ACCRETION INTO AND EMISSION FROM BLACK HOLES

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in ways beyond what I can express. It is to them that I dedicate this thesis.

ACCRETION INTO AND EMISSION FROM BLACK HOLES

Don Nelson Page

ABSTRACT

Analyses are given of various processes involving matter falling into or coming out of black holes.

A significant amount of matter may fall into a black hole in a galactic nucleus or in a binary system. There gas with relatively high angular momentum is expected to form an accretion disk flowing into the hole. In this thesis the conservation laws of rest mass, energy, and angular momentum are used to calculate the radial structure of such a disk. The averaged torque in the disk and flux of radiation from the disk are expressed as explicit, algebraic functions of radius.

Matter may be created and come out of the gravitational field of a black hole in a quantum-mechanical process recently discovered by Hawking. In this thesis the emission rates of massless particles by Hawking's process are computed numerically. The resulting power spectra of neutrinos, photons, and gravitons emitted by a nonrotating hole are given. For rotating holes, the rates of emission of energy and angular momentum are calculated for various values of the rotation parameter. The evolution of a rotating hole is followed as energy and angular momentum are given up to the emitted particles. It is found that angular momentum is lost considerably faster than energy, so that a black hole spins down to a nearly nonrotating configuration before it loses a large fraction of its mass. The implications are discussed for the lifetimes and possible present configurations of primordial black holes (the only holes small enough for the emission to be significant within the present age of the universe).

As an astrophysical application, a calculation is given of the gamma-ray spectrum today from the emission by an assumed distribution of primordial black holes during the history of the universe. Comparison with the observed isotropic gamma-ray flux above about 100 MeV yields an upper limit of approximately $10^4 pc^{-3}$ for the average number density of holes around 5×10^{14}g. (This is the initial mass of a nonrotating black hole that would just decay away in the age of the universe.) The prospects are discussed for observing the final, explosive decay of an individual primordial black hole. Such an observation could test the combined predictions of general relativity and quantum mechanics and also could provide information about inhomogeneities in the early universe and about the nature of strong interactions at high temperatures.

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PART I

INTRODUCTION

This thesis is a compilation of four papers, published or in press, that analyze physical processes in the gravitational fields of black holes and discuss possible astrophysical implications of those processes. The first paper deals with the accretion of a disk of matter into a black hole. Such disks are likely to exist around holes of stellar mass or greater which are in binary systems or galactic nuclei. The remaining three papers deal with the quantum mechanical emission of particles by black holes--a process that is significant only for holes much smaller than a stellar mass that may have been created in the early universe. The analyses of accretion and emission both use the basic background gravitational field of a Kerr black hole; but other than that similarity, the treatments and domains of applicability are quite distinct.

PART II

DISK-ACCRETION

INTO BLACK HOLES

(a) Introductory Discussion

Since black holes of stellar mass or greater cannot emit any remotely significant amounts of matter and radiation, such holes can be detected only by the effects that they have on material outside their surfaces. The best hope for detection of such holes is the observation of radiation from matter falling into them. The most promising places to look for this seem to be in galactic nuclei (Lynden-Bell 1969, Lynden-Bell and Rees 1971) or in binary systems (Pringle and Rees 1972, Shakura and Sunyaev 1973), where there is a substantial amount of material available and where a black hole might reasonably be expected to reside.

In these environments, the matter that flows into a black hole will probably have sufficiently large angular momentum to form an accretion disk (Prendergast and Burbidge 1968; cf. the four references above). The matter must give up most of its energy and angular momentum before being swallowed by the hole. It does this by viscously transferring some energy and angular momentum outward through the disk, and by emitting radiation off the faces of the disk. The ratio of energy to angular momentum carried outward by viscous stresses is a fixed function of radius, as is the ratio of energy to angular momentum carried away by radiation. These losses cause the matter to move inward through a sequence of nearly Keplerian orbits, which also have a definite relationship between energy and angular momentum. By comparing these three energy-angular momentum relations, one can calculate how much of the energy and angular momentum released by the matter must be transported outward by stresses and how much must be radiated.

Pringle and Rees (1972) and Shakura and Sunyaev (1973) did Newtonian analyses of this problem, and Novikov and Thorne (1973) made the first relativistic analysis.

Novikov and Thorne solved for the radial structure of an accretion disk and its radiation by using the conservation of rest mass, the conservation of angular momentum, and the conservation of energy as seen locally in the frame of the matter. They assumed a fixed, radius-independent rate $\dot{M}$ of mass flow inward through the disk, and they presumed that the disk was in the equatorial plane of the Kerr (1963) gravitational field outside the black hole. Their analysis yielded expressions for the vertically integrated shear stress and the flux of radiation off the disk at a given radius in terms of two integrals over functions of the Kerr metric (Novikov and Thorne 1973, Eqs. 5.4.1h,j). These two integrals had to be evaluated numerically, so the radial structure was not expressed in closed algebraic form.

In attempting to rederive the radial structure, I used the conservation of rest mass and the conservation of angular momentum in the same form as Novikov and Thorne but used the conservation of energy as seen by an observer at infinity in a different form:

$$\vec{\nabla} \cdot (-\mathbf{T} \cdot \partial/\partial t) = 0 \quad , \qquad \text{(II.1a)}$$

instead of

$$\vec{u} \cdot (\vec{\nabla} \cdot \mathbf{T}) = 0 \quad . \qquad \text{(II.1b)}$$

Here $\partial/\partial t$ is the timelike Killing vector at infinity, $\vec{u}$ is the four-velocity of the matter, and $\mathbf{T}$ is the stress-energy tensor. For matter in nearly circular motion, Eq. (II.1a) is simply a linear combination of Eq. (II.1b) and the law of angular-momentum conservation, but my formulation of the problem seemed to lead to equations different from Novikov and Thorne's. The apparent discrepancy turned out to be proportional to $dE/dr - \Omega dL/dr$, where E, L, and Ω are the energy, angular momentum, and angular velocity respectively of circular geodesic orbits at radius r. One night I discovered a proof (erroneous at first, as it turned out, but which I later replaced by a correct proof) that this quantity was identically zero. The resulting identity not only removed the apparent discrepancy between the equations of Novikov and Thorne and of mine but also allowed the results to be cast in a simpler form so that one of the integrals could be evaluated explicitly for any arbitrary stationary, axially symmetric geometry and the other could be evaluated explicitly for the case of the Kerr metric. Thus the shear stress integrated vertically through an accretion disk and the radiation flux emitted off the surface could be expressed as explicit, algebraic functions of the radius.

These results are written up in Paper I, with Kip Thorne as coauthor, who is responsible for most of the manner of presentation and for pointing out that the analysis is valid in a time-averaged sense even if the disk is highly dynamical.

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(b) Disk-Accretion onto a Black Hole. I. Time-Averaged Structure of Accretion Disk (Paper I; collaboration with K. S. Thorne, published in Ap. J. 191, 499 [1974]; copied by permission of K. S. Thorne and the University of Chicago Press).

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DISK-ACCRETION ONTO A BLACK HOLE. I. TIME-AVERAGED STRUCTURE OF ACCRETION DISK*†

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ABSTRACT

An analysis is given of the time-averaged structure of a disk of material accreting onto a black hole. The analysis is valid even if the disk is highly dynamical. It assumes only that the hole is stationary and axially symmetric (e.g., that it is a Kerr hole); that the disk lies in the equatorial plane of the hole, with its material moving in nearly geodesic circular orbits; that the disk is thin; and that radial heat transport is negligible compared with heat losses through the surface of the disk. The most important result of the analysis is an explicit, algebraic expression for the radial dependence of the time-averaged energy flux emitted from the disk's surface, $F(r)$.

Subject headings: binaries — black holes

I. INTRODUCTION

It now seems probable that some compact X-ray sources are binary systems consisting of a normal star that dumps material onto a companion black hole. The most popular current models for the mass transfer (Pringle and Rees 1972; Shakura and Sunyaev 1973; review in Novikov and Thorne 1973) presume that the transferred material forms a thin disk around the hole. Viscous stresses (magnetic and/or turbulent) transfer angular momentum outward through the disk, thereby allowing the material to spiral gradually inward. The viscous stresses, working against the disk's differential rotation, heat the disk, causing it to emit a large flux of X-rays.

Disk accretion onto a black hole may also occur in the nuclei of galaxies (Lynden-Bell 1969; Lynden-Bell and Rees 1971; review in Novikov and Thorne 1973). In this case the hole is envisaged as supermassive ($M \sim 10^7$ to $10^{11}\ M_\odot$), and the accreting material is interstellar gas and magnetic fields. The models predict significant radiation in the ultraviolet, optical, infrared, and radio regions of the spectrum, but not much X-rays.

Thus far all models for disk accretion onto black holes have been steady-state models, or at least quasi–steady-state. In this paper we ask the question: How much can be learned about the time-averaged behavior of a highly dynamical accreting disk by application of the laws of conservation of rest mass, angular momentum, and energy? (Of course, our results are also applicable to steady-state disks and quasi–steady-state disks.) We shall find that the conservation laws yield an explicit algebraic expression for the time-averaged energy flux, $F(r)$, emitted by the disk's surface, as a function of radius, and also an explicit algebraic expression for the time-averaged torque in the disk, $W_\varphi{}^r(r)$.

The precise assumptions that underlie these expressions are spelled out in § II; the expressions for $F(r)$ and $W_\varphi{}^r(r)$ are presented in § III*a* and are derived in §§ III*b* and III*c*; and some implications of these expressions for steady-state disk models are spelled out in § IV. Throughout we shall use the notation of Novikov and Thorne (1973) and of Misner, Thorne, and Wheeler (1973)—including units with $c = G = k = 1$ (k = Boltzmann constant) —except where typography limitations force changes. The main change is the use of a dagger ($E^\dagger$ and $L^\dagger$), where previous usage would be a tilde ($\tilde{E}$ and $\tilde{L}$).

II. ASSUMPTIONS AND NOTATION FOR THE ANALYSIS

In analyzing the time-averaged structure of the accretion disk, we make the following assumptions and use the following notation.

i) *Assumption*: The black hole has an external spacetime geometry in which the disk, with negligible self-gravity, resides. The external geometry is stationary, axially symmetric, asymptotically flat, and reflection-symmetric in an equatorial plane. (At the end of the analysis, and only there, we shall specialize our formulae to the Kerr geometry.)

* This paper and its companion (Thorne 1974) were cited in previous writings (e.g., in Misner, Thorne, and Wheeler 1973 ["MTW"] and in Novikov and Thorne 1973) as "K. S. Thorne, Black-Hole Models for Compact X-ray Sources, *Ap. J.*, in preparation (1973)." The research reported in these papers was completed in late 1972; it was reported by KST at the Texas Symposium on Relativistic Astrophysics in New York City, 1972 December 21, and at a variety of subsequent meetings in 1973; but it was not written up for publication until this late date (1973 November) because of a complete preoccupation with the proofs of MTW.

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Notation: In and near the equatorial plane we introduce coordinates t ("time"), r ("radius"), z ("height above equatorial plane"), φ ("azimuthal angle"), with respect to which the metric reads

$$ds^2 = -e^{2\nu}dt^2 + e^{2\psi}(d\varphi - \omega dt)^2 + e^{2\mu}dr^2 + dz^2; \tag{1a}$$

$$\nabla^2 = -e^{-2\nu}\left(\frac{\partial}{\partial t} + \omega\frac{\partial}{\partial\varphi}\right)^2 + e^{-2\psi}\left(\frac{\partial}{\partial\varphi}\right)^2 + e^{-2\mu}\left(\frac{\partial}{\partial r}\right)^2 + \left(\frac{\partial}{\partial z}\right)^2; \tag{1b}$$

$$\nu,\ \psi,\ \mu,\ \omega \text{ are functions of } r \text{ only}; \tag{1c}$$

$$\text{as } r \to \infty, \quad \psi = \ln r + O(1/r), \quad \nu \sim \mu \sim O(1/r), \quad \omega \sim O(1/r^3). \tag{1d}$$

Note that

$$(-g)^{1/2} \equiv (-\det\|g_{\alpha\beta}\|)^{1/2} = e^{\nu+\psi+\mu}. \tag{1e}$$

As one moves out of the equatorial plane, the metric coefficients acquire corrections of order $(z/r)^2$. (All such corrections will be ignored in this paper.) For proofs of the existence of such a coordinate system see Papapetrou (1966), Kundt and Trümper (1966), Carter (1969, 1970).

ii) *Assumption*: The central plane of the disk lies in the equatorial plane of the black hole.

iii) *Assumption*: The disk is thin; i.e., at radius r its thickness $\Delta z = 2h$ is always much less than r. This permits us to use the metric in its near-equatorial-plane form (1), with ν, ψ, μ, ω independent of z.

iv) *Assumption*: There exists a time interval Δt which (a) is small enough that during Δt the external geometry of the hole changes negligibly; but (b) is large enough that, for any radius r of interest, the total mass that flows inward across r during Δt is large compared with the typical mass contained between r and $2r$. *Notation*: by $\langle\ \rangle$ we denote an average over angle $\Delta\varphi = 2\pi$ and over time Δt:

$$\langle\Psi(z, r)\rangle \equiv (2\pi\Delta t)^{-1}\int_0^{\Delta t}\int_0^{2\pi}\Psi(t, r, z, \varphi)d\varphi dt. \tag{2}$$

If Ψ is a tensor field, it is to be Lie-dragged along $\partial/\partial t$ and $\partial/\partial\varphi$ during the averaging process. Equivalently, its components in the t, r, z, φ coordinate system are to be averaged.

v) *Notation*: The "local rest frame" of the baryons at an event $\mathscr{P}_0$ (the frame in which there is no net spatial baryon flux) has a 4-velocity $u^{\text{inst}}(\mathscr{P}_0)$ ("inst" means "instantaneous"). When mass-averaged over φ and Δt and height, this 4-velocity is denoted

$$\boldsymbol{u}(r) \equiv (1/\Sigma)\int_{-H}^{+H}\langle\rho_0 \boldsymbol{u}^{\text{inst}}\rangle dz. \tag{3}$$

Here ρ_0 is the density of rest mass (number density of baryons n multiplied by a standard constant, mean rest mass per baryon) as measured in the instantaneous local rest frame; Σ is the time-averaged surface density,

$$\Sigma(r) \equiv \int_{-H}^{+H}\langle\rho_0\rangle dz; \tag{4}$$

and H is the maximum half-thickness of the disk during the time Δt,

$$H \equiv \max_{\Delta t}(h). \tag{5}$$

Without making any assumptions about the types of stress-energy present (magnetic fields, viscous stresses, etc.), we algebraically decompose the stress-energy tensor $\boldsymbol{T}$ with respect to the 4-velocity field $\boldsymbol{u}$:

$$\boldsymbol{T} = \rho_0(1 + \Pi)\boldsymbol{u}\otimes\boldsymbol{u} + \boldsymbol{t} + \boldsymbol{u}\otimes\boldsymbol{q} + \boldsymbol{q}\otimes\boldsymbol{u}, \tag{6a}$$

$$\Pi = \text{"specific internal energy,"} \tag{6b}$$

$$\boldsymbol{t} = \text{"stress tensor in averaged rest frame" is a second-rank, symmetric tensor orthogonal to } \boldsymbol{u},\ \boldsymbol{t}\cdot\boldsymbol{u} = \boldsymbol{u}\cdot\boldsymbol{t} = 0, \tag{6c}$$

$$\boldsymbol{q} = \text{"energy-flow vector" is a 4-vector orthogonal to } \boldsymbol{u},\ \boldsymbol{q}\cdot\boldsymbol{u} = 0. \tag{6d}$$

We use units in which $c = G = k$ (Boltzmann constant) $= 1$.

vi) *Assumption*: When mass-averaged over φ, Δt, and height, the baryons move very nearly in equatorial, circular, geodesic orbits about the black hole. Thus,

$$\boldsymbol{u}(r) \simeq \boldsymbol{w}(r) \equiv \text{(four-velocity for a circular geodesic orbit in the equatorial plane)}. \tag{7a}$$

Such orbits have specific energy-at-infinity $E^\dagger$, specific angular momentum $L^\dagger$, and angular velocity Ω given by

$$E^\dagger(r) \equiv -w_t(r)\,, \qquad L^\dagger(r) \equiv w_\varphi(r)\,, \qquad \Omega(r) \equiv w^\varphi/w^t\,. \tag{7b}$$

Consequence of above assumption.—Physically, the mean motion can be nearly geodesic only if radial pressure forces are negligible compared with the gravitational pull of the hole:

$$\text{(radial accelerations due to pressure gradients)} \sim \left|\frac{t_{rr,r}}{\rho_0}\right| \sim \left|\left(\frac{t_{rr}}{\rho_0}\right)_{,r}\right|$$

$$\ll \text{(gravitational acceleration of hole)} \sim |E^\dagger{}_{,r}| = |(1 - E^\dagger)_{,r}|\,.$$

Integrating this inequality and using the relation (valid for any astrophysical material)

$$\text{(internal energy density)} \equiv \rho_0 \Pi \sim |t_{rr}|\,,$$

we see that

$$\Pi \ll 1 - E^\dagger\,. \tag{8}$$

We call this the "condition of negligible specific heat." It says that the internal energy is negligible compared with the gravitational potential energy. In other words, as the material of the disk spirals slowly inward, releasing gravitational energy, a negligible amount of the energy released is stored internally. Almost all energy is transported away or radiated away. In terms of temperatures, condition (8) says

$$\Pi \sim T/m_p \sim T/10^{13}K \ll 1 - E^\dagger \sim M/r\,,$$

where M is the mass of the hole and m_p is the mass of a proton.

vii) *Assumption*: Heat flow within the disk is negligible, except in the vertical direction; i.e.,

$$\langle q(r, z)\rangle \approx \langle q^z(r, z)\rangle(\partial/\partial z)\,. \tag{9a}$$

(This is a reasonable assumption in view of the thinness of the disk.)

viii) *Assumption*: The only time-averaged stress-energy that reaches out of the faces of the disk is that carried by photons. (This assumption is meant to rule out gravitational waves as well as extended magnetic fields. If magnetic fields bulge out of the disk, but do not extend to heights $|z| \sim r$, then one can "redefine them into the disk" by making the "official" disk thickness, $2H$, large enough to enclose them.) Moreover, essentially all the stress-energy carried off is borne by photons of wavelength $\lambda \ll M$ = (size of hole).[1] (This allows one to neglect coherent superposition of the radiation reaction in adjacent [different r] regions of the disk, and to neglect "black-hole superradiance effects.") In addition—and as a corollary of (9a)—the photons emitted from the disk's surface are emitted, on the average, vertically as seen in the mean local rest frame of the orbiting gas. This, together with our neglect of (typically nonvertical) reimpinging radiation—see below—means that

$$\langle t_\varphi{}^z\rangle = \langle t_t{}^z\rangle = \langle t_r{}^z\rangle = \langle q_\varphi\rangle = \langle q_t\rangle = \langle q_r\rangle = 0 \qquad \text{at } z = \pm H\,. \tag{9b}$$

ix) *Assumption*: One can neglect energy and momentum transport from one region of the disk to another by photons emitted from the disk's surface. (This assumption is *not very reasonable*; heating of the outer regions by X-rays from the inner regions may be rather important—see Shakura and Sunyaev 1973. And in the inner regions, $M \leqslant r \leqslant 10M$, intense gravitational fields may pull a non-negligible fraction of the emitted photons back onto the disk. The effects of this are currently being studied by Cunningham 1974 and by Polnarev 1974.)

III. TIME-AVERAGED RADIAL DISK STRUCTURE

a) Summary of Results

By combining the assumptions of § II with the laws of conservation of rest mass, angular momentum, and energy (§ III*c*, below) one can derive three important equations for the time-averaged radial structure of the disk. These are equations for three quantities:

$$\dot{M}_0 \equiv \frac{dM_0}{dt} \equiv \begin{pmatrix}\text{radius-independent, time-averaged rate [rate measured}\\ \text{in terms of group-theoretically defined coordinate}\\ \text{time } t\text{] at which rest mass flows inward through disk}\end{pmatrix}, \tag{10a}$$

[1] We thank Douglas M. Eardley for pointing out to us the need for this assumption.

$$F(r) \equiv \langle q^z(r, z = H)\rangle = \langle -q^z(r, z = -H)\rangle$$

$$= \begin{pmatrix}\text{time-averaged flux of radiant energy [energy per unit proper} \\ \text{time } \tau \text{ per unit proper area } A\text{] flowing out of upper face} \\ \text{of disk, as measured by an observer on the upper face who} \\ \text{orbits with the time-averaged motion of the disk's matter}\end{pmatrix}$$

$$= (\text{time-averaged flux flowing out of lower face}) , \tag{10b}$$

$$W_\varphi{}^r(r) \equiv \int_{-H}^{+H} \langle t_\varphi{}^r \rangle dz$$

$$= (g^{rr})^{1/2} \times \begin{pmatrix}\text{time-averaged torque per unit circumference acting across} \\ \text{a cylinder at radius } r\text{, due to the stresses in the disk}\end{pmatrix} . \tag{10c}$$

The equations derived are

$$\dot{M}_0 = -2\pi e^{\nu+\psi+\mu}\Sigma u^r , \tag{11a}$$

$$F(r) = (\dot{M}_0/4\pi)e^{-(\nu+\psi+\mu)}f , \tag{11b}$$

$$W_\varphi{}^r = (\dot{M}_0/2\pi)e^{-(\nu+\psi+\mu)}[(E^\dagger - \Omega L^\dagger)/(-\Omega_{,r})]f . \tag{11c}$$

Here ν, ψ, μ are the metric coefficients (functions of r) of equation (1); $E^\dagger$, $L^\dagger$, Ω are the specific energy-at-infinity specific angular momentum, and angular velocity (functions of r) defined in equations (7a, b); Σ and u^r are the surface density and radial velocity (unknown functions of r) defined in equations (3) and (4); and f is the function of radius

$$f \equiv -\Omega_{,r}(E^\dagger - \Omega L^\dagger)^{-2}\int_{r_{ms}}^{r}(E^\dagger - \Omega L^\dagger)L^\dagger{}_{,r}dr$$

$$= -(w^t{}_{,r}/w_\varphi)\int_{r_{ms}}^{r}(w_{\varphi,r}/w^t)dr . \tag{12}$$

Here r_{ms} ("ms" = "marginally stable") is the radius of the innermost stable circular geodesic orbit:

$$r_{ms} = (\text{radius at which } dE^\dagger/dr = dL^\dagger/dr = 0) . \tag{13}$$

The derivation of equations (11) will be presented in §§ III*b* and III*c*.

When specialized to the Kerr metric, the above functions have the following forms. We express them in terms of

M = mass of black hole ,

a = specific angular momentum of hole ($a > 0$ if disk orbits in same direction as hole rotates; $a < 0$ if it orbits in opposite direction) ,

$$a_* \equiv a/M \text{ (note: } -1 \leq a_* \leq +1) ,$$

$$x \equiv (r/M)^{1/2} = (\text{dimensionless radial coordinate}) ,$$

$$x_0 = (r_{ms}/M)^{1/2} ,$$

$x_1, x_2, x_3 \equiv$ the three roots of $x^3 - 3x + 2a_* = 0$; in particular ,

$$x_1 = 2\cos\left(\tfrac{1}{3}\cos^{-1}a_* - \pi/3\right) ,$$

$$x_2 = 2\cos\left(\tfrac{1}{3}\cos^{-1}a_* + \pi/3\right) ,$$

$$x_3 = -2\cos\left(\tfrac{1}{3}\cos^{-1}a_*\right) .$$

$$\mathscr{A} = 1 + a_*^2x^{-4} + 2a_*^2x^{-6} , \qquad \mathscr{B} = 1 + a_*x^{-3} ,$$

$$\mathscr{C} = 1 - 3x^{-2} + 2a_*x^{-3} , \qquad \mathscr{D} = 1 - 2x^{-2} + a_*^2x^{-4} ,$$

$$\mathscr{E} = 1 + 4a_*^2x^{-4} - 4a_*^2x^{-6} + 3a_*^4x^{-8} , \qquad \mathscr{F} = 1 - 2a_*x^{-3} + a_*^2x^{-4} ,$$

$$\mathscr{G} = 1 - 2x^{-2} + a_*x^{-3} . \tag{14}$$

They are (cf. Bardeen, Press, and Teukolsky 1972)

$$e^{2\nu} = \mathscr{A}^{-1}\mathscr{D}, \qquad e^{2\psi} = M^2x^4\mathscr{A}, \qquad e^{2\mu} = \mathscr{D}^{-1}, \tag{15a, b, c}$$

$$e^{\nu+\mu+\psi} = r = Mx^2, \qquad \omega = 2a_*M^{-1}x^{-6}\mathscr{A}^{-1}, \qquad \Omega = M^{-1}x^{-3}\mathscr{B}^{-1}, \tag{15d, e, f}$$

$$E^\dagger = \mathscr{C}^{-1/2}\mathscr{G}, \qquad L^\dagger = Mx\mathscr{C}^{-1/2}\mathscr{F}, \tag{15g, h}$$

$$E^\dagger - \Omega L^\dagger = \mathscr{B}^{-1}\mathscr{C}^{1/2}, \qquad \Omega_{,r} = -\tfrac{3}{2}M^{-2}x^{-5}\mathscr{B}^{-2}; \tag{15i, j}$$

$$x_0 = \{3 + Z_2 - \operatorname{sgn}(a_*)[(3 - Z_1)(3 + Z_1 + 2Z_2)]^{1/2}\}^{1/2}, \qquad \text{where} \tag{15k}$$

$$Z_1 \equiv 1 + (1 - a_*^2)^{1/3}[(1 + a_*)^{1/3} + (1 - a_*)^{1/3}], \qquad Z_2 \equiv (3a_*^2 + Z_1^2)^{1/2}; \tag{15l, m}$$

$$f = \frac{3}{2M}\frac{1}{x^2(x^3 - 3x + 2a_*)}\left[x - x_0 - \tfrac{3}{2}a_* \ln\left(\frac{x}{x_0}\right) - \frac{3(x_1 - a_*)^2}{x_1(x_1 - x_2)(x_1 - x_3)}\ln\left(\frac{x - x_1}{x_0 - x_1}\right) - \frac{3(x_2 - a_*)^2}{x_2(x_2 - x_1)(x_2 - x_3)}\ln\left(\frac{x - x_2}{x_0 - x_2}\right) - \frac{3(x_3 - a_*)^2}{x_3(x_3 - x_1)(x_3 - x_2)}\ln\left(\frac{x - x_3}{x_0 - x_3}\right)\right]. \tag{15n}$$

The rest of § III is a derivation of the radial-structure formulae (11), beginning (§ III*b*) with a crucial relation for geodesic orbits, and then turning (§ III*c*) to formulation and manipulation of the conservation laws.

b) The Energy–Angular-Momentum Relation for Circular Geodesic Orbits

Consider circular geodesic orbits in the equatorial plane of metric (1) with four-velocities $w(r)$ having nonzero components given by equation (7b). By combining the geodesic equation and the normalization condition,

$$\nabla_w w = 0, \qquad w \cdot w = -1, \tag{16}$$

with the symmetry of the covariant derivative and the vanishing of the radial component w_r, one obtains

$$\begin{aligned} 0 &= [\nabla_w w - \tfrac{1}{2}\nabla(w \cdot w)] \cdot (\partial/\partial r) \equiv w^\alpha w_{r;\alpha} - w^\alpha w_{\alpha;r} = w^\alpha(w_{r,\alpha} - w_{\alpha,r}) \\ &= -w^\alpha w_{\alpha,r} = -w^t w_{t,r} - w^\varphi w_{\varphi,r} = w^t(E^\dagger{}_{,r} - \Omega L^\dagger{}_{,r}). \end{aligned} \tag{17}$$

Hence, the circular geodesic orbits satisfy the fundamental relation

$$E^\dagger{}_{,r} = \Omega L^\dagger{}_{,r}. \tag{18}$$

This is a special case of the universal "energy–angular-momentum relation," $dE = \Omega dJ$ or

$$(\text{change of energy}) = (\text{angular velocity}) \cdot (\text{change of angular momentum}), \tag{19}$$

which plays a fundamental role throughout astrophysics. (See, e.g., Appendix B of Ostriker and Gunn 1969; eq. [80] of Bardeen 1970; Hartle 1970; and § 10.7 of Zel'dovich and Novikov 1971.)

c) Formulation and Manipulation of the Conservation Laws

The radial structure of the disk is governed by three conservation laws: conservation of rest mass, of angular momentum, and of energy.

In differential form the law of rest-mass conservation reads

$$\nabla \cdot (\rho_0 u^{\text{inst}}) = 0. \tag{20}$$

We convert to a more useful integral conservation law by integrating over the 3-volume of the disk between radius r and $r + \Delta r$ and over time Δt, and by then using Gauss's theorem to convert to a surface integral:

$$\begin{aligned} 0 &= \int_{\mathscr{V}} \nabla \cdot (\rho_0 u^{\text{inst}})(-g)^{1/2} dt dr dz d\varphi = \int_{\partial\mathscr{V}} \rho_0 u^{\text{inst}} \cdot d^3\Sigma \\ &= \left[\int_{-H}^{+H}\int_t^{t+\Delta t}\int_0^{2\pi} \rho_0 u^r{}_{\text{inst}}(-g)^{1/2} d\varphi dt dz\right]_r^{r+\Delta r} + [\text{total rest mass in the 3-volume}]_t^{t+\Delta t} \\ &= (\Delta t)(2\pi e^{\nu+\psi+\mu}\Sigma u^r)_{,r}\Delta r + 0. \end{aligned}$$

The second bracket, [ ], can be neglected compared with the first because of assumption (iv) of § II (mass in Δr negligible compared with mass that flows across r in time Δt). Physically the above equation says

$$\dot{M}_0 \equiv -2\pi e^{\nu+\psi+\mu}\Sigma u^r = \text{(time-averaged rate of accretion of rest mass) is independent of radius } r\,. \quad (21)$$

This is the first of our radial structure equations, equation (11a).

