# GROUPS OF AUTOMORPHISMS OF OPERATOR ALGEBRAS

By

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## PREFACE

This thesis is submitted in accordance with the regulations or the degree of Doctor of Philosophy in the University of Newcastle upon Tyne. No part of it has been previously submitted by the author for a university degree. The contents are believed to be original except where otherwise indicated, the main exception being Chapter I which is introductory.

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## ABSTRACT

An important part of the theory of locally compact groups is the study of their unitary representations. In this thesis, we study the representation of such groups, and more general topological groups, as groups of automorphisms of C*-algebras. Certain types of continuity and measurability of such representations (which we call automorphic representations) are defined and shown to be equivalent in certain cases. We consider a continuous representation, $\alpha$, of an abelian connected topological group G as a group of automorphisms of a C*-algebra $\mathfrak{A}$ acting on a Hilbert space $\mathcal{H}$. The topology on $\alpha(G)$ is that derived as a subset of the Banach space of bounded operators on $\mathfrak{A}$. Such a representation is shown to be equivalent to a norm continuous unitary representation $g \to U_g$ of G by unitaries $U_g$ in the weak operator closure of $\mathfrak{A}$, such that $\alpha(g)(A) = U_g A U_g^*$ $(g \in G,\ A \in \mathfrak{A})$. In the case of a locally compact group G and a weaker continuity condition on the representation $\alpha$, we obtain (when $\mathfrak{A}$ is a factor or a separable simple C*-algebra with unit) a necessary and sufficient condition that there exist a strongly continuous unitary representation $g \to U_g$ of G by unitaries $U_g \in \mathfrak{A}$ such that $\alpha(g)(A) = U_g A U_g^*$ $(A \in \mathfrak{A},\ g \in G)$.

If G is a group of automorphisms of a von Neumann algebra an equivalence relation can be defined, in terms of G, on the projections in $\mathfrak{R}$, which extends the usual definition of equivalence of projections. We show that certain results concerning the type of the tensor product of von Neumann algebras carry over to this more general situation.

Ergodic theory is essentially the study of groups of transformations of a measure space $(X, \mu)$. If $X$ is a locally compact space, $L^{\infty}(X, \mu)$ is an abelian von Neumann algebra. We prove that certain results concerning the existence of an equivalent measure on $X$ invariant under the transformation group carry over to the case of an amenable group $G$ of automorphisms of a general von Neumann algebra $\mathfrak{R}$. This gives a necessary and sufficient condition for the existence of a faithful normal state on $\mathfrak{R}$ invariant under $G$. We also show that a link exists between normal extremal $G$-invariant states and the ergodic action of $G$ on subalgebras of $\mathfrak{R}$ ($G$ acts ergodically if $0$ and $I$ are the only invariant projections).

# CONTENTS

# CHAPTER I

## INTRODUCTION

In this Chapter, we introduce those definitions and results required for the understanding of the work which follows. We assume that the reader is familiar with the concepts of functional analysis, elementary topology, and measure theory. Three basic texts for reference to the subject of operator algebras are the two books of Dixmier ([7], [8]) and Sakai's book ([46]). Throughout, we assume that all our linear spaces are defined over the field of complex numbers, denoted $\mathbb{C}$. $\mathbb{R}$ will denote the set of real numbers, $\mathbb{T}$ the unit circle (i.e. those elements $z \in \mathbb{C}$ such that $|z| = 1$) and $\mathbb{Z}$ the set of integers.

<u>C*-algebras</u> A linear associative algebra which is also a normed linear space relative to the norm $\|.\|$ is said to be a <u>Banach algebra</u> if

$$\text{(i)} \quad \|AB\| \leq \|A\| \, \|B\| \qquad (A,B \in \mathfrak{A}) .$$

(ii) $\mathfrak{A}$ is complete relative to the norm topology.

If $\mathfrak{A}$ has a unit, we denote it by $I_{\mathfrak{A}}$, or simply I if no confusion arises. We assume always that $\|I\| = 1$. This assumption involves no essential restriction. $\mathcal{Z}(\mathfrak{A})$ will denote the centre of $\mathfrak{A}$.

Throughout this thesis, if $\mathfrak{B}$ is a normed linear space, $\mathfrak{B}_1$ will denote the unit ball of $\mathfrak{B}$ i.e.

$$\mathfrak{B}_1 = \{x \in \mathfrak{B} ; \|x\| \leq 1\} .$$

Let $^*$ be a map from $\mathfrak{A}$ to $\mathfrak{A}$. $^*$ is called an <u>involution</u> if the following conditions are satisfied

$$\text{(i)}\quad (\lambda A + \mu B)^* = \bar{\lambda} A^* + \bar{\mu} B^* \qquad (\lambda, \mu \in \quad , \quad A, B \in \mathfrak{A})$$

$$\text{(ii)}\quad (AB)^* = B^* A^* \qquad (A, B \in \mathfrak{A})$$

$$\text{(iii)}\quad A^{**} = A \qquad (A \in \mathfrak{A}) .$$

If $\mathfrak{A}$ is an algebra with involution *, $\mathfrak{A}$ is called a *-algebra.

A Banach *-algebra is a Banach algebra with involution. A C*-algebra is a Banach *-algebra such that

$$\|A^*A\| = \|A\|^2 \qquad (A \in \mathfrak{A})$$

If $\mathfrak{A}$ is a C*-algebra, and $\mathfrak{J}$ is a closed two-sided ideal of $\mathfrak{A}$, then $\mathfrak{A}/\mathfrak{J}$ is also a C*-algebra with the involution

$$A + \mathfrak{J} \to A^* + \mathfrak{J} .$$

([8], 1.8.2).

Suppose $\mathfrak{A}$ is a C*-algebra, and $\mathfrak{B} \subset \mathfrak{A}$. $\mathfrak{B}$ is said to be selfadjoint if

$$\mathfrak{B} = \{A^*; A \in \mathfrak{B}\} = \mathfrak{B}^* .$$

If $\mathfrak{B}$ is also a subalgebra of $\mathfrak{A}$, $\mathfrak{B}$ is called a *-subalgebra of $\mathfrak{A}$. If $\mathfrak{B}$ is closed, $\mathfrak{B}$ is also a C*-algebra, called a C*-subalgebra of $\mathfrak{A}$.

Let $\mathfrak{A}$ have unit I. For $A \in \mathfrak{A}$, we define the spectrum of A, denoted $\sigma_{\mathfrak{A}}(A)$, to be the set of $\lambda \in C$ such that $A - \lambda I$ is singular. $\sigma_{\mathfrak{A}}(A)$ is a non-void compact subset of C, and if $\mathfrak{B}$ is a C*-subalgebra of $\mathfrak{A}$ containing I, and A, then

$$\sigma_{\mathfrak{B}}(A) = \sigma_{\mathfrak{A}}(A) .$$

by ([8], 1.3.10). Thus we may write the spectrum of A as $\sigma(A)$ without ambiguity as to the containing C*-algebra. An element A of $\mathfrak{A}$ is said to be

(a) <u>normal</u> if $AA^* = A^*A$

(b) <u>unitary</u> if $A^*A = AA^* = I$

(c) <u>self-adjoint</u> if $A = A^*$

(d) <u>positive</u> if $A = A^*$ and $\sigma(A) \subseteq R^+$ (the set of non-negative real numbers)

Let $\mathfrak{A}^+$ denote the set of positive elements of $\mathfrak{A}$ and $\mathfrak{A}^{s.a.}$ the set of self-adjoint elements of $\mathfrak{A}$. If $A \in \mathfrak{A}^{s.a.}$ then

$$\sigma(A) \subseteq R.$$

(by [8], 1.3.9). $\mathfrak{A}^+$ is a closed convex cone in $\mathfrak{A}^{s.a.}$ ([46], 1.4.2) thus we can define a partial ordering on $\mathfrak{A}^{s.a.}$ by saying

$$A \geq B \qquad \text{if } A - B \in \mathfrak{A}^+ .$$

Let $\mathfrak{A}^*$ be the Banach dual of $\mathfrak{A}$ i.e. $\mathfrak{A}^*$ is the set of all continuous linear functionals on $\mathfrak{A}$. If $f \in \mathfrak{A}^*$, f is said to be <u>positive</u>, written $f \geq 0$, if

$$f(A) \geq 0 \qquad (A \in \mathfrak{A}^+) .$$

f is a <u>state</u> of $\mathfrak{A}$ if also $f(I) = 1$. Each element of $\mathfrak{A}^*$ can be expressed as a linear combination of states of $\mathfrak{A}$ ([8], 2.6.4 and 1.1.10). $(\mathfrak{A}^*)_1$ is compact in the weak* (i.e. $\sigma(\mathfrak{A}^*,\mathfrak{A})$) topology and the set $E(\mathfrak{A})$ of all states of $\mathfrak{A}$ is a weak* closed convex subset of $(\mathfrak{A}^*)_1$, so $E(\mathfrak{A})$ is weak* compact. ([8], 2.5.5). By the Krein-Mil'man theorem, $E(\mathfrak{A})$ is the weak*-closed convex hull of its extreme points. Such extreme points are called <u>pure states</u>.

If $\mathfrak{A}$ and $\mathfrak{B}$ are C*-algebras, a <u>homomorphism</u> $\pi$ from $\mathfrak{A}$ to $\mathfrak{B}$ is a linear map from $\mathfrak{A}$ to $\mathfrak{B}$ such that

$$\text{(i)} \quad \pi(AB) = \pi(A)\,\pi(B) \qquad (A,B \in \mathfrak{A})$$

$$\text{(ii)} \quad \pi(A^*) = \pi(A)^* \qquad (A \in \mathfrak{A}) .$$

$\pi$ is an <u>isomorphism</u> if Ker $\pi = \{0\}$. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ the C*-algebra of all bounded operators, with the involution determined by

$$\langle Tx, y\rangle = \langle x, T^*y\rangle$$

$$(T \in \mathcal{B}(\mathcal{H}), \quad x,y \in \mathcal{H}) .$$

A homomorphism $\pi: \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ is called a <u>representation</u> of $\mathfrak{A}$ on $\mathcal{H}$. If $\pi$ is an isomorphism we say $\pi$ is a <u>faithful</u> representation. If $\mathcal{E} \subset \mathcal{B}(\mathcal{H})$, $\mathcal{F} \subset \mathcal{H}$, let

$$[\mathcal{E}\mathcal{F}]$$

denote the closed linear subspace of $\mathcal{H}$ generated by

$$\{Ex;\ E \in \mathcal{E},\ x \in \mathcal{F}\} .$$

Let it also denote the projection onto this subspace. Suppose there is an $x \in \mathcal{H}$ such that

$$\mathcal{H} = [\pi(\mathfrak{A})x]$$

then $\pi$ is said to be a <u>cyclic</u> <u>representation</u> and $x$ is a <u>cyclic</u> <u>vector</u> for the representation $\pi$. If $\mathcal{B}$ is a subset of $\mathcal{B}(\mathcal{H})$, let

$$\mathcal{B}' = \{A \in \mathcal{B}(\mathcal{H});\ AB = BA \quad (B \in \mathcal{B})\}$$

$\mathcal{B}'$ is called the <u>commutant</u> of $\mathcal{B}$. If $\mathcal{F} \subset \mathcal{H}$ is such that

$$[\mathcal{B}\mathcal{F}] = \mathcal{H}$$

$\mathcal{F}$ is called a <u>generating</u> <u>set</u> for $\mathcal{B}$. If $\{x\}$ is generating for $\mathcal{B}$, $x$ is called a <u>cyclic</u> or <u>generating</u> vector.

$\mathcal{F} \subset \mathcal{H}$ is said to be a <u>separating</u> <u>set</u> for $\mathcal{B}$ if $Bx = 0$ $(x \in \mathcal{F})$

and $B \in \mathcal{B}$ imply $B = 0$.

Let $\mathcal{B}$ be a *-subalgebra of $\mathcal{B}(\mathcal{H})$, then $\mathcal{F}$ is generating for $\mathcal{B}$ if and only if $\mathcal{F}$ is separating for $\mathcal{B}'$ ([14], Theorem 3, p.27)

If $\{x\}$ is separating for $\mathcal{B}$, $x$ is said to be a separating vector, and if $x$ is at the same time both a separating and generating vector, we say $x$ is a separating-generating vector. $\mathcal{B}$ is said to be countably decomposable if there is a countable subset $\mathcal{F}$ of $\mathcal{H}$ which is separating for $\mathcal{B}$.

For $x, y \in \mathcal{H}$, let $\omega_{x,y}$ denote the linear functional

$$T \to \langle Tx, y\rangle$$

on $\mathcal{B}(\mathcal{H})$, and write $\omega_x$ in place of $\omega_{x,x}$. If $\mathfrak{A}$ is a C*-algebra with unit, and $f \in E(\mathfrak{A})$, there is a cyclic representation $\pi_f$ of $\mathfrak{A}$, with cyclic vector $x_f$, on a Hilbert space $\mathcal{H}_f$, such that

$$f = \omega_{x_f} \circ \pi_f$$

Conversely, every cyclic representation of $\mathfrak{A}$ arises in this manner. ([8], 2.4.4 and 2.4.1). Suppose $\{\pi_\alpha\}_{\alpha \in A}$ is a family of representations of $\mathfrak{A}$ on the Hilbert spaces $\{\mathcal{H}_\alpha\}_{\alpha \in A}$. The direct sum representation, $\Sigma^\oplus \pi_\alpha$, of $\mathfrak{A}$ on $\Sigma^\oplus \mathcal{H}_\alpha$ is defined as

$$(\Sigma^\oplus \pi_\alpha)(A) = \Sigma^\oplus \pi_\alpha(A) \qquad (A \in \mathfrak{A})$$

Every non trivial representation of $\mathfrak{A}$ is in this sense a direct sum of cyclic representations ([8], 2.2.7). A representation $\pi$ of $\mathfrak{A}$ on $\mathcal{H}$ is said to be irreducible if

$$\pi(\mathfrak{A})' = \mathbb{C} I_{\mathcal{B}(\mathcal{H})}$$

This is equivalent to saying that there are no closed subspaces

F of $\mathcal{H}$ such that

$$\pi(\mathfrak{A})F \subseteq F \ .$$

([8], 2.3.1). Let $\pi_1$ and $\pi_2$ be irreducible representations of $\mathfrak{A}$ on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. $\pi_1$ is equivalent to $\pi_2$ if there is an isometric linear map U from $\mathcal{H}_1$ onto $\mathcal{H}_2$ such that

$$U\pi_1(A)U^{-1} = \pi_2(A) \qquad (A \in \mathfrak{A}) \ .$$

If f is a state of $\mathfrak{A}$ and $\pi_f$ the corresponding cyclic representation, then f is a pure state if and only if $\pi_f$ is irreducible. ([8], 2.5.4). Moreover, any irreducible representation of $\mathfrak{A}$ is equivalent to one obtained in this manner. ([8], 2.4.6, 2.5.4). The universal representation, $\pi$, of $\mathfrak{A}$ is defined as

$$\Sigma^{\oplus}\{\pi_f ; f \in E(\mathfrak{A})\} \ .$$

The Gelfand-Naimark theorem states that $\pi$ is a faithful representation of $\mathfrak{A}$ as a norm closed C*-subalgebra of $\mathcal{B}(\mathcal{H})$ where

$$\mathcal{H} = \Sigma^{\oplus}\{\mathcal{H}_f ; f \in E(\mathfrak{A})\}$$

([46], 1.16.6). The C*-algebra $\mathfrak{A}$ is simple if $\mathfrak{A}$ contains no proper closed two-sided ideals. In this case all non trivial representations of $\mathfrak{A}$ are faithful.

For a C*-algebra $\mathfrak{A}$ with unit, let $\mathcal{U}(\mathfrak{A})$ denote the group of unitary elements of $\mathfrak{A}$. An automorphism, $\gamma$, of $\mathfrak{A}$ is an isomorphism from $\mathfrak{A}$ to $\mathfrak{A}$.

WARNING: Note that

$$\gamma(A^*) = \gamma(A)^* \qquad (A \in \mathfrak{A}) \qquad (\dagger)$$

Such maps are usually referred to as *-automorphisms. However, we shall have no occasion to consider automorphisms $\gamma$ not satisfying (†).

Let aut($\mathfrak{A}$) be the group of all automorphisms of $\mathfrak{A}$ under composition of maps. Since every isomorphism between C*-algebras is isometric ([8], 1.3.7), each element of aut($\mathfrak{A}$) is isometric, thus

$$\mathrm{aut}(\mathfrak{A}) \subseteq \mathcal{B}(\mathfrak{A})$$

($\mathcal{B}(\mathfrak{A})$ = set of all bounded operators on the Banach space $\mathfrak{A}$.) aut($\mathfrak{A}$) has identity element the automorphism

$$\iota : A \to A \qquad (A \in \mathfrak{A}) .$$

If $U \in \mathcal{U}(\mathcal{B}(\mathcal{H}))$, where $\mathcal{H}$ is a Hilbert space, and $\mathfrak{A}$ is a C*-subalgebra of $\mathcal{B}(\mathcal{H})$ such that $A \in \mathfrak{A}$ implies $UAU^* \in \mathfrak{A}$ then the map

$$\gamma : A \to UAU^*$$

is an automorphism of $\mathfrak{A}$, denoted ad $U|_{\mathfrak{A}}$ (or simply ad U if no confusion arises). In this case we say that $\gamma$ is a <u>spatial automorphism</u> of $\mathfrak{A}$, and $\gamma$ is <u>implemented</u> by the unitary U. ~~If~~
If $U, V \in \mathcal{U}(\mathcal{B}(\mathcal{H}))$ and $\gamma \in \mathrm{aut}(\mathfrak{A})$ are such that

$$\gamma = \mathrm{ad}\, U|_{\mathfrak{A}} = \mathrm{ad}\, V|_{\mathfrak{A}}$$

then $$U A U^* = V A V^* \qquad (A \in \mathfrak{A})$$

so $$(V^*U)A = A(V^*U) \qquad (A \in \mathfrak{A})$$

thus $V^*U \in \mathfrak{A}'$. Hence there is a unitary $Q \in \mathfrak{A}'$ with

$$U = QV$$

If also $U, V \in \mathfrak{A}$, then $Q \in \mathcal{U}(\mathcal{Z}(\mathfrak{A}))$. If $U \in \mathcal{U}(\mathfrak{A})$, we say

$\gamma = \text{ad } U$ is an <u>inner automorphism</u> of $\mathfrak{A}$. The group inn($\mathfrak{A}$) of all inner automorphisms of $\mathfrak{A}$ is a subgroup of aut($\mathfrak{A}$).

If X is a compact Hausdorff space, the set C(X) of all complex valued continuous functions on X is an abelian C*-algebra when the algebraic structure, involution and norm are defined by

$$\begin{aligned}(\lambda f + \mu g)(x) &= \lambda f(x) + \mu g(x)\\ fg(x) &= f(x)\, g(x)\\ f^*(x) &= \overline{f(x)}\\ \|f\| &= \sup \{|f(x)|;\ x \in X\}\end{aligned}$$

$$(f,\ g \in C(X))$$

Conversely, if $\mathfrak{A}$ is an abelian C*-algebra with unit, let $\Phi_{\mathfrak{A}}$ be the set of all continuous non-zero homomorphisms from $\mathfrak{A}$ to C. $\Phi_{\mathfrak{A}}$ is a weak*-closed subset of $(\mathfrak{A}^*)_1$, hence weak*-compact. By [8], 1.4.1, $\mathfrak{A}$ is isomorphic to $C(\Phi_{\mathfrak{A}})$ via the map

$$A \to \hat{A}$$

where $\quad \hat{A}(\rho) = \rho(A) \qquad (\rho \in \Phi_{\mathfrak{A}})$ .

We see from [13], Ch. V, §8, Lemma 6 that $\Phi_{\mathfrak{A}}$ is just the set of pure states of $\mathfrak{A}$. The mapping $A \to \hat{A}$ is called the <u>Gelfand transform</u>. Since the Gelfand transform is an algebra isomorphism, for $A \in \mathfrak{A}$ we have

$$\begin{aligned}\sigma(A) &= \sigma(\hat{A})\\ &= \{\hat{A}(\rho);\ \rho \in \Phi_{\mathfrak{A}}\}\\ &= \{\rho(A);\ \rho \in \Phi_{\mathfrak{A}}\}\end{aligned}$$

Suppose now $\mathfrak{A}$ is an arbitrary C*-algebra with identity, and A is a normal element of $\mathfrak{A}$. The C*-algebra generated by $\{I, A, A^*\}$ is abelian. Denote it by $\mathfrak{A}(A)$. The map

$$\Delta : \rho \to \rho(A)$$

from $\Phi_{\mathfrak{A}(A)}$ onto $\sigma(A)$ is a continuous bijection from the compact Hausdorff space $\Phi_{\mathfrak{A}(A)}$, so $\Delta$ is a homeomorphism.

If $\cong$ denotes isomorphism of C*-algebras we have

$$\mathfrak{A}(A) \cong C(\Phi_{\mathfrak{A}(A)}) \cong C(\sigma(A))$$

The isomorphism

$$f \to f(A) : C(\sigma(A)) \to \mathfrak{A}(A)$$

is called the <u>functional calculus</u>. Note that if A is normal

$$\begin{aligned} \|A\| = \|\hat{A}\| &= \sup\{|\hat{A}(\rho)| \; ; \rho \in \Phi_{\mathfrak{A}(A)}\} \\ &= \sup\{|\rho(A)| \; ; \rho \in \Phi_{\mathfrak{A}(A)}\} \\ &= \sup\{|\lambda| \; ; \lambda \in \sigma(A)\} \end{aligned}$$

Suppose $\mathfrak{A}$ is a C*-algebra, and $\mathfrak{B}$ is a C*-subalgebra of $\mathfrak{A}$. A <u>projection of norm one</u> from $\mathfrak{A}$ onto $\mathfrak{B}$ is a linear map $\Omega$ from $\mathfrak{A}$ onto $\mathfrak{B}$ such that

(i) $\Omega(B) = B \qquad (B \in \mathfrak{B})$

(ii) $\|\Omega(A)\| \leq \|A\| \qquad (A \in \mathfrak{A})$

<u>Topologies on $\mathfrak{B}(\mathcal{H})$; von Neumann Algebras</u>

Let $\mathcal{H}$ be a Hilbert space

(a) <u>The strong operator topology ($\mathcal{T}_s$)</u>

If $x \in \mathcal{H}$, the equation

$$p_x(T) = \|Tx\|$$

defines a seminorm $p_x$ on $\mathcal{B}(\mathcal{H})$. The topology defined by the family $\{p_x : x \in \mathcal{H}\}$ is called the <u>strong operator topology</u>, denoted $\mathcal{T}_s$. For fixed S, the maps $T \to ST$, $T \to TS$ are continuous on $\mathcal{B}(\mathcal{H})$, and the map

$$(S, T) \to ST$$

is continuous on $\mathcal{B}(\mathcal{H})_1 \times \mathcal{B}(\mathcal{H})$. However the map $T \to T^*$ is not $\mathcal{T}_s$-continuous. ($\mathcal{H}$ infinite dimensional)

$\mathcal{B}(\mathcal{H})_1$ is complete for the strong operator topology, and metrizable if $\mathcal{H}$ is separable. ([7], Ch.1, §3, p.30).

<u>The ultrastrong topology $\mathcal{T}_{\sigma s}$</u>

Given a sequence $X = \{x_i\}$ of elements of $\mathcal{H}$ such that $\Sigma \|x_i\|^2 < \infty$, the equation

$$p_x(T) = (\Sigma \|Tx_i\|^2)^{\frac{1}{2}}$$

defines a seminorm on $\mathcal{B}(\mathcal{H})$. The family of all such seminorms defines the <u>ultrastrong topology</u>, denoted $\mathcal{T}_{\sigma s}$ or $\sigma s$.

Continuity of algebraic operations is the same as for $\mathcal{T}_s$. $\mathcal{T}_s$ and $\mathcal{T}_{\sigma s}$ coincide on $\mathcal{B}(\mathcal{H})_1$. ([7], Ch.1, §3, p.34).

<u>The weak operator topology $\mathcal{T}_w$</u>

For $x, y \in \mathcal{H}$, define a seminorm on $\mathcal{B}(\mathcal{H})$ by

$$p_{x,y}(T) = |\langle Tx , y\rangle| .$$

The topology defined by

$$\{p_{x,y} ; x,y \in \mathcal{H}\}$$

is called the <u>weak operator topology</u>, denoted $\mathcal{T}_w$.

The map $(S, T) \to ST$ is not $\mathcal{T}_w$-continuous, however for fixed S the maps

$$T \mapsto ST, \quad T \to ST$$

are $\mathcal{T}_w$ continuous, as is the map

$$T \to T^*$$

$\mathcal{B}(\mathcal{H})_1$ is $\mathcal{T}_w$ compact ([7], Ch.1, §3, p.32).

The ultraweak operator topology $\mathcal{T}_{\sigma w}$

If $X = \{x_i\}$ and $Y = \{y_i\}$ are sequences in $\mathcal{H}$ with

$$\Sigma\, (\|x_i\|^2 + \|y_i\|^2) < \infty$$

the equation

$$p_{x,y}(T) = |\Sigma\langle Tx_i , y_i\rangle|$$

defines a seminorm on $\mathcal{B}(\mathcal{H})$. The set of all such seminorms defines the ultraweak operator topology, denoted $\mathcal{T}_{\sigma w}$ or $\sigma w$.

Continuity properties are as for $\mathcal{T}_w$. By [7] (Ch.1, §3, p.34), $\mathcal{T}_w = \mathcal{T}_{\sigma w}$ on $\mathcal{B}(\mathcal{H})_1$.

If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on a space X, write

$$\mathcal{T}_1 < \mathcal{T}_2$$

if every $\mathcal{T}_1$-open set is $\mathcal{T}_2$-open.

Let $\mathcal{T}_n$ denote the norm topology on $\mathcal{B}(\mathcal{H})$. We have

$$\begin{array}{ccccc} \mathcal{T}_w & < & \mathcal{T}_s & < & \mathcal{T}_n \\ \wedge & & \wedge & & \\ \mathcal{T}_{\sigma w} & < & \mathcal{T}_{\sigma s} & < & \mathcal{T}_n . \end{array}$$

and on $\mathcal{B}(\mathcal{H})_1$.

$$\mathcal{T}_w = \mathcal{T}_{\sigma w} < \mathcal{T}_s = \mathcal{T}_{\sigma s} < \mathcal{T}_n \quad .$$

If $\mathcal{H}$ is separable, $\mathcal{B}(\mathcal{H})_1$ is $\mathcal{T}_w$-metrizable, hence $\mathcal{T}_w$-separable. ([7], Ch.1, §3, p.32). If $\mathcal{F} \subset \mathcal{B}(\mathcal{H})$ let $\mathcal{F}^-$ denote the ultra-strong closure of $\mathcal{F}$.

A <u>von Neumann algebra</u> $\mathcal{R}$ is defined to be a *-subalgebra of $\mathcal{B}(\mathcal{H})$ which is $\sigma s$-closed. Clearly a von Neumann algebra is a C*-algebra.

<u>Theorem</u> ([7], Ch.1, §3, Theorem 2, p.41)
("Double Commutant Theorem")

Let $\mathfrak{A}$ be a *-algebra of operators on a Hilbert space $\mathcal{H}$.

(i) The following are equivalent

(i1) (resp (i2)) $\mathfrak{A}$ (resp $\mathfrak{A}_1$) is $\mathcal{T}_w$-closed

(i3) (resp i4) $\mathfrak{A}$ (resp $\mathfrak{A}_1$) is $\mathcal{T}_s$-closed

(i5) (resp i6) $\mathfrak{A}$ (resp $\mathfrak{A}_1$) is $\mathfrak{A}_{\sigma w}$-closed

(i7) (resp i8) $\mathfrak{A}$ (resp $\mathfrak{A}_1$) is $\mathcal{T}_{\sigma s}$-closed

(ii) Suppose the conditions of (i) are satisfied then

$$E = [\mathfrak{A}\mathcal{H}]$$

is the largest projection in $\mathfrak{A}$. For each $T \in \mathfrak{A}$,

$$TE = ET = T$$

The operators in $\mathfrak{A}'' = (\mathfrak{A}')'$ are the operators $T + \lambda I_{\mathcal{B}(\mathcal{H})}$ with $T \in \mathfrak{A}$.

