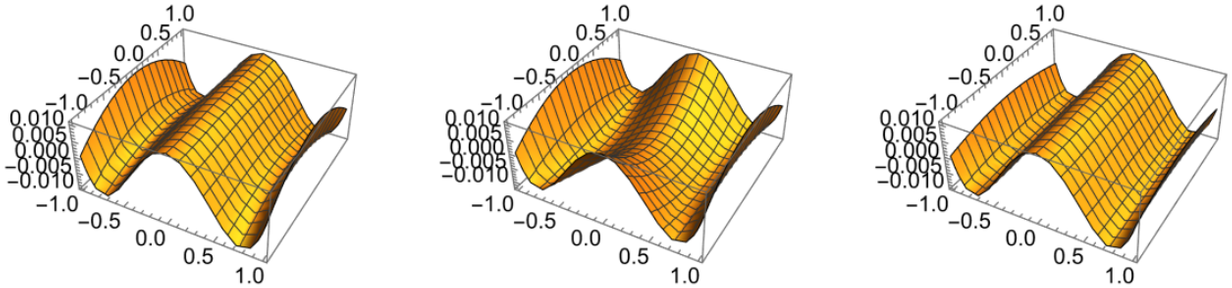


In[ ]:=
```
MoreApproximant[0.75, 0.2, g]
```

The parameter γ_1=1.02769 and for the other two parameters we have more possibilities.

Every pair (γ_2,δ) in the triangle with vertices (0.5313 ,1.446), (0.5313,1.388), (0.523878,1.45496) induces a best approximant.

Out[ ]=



6. Numerical examples

In[ ]:=
```
a_max = 1/√2;
```

In[ ]:=
```
tabel = {{"a", "(γ*_r, δ*_r)", "|g|", "r"}};
x = OneApproximantParameters[N[a_max], N[a_max], g];
d1 = Abs[g[0, 0, a_max, a_max, x[[1]], x[[1]], x[[3]]]];
AppendTo[tabel,
  {"a_max", "(" <> ToString[N[x[[1]], 5]] <> "," <> ToString[N[x[[3]], 5]] <> ")", ScientificForm[d1], "-"}];
For[i = 1, i ≤ 6, i++,
 x = OneApproximantParameters[N[1/2^i a_max], N[1/2^i a_max], g];
 d2 = Abs[g[0, 0, 1/2^i a_max, 1/2^i a_max, x[[1]], x[[1]], x[[3]]]];
 AppendTo[tabel,
  {1/2^i "a_max", "(" <> ToString[N[x[[1]], 5]] <> "," <> ToString[N[x[[3]], 5]] <> ")", ScientificForm[d2],
   Log[d1 / d2] / Log[2]}];
 d1 = d2]
```

Table 1: Optimal parameters γ_r^* and δ_r^* according to the radial error g, the corresponding radial distance $|g|$ and the numerical rate of convergence r for several parameters $a = a_{max}/2^i$, $i = 0, 1, \ldots, 6$, with $a_{max} = 1/\sqrt{2}$.

In[ ]:=
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Grid[tabel, Frame → All]
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Out[ ]=

a	(γ_r^*, δ_r^*)	$\lvert g \rvert$	r
a_{max}	(1.03065,4.32932)	8.23305×10^{-2}	–
$\frac{a_{max}}{2}$	(0.569793,1.16295)	6.99957×10^{-4}	6.87802
$\frac{a_{max}}{4}$	(0.516039,1.03329)	3.74205×10^{-5}	4.22537
$\frac{a_{max}}{8}$	(0.503932,1.00794)	2.25961×10^{-6}	4.04968
$\frac{a_{max}}{16}$	(0.500978,1.00196)	1.40049×10^{-7}	4.01207
$\frac{a_{max}}{32}$	(0.500244,1.00049)	8.73493×10^{-9}	4.003
$\frac{a_{max}}{64}$	(0.500061,1.00012)	5.45649×10^{-10}	4.00075