In differential form the law of angular-momentum conservation reads

$$\nabla\cdot \boldsymbol{J} = 0\,, \qquad \boldsymbol{J} \equiv \boldsymbol{T}\cdot\partial/\partial\varphi = \text{(density-flux 4-vector for angular momentum)}\,. \quad (22)$$

Again we convert to an integral conservation law by integrating over the 3-volume between radius r and $r + \Delta r$ and over time Δt, and by then using Gauss's theorem to convert to a surface integral. In the case of rest mass there was no flux across the upper and lower faces of the disk ($z = \pm H$), so the only contributions to the surface integral were at the outer and inner radii, $r + \Delta r$ and r, and at the hypersurfaces of constant time, $t + \Delta t$ and t. However, radiation pouring out of the disk produces an angular-momentum flux across the upper and lower faces, so in this case we get six terms in the surface integral:

$$\begin{aligned} 0 &= \int_{\mathscr{V}} \nabla\cdot \boldsymbol{J}(-g)^{1/2} dt dr dz d\varphi = \int_{\partial\mathscr{V}} \boldsymbol{J}\cdot d^3\boldsymbol{\Sigma} = \int_{\partial\mathscr{V}} T_\varphi{}^\alpha d^3\Sigma_\alpha \\ &= \left\{\int_{-H}^{+H}\int_t^{t+\Delta t}\int_0^{2\pi} [\rho_0(1+\Pi)u_\varphi u^r + t_\varphi{}^r + u_\varphi q^r + q_\varphi u^r](-g)^{1/2} d\varphi dt dz\right\}_r^{r+\Delta r} \\ &\quad + \left\{\int_r^{r+\Delta r}\int_t^{t+\Delta t}\int_0^{2\pi} [\rho_0(1+\Pi)u_\varphi u^z + t_\varphi{}^z + u_\varphi q^z + q_\varphi u^z](-g)^{1/2} d\varphi dt dr\right\}_{-H}^{+H} \\ &\quad + \{\text{total angular momentum in the 3-volume}\}_t^{t+\Delta t}\,. \end{aligned} \quad (23)$$

In the first brace, { }, we can ignore Π (negligible specific heat; eq. [8]); and we can ignore $u_\varphi q^r$ and $q_\varphi u^r$ by comparison with the $u_\varphi q^z$ of the second brace (negligible heat transport along the plane of the disk; eq. [9a]). Hence, the first brace reduces to

$$\begin{aligned} \left\{\int_{-H}^{+H} (2\pi\Delta t)[\langle\rho_0\rangle u_\varphi u^r + \langle t_\varphi{}^r\rangle](-g)^{1/2} dz\right\}_r^{r+\Delta r} &= \{(2\pi\Delta t)[\Sigma L^\dagger u^r + W_\varphi{}^r]e^{\nu+\psi+\mu}\}_r^{r+\Delta r} \\ &= \Delta t[-\dot{M}_0 L^\dagger + 2\pi e^{\nu+\psi+\mu} W_\varphi{}^r]_{,r}\Delta r\,. \end{aligned} \quad (24)$$

Here we have used equations (4), (7), and (21), and the definition

$$W_\alpha{}^\beta \equiv \int_{-H}^{+H} \langle t_\alpha{}^\beta\rangle dz\,. \quad (25)$$

In the second brace { } of formula (23) the first and last terms vanish because $u^z = 0$, and the second term vanishes by equation (9b). Hence, the second brace becomes

$$\left\{\int_r^{r+\Delta r} (2\pi\Delta t) u_\varphi \langle q^z\rangle (-g)^{1/2} dr\right\}_{-H}^{+H} = 2\Delta t[2\pi e^{\nu+\psi+\mu} L^\dagger F]\Delta r\,, \quad (26)$$

where we have used equations (7a, b), plus definition (10b) of F. The third brace { } in formula (23) can be neglected compared with the first brace because of assumption (iv). Combining a zero value for the third brace with equation (26) for the second brace and equation (24) for the first brace, we obtain

$$[\dot{M}_0 L^\dagger - 2\pi e^{\nu+\psi+\mu} W_\varphi{}^r]_{,r} = 4\pi e^{\nu+\psi+\mu} F L^\dagger\,. \quad (27)$$

This is our final form for the law of angular momentum conservation. The first term represents the angular momentum carried by the rest mass of the disk; the second term is the angular momentum transported mechanically by torques in the disk (by viscous stresses, by turbulent stresses, by magnetic stresses, etc.); the third term is the angular momentum carried away from the disk's surface by radiation.

The differential form of the law of energy conservation is

$$\nabla\cdot \boldsymbol{E} = 0\,, \qquad \boldsymbol{E} \equiv -\boldsymbol{T}\cdot\partial/\partial t = \text{(density-flux 4-vector for energy-at-infinity)}\,. \quad (28)$$

By manipulating this conservation law in precisely the same manner as we manipulated the law of angular momentum conservation (22), we arrive at the time-averaged and volume-integrated conservation law

$$[\dot{M}_0 E^\dagger + 2\pi e^{\nu+\psi+\mu} W_t{}^r]_{,r} = 4\pi e^{\nu+\psi+\mu} F E^\dagger\,. \quad (29)$$

The second term can be rewritten in terms of $W_\varphi{}^r$ by use of the orthogonality relation $u^\alpha t_\alpha{}^\beta = 0$, which implies that $u^\alpha W_\alpha{}^\beta = 0$, or

$$W_t^r = -(u^\varphi/u^t)W_\varphi{}^r = -\Omega W_\varphi{}^r .$$

The result for equation (29) is

$$[\dot{M}_0 E^\dagger - 2\pi e^{\nu+\psi+\mu} W_\varphi{}^r \Omega]_{,r} = 4\pi e^{\nu+\psi+\mu} F E^\dagger . \tag{30}$$

d) Integration of the Conservation Laws

Equations (27) and (30) can be integrated to obtain the emitted flux F and the torque per unit circumference $W_\varphi{}^r$. This is done as follows:

i) Change variables to

$$f \equiv 4\pi e^{\nu+\psi+\mu} F/\dot{M}_0 , \qquad w \equiv 2\pi e^{\nu+\psi+\mu} W_\varphi{}^r/\dot{M}_0 . \tag{31a, b}$$

ii) In terms of these variables the conservation laws (27) and (30) become

$$(L^\dagger - w)_{,r} = fL^\dagger , \qquad (E^\dagger - \Omega w)_{,r} = fE^\dagger . \tag{32a, b}$$

iii) Multiply (32a) by Ω, subtract from (32b), and use the "energy–angular-momentum relation" (18) to obtain the algebraic relation

$$w = [(E^\dagger - \Omega L^\dagger)/(-\Omega_{,r})]f . \tag{33}$$

iv) Insert this expression for w into (32a), and integrate the resulting first-order differential equation for f, making use of (18). The result is

$$\frac{(E^\dagger - \Omega L^\dagger)^2}{-\Omega_{,r}} f = \int (E^\dagger - \Omega L^\dagger) L^\dagger{}_{,r} dr + \text{const.}$$

v) To fix the constant of integration, use the following physical fact: When the accreting material reaches the innermost stable circular orbit, $r = r_{ms}$, it drops out of the disk and falls directly down the hole. Hence, just inside $r = r_{ms}$ there is negligible material to "torque up" the material just outside $r = r_{ms}$—which means that the torque $W_\varphi{}^r$, and hence w, must vanish at $r = r_{ms}$.[2] To make $w(t_{ms})$ vanish we must choose our constant of integration such that

$$\frac{(E^\dagger - \Omega L^\dagger)^2}{-\Omega_{,r}} f = \int_{r_{ms}}^{r} (E^\dagger - \Omega L^\dagger) L^\dagger{}_{,r} dr .$$

vi) Bring this result into the following alternative forms by integration by parts and use of the energy–angular-momentum relation (18):

$$\begin{aligned} f &= -\Omega_{,r}(E^\dagger - \Omega L^\dagger)^{-2} \int_{r_{ms}}^{r} (E^\dagger - \Omega L^\dagger) L^\dagger{}_{,r} dr \\ &= -\Omega_{,r}(E^\dagger - \Omega L^\dagger)^{-2} \left[E^\dagger L^\dagger - E^\dagger{}_{ms} L^\dagger{}_{ms} - 2 \int_{r_{ms}}^{r} L^\dagger E^\dagger{}_{,r} dr \right] \\ &= -\Omega_{,r}(E^\dagger - \Omega L^\dagger)^{-2} \left[-E^\dagger L^\dagger + E^\dagger{}_{ms} L^\dagger{}_{ms} + 2 \int_{r_{ms}}^{r} E^\dagger L^\dagger{}_{,r} dr \right] . \end{aligned} \tag{34}$$

Equation (34) for f and equations (31), (33) for F and $W_\varphi{}^r$ are the radial-structure equations (12) and (11b, c) quoted in § III*a*.

IV. STEADY-STATE DISK MODELS

Steady-state relativistic models for the accretion disk around a Kerr black hole have been built by Novikov and Thorne (1973). These models are patterned after Newtonian models by Shakura and Sunyaev. They include details of vertical structure (vertical force balance; vertical energy transport; etc.), as well as details of radial structure.

[2] It is conceivable that the disk material might contain extremely strong magnetic fields, and that these fields might transport a torque from the infalling material at $r < r_{ms}$ to the disk at $r \geq r_{ms}$. In this case the boundary condition at r_{ms} would be modified, and the solution for f would be changed. It seems to us unlikely that the changes would be substantial, except very near r_{ms} (i.e., at $r - r_{ms} \lesssim 0.1 r_{ms}$). But when constructing explicit disk models, one should examine this possibility carefully.

All of the quantities appearing in the Novikov-Thorne models (§§ 5.9 and 5.10 of their paper) are expressed as explicit, algebraic functions of radius, except one: the function $\mathscr{Q}(r)$. The results of this paper allow one to also express $\mathscr{Q}$ as an explicit algebraic function of r—or, equivalently, of $x = (r/M)^{1/2}$. Direct comparison of equations (5.6.14b) and (5.4.1b, c) of Novikov-Thorne with equations (11b) and (15d, n) of this paper shows that

$$\mathscr{Q} = \frac{1 + a_* x^{-3}}{(1 - 3x^{-2} + 2a_* x^{-3})^{1/2}} \frac{1}{x} \left[x - x_0 - \tfrac{3}{2} a_* \ln \left(\frac{x}{x_0} \right) - \frac{3(x_1 - a_*)^2}{x_1(x_1 - x_2)(x_1 - x_3)} \ln \left(\frac{x - x_1}{x_0 - x_1} \right) - \frac{3(x_2 - a_*)^2}{x_2(x_2 - x_1)(x_2 - x_3)} \ln \left(\frac{x - x_2}{x_0 - x_2} \right) - \frac{3(x_3 - a_*)^2}{x_3(x_3 - x_1)(x_3 - x_2)} \ln \left(\frac{x - x_3}{x_0 - x_3} \right) \right] . \tag{35}$$

(NOTE.—In equation [5.4.1h] of Novikov and Thorne there is an error, pointed out to us by Chris Cunningham: the sign in the exponential should be plus rather than minus.)

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PART III

PARTICLE EMISSION RATES FROM BLACK HOLES

(a) Historical Background

General relativity and quantum mechanics have certainly been among the most fundamental developments of physics in the twentieth century. General relativity expressed gravity in terms of curvatures of spacetime, a concept radically different from the Newtonian theory of instantaneous action at a distance. Quantum mechanics expressed physical processes in terms of amplitudes that give probabilities for certain observations rather than the deterministic evolution of all observables. Both of these new formulations could be shown to reduce to the older laws of classical physics in the realm of experience where the older laws had been strongly verified, but they predicted different phenomena in other realms. Quantum mechanics predicted new effects such as the uncertainty principle that would show up for very small objects like atoms, and general relativity predicted new effects such as collapse into black holes that would show up for very large objects like massive stars.

One would like to unify general relativity and quantum mechanics (a task not yet completed), but it appeared that, at least in the present universe, the realm where general relativity is important (very large objects) does not overlap the realm where quantum mechanics is important (very small objects). For example, general relativity is important for an object that has a linear size L not much greater than its Schwarzschild radius $2GM/c^2$ (the size of the black hole it would form), so that

$$\frac{c^2}{G} L \lesssim M \quad . \tag{III.1}$$

Alternatively, quantum mechanics is important for an object whose intrinsic size L is not much greater than its reduced Compton wavelength $\hbar/Mc$, so

$$M \lesssim \frac{\hbar}{c} L^{-1} \quad . \qquad \text{(III.2)}$$

For both general relativity and quantum mechanics to be important, the combination of (III.1) and (III.2) requires

$$L \lesssim \left(\frac{\hbar G}{c^3}\right)^{1/2} \equiv L_{Planck} = 1.62 \times 10^{-33} \text{cm} \quad , \qquad \text{(III.3)}$$

which is a length much shorter than that probed by any current experiment. Likewise, the density must be

$$\rho \simeq ML^{-3} \gtrsim \frac{c^2}{G} L^{-2} \gtrsim \frac{c^5}{\hbar G^2} \equiv \rho_{Planck} = 5.16 \times 10^{93} \text{g cm}^{-3} \quad , \qquad \text{(III.4)}$$

which is far beyond any densities observed in the present universe. Therefore, one might conclude that any significant union of general relativity and quantum mechanics could not be experimentally tested.

However, quantum effects associated with strong gravitational fields could show up observationally if they would accumulate over the age of the universe, which gives a factor of roughly 10^{61} in terms of the Planck time

$$t_{Planck} \equiv L_{Planck}/c = \left(\frac{\hbar G}{c^5}\right)^{1/2} = 5.39 \times 10^{-44} \text{s} \quad . \qquad \text{(III.5)}$$

It turns out that such effects indeed can occur for primordial black holes within a certain mass range--the effects being observable in the present universe if there are a sufficient number of such black holes. These effects involve the creation and emission of particles by black holes, which will now be discussed.

The first prediction of emission by a black hole was made by Zel'dovich (1971,1972). He pointed out on heuristic grounds that a rotating black hole should amplify certain waves and that there should be an analogous quantum effect of spontaneous radiation of energy and angular momentum. Later Misner (1972) and Starobinsky (1973) confirmed the amplification by a Kerr hole of scalar waves in the "superradiant regime" (where the angular velocity of the wave-fronts is lower than that of the hole), and Bekenstein (1973a) showed that amplification should occur for all kinds of waves with positive energy density. However, the quantum effect predicted by Zel'dovich was not universally known, and in fact Larry Ford at Princeton University and I independently rediscovered it.

The argument for this spontaneous radiation was that in a quantum analysis the amplification of waves is stimulated emission of quanta, so that even in the absence of incoming quanta one should get spontaneous emission. By using the relation between the Einstein coefficients for spontaneous and stimulated emission, one can calculate the spontaneous rate from the amplification factor, as Starobinsky (1973) noted, at least when the spontaneous emission probability is much less than unity.

A problem arose for neutrinos in that Unruh (1973) showed that their waves are never amplified. This result violated Bekenstein's conclusion and seemed to be a breakdown in the Hawking (1971) area theorem. The reason for the violation was traced to a negative local energy density of the classical neutrino waves at the horizon. However, Feynman suggested (unpublished) that the lack of amplification might be due to the Pauli exclusion principle, so that incident neutrinos suppress spontaneous emission which otherwise occurs. The amplification factor would then be less than unity, since the calculation of an unquantized neutrino wave cannot directly show the spontaneous emission but only how the emission changes as the incident flux is varied.

One might be surprised to find such a difference between integral and half-integral spins showing up in the behavior of their unquantized waves, but this is merely an illustration of the connection between spin and statistics. Pauli (1940) has shown that half-integral spins must be assigned anticommutation relations in order to get a positive energy density, which is precisely what the unquantized neutrino waves violate in not showing superradiance.

Indeed, this same behavior occurs in the Klein paradox. A scalar wave incident on an electrostatic potential step higher than the kinetic energy plus twice the mass gives a reflected current greater than the incident current. On the other hand, a Dirac wave incident on such a step gives less reflected current. (This is the result if one makes the causality requirement of the transmitted waves' having a group velocity away from the step, rather than having the momentum vector away from the step as in Bjorken and Drell 1964.) Nikishov (1970) uses field theory to calculate the pair production by a potential step of general shape with no particles incident. His results show that the expected number of particles emitted in a given Klein-paradox state is

$$\langle N\rangle = \pm (A - 1) \quad , \qquad \text{(III.6)}$$

where A is the amplification factor for the reflected wave of the unquantized Klein-Gordon (+) or Dirac (-) equation. This formula applies even if the emission probabilities are not small, so that $\langle N\rangle$ includes the possibility of emitting more than one particle (if a boson) in the same state.

Unruh (1974) made a formal calculation of second quantization of scalar and neutrino fields in the complete Kerr metric and found essentially the same results as Eq. (III.6) if he chose the initial vacuum state to correspond to no particles coming out of the past horizon. Ford (1975) quantized the massive scalar field in a somewhat different way with similar results. However, Unruh noted that the actual situation might be different, with no past horizon but the black hole formed by collapse. Nevertheless, neither he nor any of the discoverers of the spontaneous emission attempted to calculate that situation.

Meanwhile (summer 1973), Stephen Hawking at Cambridge University heard of this work through Douglas Eardley and so while in Moscow discussed it with Zel'dovich and Starobinsky. Believing in the reality of the spontaneous emission but wishing to put its derivation on a firmer footing, Hawking dared to attempt the difficult calculation of field theory during the collapse and formation of a black hole. Separating out the essential elements, Hawking found how to calculate the particle emission at late times, after the collapse had settled down to form a stationary black hole. At first Hawking got an infinite number of particles emitted, but then he discovered that the infinity corresponded to emission at a steady rate. However, the emission was not only in the superradiant states or modes but in all modes that could come from the black hole!

Hawking initially did not believe this result (a consolation to those of us who doubted it also when we first heard it). Thinking that the emission might be an artifact of the spherical symmetry he had assumed, Hawking considered nonspherical collapse and got the same emission. Then he tried putting in a cutoff on the frequencies of the modes in the initial state before the collapse, but that eliminated all the emission, including the spontaneous emission in the superradiant modes that Hawking was certain existed. Perhaps most convincing to Hawking was the fact that the emission rate was just that of a thermal body with the same absorption probabilities as the black hole and with a temperature (in geometrical units) equal to the surface gravity of the hole divided by 2π. This result held for fields of any spin and seemed to confirm some thermodynamic ideas of Bekenstein (1973b). However, before the emission process was discovered, Bardeen, Carter, and Hawking (1973) had argued against Bekenstein's suggestion of a black-hole temperature proportional to surface gravity. Thus Bekenstein's ideas were originally not a motivation for Hawking's calculation.

As word of his calculation began to spread, Hawking published a simplified version of it in Nature (1974). However, even at this stage Hawking was not certain of the result and so expressed the title as a question, "Black hole explosions?" He noted that the calculation ignored the change in the metric due to the particles created and to quantum fluctuations. One objection raised by several people was that the calculation seemed to give a very high energy flux just outside the horizon, which might prevent the black hole from forming at all. Hawking later answered this and other problems by a more detailed version of the calculation (Hawking 1975a), which showed that an infalling observer would not see many particles near the horizon. However, it might be noted that there is still some controversy about the existence of particles there. The back reaction of the particles created would, in Hawking's view, simply be to reduce the mass of the hole by the amount of the energy radiated away.

Presumably quantum fluctuations of the metric itself can give rise to the emission of gravitons in addition to the emission of other particles calculated as if the geometry were fixed. By considering linearized fluctuations in the metric about a given background, the emission of gravitons can be handled in the same manner as the emission of any other particles, though one might argue that graviton emission depends more fundamentally upon the assumed fluctuations in the metric. Therefore, any observed consequences of graviton emission can be viewed as testing whether gravity is quantized.

Hawking has argued (unpublished) that quantum mechanics allows small deviations of the action from the extremum value that gives the classical field equations for matter and gravity. Thus the classical equations can be violated in a small region near a black hole, giving rise to the emission of matter or gravitational waves, but the equations cannot be violated significantly on a very large surface surrounding the hole. Therefore, quantities determined by surface fluxes at infinity do remain conserved: energy, momentum, angular momentum, and charge. This is the basis for arguing that the emission carries away the quantities of the hole which otherwise would be constant. Note that baryon and lepton numbers are not observed to be connected with long-range fields, so they presumably cannot be determined by surface fluxes at infinity and thus would not be conserved globally by the black-hole emission process.

The thermal emission first calculated by Hawking has been verified by several subsequent calculations. Boulware (1975) and Davies (1976) have calculated the emission from a collapsing shell. Gerlach (1975) has interpreted the emission as parametric amplification of the zero-point oscillations of the field inside the collapsing object. DeWitt (1975) has given detailed derivations of both the spontaneous emission process in the complete Kerr metric (with no particles coming out of the past horizon) and of the thermal emission from a black hole formed by collapse. Unruh (1975) has found that his derivation in the complete Kerr metric will give not only the spontaneous but also the thermal emission if the boundary condition at the past horizon is changed from no particles seen by an observer at fixed radius just outside the horizon to no particles seen by an observer freely falling along the horizon. Wald (1975), Parker (1975) and Hawking (1975b) have calculated the density matrix of the emitted particles and find that it, as well as the expected number in each mode, is precisely thermal. Bekenstein (1975) has given an information-theory argument of why this should be so. Hartle and Hawking (1975) have done a path-integral calculation of the probability for a particle to propagate out of a black hole from the future singularity and show that this method also leads to the same thermal emission. In summary, the thermal emission from a black hole has been derived in a variety of ways by several people, so its prediction seems to be a clear consequence of our present theories of quantum mechanics and general relativity.

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(b) Numerical Calculations

After recognizing the existence of spontaneous emission from a rotating black hole, I intended to calculate those emission rates to show how fast a black hole would give up its angular momentum. However, after Hawking announced his result of thermal emission and after my initial objections to it were answered by the more detailed publication of his calculations, I began calculating the thermal emission rates.

The basic result from Hawking's calculation that I needed was his expression for the expected number of particles emitted in a mode in terms of the absorption probability for that mode (Eq. 3.4 of Hawking 1975). To compute the absorption probabilities for neutrino, photon, and graviton modes, I used the separated wave equation of Teukolsky (1972,1973), and I modified the computer programs initially written by Teukolsky and Press (1974) for solving the Teukolsky equation numerically. The thermal emission by Hawking's formula was then integrated over all frequency modes for a given angular mode, and finally the angular modes were summed to give the total emission.

In calculating the total number rate and power in a given angular mode, by far the most computer time is spent evaluating the absorption probabilities at the different frequencies. Therefore, it was desired to reduce the number of such evaluations needed for a given accuracy of the integrated number rate and power. To accomplish this, an algorithm was devised which would vary the step size for the integration and yet which would not waste calculated points if the attempted step size proved to be too large to give the required accuracy.

The algorithm did the integration in a sequential manner by starting at zero frequency and moving through a series of intervals to some frequency high enough that the integral from there to infinity could be accurately estimated without evaluating any more points. As a first attempt, a new interval was integrated in one step--a "step" being the basic unit of integration. The integral for each step was estimated by Bode's rule (the closed-type Newton-Cotes formula for the integral of a quartic polynomial which fits five points equally spaced between the ends of the step). An estimate for the error was made by comparing Bode's rule with Simpson's rule applied once for each half of the step, using the same five points already evaluated for Bode's rule. Thus the error estimate reflected the fourth-order error of Simpson's rule, but the integration was accurate to the sixth-order error of Bode's rule.

If the error estimate exceeded the preset tolerance criterion, the first half of the interval was taken as the next attempt for a successful step, requiring only two new points to be evaluated to apply Bode's rule again. This step-halving was iterated until the step was made sufficiently small to be successfully integrated within the error criterion. Then the integration proceeded forward with the other half of the last unsuccessful step. If it could be integrated successfully without further halving, the step size was doubled to do the second half of the next uncompleted step. As long as the integrations were successful, the step-doubling was continued until the original interval was finished. Thus the points temporarily discarded by successively halving the step were actually saved and re-used when the step size was successively doubled.

Once the step-doubling was completed (or if the interval was done successfully in one step), the algorithm proceeded to a new interval. The size of this interval was chosen so that the estimated error would be one-half the tolerance criterion if the fourth derivative stayed the same as its value estimated for the last step. The fact that this constant-fourth-derivative assumption was false led to the occasional need to halve the step size before the estimated error was reduced below the tolerance criterion.

Paper II gives the massless particle emission rates from a Schwarzschild black hole and estimates the resulting lifetimes of primordial black holes. In addition, this paper derives analytic expressions for the absorption probabilities, cross sections, and emission rates for low frequency waves in the field of a black hole of arbitrary rotation. Paper III extends the numerical calculations to rotating holes and shows how they evolve as they lose energy and angular momentum to massless or nearly massless particles. The implications for the lifetimes and present configurations of primordial black holes are discussed. Following Paper III is a listing of coefficients of polynomial fits used to calculate the angular eigenvalues for neutrinos that are needed in solving the radial Teukolsky equation.

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(c) Particle Emission Rates from a Black Hole: Massless Particles from an Uncharged, Nonrotating Hole (Paper II; published in Phys. Rev. D13, 198 [1976]; copied by permission of the American Physical Society)

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Particle emission rates from a black hole: Massless particles from an uncharged, nonrotating hole*

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Hawking has predicted that a black hole will emit particles as if it had a temperature proportional to its surface gravity. This paper combines Hawking's quantum formalism with the black-hole perturbation methods of Teukolsky and Press to calculate the emission rate for the known massless particles. Numerical results indicate that a hole of mass $M \gg 10^{17}$ g should emit a total power output of $2\times 10^{-4}\hbar c^6 G^{-2} M^{-2}$, of which 81% is in neutrinos, 17% is in photons, and 2% is in gravitons. These rates plus an estimate for the emission rates of massive particles from smaller holes allow one to infer that a primordial black hole will have decayed away within the present age of the universe if and only if its initial mass was $M < (5 \pm 1) \times 10^{14}$ g.

I. INTRODUCTION

Hawking has calculated quantum mechanically[1] that a black hole will emit particles as if it were a hot body with a temperature T proportional to its surface gravity. Since the surface gravity is inversely proportional to the black-hole mass M, and the emitting area A is proportional to M^2, the luminosity or total power emitted is proportional to AT^4 or M^{-2}. As M decreases at this rate, the black-hole lifetime will be proportional to M^3. Dimensional arguments indicate that the lifetime will be less than the age of the universe only if $M \lesssim 10^{15}$ g. Consequently, the thermal emission is insignificant for black holes formed by the stellar collapse ($M \gtrsim M_\odot$, lifetime $\gtrsim 10^{66}$ yr), but it is of crucial importance for the small primordial black holes possibly formed by fluctuations in the early universe.[2-4]

This paper reports numerical calculations of the emission rates for massless particles. The spectra from the dominant angular modes are given for neutrinos, photons, and gravitons. The spectra are integrated to give the total number rate and power emitted in the various modes. From the total power emitted in all modes, the lifetime of a black hole is predicted. Essentially, this paper gives numerical coefficients for the dimensionally determined quantities of the preceding paragraph.

To simplify the notation, dimensionless units will be used such that

$$\hbar = c = G = k \text{ (Boltzmann's constant)} = 1. \qquad (1)$$

That is, all quantities will be written in terms of the Planck mass ($[\hbar c/G]^{1/2} = 2.18\times 10^{-5}$ g), length ($[\hbar G/c^3]^{1/2} = 1.62\times 10^{-33}$ cm), time ($[\hbar G/c^5]^{1/2} = 5.39\times 10^{-44}$ sec), temperature ($[\hbar c^5/G]^{1/2}/k = 1.42\times 10^{32}$ °K), energy ($[\hbar c^5/G]^{1/2} = 1.96\times 10^{16}$ erg $= 1.22\times 10^{22}$ MeV), power ($c^5/G = 3.63\times 10^{59}$ erg sec^{-1}), charge ($[\hbar c]^{1/2} = 5.62\times 10^{-9}$ esu $= 11.7e$), etc. For example, the electron mass is $m_e = 4.19\times 10^{-23}$, the muon mass is $m_\mu = 8.65\times 10^{-21}$, the blackbody background temperature is $T_\gamma = 1.9\times 10^{-32}$, the age of the universe is $t_0 \approx 10^{61}$ (= 17 billion years), and the solar mass and luminosity are $M_\odot = 9.14\times 10^{37}$ and $L_\odot = 1.05\times 10^{-26}$, respectively.[5]

The present paper will limit itself to the known massless particles (ν_e, $\bar{\nu}_e$, ν_μ, $\bar{\nu}_\mu$, γ, and graviton) being emitted from an uncharged, nonrotating hole. Future papers in this series are being planned to consider rotating holes and the emission of massive particles. Massless particles will dominate the emission when $T \lesssim m_e$ (the smallest nonzero rest mass known). The approximation of zero rest mass should also be valid for $m_e \ll T \ll m_\mu$, in which case electrons and positrons will be emitted ultrarelativistically so that their rest mass can be ignored, whereas heavier particles will hardly be emitted at all. The approximation breaks down for the case $T \gtrsim m_\mu$ or $M \lesssim 5\times 10^{18} \approx 1\times 10^{14}$ g, which will not be considered.