(Thus we see that replacing $\mathcal{H}$ by $E\mathcal{H}$ we may assume $\mathfrak{A}$ has an identity.)

<u>Theorem</u> ([7], Ch.1, §3, Theorem 1, p.38)

Let $\mathfrak{R}$ be a von Neumann algebra acting on the Hilbert space $\mathcal{H}$, $\varphi \in \mathfrak{R}^*$.

(i) The following are equivalent

(i1) $\varphi$ is $\mathcal{T}_w$- continuous

(i2) $\varphi$ is $\mathcal{T}_s$-continuous

(i3) $\varphi = \sum_{i=1}^{n} \omega_{x_i, y_i}$ with $x_1, \ldots, x_n, \in \mathcal{H}$.

(ii) The following are equivalent

(ii1) $\varphi$ is ultraweakly continuous

(ii2) $\varphi$ is ultrastrongly continuous

(ii3) $\varphi = \sum_{i=1}^{\infty} \omega_{x_i, y_i}$ with $\Sigma \|x_i\|^2 + \|y_i\|^2 < \infty$

(ii4) (resp ii5) $\varphi|\mathfrak{R}_1$ is ultraweakly (resp. weakly) continuous.

(ii6) (resp ii7) $\varphi|\mathfrak{R}_1$ is ultrastrongly (resp strongly) continuous.

(iii) Let $\mathfrak{R}_\sim$ (resp $\mathfrak{R}_*$) denote the set of weakly continuous (resp $\mathcal{T}_{\sigma w}$-continuous) linear functionals on $\mathfrak{R}$. $\mathfrak{R}_*$ is the norm closure of $\mathfrak{R}_\sim$ in $\mathfrak{R}^*$, and $\mathfrak{R}$ can be identified as the Banach dual of $\mathfrak{R}_*$ by the duality

$$\langle A, f\rangle = f(A) \qquad (A \in \mathfrak{R},\ f \in \mathfrak{R}_*) .$$

(iv) Let $\mathcal{K}$ be a convex subset of $\mathfrak{R}$. The following are equivalent

(iv1) $\mathcal{K}$ is ultraweakly closed

(iv2) $\mathcal{K}$ is ultrastrongly closed

In particular, if $\mathfrak{R}$ acts on a separable Hilbert space $\mathcal{H}$, and $\{T_n\}_{n=1}^{\infty}$ is ultraweakly dense in $\mathfrak{R}$, then

$$\mathrm{co}(\{T_n;\ n = 1,2, \ldots\})$$

is ultrastrongly dense in $\mathfrak{R}$, so $\mathfrak{R}$ is ultrastrongly (hence strongly) separable.

<u>Theorem</u> (Kaplansky density theorem)([7], Ch. 1, §3, Theorem 3, p.43)

Let $\mathfrak{A}$ and $\mathfrak{B}$ be self-adjoint algebras of operators on the Hilbert space $\mathcal{H}$, with $\mathfrak{A} \subseteq \mathfrak{B}$, and suppose $\mathfrak{A}$ is strongly dense in $\mathfrak{B}$. Let $\mathcal{M}$ (resp $\mathcal{N}$) be the set of self-adjoint elements of $\mathfrak{A}$ (resp $\mathfrak{B}$) then $\mathfrak{A}_1$ is strongly dense in $\mathfrak{B}_1$ and $\mathcal{M}_1$ is strongly dense in $\mathcal{N}_1$.

Let $\mathfrak{A}$ be a C*-algebra a subset $\mathcal{F} \subseteq \mathfrak{A}^+$ is said to be <u>directed</u> if given A, B $\in \mathcal{F}$, there is a C $\in \mathcal{F}$ with $C \geq A$, $C \geq B$. Suppose $\mathcal{F}$ has a supremum D in $\mathfrak{A}^+$. A positive linear functional $\varphi$ on $\mathfrak{A}$ is said to be <u>normal</u> if

$$\sup \{\varphi(F);\ F \in \mathcal{F}\} = \varphi(D)$$

for each such directed set $\mathcal{F}$.

<u>Theorem</u> ([7], Ch.1, §4, Theorem 1, p.51)

Let $\mathfrak{R}$ be a von Neumann algebra. The following are equivalent

(1) $\varphi$ is a normal positive linear functional on $\mathfrak{R}$

(2) $\varphi$ is a $\sigma$w-continuous positive linear functional on $\mathfrak{R}$

It is easy to see (as shown in [44], Theorem 3, p.136) that (1) is equivalent to

(3) If $(E_\alpha)_{\alpha \in A}$ is an orthogonal family of projections in $\mathfrak{R}$, and $E = \sum_{\alpha \in A} E_\alpha$ (sum in $\mathcal{T}_s$-topology) then $\varphi(E) = \sum_{\alpha \in A} \varphi(E_\alpha)$ (we say then that $\varphi$ is <u>completely additive</u>).

If $\mathcal{R} \subseteq \mathcal{B}(\mathcal{H})$, $\mathcal{S} \subseteq \mathcal{B}(\mathcal{K})$ are von Neumann algebras define the operator $A \otimes B$ on the algebraic tensor product of $\mathcal{H}$ and $\mathcal{K}$ by $A \otimes B\ (\Sigma\, x_i \otimes y_i) = \Sigma\, Ax_i \otimes By_i$. Then $\|A \otimes B\| \leq \|A\|\,\|B\|$ ([7], Ch.1, §3, p.21) so $A \otimes B$ extends to a bounded linear operator on the completed Hilbert space $\mathcal{H} \otimes \mathcal{K}$. Linear combinations of such operators form a *-algebra $\mathcal{D}$, (since $(A \otimes B)^* = A^* \otimes B^*$). The strong operator closure of $\mathcal{D}$ is a von Neumann algebra, called the <u>tensor product</u> of $\mathcal{R}$ and $\mathcal{S}$, and denoted $\mathcal{R} \otimes \mathcal{S}$.

If $\varphi \in \mathcal{R}_*$, $\psi \in \mathcal{S}_*$, let

$$\varphi = \Sigma\, \omega_{x_i, y_i} \quad (\Sigma\, \|x_i\|^2 + \|y_i\|^2 < \infty)$$

$$\psi = \Sigma\, \omega_{a_j, b_j} \quad (\Sigma\, \|a_j\|^2 + \|b_j\|^2 < \infty).$$

Define the linear functional $\varphi \otimes \psi$ on $\mathcal{R} \otimes \mathcal{S}$ by

$$\varphi \otimes \psi = \sum_{i,j,k,l} \omega_{x_i \otimes a_j,\, y_k \otimes b_l}$$

Then $\varphi \otimes \psi \in (\mathcal{R} \otimes \mathcal{S})_*$ and $\{\varphi \otimes \psi ;\ \varphi \in \mathcal{R}_*,\ \psi \in \mathcal{S}_*\}$ is separating for $(\mathcal{R} \otimes \mathcal{S})^+$ in the sense that if $A \in (\mathcal{R} \otimes \mathcal{S})^+$ and $(\varphi \otimes \psi)(A) = 0$ for all such $\varphi$, $\psi$ then $A = 0$.

Also

$$\varphi \otimes \psi\ (A \otimes B) = \varphi(A)\, \psi(B) \qquad (A \in \mathcal{R},\ B \in \mathcal{S}) \tag{1}$$

This shows that $\varphi \otimes \psi$ is defined independently of the particular representation of $\varphi$ and $\psi$ by means of vectors.

If $\mathcal{R}$ and $\mathcal{S}$ are von Neumann algebras and $\Phi$ is an isomorphism from $\mathcal{R}$ onto $\mathcal{S}$ then $\Phi$ is $\sigma w$ and $\sigma s$ bicontinuous ([7], Ch.1, §4, Cor. 1, p.54).

Suppose $\alpha \in \text{aut}(\mathcal{R})$, $\beta \in \text{aut}(\mathcal{S})$, then there is a unique element $\alpha \otimes \beta$ of aut$(\mathcal{R} \otimes \mathcal{S})$ such that

$$\alpha \otimes \beta(A \otimes B) = \alpha(A) \otimes \beta(B) \qquad (A \in \mathcal{R}, \quad B \in \mathcal{S})$$

(by [7], Ch.1, §4, Propn. 2, p.57), since $\alpha$ and $\beta$ are $\sigma w$ and $\sigma s$ bicontinuous.

If $\mathcal{R}$ is a von Neumann algebra acting on the Hilbert space $\mathcal{H}$, and $V \in \mathcal{R}$, V is a partial isometry if $V^*V$ is a projection. In this case $VV^*$ is also a projection since $\sigma(VV^*) = \sigma(V^*V)$. $V^*V$ is called the initial projection of V and $VV^*$ the final projection of V. Let $\mathcal{P}(\mathcal{R})$ denote the set of projections in $\mathcal{R}$. If E, F $\in \mathcal{P}(\mathcal{R})$ we say

(1) E is equivalent to F (written $E \sim F$) if there is a partial isometry V in $\mathcal{R}$ with $V^*V = E$, $VV^* = F$.

(2) $E \prec F$ if there is an $E_1 \in \mathcal{P}(\mathcal{R})$ with $E \sim E_1 \leq F$.

The relation $\sim$ is an equivalence relation on $\mathcal{P}(\mathcal{R})$, and $\prec$ induces a partial order on equivalence classes. ([7], Ch.III, §1, pp.215-216).

The central carrier of an element A of $\mathcal{R}$ is denoted $C_A$ and defined to be the smallest projection Q in $\mathcal{Z}(\mathcal{R})$ such that $QA = A$. By [44], p.29,

$$C_A = [RAx; R \in \mathcal{R}, \; x \in \mathcal{H}].$$

Theorem (The Comparison Theorem) ([7], Ch.III, §1, Thm.1, p.218)

Let $\mathcal{R}$ be a von Neumann algebra with centre $\mathcal{Z}$, E and F projections in $\mathcal{R}$. There is a projection G in $\mathcal{Z}$ such that

$$FG \prec EG \; ,$$

$$E(I - G) \prec F(I - G)$$

If E is a projection in the von Neumann algebra $\mathfrak{R}$ we consider $E\mathfrak{R}E$ as a von Neumann algebra acting on $E\mathcal{H}$. Let $E \in \mathcal{P}(\mathfrak{R})$. E is <u>abelian</u> if $E\mathfrak{R}E$ is an abelian algebra. $\mathfrak{R}$ is said to be of <u>type I</u> if there is an abelian projection E in $\mathfrak{R}$ with $C_E = I_{\mathfrak{R}}$. E is <u>finite</u> if $F \in \mathcal{P}(\mathfrak{R})$ and $E \sim F \leq E$ imply $F = E$. E is <u>infinite</u> if E is not finite, and <u>purely infinite</u> if QE is infinite if for each central projection Q in $\mathfrak{R}$ with $0 \neq Q \leq C_E$. $\mathfrak{R}$ is said to be

(a) <u>continuous</u> if it contains no abelian projections

(b) <u>finite</u> (<u>infinite</u>, <u>purely infinite</u>) if $I_{\mathfrak{R}}$ is finite (infinite, purely infinite).

(c) <u>type II</u> if it is continuous and contains a finite projection E with $C_E = I$.

(d) <u>type $II_1$</u> if it is continuous and finite

(e) <u>type $II_\infty$</u> if it is type II and purely infinite

(f) <u>type III</u> if it contains no (non-zero) finite projections.

If $\mathfrak{R}$ and $\mathfrak{S}$ are von Neumann algebras with $\mathfrak{S} \subseteq \mathfrak{R}$, a projection $\Omega$ of norm one from $\mathfrak{R}$ onto $\mathfrak{S}$ is <u>normal</u> if $f \in \mathfrak{S}_*$ implies $f \circ \Omega \in \mathfrak{R}_*$.

A <u>faithful normal finite trace</u> on $\mathfrak{R}$ is a faithful normal projection $\Omega$ of norm one from $\mathfrak{R}$ onto $\mathcal{Z}(\mathfrak{R})$ such that

$$\Omega(AB) = \Omega(BA) \qquad (A,\ B \in \mathfrak{R})$$

<u>Theorem</u> ([46], 2.4.6) If $\mathfrak{R}$ is a finite von Neumann algebra, there is a faithful normal finite trace on $\mathfrak{R}$.

A trace on $\mathcal{R}$ is mapping $\tau : \mathcal{R}^+ \to [0, \infty]$ such that

(1) $\tau(A + B) = \tau(A) + \tau(B)$ $\qquad (A, B \in \mathcal{R}^+)$

(2) $\tau(\lambda A) = \lambda\tau(A)$ $\qquad (\lambda \geq 0,\ A \in \mathcal{R}^+)$

(3) If $A \in \mathcal{R}^+$, $U \in \mathcal{U}(\mathcal{R})$ then $\tau(A) = \tau(UAU^*)$.

$\tau$ is faithful if $A \in \mathcal{R}^+$ and $\tau(A) = 0$ imply $A = 0$, semifinite if for each non-zero $A \in \mathcal{R}^+$, there is a $B \in \mathcal{R}^+$ with $0 \neq B \leq A$ and $\tau(B) < \infty$.

$\tau$ is normal if for each bounded directed net $\mathcal{F} \subset \mathcal{R}^+$,

$$\tau(\sup \mathcal{F}) = \sup \{\tau(A);\ A \in \mathcal{F}\}$$

$\mathcal{R}$ is a factor if $\mathcal{Z}(\mathcal{R}) = \mathbb{C}I_{\mathcal{R}}$. If $\mathcal{R}$ is a finite factor we see there is a trace $\tau : \mathcal{R} \to [0, \infty]$. $\mathcal{R}$ is semifinite if there is a finite projection $E \in \mathcal{R}$ with $C_E = I_{\mathcal{R}}$.

Theorem ([46], 2.5.7) If $\mathcal{R}$ is a semifinite von Neumann algebra there is a semifinite faithful normal trace on $\mathcal{R}^+$.

If $E$ is a projection in a von Neumann algebra $\mathcal{R}$, $E$ is minimal if $E\mathcal{R}E = \mathbb{C}E$. When $\mathcal{R}$ is a factor, every abelian projection is minimal, since $E\mathcal{R}E = \mathcal{Z}(\mathcal{R})E = \mathbb{C}E$. For the type of tensor products, the situation is as follows (see [46], Theorem 2.6.6).

If $\mathcal{R}$ and $\mathcal{S}$ are finite (resp. semifinite) then $\mathcal{R} \otimes \mathcal{S}$ is finite (resp. semifinite). If one of $\mathcal{R}$ and $\mathcal{S}$ is type III then $\mathcal{R} \otimes \mathcal{S}$ is type III.

Note that if $E$ is a finite projection in $\mathcal{R}$ and $F$ is a finite projection in $\mathcal{S}$ then $E\mathcal{R}E$ and $F\mathcal{S}F$ are finite, so $(E \otimes F)(\mathcal{R} \otimes \mathcal{S})(E \otimes F) = E\mathcal{R}E \otimes F\mathcal{S}F$ is a finite algebra in $\mathcal{R} \otimes \mathcal{S}$.

If $\mathfrak{A}$ is a C*-algebra, two representations $\pi$, $\varphi$ are said to be quasi-equivalent if there is an isomorphism, $\beta$, between the von Neumann algebras $\pi(\mathfrak{A})^-$ and $\varphi(\mathfrak{A})^-$ such that

$$\beta(\pi(A)) = \varphi(A) \qquad (A \in \mathfrak{A}) .$$

Theorem If $\varphi$ is a faithful representation of the C*-algebra $\mathfrak{A}$, and $\pi$ is the universal representation of $\mathfrak{A}$, there is a projection Q in $\mathcal{Z}(\pi(\mathfrak{A})^-)$ such that $\varphi$ is a quasi-equivalent to to the representation

$$A \to \pi(A)Q$$

Proof Note first that if f is a state on $\pi(\mathfrak{A})$ then f o $\pi$ is a state on $\mathfrak{A}$, call it g. Let $\pi(\mathfrak{A})$ act on the Hilbert space $\mathcal{H}$, and define $y \in \mathcal{H}$ by

$$y = \Sigma^{\oplus}\{y_\varphi ; \varphi \in E(\mathfrak{A})\}$$

where

$$y_\varphi = 0 \qquad (\varphi \neq g)$$

$$y_\varphi = x_g \qquad (\varphi = g)$$

where $x_g$ is a cyclic vector for $\pi_g$. Then $f \circ \pi = g = \omega_y \circ \pi$. So $f = \omega_y$. This shows that every state of $\pi(\mathfrak{A})$ is a $\sigma$w-continuous linear functional. Since every element of $\pi(\mathfrak{A})^*$ is a linear combination of states, each element of $\pi(\mathfrak{A})^*$ is $\sigma$w-continuous.

Let $\omega$ be a $\sigma$w-continuous linear functional on $\varphi(\mathfrak{A})$, then $\omega \circ \varphi \circ \pi^{-1}$ is a norm continuous, hence $\sigma$w-continuous linear functional on $\pi(\mathfrak{A})$. Thus the isomorphism

$$\varphi\pi^{-1} : \pi(A) \to \varphi(A)$$

is σω-continuous and isometric. For $n = 1, 2, \ldots,$ let

$$\beta_n = \varphi\pi^{-1} | n\pi(\mathfrak{A})_1$$

Since $\beta_n$ is σw-continuous; $n\pi(\mathfrak{A})_1$ is σw dense in $n(\pi(\mathfrak{A})^-)_1$ (Kaplanksy density theorem) and

$$\beta_n(n\pi(\mathfrak{A})_1) = n\varphi(\mathfrak{A})_1$$

is σω dense in the compact set $n(\varphi(\mathfrak{A})^-)_1$, $\beta_n$ extends by continuity (uniquely) to a σw-continuous map $\bar{\beta}_n$

$$\bar{\beta}_n : n\pi(\mathfrak{A})_1^- \to n\varphi(\mathfrak{A})_1^- .$$

This map is onto since its range is σω-compact, and contains the dense subset $n\varphi(\mathfrak{A})_1$ of $n\varphi(\mathfrak{A})_1^-$. Uniqueness of $\bar{\beta}_n$ implies that

$$\bar{\beta}_n = \bar{\beta}_m | n\pi(\mathfrak{A})_1^- \qquad (m \geq n) .$$

Hence there is a map

$$\beta : \pi(\mathfrak{A})^- \text{ onto } \varphi(\mathfrak{A})^-$$

$$\text{s.t.} \quad \bar{\beta}_n = \beta | n\pi(\mathfrak{A})_1^- \qquad (n = 1, 2, \ldots)$$

Clearly β is a homomorphism. For each σw-continuous linear functional ω on $\varphi(\mathfrak{A})^-$, $\omega \circ \beta$ is σw-continuous on $\pi(\mathfrak{A})_1^-$ since

$$\omega \circ \beta | \pi(\mathfrak{A})_1^- = \omega \circ \bar{\beta}_1$$

Thus $\omega \circ \beta$ is σw-continuous on $\pi(\mathfrak{A})^-$. So β is σw-continuous. We have thus constructed a σw-continuous homomorphism

$$\beta : \pi(\mathfrak{A})^- \text{ onto } \varphi(\mathfrak{A})^-$$

which extends the map $\varphi \circ \pi^{-1}$

The kernel $\beta^{-1}(0)$ is a σw-closed two-sided ideal in $\pi(\mathfrak{A})^{-}$, hence

$$\beta^{-1}(0) = \pi(\mathfrak{A})^{-}(I - P)$$

for some projection P in the centre of $\pi(\mathfrak{A})^{-}$ ([7], Ch.1, §3, Corollary 3, p.42) thus

$$\alpha = \beta|\pi(\mathfrak{A})^{-}P$$

is an isomorphism, and has range $\varphi(\mathfrak{A})^{-}$ since if $A \in \pi(\mathfrak{A})^{-}$

$$\begin{aligned}\alpha(AP) &= \beta(AP) = \beta(AP + A(1 - P)) \\ &= \beta(A)\end{aligned}$$

For $A \in \mathfrak{A}$

$$\begin{aligned}\alpha(\pi(A)P) &= \beta(\pi(A)) \\ &= \varphi\pi^{-1}\pi(A) \\ &= \varphi(A)\end{aligned}$$

Thus $\varphi$ is a quasi-equivalent to the representation

$$A \rightarrow \pi(A)P$$

This completes the proof.

A topological space X is said to be <u>Stonean</u> if X is a compact Hausdorff space and the closure of each open set in X is open. A regular borel measure $\mu$ on X is said to be <u>normal</u> if the map

$$A \rightarrow \int A\,d\mu$$

is a normal linear functional on C(X). If for each non-zero $A \in C(X)^{+}$ there is a positive normal regular borel measure $\mu$ on X with $\int A\,d\mu \neq 0$, we say there is a separating family of normal measures on X. If X is Stonean and there is a separating family of normal measures on X, X is called <u>hyperstonean</u>. If $\mathfrak{R}$ is an abelian von Neumann algebra, $\Phi_{\mathfrak{R}}$ is a hyperstonean space for the weak* topology. ([9], Theorem 2).

There is another representation of an abelian von Neumann algebra ([7], Ch.1, §7, Theorem 1, p.118). Let $\mathcal{H}$ be a complex Hilbert space, $\mathcal{Z}$ an abelian von Neumann algebra acting on $\mathcal{H}$. There is a locally compact space Z, a positive regular borel measure $\mu$ on Z with support Z, and an isomorphism between the *-algebra $\mathcal{Z}$ and the *-algebra $L^{\infty}(Z, \mu)$. If $\mathcal{H}$ is separable, Z may be chosen compact and with countable base.

A <u>left Hilbert algebra</u> $\mathfrak{A}$ is an involutive algebra with involution $\xi \to \xi^{\#}$ and an inner product $\langle \xi, \eta \rangle$ which satisfy the following conditions

(i) The map $\eta \in \mathfrak{A} \to \xi\eta \in \mathfrak{A}$ is continuous for every $\xi \in \mathfrak{A}$.

(ii) $\langle \xi\eta_1, \eta_2 \rangle = \langle \eta_1, \xi^{\#}\eta_2 \rangle$ for all $\xi, \eta_1, \eta_2 \in \mathfrak{A}$

(iii) The subalgebra $\mathfrak{A}^2$ of $\mathfrak{A}$ spanned by the elements $\xi\eta$ with $\xi, \eta \in \mathfrak{A}$, is dense in $\mathfrak{A}$.

(iv) If $\mathcal{H}$ denotes the Hilbert space obtained by completion of $\mathfrak{A}$, then there is a closed linear operator from $\mathcal{H}$ to $\overline{\mathcal{H}}$ (the conjugate Hilbert space) extending the map $\xi \in \mathfrak{A} \to \xi^{\#} \in \mathfrak{A}$.

For any $\xi \in \mathfrak{A}$ we denote by $\pi(\xi)$ the unique continuous linear operator on $\mathcal{H}$ such that $\pi(\xi)\eta = \xi\eta$, for all $\eta \in \mathfrak{A}$. The von Neumann algebra generated by $\pi(\mathfrak{A})$ is denoted by $\mathcal{L}(\mathfrak{A})$. S denotes the closure of the map $\xi \in \mathfrak{A}^2 \to \xi^{\#} \in \mathfrak{A}^2$ and F the adjoint of S. By [56], Lemma 2.4, there is an isometric involution J and a non-singular positive selfadjoint operator $\Delta$ on $\mathcal{H}$ such that

$$S = J\Delta^{\frac{1}{2}}$$

<u>Theorem</u> (Tomita's Theorem) ([56], Theorem 5.3)

(i) $J\mathfrak{L}(\mathfrak{A})J = \mathfrak{L}(\mathfrak{A})'$

(ii) $\Delta^{it}\mathfrak{L}(\mathfrak{A})\Delta^{-it} = \mathfrak{L}(\mathfrak{A})$ for all $t \in R$

Let $\mathfrak{R}$ be a von Neumann algebra with a separating-generating vector $x_0$, acting on the Hilbert space $\mathfrak{H}$. Define

$$\mathfrak{A} = \{Ax_0 \ ; A \in \mathfrak{R}\}$$

We shall check that $\mathfrak{A}$ satisfies the axioms of a left Hilbert algebra, with the involution $(Ax_0)^{\#} = A^* x_0$ and $\mathfrak{R}$ can be identified with $\mathfrak{L}(\mathfrak{A})$.

<u>Lemma</u> The map

$$Ax_0 \to A^* x_0$$

is preclosed as a linear mapping from $\mathfrak{H}$ to $\bar{\mathfrak{H}}$ (the conjugate Hilbert space). If S denotes its closure and F its adjoint then $\mathfrak{R}' x_0 \subset \mathfrak{D}(F)$ (the domain of F) and

$$FA' x_0 = A'^* x_0$$

for $A' \in \mathfrak{R}'$.

Moreover S and F are involutions in the sense that $y \in \mathfrak{D}(S)$ implies $Sy \in \mathfrak{D}(S)$ and

$$SSy = y$$

and similarly for F.

<u>Proof</u> From the relations

$$\langle A^* x_0, A' x_0 \rangle = \langle x_0, AA' x_0 \rangle$$

$$= \langle A'^* x_0, Ax_0 \rangle$$

for $A \in \mathfrak{R}$, $A' \in \mathfrak{R}'$, and the fact that $\mathfrak{R}x_o$ and $\mathfrak{R}'x_o$ are dense in $\mathcal{H}$, we obtain that the map $Ax_o \in \mathfrak{R}x_o \to A^*x_o$ is preclosed as a conjugate linear operator. If its closure is denoted by S and its adjoint by F it also follows from this relation that $\mathfrak{R}'x_o \subset \mathcal{D}(F)$ and

$$FA'x_o = A'^*x_o$$

for $A' \in \mathfrak{R}'$.

Finally to prove that S and F are involutions, take $y \in \mathcal{D}(F)$, $A \in \mathfrak{R}$, then

$$\begin{aligned}\langle Fy, SAx_o\rangle &= \langle Fy, A^*x_o\rangle \\ &= \langle SA^*x_o, y\rangle \\ &= \langle Ax_o, y\rangle\end{aligned}$$

so $Fy \in \mathcal{D}(F)$ and

$$FFy = y$$

since F is the adjoint of

$$Ax_o \to A^*x_o$$

A similar argument works for S.

This shows that $\mathfrak{A}$ is a left Hilbert algebra, with multiplication

$$(Ax_o)(Bx_o) = ABx_o$$

and involution

$$(Ax_o)^\# = A^*x_o \quad .$$

We have, for $A,B \in \mathfrak{R}$,

$$(Ax_o)Bx_o = ABx_o = A(Bx_o).$$

Thus $\pi(Ax_o) = A$, so

$$\mathfrak{R} = \mathfrak{L}(\mathfrak{A}) \;.$$

It follows by Tomita's theorem that

$$J\mathcal{R}J = \mathcal{R}'$$

$$\Delta^{it}\mathcal{R}\Delta^{-it} = \mathcal{R} \qquad (t \in R)$$

Topological Groups

A <u>topological group</u> is a group G on which a topology is defined such that

(i) the map $g \to g^{-1}$ is continuous from G to G

(ii) the map $(g,h) \to gh$ is continuous from G x G to G.

If $\mathfrak{A}$ is a C*-algebra, and aut($\mathfrak{A}$) has the relative topology as a subset of the bounded operators on $\mathfrak{A}$, then aut($\mathfrak{A}$) is a topological group since

$$\|\gamma - \iota\| = \|\gamma^{-1} - \iota\|$$

and

$$\|\alpha\beta - \iota\| = \|\alpha - \beta^{-1}\|$$

$$\leq \|\alpha - \iota\| + \|\beta - \iota\|$$

We shall usually denote the identity of a group by e. If G, H are groups, a <u>homomorphism</u> $\varphi$ from G to H is a map from G to H such that $\varphi(gh) = \varphi(g)\,\varphi(h)$ $(g,h \in G)$. $\varphi$ is an <u>isomorphism</u> if $\varphi$ is injective.