Zaumen[6] and Gibbons[7] have shown that a black hole will discharge rapidly by a Schwinger-type pair-production process if

$$Q_* \equiv Q/M \gtrsim M m_e^2/e = 2.05\times 10^{-44} M$$

$$= M/5.34\times 10^5 M_\odot . \qquad (2)$$

Q_* is the charge parameter (dimensionless without setting $\hbar = 1$) that must be of order unity to affect significantly the geometry of a black hole and hence the emission of uncharged particles. Therefore, except for black holes above $10^5\ M_\odot$, which do not radiate at a significant rate anyway, the charge of the black hole can be ignored when analyzing the emission of uncharged particles. For a black hole small enough to be emitting electrons and positrons, the resulting random charge fluctuations are estimated to be of order unity. Such fluctuations do not affect the geometry significantly since only $M \gg 1 \approx 2\times 10^{-5}$ g is being considered, but they do affect the coupling of the hole to electrons and positrons so that their average emission rates may be changed by a fraction of the order of the fine-structure constant. This effect will be ignored until a future paper.

The idealization of no rotation for the black hole is much less justified than the idealization of no charge, but there are two effects that *may* tend to make the rotation small. First there is the tendency of a rotating hole to emit more particles with angular momentum in the same direction as the hole than in the opposite direction. Indeed, for a hole rotating as fast as possible for a given mass, each particle emitted must decrease the angular momentum of the hole, and it appears that this decrease is characteristic of the total emission at any finite rotation. However, the classically dimensionless (no $\hbar$'s needed to make it dimensionless) rotation parameter that determines the shape of a black hole is

$$a_* \equiv J/M^2, \tag{3}$$

where J is the magnitude of the angular momentum. For $a_*=1$ (maximum rotation), it is easy to show that the emission leads to a decrease in a_*, but for a_* near zero, it is not yet known whether the angular momentum decreases fast enough compared with the mass to keep a_* decreasing, or whether $d\ln J/d\ln M = 2$ at some finite a_*, causing a_* to approach that value asymptotically rather than continuing to decrease toward zero.

The second effect which may tend to reduce the rotation is an instability to the exponential growth of massive scalar fields in a quasibound state around a rotating hole. Eardley has suggested this effect[8] as an analog of the "black-hole bomb,"[9] in which the rest mass of the field replaces the mirror to confine the field. This instability should rapidly drain angular momentum from the hole into orbiting particles, which then decay or radiate away their energy and angular momentum by gravitational radiation,[10] *if* (1) the size of the hole is roughly the Compton wavelength of one of these scalar particles (a pion, say), (2) the size of the particle itself is not too large compared with the size of the hole, and (3) it is possible to create many particles in the same mode so that the field can grow exponentially. (One might suppose that if a scalar particle were made of Fermi constituents, the exclusion principle for the constituents would prevent the scalar particles from piling up in the same mode by coherent amplification, so the drain of angular momentum would not occur at any exponentially large rate limited by the gravitation radiation from the mode but rather at a rate limited by the decay or interaction time, which would not be much, if any, faster than the direct emission mechanisms.)

In summary, this paper will consider the emission rates from an uncharged, nonrotating hole for massless particles of spin $\frac{1}{2}$, 1, and 2. This is meant to apply to neutrinos, photons, and gravitons (and possibly ultrarelativistic electrons and positrons from a hole small enough) being emitted from a primordial black hole that has been neutralized, if necessary, by $e^{\pm}$ emission and that somehow has little angular momentum.

II. THEORETICAL FORMALISM

According to Hawking's calculation, the expected number of particles of the jth species with charge e emitted in a wave mode labeled by frequency or energy ω, spheroidal harmonic l, axial quantum number or angular momentum m, and polarization or helicity p is

$$\langle N_{j\omega lmp}\rangle = \Gamma_{j\omega lmp}\{\exp[2\pi\kappa^{-1}(\omega - m\Omega - e\Phi)] \mp 1\}^{-1}. \tag{4}$$

Here the minus sign is for bosons and the plus sign is for fermions; $\Gamma_{j\omega lmp}$ is the absorption probability for an incoming wave of that mode (i.e., minus the fractional energy gain in a scattered classical wave, $-Z$ in the calculations of Teukolsky and Press[11]); κ, Ω, and Φ are the surface gravity, surface angular frequency, and surface electrostatic potential, respectively, of the black hole. The values of κ, Ω, and Φ are linked to the hole's mass M, area A, angular momentum J, and charge Q by the first law of black-hole mechanics[12]

$$dM = \frac{\kappa}{8\pi}\, dA + \Omega dJ + \Phi dQ. \tag{5}$$

The expected number emitted in each mode remarkably is the same as that of a thermal body whose absorptivity matches that of the hole and whose temperature is

$$T = \frac{\kappa}{2\pi}, \tag{6}$$

so $\frac{1}{4}A$ can be identified as the entropy of the black hole.[1] For a Kerr-Newman black hole with the horizon at radius

$$r_+ = M + (M^2 - Q^2 - a^2)^{1/2}, \tag{7}$$

the specific expressions for κ, Ω, and Φ are[13]

$$\kappa = \frac{4\pi(r_+ - M)}{A}$$

$$= \tfrac{1}{2}M^{-1}[1 + (1 - \tfrac{1}{2}Q_*^2)(1 - Q_*^2 - a_*^2)^{-1/2}]^{-1} \to \frac{1}{4M}, \tag{8}$$

$$\Omega = \frac{4\pi a}{A}$$
$$= \frac{a_*}{M}[2 - Q_*^2 + 2(1 - Q_*^2 - a_*^2)^{1/2}]^{-1}$$
$$\to \frac{a_*}{4M}, \quad (9)$$

$$\Phi = \frac{4\pi Q r_+}{A}$$
$$= Q_* \frac{1+(1-Q_*^2 - a_*^2)^{1/2}}{2 - Q_*^2 + 2(1 - Q_*^2 - a_*^2)^{1/2}}$$
$$\to \tfrac{1}{2}Q_*. \quad (10)$$

Here the quantities after the arrows are the leading terms for $a_* \equiv a/M \equiv J/M^2 \ll 1$ and $Q_* \equiv Q/M \ll 1$.

To convert from the expected number emitted per mode to the average emission rate per frequency interval, one counts the number of modes per frequency interval with periodic boundary conditions in a large container around the black hole and divides by the time it takes a particle to cross the container, finding

$$\frac{dN}{dt} = \langle N \rangle \frac{v dk}{2\pi} = \langle N \rangle \frac{d\omega}{2\pi} \quad (11)$$

for each j, l, m, p, and frequency interval $(\omega, \omega + d\omega)$. Since each particle carries off energy ω and angular momentum m about the axis of the hole, the mass and angular momentum of the hole decrease at the rates given by the total power and torque emitted:

$$-\frac{d}{dt}\binom{M}{J} = \sum_{j,l,m,p} \frac{1}{2\pi} \int \Gamma_{j\omega l m p}\{\exp[2\pi\kappa^{-1}(\omega - m\Omega - e\Phi)] \mp 1\}^{-1}\binom{\omega}{m} d\omega. \quad (12)$$

The nontrivial part of the calculation of the power and torque is the determination of the absorption probabilities Γ. Fortunately, Teukolsky has shown[14] that the fundamental equations for gravitational, electromagnetic, and neutrino-field perturbations of an uncharged rotating black hole decouple into a single equation for each field, and furthermore that each of these equations is completely separable into ordinary differential equations. Teukolsky and Press[11] have developed analytic and numerical techniques for interpreting and solving these equations for gravitational and electromagnetic perturbations. Their techniques can be extended easily to the neutrino field, and I have simply modified their computer programs to cover neutrinos as well as gravitons and photons.

A check on the numerical computation can be given by the analytic form of Γ for small $M\omega$, which has been derived by Starobinsky and Churilov[15] for boson fields and which is extended in the Appendix to fermion fields obeying the Teukolsky equation. For a massless field with spin-s scattering off an uncharged hole, the formulas are

$$\Gamma_{s\omega l m p} = \left[\frac{(l-s)!(l+s)!}{(2l)!(2l+1)!!}\right]^2 \prod_{n=1}^{l}\left[1 + \left(\frac{\omega - m\Omega}{n\kappa}\right)^2\right] 2\left(\frac{\omega - m\Omega}{\kappa}\right)\left(\frac{A\kappa}{2\pi}\omega\right)^{2l+1}, \quad 2s \text{ even}, \quad (13)$$

$$\Gamma_{s\omega l m p} = \left[\frac{(l-s)!(l+s)!}{(2l)!(2l+1)!!}\right]^2 \prod_{n=1}^{l+1/2}\left[1 + \left(\frac{\omega - m\Omega}{n\kappa - \frac{1}{2}\kappa}\right)^2\right]\left(\frac{A\kappa}{2\pi}\omega\right)^{2l+1}, \quad 2s \text{ odd}, \quad (14)$$

with fractional errors of order $(A\kappa\omega)^{2l+1}$. Since $l \geq s$, the dominant contribution is from the $l = s$ modes, which give

$$\Gamma_{0\omega 00p} = \frac{A}{\pi}\omega^2 = 8M[M + (M^2 - a^2)^{1/2}]\,\omega^2 \quad \text{for } s = 0, \quad (15)$$

$$\Gamma_{\frac{1}{2}\omega\frac{1}{2}mp} = \frac{1}{4}\left(1 + \frac{\Omega^2}{\kappa^2}\right)\left(\frac{A\kappa}{2\pi}\omega\right)^2 = M^2\omega^2 \quad \text{for } s = \tfrac{1}{2}, \quad (16)$$

$$\Gamma_{1\omega 1mp} = \frac{4}{9}\frac{A}{\pi}[M^2 + (m^2 - 1)a^2](\omega - m\Omega)\omega^3 \quad \text{for } s = 1, \quad (17)$$

$$\Gamma_{2\omega 2mp} = \frac{16}{225}\frac{A}{\pi}[M^2 + (m^2 - 1)a^2][M^2 + (\tfrac{1}{4}m^2 - 1)a^2](\omega - m\Omega)\omega^5 \quad \text{for } s = 2. \quad (18)$$

Here only the lowest-order term in ω has been kept, except for the $\omega - m\Omega$ factor for bosons which guarantees that in the superradiant regime $\omega < m\Omega$, the absorption probability for bosons in negative. [I.e., waves are amplified rather than absorbed. The thermal factor of Eq. (12) is also negative in this regime, so the quantum emission rate remains positive.]

From the behavior of these analytic absorption probabilities at low frequencies for the various angular modes, one can get the low-frequency ($M\omega \ll 1$) absorption cross section for a massless particle of spin s averaged over all orientations of the black hole:[16]

$$\sigma_s(\omega) = \pi\omega^{-2} \sum_{l,m} \Gamma_{s\omega lmp} \underset{\omega\to 0}{\sim} \begin{cases} A, & s=0 \\ 2\pi M^2, & s=\frac{1}{2} \\ \frac{4}{9} A(3M^2 - a^2)\omega^2, & s=1 \\ \frac{16}{225} A(5M^2 + \frac{5}{2}M^2 a^2 + a^4)\omega^4, & s=2. \end{cases} \tag{19}$$

At high frequencies ($M\omega \gg 1$) the angle-averaged cross section for each kind of particle must approach the geometrical-optics limit of $27\pi M^2$ for a nonrotating hole and roughly the same value for a rotating hole.[17] Thus the cross sections are smaller at low frequencies. As the frequency is reduced to zero, the cross sections retain finite values for neutrinos and hypothetical spin-0 massless particles and go to zero as the frequency squared for photons and as the frequency to the fourth power for gravitons.

Combining the low-frequency absorption probabilities (13) and (14) with the thermal factor (4) for a black hole with negligible rotation, one gets the emission rate in a given angular and polarization eigenstate for low frequencies,

$$\frac{d}{dtd\omega} N_{s\omega lmp} = \frac{\beta}{4\pi^2} \left[\frac{(l-s)!(l+s)!}{(2l)!(2l+1)!!}\right]^2 (2M\omega)^{2l+1}, \tag{20}$$

where $\beta = 2$ for bosons and $\beta = \pi$ for fermions. The fractional errors are of order $M(\omega - m\Omega)$. Thus in each case the emission rate at low frequencies goes as ω^{2l+1}, and the power goes as ω^{2l+2}. This qualitative behavior causes the particles with lower spins (and thus lower l allowed, since $l \geq s$) to be emitted faster from a nonrotating hole, thereby dominating the low-frequency power drain from such a hole. However, the analytic expressions for low frequency break down long before the actual spectra peak, so numerical calculations are needed to determine whether and to what extent this effect holds also for the total power drain.

III. NUMERICAL CALCULATIONS

The particle emission rates were calculated by using Hawking's formula (4) and Eq. (11) with the absorption probabilities Γ computed by the method of Ref. 9, Sec. VII, using Bardeen's transformation discussed therein to allow stable integration of the Teukolsky equation from the horizon to infinity. A purely ingoing solution was chosen on the horizon, and after this solution was numerically integrated out to a sufficiently large radius, it was resolved into ingoing and outgoing waves at infinity. Then Γ was calculated as the ratio of the energy going down the hole to the energy of the ingoing wave at infinity, and the thermal factors were multiplied in to give the quantum emission rates. These rates were multiplied by the energy or angular momentum of each particle, integrated over frequency, and summed over all angular modes, polarizations, and species of particles to give the total power and torque emitted [cf. Eq. (12)].

The accuracy of the numerical result was limited by the step size in integrating the Teukolsky equation, the radius where the resolution into ingoing and outgoing waves is made, and the step size in integrating the spectra. To keep these three sources of error under control, variable step sizes were used with an error criterion for each step, and the resolution into ingoing and outgoing waves was required to be the same within a certain accuracy at two different radii. Thus the total error was governed by three accuracy criteria, and these were chosen for each mode to give roughly the same effect on the final result so that the result might have nearly the greatest accuracy possible for a given computer machine time.

The numerical calculations of the emission rates compared favorably with Eq. (20) at low frequencies, although departures from the extended Starobinsky-Churilov expression become significant at fairly small values of $M\omega$. For example, the actual value of Γ for neutrinos with $l = \frac{1}{2}$ becomes 50% larger than that given by Eq. (14) when $M\omega = 0.05$. This effect prevents one from getting an accurate estimate of the total power and torque emitted by inserting (13) and (14) into (12). [One might have expected such an estimate to be fairly accurate on grounds that the exponential of $8\pi M\omega$ (for a nonrotating hole) in the denominator of (12) might become large and make the integrand small before the expression for Γ develops serious errors.] In fact, such an estimate gave only 35% of the actual total power in neutrinos, 13% of the actual power in photons, and 5% of the actual power in gravitons, or 30% of the total in all massless particles.

IV. RESULTS

The power spectra for neutrinos, photons, and gravitons are given in Fig. 1. The integrated emission rates and power for the dominant angular modes are listed in Table I. The total in all of the known massless fields (four kinds of neutrinos with one helicity each and photons and gravitons with two helicities each) is $1.130\times10^{-3}\ c^3G^{-1}M^{-1}$ for the emission number rate and 2.011×10^{-4} $\times\hbar c^6 G^{-2}M^{-2}$ for the power. One may compare these numerical results with the naive estimates of thermal emission from cross sections σ that are assumed to be independent of frequency. Then the power would be

$$P=acT^4\left[\tfrac{7}{16}\sigma(\nu_e)+\tfrac{7}{16}\sigma(\overline{\nu}_e)+\tfrac{7}{16}\sigma(\nu_\mu)+\tfrac{7}{16}\sigma(\overline{\nu}_\mu)+\sigma(\gamma)+\sigma(g)\right] \tag{21}$$

for emission of ν_e, $\overline{\nu}_e$, ν_μ, $\overline{\nu}_\mu$, γ, and g (gravitons). Here

$$a=\frac{\pi^2k^4}{15\hbar^3c^3} \tag{22}$$

is the radiation density constant,[5] and T is the temperature of the black hole, given by Eq. (6). If we take the high-frequency limit, all the cross sections go to $27\pi G^2M^2/c^4$, and the power estimate becomes $5.246\times10^{-4}\,\hbar c^6G^{-2}M^{-2}$, which is a factor of 2.6 too large. If we take the low-frequency limit, Eq. (19) shows that the photon and graviton cross sections go to zero, whereas the neutrino cross sections go to $2\pi G^2M^2/c^4$, so the power estimate becomes $0.181\times10^{-4}\,\hbar c^6G^{-2}M^{-2}$, which is a factor of 11 too small. (The thermally averaged cross sections turn out to be $18.05\pi M^2$ for photons, $6.492\pi M^2$ for photons and $0.742\pi M^2$ for gravitons.)

If the black hole is small enough that electrons and positrons are emitted ultrarelativistically (and thus at the same rate for each helicity as neutrinos) but not small enough for heavier particles to be emitted at a significant rate, the power is $3.65\times10^{-4}\,\hbar c^6G^{-2}M^{-2}$. The peak in the neutrino power spectrum (which should be the same as that for ultrarelativistic electrons) is at $\omega=0.18M^{-1}$; therefore, the assumption of only ultrarelativistic $e^\pm$ applies for

$$m_e=4.19\times10^{-23}\ll0.18M^{-1}\ll m_\mu=8.65\times10^{-21}, \tag{23}$$

which is true for the mass range

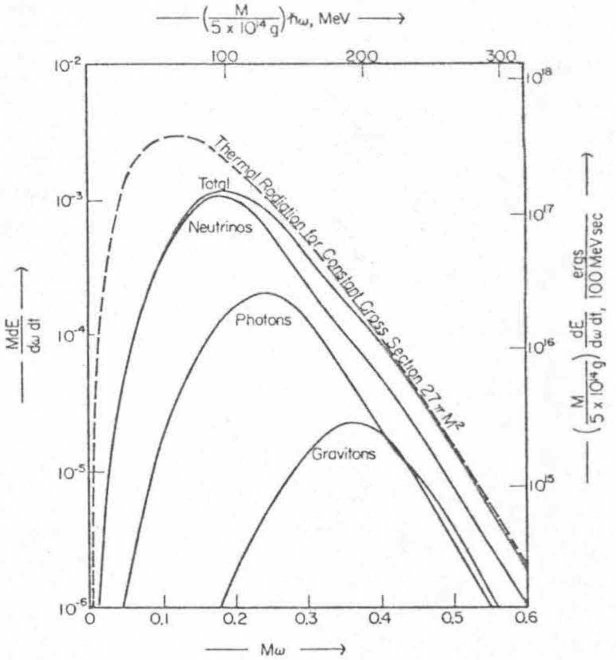

FIG. 1. Power spectra from a black hole, obtained by adding all angular modes for four kinds of neutrinos and for two polarization states (helicities) each of photons and gravitons. The lowest angular modes, $l=s$, dominate, but the $l=s+1$ modes can be seen coming in with a small "bump" in the neutrino spectrum at $M\omega\approx0.4$ and in the photon spectrum at $M\omega\approx0.5$. The total power spectrum can be seen at high frequencies to approach that of a thermal body with a cross section of $27\pi M^2$, but at low frequencies the spectrum drops below the Planck form as the cross section of the black hole is reduced.

$$2.1\times10^{19}=4.5\times10^{14}\,\mathrm{g}\ll M\ll4.3\times10^{21}=9.4\times10^{16}\,\mathrm{g}. \tag{24}$$

A black hole with $M\gg10^{17}$ g would emit virtually no known massive particles, and a hole with $M\lesssim5\times10^{14}$ g would emit muons and heavier particles at a significant rate.

Knowing the expression for the total power emitted from a nonrotating black hole, one can calculate the lifetime of such a hole. The power emitted causes the mass to decrease at the rate

$$\frac{dM}{dt}=-\frac{\hbar c^4}{G^2}\frac{\alpha}{M^2}, \tag{25}$$

where α is a numerical coefficient (see above) that depends on which particle species can be emitted at a significant rate. Since most of the decay time of the hole is spent near the original mass M_0, α can be taken to be its value α_0 at

TABLE I. Emission rates and powers for the dominant angular modes.

$2s$ [a]	$2l$ [b]	δ [c]	ϵ [d]	ζ [e]	For each mode rate [f]	For each mode power [g]	g [h]	For each (s,l) rate [f]	For each (s,l) power [g]
1	1	6	4	8	1.191×10^{-4}	1.969×10^{-5}	8	9.531×10^{-4}	1.575×10^{-4}
1	3	5	3	8	1.12×10^{-6}	3.75×10^{-7}	16	0.180×10^{-4}	0.060×10^{-4}
1	5	4	2	9	9.5×10^{-9}	4.9×10^{-9}	24	0.002×10^{-4}	0.001×10^{-4}
2	2	6	3	8	2.44×10^{-5}	5.49×10^{-6}	6	1.463×10^{-4}	0.330×10^{-4}
2	4	5	2.4	8	1.63×10^{-7}	6.67×10^{-8}	10	0.016×10^{-4}	0.007×10^{-4}
2	6	4	2	9.4	1.1×10^{-9}	6.5×10^{-10}	14	0.0001×10^{-4}	0.0001×10^{-4}
4	4	5	2.4	8	1.10×10^{-6}	3.81×10^{-7}	10	0.110×10^{-4}	0.038×10^{-4}
4	6	4	2	9.4	4.7×10^{-9}	2.6×10^{-9}	14	0.0007×10^{-4}	0.0004×10^{-4}
Total rate and power for all modes								1.130×10^{-3}	2.011×10^{-4}

[a] s is the spin of the field, here doubled to give an integer; i.e., $2s=1$ for neutrinos, $2s=2$ for photons, and $2s=4$ for gravitons.

[b] l is the total angular momentum of the mode.

[c] $10^{-\delta}$ is the fractional error criterion for each step in the radial integration of the Teukolsky equation.

[d] $10^{-\epsilon}$ is the fractional error criterion for the resolution of a numerical solution of the Teukolsky equation into ingoing and outgoing waves.

[e] $10^{-\zeta}$ is the absolute error criterion for the integration over frequencies.

[f] Rate in units of $c^3G^{-1}M^{-1}=4.038\times10^{38}(M/\mathrm{g})^{-1}\ \mathrm{sec}^{-1}$.

[g] Power in units of $\hbar c^6G^{-2}M^{-2}=1.719\times10^{50}(M/\mathrm{g})^{-2}\ \mathrm{erg\ sec}^{-1}$.

[h] g is the number of modes for a given l and s, $(2l+1)\times$(number of particle species with the given s)$\times$(number of polarizations or helicities for each species).

that mass, if $\alpha(M)$ does not change rapidly with mass near M_0 (as it might for $M_0 \lesssim 5\times10^{14}$ g). Then the lifetime of the hole is

$$\tau \approx \frac{G^2}{\hbar c^4}\frac{M_0^3}{3\alpha_0}. \tag{26}$$

For $M \gg 10^{17}$ g, $\alpha=2.011\times10^{-4}$, so

$$\tau = 8.66\times10^{-27}(M_0/\mathrm{g})^3\ \mathrm{sec}$$
$$= 2.16\times10^{66}(M_0/M_\odot)^3\ \mathrm{yr}. \tag{27}$$

For $5\times10^{14}\ \mathrm{g} \ll M \ll 10^{17}$ g, $\alpha=3.6\times10^{-4}$, so

$$\tau \approx 4.8\times10^{-27}\ (M_0/\mathrm{g})^3\ \mathrm{sec} = 1.5\times10^{-34}\ (M_0/\mathrm{g})^3\ \mathrm{yr}. \tag{28}$$

Since the lifetime of a black hole of stellar mass is so enormous, the decay is important only for black holes of much smaller mass, which cannot be formed by any processes (except for extremely rare quantum tunneling) that we know of in the present universe but which might have formed in the early universe.[2-4] It is of interest to determine what initial masses should have decayed away and what masses should still be around. Taking the lifetime of the black hole as the present age of the universe, say 16 billion years,[18] one finds that if only the known massless particles are emitted, $M_0=3.9\times10^{14}$ g. This is inconsistent with negligible emission of massive particles, so one must add ultrarelativistic $e^{\pm}$ emission, getting $M_0=4.7\times10^{14}$ g. This is at the mass where muon and pion emission are beginning to become important, so a somewhat larger mass should have decayed by now. However, unless the power is increased more than a factor of 2 due to the emission of muons and heavier particles (unlikely) and unless the universe age is outside 8–18 billion years[18] (also unlikely), probably $M_0=(5\pm1)\times10^{14}$ g is the initial mass of a primordial nonrotating, uncharged black hole that just decays away at the present age of the universe by the emission of the known elementary particles.

In conclusion, the power emitted from an uncharged, nonrotating black hole of mass $M \gg 10^{17}$ g is

$$P = 2.011\times10^{-4}\,\hbar c^6 G^{-2}M^{-2}$$
$$= 3.458\times10^{46}(M/\mathrm{g})^{-2}\ \mathrm{erg\ sec}^{-1}$$
$$= 2.28\times10^{-54}\,L_\odot(M/M_\odot)^{-2}, \tag{29}$$

of which 81.4% is in the four kinds of neutrinos, 16.7% is in photons, and 1.9% is in gravitons, assuming these are the only massless particles. For $5\times10^{14}\ \mathrm{g} \ll M \ll 10^{17}$ g,

$$P \approx 3.6\times10^{-4}\,\hbar c^6G^{-2}M^{-2}$$
$$= 6.3\times10^{16}(M/10^{15}\ \mathrm{g})^{-2}\ \mathrm{erg\ sec}^{-1}, \tag{30}$$

of which 45% is in electrons and positrons, 45%

is in neutrinos, 9% is in photons, and 1% is in gravitons. This assumes electrons and muons are the lightest particles with rest mass. The emission of particles is unimportant for stellar-mass black holes but should have caused any primordial black hole with an initial mass less than 4×10^{14} g (and perhaps somewhat greater values) to decay away by now.

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APPENDIX

The absorption probability Γ at low frequencies can be calculated by analytically solving the Teukolsky equation with the approximation $M\omega \ll 1$ and finding what fraction of any ingoing wave from infinity gets reflected back out. In Boyer-Lindquist coordinates for an uncharged hole, a massless field of spin-weight s, frequency ω, and axial quantum number m obeys the radial Teukolsky equation[19]

$$\Delta^{-s+1}\frac{d}{dr}\left(\Delta^{s+1}\frac{dR}{dr}\right)+\left[(r^2+a^2)^2\omega^2-4aMr\omega m+a^2m^2+2ia(r-M)ms-2iM(r^2-a^2)\omega s+(2ir\omega s-\lambda)\Delta\right]R=0. \quad \text{(A1)}$$

Here

$$\Delta \equiv r^2-2Mr+a^2\equiv(r-r_+)(r-r_-), \quad \text{(A2)}$$

and λ is an eigenvalue of the angular equation

$$\frac{1}{\sin\Theta}\frac{d}{d\Theta}\left(\sin\Theta\frac{dS}{d\Theta}\right)+\left[(s-a\omega\cos\Theta)^2-\left(\frac{m+s\cos\Theta}{\sin\Theta}\right)^2-s(s-1)+\lambda-a^2\omega^2\right]S=0. \quad \text{(A3)}$$

(λ is the same as in Ref. 15 and is the same as $\lambda+2am\omega$ in Ref. 11.)

Following Starobinsky and Churilov[15] generally, define

$$x\equiv\frac{r-r_+}{2(r_+-M)}=\frac{r-M-(M^2-a^2)^{1/2}}{2(M^2-a^2)^{1/2}}, \quad \text{(A4)}$$

$$Q\equiv\frac{m\Omega-\omega}{2\kappa}=\frac{Mr_+}{r_+-M}(m\Omega-\omega), \quad \text{(A5)}$$

$$k\equiv2\omega(r_+-M)=2M\omega(1-a_*{}^2)^{1/2}. \quad \text{(A6)}$$

Then small $M\omega$ implies that the radial equation can be approximated as

$$x^2(x+1)^2\frac{d^2R}{dx^2}+(s+1)x(x+1)(2x+1)\frac{dR}{dx}+\left[k^2x^4+2iskx^3-\lambda x(x+1)+isQ(2x+1)+Q^2\right]R=0, \quad \text{(A7)}$$

with $k\ll1$. Small $M\omega$ also implies $a^2\omega^2\ll1$, in which case the angular eigenvalue becomes very nearly

$$\lambda=(l-s)(l+s+1), \quad \text{(A8)}$$

where $l-s$ is a non-negative integer. (In the limit of $a_*\to0$, l is the total angular momentum of the mode.)

For $kx\ll l+1$, the first two terms inside the square brackets of Eq. (A7) can be dropped, leading to an equation with three regular singular points. A solution obeying the ingoing boundary conditions at the horizon[19] is[20]

$$R=x^{-s+iQ}(x+1)^{-s-iQ}\,{}_2F_1(-l-s,\ l-s+1;\ 1-s+2iQ;\ -x). \quad \text{(A9)}$$

Here ${}_2F_1(a,b;c;z)$ is the hypergeometric function. For $x\gg|Q|+1$, the last two terms inside the square brackets can be dropped, and $x+1$ can be replaced by x, leading to an equation with one regular and one irregular singular point. The solution is, if $2l$ is not an integer,[21]

$$R = C_1 e^{-ikx} x^{l-s} {}_1F_1(l-s+1;\ 2l+2;\ 2ikx) + C_2 e^{-ikx} x^{-l-s-1} {}_1F_1(-l-s;\ -2l;\ 2ikx). \tag{A10}$$

Here ${}_1F_1(a; c; z)$ is the confluent hypergeometric function. [To avoid solutions with logarithmic terms, and to simplify the matching procedure, we will henceforth assume $2l$ is nearly, but not exactly, integral. This is actually the case when $a^2\omega^2 \neq 0$ if we use Eq. (A8) as the definition of l when λ is given from Eq. (A3) rather than as an approximate formula for λ when $l-s$ is given as a non-negative integer.]

By matching the two solutions in the overlap region $|Q|+1 \ll x \ll (l+1)/k$, one can get

$$C_1 = \frac{\Gamma(2l+1)\,\Gamma(1-s+2iQ)}{\Gamma(l-s+1)\Gamma(l+1+2iQ)},\quad C_2 = \frac{\Gamma(-2l-1)\Gamma(1-s+2iQ)}{\Gamma(-l-s)\Gamma(-l+2iQ)}. \tag{A11}$$

Then the asymptotic form of the confluent hypergeometric functions can be used to get the solution in the form

$$R = Y_{\text{in}}\, e^{-ikx} r^{-1} + Y_{\text{out}}\, e^{ikx} r^{-2s-1} \tag{A12}$$

for $kx \gg 1$, where

$$Y_{\text{in}} = \frac{\Gamma(2l+1)\Gamma(2l+2)\Gamma(1-s+2iQ)}{\Gamma(l-s+1)\Gamma(l+s+1)\Gamma(l+1+2iQ)}\frac{k}{\omega}(-2ik)^{-l+s-1} + \frac{\Gamma(-2l)\Gamma(-2l-1)\Gamma(1-s+2iQ)}{\Gamma(-l-s)\Gamma(-l+s)\Gamma(-l+2iQ)}\frac{k}{\omega}(-2ik)^{l+s},$$

$$Y_{\text{out}} = \frac{\Gamma(2l+1)\Gamma(2l+2)\Gamma(1-s+2iQ)}{[\Gamma(l-s+1)]^2\,\Gamma(l+1+2iQ)}\left(\frac{k}{\omega}\right)^{2s+1}(2ik)^{-l-s-1} + \frac{\Gamma(-2l)\Gamma(-2l-1)\Gamma(1-s+2iQ)}{[\Gamma(-l-s)]^2\,\Gamma(-l+2iQ)}\left(\frac{k}{\omega}\right)^{2s+1}(2ik)^{l-s}. \tag{A13}$$

To obtain the ratio of outgoing to ingoing fluxes, one can either calculate the normalization factors of Ref. 11 to apply to $|Y_{\text{out}}/Y_{\text{in}}|^2$, or one can use the following trick: Solve the radial equation with s replaced by $-s$ to get the asymptotic form

$$R_{-s} = Z_{\text{in}}\, e^{-ikx} r^{-1} + Z_{\text{out}} e^{ikx} r^{2s-1} \tag{A14}$$

[i.e., Z_{in} and Z_{out} are the same as Y_{in} and Y_{out}, respectively, in Eq. (A13) above with s replaced by $-s$]. Then the reflection coefficient is (cf. Ref. 11)

$$1 - \Gamma = \frac{dE_{\text{out}}}{dE_{\text{in}}} = \left|\frac{Y_{\text{out}}\, Z_{\text{out}}}{Y_{\text{in}}\ Z_{\text{in}}}\right|. \tag{A15}$$

After some algebra, one finds that with a fractional error of order k^{2l+1},

$$\Gamma_{s\omega lmp} = \operatorname{Re}\left\{4e^{i\pi(s-1/2)}\cos[\pi(l-s)]\,\frac{\Gamma(-2l)\Gamma(-2l-1)}{\Gamma(2l+1)\Gamma(2l+2)}\left[\frac{\Gamma(l-s+1)}{\Gamma(-l-s)}\right]^2\frac{\Gamma(l+1+2iQ)}{\Gamma(-l+2iQ)}(2k)^{2l+1}\right\}. \tag{A16}$$

Now one can keep $2s$ exactly integral and take the limit as $l-s$ approaches a non-negative integer. Then

$$\Gamma_{s\omega lmp} = \operatorname{Re}\left\{\left[\frac{(l-s)!(l+s)!}{(2l)!(2l+1)!}\right]^2\frac{\Gamma(l+1+2iQ)}{\Gamma(-l+2iQ)}(2ik)^{2l+1}\right\}. \tag{A17}$$

Taking the cases of integral or half-integral spins separately (corresponding to $2l$ even or odd, respectively) to express the quotient of the two Γ functions as a finite product, and rewriting Q and k in terms of ω, m, Ω, κ, and A, one obtains Eqs. (13) and (14). The result for integral spins was given by Starobinsky and Churilov, though not the result for half-integral spins.

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(d) Particle Emission Rates from a Black Hole.
II. Massless Particles from a Rotating Hole
(Paper III; submitted to <u>Phys. Rev.</u> D)

Particle Emission Rates from a Black Hole.

II. Massless Particles from a Rotating Hole*

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ABSTRACT

The calculations of the first paper of this series (for non-rotating black holes) are extended to the emission rates of massless or nearly massless particles from a rotating hole and the consequent evolution of the hole. The power emitted increases as a function of the angular momentum of the hole, for a given mass, by factors of up to 13.35 for neutrinos, 107.5 for photons, and 26380 for gravitons. Angular momentum is emitted several times faster than energy, so a rapidly rotating black hole spins down to a nearly nonrotating state before most of its mass has been given up. The third law of black hole mechanics is proved for small perturbations of an uncharged hole, showing that it is impossible to spin up a hole to the extreme Kerr configuration. If a hole is rotating fast enough, its area and entropy initially increase with time (at an infinite rate for extreme Kerr) as heat flows into the hole from particle pairs created in the ergosphere. As the rotation decreases, the thermal emission becomes dominant,

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drawing heat out of the hole and decreasing its area. The lifetime of a black hole of a given mass varies with the initial rotation by a factor of only 2.0-2.7 (depending upon which particle species are emitted). If a nonrotating primordial black hole with initial mass 5×10^{14}g would have just decayed away within the present age of the universe, a hole created maximally rotating would have just died if its initial mass were about 7×10^{14}g. Primordial black holes created with larger masses would still exist today, but they would have a maximum rotation rate determined uniquely by the present mass. If they are small enough today to be emitting many hadrons, they are predicted to be very nearly nonrotating.

I. INTRODUCTION

Black holes, as Hawking and others have shown,[1-6] emit particles like thermal bodies. Paper I[7] reported numerical calculations of the emission rates from a nonrotating black hole. This paper gives the rates for the known particles of zero or negligible rest mass from a rotating (Kerr) black hole and shows how such a hole would evolve as it emitted these particles. These results are of interest in testing the validity of the simplifying assumption that most black holes which emit significantly today are not rotating (see, for example, Refs. 7-9).

Paper I noted that although a small black hole will quickly give up its electric charge,[10-12] it is much less certain whether the rotation will also become small. The main difference in the time scales of the two processes can be seen in the following way (using henceforth the dimensionless Planck units spelled out in Paper I):

The parameters that determine the shape of a black hole are

$$a_* \equiv J/M^2 \qquad \text{and} \qquad Q_* \equiv Q/M \quad , \tag{1}$$

where J is the angular momentum, Q is the charge, and M is the mass (which sets the scale of the size). These quantities have a domain limited by the constraint

$$a_*^2 + Q_*^2 \leq 1 \quad . \tag{2}$$

Only black holes which emit quanta of much smaller energy than the hole mass will be considered, so that the adiabatic approximation used in the quantum calculations of the emission[1-6] will be valid. The quanta emitted have typical energies of the order of the black hole temperature or of M^{-1} (with $[10^{15}g]^{-1}$ = 266 MeV in conventional units), which we want much less than M, so we need

$$M \gg 1 \quad (\text{"Planck mass"}) = 2.18 \times 10^{-5} g \quad . \tag{3}$$

Then roughly M^2 quanta are needed to carry away the energy of the hole; i.e., the entropy in the radiation, which is roughly the number of quanta when thermally distributed, is of the same order as the initial entropy of the hole, which is one-fourth the area[13] or roughly M^2.

When a black hole is charged and/or rotating so that Q_* and/or a_* are significantly different from zero, and when it has temperature or electrostatic potential high enough to permit emission of electrons or positrons, it tends to emit most of its quanta with the same sign of the charge and/or angular momentum as the hole. A charged particle carries off charge

$$|\Delta Q| = e = 0.0854 \quad , \tag{4}$$

which is roughly of order unity, and a typical quantum also carries off an angular momentum

$$-\Delta J = m \tag{5}$$

of order unity. Since a_* and Q_* must have absolute values not greater than unity, the number of charged particles needed to neutralize the hole is Q/e, which is only of order M, whereas the number of particles needed to carry off the angular momentum J is of order M^2. Thus the charge can be emitted fairly quickly, but the loss of angular momentum requires roughly the same number of particles as the loss of mass. Therefore, in this paper we will assume that the charge neutralization has already occurred but that the angular momentum may still be significant.

Though one expected a black hole to give up its angular momentum in the same order of time as it gives up its mass, it has not been known whether a_* tends to zero as the black hole evolves. Carter[12] argued that it would tend asymptotically toward a fixed value less than unity, but he gave no indication of what that value would be. Numerical calculations were needed to show whether $\ln J$ always decreases faster than $\ln M^2$, pushing a_* toward zero, or whether these two quantities decrease equally fast at some nonzero limiting value for a_*. There is some indirect evidence, to be given below, that if there were a large enough number of massless scalar fields (unknown at present and therefore not calculated in this paper) to dominate the emission, a_* might indeed get hung up at some nonzero value. However, this paper shows that emission of the known massless fields can only decrease a_* toward zero, and that in fact the decrease is rather rapid compared with the mass decrease.

Because black holes that died in recent epochs or that are emitting significantly today spend almost all their lives with temperatures of order 20 MeV, which is well above the mass of the electron but well below that of each known heavier particle, it is reasonable to do the calculations for the idealized case of emission of a fixed set of species with negligible rest mass. For example, the "canonical combination" used below is the set of known species with masses less than 20 MeV: gravitons, photons, electron and muon neutrinos with one helicity each, electrons, and the corresponding antileptons. However, the results will also be given for other sets of species, to include some of the possibilities (to be discussed below) of other near-massless particles in nature or of the emission from black holes too cold to emit electrons and positrons.

The quantities to be calculated in this paper are the rates at which energy and angular momentum are radiated; the evolution of the mass, rotation parameter, and area of the hole; the lifetimes of holes with different initial angular momenta; the masses of primordial black holes (PBHs) that would be just disappearing today; and the maximum rotation parameters that PBHs of various masses today could have. The remainder of the paper will derive the mathematical formulas for the quantities desired, describe the numerical methods used to calculate them, give the results, and discuss their properties.

II. MATHEMATICAL FORMULAS

Since the total number of particles emitted during the black-hole evolution, roughly M^2 , is assumed to be very large, the emission may be approximated as a continuous process with negligible fluctuations due to particle discreteness. Then the rates are well-determined functions of M and a_* alone (assuming $Q_* = 0$, which was justified above). The rest masses of the particles emitted are assumed to be negligible, and the particle species emitted are assumed to be fixed (independent of M), so the only scale in the problem (other than the Planck units, which are here defined to be unity) is determined by M. All quantities to be calculated scale as some power of M and can therefore be put into a scale-invariant form (e.g., depending only on a_*) by dividing out this power of M--or, when one calculates the evolution of a hole (Eqs. [12] ff. below), by dividing out the value of M at some particular point on the evolutionary track.

First, let us consider the rates at which the mass and angular momentum of a black hole decrease, which are given in Paper I by Eq. (I:12). Since the time t scales as M^3, we may define the scale-invariant quantities

$$f \equiv -M^3 d \ln M/dt = -M^2 dM/dt \quad , \tag{6}$$

$$g \equiv -M^3 d \ln J/dt = -M a_*^{-1} dJ/dt \quad . \tag{7}$$

These can be seen to be functions of a_* alone:
If we define the scale-invariant energy of an emitted particle as

$$x \equiv M\omega \quad , \tag{8}$$

then Eq. (I:12) gives

$$\binom{f}{g} = \sum_{j,\ell,m,p} \frac{1}{2\pi} \int_0^\infty dx \, \langle N_{jx\ell mp} \rangle \binom{x}{m a_*^{-1}} \quad , \tag{9}$$

where the expected number of particles of the j^{th} species of spin s

emitted in the mode or state with energy $M^{-1}x$, spheroidal harmonic ℓ, axial angular momentum m, and polarization p is

$$\langle N_{jx\ell mp}\rangle = \frac{\Gamma_{j\ell mp}(a_*,x)}{e^{4\pi[1+(1-a_*^2)^{-1/2}]x - 2\pi a_*(1-a_*^2)^{-1/2}m} - (-1)^{2s}} \quad . \tag{10}$$

Here Eq. (I:4) has been used, with the values of the surface gravity, angular frequency, and electrostatic potential of the hole obtained from Eqs. (I:8), (I:9), and (I:10). $\Gamma_{j\ell mp}$ is the absorption probability for an incoming wave of the mode considered and can be found by numerically solving the Teukolsky equation.[14,15] It can easily be seen to depend only on a_* and x in addition to the subscripts. The dependence on the species j and polarization p is only through the spin s (assumed positive) and the number of polarizations p that the species has; then ℓ and m can take on any values such that $\ell - s$ and $\ell - |m|$ are nonnegative integers.

Next, let us consider the evolution of the black hole. Equations (6) and (7) give the rates of change of M and J with respect to time once f and g have been calculated. Since f and g are functions of a_*, however, it is easier to solve the equations if a_* is considered as the independent variable. Furthermore, dividing Eq. (7) by Eq. (6) shows us that

$$\frac{d\,\ell n\, a_*}{d\,\ell n\, M} = \frac{d\,\ell n\, J}{d\,\ell n\, M} - 2 = \frac{g}{f} - 2 \equiv h(a_*) \quad , \tag{11}$$

which approaches a constant value as a_* approaches zero (assuming the value is positive so indeed $a_* \to 0$ as $M \to 0$). Because of the logarithms in Eq. (11), it is convenient to define the independent variable to be

$$y \equiv -\ln a_* \quad . \tag{12}$$

To cover the greatest range of possibilities, the evolution will first be calculated from $a_* = 1$ or $y = 0$ to $a_* = 0$ or $y = \infty$; a black hole starting at a different value of a_* will simply follow the evolutionary track from that point onward.

Now the object is to find how the mass and time vary with y. Let the starting mass at $a_* = 1$ be

$$M_1 \equiv M(y = 0) \quad ; \tag{13}$$

this will be the mass that sets the scale. With an eye back on Eq. (11), set

$$z \equiv -\ln(M/M_1) \quad , \tag{14}$$

which has the initial value

$$z(0) = 0 \tag{15}$$

and evolves according to the reciprocal of Eq. (11) as

$$dz/dy = 1/h = f/(g - 2f) \quad . \tag{16}$$

It has been noted that the time scales as the mass cubed, so define the scale-invariant time parameter as

$$\tau \equiv M_1^{-3}\, t \tag{17}$$

with initial value

$$\tau(0) = 0 \quad . \tag{18}$$

Then Eq. (6) combined with Eq. (16) gives

$$\frac{d\tau}{dy} = \frac{e^{-3z}}{fh} = \frac{e^{-3z}}{g-2f} \quad . \tag{19}$$

From the solutions $z(y)$ and $\tau(y)$ of the coupled differential equations (16) and (19), one can get $y(\tau)$ and $z(\tau)$, and hence a_* and M/M_1, as a function of time. From these, one can find how other quantities evolve, such as the area

$$A = 8\pi M^2[1 + (1-a_*^2)^{1/2}] \quad . \tag{20}$$

Once one has the evolution of a black hole from $a_* = 1$, one can consider holes with other initial values a_{*i} of the rotation parameter. They will follow the same solution $z(y)$ and $\tau(y)$ but with different initial values:

$$y_i \equiv -\ell n\, a_{*i} \quad , \tag{21}$$

$$z_i \equiv z(y_i) = -\ell n(M_i/M_1) \quad , \tag{22}$$

$$\tau_i \equiv \tau(y_i) = M_1^{-3}\, t_i \quad . \tag{23}$$

These equations determine M_1 and t_i such that the hole would have mass M_i and rotation a_{*i} at time t_i if it had started with $M = M_1$ and $a_* = 1$ at time $t = 0$. In terms of M_i and a_{*i}, the evolution follows

$$M = M_1 e^{-z} = M_i e^{z_i - z} \quad , \tag{24}$$

$$t - t_i = M_1^3(\tau-\tau_i) = M_i^3\, e^{3z_i}(\tau-\tau_i) \quad . \tag{25}$$

Equation (25) and the "standard evolution law" $z(y)$ and $\tau(y)$ can be inverted to get τ and hence y and a_* as functions of time, and then Eq. (24) gives the mass.

A particular quantity desired is the lifetime $T(M_i, a_{*i})$ of a black hole with initial mass M_i and rotation parameter a_{*i}. It can be seen from Eq. (25), assuming that the black hole does evolve to $a_* \to 0$ or $y \to \infty$ as $M \to 0$, that this is

$$T(M_i, a_{*i}) \equiv t(y{=}\infty) - t_i = M_i^3\, e^{3z_i}(\tau_f-\tau_i) \quad , \tag{26}$$

where

$$\tau_f \equiv \tau(y{=}\infty) \tag{27}$$

is the lifetime in units of M_1^3 of a hole that started with $a_* = 1$. The mass dependence of the lifetime can be divided out to get the scale-invariant quantity

$$\theta_i \equiv M_i^{-3}\, T(M_i, a_{*i}) = e^{3z_i}(\tau_f - \tau_i) \ , \tag{28}$$

thus written in terms of quantities previously calculated. Once the lifetime of any black hole is known, one can calculate the initial mass of a primordial black hole that has just disappeared within the present age t_o of the universe:

$$M_i(a_{*i}, t_o) = t_o^{1/3}\, \theta_i^{-1/3} = t_o^{1/3}\, e^{-z_i}(\tau_f-\tau_i)^{-1/3} . \tag{29}$$

Since PBHs would have been spinning down since their creation at time t_o ago, their present values of a_* should have an upper limit $a_{*max}(M, t_o)$ less than unity, depending upon the present mass M. It is simpler to solve for the inverse function $M_{min}(a_*, t_o)$, the minimum mass of a PBH with a_* today. By combining Eqs. (24) and (25) with $t - t_i = t_o$, one finds that

$$M = t_o^{1/3}(\tau - \tau_i)^{-1/3} e^{-z} \quad , \tag{30}$$

where τ and z are evaluated at the present value of y or a_*. Clearly, the minimum occurs at the smallest value of τ_i, which is zero if PBHs can be created with a_{*i} up to unity, so in that case

$$M_{min}(a_*, t_o) = t_o^{1/3}[\tau(-\ln a_*)]^{-1/3} e^{-z(-\ln a_*)} \quad , \tag{31}$$

where $-\ln a_*$ is shown explicitly as the argument of $\tau(y)$ and $z(y)$. If a_{*i} has a smaller maximum value, the corresponding minimum for τ_i is to be used in Eq. (30) to give $M_{min}(a_*, t_o)$. One can see that for fixed τ_i, M in Eq. (30) is a monotonically increasing function of a_*, assuming $g - 2f$ is always positive so that τ is a decreasing function of a_* by Eq. (19). Then the inverse $a_{*max}(M, t_o)$ is uniquely defined and is a monotonically increasing function of M .

III. NUMERICAL METHODS

The major part of the numerical calculations consisted of computing the functions $f(a_*)$ and $g(a_*)$ by Eqs. (9) and (10), which was done at 14 values of a_* from 0.01 to 0.99999 to an accuracy of one part in roughly 10^4 or better at low a_* and 10^3 at high a_*. The basic method is briefly summarized in §III of Paper I. In order to cover different possibilities for the set of particle species, the contributions to f and g from each species were calculated separately. Thus $f_{1/2}$ and $g_{1/2}$, f_1 and g_1, and f_2 and g_2 were calculated as the contributions from one species with two polarizations of spin $\frac{1}{2}$, 1, and 2, respectively:

$$\begin{pmatrix} f_s(a_*) \\ g_s(a_*) \end{pmatrix} = \sum_{\ell,m} \frac{1}{\pi} \int_0^\infty dx \langle N_{sx\ell m}(a_*) \rangle \begin{pmatrix} x \\ m a_*^{-1} \end{pmatrix} . \tag{32}$$

Here the dependence on the species is only through its spin s, and the sum over the two polarizations has already been taken, since the expected number emitted in a mode labeled by x, ℓ, and m is independent of the polarization. Then,

$$\begin{pmatrix} f \\ g \end{pmatrix} = n_{1/2} \begin{pmatrix} f_{1/2} \\ g_{1/2} \end{pmatrix} + n_1 \begin{pmatrix} f_1 \\ g_1 \end{pmatrix} + n_2 \begin{pmatrix} f_2 \\ g_2 \end{pmatrix} , \tag{33}$$

where $n_{1/2}$, n_1, and n_2 are the number of species with spin $\frac{1}{2}$, 1, and 2, respectively, assuming that there are no massless particles of other spins.

A total of 463 angular modes (a combination of s, ℓ, m, and a_*) were calculated and integrated over frequency: 170 modes for $s = 1/2$, 155 for $s = 1$, and 138 for $s = 2$. For example, at low a_* all the modes up through $\ell = 5/2$ for $s = 1/2$ and through $\ell = 3$ for $s = 1$ and $s = 2$ were calculated. At high a_* the $\ell = m$ modes were calculated up to $\ell = 25/2$ for $s = 1/2$, $\ell = 11$ for $s = 1$, and $\ell = 9$ for $s = 2$, and several $\ell = m+1$ modes were calculated (with considerably smaller results), but no modes with $\ell - m > 1$. At intermediate values of a_*, some combination between these two extremes was taken. The modes calculated appeared to include nearly all of the radiation, though estimates for the small contributions of all the other modes were added in, assuming that the sum over m dropped off exponentially in ℓ roughly as the calculated modes did.

Once the functions f_s and g_s were found at 14 values of a_*, an interpolation algorithm was needed to evaluate them at other values of a_* or y. These functions varied by factors of up to 25000 from $a_* = 0.01$ to $a_* = 0.99999$, and the variation with a_* was particularly rapid at the upper end. To find smooth relationships, various functions of the f's and g's were plotted against various functions of a_*. Of the combinations tried, a small fractional power of the f's and g's versus the surface gravity κ of the hole was the most linear. Therefore, cubic spline fits,[16] minimizing the sum of the squares of the third derivative discontinuities at the 14 values of a_*, were made of $f_s^{0.4}$ and $g_s^{0.4}$ versus

$$4M\kappa = 2[1 + (1-a_*^2)^{-1/2}]^{-1} , \tag{34}$$

which varies from 0 at $a_* = 1$ or $y = 0$ to 1 at $a_* = 0$ or $y = \infty$. The fits of these variables indeed were quite smooth, with the slopes never changing by a factor of more than 3.6 (even though the values of the fractional powers themselves changed by factors exceeding 50) and with only four of the eighty-four values of the second derivatives of the splines at the knots exceeding unity in magnitude.

The functions f_s and g_s were evaluated at 363 values of a_* from 1 down to 0.0005 by the cubic spline interpolation algorithm and then were combined by Eq. (33) for some combination of n's to get f and g at each point. A fourth-order Runga-Kutta method was used to integrate Eqs. (16) and (19) simultaneously over the corresponding range of y with the initial values set by Eqs. (15) and (18). At every other point (since the integration requires two points per step), the values of M/M_1, θ , $M_i(a_{*i},t_o)$, $M_{min}(a_*,t_o)$, and A/A_1 (where $A_1 = 8\pi M_1^2$ was the area at $a_*= 1$) were calculated by Eqs. (24), (28), (29), (31), and (20). As a check on the accuracy of the numerical integration, the step size was halved, which resulted in agreement to four or five decimal places.

For a_* smaller than 0.0005, the values of f and g at $a_*= 0$ or $y = \infty$ were used:

$$\alpha \equiv f(a_*= 0) \quad , \tag{35}$$

$$\beta \equiv g(a_*= 0) \quad . \tag{36}$$

Then Eqs. (16) and (19) become

$$dz/dy \underset{y \to \infty}{\sim} \alpha/(\beta-2\alpha) \equiv \gamma \quad , \tag{37}$$

$$d\tau/dy \underset{y \to \infty}{\sim} e^{-3z}/(\beta-\alpha) \quad , \tag{38}$$

so the solution is

$$z \sim \gamma y + \delta \quad , \tag{39}$$

$$\tau \sim \tau_f - \frac{1}{3}\alpha^{-1} e^{-3z} \quad , \tag{40}$$

where

$$\delta \equiv \int_0^\infty \left(\frac{f}{g-2f} - \gamma\right) dy \tag{41}$$

is a constant that was simply estimated as $z - \gamma y$ at $a_* = 0.0005$. The solution for large y or small a_* gives the asymptotic forms

$$M/M_1 \sim e^{-\gamma y - \delta} = e^{-\delta} a_*^\gamma \sim [3\alpha(\tau_f - \tau)]^{1/3} \quad , \tag{42}$$

$$\theta \sim \frac{1}{3}\alpha^{-1} \tag{43}$$

[cf. Eq. (I:26), where $M_o = M_i$ and $\tau = T(M_i, a_{*i} = 0)$],

$$M_i(a_{*i} \to 0, t_o) \sim (3\alpha\ t_o)^{1/3} \quad , \tag{44}$$

$$M_{min}(a_*, t_o) \sim (3\alpha\ t_o)^{1/3} (3\alpha\tau_f e^{3\delta} a_*^{-3\gamma} - 1)^{-1/3}$$

$$\sim (t_o/\tau_f)^{1/3} e^{-\delta} a_*^\gamma = M_i(a_{*i} = 1, t_o) e^{-\delta} a_*^\gamma, \tag{45}$$

$$A/A_1 \sim 2M^2/M_1^2 \sim 2[3\alpha(\tau_f - \tau)]^{2/3} \quad . \tag{46}$$

IV. RESULTS

The values of f_s, the scale-invariant power in a two-helicity particle species of spin s, and of g_s, the scale-invariant torque per angular momentum of the hole, are listed at the 14 values of a_*

in Table I, along with the extrapolated values for $a_* = 1$. The cubic spline interpolations are graphed in Figs. 1 and 2, which show that below roughly $a_* = 0.6$ the neutrino ($s = 1/2$) power dominates, followed by photons ($s = 1$) and finally gravitons ($s = 2$). However, at greater values of a_* the order is reversed, with gravitons dominating the emission and photons and neutrinos coming second and third, respectively.

This behavior can be explained qualitatively in the following way: For a slowly rotating hole, the coupling depends most strongly on the spheroidal harmonic index ℓ (which reduces to the total, not the orbital, angular momentum when $a_* = 0$) rather than on the axial angular momentum m or the spin s. The coupling is greater at lower ℓ values (e.g., Paper I showed that the emission rate at low frequencies goes as $\omega^{2\ell+1}$), but $\ell \geq s$, so the emission is greater at lower values of s, which allow lower values of ℓ. On the other hand, a rapidly rotating hole couples strongly with the axial angular momentum and also with the spin,[17] so the $s = \ell = m$ angular mode dominates greatly and now has an effect increasing with s.

It is of interest to note that as $a_* \to 1$, the surface gravity and hence temperature of the black hole go to zero, but the emission does not. In fact, Eq. (10) becomes

$$\langle N_{jx\ell mp}\rangle \underset{a_*\to 1}{\sim} (-1)^{2s+1}\, \Gamma_{j\ell mp}(a_*,x)\, H(m-2x) , \qquad (47)$$

where $H(m-2x)$ is the Heaviside step function (0 if $m-2x < 0$, 1 if $m-2x > 0$), so one gets simply the spontaneous emission (first discovered by Zel'dovich[18]) in the superradiant regime where the angular velocity ω/m of the wave is lower than the angular velocity $\Omega_{a_*\to 1} \sim \frac{1}{2} M^{-1}$ of the hole. For bosons (2s even), Γ is negative in the superradiant regime, as predicted by Zel'dovich[18,19] and confirmed by Misner,[20] Starobinsky,[21] and Press and Teukolsky[22] for scalar waves, and by Teukolsky[15] and Starobinsky and Churilov[23] for electromagnetic and gravitational waves. That is, the waves gain amplitude on reflection and extract rotational energy from the hole in the wave analogue of the Penrose process.[24] Bekenstein[25] has shown that this result follows from Hawking's area theorem[26] for waves with positive definite energy density. For fermions (2s odd), Γ is always positive, as Unruh[27] has shown for the classical neutrino field, which has a negative energy density near the hole in the superradiant regime. In the quantum analysis, the amplification of a boson wave corresponds to stimulated emission, whereas the Pauli exclusion principle prevents fermions from being amplified. The fact that this behavior shows up in the solutions of the classical wave equations is a manifestation of the connection between spin and statistics.[28] Field theoretic derivations of the spontaneous emission from a rotating black hole with the appropriate initial state for no thermal emission have been given by Unruh[29] and Ford,[30] but one must remember that a black hole formed by collapse has a nonzero temperature (except when $a_* = 1$) and thus emits at a greater rate.[1-6]

Figures 1 and 2 also show the power and relative torque for various combinations $(n_{1/2}, n_1, n_2)$ of the numbers of species emitted with spin $\frac{1}{2}$, 1, and 2, respectively. Since photons and gravitons are the only massless bosons known or commonly theorized to exist as free particles (thus excluding gluons, which are conjectured to exist only in color singlet configurations[31]), only combinations with $n_1 = n_2 = 1$ have been given. There is a greater uncertainty about $n_{1/2}$, the number of 2-helicity spin-1/2 species. The simplest picture consistent with experiment is that $n_{1/2} = 2$, corresponding to the $(\nu_e, \overline{\nu}_e)$ species with a left-handed electron neutrino and its right-handed antiparticle, and the $(\nu_\mu, \overline{\nu}_\mu)$ species with its muon neutrino and antineutrino. However, both of these species may also have the opposite helicity states, which would couple to gravity even if the V-A weak interaction[32] didn't couple them to other leptons, thus making $n_{1/2} = 4$. Indeed, vectorlike gauge theories of elementary particles have been made[33-35] in which there are additional neutrino states. Furthermore, black holes small enough to evaporate within the present age of the universe are hot enough to emit ultrarelativistic electrons and positrons,[7] each with two spin states, so $n_{1/2}$ must be augmented by 2 over the number for neutrinos if we consider all rest masses less than 20 MeV as negligible. Therefore, curves are given for $n_{1/2}$ from 2 to 10. In Figs. 3, 4, 5, 6, 7, and 8, the "canonical combination" $(n_{1/2}, n_1, n_2) = (4,1,1)$ is labeled as "everything emitted," meaning all of the presently known species with rest mass below 20 MeV, as listed explicitly in Fig. 9.