A <u>locally compact</u> group is a topological group G such that the topology of G makes G into a locally compact space. If G is a locally compact group, a <u>left invariant measure</u> on G is a positive regular borel measure $\mu$ on G such that if E is a borel subset of G, then

$$\mu(gE) = \mu(E) \qquad \text{for all } g \in G.$$

(If H, K are subsets of a group G we denote by HK the set

$\{hk;\ h \in H,\ k \in K\}$).

Such a measure $\mu$ is called a left Haar measure on G. $\mu$ is unique up to multiplication by a scalar, and every locally compact group possesses a left Haar measure ([33], Ch. VI).

If G is a locally compact abelian group, let $\hat{G}$ denote the set of continuous homomorphisms from G to $\mathbb{T}$. Define a topology on $\hat{G}$ by $\rho_\alpha \to \rho$ in $\hat{G}$ if and only if

$$\sup_{g \in C} |\rho_\alpha(g) - \rho(g)| \to 0$$

for each compact set C in G. With this topology $\hat{G}$ is a locally compact abelian group, so we can consider

$$\hat{\hat{G}} = (\hat{G})^\wedge$$

Pontryagin's duality theorem asserts that G is isomorphic and homeomorphic to $\hat{\hat{G}}$ via the map

$$g \to \hat{g}$$

where $$\hat{g}(\rho) = \rho(g) \qquad (\rho \in \hat{G})$$

([33], §37.D, p.151).

Suppose now G is a discrete group, and let B(G) be the set of all bounded complex valued functions on G. B(G) is a Banach space with the supremum norm. Let

$$\rho(g)(f)(h) = f(gh)$$

and

$$\lambda(g)(f)(h) = f(hg) \qquad (f \in B(G),\ g \in G)$$

The sets $\{\rho(g);\ g \in G\}$ and $\{\lambda(g)\ ;\ g \in G\}$ are groups of transformations of B(G) called the group of left (resp. right) translations. An invariant mean $\Omega$ on G is a map $\Omega \in B(G)^*$ such that for $f \in B(G)$

(i) $|\Omega(f)| \leq \sup\{|f(g)|;\ g \in G\}$

(ii) If 1 denotes the function taking the value 1 everywhere on G, then $\Omega(1) = 1$.

(iii) $f \geq 0$ implies $\Omega(f) \geq 0$.

(iv) $\Omega(\rho(g)(f)) = \Omega(f) = \Omega(\lambda(g)(f)) \quad (g \in G)$

If a discrete group G has an invariant mean, we say G is an <u>amenable group</u>.

If G is a group, $\mathfrak{A}$ a C*-algebra, a <u>representation of G on $\mathfrak{A}$</u> (or a <u>representation of G as automorphisms of $\mathfrak{A}$</u>) is a homomorphism from G to aut($\mathfrak{A}$). Let $\mathcal{H}$ be a Hilbert space. A <u>unitary representation</u> of G on $\mathcal{H}$ is a homomorphism from G to $\mathcal{U}(\mathcal{B}(\mathcal{H}))$. Suppose G is a topological group, U: $g \to U_g$ a representation of G on $\mathcal{H}$. We say that U is

(a) <u>strongly</u> (resp. <u>weakly</u>) <u>continuous</u> if U is a continuous map from G to $\mathcal{U}(\mathcal{B}(\mathcal{H}))$ with the strong operator (resp. weak operator) topology.

(b) <u>norm continuous</u> if U is a continuous map from G to $\mathcal{U}(\mathcal{B}(\mathcal{H}))$ with the norm topology as operators on $\mathcal{H}$.

Weak continuity of U is equivalent to the map

$$g \to \langle U_g x, x\rangle$$

being continuous for each $x \in \mathcal{H}$.

Strong continuity is equivalent to saying that

$$\|U_g x - U_h x\| \to 0 \quad \text{as} \quad g \to h$$

for any $x \in \mathcal{H}$, $h \in G$.

Since $\mathcal{T}_w < \mathcal{T}_s$, strong continuity implies weak continuity. However if U is weakly continuous

$$\|U_g x - U_h x\|^2 = 2\|x\|^2 - \langle U_g^* U_h x, x\rangle - \langle U_h^* U_g x, x\rangle \to 0 \quad \text{as} \quad g \to h ,$$

so strong and weak continuity coincide.

Norm continuity of U is equivalent to

$$\|U_g - U_h\| \to 0 \qquad \text{as } g \to h \text{ in } G$$
$$\text{for any } h \text{ in } G.$$

If G is a locally compact group, with left Haar measure m, and U is a unitary representation of G on the Hilbert space $\mathcal{H}$, U is said to be <u>weakly measurable</u> if the map

$$g \to \langle U_g x, x\rangle$$

is m-measurable for each $x \in \mathcal{H}$. If $\mathcal{H}$ is separable, a weakly measurable unitary representation is strongly continuous ([18], Theorem 22.20(b), p.347).

<u>The Bochner Integral</u> Let C be a measure space, X a Banach space. A countably valued function $x: C \to X$ is <u>Bochner Integrable</u> if the map $\sigma \to \|x(\sigma)\|$ is Lebesgue measurable and the inverse image under x of each element in the range of x is a measurable set in C. A function $x: C \to X$ is <u>Bochner integrable</u> if and only if there is a sequence $\{x_n\}$ of countably valued Bochner integrable functions such that

$$\|x_n(\sigma) - x(\sigma)\| \to 0 \quad \text{for almost all } \sigma$$

$$\lim_{n \to \infty} \int_C \|x(\sigma) - x_n(\sigma)\| \, dm = 0$$

$$(m = \text{Lebesgue measure on } R)$$

# CHAPTER II

## CONTINUITY OF AUTOMORPHIC REPRESENTATIONS ([38])

2.1 Definition  Let $\mathfrak{A}$ be a C*-algebra acting on a Hilbert space $\mathcal{H}$, G a locally compact group, and $\alpha$ a representation of G on $\mathfrak{A}$. $\alpha$ is weakly measurable if for each $x \in \mathcal{H}$, and $T \in \mathfrak{A}$, the map $g \to \langle \alpha(g)(T)x, x \rangle$ is measurable with respect to left haar measure on G. Let $\tau$ denote one of the operator topologies on $\mathfrak{A}$. $\alpha$ is $\tau$-continuous if the map $g \to \alpha(g)(T)$ is continuous: $G \to \mathfrak{A}$, where $\mathfrak{A}$ has the $\tau$-topology.

We shall show that if $\mathcal{R}$ is a von Neumann algebra, acting on a separable Hilbert space, then a weakly measurable representation of the locally compact group G on $\mathcal{R}$ is actually ultraweakly continuous. From this we can deduce that if $\mathfrak{A}$ is a C*-algebra acting on separable Hilbert space, and $\alpha$ is a weakly measurable representation of the locally compact group G on $\mathfrak{A}$, such that each $\alpha(g)$ extends to an automorphism $\beta(g)$ of $\mathfrak{A}^-$, then $\beta: g \to \beta(g)$ is an ultraweakly continuous representation of G on $\mathfrak{A}^-$. This extends known results of J.F. Aarnes ([1], Theorem 8, p.31) and R.R. Kallman ([24], Theorem). We also show that if $\mathcal{R}$ and $\mathcal{S}$ are von Neumann algebras acting on separable Hilbert space and $\alpha$, $\beta$ are weakly measurable representations of G on $\mathcal{R}$ and $\mathcal{S}$ respectively, then there is a strongly continuous representation $\alpha \otimes \beta$ of G on $\mathcal{R} \otimes \mathcal{S}$ such that $\alpha \otimes \beta\ (g)(A \otimes B) = \alpha(g)(A) \otimes \beta(g)(B)$ $(A \in \mathcal{R},\ B \in \mathcal{S})$.

The main result was suggested by the corresponding theorem for unitary representations of groups which states: Let G be a locally compact group, $U: g \to U_g$ a unitary representation of G on a separable Hilbert space $\mathcal{H}$, such that U is weakly measurable, then U is strongly continuous. We use some of the ideas involved in proving the above result (see [18], Theorem 22.20 (b)). We shall show below that if $\alpha$ is a representation of the group G on the von Neumann algebra $\mathfrak{R}$, and $\nu(g) = \alpha(g)^*$, the adjoint of $\alpha(g)$ then $\nu(g)\colon \mathfrak{R}_* \to \mathfrak{R}_*$ and if $\mathfrak{R}$ acts on separable Hilbert space, $\mathfrak{R}_*$ is a separable Banach space. Thus our result could be obtained from [5], Section 4, No.7, p.171, Corollary 2, by considering $\nu$ as a representation of G by operators on $\mathfrak{R}_*$. However, this proof is long and complicated by measure theoretic considerations. We give a simple, direct proof.

<u>2.2 Theorem</u> Let $\mathfrak{R}$ be a von Neumann algebra acting on separable Hilbert space $\mathcal{H}$, G a locally compact group, and $\alpha$ a weakly measurable representation of G on $\mathfrak{R}$, then

$$\|f \circ \alpha(g) - f \circ \alpha(h)\| \to 0 \quad \text{as} \quad g \to h \text{ in } G$$

for any $h \in G$, $f \in \mathfrak{R}_*$.

<u>Proof</u> By [7] (Theorem 1, Ch.1, §4, p.51) every positive element of $\mathfrak{R}_*$ is of the form $\sum_{i=1}^{\infty} \omega_{y_i}$ with $\sum_{i=1}^{\infty} \|y_i\|^2 < \infty$, and every element of $\mathfrak{R}_*$ is a finite linear combination of positive elements. Thus if $f = \sum_{i=1}^{\infty} \omega_{y_i}$ is a positive element of $\mathfrak{R}_*$, and $T \in \mathfrak{R}$, the map $g \to f(\alpha(g)(T))$ is the pointwise limit of the measurable maps $g \to \sum_{i=1}^{n} \omega_{y_i}(\alpha(g)(T))$ and hence is m-measurable where m denotes left haar measure on G. It follows that $g \to f(\alpha(g)(T))$ is m-measurable for all $f \in \mathfrak{R}_*$, $T \in \mathfrak{R}$.

Let $\{x_n\}$ be a countable dense subset of $\mathcal{H}$, and $\mathcal{S}$ denote the set of all finite sums of $\omega_{x_j}$'s. It is easy to check that the linear hull of $\mathcal{S}$ is dense in $\mathfrak{R}_*$, hence $\mathfrak{R}_*$ is separable. $\mathfrak{R}_1$, the unit ball of $\mathcal{R}$, is weakly compact and metrizable ([7], Ch.1, §3, p.32), hence weakly separable. Let $\{T_n\}$ be a countable weakly dense subset of $\mathcal{R}_1$. Since the weak and ultraweak topologies coincide on $\mathcal{R}_1$, ([7], Ch.1, §3, p.34), $\{T_n\}$ is ultraweakly dense in $\mathcal{R}_1$.

Define $\nu(g): \mathcal{R}^* \to \mathcal{R}^*$ by $\nu(g)(f) = f \circ \alpha(g)^{-1}$, then $\nu(g)$ is the dual of the isometric linear map $\alpha(g)^{-1}$, so $\nu(g)$ is an isometric linear map. Since each $\alpha(g)$ is ultraweakly continuous ([7], Ch.1, §4, Theorem 2, p.53), $\nu(g): \mathcal{R}_* \to \mathcal{R}_*$. We have also

$$\nu(gh)(f) = f \circ \alpha(gh)^{-1} = f \circ \alpha(h)^{-1} \circ \alpha(g)^{-1}$$
$$= \nu(g)\, \nu(h)(f) \quad .$$

Let $f \in \mathcal{R}_*$, and $\varepsilon > 0$. Define

$$W = \{g \in G;\ \|\nu(g)(f) - f\| \leq \varepsilon/2\}$$

then $\nu(g)(f) - f \in \mathcal{R}_*$, so

$$W = \bigcap_{n=1}^{\infty} \{g \in G;\ |f(\alpha(g)(T_n)) - f(T_n)| \leq \varepsilon/2\}$$

Hence W is m-measurable. Since $\nu(g)$ is isometric, $W = W^{-1}$ and

$$W^2 \subset \{g \in G;\ \|\nu(g)(f) - f\| \leq \varepsilon\}$$

Let $\mathcal{T} = \{\nu(g)(f);\ g \in G\}$. $\mathcal{R}_*$ is separable, so $\mathcal{T}$ is separable, hence there is a countable dense subset $\{\nu(g_n)(f)\}$ of $\mathcal{T}$. If $g \in G$, there is a $g_n$ with $\|\nu(g)(f) - \nu(g_n)(f)\| \leq \varepsilon/2$, hence $\|f - \nu(g_n^{-1}g)(f)\| \leq \varepsilon/2$, so $g_n^{-1}g \in W$, $g \in g_nW$, and $G = \bigcup_{n=1}^{\infty} g_nW$. By left invariance of haar measure, W contains a compact set C of positive measure. Then $CC^{-1}$ contains a neighbourhood N of e ([18], 20.17, Corollary, p.296), and we have

$N \subset CC^{-1} \subset W^2 \subset \{g \in G;\ \|\nu(g)(f) - f\| \leq \epsilon\}$. Hence $g \to \nu(g)(f)$ is continuous at $e$, and so, by translation, is continuous everywhere on G. This completes the proof.

<u>2.4 Corollary</u> If $\mathfrak{R}$ is a von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$, G is a locally compact group and $\alpha$ is a weakly measurable representation of G on $\mathfrak{R}$, then $\alpha$ is ultraweakly continuous.

<u>Proof</u> Let $f \in \mathfrak{R}_*$, $T \in \mathfrak{R}$. By theorem 2.2, if $h \in G$,

$$|f(\alpha(g)(T)) - f(\alpha(h)(T))|$$

$$\leq \|T\| \, \|f \circ \alpha(g) - f \circ \alpha(h)\| \to 0 \quad \text{as} \quad g \to h$$

Hence $f(\alpha(g)(T)) \to f(\alpha(h)(T))$ as $g \to h$, for each $f \in \mathfrak{R}_*$. Since the ultraweak topology on $\mathfrak{R}$ coincides with $\sigma(\mathfrak{R},\mathfrak{R}_*)$ (by [7], Theorem 1, Ch.1, §4, p.51), this gives the result.

<u>2.5 Remark</u> An examination of the proof of the theorem shows that the same conclusion will hold whenever we can show that the set W is m-measurable, and the set $\mathcal{V} = \{\nu(g)(f);\ g \in G\backslash N\}$ is separable, for some set N in G of measure zero.

<u>2.6 Corollary</u> Let G be a locally compact group, $\mathfrak{R}$ a von Neumann algebra. Let $\alpha$ be a representation of G on $\mathfrak{R}$ such that the map $\varphi: g \to \nu(g)(f) = f \circ \alpha(g)^{-1}$ is Bochner integrable for each $f \in \mathfrak{R}_*$, then $\varphi$ is continuous: $G \to \mathfrak{R}_*$ i.e. $\|f \circ \alpha(g) - f \circ \alpha(h)\| \to 0$ as $g \to h$ in G for each $f \in \mathfrak{R}_*$.

Proof  $\varphi$ is Bochner integrable, so $\varphi$ is the pointwise limit in G of countably valued integrable functions on G. Hence W is measurable. Also, by [20] (Theorem 3.5.3), $\varphi$ is almost everywhere separably valued i.e. there is a null set N such that $\{f \circ \alpha(g);\ g \in G\setminus N\}$ is separable. The result follows by Remark 2.5.

Suppose now $\mathfrak{A}$ is a C*-algebra acting on a separable Hilbert space $\mathcal{K}$, G is a locally compact group, and $\alpha$ is a weakly measurable representation of G on $\mathfrak{A}$, such that each $\alpha(g)$ is extendable to an automorphism, $\beta(g)$, of $\mathfrak{A}^-$, the weak operator closure of $\mathfrak{A}$. (Equivalently each $\alpha(g)$ is an ultraweakly bicontinuous automorphism of $\mathfrak{A}$.) It is clear that $\beta: g \to \beta(g)$ is a representation of G on $\mathfrak{A}^-$.

2.7 Corollary  With notation as above $\| f \circ \beta(g) - f \circ \beta(h)\| \to 0$ as $g \to h$ in G for any $h \in G$, $f \in (\mathfrak{A}^-)_*$.

Proof  Let $T \in \mathfrak{A}^-$. Since $\mathcal{K}$ is separable, the ball radius $\|T\|$ in $\mathfrak{A}^-$ weakly metrizable ([7], Ch.1, §3, p.32). Hence by Kaplansky's density theorem ([7], Ch.1, §3, Theorem 3, p.43) there is a sequence $\{S_n\}$ in $\mathfrak{A}$ with $S_n \to T$ weakly. Thus if $x \in \mathcal{K}$, $\omega_x(\beta(g)(T)) = \lim \omega_x(\alpha(g)(S_n))$. Now each map $g \to \omega_x(\alpha(g)(S_n))$ is m-measurable by hypothesis, so $g \to \omega_x(\beta(g)(T))$ is m-measurable for all $x \in \mathcal{K}$, $T \in \mathfrak{A}^-$. The result follows by Theorem 2.3.

2.8 Corollary  Let $\mathfrak{A}$ be a C*-algebra acting on a separable Hilbert space $\mathcal{K}$, G a locally compact group $\alpha$ a weakly measurable representation of G on $\mathfrak{A}$ such that each $\alpha(g)$ extends to an automorphism $\beta(g)$ of $\mathfrak{A}^{-}$. Then $\beta: g \to \beta(g)$ is an ultraweakly continuous representation of G on $\mathfrak{A}^{-}$.

Proof This follows immediately from Corollary 2.7.

2.9 Remark Since every weakly continuous representation is weakly measurable, Corollary 2.8 is an extension of results due to Aarnes ([1], Theorem 8, p.31) and Kallman ([24], Theorem).

Suppose now that $\mathfrak{R}$ and $\mathfrak{S}$ are von Neumann algebras acting on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively. Let $\alpha$ be a representation of G on $\mathfrak{R}$, $\beta$ a representation of G on $\mathfrak{S}$. Denote by $\mathfrak{B}$ the set of all finite linear combinations of elements $R \otimes S$ with $R \in \mathfrak{R}$, $S \in \mathfrak{S}$, so that $\mathfrak{R} \otimes \mathfrak{S}$ is the weak closure of $\mathfrak{B}$. $\mathcal{H} \otimes \mathcal{K}$ denotes the completion of the algebraic tensor product. $\mathcal{H}_o$, of $\mathcal{H}$ and $\mathcal{K}$. For each $g \in G$, there is a unique automorphism $\alpha(g) \otimes \beta(g)$ on $\mathfrak{R} \otimes \mathfrak{S}$ such that $(\alpha(g) \otimes \beta(g))(R \otimes S) = \alpha(g)(R) \otimes \beta(g)(S)$ for all $R \in \mathfrak{R}$, $S \in \mathfrak{S}$. ([7], Ch. 1, §4, Proposition 2, p.56). Clearly $\alpha \otimes \beta: g \to \alpha(g) \otimes \beta(g)$ is a representation of G on $\mathfrak{R} \otimes \mathfrak{S}$.

2.10 Corollary With the above notation, suppose $\mathcal{H}$ and $\mathcal{K}$ are both separable, and $\alpha$, $\beta$ are both weakly measurable representations, then

$$\| f \circ (\alpha(g) \otimes \beta(g)) - f \circ (\alpha(h) \otimes \beta(h))\| \to 0$$

as $g \to h$ in G for each $h \in G$, $f \in (\mathfrak{R} \otimes \mathfrak{S})_*$.

Proof If $\{x_n\}$ is dense in $\mathcal{H}$, and $\{y_n\}$ dense in $\mathcal{K}$, then linear combinations of elements $\{x_n \otimes y_m; n, m \in N\}$ are dense in $\mathcal{H} \otimes \mathcal{K}$, so $\mathcal{H} \otimes \mathcal{K}$ is separable. Thus, by the theorem, it suffices to show that the map $\Theta$: $g \to \langle \alpha(g) \otimes \beta(g)(A)x, x\rangle$ is m-measurable for each $A \in \mathcal{R} \otimes \mathcal{S}$, and $x \in \mathcal{H} \otimes \mathcal{K}$.

Suppose first that $A = B \otimes C$ with $B \in \mathcal{R}$, $C \in \mathcal{S}$ and $x = \sum_{i=1}^{n} x_i \otimes y_i \in \mathcal{H}_0$. Then

$$\langle \alpha(g) \otimes \beta(g)(A)x, x\rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} \langle \alpha(g)(B)x_i, x_j\rangle\langle \beta(g)(C)y_i, y_j\rangle$$

Hence $g \to \omega_x(\alpha(g) \otimes \beta(g)(A))$ is m-measurable for A and x of this form, so $\Theta$ is m-measurable for $A \in \mathcal{B}$, $x \in \mathcal{H}_0$. If now $A \in \mathcal{R} \otimes \mathcal{S}$, then since $\mathcal{H} \otimes \mathcal{K}$ is separable, a similar argument to the foregoing, using the Kaplansky density theorem shows that there is a sequence $A_n \in \mathcal{B}$ with $A_n \to A$ weakly. Hence $g \to \omega_x(\alpha(g) \otimes \beta(g)(A))$ is m-measurable for $A \in \mathcal{R} \otimes \mathcal{S}$, $x \in \mathcal{H}_0$. Such x are norm dense in $\mathcal{H} \otimes \mathcal{K}$, so the result follows.

An immediate consequence of this is the following:

<u>2.11 Corollary</u> Let $\mathcal{R}$ and $\mathcal{S}$ be von Neumann algebras acting on separable Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively, G a locally compact group. If $\alpha$ is an ultraweakly continuous representation of G on $\mathcal{R}$, and $\beta$ is an ultraweakly continuous representation of G on $\mathcal{S}$, then $\alpha \otimes \beta$: $g \to \alpha(g) \otimes \beta(g)$ is an ultraweakly continuous representation of G on $\mathcal{R} \otimes \mathcal{S}$.

# CHAPTER III

## COVARIANT REPRESENTATIONS, PERMANENTLY SPATIAL AUTOMORPHISMS

Let G be a group of automorphisms of a C*-algebra $\mathfrak{A}$.

<u>3.1 Definition</u> Let $\mathcal{S} \subseteq E(\mathfrak{A})$. $\mathcal{S}$ is <u>G-closed</u> if $f \in \mathcal{S}$ implies $f \circ g \in \mathcal{S}$ for all $g \in G$.

<u>3.2 Definition</u> Let $\pi$ be a representation of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}_\pi$. $\pi$ is <u>covariant</u> if there is a homomorphism $U: g \to U_g$, from G to the unitary group of $\mathcal{B}(\mathcal{H}_\pi)$, with $U_g\pi(A)U_g^* = \pi(g(A))$, $(A \in \mathfrak{A},\ g \in G)$.

Covariant representations have been studied by several authors (see for example [11], [3], [53]). It is a natural objective to obtain a representation of the algebra in which the group of automorphisms is implemented by a unitary group, since this is a simpler and better known situation. The papers referred to above concern covariant representations where G is a topological group, and some continuity condition on the map $g \to U_g$ is demanded. We shall be concerned here, however, only with discrete groups. One of the consequences of our results is that if G is a group of automorphisms of a C*-algebra $\mathfrak{A}$, then $\mathfrak{A}$ has a faithful covariant representation, $\pi$ say. Moreover, if $\mathfrak{A}$ is a von Neumann algebra, we may choose $\pi$ so that $\pi(\mathfrak{A})$ is also a von Neumann algebra. Another consequence is that every automorphism of the universal representation of a C*-algebra is spatial, and we use this to obtain a result about single automorphisms of a C*-algebra. To discuss this we need the following definitions.

<u>3.3 Definition</u> Let $\alpha$ be an automorphism of a C*-algebra $\mathfrak{A}$. $\alpha$ is <u>permanently centre fixing</u> if for each faithful representation $\pi$ of $\mathfrak{A}$, the automorphism $\pi\alpha\pi^{-1}$ extends to an automorphism of $\pi(\mathfrak{A})^{-}$ which fixes the centre of $\pi(\mathfrak{A})^{-}$. $\alpha$ is <u>permanently spatial</u> if for each such representation $\pi$, there is a representation $\psi$ quasi-equivalent to $\pi$, such that $\psi\alpha\psi^{-1}$ is a spatial automorphism. We say that $\alpha$ is <u>permanently weakly inner</u> if for each such $\pi$, $\pi\alpha\pi^{-1}$ extends to an inner automorphism of $\pi(\mathfrak{A})^{-}$.

It is a reasonable conjecture that if $\alpha$ is permanently centre fixing, then $\alpha$ is permanently weakly inner. We shall show that if $\alpha$ is permanently centre fixing, then $\alpha$ is permanently spatial. Permanently weakly inner automorphisms have been discussed in [23], [32], [31], [7]. The main result of [23] is that if $\mathfrak{A}$ is a C*-algebra, and aut $(\mathfrak{A})$ has the norm topology as bounded operators on $\mathfrak{A}$, then every element of the connected component of the identity in aut $(\mathfrak{A})$ is permanently weakly inner.

<u>3.4 Theorem</u> Let G be a group of automorphisms of a C*-algebra $\mathfrak{A}$. If $\mathcal{S} \subseteq E(\mathfrak{A})$ is G-closed, then

$$\pi = \overset{\oplus}{\Sigma}\{\pi_f ; f \in \mathcal{S}\}$$

is a covariant representation.