Figure 3 graphs the lifetime of a black hole--in units of its initial mass cubed--(i.e., θ_i of Eq. [28]), versus the initial rotation parameter a_{*i}, for various combinations of $(n_{1/2}, n_1, n_2)$. (The conversion factor in cgs units is $[10^{15}g]^3 = 5.23 \times 10^{15} s = 1.66 \times 10^8 yr.$) Then Fig. 4 gives the initial mass of a PBH that just evaporates today (Eq. [29]), assuming the present age of the universe is

$$t_o = 16 \times 10^9 yrs = 9.37 \times 10^{60} \quad , \tag{48}$$

so that

$$t_o^{1/3} = 2.11 \times 10^{20} = 4.59 \times 10^{15} g \quad . \tag{49}$$

For example, a PBH emitting the canonical combination of all known species (except for the small amount of muons and heavier particles emitted) would have just given up all its mass by now if its initial mass had been 4.73×10^{14}g if nonrotating or 6.26×10^{14}g if initially maximally rotating. The curves marked "neutrinos only emitted" in Figs. 3 and 4, as in Figs. 1 and 2, give the results if only one species of neutrinos are emitted; for successive graphs it does not matter how many neutrino species there are for the curves labeled "neutrinos only," since those graphs have the rates scaled out and depend only on the ratios of f's and of g's at different values of a_* .

The time evolution of the mass and rotation parameter are shown in Figs. 5 and 6. The curves for neutrinos, photons, or gravitons only cover the purely hypothetical cases in which the black hole emits only particles of one spin; they are included to illustrate the different behavior that would result. For example, gravitons cause the mass and particularly a_* to decrease more rapidly at large a_*, as compared with the behavior at small a_*, than photons or neutrinos do. Only one combination with species of all three spins being emitted is included, $(n_{1/2}, n_1, n_2) = (4,1,1)$, since other combinations gave curves only slightly different. One can see that for this canonical combination, a black hole which started at $a_* = 1$ will lose half its initial mass in 71% of its lifetime but half its initial a_* in only 21% of its lifetime. (Half the angular momentum $J = M^2 a_*$ is lost in only 6.7% of the lifetime.)

Figure 7 shows how a_* varies with the mass as the black hole gives up its angular momentum and energy. The emission of gravitons causes a_* to decrease at the fastest rate compared with M, essentially because gravitons have the greatest spin and thus carry off the most angular momentum per quantum. For the canonical combination of species, Fig. 6 showed that a_* is reduced to 0.19 after half of the lifetime from $a_* = 1$, but since it takes 71% of the lifetime to reduce M to half its original value, a_* is further reduced to 0.06 by then, as Fig. 7 illustrates directly. A check of the values represented by Fig. 1 reveals that f is then only 1% greater than its value at $a_* = 0$. Therefore, a black hole decaying by the emission of gravitons, photons, the presently known neutrinos, and ultrarelativistic electrons and positrons will emit more than 50% of its energy when it is so slowly rotating that its power is within 1% of the Schwarzschild value given in Paper I. This result gives a fairly strong justification for the usual simplifying assumption, mentioned in the Introduction, that emitting black holes are not rotating.[7-9]

One might note that this result was not apparent _a priori_, since $h(a_*)$ in Eq. (11) might have gone to zero at a nonzero value of a_*, in which case the curves in Fig. 7 would have leveled out at that value of a_* as M decreased. In fact, although the calculations have not been made for hypothetical massless spin-0 particles, there are two reasons for suspecting that h might indeed go to zero somewhere if the emission were predominantly in scalar radiation:

(1) If one defines $h_s(a_*)$ by Eq. (11) with f and g replaced by f_s and g_s, one has the logarithmic slope of a_* vs. M in the curves for only one spin emitted in Fig. 7. These curves thus have a_* going as some power of M for small a_*, where the power is $h_s(a_*=0)$. The numerical calculations indicate that there is a remarkably linear relationship between $h_s(a_*=0)$ and the spin s for $s = \frac{1}{2}$, 1, and 2:

$$h_s(a_*=0) \simeq 13.4464s - 1.1948 \qquad (50)$$

is accurate to one part in 10^4 for all three values, roughly the accuracy of the numerical calculations. Although there is no apparent theoretical reason to suspect such a highly linear relationship, which comes only after one does integrals over frequency and sums over angular modes in Eq. (32) and therefore seems to be accidental, it is tempting to extrapolate it to $s = 0$ to get a negative value for $h_0(a_*=0)$. One can easily see that the emission of any species makes $h(a_*=1) > 0$, since Eq. (47) says that the emission from a maximally rotating hole is entirely in the superradiant regime where each quantum contributes

$$\frac{\Delta \ln a_*}{\Delta \ln M} = \frac{\Delta \ln J}{\Delta \ln M} - 2 \approx \frac{M\Delta J}{J\Delta M} - 2 = \frac{1}{Ma_*}\frac{m}{\omega} - 2 = \frac{1}{a_*}(\frac{m}{x} - 2a_*) > 0 \ . \quad (51)$$

Therefore, if $h_o(a_*)$ is continuous and is negative at $a_* = 0$, it must become zero at some intermediate a_* .

(2) The dominant angular mode at small a_* is presumably the $\ell = s$ mode, as it is for $s = \frac{1}{2}$, 1, and 2. For $s = 0$ that mode carries off energy but no angular momentum; so unless higher angular modes contribute significantly, one would expect $g_o(a_*= 0)$ to be roughly zero and hence $h_o(a_*= 0)$ to be roughly -2. The higher angular modes would raise $h_o(a_*= 0)$ above -2 (conceivably to the value -1.1948 predicted by Eq. [50]!) but would probably leave it negative, so again one deduces that $h_o(a_*)$ may be zero for some a_* between zero and one.

If either (1) or (2) is valid and if scalar radiation dominates sufficiently at low a_* for the total radiation to give $h(a_*= 0) < 0$, then the black hole will spin down only to the nonzero value of a_* at which $h(a_*) = 0$. This does not occur for emission of the canonical combination of species, which causes the hole to spin down rapidly toward $a_* = 0$, as shown in the curve marked "everything" in Fig. 7. Once a_* is reduced to a small value, it decreases as a power law of M , with the exponent being

$$h(a_*= 0) = 6.3611 \quad (52)$$

in the canonical case.

Another interesting result is the evolution of the black-hole area A, which is illustrated in Fig. 8. The area first increases with time at large a_* and then decreases to zero along with a_* and the mass. This can be seen formally by using Eq. (11) to differentiate Eq. (20):

$$\frac{d\,\ln A}{d\,\ln a_*} = \frac{g}{g-2f} - (1-a_*^2)^{-1/2} \quad . \tag{53}$$

One may further use Eq. (6) to express the time derivative as

$$\frac{dA}{dt} = AM^{-3}[(1-a_*^2)^{-1/2}(g-2f) - g] \quad . \tag{54}$$

For small a_*, the right hand side of Eq. (54) becomes $-2AM^{-3}f$. This means that the area decreases logarithmically at twice the rate the mass does from Eq. (6), which is obvious since at small a_* the area is simply proportional to M^2. At large a_*, it was shown above that $h > 0$, and hence $g-2f > 0$ since $f > 0$. But $(1-a_*^2)^{-1/2}$ diverges as $a_* \to 1$, so dA/dt becomes positive and even goes infinite as $a_* \to 1$ (cf. the vertical behavior of the curves at the right edge of Fig. 8). The area is at a maximum where

$$2f = [1-(1-a_*^2)^{1/2}]g \quad . \tag{55}$$

For the canonical combination of species, this occurs at $a_* = 0.8868$, where the area is 17.3% greater than the original value, after a time of only 6.729 M_i^3 or 1.7% of the total lifetime 394.5 M_i^3 of a hole with $a_{*i} = 1$.

Physically, the behavior of the area can be understood by thermodynamic arguments, since the area is proportional to the entropy of the black hole (as was first suggested by Bekenstein,[36] though there were problems with this interpretation for a black hole immersed in a background of very low temperature until Hawking discovered that black holes not only absorb but also emit thermal radiation[13]). At high values of a_*, the emission is primarily the spontaneous emission discovered by Zel'dovich[18] that corresponds to the stimulated emission of superradiant scattering. In this process, pairs are created in the ergosphere with particles (say) being emitted to infinity with positive energies and their antiparticles going down the hole with negative energies as measured at infinity but positive energies as measured locally. In fact, the antiparticles can even be on classical trajectories at the horizon. Thus heat flows down the hole as well as out to infinity, increasing the entropy of both. On the other hand, at lower values of a_* the emission is primarily thermal, drawing entropy out of the hole. The process may still be regarded as the creation of pairs, with antiparticles going down the hole having negative energies with respect to infinity, but outside the superradiant regime (which becomes negligible at small a_*), the antiparticles also have negative energy locally at the horizon and therefore cannot be on classical trajectories. Instead, they are tunneling through a classically forbidden region in virtual states that actually bring heat out of the hole.

There is still some entropy produced by the partial scattering off the gravitational potential barrier surrounding the hole, but outside the superradiant regime this can only partially cancel the entropy flow out of the hole and serves in effect to increase the entropy emitted to the surrounding region for a given entropy loss by the hole. For example, numerical calculations for a nonrotating hole show that the emission of $s = 1/2$ particles into empty space increases the external entropy by 1.6391 times the entropy drawn out of the hole, $s = 1$ particles increase it by a factor of 1.5003, $s = 2$ particles by 1.3481, and the canonical combination of species gives 1.6233 times as much entropy in radiation as the entropy decrease of the hole.

The fact that $g - 2f > 0$ at $a_* = 1$ allows one to prove the third law of black hole mechanics[37] for small perturbations of an uncharged black hole. (Similar reasoning can presumably be made also for an electrically charged hole). The third law states that it is impossible to reduce the surface gravity κ to zero by a finite sequence of operations: Using Eqs. (6) and (7) to differentiate the expression for κ in Eq. (I:8) (cf. Eq. [34]), one finds that the emission of particles makes

$$\frac{d\kappa}{dt} = \frac{[1 - (1 - a_*^2)^{1/2}]\,(g - 2f) + (1 - a_*^2)f}{2M^4(1 - a_*^2)^{1/2}[1 + (1 - a_*^2)^{1/2}]} \underset{\kappa \to 0}{\sim} \frac{g - 2f}{4M^5\kappa}, \quad (56)$$

which diverges as $a_* \to 1$ or $\kappa \to 0$. Incident particles can only decrease κ at a finite rate (which even gets smaller as κ is reduced[37]), so eventually the emission dominates and keeps κ away from zero. Thus it is impossible to spin up a black hole to the extreme Kerr configuration.

Figure 9 gives the maximum present value of the rotation parameter a_* for a PBH with present mass M that was created 16 billion years ago, assuming no spin up from incident particles. The curves resulting from the emission of neutrinos, photons, or gravitons only are purely illustrative; the true maximum is probably near or somewhat below the curve for the canonical combination of particle species, since those species and possibly a few others are the ones predominantly emitted for the mass range shown. For example, electrons, positrons and all lighter particles will be emitted with negligible effects from their rest masses over the whole range shown, and muons and heavier particles will also be emitted at a significant rate for $M < 5 \times 10^{14}$g, as Paper I pointed out. The graph shows that a PBH with $M < 10^{15}$g should have $a_* < 0.64$ today.

The asymptotic behavior of the graphs in Figs. 1-9 as small a_* was given in functional form by Eqs. (35-46), and the parameters α, β, γ, δ, τ_f, and $M_i(a_{*i} = 0, t_o)$ are given in Table II for the various combinations $(n_{1/2}, n_1, n_2)$ of species of spin $\frac{1}{2}$, 1, and 2. Note that

$$\left(\frac{d\ \ell n\ M}{d\ \ell n\ a_*}\right)_{a_*=0} \equiv \gamma \equiv \frac{\alpha}{\beta - 2\alpha} \equiv \frac{f(a_*= 0)}{g(a_*=0) - 2f(a_*=0)} = \frac{1}{h(a_*=0)} \quad (57)$$

is the reciprocal of the exponent of the power-law behavior of a_* versus M at the lower left edge of Fig. 7. The ratio of the lifetime of a black hole with $a_{*i}= 1$ to one of the same initial mass with $a_{*i}= 0$ is $3\alpha\tau_f$, so Eq. (29) gives the initial mass of a PBH with $a_{*i}= 1$ that would just go away today as

$$M_i(a_{*i}=1,t_o) = (t_o/\tau_f)^{1/3} = (3\alpha\tau_f)^{-1/3}\, M_i(a_{*i}=0,t_o), \tag{58}$$

here written in terms of the parameters in Table II. Then $M_{min}(a_*,t_o)$ can be directly evaluated from the last quantity in Eq. (45) at small a_*. One can invert this asymptotic formula to obtain

$$a_{*max}(M,t_o) \sim [e^{\delta} M/M_i(a_{*i}=1,t_o)]^{1/\gamma} \tag{59}$$

for $M << M_i(a_{*i}=1,t_o)$. For example, the canonical combination of species gives

$$a_{*max}(M,t_o) \sim (M/4.870\times 10^{14}g)^{6.361} = 4.234\times 10^{-5}(M/10^{14}g)^{6.361}. \tag{60}$$

The actual maximum is almost certainly somewhat lower than this, since muons and other particles omitted in the calculation will have decreased the spin even more, and the upper limit on a_{*i} may be lower than unity; but unless small black holes were formed significantly more recently than 16 billion years ago, one may predict that any black hole found today with $M < 10^{14}g$ will have $a_* < 0.0000423$.

One can also get asymptotic forms near $a_*=1$. The lifetime has already been given by Eq. (26) with $z_i=0$, $\tau_i=0$, and τ_f listed in Table II; and $M_i(a_{*i}=1,t_o)$ was given by Eq. (58). If we set

$$\alpha_1 \equi f(a_*=1) \quad , \tag{61}$$

$$\beta_1 \equiv g(a_*=1) \quad , \tag{62}$$

which can be evaluated by combining the numbers of the last row of Table I according to Eq. (33), then integrating Eqs. (6) and (7) for a small time $t \ll M_1^3$ from $t = 0$ at $a_* = 1$ and $M = M_1$ gives

$$M \sim M_1(1 - \alpha_1 M_1^{-3} t) \quad , \tag{63}$$

$$J \sim M_1^2(1 - \beta_1 M_1^{-3} t) \quad , \tag{64}$$

$$a_* \equiv J/M^2 \sim 1 - (\beta_1 - 2\alpha_1)M_1^{-3} t \quad . \tag{65}$$

Since the mass decreases only infinitesimally within the age of the universe if $M_1^3 \gg \alpha_1 t_o$, one can use Eq. (65) with $M = M_1$ and $t = t_o$ as an asymptotic approximation to $a_{*max}(M,t_o)$ for large M. For example, the canonical combination of species gives

$$a_{*max}(M, t_o = 16 \times 10^9 \text{yrs}) \sim 1 - (M/1.500 \times 10^{15} \text{g})^{-3}$$

$$= 1 - 0.003378(M/10^{16}\text{g})^{-3} \quad . \tag{66}$$

This formula depends only weakly on the number of spin-$\frac{1}{2}$ species, since gravitons dominate the emission. However, since f and g change so rapidly with a_* near one (e.g., decreasing roughly 10% between $a_* = 1$ and $a_* = 0.9999$), these asymptotic formulas are only accurate very near $a_* = 1$.

V. CONCLUSIONS

The power emitted from a black hole in particles of negligible mass and of spin $\frac{1}{2}$, 1, and 2 are strongly increasing functions of the rotation parameter $a_* = J/M^2$, varying in the range $a_* = 0$ to $a_* = 1$ by factors of 13.35 for spin $\frac{1}{2}$, 107.5 for spin 1, and 26380 for spin 2.

The power increases 299.3 times for the "canonical combination" of 4 spin-$\frac{1}{2}$, 1 spin-1, and 1 spin-2 species that represent all of the presently known particles with rest masses less than 20 MeV. The power is greatest in spin-$\frac{1}{2}$ particles for $a_* \lesssim 0.6$, followed by spin 1 and then spin 2; but for $a_* \gtrsim 0.6$ the order is reversed.

The emission of angular momentum also increases greatly with a_*, even after the linear dependence expected at small a_* is factored out to get the relative torque or logarithmic rate of decrease in the angular momentum of the hole. The relative torque g behaves similar to the relative power f with respect to spin and a_*, but it is always sufficiently greater than 2f, for the three spins calculated, that a black hole spins down toward a Schwarzschild configuration much faster than it loses energy. More than half of the energy is emitted after a_* is reduced below a small value, less than 0.06 for the canonical combination of species. At this point the power is within 1% of its Schwarzschild value, so the assumption that decaying black holes have negligible rotation is generally valid.

Even though the power emitted is such a strong function of a_*, the fact that a black hole loses a_* so rapidly means that the total lifetime for a given mass varies only by a factor between 2.02 (for the emission of spin $\frac{1}{2}$ only) and 2.67 (for spin 2 only) over all a_{*i}. A black hole emitting the canonical species has a lifetime 2.32 times as long if initially nonrotating as one the same mass maximally rotating initially. The initial mass of a PBH created 16 billion years ago that just disappears today varies from 4.73×10^{14}g for a

Schwarzschild hole to 6.26×10^{14}g for an extreme Kerr hole initially. (This is for the emission of the canonical species; the emission of muons and heavier particles will make these masses somewhat greater, say 5×10^{15}g and 6.6×10^{14}g respectively.)

A black hole evolving from $a_{*i} = 1$ initially has its area and entropy increase as heat flows into the hole from particle pairs created in the ergosphere. Then as a_* falls low enough (below 0.89 for the canonical species), the non-superradiant thermal emission begins to dominate, taking heat out of the hole and thus causing the entropy and area to decrease. The maximum increase in the area is about 17.3% for the canonical emission. For a Schwarzschild hole that emits its energy into the canonical species in empty space, the emission process increases the entropy of the universe ($\frac{1}{4}A +$ entropy outside) by 62.3% of the black hole's initial entropy.

Finally, it was shown that a black hole cannot be spun up to $a_* = 1$. A PBH today is predicted to have a maximum rotation parameter as a function of mass that is given by Fig. 9 for $10^{14}\text{g} < M < 10^{16}\text{g}$ and by Eqs. (59) and (66) for larger and smaller values of the mass. Black holes that are small enough to emit many muons and heavier particles today are seen to be very nearly nonrotating.

This work was aided by discussions with many colleagues. Special thanks are due S. A. Teukolsky and W. H. Press for making available to me their computer programs to calculate the absorption probabilities for gravitons and photons. F. J. Nagy gave continued guidance on the use of the Lawrence Berkeley CDC-7600 computer. K. S. Thorne offered constructive suggestions on the manuscript.

TABLE I. Power and torque emitted by a black hole. For each spin s, $f_s(a_*)$ and $g_s(a_*)$ are the contributions of one species with two polarizations to $f \equiv -M^3 d\,\ln M/dt$ and $g \equiv -M^3 d\,\ln J/dt$ at that value of the rotation parameter a_*. The first 14 rows were calculated by Eq. (32); the last row came from a cubic spline extrapolation. The values for $a_* = 0$ are nearly the same as for $a_* = 0.01$; see Table II for more precise values.

$a_* = J/M^2$	$f_{1/2}(a_*)$	$f_1(a_*)$	$f_2(a_*)$	$g_{1/2}(a_*)$	$g_1(a_*)$	$g_2(a_*)$
.01000	8.18×10^{-5}	3.37×10^{-5}	3.84×10^{-6}	6.16×10^{-4}	4.79×10^{-4}	1.06×10^{-4}
.10000	8.34×10^{-5}	3.58×10^{-5}	4.68×10^{-6}	6.17×10^{-4}	4.89×10^{-4}	1.17×10^{-4}
.20000	8.83×10^{-5}	4.27×10^{-5}	7.73×10^{-6}	6.22×10^{-4}	5.21×10^{-4}	1.51×10^{-4}
.30000	9.67×10^{-5}	5.52×10^{-5}	1.49×10^{-5}	6.30×10^{-4}	5.76×10^{-4}	2.23×10^{-4}
.40000	1.09×10^{-4}	7.57×10^{-5}	3.12×10^{-5}	6.43×10^{-4}	6.60×10^{-4}	3.60×10^{-4}
.50000	1.26×10^{-4}	1.08×10^{-4}	6.82×10^{-5}	6.63×10^{-4}	7.84×10^{-4}	6.24×10^{-4}
.60000	1.49×10^{-4}	1.59×10^{-4}	1.57×10^{-4}	6.95×10^{-4}	9.67×10^{-4}	1.16×10^{-3}
.70000	1.80×10^{-4}	2.45×10^{-4}	3.91×10^{-4}	7.46×10^{-4}	1.24×10^{-3}	2.32×10^{-3}
.80000	2.28×10^{-4}	4.01×10^{-4}	1.10×10^{-3}	8.37×10^{-4}	1.71×10^{-3}	5.29×10^{-3}
.90000	3.19×10^{-4}	7.52×10^{-4}	4.11×10^{-3}	1.03×10^{-3}	2.63×10^{-3}	1.54×10^{-2}
.96000	4.57×10^{-4}	1.31×10^{-3}	1.30×10^{-2}	1.34×10^{-3}	3.98×10^{-3}	4.06×10^{-2}
.99000	6.71×10^{-4}	2.15×10^{-3}	3.58×10^{-2}	1.81×10^{-3}	5.83×10^{-3}	9.55×10^{-2}
.99900	9.25×10^{-4}	3.06×10^{-3}	7.25×10^{-2}	2.34×10^{-3}	7.72×10^{-3}	1.75×10^{-1}
.99999	1.07×10^{-3}	3.56×10^{-3}	9.79×10^{-2}	2.64×10^{-3}	8.73×10^{-3}	2.27×10^{-1}
1.00000	1.09×10^{-3}	3.62×10^{-3}	1.01×10^{-1}	2.68×10^{-3}	8.85×10^{-3}	2.34×10^{-1}

TABLE II. Parameters in the asymptotic behavior of a slowly rotating black hole: $\alpha \equiv f(a_*=0) \sim -M^2 dM/dt$, $\beta \equiv g(a_*=0) \sim -Ma_*^{-1}\, dJ/dt$, $\gamma \equiv \alpha/(\beta-2a) \sim d\,\ell n\, M/d\,\ell n\, a_*$, $\delta \equiv \int_0^1 (h^{-1}-\gamma)a_*^{-1}da_* \sim \gamma\,\ell n\, a_* - \ell n(M/M_1)$, $\tau_f \equiv \tau(a_*=0)$ = (lifetime from $a_*=1$)/(initial mass $M_1)^3$, and $M_i(a_{*i}=0,t_o)$ ≡ (initial mass of a Schwarzschild hole with lifetime t_o) = $(3\alpha t_o)^{1/3}$. These are given for various combinations of $n_{1/2}$ spin-$\frac{1}{2}$, n_1 spin-1, and n_2 spin-2 species.

$(n_{1/2},n_1,n_2)$	$10^4\alpha$	$10^4\beta$	γ	δ	τ_f	$M_i(a_{*i}=0,t_o)$
(1,0,0)	0.81830	6.16108	0.18086	0.32338	2011.52	2.8728×10^{14}g
(0,1,0)	0.33638	4.79364	0.08163	0.29539	3856.63	2.1360×10^{14}g
(0,0,1)	0.03836	1.06265	0.03891	0.28073	32560.0	1.0359×10^{14}g
(0,1,1)	0.37475	5.85629	0.07338	0.26405	3449.2	2.2143×10^{14}g
(1,1,1)	1.19304	12.01736	0.12387	0.28214	1159.5	3.2575×10^{14}g
(2,1,1)	2.01133	18.17843	0.14209	0.24084	695.08	3.8770×10^{14}g
(3,1,1)	2.82963	24.33952	0.15148	0.24551	502.44	4.3442×10^{14}g
(4,1,1)	3.64793	30.50059	0.15721	0.25047	394.50	4.7280×10^{14}g
(5,1,1)	4.46623	36.66167	0.16107	0.25506	325.27	5.0580×10^{14}g
(6,1,1)	5.28452	42.82275	0.16384	0.25917	277.01	5.3497×10^{14}g
(8,1,1)	6.92111	55.14490	0.16757	0.26610	214.02	5.8531×10^{14}g
(10,1,1)	8.55771	67.46706	0.16996	0.27164	174.65	6.2823×10^{14}g

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FIGURE CAPTIONS

Fig. 1. Power emitted in various combinations of species by a rotating black hole, expressed in a scale-invariant way by f. The symbol $(n_{1/2},n_1,n_2)$ denotes a combination of $n_{1/2}$ spin-$\frac{1}{2}$, n_1 spin-1, and n_2 spin-2 species, where each species is assumed to have two polarizations (e.g., left-handed neutrino plus right-handed anti-neutrino).

Fig. 2. Relative torque emitted by a black hole (i.e., the rate of emission of angular momentum, divided by the angular momentum of the hole), expressed in a scale-invariant form by g.

Fig. 3. Lifetime of a black hole, scaled by the initial mass cubed to give θ_i, versus the initial rotation parameter a_{*i}. "Everything emitted" means the canonical combination (4,1,1) of all known particles with masses less than 20 MeV.

Fig. 4. Initial mass of a primordial black hole created with rotation parameter a_{*i} 16×10^9yrs ago that just goes away today, assuming that it emits the combination $(n_{1/2},n_1,n_2)$ of species with negligible rest mass. The emission of all known particles, including those with masses greater than 20 MeV, would give a curve slightly above the (4,1,1) curve labeled "Everything emitted." However, if there are additional neutrino states, the true curve would be slightly above one of the higher curves shown.

Fig. 5. Time evolution of the mass of a black hole which started out maximally rotating. The vertical and horizontal axes have been scaled by the initial mass $M_i = M_1$ and lifetime $M_i^3\,\theta_i = M_1^3\,\tau_f$.

For a black hole that starts with $a_{*i} < 1$, one can use one of the same curves but shrink the axes so that the upper left corner of the graph is on the curve at a later point (to be determined from the value of t/lifetime at $a_* = a_{*i}$ in Fig. 6) and the lower right corner stays fixed, at the endpoint of the curve.

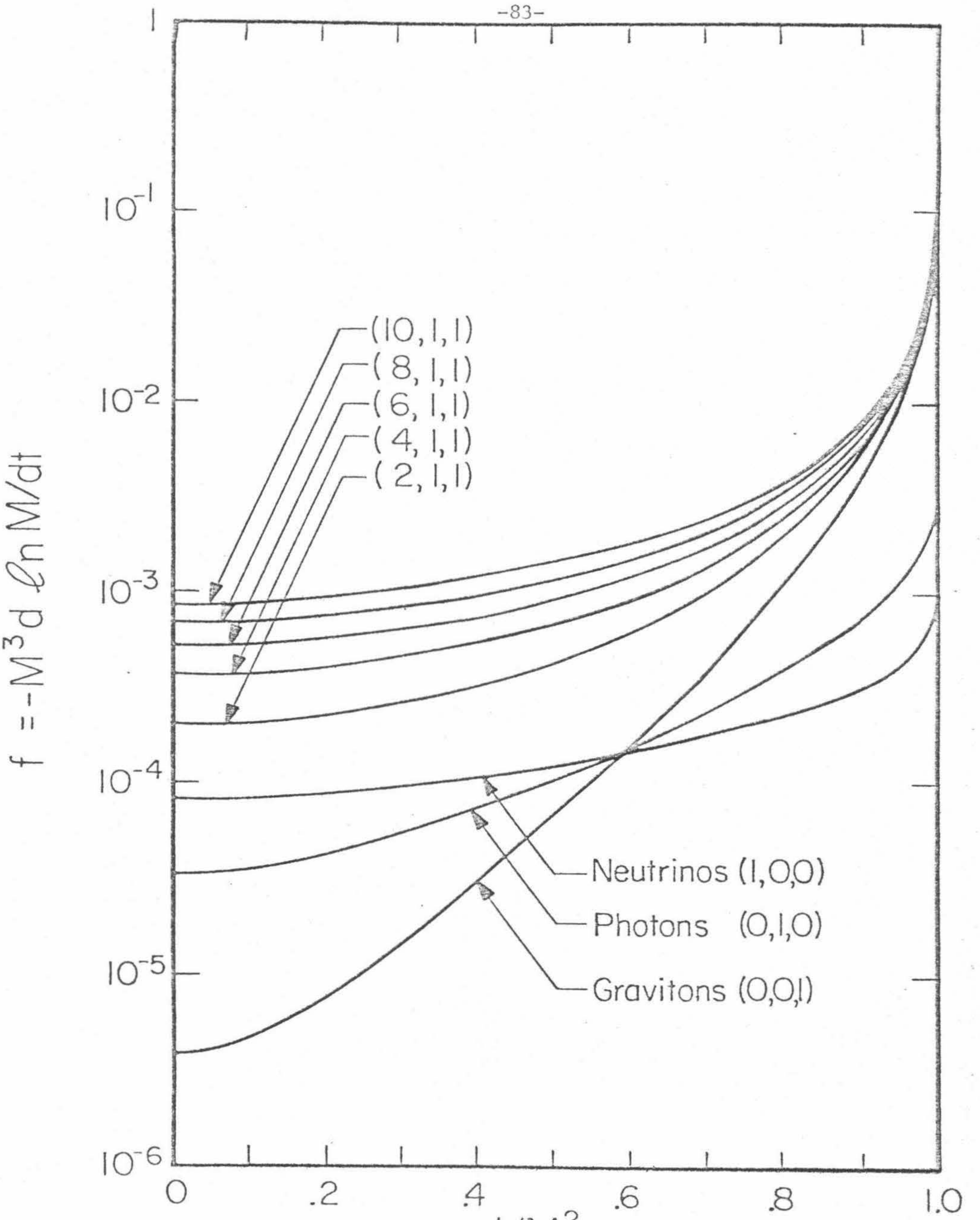
Fig. 6. Time evolution of the rotation parameter a_* of a black hole that started with $a_{*i} = 1$. For any given curve representing the emission of an assumed combination of species, the evolution from $a_{*i} < 1$ can be gotten by moving the left vertical axis to the right until it intersects the curve at $a_* = a_{*i}$, meanwhile shrinking the horizontal axis appropriately to leave its right end fixed.

Fig. 7. Variation of the rotation parameter with the mass during the evolution of a black hole, which proceeds from the upper right to the lower left corner. The evolution from $a_{*i} < 1$ can be gotten by keeping the left end of the horizontal axis fixed and shrinking the scale so that $M/M_i = 1$ falls at $a_* = a_{*i}$ on the curve considered.

Fig. 8. Evolution of the area A of a black hole, scaled by the initial area A_i in the case $a_{*i} = 1$. For general a_{*i} , the evolution starts at that value of a_* with the vertical axis rescaled to give $A/A_i = 1$ there, and proceeds to the left along the appropriate curve as a_* decreases with time. The evolution of A is plotted versus a_* rather than time to spread out the very rapid changes near $a_* = 1$, where A actually increases with

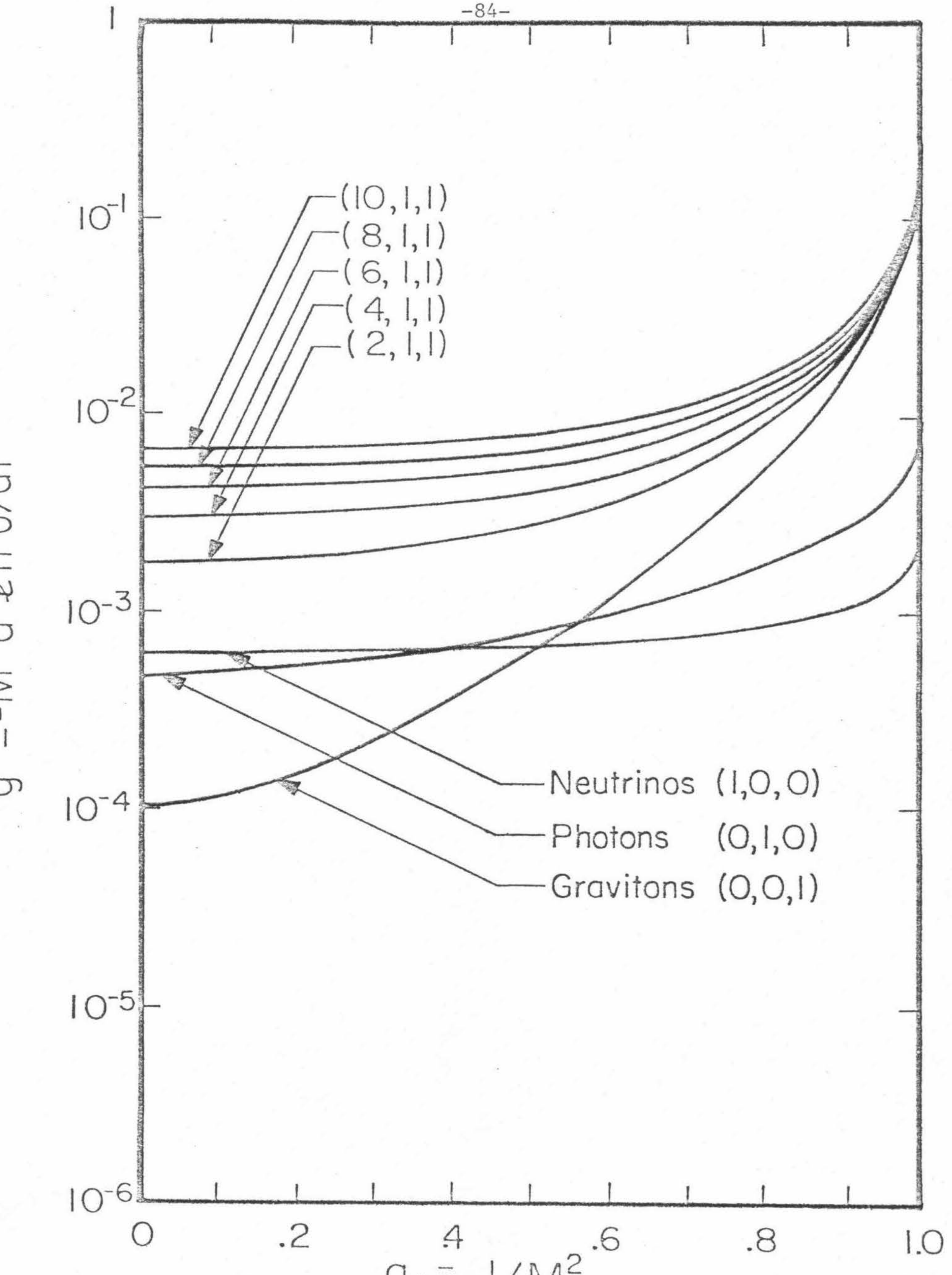
time. Since the area is four times the entropy of the hole, these curves can also be viewed as giving the evolution of the entropy.

Fig. 9. Maximum present rotation parameter a_* of a primordial black hole with mass M today, assuming it was created 16 billion years ago with unity as the upper limit on the rotation parameter then. Under these assumptions, the actual maximum is probably near (particularly for $M > 10^{15}$g) or somewhat below (particularly for $M < 10^{15}$g) the bottom curve given, depending upon the additional emitted species not covered in the canonical combination.



1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
f = -M^{3}d ℓn M/dt
(10, 1, 1)
(8, 1, 1)
(6, 1, 1)
(4, 1, 1)
(2, 1, 1)
Neutrinos (1,0,0)
Photons (0,1,0)
Gravitons (0,0,1)
0
.2
.4
.6
.8
1.0
$a_* = J/M^2$

Fig. 1



1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
(10,1,1)
(8,1,1)
(6,1,1)
(4,1,1)
(2,1,1)
Neutrinos (1,0,0)
Photons (0,1,0)
Gravitons (0,0,1)
$g = -M^3 d\, \ln J/dt$
0
.2
.4
.6
.8
1.0
$a_* = J/M^2$

Fig. 2

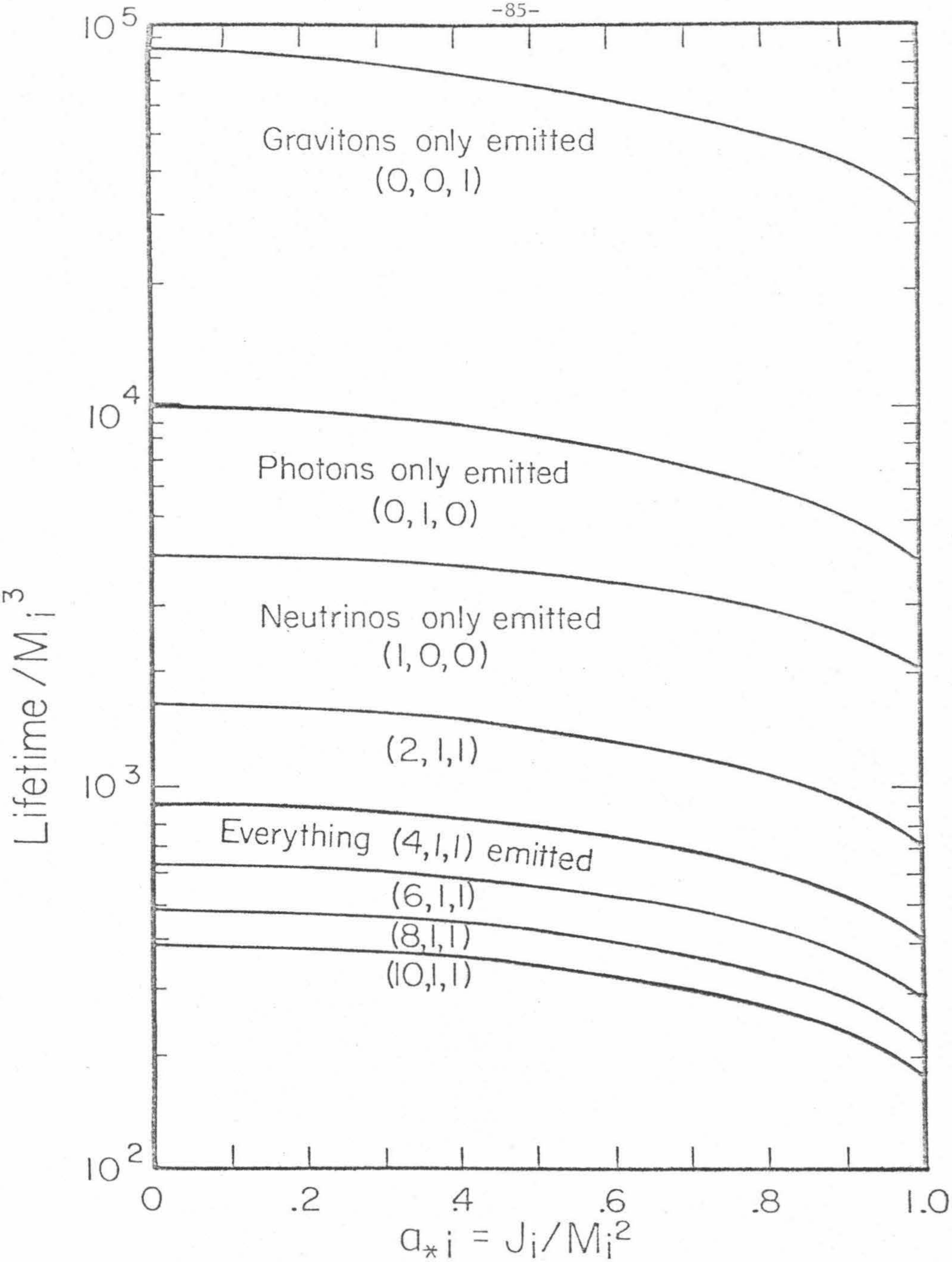

Gravitons only emitted
(0, 0, 1)
Photons only emitted
(0, 1, 0)
Neutrinos only emitted
(1, 0, 0)
(2, 1, 1)
Everything (4, 1, 1) emitted
(6, 1, 1)
(8, 1, 1)
(10, 1, 1)
Lifetime / M_i^3
10^5
10^4
10^3
10^2
0
.2
.4
.6
.8
1.0
$a_{*i} = J_i / M_i^2$

Fig. 3

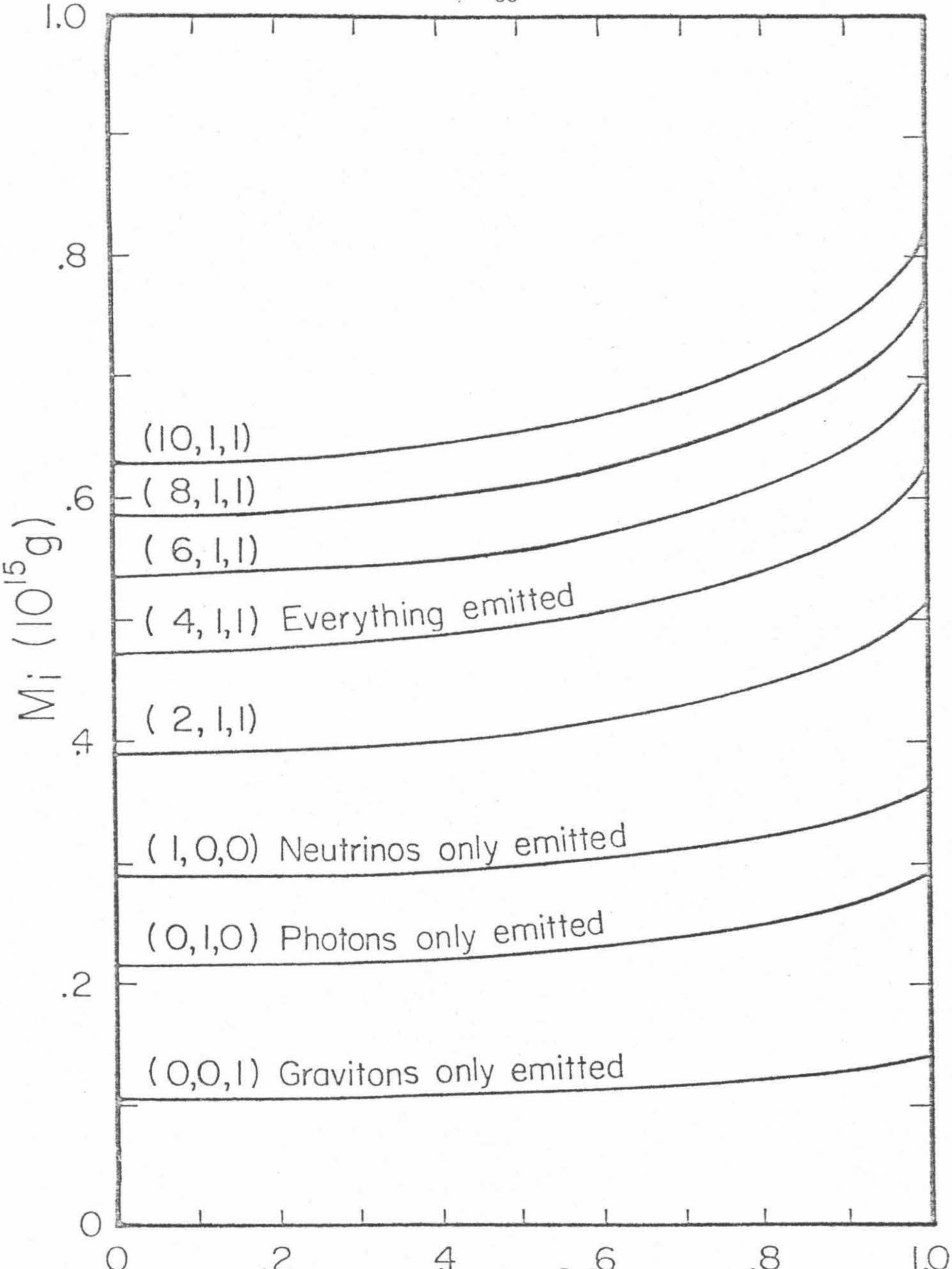
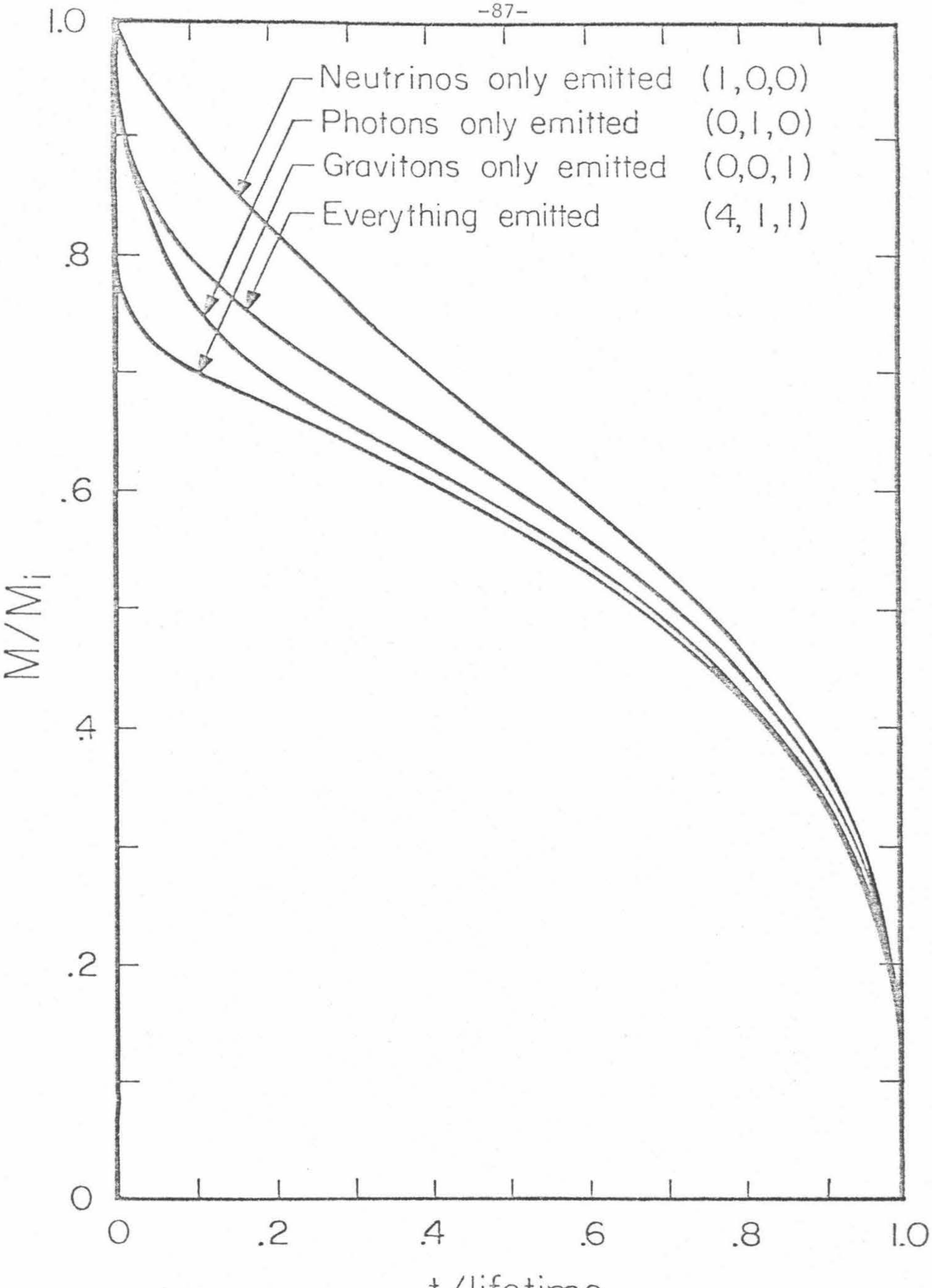
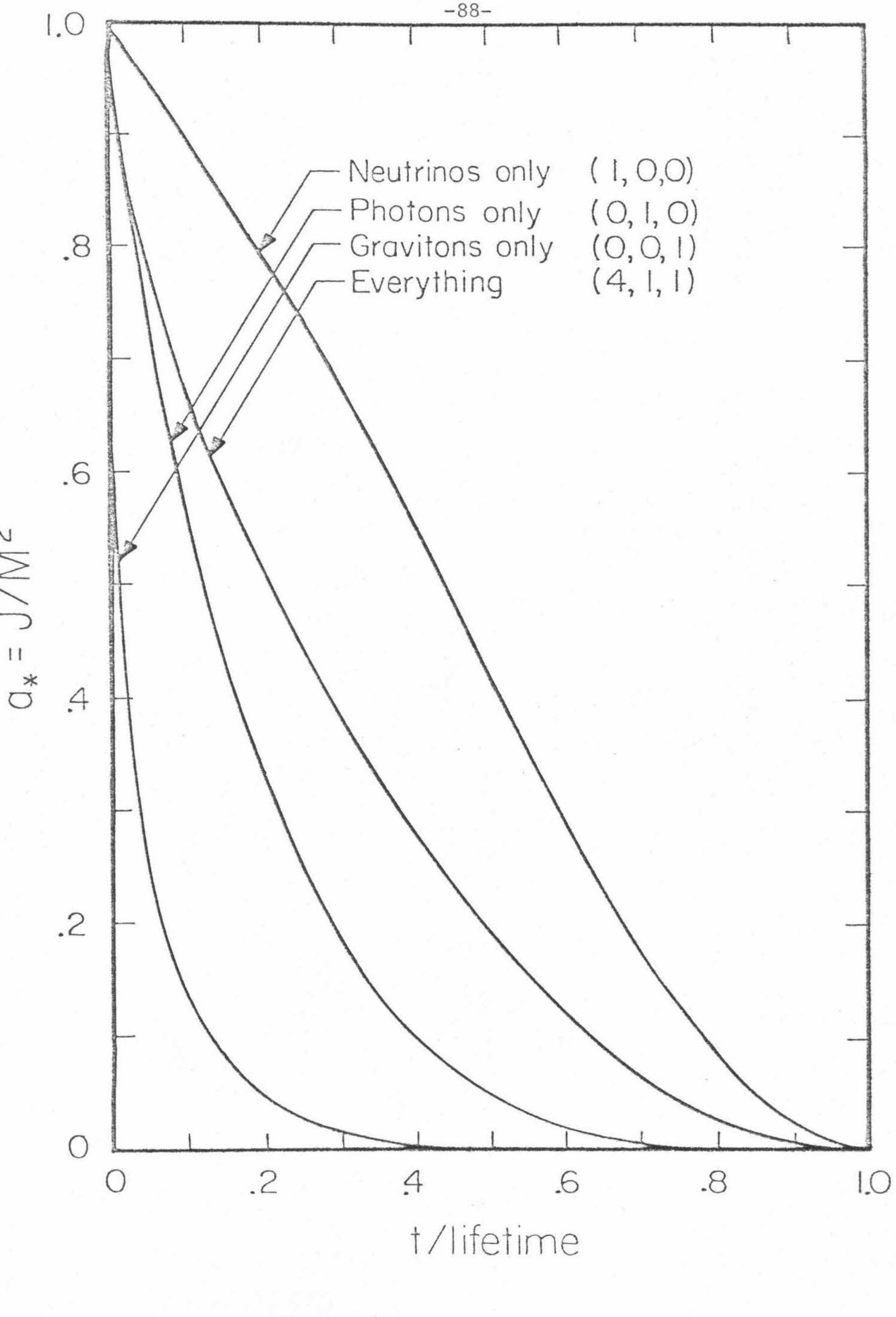

1.0
.8
.6
.4
.2
0
(10,1,1)
(8,1,1)
(6,1,1)
(4,1,1) Everything emitted
(2,1,1)
(1,0,0) Neutrinos only emitted
(0,1,0) Photons only emitted
(0,0,1) Gravitons only emitted
0
.2
.4
.6
.8
1.0
M_i (10^{15}g)
$a_{*i} = J_i/M_i^2$

Fig. 4



Neutrinos only emitted (1,0,0)
Photons only emitted (0,1,0)
Gravitons only emitted (0,0,1)
Everything emitted (4,1,1)
M/Mi
t/lifetime
1.0
.8
.6
.4
.2
0
0
.2
.4
.6
.8
1.0

Fig. 5



1.0
.8
.6
.4
.2
0
$a_* = J/M^2$
Neutrinos only (1, 0,0)
Photons only (0, 1, 0)
Gravitons only (0, 0, 1)
Everything (4, 1, 1)
0
.2
.4
.6
.8
1.0
t/lifetime

Fig. 6

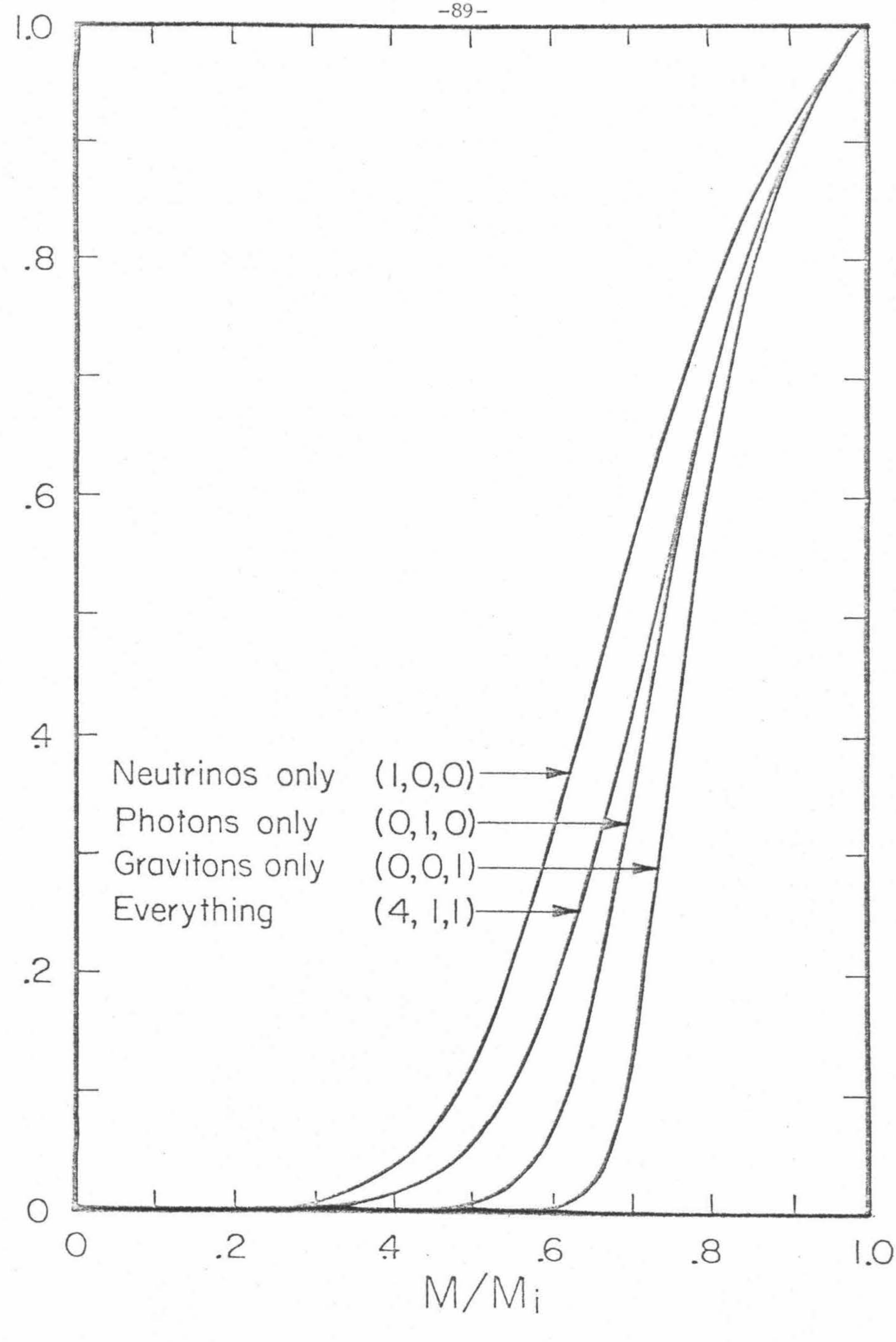
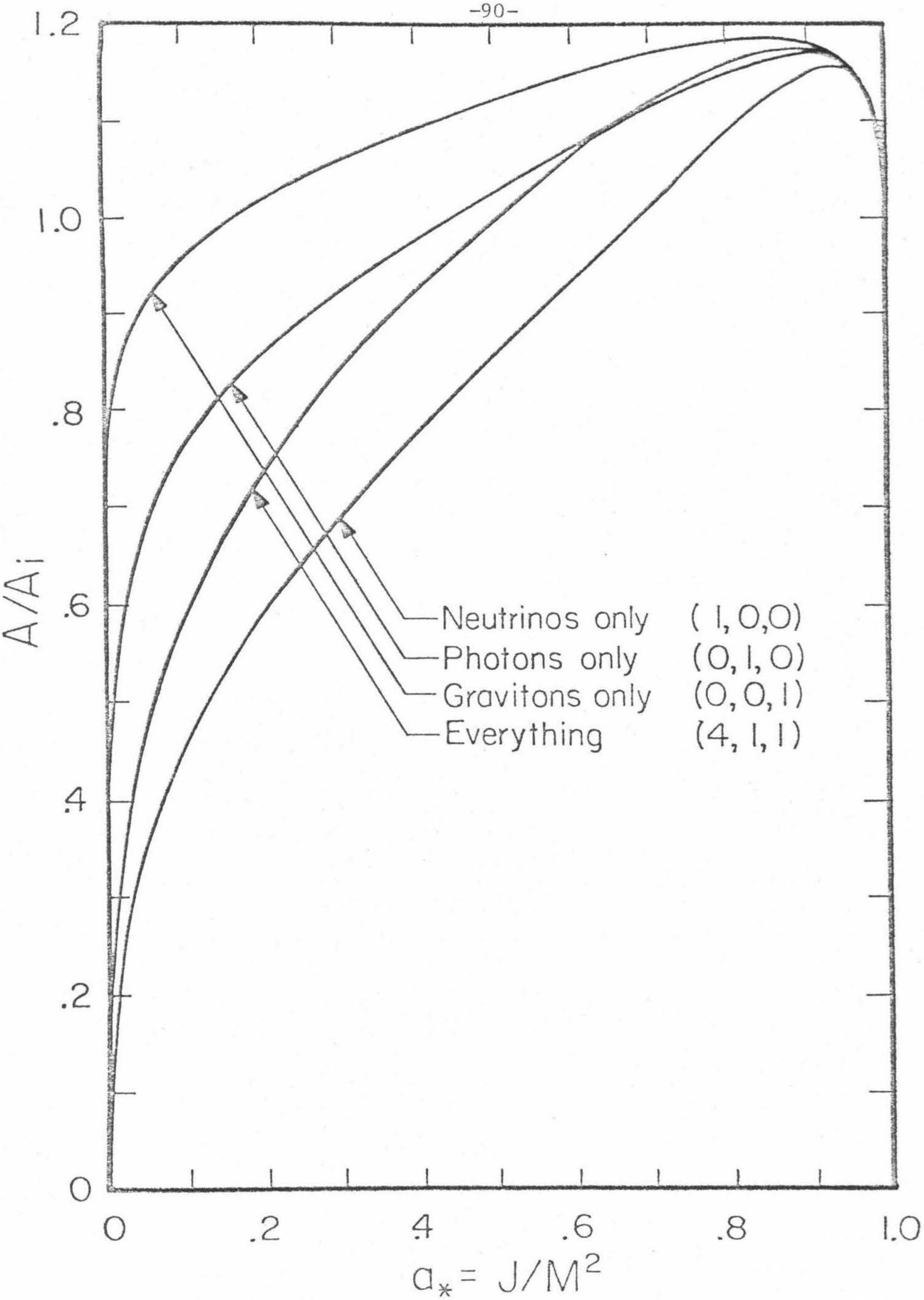