<u>Proof</u> For $f \in \mathcal{S}$, let $\pi_f$ denote the representation of $\mathfrak{A}$ corresponding to f, acting on the Hilbert space $\mathcal{H}_f$, and $x_f$ a

cyclic vector in $\mathcal{H}_f$ with $f = \omega_{x_f} \circ \pi_f$. Let

$$\overset{\circ}{\mathcal{H}}_f = \{\pi_f(A)x_f ; A \in \mathfrak{A}\}.$$

Then if $\pi(\mathfrak{A})$ acts on the Hilbert space $\mathcal{H}_\pi$, we have

$$\mathcal{H}_\pi = \sum_f{}^{\oplus} \bar{\overset{\circ}{\mathcal{H}}}_f = \overline{\sum_f{}^{\oplus} \overset{\circ}{\mathcal{H}}_f}$$

Let $\overset{\circ}{\mathcal{H}} = \Sigma^{\oplus} \overset{\circ}{\mathcal{H}}_f$, and $x = \sum_f{}^{\oplus} \pi_f(A_f)x_f \in \overset{\circ}{\mathcal{H}}$. Then

$$\| \Sigma^{\oplus} \pi_f(g(A_{f \circ g}))x_f \|^2$$

$$= \langle \sum_f{}^{\oplus} \pi_f(g(A_{f \circ g}))x_f, \ \sum_f{}^{\oplus} \pi_f(g(A_{f \circ g}))x_f \rangle$$

$$= \sum_f \langle \pi_f(g(A_{f \circ g}))x_f, \ \pi_f(g(A_{f \circ g}))x_f \rangle$$

$$= \sum_f (f \circ g)(A^*_{f \circ g} A_{f \circ g})$$

$$= \sum_f f(A^*_f A_f)$$

since $\mathcal{S}$ is G-closed,

$$= \sum_f \langle \pi_f(A_f)x_f, \ \pi_f(A_f)x_f \rangle$$

$$= \langle \Sigma^{\oplus} \pi_f(A_f)x_f, \ \Sigma^{\oplus} \pi_f(A_f)x_f \rangle$$

$$= \| x \|^2$$

Thus we may define unambiguously an isometric mapping, $U_g$, from $\overset{\circ}{\mathcal{H}}$ to $\overset{\circ}{\mathcal{H}}$ by

$$U_g(\Sigma^{\oplus}\pi_f(A_f)x_f) = \Sigma^{\oplus}\pi_f(g(A_{f\circ g}))x_f .$$

$U_g$ is linear on $\overset{\circ}{\mathcal{H}}$ :

Let $x = \sum_f^{\oplus}\pi_f(A_f)x_f$, $y = \Sigma^{\oplus}\pi_f(B_f)x_f$, then

$$U_g(\lambda x + \mu y) = U_g(\sum_f^{\oplus}\lambda\pi_f(A_f)x_f + \mu\pi_f(B_f)x_f)$$

$$= U_g(\sum_f^{\oplus}\pi_f(\lambda A_f + \mu B_f)x_f)$$

$$= \sum_f^{\oplus}\pi_f(\lambda g(A_{f\circ g}) + \mu g(B_{f\circ g}))x_f$$

$$= \lambda\sum_f^{\oplus}\pi_f(g(A_{f\circ g}))x_f + \mu\sum_f^{\oplus}\pi_f(g(B_{f\circ g}))x_f$$

$$= \lambda U_g x + \mu U_g y .$$

$U_g U_h = U_{gh}$ on $\overset{\circ}{\mathcal{H}}$ $(g, h \in G)$ :

$$U_g U_h(\Sigma^{\oplus}\pi_f(A_f)x_f) = U_g(\Sigma^{\oplus}\pi_f(h(A_{f\circ h}))x_f)$$

$$= \Sigma^{\oplus}\pi_f(gh(A_{f\circ gh}))x_f$$

$$= U_{gh}(\Sigma^{\oplus}\pi_f(A_f)x_f .$$

Thus each $U_g$ extends to a unitary on $\mathcal{H}_\pi$, and they satisfy $U_g U_h = U_{gh}$ $(g, h \in G)$. Clearly $U_e = I$, with e the identity of G, so $g \to U_g$ is a group homomorphism. We claim that

$$\pi(g(A)) = U_g\pi(A)U_g^* \qquad (A \in \mathfrak{A}, g \in G) .$$

Let $y = \Sigma^{\oplus} \pi_f(B_f)x_f \in \overset{o}{\mathfrak{H}}$, $A \in \mathfrak{A}$, $g \in G$, then

$$\langle U_g^* \pi(A) U_g y, y \rangle = \langle \pi(A) U_g y, U_g y \rangle$$

$$= \sum_f \langle \pi_f(A) \pi_f(g(B_{f \circ g}))x_f, \ \pi_f(g(B_{f \circ g}))x_f \rangle$$

$$= \sum_f \langle \pi_f(g(B^*_{f \circ g})A \ g(B_{f \circ g}))x_f, x_f \rangle$$

$$= \sum_f (f \circ g)(B^*_{f \circ g} \ g^{-1}(A) \ B_{f \circ g})$$

$$= \sum_f f(B_f^* g^{-1} (A) B_f)$$

since $\mathcal{S}$ is G-closed. Thus

$$\langle U_g^* \pi(A) U_g y, y \rangle = \sum_f \langle \pi_f(B_f^* g^{-1} (A) B_f)x_f, x_f \rangle$$

$$= \sum_f \langle \pi_f(g^{-1}(A)) \pi_f(B_f)x_f, \ \pi_f(B_f)x_f \rangle$$

$$= \langle \pi(g^{-1}(A))y, y \rangle .$$

Such y are dense in $\mathfrak{H}_\pi$, so the result follows.

<u>3.5 Corollary</u> Let G be a group of automorphisms of a C*-algebra $\mathfrak{A}$, then $\mathfrak{A}$ has a faithful covariant representation.

<u>Proof</u> Let $\pi$ be the universal representation of $\mathfrak{A}$. $\pi = \Sigma^{\oplus}\{\pi_f ; f \in E(\mathfrak{A})\}$ and $E(\mathfrak{A})$ is G-closed. The result follows by Theorem 3.4.

<u>3.6 Corollary</u> Let G be a group of automorphisms of a von Neumann algebra $\mathfrak{R}$, then there is a faithful covariant representation $\psi$ of

$\mathfrak{R}$ such that $\psi(\mathfrak{R})$ is a von Neumann algebra.

Proof   Let $\mathfrak{S}$ denote the set of all normal states of $\mathfrak{R}$. Since each automorphism of $\mathfrak{R}$ is ultraweakly continuous, $\mathfrak{S}$ is G-closed. Thus $\psi = \Sigma^{\oplus}\{\pi_f;\ f \in \mathfrak{S}\}$ is a covariant representation, by Theorem 3.4. If f is a normal state, $\pi_f(\mathfrak{R})$ is a von Neumann algebra, by [7, Ch. 1, §4, Proposition 1, p.54], thus $\psi(\mathfrak{R})$ is a von Neumann algebra. $\psi$ is faithful since $\mathfrak{R}$ is isomorphic to the dual of $\mathfrak{R}_*$ and the normal states span $\mathfrak{R}_*$.

3.6 Corollary   Let $\mathfrak{A}$ be a C*-algebra, $\alpha$ an automorphism of $\mathfrak{A}$, $\pi$ the universal representation of $\mathfrak{A}$, then there is a unitary U on the Hilbert space $\mathcal{H}_\pi$ on which $\pi(\mathfrak{A})$ acts, such that

$$U^* \pi(A) U = \pi(\alpha(A)) \qquad (A \in \mathfrak{A}) .$$

Proof   This is a special case of Corollary 3.5.

3.7 Corollary   Let $\alpha$ be an automorphism of a C*-algebra $\mathfrak{A}$. If $\alpha$ is permanently centre fixing, then $\alpha$ is permanently spatial.

Proof   Let $\pi$ be the universal representation of $\mathfrak{A}$. By Corollary 3.6, there is a unitary U on $\mathcal{H}_\pi$ with $\pi\alpha\pi^{-1}(A) = UAU^*$ $(A \in \pi(\mathfrak{A}))$. Since $\alpha$ is permanently centre fixing, we have $UPU^* = P$ for all central projections P in $\pi(\mathfrak{A})^-$. If $\varphi$ is a faithful representation of $\mathfrak{A}$, then there is a projection Q in the centre of $\pi(\mathfrak{A})^-$ with $\varphi(\mathfrak{A})^-$ isomorphic to $\pi(\mathfrak{A})^- Q$. If $\beta$ denotes this isomorphism, then $\beta(\varphi(A)) = \pi(A)Q$ $(A \in \mathfrak{A})$. Now if $\Omega$ denotes the faithful representation $A \to \pi(A)Q$, then $\Omega$ is quasi-equivalent to $\varphi$, and $\Omega\alpha\Omega^{-1}$ is implemented by the unitary $U\Omega$. This completes the proof.

# CHAPTER IV

## ON A G-TWISTED EQUIVALENCE RELATION FOR VON NEUMANN ALGEBRAS ([39])

Let $\mathcal{R}$ be a von Neumann algebra, G a group of automorphisms of $\mathcal{R}$.

<u>4.1 Definition</u> Let E and F be projections in $\mathcal{R}$. E and F are <u>G-equivalent</u>, written $E \underset{G}{\sim} F$, if there is for each $g \in G$, an element $A_g \in \mathcal{R}$ with $E = \sum_{g \in G} A_g^* A_g$, $F = \sum_{g \in G} g(A_g A_g^*)$. Write $E \underset{G}{\prec} F$ if there is a projection $F_o$ in $\mathcal{R}$ with $E \underset{G}{\sim} F_o \leq F$.

F is $\underset{G}{\sim}$-finite if $E \leq F$ and $E \underset{G}{\sim} F$ imply $E = F$. $\mathcal{R}$ is said to be $\underset{G}{\sim}$-finite if the identity of $\mathcal{R}$ is a $\underset{G}{\sim}$-finite projection. $\mathcal{R}$ is said to be $\underset{G}{\sim}$-semifinite if every non-zero projection in $\mathcal{R}$ majorises a non-zero $\underset{G}{\sim}$-finite projection in $\mathcal{R}$. Let

$$C^G = \{A \in \mathcal{Z}(\mathcal{R}); \qquad g(A) = A \text{ for all } g \in G\}$$

If E is a projection in $\mathcal{R}$ let D(E) denote the smallest projection F in $C^G$ with $F \geq E$ (this exists since $C^G$ is abelian.) A projection E in $\mathcal{R}$ is said to be $\underset{G}{\sim}$-abelian if $E\mathcal{R}E = C^G E$.

If G is the group consisting of the identity automorphism, the above definition coincides with the usual equivalence between projections, by [22], (Theorem 4.1). Further, if $E \sim F$, then $E \underset{G}{\sim} F$ for any group G, clearly. In particular, this shows that a $\underset{G}{\sim}$-finite projection is finite, and a $\underset{G}{\sim}$-abelian projection is abelian. Hence if $\mathcal{R}$ is $\underset{G}{\sim}$-finite (resp. $\underset{G}{\sim}$-semifinite) then $\mathcal{R}$ is finite (resp. semifinite), so $\mathcal{R}$ posesses a trace. In [50] it is shown (Theorem 2) that $\mathcal{R}$ is $\underset{G}{\sim}$-semifinite if and only if $\mathcal{R}^+$ has a faithful normal G-invariant semifinite trace and, (Theorem 3), $\mathcal{R}$ is $\underset{G}{\sim}$-finite and countably decomposable if and only if there is a scalar valued faithful normal finite G-invariant trace on $\mathcal{R}$. If should be noted that the results of [50] go over to the case of a group of automorphisms of $\mathcal{R}$ not necessarily implemented by a group of unitaries. This follows by Corollary 3.6 of this thesis.

We define $\mathcal{R}$ to be $\underset{G}{\sim}$-type III if $\mathcal{R}$ contains no $\underset{G}{\sim}$-finite projections. It is easy to see (as we shall show below) that $\mathcal{R}$ is $\underset{G}{\sim}$-type III if and only if $\mathcal{R}Q$ is not $\underset{G}{\sim}$-semifinite for any projection Q in $C^G$. Suppose now that $\mathcal{R}$ (resp. $\mathcal{S}$) is a von Neumann algebra and G (resp. H) is a group of automorphisms of $\mathcal{R}$ (resp. $\mathcal{S}$). If $g \in G$, $h \in H$, then there is a unique automorphism, which we shall denote by $g \otimes h$, of $\mathcal{R} \otimes \mathcal{S}$ such that $g \otimes h(A \otimes B) = g(A) \otimes h(B)$ ($A \in \mathcal{R}$, $B \in \mathcal{S}$), by [7], (p.56 Proposition 2). The map $(g,h) \to g \otimes h$ is a group homomorphism identifying the direct product, G x H, of G and H as a group of automorphisms of $\mathcal{R} \otimes \mathcal{S}$. We shall show that if either $\mathcal{R}$ is $\underset{G}{\sim}$-type III or $\mathcal{S}$ is $\underset{H}{\sim}$-type III, then $\mathcal{R} \otimes \mathcal{S}$ is $\underset{G\times H}{\sim}$-type III. The motivation for this result is a theorem of Sakai ([46], Theorem 2.6.4), and if G and H are both the trivial group consisting of the identity automorphism, then our result coincides with Sakai's result. Sakai also shows in [46], Proposition 2.6.1, that the tensor product of two finite (resp. semifinite) von Neumann algebras is finite (resp. semifinite). We shall show, in Corollaries 4.6 and 4.7 below that if $\mathcal{R}$ is $\underset{G}{\sim}$-finite (resp. $\underset{G}{\sim}$-semifinite) and $\mathcal{S}$ is $\underset{H}{\sim}$-finite (resp. $\underset{H}{\sim}$-semifinite), then $\mathcal{R} \otimes \mathcal{S}$ is $\underset{G\times H}{\sim}$-finite (resp. $\underset{G\times H}{\sim}$-semifinite).

We now describe the crossed product algebra, outlined in [41], but in a more complete and detailed form, using tensor product notation. Let $\mathcal{R}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, G a group of automorphisms of $\mathcal{R}$. Denote by $\epsilon_g$ the function which takes the value 1 at g and is zero elsewhere on G. $\{\epsilon_g; g \in G\}$ is an orthonormal basis for $\ell^2(G)$. If $g \in G$, $x \in \mathcal{H}$, define

$$U_h(x \otimes \epsilon_g) = x \otimes \epsilon_{gh^{-1}} \qquad (h \in G)$$

and

$$\Phi(A)(x \otimes \epsilon_g) = g(A)x \otimes \epsilon_g , \qquad (A \in \mathcal{R}, \ g \in G)$$

Then $U_h$ extends to a unitary on $\mathcal{H} \otimes \ell^2(G)$, and

$$\begin{aligned} U_h U_k(x \otimes \epsilon_g) &= U_h(x \otimes \epsilon_{gk^{-1}}) \\ &= x \otimes \epsilon_{gk^{-1}h^{-1}} \\ &= U_{hk}(x \otimes \epsilon_g) \end{aligned}$$

Also, $\Phi(A)$ extends to a bounded linear operator on $\mathcal{H} \otimes \ell^2(G)$, and

$$\begin{aligned} U_h\Phi(A)U_{h^{-1}}(x \otimes \epsilon_g) &= U_h\Phi(A)(x \otimes \epsilon_{gh}) \\ &= U_h(gh(A)x \otimes \epsilon_{gh}) \\ &= gh(A)x \otimes \epsilon_g \\ &= g(h(A))x \otimes \epsilon_g \\ &= \Phi(h(A))(x \otimes \epsilon_g) \end{aligned}$$

Thus, since linear combinations of $x \otimes \epsilon_g$, $(x \in \mathcal{H}, g \in G)$, are dense in $\mathcal{H} \otimes \ell^2(G)$, $g \to U_g$ is a unitary representation of G with

$$U_g\Phi(A)U_{g^{-1}} = \Phi(g(A)) \qquad (g \in G, \ A \in \mathcal{R}) .$$

$\Phi$ is an ultraweakly continuous *-isomorphism of $\mathfrak{R}$, so $\Phi(\mathfrak{R})$ is a von Neumann algebra. We define R x G (the crossed product algebra) to be the von Neumann algebra generated by $\{\Phi(A), U_g; A \in \mathfrak{R}, g \in G\}$. Since $(\Phi(A)U_g)^* = U_{g^{-1}}\Phi(A^*) = \Phi(g^{-1}(A^*))U_{g^{-1}}$, finite sums $\sum_i \Phi(A_i)U_{g_i}$ form a *-algebra weakly dense in $\mathfrak{R} \times G$. We call this *-algebra $(\mathfrak{R} \times G)_o$. The map $x \to x \otimes \epsilon_g$ is an isometric linear map from $\mathcal{H}$ to a closed linear subspace $\mathcal{H}_g$ of $\mathcal{K} = \mathcal{H} \otimes \ell^2(G)$. The $\mathcal{H}_g$'s are orthogonal since the $\epsilon_g$'s are orthonormal, and the linear subspace generated by $\{\mathcal{H}_g; g \in G\}$ is dense in $\mathcal{K}$. It follows that $\mathcal{K}$ is the direct sum of the $\mathcal{H}_g$'s, and every element x of $\mathcal{K}$ can be represented uniquely in the form $x = \sum_{g \in G} x_g \otimes \epsilon_g$, where $\{x_g; g \in G\}$ is any family of elements of $\mathcal{H}$ such that $\sum \|x_g\|^2 < \infty$. We have also $\|x\|^2 = \sum \|x_g\|^2$.

Let $E_g$ denote the projection onto $\mathcal{H}_g$, and let $B = \sum U_g \Phi(A_g)$ be an element of $(\mathfrak{R} \times G)_o$. Then

$$
\begin{aligned}
E_s B E_t (\Sigma\, x_g \otimes \epsilon_g) &= E_s B (x_t \otimes \epsilon_t) \\
&= E_s (\Sigma\, U_g \Phi(A_g)(x_t \otimes \epsilon_t)) \\
&= E_s\, \Sigma\, U_g (t(A_g) x_t \otimes \epsilon_t)) \\
&= E_s\, \Sigma (t(A_g) x_t \otimes \epsilon_{tg^{-1}}) \\
&= t(A_{s^{-1}t}) x_t \otimes \epsilon_s \\
&= E_s (t(A_{s^{-1}t}) x_t \otimes \epsilon_s) \\
&= E_s U_{s^{-1}t} (t(A_{s^{-1}t}) x_t \otimes \epsilon_t) \\
&= E_s\, U_{s^{-1}t} \Phi(A_{s^{-1}t}) E_t (\Sigma\, x_g \otimes \epsilon_g)
\end{aligned}
$$

Thus

$$E_sBE_t = E_sU_{s^{-1}t}\Phi(A_{s^{-1}t})E_t \ .$$

If now $B \in \mathfrak{R} \times G$, then by the Kaplansky density theorem, there is a net $(B_\alpha)_{\alpha \in A}$, with $B_\alpha \to B$ ultraweakly, $B_\alpha \in (\mathfrak{R} \times G)_o$, and $\|B_\alpha\| \leq \|B\|$ for all $\alpha \in A$. Then $E_tBE_s = \lim E_tB_\alpha E_s$, with $B_\alpha$ of the form $B_\alpha = \Sigma\, U_g\Phi(T_g^\alpha)$. Now

$$E_tB_\alpha E_s = E_t \cdot U_{t^{-1}s}\Phi(T^\alpha_{t^{-1}s})E_s$$

Since $\Phi$ and $U_g$ are isometric, $\|T^\alpha_{t^{-1}s}\| = \|U_{t^{-1}s}\Phi(T^\alpha_{t^{-1}s})\|$ $= \sup_{s,t} \|E_tU_{t^{-1}s}\Phi(T^\alpha_{t^{-1}s})E_s\| = \sup_{s,t} \|E_tB_\alpha E_s\| \leq \|B\|$. The ball radius $\|B\|$ in $\mathfrak{R}$ is $\sigma w$-compact, thus there is a $\sigma w$-convergent subnet $(T^{\alpha'}_{t^{-1}s})$ of $(T^{\alpha}_{t^{-1}s})$. Let $(T^{\alpha'}_{t^{-1}s})$ converge ultraweakly to $T_{t^{-1}s} \in \mathfrak{R}$. Then $E_tU_{t^{-1}s}\Phi(T^{\alpha'}_{t^{-1}s})E_s$ converges ultraweakly to $E_tBE_s$. Thus

$$E_tBE_s = E_tU_{t^{-1}s}\Phi(T_{t^{-1}s})E_s \ .$$

In particular, with s=t=e,

$$E_eBE_e = E_e\Phi(T_e)E_e \ ,$$

and $E_e\Phi(T_e)E_e\,(\Sigma\, x_g \otimes \epsilon_g)$

$$= E_e\Phi(T_e)(x_e \otimes \epsilon_e)$$

$$= E_e(T_ex_e \otimes \epsilon_e)$$

$$= T_ex_e \otimes \epsilon_e \ .$$

If S, T are elements of $\mathfrak{R}$, such that $E_eBE_e(\Sigma\, x_g \otimes \epsilon_g) = Sx_e \otimes \epsilon_e$ $= Tx_e \otimes \epsilon_e$, then $Sx = Tx$ for all $x \in \mathcal{H}$, so $S = T$. This shows that $T_e$ is unique. Define $\Gamma(B) = T_e$. Then $\Gamma$ is a mapping from $\mathfrak{R} \times G$ into $\mathfrak{R}$. Clearly $\Gamma$ is linear, and $\Gamma(\Phi(T)) = T \quad (T \in \mathfrak{R})$. If $(B_\alpha)_{\alpha \in A} \in \mathfrak{R} \times G$, and $B_\alpha \to B$ ultraweakly, then $E_e B_\alpha E_e \to E_e B E_e$ ultraweakly, so $\Gamma$ is normal. If we identify $\mathfrak{R}$ with its image $\Phi(\mathfrak{R})$ in $\mathfrak{R} \times G$, this shows that $\Gamma$ is a normal projection of norm one from $\mathfrak{R} \times G$ onto $\mathfrak{R}$. Let $B \in \mathfrak{R} \times G$, with

$$E_t B E_s = E_t U_{t^{-1}s} \Phi(T_{t^{-1}s}) E_s \qquad (s,t \in G)$$

then

$$E_s B E_s = E_s \Phi(T_e) E_s \quad .$$

Thus if $\Gamma(B) = 0$, then $E_s B E_s = 0 \quad (s \in G)$. If $B \geq 0$, this implies that $B^{\frac{1}{2}} E_s = 0$ for all $s \in G$. Now $\sum_{s \in G} E_s = I$, thus $B^{\frac{1}{2}} = 0$ and $B = 0$. Hence $\Gamma$ is faithful.

The following Lemma is part of the proof of Lemma 10 in [50].

<u>4.3 Lemma</u> Let $\mathfrak{R}$ be a von Neumann algebra, $G$ a group of automorphisms of $\mathfrak{R}$. Suppose $\mathfrak{R}$ contains no non-zero $\underset{G}{\sim}$-abelian projections, yet contains a countably decomposable $\sim$-finite projection, then there is a non-zero projection $Q \in C^G$ such that $\mathfrak{R}Q$ is $\underset{G}{\sim}$-semifinite.

<u>Proof</u> Let $E$ be a countably decomposable $\underset{G}{\sim}$-finite projection in $\mathfrak{R}$. Since $E$ is not $\underset{G}{\sim}$-abelian, $E\mathfrak{R}E \neq C^G E$, thus there is a projection $H$ in $E\mathfrak{R}E$ with $H \neq ED(H)$. Let $F = H + (1 - D(H))E$. Then $F \leq E$, $F \neq E$, and

$$D(F) = D(H) + (1 - D(H))D(E) = D(E)$$

Suppose $\Phi(F)$ is a purely infinite projection in $\mathfrak{R} \times G$. Then $\Phi(E)$ is countably decomposable in $\mathfrak{R} \times G$ by Lemma 5 of [50], and $\Phi(F) \leq \Phi(E)$, thus $\Phi(F) \sim \Phi(E)$ in $\mathfrak{R} \times G$.

Hence by Lemma 1 of [50], $F \underset{G}{\sim} E$, contradicting $\underset{G}{\sim}$-finiteness of E. It follows that for some non-zero central projection P in $\mathcal{Z}(\mathcal{R} \times G)$, $\Phi(F)P$ is a finite non-zero projection. Noting that $\Phi(D(F)) = C_{\Phi(F)}$ by Lemma 3 of [50] we have

$$(\mathcal{R} \times G)\Phi(D(E))P = (\mathcal{R} \times G)\Phi(D(F))P$$

$$= (\mathcal{R} \times G)C_{\Phi(F)}P$$

$$= (\mathcal{R} \times G)C_{P\Phi(F)}$$

Since $P\Phi(F)$ is finite, it follows that $(\mathcal{R} \times G)\Phi(D(E))P$ is semifinite, and non-zero. Let $\varphi$ be a normal semifinite scalar valued trace on $\mathcal{R} \times G$ with support $P\Phi(D(E))$. For $A \in \mathcal{R}^+$, define $\tau(A) = \varphi(\Phi(A))$. Clearly $\tau$ is normal, and

$$\tau(g(A)) = \varphi(U_g\Phi(A)U_g^*)$$

$$= \varphi(\Phi(A))$$

$$= \tau(A) .$$

So $\tau$ is G-invariant. Since $\tau(F) < \infty$, there is a non-zero central projection Q in $\mathcal{R}$ such that $\tau$ is faithful and semifinite on $\mathcal{R}Q$ ([7], Ch.1 §6, Corollary 2). Since $\tau$ is G-invariant, Q must be G-invariant, by uniqueness of Q. Thus $Q \in C^G$, and the Lemma is proved.

<u>4.4 Proposition</u> Let G be a group of automorphisms of a von Neumann algebra $\mathcal{R}$, then $\mathcal{R}$ is $\underset{G}{\sim}$-type III if and only if $\mathcal{R}Q$ is not $\underset{G}{\sim}$-semifinite for any non-zero projection Q in $C^G$.

Proof By Lemma 6 of [50], there is a $\underset{G}{\sim}$-abelian projection E in $\mathcal{R}$ such that

$$\mathcal{R} = \mathcal{R}D(E) + \mathcal{R}(1 - D(E))$$

and $\mathcal{R}(1 - D(E))$ has no non-zero $\underset{G}{\sim}$-abelian projections. By Lemma 9 of [50], $\mathcal{R}D(E)$ has a faithful normal semifinite G-invariant trace, so by Theorem 2 of [50], $\mathcal{R}D(E)$ is $\underset{G}{\sim}$-semifinite. Let $\mathcal{S} = \mathcal{R}(1 - D(E))$, then $\mathcal{S}$ contains no $\underset{G}{\sim}$-abelian projections. Suppose $\mathcal{S}$ contains a $\underset{G}{\sim}$-finite projection E, then if F is a countably decomposable subprojection of E, F is also $\underset{G}{\sim}$-finite. Hence by the preceding Lemma we have that there is a projection Q in $C^G$ such that $\mathcal{R}Q$ is $\underset{G}{\sim}$-semifinite. Thus we have that if $\mathcal{R}Q$ is not $\underset{G}{\sim}$-semifinite for any $Q \in C^G$, $\mathcal{R}$ must be $\underset{G}{\sim}$-type III. Conversely if $\mathcal{R}Q$ is $\sim$-semifinite for some $Q \in C^G$, let E be a $\underset{G}{\sim}$-finite projection in $\mathcal{R}Q$, then E is $\underset{G}{\sim}$-finite in $\mathcal{R}$. This proves the result.

Let $\mathcal{R}$ (resp. $\mathcal{S}$) be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ (resp. $\mathcal{K}$) and G (resp. H) be a group of automorphisms of $\mathcal{R}$ (resp. $\mathcal{S}$). Denote by $V_h$ (resp $W_{(g,h)}$) the group of unitaries in the crossed product algebra $(\mathcal{S} \times H)$ (resp. $\mathcal{R} \otimes \mathcal{S} \times (G \times H)$ corresponding to the $U_g$'s defined above for $\mathcal{R}$ and G.

4.5 Lemma With notation as above, $(\mathcal{R} \times G) \otimes (\mathcal{S} \times H)$ is spatially *-isomorphic to $\mathcal{R} \otimes \mathcal{S} \times (G \times H)$.

Proof Let $(x_\alpha)$, $(y_\beta)$ be orthonormal bases for $\mathcal{H}$, $\mathcal{K}$ respectively, and $\{\epsilon_g; g \in G\}$, $\{\epsilon_h; h \in H\}$, $\{\epsilon_{(g,h)}; (g,h) \in G \times H\}$ the orthonormal bases described above for $\ell^2(G)$, $\ell^2(H)$ and $\ell^2(G \times H)$ respectively. Define

$$V((x_\alpha \otimes y_\beta) \otimes \epsilon_{(g,h)}) = (x_\alpha \otimes \epsilon_g) \otimes (y_\beta \otimes \epsilon_h)$$

for $g \in G$, $h \in H$. Then $V$ extends to a unitary transformation between $(\mathcal{H} \otimes \mathcal{K}) \otimes \ell^2(G \times H)$ and $(\mathcal{H} \otimes \ell^2(G)) \otimes (\mathcal{K} \otimes \ell^2(H))$. If $A \in \mathcal{R}$, $B \in \mathcal{S}$, $x_\alpha \in \mathcal{H}$, $y_\beta \in \mathcal{K}$, and $(g,h)$, $(k,\ell) \in G \times H$, we have

$$V^{-1}(\Phi(A)U_g \otimes \Phi(B)V_h)V((x_\alpha \otimes y_\beta) \otimes \epsilon_{(k,\ell)})$$

$$= V^{-1}(\Phi(A)U_g \otimes \Phi(B)V_h)((x_\alpha \otimes \epsilon_k) \otimes (y_\beta \otimes \epsilon_\ell))$$

$$= V^{-1}(\Phi(A)(x_\alpha \otimes \epsilon_{kg^{-1}}) \otimes \Phi(B)(y_\beta \otimes \epsilon_{\ell h^{-1}}))$$

$$= V^{-1}((kg^{-1}(A)x_\alpha \otimes \epsilon_{kg^{-1}}) \otimes (\ell h^{-1}(B)y_\beta \otimes \epsilon_{\ell h^{-1}}))$$

$$= (kg^{-1}(A)x_\alpha \otimes \ell h^{-1}(B)y_\beta) \otimes \epsilon_{(kg^{-1},\, \ell h^{-1})}$$

$$= \Phi(A \otimes B)((x_\alpha \otimes y_\beta) \otimes \epsilon_{(kg^{-1},\, \ell h^{-1})})$$

$$= \Phi(A \otimes B)W_{(g,h)}(x_\alpha \otimes y_\beta) \otimes \epsilon_{(k,\ell)}) .$$

Since linear combinations of elements $(x_\alpha \otimes y_\beta) \otimes \epsilon_{(k,\ell)}$ are dense in $\mathcal{H} \otimes \mathcal{K} \otimes \ell^2(G \times H)$, the following identity gives rise to a mapping from $(\mathcal{R} \times G)_0 \otimes (\mathcal{S} \times H)_0$ onto $(\mathcal{R} \otimes \mathcal{S} \times (G \times H))_0$:

$$V^{-1}(\Phi(A)U_g \otimes \Phi(B)V_h)V = \Phi(A \otimes B)W_{(g,h)} .$$

This map then extends to the required spatial *-automorphism.

<u>4.6 Corollary</u> Let G be a group of automorphisms of a von Neumann algebra $\mathfrak{R}$, H a group of automorphisms of a von Neumann algebra $\mathfrak{S}$. If $\mathfrak{R}$ is $\underset{G}{\sim}$-finite and $\mathfrak{S}$ is $\underset{H}{\sim}$-finite, then $\mathfrak{R} \otimes \mathfrak{S}$ is $\underset{G \times H}{\sim}$-finite.

<u>Proof</u> Let I denote the identity of $\mathfrak{R}$. Then I is a $\underset{G}{\sim}$-finite projection in $\mathfrak{R}$, so $\Phi(I)$ is a finite projection in $\mathfrak{R} \times G$. ([41] Theorem 4.1). Clearly $\Phi(I)$ is the identity of $\mathfrak{R} \times G$, so $\mathfrak{R} \times G$ is finite. Similarly $\mathfrak{S} \times H$ is finite. Hence $(\mathfrak{R} \times G) \otimes (\mathfrak{S} \times H)$ is finite by [46], Proposition 2.6.1, and so $(\mathfrak{R} \otimes \mathfrak{S}) \times (G \times H)$ is finite by the Lemma. Using [41], Theorem 4.1, again, we see that the identity of $\mathfrak{R} \otimes \mathfrak{S}$ is a $\underset{G \times H}{\sim}$-finite projection, giving the result.

<u>4.7 Corollary</u> Let G be a group of automorphisms of a von Neumann algebra $\mathfrak{R}$, H a group of automorphisms of a von Neumann algebra $\mathfrak{S}$. If $\mathfrak{R}$ is $\underset{G}{\sim}$-semifinite and $\mathfrak{S}$ is $\underset{H}{\sim}$-semifinite, then $\mathfrak{R} \otimes \mathfrak{S}$ is $\underset{G \times H}{\sim}$-semifinite.

<u>Proof</u> Let $(E_\alpha)$ be a maximal orthogonal family of $\underset{G}{\sim}$-finite projections in $\mathfrak{R}$. Let $E = \Sigma E_\alpha$. If $E \neq I$, there is a non-zero $\underset{G}{\sim}$-finite projection P with $P \leq I - E$. Then P can be added to the family, contradicting maximality. Hence $\Sigma E_\alpha = I$. Similarly there is an orthogonal family $(F_\beta)$ of $\underset{H}{\sim}$-finite projections in $\mathfrak{S}$ with $I = \Sigma F_\beta$. Now suppose $\mathfrak{R} \otimes \mathfrak{S}$ is not $\underset{G \times H}{\sim}$-semifinite. If $(Q_\alpha)$ is an increasing family of projections in $C^{G \times H}$ such that $(\mathfrak{R} \otimes \mathfrak{S})Q_\alpha$ is $\underset{G \times H}{\sim}$-semifinite, and $Q = \bigvee_\alpha Q_\alpha$, let E be a non-zero

$\text{proj}^n$ in $(\mathcal{R} \otimes \mathcal{S})Q$, then $EQ_\alpha \neq 0$ for some $\alpha$, thus there is a $\underset{G \times H}{\sim}$-finite projection F with $0 \neq F \leq EQ_\alpha \leq E$. Hence $(\mathcal{R} \otimes \mathcal{S})Q$ is $\underset{G \times H}{\sim}$-semifinite. Thus by Zorn's Lemma, there is a maximal projection Q in $C^{G \times H}$ with $(\mathcal{R} \otimes \mathcal{S})Q$ being $\underset{G \times H}{\sim}$-semifinite. Clearly, by Proposition 4.4, $(\mathcal{R} \otimes \mathcal{S})(1 - Q)$ is $\underset{G \times H}{\sim}$-type III. Now $E_\alpha \otimes F_\beta \to I$ ultraweakly, thus for some $\alpha$ and $\beta$, $(I - Q)(E_\alpha \otimes F_\beta) \neq 0$. If $\Omega$ denotes the *-isomorphism of Lemma 4.5

$$\Phi(E_\alpha) \otimes \Phi(F_\beta) = \Omega^{-1}(\Phi(E_\alpha \otimes F_\beta))$$

$E_\alpha$ is $\underset{G}{\sim}$-finite and $F_\beta$ is $\underset{H}{\sim}$-finite, thus by [41], Theorem 4.1, $\Phi(E_\alpha)$ and $\Phi(F_\beta)$ are finite projections, hence $\Phi(E_\alpha) \otimes \Phi(F_\beta)$ is finite, and so $\Phi(E_\alpha \otimes F_\beta)$ is a finite projection. Using [41] Theorem 4.1 again, we see that $E_\alpha \otimes F_\beta$ is a $\underset{G \times H}{\sim}$-finite projection in $\mathcal{R} \otimes \mathcal{S}$. Let

$$(1 - Q)(E_\alpha \otimes F_\beta) \underset{G}{\sim} F_0 \leq (1 - Q)(E_\alpha \otimes F_\beta)$$

Then

$$E_\alpha \otimes F_\beta \underset{G}{\sim} F_0 + Q(E_\alpha \otimes F_\beta) \leq E_\alpha \otimes F_\beta$$

Thus

$$F_0 + Q(E_\alpha \otimes F_\beta) = E_\alpha \otimes F_\beta \ ,$$

so $(1 - Q)(E_\alpha \otimes F_\beta)$ is a $\underset{G \times H}{\sim}$-finite projection in $(\mathcal{R} \otimes \mathcal{S})(1 - Q)$, contradicting the fact that $(\mathcal{R} \otimes \mathcal{S})(1 - Q)$ is $\underset{G \times H}{\sim}$-type III. Hence $\mathcal{R} \otimes \mathcal{S}$ is $\underset{G \times H}{\sim}$-semifinite.

The following Lemma is part of the proof of Theorem 2.6.4 of [46], in more detailed form.

<u>4.7 Lemma</u> Let $\mathcal{R}$ and $\mathcal{S}$ be von Neumann algebras, then for each normal state $\varphi$ of $\mathcal{S}$, there is a mapping $P_\varphi$ from $\mathcal{R} \otimes \mathcal{S}$

onto $\mathfrak{R}$ satisfying the following conditions:

1. $P_\varphi(I \otimes I) = I$

2. $\|P_\varphi(A)\| \leq \|A\|$

3. $P_\varphi(H) \geq 0 \qquad (H \geq 0)$

4. $P_\varphi(AXB) = AP_\varphi(X)B \qquad (A,B \in \mathfrak{R},\ X \in \mathfrak{R} \otimes \mathfrak{S})$

5. $P_\varphi(X)^* P_\varphi(X) \leq P_\varphi(X^*X)$

6. If $P_\varphi(X^*X) = 0$ for all normal states $\varphi$ on $\mathfrak{S}$, then $X = 0$.

7. $P_\varphi$ is ultraweakly and ultrastrongly continuous.

Proof  Let $\varphi$ be a normal state on $\mathfrak{S}$. For $A \in \mathfrak{R} \otimes \mathfrak{S}$, put $T_A(f) = (f \otimes \varphi)(A) \quad (f \in \mathfrak{R}_*)$. Then $|T_A(f)| \leq \|f\|\ \|A\|$, thus $T_A \in (\mathfrak{R}_*)^* = \mathfrak{R}$. Hence there is a unique element $P_\varphi(A)$ in $\mathfrak{R}$ with

$$(f \otimes \varphi)(A) = f(P_\varphi(A)) \qquad (A \in \mathfrak{R} \otimes \mathfrak{S})$$

$$(f \otimes \varphi)(I \otimes I) = f(I) \qquad (f \in \mathfrak{R}_*)$$

Since $\mathfrak{R}_*$ separates the points of $\mathfrak{R}$, this proves part 1.

$$\|P_\varphi(A)\| = \sup\{|f(P_\varphi(A)|\ ;\ f \in \mathfrak{R}_*,\ \|f\| = 1\}$$

This proves 2.

If $H \in \mathfrak{R} \otimes \mathfrak{S}$, $H \geq 0$, then for each normal state $f$ of $\mathfrak{R}$,

$$0 \leq (f \otimes \varphi)(H) = f(P_\varphi(H))$$

Hence $P_\varphi(H) \geq 0$. This proves 3. For $f \in \mathcal{R}_*$, $X \in \mathcal{R}$, let $(L_X f): A \to f(XA)$ $(A \in \mathcal{R})$, and $(R_X f): A \to f(AX)$ $(A \in \mathcal{R})$. Then clearly, $L_X f$, $R_X f$ lie in $\mathcal{R}_*$. If $X, Y \in \mathcal{R}$, $A \in \mathcal{R} \otimes \mathcal{S}$, $f \in \mathcal{R}_*$,

$$
\begin{aligned}
(f \otimes \varphi)(XAY) &= (L_X R_Y f) \otimes \varphi(A) \\
&= L_X R_Y f(P_\varphi(A)) \\
&= f(XP_\varphi(A)Y)
\end{aligned}
$$

This shows 4.

Let

$$\psi \geq 0, \quad \psi \in \mathcal{R}_* .$$

Then

$$
\begin{aligned}
(\psi \otimes \varphi)(P_\varphi(A)^* P_\varphi(A)) &= (\psi \otimes \varphi)(P_\varphi(A^* P_\varphi(A))) \text{ by part 4} \\
&= \psi(P_\varphi(A^* P_\varphi(A))) \\
&= (\psi \otimes \varphi)(A^* P_\varphi(A)) \\
&\leq (\psi \otimes \varphi)(A^*A)^{\frac{1}{2}} (\psi \otimes \varphi)(P_\varphi(A)^* P_\varphi(A))^{\frac{1}{2}}
\end{aligned}
$$

by the Cauchy-Schwartz Theorem. Hence

$$(\psi \otimes \varphi)(P_\varphi(A)^* P_\varphi(A))^{\frac{1}{2}} \leq (\psi \otimes \varphi)(A^*A)^{\frac{1}{2}}$$

thus

$$(\psi \otimes \varphi)(P_\varphi(A)^* P_\varphi(A)) \leq (\psi \otimes \varphi)(A^*A)$$

i.e.

$$\psi(P_\varphi(A)^* P_\varphi(A)) \leq \psi(P_\varphi(A^*A))$$

This holds for all such $\psi$, so

$$P_\varphi(A)^* P_\varphi(A) \leq P_\varphi(A^*A)$$

This proves part 5.

Let $A_\alpha \to A$ ultraweakly, with $A_\alpha$, $A$ in $\mathcal{R} \otimes \mathcal{S}$. Let $f \in \mathcal{R}_*$. Then

$$f(P_\varphi(A_\alpha)) = (f \otimes \varphi)(A_\alpha) \to (f \otimes \varphi)(A) = f(P_\varphi(A)) .$$

Thus $P_\varphi$ is ultraweakly continuous. If $A_\alpha \to A$ ultrastrongly, then $A_\alpha^* A_\alpha \to A^*A$ ultraweakly. Thus part 7 follows by part 5.

If $P_\varphi(A^*A) = 0$ for all normal states $\varphi$ on $\mathcal{S}$, then $(f \otimes \varphi)(A^*A) = 0$ for all $f \in \mathcal{R}_*$, $\varphi \in \mathcal{S}_*$, since the normal states span $\mathcal{S}_*$. Such elements are separating for $\mathcal{R} \otimes \mathcal{S}$ hence $A^*A = 0$ and $A = 0$. This proves 6, and completes the proof.

<u>4.8 Theorem</u> Let $\mathcal{R}$ and $\mathcal{S}$ be von Neumann algebras acting on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively. Let G (resp. H) be a group of automorphisms of $\mathcal{R}$ (resp. $\mathcal{S}$). If either $\mathcal{R}$ is $\underset{G}{\sim}$-type III or $\mathcal{S}$ is $\underset{H}{\sim}$-type III, then $\mathcal{R} \otimes \mathcal{S}$ is $\underset{G \times H}{\sim}$-type III.

<u>Proof</u> Let $\{\psi_\varphi ; \varphi \in \mathcal{S} \times H)_*\}$ be the projections of norm one from $(\mathcal{R} \times G) \otimes (\mathcal{S} \times H)$ onto $(\mathcal{R} \times G)$ constructed in Lemma 4.4, and suppose $\mathcal{R}$ is $\underset{G}{\sim}$-type III. Denote by $\Omega$ the *-isomorphism of Lemma 4.3 above. Let E be a $\underset{G \times H}{\sim}$-finite projection in $\mathcal{R} \otimes \mathcal{S}$. We have to prove $E = 0$. Suppose $E \neq 0$, then $0 \neq \Phi(E)$ is a finite projection in $\mathcal{R} \otimes \mathcal{S} \times (G \times H)$ by [41] Theorem 4.1, so $F = \Omega^{-1}\Phi(E)$ is a finite projection in $(\mathcal{R} \times G) \otimes (\mathcal{S} \times H)$. Since $\Phi(E) \in \Phi(\mathcal{R} \otimes \mathcal{S})$, $\Phi(E)$ is the ultraweak limit of elements of the form $\Phi(A_\gamma)$, with $A_\gamma = \sum_i \lambda_i C_i \otimes D_i$ $(C_i \in \mathcal{R}, D_i \in \mathcal{S})$. Now

$$\Omega^{-1}\Phi(A_\gamma) = \Sigma\, \lambda_i \Omega^{-1}\Phi(C_i \otimes D_i)$$

$$= \Sigma\, \lambda_i \Phi(C_i) \otimes \Phi(D_i)$$

Thus

$$\Omega^{-1}\Phi(A_\gamma) \in \Phi(\mathcal{R}) \otimes \Phi(\mathcal{S})$$

for all such $A_\gamma$. Since $\Omega$ is ultraweakly bicontinuous ([7], Ch.1, §4, Theorem 2, p.53), F is the ultraweak limit of elements of the form $\Omega^{-1}\Phi(A_\gamma)$. Now $\Phi$ is an ultraweakly continuous *-isomorphism, so $\Phi(\mathcal{R}) \otimes \Phi(\mathcal{S})$ is ultraweakly closed, thus $F \in \Phi(\mathcal{R}) \otimes \Phi(\mathcal{S})$. If $f \in (\mathcal{R} \times G)_*, \varphi \in (\mathcal{S} \times H)_*$, $A \in \Phi(\mathcal{R})$, $B \in \Phi(\mathcal{S})$, then

$$f(\psi_\varphi(A \otimes B)) = (f \otimes \varphi)(A \otimes B)$$

$$= f(A)\,\varphi(B)$$

$$= f(\varphi(B) A)$$

Since $(\mathcal{R} \times G)_*$ separates the points of $\mathcal{R} \times G$, we have

$$\psi_\varphi(A \otimes B) = \varphi(B)\, A\, .$$

Suppose now $B_\gamma \rightarrow F$ ultraweakly, with $B_\gamma$ of the form $B_\gamma = \Sigma\, \lambda_i M_i \otimes N_i$ ($M_i \in \Phi(\mathcal{R})$, $N_i \in \Phi(\mathcal{S})$), then $\psi_\varphi(F)$ is the ultraweak limit of $\psi_\varphi(B_\gamma)$, since each $\psi_\varphi$ is normal, and

$$\psi_\varphi(B_\gamma) = \Sigma\, \lambda_i \psi_\varphi(M_i \otimes N_i)$$

$$= \Sigma\, \lambda_i \varphi(N_i)\, M_i$$

Thus $\psi_\varphi(B_\gamma) \in \Phi(\mathcal{R})$, and hence

$$\psi_\varphi(F) \in \Phi(\mathcal{R}) \qquad \text{for all states } \varphi \in (\mathcal{S} \times H)_*$$

Choose a $\varphi_o$ with $\psi_{\varphi_o}(F) \neq 0$. Let P be a spectral projection of $\psi_{\varphi_o}(F)$ with $0 < \lambda P < \psi_{\varphi_o}(F)$ for some $\lambda > 0$ (this is possible since $\psi_{\varphi_o}(F) \geq 0$). P is a projection in $\Phi(\mathfrak{R})$ since $\Phi(\mathfrak{R})$ is a von Neumann algebra. Let $(A_\beta)$ be a net in $P(\mathfrak{R} \times G)P$ with $\|A_\beta\| \leq 1$, and $A_\beta \to 0$ ultrastrongly, then $A_\beta F \to 0$ ultrastrongly. Let I denote the identity of $\mathfrak{S} \times H$. (The map $A \to A \otimes I$ is a *-isomorphism between $\mathfrak{R} \times G$ and $(\mathfrak{R} \times G) \otimes I \subset (\mathfrak{R} \times G) \otimes (\mathfrak{S} \times H)$. Thus we shall consider $\mathfrak{R} \times G$ as a subalgebra of $(\mathfrak{R} \times G) \otimes (\mathfrak{S} \times H)$). Since F is finite, $(A_\beta F)^* = FA_\beta^* \to 0$ ultrastrongly ([46], p.97, Theorem 2.5.6). Since each $\psi_\varphi$ is ultrastrongly continuous,

$$\psi_{\varphi_o}(FA_\beta^*) = \psi_{\varphi_o}(F)A_\beta^* \to 0 \text{ ultrastrongly}$$

It follows that

$$A_\beta^* = \{P\psi_{\varphi_o}(F)P + 1 - P\}^{-1}P\psi_{\varphi_o}(F)A_\beta^* \to 0 \quad \text{ultrastrongly}$$

This shows that the *-operation is ultrastrongly continuous on bounded spheres of $P(\mathfrak{R} \times G)P$, so P is a finite projection in $\mathfrak{R} \times G$. ([46], p.97 Theorem 2.5.6). Hence $\Phi^{-1}(P)$ is a $\underset{G}{\sim}$-finite non-zero projection in $\mathfrak{R}$ ([41], Theorem 4.1), a contradiction since $\mathfrak{R}$ is $\underset{G}{\sim}$-type III. It follows that $E = 0$ and $\mathfrak{R} \otimes \mathfrak{S}$ is $\underset{G \times H}{\sim}$-type III.

The above result leaves open the following question:

<u>4.9 Conjecture</u> Let G be a group of automorphisms of a von Neumann algebra $\mathfrak{R}$. Then $\mathfrak{R} \times G$ is type III $\Leftrightarrow$ $\mathfrak{R}$ is $\underset{G}{\sim}$-type III

We can prove this in a certain special case.

<u>4.10 Definition</u> $\mathfrak{R}$ is <u>G-finite</u> if the normal G-invariant

states on $\mathfrak{R}$ separate the points of $\mathfrak{R}^+$.

<u>4.11 Proposition</u> Let G be a group of centre fixing automorphisms of a von Neumann algebra $\mathfrak{R}$. Suppose $\mathfrak{R}$ is G-finite, then

$$\mathfrak{R} \text{ is } \underset{G}{\sim}\text{-type III} \Leftrightarrow \mathfrak{R} \times G \text{ is type III}$$

<u>Proof</u> If $\mathfrak{R}$ is not $\underset{G}{\sim}$-type III, then $\mathfrak{R}$ contains a non-zero $\underset{G}{\sim}$-finite projection E. Then $\Phi(E)$ is a finite projection in $\mathfrak{R} \times G$ ([ ], Theorem 4.1), so $\mathfrak{R} \times G$ is not type III.

Conversely, if $\mathfrak{R} \times G$ is not type III, there is a faithful normal projection $\Gamma$ of norm one from $\mathfrak{R} \times G$ to $\mathfrak{R}$, so $\mathfrak{R}$ is not type III. ([46], p. 100, Lemma 2.6.5).

Hence $\mathfrak{R}Z$ is semifinite for some central projection Z in $\mathfrak{R}$. By [52] (Lemma 2.1), $\mathfrak{R}Z$ is $\underset{G}{\sim}$-semifinite, so $\mathfrak{R}$ is not $\underset{G}{\sim}$-type III.

<u>4.12 Corollary</u> Let G be a compact group, $\alpha$ a weakly continuous representation of G by centre fixing automorphisms on the von Neumann algebra $\mathfrak{R}$, then $\mathfrak{R}$ is G-finite and hence $\mathfrak{R}$ is $\underset{G}{\sim}$-type III $\Leftrightarrow \mathfrak{R} \times G$ is type III.

<u>Proof</u> Let f be a normal state on $\mathfrak{R}$. The map $\nu_g: f \to f \circ \alpha(g)$ is continuous from $\mathfrak{R}_* \to \mathfrak{R}_*$, (since each $\alpha(g)$ is ultraweakly continuous), when $\mathfrak{R}_*$ has the $\sigma(\mathfrak{R}_*,\mathfrak{R})$ topology. G is compact and the map $g \to \nu_g(f)$ is continuous from G to $\mathfrak{R}_*$ with the $\sigma(\mathfrak{R}_*,\mathfrak{R})$ topology, hence $\{f \circ \alpha(g);\ g \in G\}$ is $\sigma(\mathfrak{R}_*,\mathfrak{R})$ compact in $\mathfrak{R}_*$. Thus $\mathcal{E} = \overline{co}\ \{ f \circ \alpha(g);\ g \in G\}$ (closure in $\sigma(\mathfrak{R}_*,\mathfrak{R})$ topology) is $\sigma(\mathfrak{R}_*,\mathfrak{R})$ compact by the Krein-Smulian theorem ([13], Theorem 4, p.434).

$\{\nu_g;\ g \in G\}$ is a group of continuous non-contracting maps from $\mathcal{E} \to \mathcal{E}$, so by the Ryll-Nardzewski fixed point theorem (see Appendix A ) there is an $h \in \mathcal{E}$ with $h \circ \alpha(g) = h \quad (g \in G)$.

Let $E_h = \operatorname{supp} h$. Then $E_h$ is G-invariant, thus $f(E_h) = h(E_h) = 1$, so $E_f \leq E_h$. It follows from this that $\mathcal{R}$ must be G-finite, and the result is proved.

# CHAPTER V

## CONNECTED TOPOLOGICAL GROUPS ACTING ON VON NEUMANN ALGEBRAS ([37])

In [23], R.V. Kadison and J.R. Ringrose showed that if G is a connected topological group and $\alpha$ is a representation of G as automorphisms of a C*-algebra $\mathfrak{A}$, acting on a Hilbert space, such that $\|\alpha(g) - \iota\| \to 0$ as $g \to e$, then each $\alpha(g)$ is weakly inner. Hence there is a unitary $W_g$ in $\mathfrak{A}^-$, the weak operator closure of $\mathfrak{A}$, such that $\alpha(g)(A) = W_g A W_g^* \ (A \in \mathfrak{A},\ g \in G)$. We shall show that if G is an abelian connected topological group, then there is a <u>unitary representation</u> $g \to U_g$ of G by unitaries in $\mathfrak{A}^-$, such that $\alpha(g)(A) = U_g A U_g^*$ $(g \in G,\ A \in \mathfrak{A})$ and $\|U_g - I\| \to 0$ as $g \to e$.

It is easy to see, as shown in [23], proof of Lemma 2, that the result is true if G is the real line, by using the theorem of Kadison-Sakai that all derivations of a von Neumann algebra are inner ([21], [47]). As a consequence of this, J. Dixmier showed, in [10], that the same result holds if G is a simply connected Lie group. However, we shall give an example to show that there is no hope of extending the result, even in a 'local' sense, to a general non-abelian connected group. We also obtain some results on invariant states associated with the representation $\alpha: G \to \mathrm{aut}(\mathfrak{A})$. In particular, we show in the case of an abelian connected topological group, that every extreme point of the set of $\alpha$-invariant states is actually a pure state, and hence at least one $\alpha$-invariant pure state must exist.

<u>5.1 Lemma</u> Let $\lambda, \mu \in \mathbb{C}$, $|\lambda| = |\mu| = 1$, Re $\mu > 0$, $\lambda \neq 1$, then there is an integer n such that Re $\lambda^n\mu < 0$.

<u>Proof</u> Let $\lambda = e^{2\pi i\Theta}$.

<u>Case 1</u> $\Theta$ irrational, then $\{\lambda^n; n \in \mathbb{Z}\}$ is dense in the unit circle, and $|\lambda^n\mu + \mu| = |\lambda^n + 1|$. We can thus choose $\lambda^n$ sufficiently close to $-1$ so that Re $\lambda^n\mu < 0$, (since Re$(-\mu) < 0$).

<u>Case 2</u> $\Theta$ is rational, $\Theta = p/q$ in lowest terms, then there are integers m, n such that $mp + nq = 1$, thus $mp/q = 1/q - n$. Hence $\lambda^m = e^{2\pi imp/q} = e^{2\pi i/q}$. So we may assume $p = 1$

(i) $q = 2n$ for some integer n, then $\lambda^n = e^{\pi i} = -1$, thus Re $(\lambda^n\mu) =$ Re $(-\mu) < 0$.

(ii) $q = 2n + 1$ for some integer n, then $\{(e^{2\pi i/q})^r\mu\}_{1 \leq r \leq q}$ are the vertices of a regular polygon of k sides, where $k \geq 3$ is an odd number, and with one vertex at $\mu$. Hence for some r, Re $(\lambda^r\mu) < 0$. This completes the proof.

<u>5.2 Theorem</u> Let $\mathfrak{R}$ be a von Neumann algebra, G an abelian connected topological group, and $\alpha$ a representation of G as automorphisms of $\mathfrak{R}$ such that $\|\alpha(g) - \iota\| \to 0$ as $g \to e$, then there is a unitary representation $g \to U_g$ of G by unitaries in $\mathfrak{R}$ with $\alpha(g)(A) = U_g A U_g^*$ $(A \in \mathfrak{R},\ g \in G)$ and $\|U_g - I\| \to 0$ as $g \to e$.

<u>Proof</u> By [23] (Theorem 7), $\alpha(g)$ is an inner automorphism of $\mathfrak{R}$ for all g in G. For each g in G choose a unitary $W_g$ in $\mathfrak{R}$ which implements $\alpha(g)$. Let e be the identity of G, and W a neighbourhood of e in G such that if $g \in W$ then $\|\alpha(g) - \iota\| < 2$. If $g \in W$,

we may choose a unitary $W_g$ in $\mathfrak{R}$ implementing $\alpha(g)$ with

$$\sigma(W_g) \subseteq \{z; \operatorname{Re} z \geq \tfrac{1}{2} (4 - \|\alpha(g) - \iota\|^2)^{\frac{1}{2}}\}$$

by [23, Lemma 5, Theorem 7].