1.0
.8
.6
.4
.2
0
$a_* = J/M^2$
Neutrinos only (1,0,0)
Photons only (0,1,0)
Gravitons only (0,0,1)
Everything (4,1,1)
0
.2
.4
.6
.8
1.0
M/M_i

Fig. 7



1.2
1.0
.8
.6
.4
.2
0
A/Ai
Neutrinos only (1,0,0)
Photons only (0,1,0)
Gravitons only (0,0,1)
Everything (4,1,1)
0
.2
.4
.6
.8
1.0
a* = J/M2

Fig. 8

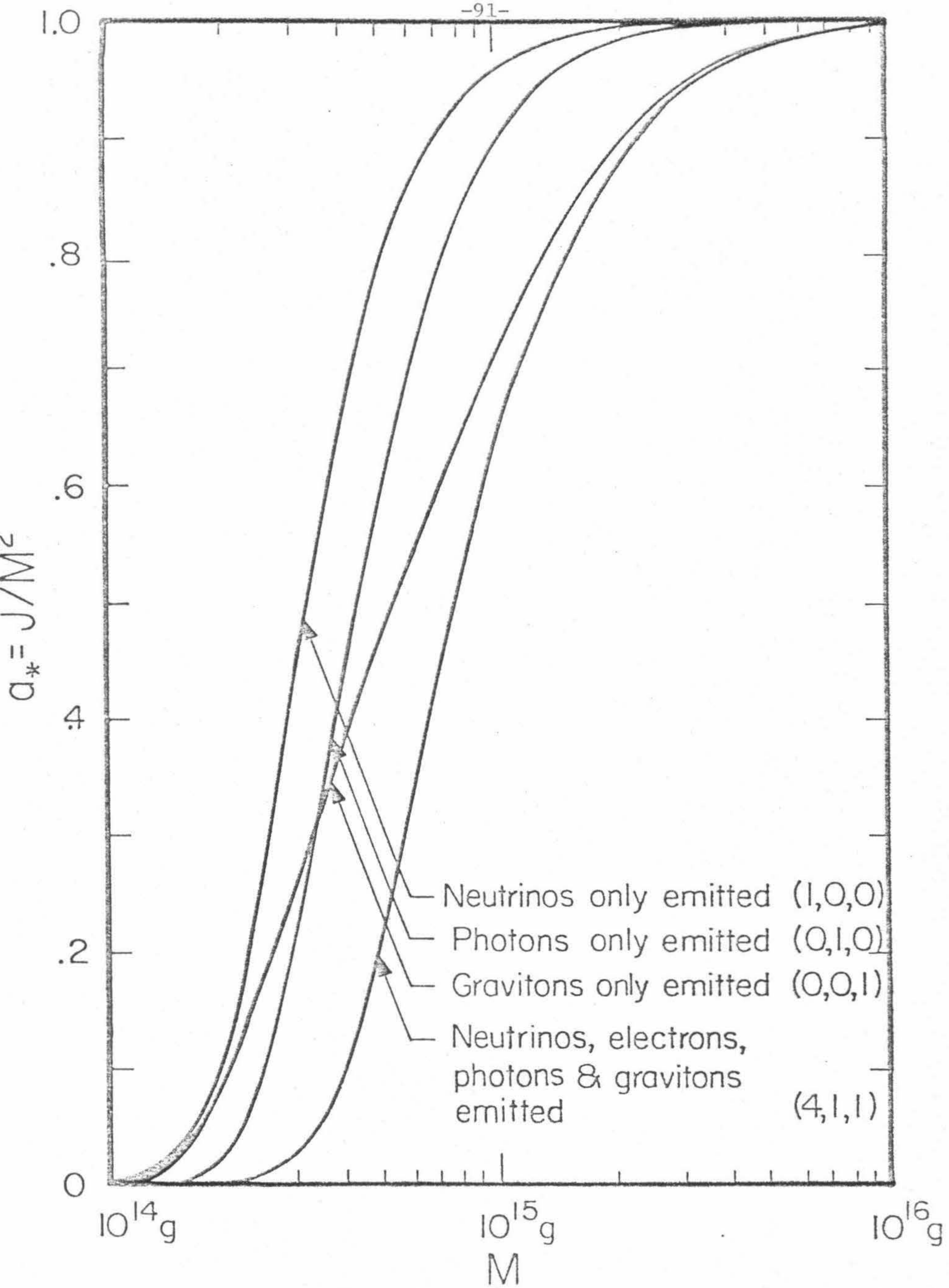

1.0
.8
.6
.4
.2
0
$a_* = J/M^2$
10^{14}g
10^{15}g
10^{16}g
M
Neutrinos only emitted (1,0,0)
Photons only emitted (0,1,0)
Gravitons only emitted (0,0,1)
Neutrinos, electrons, photons & gravitons emitted (4,1,1)

Fig. 9

(e) Neutrino Angular Eigenvalues

Press and Teukolsky (1973) have described a continuation method for calculating the eigenvalues of the angular Teukolsky equation and have given polynomial approximations for the case of gravitons. Teukolsky and Press (1974) gave polynomial fits for the angular eigenvalues of electromagnetic perturbations. Since Paper III includes the emission of neutrinos as well, their angular eigenvalues were also calculated by the continuation method and then fit to sixth-degree polynomials in $a\omega$ whose coefficients are listed below in Table I. The optimal polynomial for each angular mode (ℓ, m) was chosen to give the best least-squares fit to the eigenvalues at 21 values of $a\omega$ evenly spaced from 0 to 1 or to $m/2$, whichever was larger. (An exception is $\ell = 3.5$, $m = 1.5$, which used values of $a\omega$ up to $m/2 = .75$.) To six decimal places, the constant term always agreed with $\ell(\ell+1)$, so that term is not listed.

TABLE I. Polynomial Approximations for Neutrino Angular Eigenvalues

$$_{1/2}E_{\ell m}(a\omega) \simeq \ell(\ell+1) + \sum_{n=1}^{6} c_n (a\omega)^n \quad \text{in} \quad 0 \leq a\omega \leq \mathrm{Max}(1, m/2)$$

ℓ	m	c_1	c_2	c_3	c_4	c_5	c_6
0.5	0.5	-.333338	-.407327	-.053178	-.018980	-.003258	.002312
0.5	-0.5	.333335	-.407434	.052844	-.021015	.001836	-.000327
1.5	1.5	-.200001	-.231992	-.017598	-.006908	-.000915	-.000096
1.5	0.5	-.066662	-.440675	.044407	.007558	.004320	-.002321
1.5	-0.5	.066665	-.440570	-.044042	.009534	-.002789	.000226
1.5	-1.5	.200000	-.231999	.017550	-.007009	.000755	-.000155
2.5	2.5	-.142857	-.160353	-.007600	-.002912	-.000220	-.000086
2.5	1.5	-.085714	.368589	.009972	.000645	.000997	.000096
2.5	0.5	-.028571	-.472701	.005738	.002930	-.000882	-.000037
2.5	-0.5	.028571	-.472694	-.005764	.003053	.000783	.000120
2.5	-1.5	.085714	-.368586	-.009922	.000682	-.000837	.000089
2.5	-2.5	.142857	-.160349	.007613	-.002867	.000248	-.000050
3.5	3.5	-.111110	-.122097	-.003901	-.001475	-.000051	-.000041
3.5	2.5	-.079365	-.303294	.002667	-.000654	.000020	.000073
3.5	1.5	-.047619	-.424108	.003814	.000742	-.000067	.000006

TABLE I - Continued

ℓ	m	c_1	c_2	c_3	c_4	c_5	c_6
4.5	4.5	-.090906	-.098444	-.002238	-.000852	-.000005	-.000020
4.5	3.5	-.070709	-.255061	.000685	-.000696	.000026	.000030
4.5	2.5	-.050505	-.372566	.001876	-.000107	.000036	.000003
5.5	5.5	-.076918	-.082414	-.001389	-.000538	.000006	-.000011
5.5	4.5	-.062940	-.219262	.000065	-.000559	-.000009	.000013
5.5	3.5	-.048951	-.328804	.000910	-.000331	.000024	.000004

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PART IV

GAMMA RAYS FROM PRIMORDIAL BLACK HOLES

(a) Introductory Discussion

The quantum-mechanical emission from black holes is important in principle when one considers the final state of a system that has undergone gravitational collapse. However, in practice the emission would not be observable for a hole of a solar mass or greater, since the emission temperature would be less than 10^{-7}°K, the lifetime for decay would be greater than 10^{66}yrs, and the absorption of background radiation would dominate the emission in the present epoch. The present universe is not likely to produce black holes of mass smaller than the sun, so the only hope for observable quantum effects is from holes possibly formed in the early stages of the universe. Such holes are known as primordial black holes (PBHs).

Stephen Hawking (1971) first suggested the possibility of PBHs, which might result from fluctuations in the early universe such that there would be regions with deficient energy which would collapse gravitationally. Zel'dovich and Novikov (1967) had previously made an analysis indicating that condensed objects in the early universe should accrete matter rapidly and grow as fast as the particle horizon. Thus the absence of any such enormous condensation in our observable universe seemed to be evidence against PBHs. However, Carr and Hawking (1974) found that the situation analyzed by Zel'dovich and Novikov corresponded to the special initial conditions of having everything thrown at the hole from the beginning, and that a PBH formed locally would not grow so fast but could remain small (unless the equation of state were as stiff as causality allowed; cf. Lin, Carr, and Fall 1976). Carr (1975) has gone on to analyze the mass spectrum of PBHs that could be produced by various conditions in the early universe, finding that under certain reasonable assumptions the spectrum might go as a certain power law in the mass. The detection of radiation from these PBHs would be not only a confirmation of quantum effects in strong gravitational fields but also an indication of the degree of inhomogeneity in the early universe.

The possibility of detecting emission from PBHs depends on their number density. Hawking (1971) noted that measurements of the deceleration parameter of the universe set an upper limit on the PBH mass density. Chapline (1975) obtained a smaller limit for holes near 10^{15}g from the isotropic X-ray background above 10 MeV. Carr (1975) deduced similar limits for PBHs in this mass range and other limits for different mass regimes. Later Carr (1976) showed in a qualitative manner what the shape of the photon spectrum from PBHs would be without absorption and how it compared with the observed isotropic spectrum.

In Paper IV below, Hawking and I looked in somewhat more detail at the possibilities for observing hard gamma rays from PBHs. Hawking originated most of the ideas concerning bursts from dying PBHs, and I derived the formulas and did the numerical integrations for the background spectrum from all decaying PBHs. The main bases for these latter calculations were Carr's power-law mass spectra of PBHs and my calculations of the total power and photon spectrum from a nonrotating black hole (Paper II, with Paper III justifying the neglect of possible rotation). The photon spectrum is listed in Table II below. We found that it might be possible to detect exploding PBHs without too much difficulty if conditions were the optimum consistent with present observations. On the other hand, they might be indefinitely more difficult to detect. However, the payoff of a positive detection could be tremendous in terms of knowledge about fundamental physics and conditions of the early universe.

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(b) Photon Spectrum from a Single Black Hole

Paper II gives the photon power spectrum from a nonrotating black hole in graphical form (Fig. 1). For that paper, the emission rates were calculated at frequency values that were unevenly spaced in order to optimize the integration over frequency of the total rate and power. However, for the calculations of Paper IV, it was desired to have the spectrum evaluated at evenly spaced frequencies, so the cross section σ and the rate and power spectra were calculated for 100 values of

$$x \equiv M\omega \tag{IV.1}$$

from 0.01 to 1.00. The cross section in units of the high-frequency cross section $\sigma_\infty = 27\pi M^2$ is (cf. Paper II, Eq. 19)

$$S(x) \equiv \frac{\sigma}{\sigma_\infty} = \frac{1}{27x^2} \sum_{\ell,m} \Gamma_{\ell mp}(x) \quad , \tag{IV.2}$$

and the photon emission rate is (cf. Paper IV, Eq. 3)

$$f(x) \equiv \frac{dN_\gamma}{dt\ d\omega} = \frac{1}{2\pi} \sum_{\ell,m,p} \frac{\Gamma_{\ell mp}(x)}{e^{8\pi x} - 1} = \frac{27x^2\ S(x)}{\pi(e^{8\pi x} - 1)} \ . \tag{IV.3}$$

The power spectrum is

$$xf(x) = \frac{MdE}{dt\ d\omega} \ , \tag{IV.4}$$

which is what is plotted in Fig. 1 of Paper II.

The values of x , $S(x)$, $f(x)$, and $xf(x)$ calculated for Paper IV are listed in Table II, along with the estimated errors in solving the Teukolsky equation and resolving the solution into ingoing and outgoing waves at large radius. The calculations were estimated to be accurate to one part in roughly 10^5 below $x = 0.4$, roughly 10^4 up to $x = 0.7$, and roughly 10^3 over the rest of the range to $x = 1$.

TABLE II. Photon Cross Section and Spectrum from a Nonrotating Black Hole

$x \equiv M\omega$	$S(x) \equiv \sigma/27\pi M^2$	$10^4 f(x) \equiv 10^4 \frac{dN_\gamma}{dt\,d\omega}$	$10^4 xf(x) \equiv 10^4 \frac{M\,dE}{dt\,d\omega}$
0.01	.000085 ± .000000	.002545 ± .000000	.000025 ± .000000
0.02	.000366 ± .000000	.019249 ± .000000	.000385 ± .000000
0.03	.000896 ± .000000	.061614 ± .000000	.001848 ± .000000
0.04	.001750 ± .000000	.138865 ± .000001	.005555 ± .000000
0.05	.003023 ± .000000	.258439 ± .000002	.012922 ± .000000
0.06	.004847 ± .000000	.426341 ± .000004	.025580 ± .000000
0.07	.007392 ± .000000	.647399 ± .000006	.045318 ± .000000
0.08	.010882 ± .000000	.925440 ± .000008	.074035 ± .000001
0.09	.015611 ± .000000	1.263397 ± .000012	.113706 ± .000001
0.10	.021957 ± .000000	1.663329 ± .000016	.166333 ± .000002
0.11	.030411 ± .000000	2.126357 ± .000021	.233899 ± .000002
0.12	.041596 ± .000000	2.652469 ± .000026	.318296 ± .000003
0.13	.056304 ± .000001	3.240147 ± .000033	.421219 ± .000004
0.14	.075516 ± .000001	3.885769 ± .000040	.544008 ± .000006
0.15	.100426 ± .000001	4.582687 ± .000048	.687403 ± .000007
0.16	.132433 ± .000001	5.319932 ± .000057	.851189 ± .000009
0.17	.173094 ± .000002	6.080519 ± .000066	1.033688 ± .000011
0.18	.223991 ± .000003	6.839484 ± .000075	1.231107 ± .000013
0.19	.286479 ± .000003	7.562087 ± .000084	1.436797 ± .000016
0.20	.361281 ± .000004	8.203066 ± .000092	1.640613 ± .000018
0.21	.447935 ± .000005	8.708384 ± .000098	1.828761 ± .000021
0.22	.544219 ± .000006	9.021066 ± .000100	1.984634 ± .000023
0.23	.645791 ± .000007	9.091843 ± .000100	2.091124 ± .000024
0.24	.746384 ± .000009	8.892828 ± .000100	2.134279 ± .000025
0.25	.838750 ± .000010	8.429183 ± .000099	2.107296 ± .000025
0.26	.916195 ± .000011	7.742420 ± .000091	2.013029 ± .000024
0.27	.974093 ± .000012	6.902077 ± .000082	1.863561 ± .000022
0.28	1.010737 ± .000012	5.988895 ± .000072	1.676890 ± .000020
0.29	1.027220 ± .000013	5.077117 ± .000062	1.472364 ± .000018

TABLE II - Continued

$x \equiv M\omega$	$S(x) \equiv \sigma/27\pi M^2$	$10^4 f(x) \equiv 10^4 \frac{dN_\gamma}{dt d\omega}$	$10^4 x f(x) \equiv 10^4 \frac{MdE}{dt d\omega}$
0.30	1.026618 ± .000013	4.222725 ± .000052	1.266818 ± .000016
0.31	1.012944 ± .000013	3.459785 ± .000044	1.072533 ± .000014
0.32	.990308 ± .000013	2.802983 ± .000037	.896955 ± .000012
0.33	.962408 ± .000012	2.252973 ± .000029	.743481 ± .000010
0.34	.932344 ± .000013	1.801891 ± .000024	.612643 ± .000008
0.35	.902633 ± .000013	1.437716 ± .000020	.503200 ± .000007
0.36	.875332 ± .000013	1.147186 ± .000017	.412987 ± .000006
0.37	.852122 ± .000011	.917498 ± .000012	.339474 ± .000005
0.38	.834518 ± .000013	.737129 ± .000011	.280109 ± .000004
0.39	.823814 ± .000014	.596131 ± .000010	.232491 ± .000004
0.40	.821103 ± .000016	.486123 ± .000010	.194449 ± .000004
0.41	.827128 ± .000019	.400142 ± .000009	.164058 ± .000004
0.42	.842044 ± .000022	.332471 ± .000009	.139638 ± .000004
0.43	.865137 ± .000026	.278477 ± .000008	.119745 ± .000004
0.44	.894598 ± .000030	.234503 ± .000008	.103181 ± .000003
0.45	.927537 ± .000034	.197797 ± .000007	.089009 ± .000003
0.46	.960344 ± .000038	.166439 ± .000007	.076562 ± .000003
0.47	.989362 ± .000042	.139224 ± .000006	.065435 ± .000003
0.48	1.011648 ± .000045	.115484 ± .000005	.055432 ± .000003
0.49	1.025498 ± .000047	.094883 ± .000004	.046493 ± .000002
0.50	1.030575 ± .000049	.077220 ± .000004	.038610 ± .000002
0.51	1.027657 ± .000051	.062309 ± .000003	.031777 ± .000002
0.52	1.018228 ± .000053	.049918 ± .000003	.025958 ± .000001
0.53	1.004070 ± .000055	.039772 ± .000002	.021079 ± .000001
0.54	.986987 ± .000058	.031565 ± .000002	.017045 ± .000001
0.55	.968656 ± .000061	.024995 ± .000002	.013747 ± .000001
0.56	.950585 ± .000066	.019778 ± .000001	.011075 ± .000001
0.57	.934124 ± .000072	.015661 ± .000001	.008927 ± .000001
0.58	.920500 ± .000080	.012428 ± .000001	.007208 ± .000001
0.59	.910823 ± .000090	.009897 ± .000001	.005839 ± .000001

TABLE II - Continued

$x \equiv M\omega$	$S(x) \equiv \sigma/27\pi M^2$	$10^4 f(x) \equiv 10^4 \frac{dN_\gamma}{dtd\omega}$	$10^4 xf(x) \equiv 10^4 \frac{M\ dE}{dtd\omega}$
0.60	.906059 ± .000100	.007919 ± .000001	.004751 ± .000001
0.61	.906933 ± .000120	.006372 ± .000001	.003887 ± .000001
0.62	.913765 ± .000140	.005159 ± .000001	.003198 ± .000000
0.63	.926270 ± .000160	.004199 ± .000001	.002646 ± .000000
0.64	.943398 ± .000180	.003433 ± .000001	.002197 ± .000000
0.65	.963312 ± .000160	.002812 ± .000000	.001828 ± .000000
0.66	.983647 ± .000180	.002303 ± .000000	.001520 ± .000000
0.67	1.001912 ± .000190	.001880 ± .000000	.001260 ± .000000
0.68	1.016037 ± .000210	.001527 ± .000000	.001039 ± .000000
0.69	1.024740 ± .000220	.001234 ± .000000	.000851 ± .000000
0.70	1.027644 ± .000230	.000990 ± .000000	.000693 ± .000000
0.71	1.025145 ± .000230	.000790 ± .000000	.000561 ± .000000
0.72	1.018153 ± .000240	.000628 ± .000000	.000452 ± .000000
0.73	1.007829 ± .000250	.000497 ± .000000	.000363 ± .000000
0.74	.995399 ± .000260	.000392 ± .000000	.000290 ± .000000
0.75	.982044 ± .000270	.000309 ± .000000	.000232 ± .000000
0.76	.968866 ± .000290	.000244 ± .000000	.000185 ± .000000
0.77	.956886 ± .000310	.000192 ± .000000	.000148 ± .000000
0.78	.947051 ± .000340	.000152 ± .000000	.000118 ± .000000
0.79	.940236 ± .000380	.000120 ± .000000	.000095 ± .000000
0.80	.937188 ± .000430	.000096 ± .000000	.000076 ± .000000
0.81	.938445 ± .000500	.000076 ± .000000	.000062 ± .000000
0.82	.944190 ± .000570	.000061 ± .000000	.000050 ± .000000
0.83	.954091 ± .000660	.000049 ± .000000	.000041 ± .000000
0.84	.967201 ± .000750	.000040 ± .000000	.000033 ± .000000
0.85	.982008 ± .000850	.000032 ± .000000	.000027 ± .000000
0.86	.996648 ± .000940	.000026 ± .000000	.000022 ± .000000
0.87	1.009284 ± .001000	.000021 ± .000000	.000018 ± .000000
0.88	1.018489 ± .001100	.000017 ± .000000	.000015 ± .000000
0.89	1.023473 ± .001200	.000013 ± .000000	.000012 ± .000000

TABLE II - Continued

$x \equiv M\omega$	$S(x) \equiv \sigma/27\pi M^2$	$10^4 f(x) \equiv 10^4 \frac{dN_\gamma}{dt\,d\omega}$	$10^4 xf(x) \equiv 10^4 \frac{M\ dE}{dt\,d\omega}$
0.90	1.024105 ± .001200	.000011 ± .000000	.000010 ± .000000
0.91	1.020801 ± .001300	.000008 ± .000000	.000008 ± .000000
0.92	1.014308 ± .001300	.000007 ± .000000	.000006 ± .000000
0.93	1.005524 ± .001400	.000005 ± .000000	.000005 ± .000000
0.94	.995383 ± .001400	.000004 ± .000000	.000004 ± .000000
0.95	.984793 ± .001500	.000003 ± .000000	.000003 ± .000000
0.96	.974603 ± .001700	.000003 ± .000000	.000002 ± .000000
0.97	.965618 ± .001900	.000002 ± .000000	.000002 ± .000000
0.98	.958600 ± .002100	.000002 ± .000000	.000002 ± .000000
0.99	.954236 ± .002400	.000001 ± .000000	.000001 ± .000000
1.00	.953086 ± .002800	.000001 ± .000000	.000001 ± .000000

(c) Gamma Rays from Primordial Black Holes (Paper IV; collaboration with S. W. Hawking, to be published in _Ap. J._, May 15, 1976; copied by permission of S. W. Hawking and the University of Chicago press)

GAMMA RAYS FROM PRIMORDIAL BLACK HOLES*

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ABSTRACT

This paper examines the possibilities of detecting hard gamma rays produced by the quantum mechanical decay of small black holes created by inhomogeneities in the early universe. Observations of the isotropic gamma-ray background around 100 MeV place an upper limit of 10^4 pc^{-3} on the average number density of primordial black holes with initial masses around 10^{15} g. The local number density could be greater than this by a factor of up to 10^6 if the black holes were clustered in the halos of galaxies. The best prospect for detecting a primordial black hole seems to be to look for the burst of hard gamma rays that would be expected in the final stages of the evaporation of the black hole. Such observations would be a great confirmation of general relativity and quantum theory and would provide information about the early universe and about strong interaction physics.

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I. INTRODUCTION

The aim of this paper is to discuss the possibilities of detecting high-energy gamma rays produced by the quantum mechanical decay of small black holes created in the early universe. Recently it has been shown (Hawking 1974, 1975a,b; Wald 1975; Parker 1975; DeWitt 1975) that the strong gravitational fields around black holes cause particle creation and that the black holes emit all species of particles thermally with a temperature of about $1.2 \times 10^{26}\ M^{-1}$ °K where M is the mass in grams of the black hole. One can think of this emission as arising from the spontaneous creation of pairs of particles near the event horizon of the black hole. One particle, having a positive energy, can escape to infinity. The other particle has negative energy and has to tunnel through the horizon into the black hole where there are particle states with negative energy with respect to infinity. Equivalently, one can regard the particles as coming from the singularity inside the black hole and tunneling out through the event horizon to infinity (Hartle and Hawking 1975). As black holes emit particles, they lose mass and so will evaporate completely and disappear in a time of the order of $10^{-26}\ M^3$ sec (Page 1976). (For $M < 10^{14}$ g this lifetime may be shortened by strong interaction effects discussed in § III.)

It would be practically impossible to detect particle emission from black holes of stellar mass because the temperature would be less than 10^{-7} °K. One does not know of any process that could produce black holes in the present epoch with mass substantially less than a stellar mass and therefore with higher temperatures. However, one would expect that small black holes would have been created in the early universe if at these epochs the universe was chaotic or had a soft equation of state (Hawking 1971; Carr and Hawking 1974; Carr 1975a). Such black holes will be referred to as primordial. If their original mass was less than $M_* \simeq 5 \times 10^{14}$ g (Page 1976), they would have completely evaporated by now. Primordial black holes of slightly greater initial mass would by now have decayed to a mass of around 5×10^{14} g and would have a temperature of about 2.5×10^{11} °K = 20 MeV. Calculations by Page (1976) indicate that such a black hole would radiate energy at the rate of 2.5×10^{17} erg sec^{-1} of which 1 percent is in gravitons, 45 percent is in neutrinos, 45 percent is in electrons and positrons, and 9 percent is in photons. (At this temperature there will also be some emission of muons and pions which is not included in the energy rate above.) It would be very difficult to detect the gravitons or neutrinos because they have such small interaction cross sections. The charged particles would be deflected by magnetic fields and so would not propagate freely to the earth. On the other hand, the photons, whose number spectrum would be peaked at about 120 MeV, could reach us from anywhere in the observable universe. There are three possibilities for detecting these photons.

(1) One could look in the isotropic gamma-ray background for the integrated emission of all the primordial black holes in the universe. As shown in § II, a uniform distribution of primordial black holes would give a background number spectrum of gamma rays with a logarithmic slope of −3 above 120 MeV. Below 120 MeV the spectrum may be flatter depending on the slope of the number spectrum of black holes. Observations of background gamma rays (Fichtel et al. 1975) show no indication of a break in the spectrum at 120 MeV. This puts an upper limit of about 3×10^{-52} cm^{-3} or about 10^4 pc^{-3} on $dn/d\ell nM$ at M_* where n is the original number per comoving volume of primordial black holes with original masses less than M. [Similar upper limits have been placed by Chapline (1975) and Carr (1975b)]. The upper limit on the local number density might be increased by a factor of up to 10^6 if the

black holes were clustered in the halos of galaxies rather than being uniformly distributed throughout the universe.

(2) One might hope to detect the steady emission from a primordial black hole sufficiently near the earth. However, the upper limit from the gamma-ray background indicates that the nearest primordial black hole is probably not closer than 10^{15} cm, about one and a half times the distance to Pluto. To obtain a counting rate of one photon per thousand seconds would require a detector with directional resolution (to overcome background) and an effective area of at least 10^8 cm^2.

(3) As the black hole loses mass, its temperature will rise and the black hole will begin emitting particles of higher rest mass. In the statistical bootstrap (Hagedorn 1973, Frautschi 1971) or dual resonance models of strong interactions (Huang and Weinberg 1970) the number of species of particles rises exponentially with mass. This might cause a black hole to emit all its remaining mass in a very short time when it got down to a mass of about $M_H = 6.6 \times 10^{13}$ gm corresponding to the Hagedorn limiting temperature of about 160 MeV. The heavy hadrons emitted by the hole would decay rapidly and one might expect about 10-30 percent of their energy or about 10^{34} ergs to emerge as a short burst of hard gamma rays between 100 and 1000 MeV. These bursts cannot be connected with those reported by Klebesadel (1973) which were very soft ($\sim$ 150 keV). If the number density of primordial black holes were near the upper limit set by the gamma-ray background, one would expect one burst per month within a distance of 200 pc if the black holes were uniformly distributed or about 2 pc if they were clustered in the halos of galaxies. To detect such a burst one would need a detector with an effective area of greater than 4×10^5 cm^2 in the former case and 40 cm^2 in the latter. The

burst could be distinguished from the gamma-ray background by the arrival of several photons in a short period of time or from within a small solid angle. Even if black holes do not explode at M_H they will have a very rapid final burst of emission when their mass gets down to some value M_B between M_H and 10^{10} g. In this case one would expect the number of photons to be reduced by a factor q^{-2} and the energy of these photons to increase by a factor q where $q = M_H/M_B$. To observe such bursts would require detectors with areas q^2 times the areas given above.