We next show that $W_g W_h = W_h W_g$ for all $g, h \in G$. Let $\pi$ be an irreducible representation of $\mathfrak{R}$, and let $\gamma(g)$ denote the automorphism

$$\pi(A) \to \pi(\alpha(g)(A)) .$$

Clearly $g \to \gamma_g$ is a group homomorphism from $G$ to aut $(\pi(\mathfrak{R}))$. Now let $V$ be a neighbourhood of $e$ in $G$ such that if $g \in V$ then $\|\alpha(g) - \iota\| < 1$, so that $V^2 \subseteq W$. Let $g, h \in V$, then $\pi(W_g W_h)$, $\pi(W_h W_g)$ and $\pi(W_{gh})$ all implement the automorphism $\gamma(gh) = \gamma(hg)$. Thus $\pi(W_g W_h)\ \pi(W_h W_g)^* \in \pi(\mathfrak{R})' = \mathbb{C}I$, so there is a scalar $\lambda$, $|\lambda| = 1$, with

$$\pi(W_g W_h) = \lambda\, \pi\, (W_h W_g).$$

Similarly, there is a scalar $\mu$, $|\mu| = 1$, with

$$\pi(W_{gh}) = \mu\, \pi\, (W_g W_h).$$

Then if $\sigma(A)$ denotes the spectrum of $A$, we have

$$\begin{aligned} \sigma(\pi(W_{gh})) &= \mu\sigma(\pi(W_g W_h)) \\ &= \mu\sigma(\pi(W_g)\ \pi(W_h)) \\ &= \mu\sigma(\pi(W_h)\ \pi(W_g)) \end{aligned}$$

(by [46, Proposition 1.1.8]). Note that $\pi(W_g)\ \pi(W_h)$ is invertible so 0 is not in the spectrum. Thus

$$\begin{aligned} \sigma(\pi(W_{gh})) &= \mu\bar{\lambda}\sigma(\pi(W_g)\ \pi(W_h)) \\ &= \bar{\lambda}\sigma(\pi(W_{gh})). \end{aligned}$$

So $\sigma(\pi(W_{gh})) = \lambda^n \sigma(\pi(W_{gh}))$ for all integers n. Choose $\mu \in \sigma(\pi(W_{gh}))$. Now $g, h \in V$, so $gh \in W$, so $\sigma(\pi(W_{gh})) \subseteq \sigma(W_{gh})$, and the right hand side is a subset of the open right halfplane. Thus Re $\mu > 0$. If $\lambda \neq 1$, then by Lemma 5.1 for some n, Re $\lambda^n \mu < 0$. A contradiction, since $\lambda^n \mu \in \sigma(\pi(W_{gh}))$. Thus $\lambda = 1$, and hence we have $\pi(W_g W_h) = \pi(W_h W_g)$ $(g, h \in V)$. G is connected, so V generates G ([42, Theorem 14, p. 129]). Let $g, h \in G$, then there are $g_i, h_j \in V$ with $g = g_1 \ldots g_n$, $h = h_1 \ldots h_m$, hence $\pi(W_g)$ and $\pi(W_{g_1} \ldots W_{g_n})$ implement $\gamma_g = \gamma_{g_1} \ldots \gamma_{g_n}$; $\pi(W_h)$ and $\pi(W_{h_1} \ldots W_{h_m})$ implement $\gamma_h = \gamma_{h_1} \ldots \gamma_{h_m}$. It follows that there are scalars, $\theta$, $\varphi$, with

$$\pi(W_g) = \theta \pi (W_{g_1} \ldots W_{g_n})$$
$$= \theta \pi (W_{g_1}) \ldots \pi(W_{g_n})$$

and

$$\pi(W_h) = \varphi \pi (W_{h_1}) \ldots \pi(W_{h_m})$$

so $\pi(W_g W_h) = \pi(W_h W_g)$ $(g, h \in G)$. This is true for all irreducible representations $\pi$, and such representations separate the points of $\mathfrak{R}$, thus

$$W_g W_h = W_h W_g \quad (g, h \in G). \tag{1}$$

Let $\mathfrak{Z}$ denote the centre of $\mathfrak{R}$, and $\mathfrak{S}$ the von Neumann algebra generated by $\{W_g; g \in G\} \cup \mathfrak{Z}$. By (1), $\mathfrak{S}$ is abelian. Let $\Phi_{\mathfrak{S}}$ (resp. $\Phi_{\mathfrak{Z}}$) denote the carrier space of $\mathfrak{S}$ (resp. $\mathfrak{Z}$). Since the elements of $\Phi_{\mathfrak{S}}$ and $\Phi_{\mathfrak{Z}}$ are precisely the multiplicative linear functionals on $\mathfrak{S}$ and $\mathfrak{Z}$, respectively, [8, 2.5.2], it results from [8, 2.10.1] that the restriction map $\varphi \to \varphi|\mathfrak{Z}$ from $\Phi_{\mathfrak{S}}$ to $\Phi_{\mathfrak{Z}}$ is a continuous surjection. Now $\mathfrak{S}$ and $\mathfrak{Z}$ are von Neumann algebras so $\Phi_{\mathcal{S}}$ is a compact totally disconnected space and $\Phi_{\mathcal{Z}}$ is Stonian by [9], Theorem 2. By Appendix C there is a continuous mapping f from $\Phi_{\mathcal{Z}}$ into $\Phi_{\mathcal{S}}$ such that $f(\varphi)|\mathcal{Z} = \varphi$, $(\varphi \in \Phi_{\mathcal{Z}})$. Define $\hat{U}_g$ by

$$\hat{U}_g(\varphi) = \overline{\hat{W}_g(f(\varphi|\mathcal{Z}))}\, \hat{W}_g(\varphi) \qquad (\varphi \in \Phi_{\mathcal{S}}) .$$

$\hat{U}_g$ is a continuous map from $\Phi_{\mathcal{S}}$ to the unit circle, and hence is the Gelfand transform of a unitary element $U_g$ of $\mathcal{S}$. Define an equivalence relation R on $\Phi_{\mathcal{S}}$ by $\varphi_1 R \varphi_2$ if and only if $\varphi_1|\mathcal{Z} = \varphi_2|\mathcal{Z}$. $\mathcal{Z}$ is a closed $*$-subalgebra of $\mathcal{S}$ containg the constants, so by the extended Stone-Weierstrass theorem (see for example [12, Theorem 2.47 ]), $\hat{\mathcal{Z}}$ is the set of continuous complex valued functions on $\Phi_{\mathcal{S}}$ which are constant on each equivalence class. Let

$$\hat{V}_g(\varphi) = \overline{\hat{W}_g(f(\varphi|\mathcal{Z}))} \qquad (\varphi \in \Phi_{\mathcal{S}}) .$$

$\hat{V}_g$ is a continuous map from $\Phi_{\mathcal{S}}$ to the unit circle and is constant on each equivalence class, so $\hat{V}_g$ is the Gelfand transform of a unitary in $\mathcal{Z}$. Thus $U_g = V_g W_g$ where $V_g$ is a unitary in $\mathcal{Z}$. It follows that $U_g$ implements $\alpha(g)$, i.e. $\alpha(g)(A) = U_g A U_g^*$ $(A \in \mathcal{R}$, $g \in G)$. By the above, $U_g U_h$ and $U_{gh}$ are unitaries in $\mathcal{R}$ implementing the automorphism $\alpha(gh)$, so $R_{g,h} = U_g U_h U_{gh}^*$ is a unitary in $\mathcal{Z}$, for each $g, h \in G$. If $\varphi \in \Phi_{\mathcal{Z}}$, then since $f(\varphi)|\mathcal{Z} = \varphi$

$$\hat{R}_{g,h}(\varphi) = \hat{R}_{g,h}(f(\varphi))$$

$$= \hat{U}_g(f(\varphi))\, \hat{U}_h(f(\varphi))\, \overline{\hat{U}_{gh}(f(\varphi))}.$$

Now if $g \in G$,

$$\hat{U}_g(f(\varphi)) = \overline{\hat{W}_g(f(f(\varphi)|\mathcal{Z}))}\ \hat{W}_g(f(\varphi))$$

$$= \overline{\hat{W}_g(f(\varphi))}\ \hat{W}_g(f(\varphi))$$

$$= 1.$$

Hence $R_{g,h}(\varphi) = 1 \quad (\varphi \in \Phi_{\mathcal{Z}})$ so $R_{g,h} = I \quad (g,h \in G)$ and $U_g U_h = U_{gh} \quad (g,h \in G)$ showing that $g \to U_g$ is unitary representation of G. It remains only to prove norm continuity of $g \to U_g$.

Let $\varphi \in \Phi_g$, then

$$|\hat{U}_g(\varphi) - 1| = |\hat{W}_g(\varphi)\ \overline{W_g(f(\varphi|\mathcal{Z}))} - 1|$$

$$\leq |\hat{W}_g(\varphi)\overline{\hat{W}_g(f(\varphi|\mathcal{Z}))} - \overline{\hat{W}_g(f(\varphi|\mathcal{Z}))}|$$

$$+ |\hat{W}_g(f(\varphi|\mathcal{Z})) - 1|$$

$$= |\hat{W}_g(\varphi) - 1| + |\hat{W}_g(f(\varphi|\mathcal{Z})) - 1|$$

Now $W_g - I$ is normal, so the above is less than $2\|W_g - I\|$. Thus $\|U_g - I\| \leq 2\|W_g - I\|$. Let W be a neighbourhood of e in G such that if $g \in W$ then $\|\alpha(g) - \iota\| < 2$. If $g \in W$, we have $\sigma(W_g) \subseteq \{z;\ \mathrm{Re}\ z \geq \beta_g\}$ where $\beta_g = \frac{1}{2}\ (4 - \|\alpha(g) - \iota\|^2)^{\frac{1}{2}}$. Note that $\beta_g \to 1$ as $g \to e$. If $g \in W$,

$$\|W_g - I\| = \sup\ \{|\lambda - 1|\ ;\ \lambda \in \sigma(W_g)\}$$

$$\leq \sqrt{2 - 2\beta_g}$$

Thus if $g \in W$

$$\|U_g - I\| \leq 2\sqrt{2 - 2\beta_g} \to 0 \quad \text{as} \quad g \to e\ .$$

This completes the proof.

<u>5.3 Theorem</u> Let $\mathfrak{A}$ be a C*-algebra acting on a Hilbert space, G an abelian connected topological group, $\alpha$ a representation of G as automorphisms of $\mathfrak{A}$ such that $\|\alpha(g) - \iota\| \to 0$ as $g \to e$, then there is a unitary representation $g \to U_g$ of G by unitaries in $\mathfrak{A}^-$ with $\alpha(g)(A) = U_g A U_g^*$ $(A \in \mathfrak{A},\ g \in G)$ and $\|U_g - I\| \to 0$ as $g \to e$.

<u>Proof</u> Let W be a neighbourhood of e in G such that if $g \in W$ then $\|\alpha(g) - \iota\| < 2$. If $g \in W$ then $\alpha(g)$ extends to an inner automorphism $\beta(g)$ of $\mathfrak{A}^-$ ([23, Theorem 7]). W generates G, since G is connected ([42, Theorem 14, p.129]) so $\alpha(g)$ extends to an inner automorphism $\beta(g)$ of $\mathfrak{A}^-$ for all $g \in G$. By Kaplansky's density theorem $\|\beta(g) - \iota\| = \|\alpha(g) - \iota\| \to 0$ as $g \to e$ and clearly $g \to \beta(g)$ is a representation of G as automorphisms of $\mathfrak{A}^-$. The result follows by applying Theorem 1 to $\mathfrak{A}^-$.

If $\alpha$ is a representation of the group G as automorphisms of a C*-algebra $\mathfrak{A}$, and f is an $\alpha$-invariant state of $\mathfrak{A}$, let $\pi = \pi_f$ be the representation of $\mathfrak{A}$ corresponding to f, on the Hilbert space $\mathcal{H}$, and $x \in \mathcal{H}$ the corresponding cyclic vector. If $g \in G$, we may define the map $U_g$ by

$$U_g \pi(A)x = \pi(\alpha(g)(A))x \quad (A \in \mathfrak{A})$$

since

$$\begin{aligned} \|U_g \pi(A)x\|^2 &= \langle \pi(\alpha(g)(A))x, \pi(\alpha(g)(A))x \rangle \\ &= \langle \pi(\alpha(g)(A^*A))x, x \rangle \\ &= f(\alpha(g)(A^*A)) \\ &= f(A^*A) \\ &= \langle \pi(A^*)\pi(A)x, x \rangle \\ &= \|\pi(A)x\|^2 , \end{aligned}$$

showing that $U_g$ is well defined. Now $\{\pi(A)x;\ A \in \mathfrak{A}\}$ is dense in $\mathcal{H}$, thus $U_g$ extends to a unitary on $\mathcal{H}$. If $g = e$, then $U_g = I$ clearly. If $g,h \in G$, $A \in \mathfrak{A}$

$$\begin{aligned} U_gU_h(\pi(A)x) &= U_g\ \pi(\alpha(h)(A))x \\ &= \pi(\alpha(g)\alpha(h)(A))x \\ &= \pi(\alpha(gh)(A)x \\ &= U_{gh}\ \pi(A)x \ . \end{aligned}$$

Thus $U_gU_h = U_{gh}$, and $g \to U_g$ is a unitary representation of G. If $B,A \in \mathfrak{A}$, then

$$\begin{aligned} U_g\ \pi(A)U_g^*\ \pi(B)x &= U_g\ \pi(A)\ \pi(\alpha(g)^{-1}(B))x \\ &= U_g\ \pi(A\ \alpha(g)^{-1}(B))x \\ &= \pi(\alpha(g)(A))\ \pi(B)x \end{aligned}$$

Thus

$$U_g\ \pi(A)U_g^* = \pi(\alpha(g)(A)) \qquad (g \in G,\ A \in \mathfrak{A})$$

Suppose now that G is a topological group, and the map $g \to f(B \cdot \alpha(g)(A)C)$ is continuous for $A,B,C \in \mathfrak{A}$. Let $g \to h$ in G, then if $A \in \mathfrak{A}$ ,

$$\begin{aligned} \|U_g\pi(A)x - U_h\pi(A)x\|^2 &= \|U_{h^{-1}g}\pi(A)x - \pi(A)x\|^2 \\ &= \|\pi(\alpha(h^{-1}g)(A))x - \pi(A)x\|^2 \\ &= \langle(\pi(\alpha(h^{-1}g)(A)) - \pi(A))^*\pi(\alpha(h^{-1}g)(A) - \pi(A))x, x\rangle \\ &= f((\alpha(h^{-1}g)(A) - A)^*(\alpha(h^{-1}g)(A) - A) \\ &= f(\alpha(h^{-1}g)(A^*A)) - f(A^*\alpha(h^{-1}g)(A)) \\ &\quad - f(\alpha(h^{-1}g)(A)^*A) + f(A^*A) \ . \end{aligned}$$

$$= 2\, f(A^*A) - f(A^*\alpha(h^{-1}g)(A)) - f(\alpha(h^{-1}g)(A^*)A)$$

since f is $\alpha$-invariant.

Now $f(A^*\alpha(h^{-1}g)(A)) \to f(A^*A)$ as $g \to h$, by hypothesis, and $f(\alpha(h^{-1}g)(A^*)A) \to f(A^*A)$, hence for each $A \in \mathfrak{A}$,

$$\|U_g\, \pi(A)x - U_h\, \pi(A)x\| \to 0 \quad \text{as } g \to h$$

It follows that $\|U_g y - U_h y\| \to 0$ as $g \to h$ in G for each $y \in \mathcal{H}$, thus $g \to U_g$ is a strongly continuous unitary representation. We call $\{U_g;\ g \in G\}$ the Segal unitaries associated with f.

<u>5.4 Theorem</u> Let G be an abelian connected topological group, $\mathfrak{R}$ a von Neumann algebra, and $\alpha$ a representation of G, as automorphisms of $\mathfrak{R}$, such that $\|\alpha(g) - \iota\| \to 0$ as $g \to e$, then

(1) There is an $\alpha$-invariant state on $\mathfrak{R}$.

(2) Let $\rho$ be an extreme point of the set of $\alpha$-invariant states, $\{V_g;\ g \in G\}$ the corresponding group of Segal unitaries, and $\pi$ the cyclic representation corresponding to $\rho$, then

(a) $V_g \in \pi(\mathfrak{R}) \quad (g \in G)$

(b) $\|V_g - I\| \to 0 \quad$ as $g \to e$

(c) $\rho$ is a pure state.

<u>Proof</u> If $f \in \mathfrak{R}^*$, define $\nu(g)(f) = f \circ \alpha(g)$ then $\{\nu(g);\ g \in G\}$ is a commuting group of weak*-continuous linear maps: $\mathfrak{R}^* \to \mathfrak{R}^*$. Let f be a state of $\mathfrak{R}$, and let

$$\mathcal{E} = \overline{co}^{W*}\{\nu(g)(f);\quad g \in G\}\ .$$ $\mathcal{E}$ is weak*-closed in $E(\mathfrak{R})$,

thus $\mathcal{E}$ is weak*-compact, convex and $\nu(g): \mathcal{E} \to \mathcal{E}$ $(g \in G)$, so by the Markov-Kakutani fixed point theorem ([13], p.456 ) there is an $h \in \mathcal{E}$ with

$$h \circ \alpha(g) = h \quad (g \in G) .$$

This proves (1).

By the Krein-Mil'man theorem, the set of $\alpha$-invariant states has an extreme point. Let $\rho$ be one such, and let $\pi = \pi_\rho$ be the representation corresponding to $\rho$. Let $\mathfrak{U} = \{V_g ; g \in G\}$ be the corresponding Segal unitaries of $\rho$. The extremal property of $\rho$ is equivalent to

$$(\pi(\mathfrak{R}) \cup \mathfrak{U})' = \mathbb{C}I$$

(as shown in [45, Theorem 6.3.3]). By Theorem 5.2, there is a unitary representation $g \to U_g$ of $G$ by unitaries in $\mathfrak{R}$ such that $\alpha(g)(A) = U_g A U_g^*$ $(g \in G,\ A \in \mathfrak{R})$. Let $g, h \in G$, $A \in \mathfrak{R}$, and $x$ the cyclic vector for the representation $\pi$, then

$$\begin{aligned}
\pi(U_g)V_h\, \pi(A)x &= \pi(U_g)\, \pi(\alpha(h)(A))x \\
&= \pi(U_g)\, \pi(U_h A U_h^*)\, x \\
&= \pi(U_{gh} A U_h^*)x \\
&= \pi(U_{hg} A U_h^*)\, x \\
&= \pi(U_h U_g A U_h^*)\, x \\
&= \pi(\alpha(h)(U_g A))x \\
&= V_h\, \pi(U_g)\, \pi(A)x
\end{aligned}$$

Now $\mathcal{H} = \{\pi(A)x ;\ A \in \mathfrak{R}\}^-$ so $\pi(U_g)V_h = V_h\, \pi(U_g)$ $(g, h \in G)$. Since $g \to V_g$ is a group homomorphism, all the $V_h$'s commute with each other and by the above, $\pi(U_g) \in \mathfrak{U}'$, so $\pi(U_g)V_{g^{-1}} \in \mathfrak{U}'$ $(g \in G)$. Now $\pi(U_g)$ and $V_g$ both implement the automorphism $\pi(A) \to \pi(\alpha(g)(A))$

thus $\pi(U_g)V_{g^{-1}} \in \pi(\mathfrak{R})'$ $(g \in G)$. Hence $\pi(U_g)V_{g^{-1}} \in (\pi(\mathfrak{R}) \cup \mathfrak{U})' = \mathbb{C}I$ $(g \in G)$.

Thus there is a scalar $\lambda_g$, $|\lambda_g| = 1$, with $\pi(\mathfrak{U}_g) = \lambda_g V_g$, showing that $V_g = \pi(\bar{\lambda}_g \mathfrak{U}_g) \in \pi(\mathfrak{R})$ $(g \in G)$.

Let $x \in \mathcal{H}$, $\|x\| = 1$, and $W = \{g \in G;\ \|\alpha(g) - \iota\| < 2\}$. If $g \in W$, then

$$\begin{aligned} |1 - \bar{\lambda}_g| &= \|x - \bar{\lambda}_g x\| \\ &\leq \|x - V_g x\| + \|V_g x - \bar{\lambda}_g x\| \\ &= \|x - V_g x\| + \|\bar{\lambda}_g \pi(\mathfrak{U}_g)x - \bar{\lambda}_g x\| \\ &\leq \|x - V_g x\| + \|\mathfrak{U}_g - I\|. \end{aligned}$$

Hence

$$\begin{aligned} \|V_g - I\| &\leq \|V_g - \bar{\lambda}_g I\| + |1 - \bar{\lambda}_g| \\ &= \|\bar{\lambda}_g \pi(\mathfrak{U}_g) - \bar{\lambda}_g I\| + |1 - \bar{\lambda}_g| \\ &\leq 2\|\mathfrak{U}_g - I\| + \|x - V_g x\| \\ &\leq 4\sqrt{2 - 2\beta_g} + \|x - V_g x\| \end{aligned}$$

(as in the proof of Theorem 5.2) and the right hand side goes to zero as g approaches e in G.

Now
$$(\pi(\mathfrak{R}))' = (\pi(\mathfrak{R}) \cup \mathfrak{U})' = \mathbb{C}I$$

since $\mathfrak{U} \subseteq \pi(\mathfrak{R})$, so $\pi$ is irreducible, hence $\rho$ is a pure state. This finishes the proof.

Let G be a topological group, $\mathfrak{R}$ a von Neumann algebra and $\alpha$ a representation of G as *-automorphisms of $\mathfrak{R}$. A <u>local lifting</u> for $\alpha$ is described as follows: there is a neighbourhood N of e in G and a map $U\colon N \to \mathfrak{U}(\mathfrak{R})$, the unitary group of $\mathfrak{R}$, such that if g, h, gh are in N, then $U_g U_h = U_{gh}$, and $U_g A U_g^* = \alpha(g)(A)$. We say the lifting is norm continuous if $\|U_g - I\| \to 0$ as $g \to e$ $(g \in N)$.

Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space, $\mathcal{U}$ the unitary group of $\mathcal{B}(\mathcal{H})$, with the norm topology as operators on $\mathcal{H}$. Since the centre of $\mathcal{B}(\mathcal{H})$ is $\{\lambda I; \lambda \in \mathbb{C}\}$, the centre of $\mathcal{U}$ is $\mathcal{T} = \{\lambda I; |\lambda| = 1\}$, which is isomorphic to the circle group. If U, V are in $\mathcal{U}$, then by [43, p.279], there are self-adjoint operators H, K in $\mathcal{B}(\mathcal{H})$ with $U = e^{iH}$, $V = e^{iK}$. Let $\varphi(\lambda) = e^{i(\lambda H + (1-\lambda)K)}$ then $\varphi(0) = V$, $\varphi(1) = U$, so $\varphi$ defines a continuous path in $\mathcal{U}$ from V to U. Thus $\mathcal{U}$ is arcwise connected (and, in particular, connected). If $U \in \mathcal{U}$, $\|U - I\| < 1$, then $|\lambda - 1| < 1$ for all $\lambda \in \sigma(U)$. Thus $\sigma(U)$ is contained in $\{z; |z| = 1, \operatorname{Re} z > 0\}$, so the log function is analytic on $\sigma(U)$. By the analytic functional calculus, we can define $\log U \in \mathcal{B}(\mathcal{H})$. If $\|H\|$ is small and $U = e^{iH}$, then $\log U = H$. Thus if $\mathcal{B}(\mathcal{H})^s$ denotes all the self-adjoint operators in $\mathcal{B}(\mathcal{H})$, and $\mathcal{N} = \{U \in \mathcal{U}; \|U - I\| < 1\}$, then the map $U \to \log U$ is a homeomorphism from $\mathcal{N}$ into $\mathcal{B}(\mathcal{H})^s$. Since $\mathcal{B}(\mathcal{H})^s$ is locally arcwise connected in the norm topology,

there is an open, arcwise connected convex set $\mathcal{M}$ with $\mathcal{M} \subset \mathcal{N}$. If $V \in \mathcal{U}$, let

$$P = \{U \in \mathcal{U}; \|U - V\| < 1\},$$

then $U \in P$ if and only if $\|UV^* - I\| < 1$, thus $P = \mathcal{N}V$, and P contains the arcwise connected set $\mathcal{M}V$. This shows that $\mathcal{U}$ is locally arcwise connected.

Let G denote the group $\mathcal{U}/\mathcal{T}$, with the quotient topology, and q the quotient map: $\mathcal{U} \to \mathcal{U}/\mathcal{T}$. G is the continuous image of $\mathcal{U}$, thus G is a connected topological group. Now $\mathcal{B}(\mathcal{H})$ is a type I factor, thus by [7] (Ch. III, §3, Corollary 2, p.241), every automorphism of $\mathcal{B}(\mathcal{H})$ is spatial. It follows that the map

$\alpha: U \to \alpha(U)$ where $\alpha(U)(A) = UAU^*$ $(A \in \mathcal{B}(\mathcal{H}))$, is a group homomorphism from $\mathcal{U}$ onto $\mathrm{aut}(\mathcal{B}(\mathcal{H}))$. If $\alpha(U) = \alpha(V)$, then $UV^* \in \mathcal{T}$, thus $\mathcal{T}$ is the kernel of $\alpha$, and $\alpha$ defines an isomorphism, also denoted $\alpha$, between G and $\mathrm{aut}(\mathcal{B}(\mathcal{H}))$, such that $\|\alpha(g) - \iota\| \to 0$ as $g \to e$ in G. The above shows that $\alpha$ is a representation of the connected group G on $\mathcal{B}(\mathcal{H})$, such that $\|\alpha(g) - \iota\| \to 0$ as $g \to e$ in G.

<u>5.5 Proposition</u> There is no norm continuous local lifting for $\alpha$.

<u>Proof</u> Suppose there is a norm continuous local lifting $\varphi: \mathcal{U}/\mathcal{T} \to \mathcal{U}$, on a neighbourhood N of $q(I)$ in G. Then if $U \in \mathcal{U}$, $\varphi(U\mathcal{T})$ implements $\alpha(U)$, so there is a scalar $\lambda_U$ of modulus one, with $\varphi(U\mathcal{T}) = U\lambda_U$. Let $U, V \in q^{-1}(N)$ be such that $UV \in q^{-1}(N)$, then

$$\begin{aligned} UV\lambda_{UV} &= \varphi\,(UV\mathcal{T}) \\ &= \varphi(U\mathcal{T}\, V\, \mathcal{T}) \\ &= \varphi(U\mathcal{T})\,\varphi(V\mathcal{T}) \\ &= U\lambda_U V\, \lambda_V \\ &= UV\lambda_U\lambda_V \end{aligned}$$

Hence

$$\lambda_{UV} = \lambda_U\lambda_V$$

Define $p(U) = \lambda_U$, then p is a local homomorphism from $q^{-1}(N)$ to the unit circle, $\mathbb{T}$, in the sense of [42] ( p.140, Para. K)

$$p(U) = \varphi(q(U))U^{-1},$$

so p is continuous on $q^{-1}(N)$. We have seen above that $\mathcal{U}$ is arcwise connected and locally arcwise connected. Also, $\mathcal{U}$ is contractible ([29, Theorem 3]), hence simply connected, so by Appendix D, p extends to a continuous character, $\bar{p}$, on $\mathcal{U}$, continuous for the norm topology. But by [14, Theorem 4, p.527], no such character exists. This contradiction shows that there can be no local lifting.

# CHAPTER VI

## IMPLEMENTING A GROUP OF AUTOMORPHISMS BY A UNITARY REPRESENTATION

6.1 Definition  Let $\mathfrak{A}$ be a C*-algebra acting on Hilbert space $\mathcal{H}$, and G a group. If $\alpha$ is a representation of G on $\mathcal{R}$ and $U: g \to U_g$ is a unitary representation of G on $\mathcal{H}$, then we say that U induces (or implements) $\alpha$ if

$$\alpha(g)(A) = U_g A U_g^* \qquad (A \in \mathfrak{A},\ g \in G) .$$

In certain cases, we shall show that if G is an abelian, locally compact group and $\alpha$ is a strongly continuous representation of G as inner automorphisms of $\mathfrak{A}$, say $\alpha(g) = \text{ad } W_g$ with $W_g \in \mathcal{U}(\mathfrak{A})$, then a necessary and sufficient condition for the existence of a strongly continuous unitary representation $U: g \to U_g$ of G by unitaries $U_g \in \mathcal{U}(\mathfrak{A})$ such that U induces $\alpha$ is that the $W_g$'s commute i.e.

$$W_g W_h = W_h W_g \qquad (g,h \in G)$$

If $\mathcal{R}$ is a von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$, denote by $\text{inn}(\mathcal{R})$ the group of inner automorphisms of $\mathcal{R}$. Let $\mathcal{Z}$ denote the centre of $\mathcal{R}$, and $\mathcal{U}(\mathcal{R})$ (resp. $\mathcal{U}(\mathcal{Z})$) the group of all unitaries in $\mathcal{R}$ (resp. $\mathcal{Z}$). There is a natural homomorphism $\psi': U \to \text{ad } U$ from $\mathcal{U}(\mathcal{R})$ to $\text{inn}(\mathcal{R})$. If V lies in the kernel of this map then $UVU^* = V$ $(U \in \mathcal{U}(\mathcal{R}))$, thus $AV = VA$ $(A \in \mathcal{R})$ since $\mathcal{U}(\mathcal{R})$ spans $\mathcal{R}$ linearly, so $V \in \mathcal{U}(\mathcal{Z})$. It follows that $\psi'$ induces a group isomorphism

$$\psi : \mathcal{U}(\mathcal{R}) / \mathcal{U}(\mathcal{Z}) \to \text{inn}(\mathcal{R})$$

Recall the following definitions: A borel space is a set E together with a family, $\mathcal{B}$, of subsets of E having the following properties: $E \in \mathcal{B}$, $\mathcal{B}$ is stable under countable union and intersection, and if $Y \in \mathcal{B}$ then $E \setminus Y \in \mathcal{B}$. The elements of $\mathcal{B}$ are called the borel sets of E. Let C be a family of subsets of E. Among the families $\mathcal{B}$ of subsets of E such that $\mathcal{B} \supset C$ and $(E, \mathcal{B})$ is a borel space, there is a smallest, $\mathcal{B}_o$. We say that $\mathcal{B}_o$ is the borel structure generated by C.

A topological space X is said to be a polish space if it has a countable base and the topology of X is given by a complete metric. It is known that if $\mathcal{B}$ is closed or open in X and X is a polish space, then $\mathcal{B}$ is also a polish space ([4], Ch.9, §6, No.1, Propositions 1 and 2). By [4], Ch.9, §6, No.1, Theorème 1, if B is a $G_\delta$-subset of X (i.e. B is a countable intersection of open subsets of X) and X is a polish space, then B is a polish space. If E is a borel space, then E is said to be standard if the borel structure of E is generated by the open sets for some topology on E with respect to which E is a polish space. By following the methods of [26], pp.508-509, we shall show that inn($\mathcal{R}$) is a standard borel space with respect to a natural borel structure, that $\mathcal{U}(\mathcal{R})$ is a polish group, and there is a borel cross section for inn($\mathcal{R}$) in $\mathcal{U}(\mathcal{R})$. (This means that if we identify inn($\mathcal{R}$) and $\mathcal{U}(\mathcal{R})/\mathcal{U}(\mathcal{Z})$, there is a borel map

$$\eta : \mathcal{U}(\mathcal{R})/\mathcal{U}(\mathcal{Z}) \to \mathcal{U}(\mathcal{R})$$

such that $\pi \circ \eta$ = identity where $\pi$ is the quotient map:

$$\mathcal{U}(\mathcal{R}) \to \mathcal{U}(\mathcal{R})/\mathcal{U}(\mathcal{Z})$$

We need these results in the proofs of the theorems in this Chapter.

Let $\mathcal{U}(\mathcal{B}(\mathcal{H}))$ be the group of all unitaries on the Hilbert space $\mathcal{H}$. The weak and strong operator topologies coincide on $\mathcal{U}(\mathcal{B}(\mathcal{H}))$ for if $U_\alpha$ and $U$ are unitaries with $U_\alpha \to U$ weakly, then

$$(U_\alpha - U)^*(U_\alpha - U) = 2I - U^*U_\alpha - U_\alpha^*U \to 0 \text{ weakly},$$

so $U_\alpha \to U$ strongly. The weak operator topology thus gives $\mathcal{U}(\mathcal{R})$ the structure of a topological group. If $\mathcal{R}$ is a von Neumann algebra acting on $\mathcal{H}$, the same is true of $\mathcal{U}(\mathcal{R})$ since $\mathcal{U}(\mathcal{R}) = \mathcal{U}(\mathcal{B}(\mathcal{H})) \cap \mathcal{R}$. If $\mathcal{Z}$ is the centre of $\mathcal{R}$ and $\mathcal{U}(\mathcal{Z})$ the group of unitaries in $\mathcal{Z}$, then $\mathcal{U}(\mathcal{Z})$ is a closed normal subgroup of $\mathcal{U}(\mathcal{R})$. Let $\pi$ denote the quotient map: $\mathcal{U}(\mathcal{R}) \to \mathcal{U}(\mathcal{R})/_{\mathcal{U}(\mathcal{Z})}$ and give $\mathcal{U}(\mathcal{R})/_{\mathcal{U}(\mathcal{Z})}$ the quotient topology. Then $\pi$ is continuous and open for if $V$ is an open set in $\mathcal{U}(\mathcal{R})$, and $W$ is the saturation of $V$,

$$W = \bigcup_{U \in \mathcal{U}(\mathcal{Z})} UV$$

Thus $W$ is open.

Suppose now that $\mathcal{H}$ is separable, and let $\{x_n\}_{n=1}^{\infty}$ be a countable dense subset of $\mathcal{H}$. Define

$$\delta(U,V) = \sum_{n=1}^{\infty} \frac{\|(U-V)x_n\|}{\|x_n\|2^n} + \sum_{n=1}^{\infty} \frac{\|(U^*-V^*)x_n\|}{\|x_n\|2^n}$$

$$(U, V \in \mathcal{U}(\mathcal{R}))$$

$\delta$ is a metric on $\mathcal{U}(\mathcal{R})$. Note that if $W \in \mathcal{U}(\mathcal{Z})$ then

$$\delta(WU, WV) = \delta(U, V)$$

$$= \delta(U^*, V^*)$$

$$= \delta(W^*U^*, W^*V^*)$$

$$= \delta(UW, VW)$$

Clearly $\delta(U_n, U) \to 0$

$\Leftrightarrow U_n x_k \to U x_k$ and $U_n^* x_k \to U^* x_k$ for all $x_k$

$\Leftrightarrow U_n x \to Ux$ and $U_n^* x \to U^* x$ for all $x \in \mathcal{H}$

$\Leftrightarrow U_n \to U$ strongly and $U_n^* \to U^*$ strongly,

$\Leftrightarrow U_n \to U$ strongly.

Thus $\delta$ is a metric on $\mathcal{U}(\mathcal{R})$ compatible withthe strong (=weak) operator topology on $\mathcal{U}(\mathcal{R})$.

Suppose now that $\{U_n\}$ is a Cauchy sequence in $(\mathcal{U}(\mathcal{R}), \delta)$, then

$$\delta(U_n, U_m) \to 0 \quad \text{as } n, m \to \infty$$

and

$$\delta(U_n^*, U_m^*) \to 0 \quad \text{as } n, m \to \infty$$

For each $x \in \mathcal{H}$, the sequences $\{U_n x\}$ and $U_n^* x\}$ are Cauchy, so there are vectors, denoted Ux, and Vx with $U_n x \to Vx$, $U_n^* x \to Wx$. Then $V: x \to Vx$ is an isometric linear mapping: $\mathcal{H} \to \mathcal{H}$ with $U_n \to V$ strongly. Similarly, $W: x \to Wx$ is an isometric linear mapping $\mathcal{H} \to \mathcal{H}$ with $U_n^* \to W$ strongly. But then $U_n^* \to W$ weakly, so $U_n \to W^*$ weakly. Thus $W^* = V$. This shows that V and $V^*$ are both isometric, so $V \in \mathcal{U}(\mathcal{R})$ and $\delta(U_n, V) \to 0$. It follows that $(\mathcal{U}(\mathcal{R}), \delta)$ is a complete metric space. Since $\mathcal{H}$ is separable, $(\mathcal{U}(\mathcal{R}), \delta)$ must also be separable since $\delta$ is compatible with the weak operator topology and $\mathcal{R}_1$, the unit ball of $\mathcal{R}$, is a separable metric space for this topology. We have shown that $\mathcal{U}(\mathcal{R})$, with the weak operator topology, is a polish space. We assert that $\mathcal{U}(\mathcal{R})/\mathcal{U}(\mathcal{J})$ is a polish space with the quotient topology. To see this, let $\mathcal{W} = \mathcal{U}(\mathcal{J})$, and define $d(U\mathcal{W}, V\mathcal{W}) = \inf_{Z \in \mathcal{W}} \delta(UZ, V)$ $(U, V \in \mathcal{U}(\mathcal{R}))$. d is a well-defined pseudometric on $\mathcal{U}(\mathcal{R})/\mathcal{W}$, since $\delta$ is invariant, and if $d(U\mathcal{W}, V\mathcal{W}) = 0$ then there is a sequence $Z_n \in \mathcal{W}$ with $\delta(UZ_n, V) \to 0$.

Thus $\quad UZ_n \to V \quad$ strongly

So $\quad Z_n \to U^*V \quad$ strongly

Hence $U^*V \in \mathcal{W}$, and $U\mathcal{W} = V\mathcal{W}$. This shows that d is a metric on $\mathcal{U}(\mathcal{R})/\mathcal{W}$. Clearly the topology defined by d is equivalent to the quotient topology. If $\{\mathcal{E}_n\}$ is a countable base of open sets of $\mathcal{U}(\mathcal{R})$, then $\{\pi(\mathcal{E}_n)\}$ is a countable base of open sets for $\mathcal{U}(\mathcal{R})/\mathcal{W}$ since $\pi$ is an open mapping. It remains to show that $(\mathcal{U}(\mathcal{R})/\mathcal{W}, d)$ is a complete metric space. Let $\{U_n\}$ be a Cauchy sequence in $(\mathcal{U}(\mathcal{R})/\mathcal{W}, d)$. It suffices to show that this has a convergent subsequence. Choose a subsequence $\{V_n\mathcal{W}\}$ of the original sequence such that

$$d(V_n\mathcal{W}, V_{n+1}\mathcal{W}) < \frac{1}{2^n}$$

We shall show that this sequence is convergent. Choose $Y_1 \in V_1\mathcal{W}$.

Now

Choose $\quad Y_2 \in V_2\mathcal{W} \quad$ with $\quad \delta(Y_1, Y_2) < \frac{1}{2}$

Choose $\quad Y_3 \in V_3\mathcal{W} \quad$ with $\quad \delta(Y_1, Y_2) < \frac{1}{2^2}$, etc.

By induction there is a sequence $\{Y_n\}$ with $Y_n \in V_n\mathcal{W}$ and $\delta(Y_n, Y_{n+1}) < \frac{1}{2^n}$. Clearly $\{Y_n\}$ is a Cauchy sequence in $(\mathcal{U}(\mathcal{R}), \delta)$, thus there is a $Y \in \mathcal{U}(\mathcal{R})$ with $\delta(Y_n, Y) \to 0$. But then

$$d(Y_n\mathcal{W}, Y\mathcal{W}) \leq \delta(Y_n, Y) \to 0 \quad ,$$

so $\mathcal{U}(\mathcal{R})/_{\mathcal{W}}$ is complete. Thus $\mathcal{U}(\mathcal{R})/_{\mathcal{W}}$ is a polish space. If we now endow $\mathcal{U}(\mathcal{R})/_{\mathcal{W}}$ with the quotient borel structure, (Q), in which a set F is borel if $\pi^{-1}(F)$ is a borel set in $\mathcal{U}(\mathcal{R})$, we assert that $\mathcal{U}(\mathcal{R})/_{\mathcal{W}}$ is a standard Q-borel space. We know that $\mathcal{U}(\mathcal{R})/_{\mathcal{W}}$ is a standard borel space for the quotient topology, and the Q-borel structure is the quotient borel structure relative to the standard borel space $\mathcal{U}(\mathcal{R})$. If U is open in $\mathcal{U}(\mathcal{R})/_{\mathcal{W}}$ for the quotient topology, then $\pi^{-1}$ (U) is open in $\mathcal{U}(\mathcal{R})$, so U is a Q-borel set. Denote the borel structure generated by the quotient topology as $\Sigma$. Then every $\Sigma$-borel set is Q borel. Thus the identity map on $\mathcal{U}(\mathcal{R})/_{\mathcal{W}}$ is a bijective borel map from $(\mathcal{U}(\mathcal{R})/_{\mathcal{W}}, Q)$ to $(\mathcal{U}(\mathcal{R})/_{\mathcal{W}}, \Sigma)$ in an obvious notation. It follows by [$\mathfrak{S}$], Appendix B, No. 22 that the two borel structures coincide.

Identifying inn($\mathcal{R}$) with $\mathcal{U}(\mathcal{R})/_{\mathcal{W}}$ by means of the map

$$\psi : \mathcal{U}(\mathcal{R})/_{\mathcal{W}} \rightarrow \text{inn}(\mathcal{R}),$$

we give inn($\mathcal{R}$) the final topology and borel structure relative to $\psi$ i.e. a set $F \subset \text{inn}(\mathcal{R})$ is open (resp. borel) if $\psi^{-1}(F)$ is open (resp. borel) in $\mathcal{U}(\mathcal{R})/_{\mathcal{W}}$. With this structure, inn($\mathcal{R}$) is a polish space and a standard borel space.

<u>6.4 Lemma</u> Let $T \in \mathcal{R}$, $x, y \in \mathcal{H}$, then the map

$$\alpha \rightarrow \langle \alpha(T)\, x, y\rangle$$

is continuous on inn($\mathcal{R}$).

<u>Proof</u> The map

$$f : U \rightarrow \langle U T U^{*} x, y\rangle$$

from $\mathcal{U}(\mathcal{R})$ to C is continuous on $\mathcal{U}(\mathcal{R})$. Since

$$\langle UTU^* x, y\rangle = \langle VTV^* x, y\rangle$$

if $U\mathcal{W} = V\mathcal{W}$, f gives rise to a continuous map

$$g : \mathcal{U}(\mathcal{R})/_{\mathcal{W}} \to C ,$$

$$g(\pi(U)) = \langle UTU^* x, y\rangle .$$

Now the map $\alpha \to \langle\alpha(T)x, y\rangle$ is just the map $g \circ \overset{-1}{\psi}(\alpha)$, hence is continuous.

<u>6.5 Proposition</u> The above borel structure on inn($\mathcal{R}$) is the smallest such that the mappings

$$\alpha \to \langle\alpha(T)x, y\rangle \qquad (T \in \mathcal{R}, \quad x, y \in \mathcal{H})$$

are continuous on inn($\mathcal{R}$).

<u>Proof</u> Let $\mathcal{B}_1$ be the borel structure contructed above for inn($\mathcal{R}$) and let $\mathcal{B}_2$ be the smallest borel structure on inn($\mathcal{R}$) such that the mappings $\alpha \to \langle\alpha(T)x, y\rangle$ are continuous. This means that $\mathcal{B}_2$ is the borel structure generated b, the open sets of the coarsest topology for which these maps are continuous. By Lemma 6.4, $\mathcal{B}_1 \supset \mathcal{B}_2$. Let $\{x_j\}_{j=1}^{\infty}$ be a sense subset of $\mathcal{H}$, and $\{T_j\}_{j=1}^{\infty}$ be weakly dense in $\mathcal{R}$. Define

$$\varphi_{ijk}(\alpha) = \langle\alpha(T_i)x_j, x_k\rangle ,$$

and let $(\Omega_n)_{n=1}^{\infty}$ be a countable basis for the topology of C. If

$$\mathcal{E}_{ijk\ell} = \varphi_{ijk}^{-1}(\Omega_\ell) ,$$

the sets $\mathcal{E}_{ijk\ell}$ form a separating family of borel subsets of inn($\mathcal{R}$) (relative to $\mathcal{B}_2$) in the sense that if $\alpha, \beta \in$ inn($\mathcal{R}$) and $\alpha \neq \beta$, then for some i, j, k, $\ell$, $\alpha \in \mathcal{E}_{ijk\ell}$ and $\beta \notin \mathcal{E}_{ijk\ell}$. By Theorem 3.3 of [35], the sets $\mathcal{E}_{ijk\ell}$ generate the $\mathcal{B}_1$-borel structure, thus $\mathcal{B}_1 = \mathcal{B}_2$.

<u>6.6 Corollary</u> The map

$$U \to \operatorname{ad} U$$

from $\mathcal{U}(\mathcal{R})$ to $\operatorname{inn}(\mathcal{R})$, is borel.

<u>Proof</u> If $T \in \mathcal{R}$, $x, y \in \mathcal{H}$, the map

$$U \to \langle UTU^* x, y \rangle$$

is continuous. The result follows by Proposition 6.5.

<u>6.7 Definition</u> A <u>borel cross-section</u> for $\mathcal{W}$ is a borel map

$$\eta : \mathcal{U}(\mathcal{R})/\mathcal{W} \to \mathcal{U}(\mathcal{R})$$

such that $\pi \circ \eta$ = identity.

<u>6.8 Proposition</u> There is a borel cross-section for $\mathcal{W}$

<u>Proof</u> By [6], Lemma 3, there is a borel subset B of $\mathcal{U}(\mathcal{R})$ which meets each coset of $\mathcal{W}$ in $\mathcal{U}(\mathcal{R})$ in exactly one point. By Corollary 1 to Theorem 3.2 of [35], B is a standard borel space, and $\pi_{|B}$ is bijective. The sets $\psi^{-1}(\mathcal{E}_{ijk\ell})$ form a separating family of borel subsets of $\mathcal{U}(\mathcal{R})/\mathcal{W}$, thus by Theorem 3.2 of [35], $\pi_{|B}$ is a borel isomorphism. Let $\eta = (\pi_{|B})^{-1}$, then $\eta$ is the required borel cross-section.

<u>6.9 Lemma</u> Let G be a locally compact group, $\mathcal{R}$ a von Neumann

algebra acting on a separable Hilbert space $\mathcal{H}$, and $\alpha$ a strongly continuous representation of G on $\mathfrak{R}$. If $\alpha(g)$ is an inner automorphism of $\mathfrak{R}$ for each $g \in G$, there is a borel map

$$\varphi : G \to \mathcal{U}(\mathfrak{R})$$

such that

$$\alpha(g) = \text{ad}\,\varphi(g) \qquad (g \in G)$$

Proof Let $\eta$ be a borel cross-section for $\mathcal{U}(\mathcal{Z})$ in $\mathcal{U}(\mathfrak{R})$, where $\mathcal{Z}$ denotes the centre of $\mathfrak{R}$. Denote by $\pi$ the quotient map $\mathcal{U}(\mathfrak{R}) \to \mathcal{U}(\mathfrak{R})/\mathcal{U}(\mathcal{Z})$ and by $\psi$ the group isomorphism

$$\mathcal{U}(\mathfrak{R})/\mathcal{U}(\mathcal{Z}) \to \text{inn}(\mathfrak{R})$$

Let

$$\alpha(g) = \text{ad}\,W_g$$

with $W_g \in \mathcal{U}(\mathfrak{R})$ and define

$$U_g = \eta(W_g\,\mathcal{U}(\mathcal{Z})) ,$$

Then

$$U_g = (\eta \circ \psi^{-1} \circ \alpha)(g)$$

Let E be a basic open subset of the identity of $\text{inn}(\mathfrak{R})$, then E is of the form

$$E = \{\gamma \in \text{inn}(\mathfrak{R}); \ |\langle\gamma(T_j)x_k, x_\ell\rangle - \langle T_j x_k, x_\ell\rangle| < 1$$

$$1 \le j,k,\ell \le n\}$$

Since $\alpha$ is strongly (hence weakly) continuous, $\alpha^{-1}(E)$ is a borel subset of G, so $\alpha$ is a borel mapping. $\eta$ and $\psi^{-1}$ are also borel, thus the map $g \to U_g$ is a borel map: $G \to \mathcal{U}(\mathfrak{R})$. Since $\pi(U_g)=\pi(W_g)$, we have $\alpha(g) = \text{ad}\,W_g = \text{ad}\,U_g$ $(g \in G)$. This completes the proof.

We can now prove our first result.

6.10 Lemma Let $\mathfrak{A}$ be a norm separable simple C*-algebra with identity acting on a separable Hilbert space $\mathcal{H}$. Let G be a locally compact abelian group and $\alpha$ a strongly continuous representation of G by inner automorphisms of $\mathfrak{A}$, so that $\alpha(g) = \text{ad } W_g$, $W_g \in \mathcal{U}(\mathfrak{A})$. There is a strongly continuous unitary representation $U : g \to U_g$ of G by unitaries $U_g \in \mathfrak{A}$ implementing $\alpha$ if and only if

$$W_g W_h = W_h W_g \qquad (g, h \in G) .$$

<u>Proof</u> ($\Rightarrow$) Suppose there is a strongly continuous unitary representation $U : g \to U_g$ of G with $U_g \in \mathfrak{A}$ and $\alpha(g) = \text{ad } U_g$. Then for each $g \in G$, there is a unitary $Q_g \in \mathcal{Z}(\mathfrak{A})$ with $U_g = Q_g W_g$. The $U_g$'s commute since G is abelian, thus the $W_g$'s must commute.

($\Leftarrow$) Let $\pi$ be an irreducible representation of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}_\pi$. If $x_o \in \mathcal{H}_\pi$ is a generating vector for $\pi$, and $\{A_n\}_{n=1}^{\infty}$ is a countable dense subset of $\mathfrak{A}$, then $\{\pi(A_n)x_o\}_{n=1}^{\infty}$ is a dense subset of $\mathcal{H}_\pi$, so $\mathcal{H}_\pi$ is separable. Since $\alpha$ is strongly (hence weakly) continuous, the map

$$g \to \omega_x(\alpha(g)(A))$$

is continuous for $x \in \mathcal{H}$ and $A \in \mathfrak{A}$. Now

$$\{\omega_x ; \ x \in \mathcal{H}, \ \ \|x\| = 1\}$$

is a full set of states of $\mathfrak{A}$, in the sense that if $A \in \mathfrak{A}$, and $\omega_x(A) \geq 0$ $(x \in \mathcal{H})$, then $A \geq 0$. Thus by ([8], Lemma 3.4.1),

$$\text{co } \{\omega_x ; \ x \in \mathcal{H}, \ \ \|x\| = 1\}$$

is weak*-dense in $E(\mathfrak{A})$. It follows by ([4], §4, No.7, p.171, Corollaire 2) that

$$\|\alpha(g)(A) - \alpha(h)(A)\| \to 0 \quad \text{as} \quad g \to h \text{ in } G ,$$

($h \in G$, $A \in \mathfrak{A}$). Since $\mathfrak{A}$ is simple, Ker $\pi = \{0\}$, thus $\pi$ is faithful and so isometric. Hence

$$\|\pi \circ \alpha(g) \circ \pi^{-1}(\pi(A)) - \pi(A)\| = \|\pi(\alpha(g)(A) - A)\|$$
$$= \|\alpha(g)(A) - A\| \to 0 \text{ as } g \to e \text{ in } G.$$

It follows that $\beta: g \to \beta(g) = \pi \circ \alpha(g) \circ \pi^{-1}$ is a strongly continuous representation of G on $\pi(\mathfrak{A})$. Each $\beta(g)$ is an inner automorphism of $\pi(\mathfrak{A})$, (being induced by $\pi(W_g)$) thus $\beta(g)$ extends to an automorphism $\gamma(g)$ of $\pi(\mathfrak{A})^-$. Theorem 2.8 of Chapter I shows that $\gamma: g \to \gamma(g)$ is a strongly continuous representation of G on $\pi(\mathfrak{A})^-$. Note that $\pi(\mathfrak{A})^- = \mathcal{B}(\mathcal{H}_\pi)$ since $\pi$ is irreducible.

By Lemma 6.9, we may choose unitaries $V_g \in \mathcal{B}(\mathcal{H}_\pi)$ such that

$$\gamma(g) = \text{ad } V_g$$

and $g \to V_g$ is a Borel map from $G \to \mathcal{U}(\mathcal{B}(\mathcal{H}_\pi))$

for the weak operator topology on $\mathcal{U}(\mathcal{B}(\mathcal{H}_\pi))$. Now

$$\beta_g = \text{ad } V_g = \text{ad } \pi(W_g) ,$$

thus $V_g \pi (W_g)^* \in \pi(\mathfrak{A})' = \mathbb{C}I$. Hence there is a scalar $\lambda_g$, $|\lambda_g| = 1$, with

$$V_g = \lambda_g \pi(W_g)$$

so

$$V_g \in \pi(\mathfrak{A}) \text{ for } g \in G .$$

Since the $W_g$'s commute, so do the $V_g$'s. Let $\mathcal{B}$ be the C*-algebra generated by $\{V_g; g \in G\}$, then $\mathcal{B}$ is an abelian C*-subalgebra of $\pi(\mathfrak{A})$. Let $\rho$ be a character on $\mathcal{B}$ and set

$$U_g = \overline{\rho(V_g)} V_g .$$

Since $\rho(V_g)$ lies in the unit circle, $\gamma(g) = \text{ad } V_g = \text{ad } U_g$.

Now for each $x \in \mathcal{H}_\pi$, the map $g \to \omega_x(V_g)$ is borel. As above, $\text{co}\{\omega_x;\ x \in \mathcal{H}_\pi,\ \|x\| = 1\}$ is weak* dense in $E(\pi(\mathfrak{A}))$. Since $\mathfrak{A}$ is norm separable, $\pi(\mathfrak{A})$ is norm separable, thus $\mathfrak{B}$ is norm separable, so the weak* topology on $E(\mathfrak{B})$ is metrizable. (To see this - let $\{B_n\}_{n=1}^{\infty}$ be dense in $\mathfrak{B}$, and $f, g \in E(\mathfrak{B})$. Define $\delta(f, g) = \sum_{n=1}^{\infty} \frac{(f-g)(B_n)}{\|B_n\|2^n}$. Then $\delta$ is a metric on $E(\mathfrak{B})$, and defines a Hausdorff topology on $E(\mathfrak{B})$ which is coarser than the weak* topology. Since $E(\mathfrak{B})$ is weak* compact, the two topologies must coincide). Hence we may choose a sequence $f_n$ in

$$\text{co}\{\omega_x;\ x \in \mathcal{H}_\pi,\ \|x\| = 1\}$$

such that

$$f_n \to \rho \quad \text{weak*}.$$

If
$$f_n = \Sigma \lambda_j \omega_{x_j},$$

then
$$f_n(V_g) = \Sigma \lambda_j\, x_j(V_g)$$

and
$$g \to f_n(V_g) \text{ is a borel map}.$$

The map $g \to \overline{\rho(V_g)}$ is the pointwise limit of the borel maps $g \to \overline{f_n(V_g)}$ hence is borel. Let $\Theta(g) = \overline{\rho(V_g)}$.

Since the pointwise product of two borel maps is again a borel map, for each $x \in \mathcal{H}$, the map

$$g \to \Theta(g)\langle V_g x, x\rangle = \langle U_g x, x\rangle$$

is Borel. However if $g, h \in G$,

$$\alpha(gh) = \text{ad } U_g U_h = \text{ad } U_{gh}$$

so there is a scalar $\mu$, $|\mu| = 1$, with

$$U_g U_h = \mu U_{gh}$$

But $$\rho(U_g) = 1 \text{ for all } g \in G ,$$

so
$$\mu = \rho(U_g U_h U^*_{gh})$$
$$= \rho(U_g)\rho(U_h)\rho\overline{(U_{gh})}$$
$$= 1$$

This shows that $U: g \to U_g$ is a weakly measurable unitary representation of G on $\mathcal{H}_\pi$. Let

$$Y_g = \pi^{-1}(U_g) .$$

$Y_g$ is a unitary in $\mathfrak{A}$, and since

$$\pi \circ \alpha(g) \circ \pi^{-1} = \text{ad } U_g ,$$

we have

$$\alpha(g) = \text{ad } Y_g$$

Clearly $Y: g \to Y_g$ is a unitary representation of G. It remains to prove that Y is strongly continuous. Since $\mathcal{H}$ is separable it suffices to prove that the map

$$g \to \langle V_g x, x\rangle .$$

is borel for $x \in \mathcal{H}$, by [18], (Theorem 22.20(b), p.347). If $\|x\| = 1$, then $\omega_x \circ \pi^{-1}$ is a state of $\pi(\mathfrak{A})$. As above

$$\text{co } \{\omega_y; \|y\| = 1, y \in \mathcal{H}_\pi\}$$

is weak* dense in $E(\pi(\mathfrak{A}))$, and the weak* topology is metrizable on $E(\pi(\mathfrak{A}))$ since $\pi(\mathfrak{A})$ is norm separable, so there is a sequence

$f_n \to \omega_x \circ \pi^{-1}$, weak*, with each $f_n$ of the form

$$f_n = \Sigma \lambda_j \omega_{y_j} \qquad (\lambda_j \geq 0, \ \Sigma \lambda_j = 1, \ y_j \in \mathcal{H}_\pi) .$$

The maps

$$g \to f_n(U_g)$$

are borel, so the map

$$g \to \omega_x \circ \pi^{-1}(U_g)$$

is borel, being the pointwise limit of a sequence of borel maps. But

$$\omega_x \circ \pi^{-1}(U_g) = \omega_x(Y_g) .$$

So the result is proved.