A definite observation of gamma rays from a primordial black hole would be a tremendous vindication of general relativity and quantum theory and would give us important information about the early universe and strong interactions at high energies, information that probably could not be obtained in any other way. On the other hand, negative observations which placed a strong upper limit on the density of primordial black holes would also give us valuable information because they would indicate that the early universe was probably nearly homogeneous and isotropic with a hard equation of state. The best experimental prospect would seem to be to look for bursts using large-area wide-angle detectors with either good time or good angular resolution. Such detectors could be flown on constant-pressure balloons or on the space shuttle. If the particles and photons from the burst were of sufficiently high energy, it might be possible to detect them from the ground either by air showers or by Cerenkov radiation in the upper atmosphere.

In § II we compute the background gamma-ray spectrum that would be produced by a uniform distribution of primordial black holes with a power-law spectrum of masses. In § III we consider the final burst of emission on the basis of various theories of strong interactions. Where convenient, we use dimensionless units in which $G = c = \hbar = k = 1$.

II. THE GAMMA-RAY BACKGROUND

In this section we shall calculate the present number flux $dJ/d\omega_o$ of gamma rays of frequency ω_o from primordial black holes. (Henceforth we shall use the abbreviation pbh.) To do this, we must integrate the contributions over the cosmological time t and at each t integrate over all pbh masses M the emission at the blue-shifted angular frequency

$$\omega = (1 + Z)\omega_o = (R_o/R)\ \omega_o \quad . \tag{1}$$

R is the expansion parameter of the universe at time t, and the subscript o denotes the value of a quantity at the present epoch. The interactions of the gamma rays with the other matter of the universe will be taken into account by putting in a factor $e^{-\tau}$ for the probability of a photon's propagating without energy loss through absorption optical depth τ from t to t_o, but the effect of the absorbed radiation will not be considered.

Consider a uniform distribution of pbh's created shortly after $t = 0$ in a nearly Friedman universe. Let $n(M_i)$ denote the original number per comoving volume of pbh's with original masses less than M_i. One can express n as

$$n(M_i) \equiv \eta \int_0^{M_i/M_*} s(y)\ dy \ . \tag{2}$$

s is a dimensionless function with $s(1) = 1$ and M_* is the original mass of a pbh that would just have evaporated by the present time t_o. We shall assume that, apart from statistical fluctuations, the pbh's are uncharged and nonrotating. Any charge would be rapidly neutralized by the preferential emission of electrons or positrons (Carter 1974, Gibbons 1975). Pbh's would also lose angular momentum but only slowly. One would not expect them to be formed with large angular momenta.

A pbh will emit photons at a rate

$$f(x) \equiv \frac{dN_\gamma}{dt\ d\omega} = \frac{1}{2\pi} \sum_{\ell,m,p} \frac{\Gamma_{\ell mp}(x)}{e^{8\pi x} - 1} \quad . \tag{3}$$

Here $\Gamma_{\ell mp}(x)$ is the absorption probability for photons of total angular momentum ℓ, axial angular momentum m, polarization or helicity p, and frequency $\omega = M^{-1}x$ where $M = M(M_i, t)$ is the mass to which a pbh of original mass M_i has been reduced by time t.

In terms of these functions the specific number flux today of gamma rays from pbh's is

$$\frac{dJ}{d\omega_o} \equiv \frac{d(\text{number of photons})}{d(\text{area})d(\text{time})d(\text{solid angle})d\omega_o}$$

$$= \frac{\eta}{4\pi} \int_0^{t_o} dt(1+Z)\ e^{-\tau} \int dy\ s(y)\ f(x) \quad . \tag{4}$$

Here $y \equiv M_i/M_*$ is to be integrated over all values of the initial mass of pbh's that do not disappear by time t, and x is the value of $M\omega$ at that t and y.

To calculate $dJ/d\omega_o$, one needs a specific model for $M(M_i,t)$, $s(y)$, $R(t)$, and $\tau(\omega_o,t)$, as well as the numerical results for $f(x)$ (Page 1976). As long as a pbh emits predominantly a fixed number of particle species at ultrarelativistic energies (i.e., with negligible effects from the rest mass),

$$\frac{dM}{dt} \simeq -\frac{\alpha}{M^2} \quad . \tag{5}$$

Page (1976) showed that for an uncharged, nonrotating hole emitting only known particles, $\alpha = 2.011 \times 10^{-4}$ for $M \gg 10^{17}$ g (emitting massless particles only), and $\alpha = 3.6 \times 10^{-4}$ for 5×10^{14} g $\ll M \ll 10^{17}$ g (emitting predominantly massless particles and ultrarelativistic electrons and positrons).

This implies that

$$M_* = (3\alpha t_o)^{1/3} \simeq 2.1 \times 10^{19} = 5 \times 10^{14} \text{ g} . \qquad (6)$$

Since the important part of the spectrum comes from $M \simeq M_*$, and since α is not known for $M \lesssim M_*$ anyway, we shall take $\alpha = 3.6 \times 10^{-4}$. Then our model for the mass evolution is

$$M = (M_i^3 - 3\alpha t)^{1/3} = M_*(y^3 - t/t_o)^{1/3} \quad . \qquad (7)$$

If we use this expression to solve for y in terms of x at some t, we find that

$$\frac{dJ}{d\omega_o} = \frac{\eta}{4\pi M_*^3}\, \omega_o^{-3} \int_0^{t_o} dt\, r^2\, e^{-\tau} \int_0^{\infty} dx\, x^2 f(x) \left(\frac{t}{t_o} + \frac{r^3 x^3}{M_*^3\, \omega_o^3}\right)^{-2/3} s\left[\left(\frac{t}{t_o} + \frac{r^3 x^3}{M_*^3\, \omega_o^3}\right)^{1/3}\right] \qquad (8)$$

where we have introduced $r \equiv R/R_o = (1+Z)^{-1}$, a function of t.

One can see from this formula that if $\omega_o \gg (t_o/t)^{1/3}\, rx/M_*$ and if $e^{-\tau}$ is insensitive to ω_o over the dominant part of the integral (generally $t \sim t_o$, $r \sim 1$, $x \sim 0.2$, and $e^{-\tau} \sim 1$, so $\omega_o \gg 0.2\, M_*^{-1} \simeq 120$ MeV), then the integral has no dependence upon ω_o and $dJ/d\omega_o$ is proportional to ω_o^{-3}, independent of the form of $s(y)$, $R(t)$, and $\tau(\omega_o,t)$ except for pathological cases. For small values of ω_o the integral is cut off by redshift and opacity factors. This means that the function $s(y)$, which determines the shape of the initial number spectrum of pbh's, is important only in the region near $y = 1$. We shall assume that in this region it has a power-law form:

$$s(y) = y^{-\beta} \qquad (9)$$

Such a form for $s(y)$ is supported by the work of Carr (1975a) who finds that a certain reasonable class of density fluctuations in the early universe favors pbh formation with a power-law spectrum where the exponent β is related to the ratio γ of pressure to energy density in the early universe by

$$\beta = \frac{2+4\gamma}{1+\gamma} \quad . \tag{10}$$

(A very soft equation of state with $\gamma = 0$ gives $\beta = 2$; a non-interacting relativistic gas with $\gamma = 1/3$ gives $\beta = 2.5$; and a stiff equation of state with $\gamma = 1$ gives $\beta = 3$.)

As a model for $R(t)$, we shall take a standard Friedman model with non-interacting dust and radiation obeying Einstein's field equations with cosmological constant $\Lambda = 0$. Such a model may be labeled by the Hubble constant H_o to set the scale and by two dimensionless parameters to determine the matter and radiation content:

$$\Omega_m = \frac{8\pi\rho_{matter}}{3H_o^2} \quad , \quad \Omega_r = \frac{8\pi\rho_{radiation}}{3H_o^2} \quad , \tag{11}$$

where the densities are measured at the present epoch. We take H_o to be 60 km s^{-1} Mpc^{-1}. We take Ω_r to be 0.0001 on the basis of observations of the microwave background and the assumption that nondegenerate electron and muon neutrinos were in thermal equilibrium with photons in the early stages of the universe. The value of Ω_m is not well known. A lower limit seems to be 0.0013, and Gott et al. (1974) suggest that 0.06 is the most probable value but the observations do not completely rule out $\Omega_m \geq 1$. In fact the value of Ω_m makes very little difference to the predicted gamma-ray spectrum except below about 10 MeV where it is strongly influenced by the opacity in the universe at redshifts $Z \gtrsim 100$. This opacity arises mainly from pair production caused by high-energy gamma rays striking neutral hydrogen or helium atoms. We have used the cross sections derived by Bethe and Heitler (1934) with corrections by Wheeler and Lamb (1939) for hydrogen and by Knasel (1968) for helium. They find that the total cross section for pair production in hydrogen is 0.0124 cm^2/g and in helium is 0.0083 cm^2/g independent of the gamma-ray energy provided it is above about 100 MeV. A primordial abundance of 70 percent hydrogen and 30 percent helium by mass was assumed (cf. Danziger 1970), making the opacity 0.0112 cm^2/g at high energies, and a crude correction for lower energies was made.

Figure 1 shows the predicted background gamma-ray spectrum from primordial black holes for $\Omega_r = 0.0001$ and $\Omega_m = 0.06$ and for various values of the exponent β in the initial number spectrum of the primordial black holes. Figure 2 shows the spectra with $\Omega_m = 1$ (approximately a $k = 0$ cosmology). As expected, the curves all agree more or less above 120 MeV and have a logarithmic slope of -3. Below 120 MeV the curves differ for different values of β but they all flatten and turn over at about 10 MeV. All the curves can be moved up or down by adopting different values of the constant η that multiplies the factor $s(y)$ in the initial pbh number spectrum $M_* dn/dM$. In figures 1 and 2 the value of η was chosen as 1×10^4 pc^{-3} to be consistent with the upper limit set by the observations which are shown in figure 3. These seem to fit roughly a power-law spectrum with exponent -2.4 from about 0.3 MeV to 200 MeV. There is no evidence of a break in the spectrum at 120 MeV but the observations in this region, which were by Fichtel et al. (1975), were statistical in nature and were fitted to an assumed power-law spectrum. Nevertheless, it is clear that $dn/d\ell nM$ at $M = M_*$ cannot be greater than 10^4 pc^{-3} and that this is only an upper limit.

The considerations above have been based on the assumption of a uniform distribution of primordial black holes throughout the universe. The observed matter in the universe, however, is strongly concentrated in galaxies and possibly in halos. Any initial velocity with which primordial black holes were created would have been reduced almost to zero by the expansion of the universe. Thus one might expect that they would be concentrated in the gravitational potential wells of galaxies. Unlike the gas, they would encounter very little friction in passing through the plane of the galaxy and so they would be distributed throughout the halo. If we assume that the primordial black holes are concentrated in halos of the order of 40 kpc around each galaxy with an r^{-2} density distribution instead of being spread uniformly, the upper limit on the number density of pbh's averaged over the whole universe would be about the same but the local density would be about a factor of 10^6 greater.

III. BURSTS

In the calculations of Page (1976) it was assumed that the emitted particles interacted only with the gravitational field and not with each other. This should be a good approximation for the emission of gravitons, photons, and leptons. It will break down when the mass of the black hole falls below about 2×10^{14} g corresponding to a temperature of about 50 MeV at which pions, the lightest hadrons, would begin to be emitted in significant numbers. Although the present field theory derivations of particle creation by black holes break down when strong interactions become important, there are thermodynamic and statistical arguments which indicate that a black hole would still emit thermal radiation with a temperature related to the mass in the same way as before (Hawking 1975c). The problem is to calculate what thermal radiation consists of in the presence of strong interactions and how it decays as it moves away from the black hole. At present there is little experimental knowledge on either of these questions so one has to resort to theoretical models. Possibly the simplest of such models is the statistical bootstrap theory (Hagedorn 1965; Frautschi 1971; Hagedorn 1973). In this approach one considers the eigenstates of the strongly-interacting fields contained within some box of volume V. Let $\sigma(E,V)$ be the number of eigenstates with energy between E and $E + dE$. One can define a quantity $\rho(m,V)$ such that $\sigma(E,V)$ is equal to the number of eigenstates of a system of non-interacting particles with mass spectrum $\rho(m,V)$ and total energy between E and $E + dE$ contained in a box of volume V. One regards $\rho(m,V)$ as representing the spectrum of resonances in the strongly interacting fields. If one neglects long-range gravitational and electromagnetic fields, one might expect $\rho(m,V)$ would be independent of V for V greater than a hadron volume $V_h \simeq 10^{-39}$ cm^3 because the strong interactions have a range of order 10^{-13} cm. One then makes the bootstrap assumption that the density of energy levels in a volume V_h is just given by this mass spectrum, i.e.,

$$\rho(E) = \sigma(E,V_h) \quad . \tag{12}$$

This gives an effective mass spectrum of the form

$$\rho(m) = am^{-b} \exp(m/c) \tag{13}$$

where $5/2 \leq b \leq 7/2$ and $c \simeq 160$ MeV.

Similar mass spectra are obtained from dual-resonance models of strong interactions (Fubini and Veneziano 1969; Huang and Weinberg 1970).

If one regards this mass spectrum as representing different species of non-interacting particles all of which a black hole would emit thermally like point particles, the rate of energy emission would become infinite when the black hole got down to a mass M_H of about 7×10^{13} g corresponding to a temperature of about 160 MeV because the Boltzmann factor $\exp(-E/T)$ would be cancelled out by the exponential in the mass spectrum. The black hole would convert itself into a fireball of very heavy hadrons. In the conventional statistical bootstrap theory, which neglects gravitational interactions, these heavy hadrons would decay slowly with lifetimes of the order of 10^{13} sec (Carlitz, Frautschi, and Nahm 1973). However, gravitational interactions between the hadrons would be significant compared to thermal energies for particle masses above 10^{-5} g. They would increase the rate of collisions between such heavy hadrons and hence, by detailed balance, the rate at which they can emit lighter hadrons and decay. Thus the fireball could probably be treated as a pressureless fluid which maintained itself in thermal equilibrium at a temperature of about 160 MeV as it expanded with parabolic velocity (cf. Carter et al. 1975). One would expect the fireball to radiate electrons, positrons, muons, photons, and perhaps neutrinos thermally from its surface. It would radiate away all its energy in a time of about 10^{-7} sec giving a burst of gamma rays peaked around 250 MeV with total energy about 10^{34} ergs.

This picture can be criticized on the ground that, even if there were in some sense an exponential mass spectrum of hadrons, they would be of the same size as the black hole or larger and thus would not be emitted as point particles. One might regard hadrons as composite bodies made up from quarks and gluons which are point particles and which are asymptotically free at small distances but are strongly bound at separations greater than 10^{-13} cm (Gross and Wilczek 1973; Politzer 1973,1974). In this case it might be that black holes smaller than 10^{-14} cm or 10^{14} g would emit individual quarks and gluons as non-interacting point particles. When they had traveled a distance of about 10^{-13} cm from the black hole, they would feel the interaction with other quarks and gluons and would join up with them to form hadrons which would then decay into lighter particles. The rate of energy emission would be about $10^{46}\ \eta(M/g)^{-2}$ ergs/sec where η is number of species of quarks, gluons, leptons, photons and gravitons with rest mass less than the black-hole temperature. In the original quark-gluon theory (Fritzch and Gell-Mann 1972) there were 18 species of quarks (three flavors, three colors and their antiparticles) and 8 species of gluons. Thus η would be 36. About 1.5 percent of the rest-mass energy of the black hole would be emitted directly in high-energy photons and further photons would arise from the decay of highly-relativistic hadrons. One might therefore expect that between 10 and 30 percent of the rest-mass energy of the black hole would emerge as photons at around 500 $(M/10^{14}\ g)^{-1}$ MeV. The emission would become very rapid when the mass of the black hole got down to 10^{10} g giving a burst of about 10^{30} photons at around 5×10^{6} MeV.

The recent discovery of the J or Ψ particles (Aubert et al. 1974; Augustin et al. 1974) suggests that there may be a fourth flavor of quark with a rather higher mass. It also seems that it may be necessary to postulate a fifth and a sixth flavor to explain the electron-positron annihilation cross section into hadrons. It is therefore possible that there is an infinite sequence of quarks with higher and higher masses. These higher mass quarks would increase the rate of energy loss of a black hole hot enough to emit them. The final burst of very rapid emission could therefore come at some mass between the Hagedorn mass M_H and 10^{10} g. In this case one would expect the number of photons to be q^{-2} times the number in the statistical bootstrap picture and the energy of each photon to be q times greater, where q is the ratio of the Hagedorn mass to the mass at which the burst occurs.

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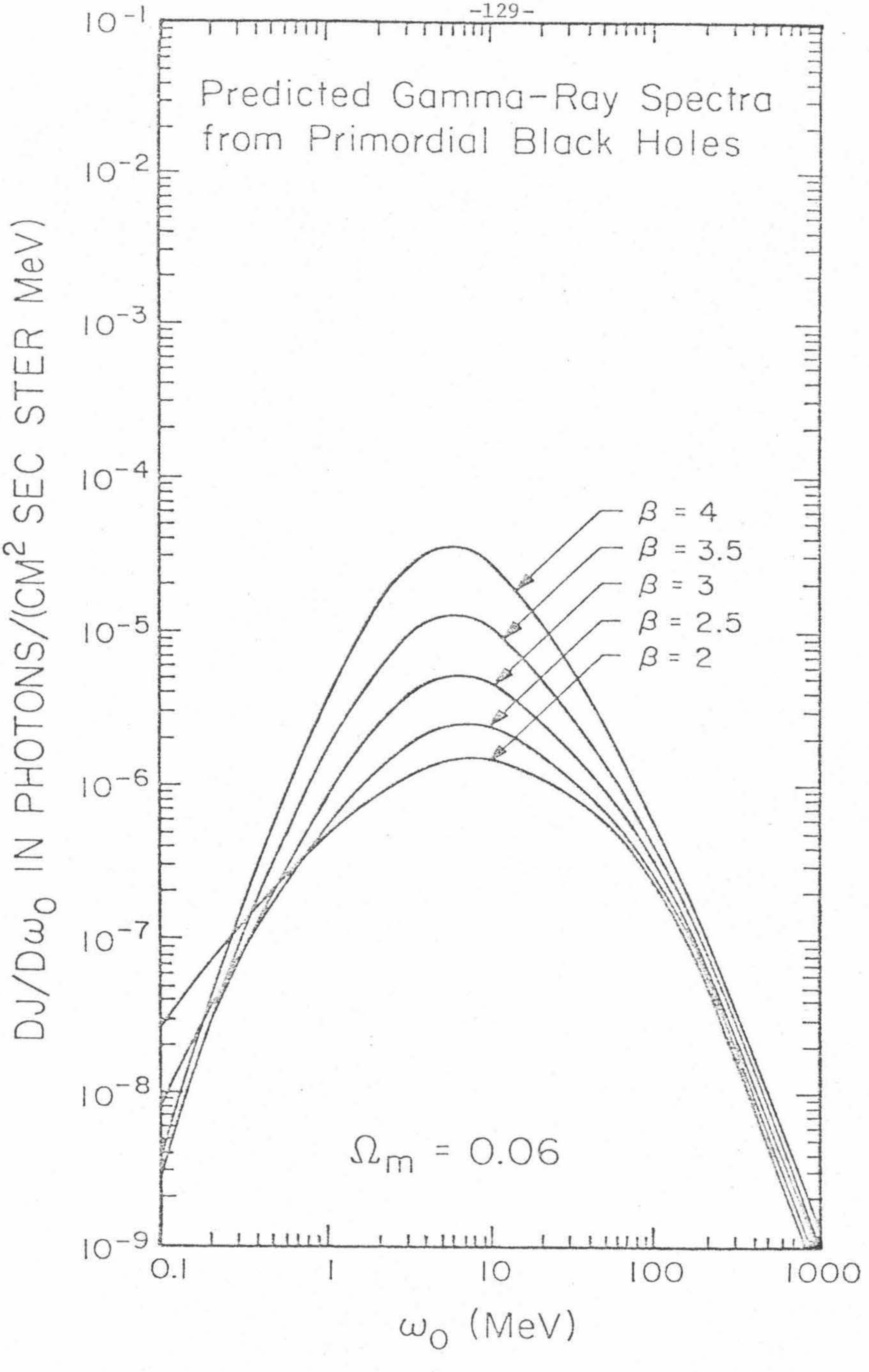
FIGURE CAPTIONS

Figs. 1 and 2. Predicted number spectra $dJ/d\omega_o$ of gamma rays from primordial black holes having initial number spectra $dn/d(M_i/M_*) = 10^4\ pc^{-3}(M_i/M_*)^{-\beta}$ for initial masses M_i around $M_* \simeq 5 \times 10^{14}$ g, where β is given values from 2 (bottom curve) to 4 (top curve) in steps of 1/2. Fig. 1 assumes the present matter density is 0.06 of the critical value for closure of the universe; Fig. 2 assumes it is at the critical value.

Fig. 3. Observed diffuse gamma ray spectrum as reported in Fichtel et al. (1975). The shaded region represents their SAS-2 measurements and uncertainties; the other points represent previous measurements they enumerate and reference.



Predicted Gamma-Ray Spectra from Primordial Black Holes
DJ/Dω0 IN PHOTONS/(CM2 SEC STER MeV)
10-1
10-2
10-3
10-4
10-5
10-6
10-7
10-8
10-9
β = 4
β = 3.5
β = 3
β = 2.5
β = 2
Ωm = 0.06
0.1
1
10
100
1000
ω0 (MeV)

Fig. 1

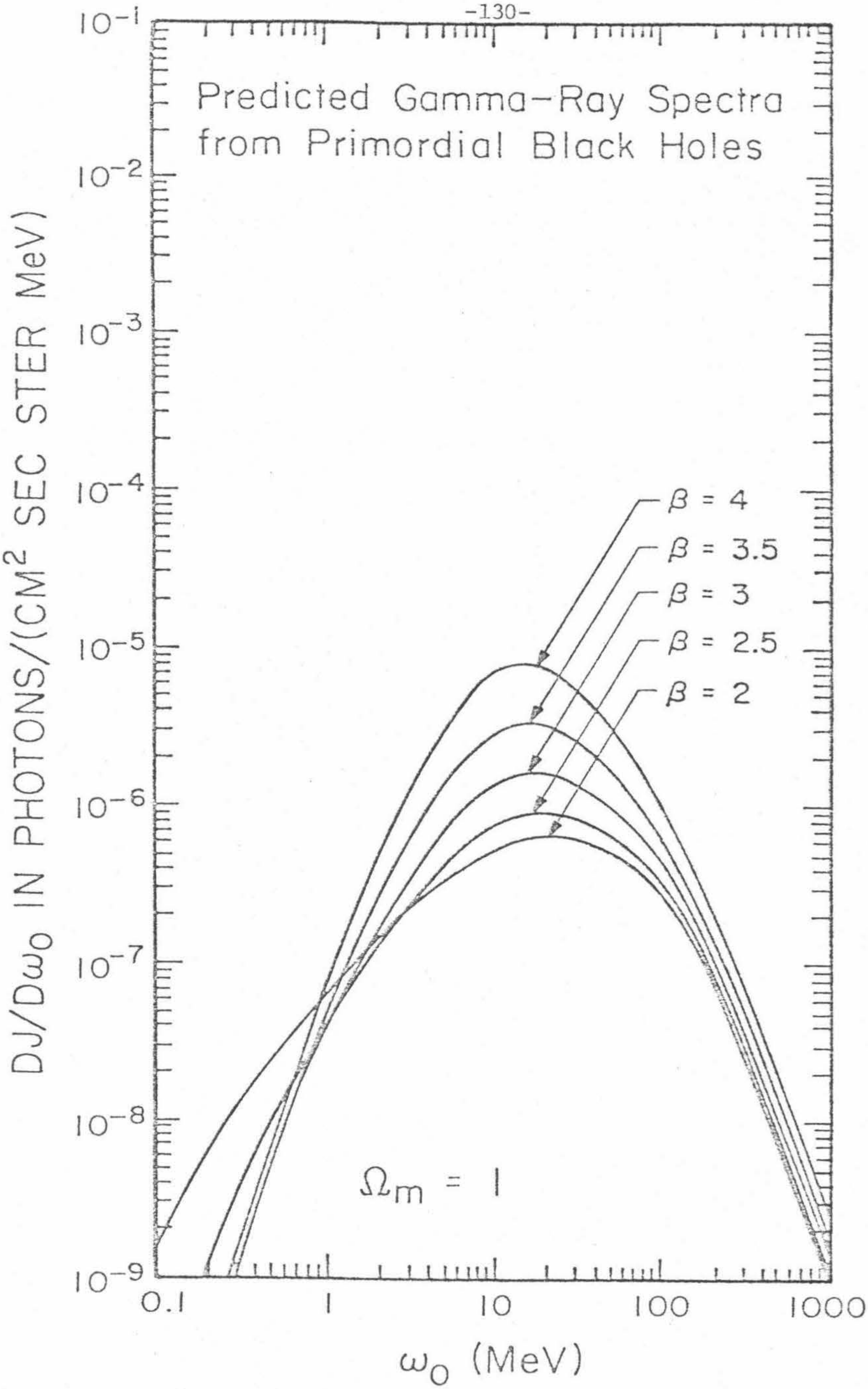

Predicted Gamma-Ray Spectra
from Primordial Black Holes
DJ/Dω₀ IN PHOTONS/(CM² SEC STER MeV)
10⁻¹
10⁻²
10⁻³
10⁻⁴
10⁻⁵
10⁻⁶
10⁻⁷
10⁻⁸
10⁻⁹
β = 4
β = 3.5
β = 3
β = 2.5
β = 2
Ωm = 1
0.1
1
10
100
1000
ω₀ (MeV)

Fig. 2

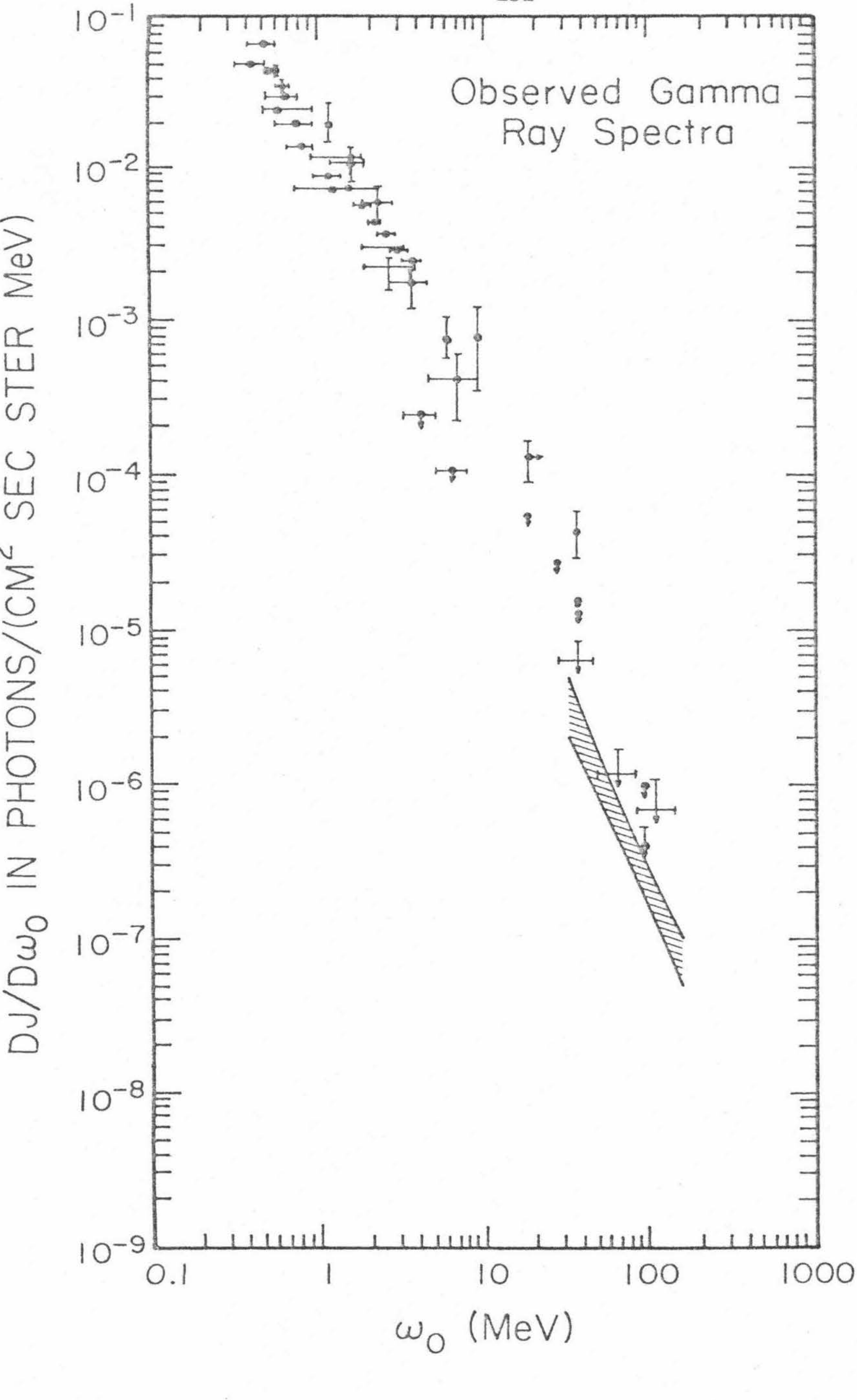

Observed Gamma
Ray Spectra
DJ/Dω₀ IN PHOTONS/(CM² SEC STER MeV)
10⁻¹
10⁻²
10⁻³
10⁻⁴
10⁻⁵
10⁻⁶
10⁻⁷
10⁻⁸
10⁻⁹
0.1
1
10
100
1000
ω₀ (MeV)

Fig. 3