<u>6.11 Theorem</u> Let $\mathfrak{A}$ be a norm separable simple C*-algebra, acting on a separable Hilbert space $\mathcal{H}$, G a locally compact abelian group, $\alpha$ a representation of G on $\mathfrak{A}$ by inner automorphisms, say

$$\alpha(g) = \text{ad } W_g, \qquad W_g \in \mathcal{U}(\mathfrak{A}) ,$$

such that

$$W_g W_h = W_h W_g \qquad (g, h \in G) .$$

Suppose $\alpha$ is a weakly measurable representation, then

$$\|\alpha(g) - \iota\| \to 0 \quad \text{as } g \to e \text{ in } G$$

and there is a norm continuous unitary representation $g \to U_g$ by unitaries $U_g \in \mathfrak{A}$, such that

$$\alpha(g) = \text{ad } U_g \qquad (g \in G) .$$

<u>Proof</u> Since each $\alpha(g)$ is an inner automorphism of $\mathfrak{A}$, it

extends to an inner automorphism, $\beta(g)$, of $\mathfrak{A}^-$. By Theorem 2.8, Chapter II, $\beta : g \to \beta(g)$ is a strongly continuous representation, thus $\alpha$ is also a strongly continuous representation. Since $\mathfrak{A}$ is norm separable, $E(\mathfrak{A})$ is metrizable for the weak* topology, as remarked above, and

$$\mathrm{co}\ \{\omega_x ;\ x \in \mathcal{H},\ \|x\| = 1\}$$

is weak*-dense in $E(\mathfrak{A})$. If $f \in E(\mathfrak{A})$, $f$ is thus the weak* limit of a sequence $\{f_n\}$, where each $f_n$ is a convex combination of vector states. By Lemma 6.10, there is a unitary representation, $g \to U_g$, of $G$ with

$$\alpha(g) = \mathrm{ad}\ U_g\ , \qquad U_g \in \mathfrak{u}(\mathfrak{A})$$

and $\qquad g \to \omega_x(U_g)$ continuous for each $x \in \mathcal{H}$ .

Thus the map

$$g \to f_n(U_g)$$

is a borel for each n, so

$$g \to f(U_g)$$

is a borel map. Now $E(\mathfrak{A})$ spans $\mathfrak{A}^*$ algebraically, so

$$g \to f(U_g)$$

is a borel map for each $f \in \mathfrak{A}^*$. The proof is now completed by using a similar argument to that of Theorem 2.2, Chapter II. Let $\{A_n\}_{n=1}^{\infty}$ be a countable dense subset of $\mathfrak{A}$, and $\{x_n\}$ be a dense subset of the unit ball of $\mathcal{H}$. Let $\epsilon > 0$. Define

$$\mathcal{W} = \{g \in G;\ \|U_g - I\| \le \epsilon/2\}\ .$$

Now
$$\|U_g - I\| = \|U_g^* - I\| = \|U_{g^{-1}} - I\|$$

and
$$\|U_{gh} - I\| = \|U_g U_h - I\| \leq \|U_g - I\| + \|U_h - I\|$$

Thus
$$\mathcal{W} = \mathcal{W}^{-1}$$

and
$$\mathcal{W}^2 \subseteq \{g \in G;\ \|U_g - I\| \leq \epsilon\}.$$

Let m denote haar measure on G.

$$\mathcal{W} = \bigcap_{n,m} \{g \in G;\ |\langle U_g x_n, x_m\rangle - \langle x_n, x_m\rangle| \leq \epsilon/2\}$$

so $\mathcal{W}$ is m-measurable. Let

$$\mathcal{T} = \{U_g;\ g \in G\}$$

Then $\mathcal{T} \subseteq \mathfrak{U}$, and $\mathfrak{U}$ is separable, so $\mathcal{T}$ is separable. Let $\{U_{g_n}\}$ be a countable dense subset of $\mathcal{T}$. If $g \in G$, there is a $g_n$ with

$$\|U_g - U_{g_n}\| \leq \epsilon/2.$$

Thus
$$\|U_{g_n^{-1} g} - I\| \leq \epsilon/2.$$

so $g_n^{-1} g \in \mathcal{W}$ and $g \in g_n\mathcal{W}$. Hence

$$G = \bigcup_n g_n \mathcal{W}.$$

By invariance of haar measure, $\mathcal{W}$ contains a compact C of positive measure. Then $CC^{-1}$ contains a neighbourhood N of e ([18], 20.17 Corollary, p.296), and we have

$$N \subseteq CC^{-1} \subseteq \mathcal{W}\mathcal{W}^{-1}$$

$$\subseteq \{g \in G;\ \|U_g - I\| \leq \epsilon\}\ .$$

This shows that

$$\|U_g - I\| \to 0 \text{ as } g \to e$$

so $g \to U_g$ is a norm continuous unitary representation, and

$$\|\alpha(g) - \iota\| = \sup\{\|U_g A U_g^* - A\|;\ A \in \mathfrak{A},\ \|A\| \leq 1\}$$

$$\leq 2\|U_g - I\| \to 0 \text{ as } g \to e\ .$$

This completes the proof.

<u>6.12 Corollary</u> Let $t \to \alpha_t$ be a weakly measurable representation of $R$ as inner automorphisms of the norm separable simple C*-algebra $\mathfrak{A}$, and suppose $\mathfrak{A}$ acts on a separable Hilbert space $\mathcal{H}$, then there is a norm continuous unitary representation $t \to U_t$ of $R$ with $U_t \in \mathcal{U}(\mathfrak{A})$, such that

$$\alpha_t = \text{ad } U_t\ ,$$

$$\|\alpha_t - \iota\| \to 0 \text{ as } t \to 0\ .$$

<u>Proof</u> By Theorem 6.11, it suffices to prove that if

$$\alpha_t = \text{ad } W_t\ , \qquad W_t \in \mathcal{U}(\mathfrak{A})\ ,$$

then $$W_t W_s = W_s W_t \qquad (s,t \in R)\ .$$

Let $\pi$ be an irreducible representation of $\mathfrak{A}$ on a Hilbert space

$\mathcal{H}_\pi$. $\mathcal{H}_\pi$ is separable, as mentioned before, and as in the proof of Lemma 6.10, $\beta: t \to \beta(t) = \pi \circ \alpha(t) \circ \pi^{-1}$ is a strongly continuous representation of R on $\pi(\mathfrak{A})$ and extends to a strongly continuous representation $\gamma$ of R on

$$\pi(\mathfrak{A})^- = \mathcal{B}(\mathcal{H}_\pi)$$

by the inner automorphisms ad $\pi(W_g)$ of $\mathcal{B}(\mathcal{H}_\pi)$. By [25] Theorem 0.1 there is a unitary representation $t \to Y_t$ of R on $\mathcal{H}_\pi$ with

$$\gamma(t) = \text{ad } Y_t = \text{ad } \pi(W_t)$$

So there is a scalar $\lambda_t$, $|\lambda_t| = 1$, such that

$$Y_t = \lambda_t \, \pi(W_t) .$$

The $Y_t$'s commute, so

$$\pi(W_t W_s) = \pi(W_s W_t) \qquad (s,t \in R) .$$

$\pi$ is a faithful representation, since $\mathfrak{A}$ is simple, thus the result follows.

After proving Theorem 5.2, Chapter V and the above results, I conjectured that the following result is true.

<u>6.13 Conjecture</u> Let G be a locally compact abelian group, $\mathcal{R}$ a von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$, and $\alpha: g \to \alpha(g)$ a strongly continuous representation of G on $\mathcal{R}$ by inner automorphisms, say

$$\alpha(g) = \text{ad } W_g \quad \text{with } W_g \in \mathcal{U}(\mathcal{R}) .$$

Then there is a strongly continuous unitary representation $U: g \to U_g$ of G by unitaries $U_g \in \mathcal{R}$ such that

$$\alpha(g) = \text{ad } U_g$$

$$\Longleftrightarrow$$

$$W_g W_h = W_h W_g \qquad (g, h \in G) .$$

A proof was sketched for the case G a separable locally compact abelian group, and $\mathcal{R}$ a factor, by C.C. Moore and communicated to me by Dr. A. Connes. I shall give this proof below in full detail. It is not clear as yet whether the result extends to the case of a von Neumann algebra, using standard direct integral methods.

Before commencing the proof we need some preliminary remarks on group extensions. A group G is called an <u>extension</u> of the group C by the group B if C is a normal subgroup of G (up to isomorphism) and $G/C$ is isomorphic to B. From now on we shall deal only with the case where C is an abelian group.

Suppose G is an extension of C by B. In every coset gC of C choose an element $g_\alpha$ where gC corresponds to the element $\alpha$ of B. Now

$$g_\alpha g_\beta \in g_{\alpha\beta} C$$

so there is an element $\eta(\alpha, \beta)$ of C such that

$$g_\alpha g_\beta = g_{\alpha\beta}\, \eta(\alpha, \beta) .$$

If $\alpha = \beta = e$, then

$$\eta(e, e) = g_e .$$

We shall assume always that $g_e = e$ (e will always denote the unit element of the appropriate group).

The map

$$a \rightarrow g_\alpha a g_\alpha^{-1}$$

induces an automorphism $\gamma(\alpha)$ of C, and

$$\gamma(\alpha)\gamma(\beta)(a) = g_\alpha g_\beta a g_\beta^{-1} g_\alpha^{-1}$$

$$= g_{\alpha\beta} a g_{\alpha\beta}^{-1}$$

$$= \gamma(\alpha\beta)(a)$$

since C is abelian, so $\gamma: \alpha \to \gamma(\alpha)$ is a group homomorphism identifying B as a group of automorphisms of C. $\eta$ is a map from $B \times B$ into C such that $\eta(e, e) = e$. Simple arguments (see for example [30], p.122) show that $\eta$ also satisfies

$$\eta(y_1, y_2)\eta(y_1 y_2, y_3) = \gamma(y_1)(\eta(y_2, y_3))\eta(y_1, y_2 y_3) \qquad (*)$$

$$y_1, y_2, y_3 \in B .$$

Such a map $\eta$ is called a system of factors.

Recall that if $(A_j)_{j=1}^{\infty}$ are sets and $f_j$ are maps from $A_{j-1}$ to $A_j$, then the sequence

$$\ldots A_{j-1} \xrightarrow{f_j} A_j \xrightarrow{f_{j+1}} A_{j+1} \to \ldots$$

is said to be exact if

$$\text{Ker } f_{j+1} = \mathcal{I}m\, f_j .$$

If A, B, C are groups, $\gamma$ is an injective homomorphism from A to B, and $\delta$ is a surjective homomorphism from B onto C, such that

$$\mathcal{I}m\, \gamma = \text{Ker } \delta ,$$

this is representable as the short exact sequence

$$0 \to A \xrightarrow{\gamma} B \xrightarrow{\delta} C \to 0$$

Thus if G is an extension of C by B, the sequence

$$0 \to C \xrightarrow{i} G \xrightarrow{\pi} B \to 0$$

is a short exact sequence where $i: C \to G$ is the identity map and $\pi: G \to B$ is the quotient mapping from G to $G/_C$ (up to isomorphism). Conversely, given a short exact sequence of groups

$$0 \to C \xrightarrow{\gamma} G \xrightarrow{\delta} B \to 0 .$$

If we identify C with its isomorphic image $\gamma(C)$ in G, we see that $G/_C$ is isomorphic to B, so G is nothing but an extension of C by B.

Suppose that G and H are extensions of C by B, defined by the short exact sequences

$$0 \to C \xrightarrow{\alpha} G \xrightarrow{\beta} B \to 0$$

$$0 \to C \xrightarrow{\alpha'} H \xrightarrow{\beta'} B \to 0$$

G and H are said to be equivalent if there is an isomorphism $\theta: G \in H$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \to & C & \xrightarrow{\alpha} & G & \xrightarrow{\beta} & B & \to & 0 \\ & & \downarrow id & & \downarrow \theta & & \downarrow id & & \\ 0 & \to & C & \xrightarrow{\alpha'} & H & \xrightarrow{\beta'} & B & \to & 0 \end{array}$$

If we are given an abelian group K, and an arbitrary group Q, and we have maps $\gamma$, $\eta$, such that $\gamma$ is a group homomorphism from Q to the group of all automorphisms of K, and $\eta$ is a map from $Q \times Q$ into K such that $\eta(e, e) = e$ and (*) holds, define a multiplication on $K \times Q$ by

$$(\xi_1, y_1) \circ (\xi_2, y_2) = (\xi_1 \gamma(y_1)(\xi_2)\, \eta\, (y_1, y_2), y_1 y_2)$$

$$(\xi_1, \xi_2 \in K, \quad y_1, y_2 \in Q) .$$

Then $K \times Q$ is a group with this multiplication ([30], pp.123-124). Moreover, the map

$$\delta : (\xi, y) \rightarrow y$$

is a homomorphism from $K \times Q$ onto $Q$, and the map

$$\theta : \xi \rightarrow (\xi, e)$$

is an injection from $K$ into $K \times Q$, with

$$\mathcal{I}m\ \theta = \text{Ker}\ \delta$$

So the sequence

$$0 \rightarrow K \overset{\theta}{\rightarrow} K \times Q \overset{\delta}{\rightarrow} Q \rightarrow 0$$

is short exact, and $K \times Q$ with this multiplication is an extension of $K$ by $Q$. We denote this extension by $K \eta Q$.

Suppose now that $S$ and $T$ are borel spaces, then we can define the product borel structure on $S \times T$ as being the borel structure generated by the sets $E \times F$ where $E$ (resp. $F$) is a borel set in $S$ (resp. $T$). If $Q$ and $K$ are groups with borel structures, we say that $\eta$ is a borel system of factors for $Q$ and $K$ if $\eta$ is a borel mapping from $Q \times Q$ (with the product borel structure) to $K$.

Now let $K$ and $Q$ be separable locally compact groups, with $K$ also abelian. Let $\gamma$ be a homomorphism from $Q$ into the group of automorphisms of $K$, such that the map

$$(x, y) \rightarrow \gamma(y)(x)$$

is continuous: $K \times Q \rightarrow K$, and let $\eta$ be a borel system of factors for $Q$ and $K$. We have the following fundamental result of Mackey ([36], Théorème 2).

<u>6.14 Theorem</u> There exists in the group extension $K \eta Q$ a unique locally compact topological structure with regard to which $K \eta Q$ is a topological group, such that the identity map from $K \times Q$ (with the product borel structure) into $K \eta Q$ is a borel mapping. The map $\theta: \xi \to (\xi, e)$ is an isomorphism from K onto a closed normal subgroup of $K \eta Q$, and $\theta$ is bicontinuous. The isomorphism from $K \eta Q/_{\theta(K)}$ onto Q defined by the mapping $\delta: (\xi, y) \to y$ from $K \eta Q$ onto Q, is also bicontinuous. Moreover, $K \eta Q$ is a separable group.

We are now in a position to prove the result discussed earlier.

<u>6.15 Theorem</u> Let $\mathfrak{R}$ be a factor acting on a separable Hilbert space $\mathcal{H}$, G a separable locally compact abelian group, and $\alpha$ a strongly continuous representation of G on $\mathfrak{R}$ by inner automorphisms, say

$$\alpha(g) = \text{ad } W_g, \text{ with } W_g \in \mathcal{U}(\mathfrak{R}) .$$

Then there is a strongly continuous unitary representation

$$U : g \to U_g$$

of G with $U_g \in \mathcal{U}(\mathfrak{R})$ such that

$$\alpha(g) = \text{ad } U_g$$

if and only if

$$W_g W_h = W_h W_g \qquad (g, h \in G) .$$

<u>Proof</u> ($\Rightarrow$) Suppose there is such a unitary representation U. There for each $g \in G$,

$$\alpha(g) = \text{ad } U_g = \text{ad } W_g ,$$

so, since $\mathfrak{R}$ is a factor, there is a scalar $\lambda_g$, $|\lambda_g| = 1$, with

$U_g = \lambda_g W_g$. Since G is abelian, the $U_g$'s commute, so the $W_g$'s must commute.

(⇐) Let

$$N = \text{Ker}\,\alpha = \{g \in G;\ \alpha(g) = i\}$$

Replacing G by $G/_N$, (which is also a locally compact separable abelian group), we may assume that

$$\alpha(g) = \alpha(h) \text{ implies } g = h .$$

By Lemma 6.9, we may choose $Y_g \in \mathcal{U}(\mathcal{R})$ such that

$$\alpha(g) = \text{ad}\, Y_g$$

and

$$g \to Y_g$$

is a borel mapping from G to $\mathcal{U}(\mathcal{R})$, with the weak operator topology. Denote by $\mathbb{T}$ the unit circle. Since $\mathcal{R}$ is a factor, $\mathcal{U}(\mathcal{Z}(\mathcal{R}))$ is isomorphic to $\mathbb{T}$. We assert that

$$\Gamma = \{\lambda Y_g;\ \lambda \in \mathbb{T},\ g \in G\}$$

is an abelian subgroup of $\mathcal{U}(\mathcal{R})$. To see this, note first that

$$\alpha(e) = \text{ad}\, I = \text{ad}\, Y_e ,$$

so

$$I = \lambda Y_e$$

for some $\lambda \in \mathbb{T}$, thus

$$I \in \Gamma .$$

If $\lambda Y_g$, $\mu Y_h$ lie in $\Gamma$, then

$$\alpha(gh) = \text{ad}\, Y_g Y_h$$

$$= \text{ad}\, Y_{gh} ,$$

so there is a $\theta \in \mathbb{T}$ with

$$Y_g Y_h = \theta Y_{gh} .$$

Thus

$$\lambda Y_g \mu Y_h = \lambda \mu \theta Y_{gh} \in \Gamma .$$

If $g \in G$, $\lambda \in \mathbb{T}$

$$\alpha(e) = \text{ad } I$$

$$= \text{ad } Y_g Y_{g^{-1}}$$

so there is a $\xi \in \mathbb{T}$ with

$$Y_g Y_{g^{-1}} = \xi I .$$

Then $$\lambda Y_g (\bar{\lambda} \bar{\xi} Y_{g^{-1}}) = I .$$

Thus $$\lambda Y_g \text{ has inverse } \bar{\lambda} \bar{\xi} Y_{g^{-1}} \in \Gamma .$$

This shows that $\Gamma$ is a group. Since

$$\alpha(g) = \text{ad } W_g = \text{ad } Y_g ,$$

there are scalars $\psi_g \in \mathbb{T}$ with

$$Y_g = \psi_g W_g .$$

The $W_g$'s commute, so the $Y_g$'s commute, and hence $\Gamma$ is abelian.

Define $$\gamma(\lambda) = \lambda I \qquad (\lambda \in \mathbb{T})$$

and $$\eta(\lambda W_g) = g \qquad (\lambda \in \mathbb{T},\ g \in G) .$$

$\eta$ is well-defined since

$$\lambda W_g = \mu W_h \Rightarrow \alpha(g) = \alpha(h) \Rightarrow g = h .$$

$\gamma$ is an injective homorphism, $\eta$ is a surjective homomorphism, and

$$\begin{aligned} \operatorname{Ker} \eta &= \{\lambda W_g ; \lambda \in \mathbb{T}, g = e\} \\ &= \mathbb{T} \\ &= \mathcal{I}m\, \gamma \ . \end{aligned}$$

Thus the sequence

$$0 \to \mathbb{T} \xrightarrow{\gamma} \Gamma \xrightarrow{\eta} G \to 0$$

is short exact, and $\Gamma$ is an extension of $\mathbb{T}$ by G. We can identify $\Gamma$ with the extension $\mathbb{T}\eta' G$ where

$$\eta'(\lambda, g) = \eta(\lambda Y_g) = g$$

via the map

$$(\lambda, g) \to \lambda Y_g \ .$$

Now G is a standard borel space by [35], Theorem 2, p.148, and $\mathcal{U}(\mathcal{R})$ is also a standard borel space, so $Y^{-1}: Y_g \to g$ is a borel mapping by [35], Theorem 3.2. The map $\eta'$ is thus a borel map from $\mathbb{T} \times G$ onto G, and is a borel system of factors for $\mathbb{T}\eta' G$. By Theorem 6.14 there is a locally compact topology on $\Gamma$ relative to which it is a separable locally compact abelian group. $\Gamma$ also contains $\mathbb{T}$ as a closed normal subgroup.

We now wish to use the fact that if H is a closed subgroup of a locally compact abelian group K, and $\rho$ is a character on H (i.e. a continuous homomorphism: $H \to \mathbb{T}$) then $\rho$ extends to a character on K.

For a proof of this, let $\hat{K}$ be the group of characters on K under pointwise multiplication. It is a well known theorem of Pontryagin ([33], § 37.D, p. 51), that $\hat{K}$ may be given a locally compact topology relative to which it is a locally compact abelian group, If $g \in K$, define

$$\omega(g)(\rho) = \rho(g) \qquad (\rho \in \hat{K})$$

then $\omega$: $g \to \omega(g)$ is an isomorphism and homeomorphism between $K$ and $\hat{\hat{K}}$. Let

$$L = \{\rho \in \hat{K};\ \rho(H) = 1\}$$

then $(\hat{K}/L)^{\wedge} = H$ clearly .

By Pontryagin's theorem, $\hat{K}/L$ is isomorphic to $\hat{H}$. Let $\varphi$ denote the isomorphism

$$\hat{H} \to \hat{K}/L\ .$$

and let $\psi$ be the restriction map

$$\psi : \rho \to \rho|_H : \hat{K} \to \hat{H}\ .$$

Now Ker $\psi = L$, so $\psi$ defines an isomorphism $\bar{\psi}$

$$\bar{\psi} : \hat{K}/L \to \hat{H}\ .$$

Thus $$\bar{\psi} = \varphi^{-1}\ .$$

Let $\Omega$ be the quotient map

$$\hat{K} \to \hat{K}/L\ ,$$

and choose

$$\sigma \in \Omega^{-1}(\psi^{-1}(\gamma))$$

then $$\sigma \in \hat{K}$$

and $$\sigma|_H = \psi(\sigma)$$
$$= \bar{\psi}(\sigma L)$$

$$= \bar{\psi}(\Omega(\sigma))$$

$$= \gamma .$$

Suppose now $\rho$ is the identity character on $\mathbb{T}$. By the above remarks, $\rho$ extends to a character $\sigma$ of $\Gamma$. Define

$$U_g = \sigma \overline{(Y_g)}\, Y_g$$

Then
$$\sigma\overline{(Y_g)} \in \mathbb{T} ,$$

so
$$\alpha(g) = \text{ad } Y_g = \text{ad } U_g \qquad (g \in G)$$

If we identify $\Gamma$ with $\mathbb{T}\eta' G$, and j is the identity map from $\mathbb{T} \times G$ into $\mathbb{T}\eta' G$, then

$$Y_g = j(1, g)$$

j is a borel map from $\mathbb{T} \times G$ into $\mathbb{T}\eta' G$, by Theorem 6.14, and the identification between $\Gamma$ and $\mathbb{T}\eta' G$ is a borel isomorphism, (by construction), so the map

$$Y : g \to Y_g$$

is a borel map from G to $\Gamma$. Now $\sigma \in \hat{\Gamma}$, so $\sigma$ is a continuous map from $\Gamma$ to $\mathbb{T}$, thus

$$\sigma \circ Y : g \to \sigma(Y_g) .$$

is a borel map from G to $\mathbb{T}$. If $x \in \mathcal{H}$, then

$$\langle U_g x, x\rangle = \overline{\sigma (Y_g)}\, \langle Y_g x, x\rangle$$

The map

$$g \to \langle Y_g x, x\rangle .$$

is borel, so the map

$$g \to \langle U_g x, x\rangle$$

is borel. Now if g,h ∈ G, then

$$\alpha(gh) = \text{ad } U_g U_h = \text{ad } U_{gh}$$

so there is a $\mu \in \mathbb{T}$ with

$$U_g U_h = \mu U_{gh}$$

But

$$\mu = \sigma(U_g U_h U^*_{gh})$$

$$= \sigma(U_g)\ \sigma\ (U_h)\ \overline{\sigma\ (U_{gh})}$$

$$= 1 .$$

Thus $$U_g U_h = U_{gh} \qquad (g,h \in G) .$$

This shows that $$U : g \to U_g$$

is a weakly measurable unitary representation of G on $\mathcal{H}$ by unitaries $U_g \in \mathcal{U}(\mathcal{R})$ such that

$$\alpha(g) = \text{ad } U_g$$

By [18], Theorem 22.20(b), p.347, U is a strongly continuous unitary representation, since $\mathcal{H}$ is separable. This completes the proof.

We finish with two counterexamples to possible extensions of Corollary 6.12 and Theorem 6.15.

Example 1 Let $\mathcal{H}$ be an infinite dimensional Hilbert space (so that $\mathcal{B}(\mathcal{H})$ is not norm separable). Let $\{Q_n\}_{n=1}^{\infty}$ be an orthogonal sequence of projections on $\mathcal{H}$ with $\Sigma Q_n = I$. Define a 1-parameter unitary group (i.e. a unitary representation of R ) on $\mathcal{H}$ by

$$U_t(\Sigma Q_n x) = \Sigma e^{int} Q_n x \qquad (x \in \mathcal{H}, \quad t \in R)$$

and let $$\alpha(t) = \text{ad } U_t .$$

Clearly $t \to U_t$ is a unitary representation of R on $\mathcal{H}$, and if

$x \in \mathcal{H}$, $t_n \to t$, then

$$\langle U_{t_n} x, x \rangle = \sum_k \langle U_{t_n} Q_k x, Q_k x \rangle$$

$$= \Sigma \langle e^{ikt_n} Q_k x, Q_k x \rangle$$

$$\to \langle U_t x, x \rangle$$

so $t \to U_t$ is a strongly continuous representation. If $A \in \mathcal{B}(\mathcal{H})$, and $t_n \to t$ in $R$, then

$$\langle \alpha(t_n)(A) x, x \rangle = \langle U_{t_n} A U_{t_n}^{-1} x, x \rangle$$

$$= \langle A U_{t_n^{-1}} x, U_{t_n^{-1}} x \rangle$$

$$\to \langle \alpha(t)(A) x, x \rangle .$$

Thus $\alpha : t \to \alpha(t)$ is a weakly continuous representation of $R$ on $\mathcal{B}(\mathcal{H})$ by inner automorphisms.

For each $n$, choose projections $E_n \leq Q_n$, $F_n \leq Q_{2n}$ with $E_n \sim F_n$. Let $V_n$ be a partial isometry taking $E_n$ to $F_n$, then

$$A = \sum_n V_n$$

exists in the strong operator topology and defines an element of $\mathcal{B}(\mathcal{H})$.

Now
$$U_t V_n U_t^* = U_t F_n V_n E_n U_t^*$$
$$= U_t F_n V_n (U_t E_n)^*$$
$$= e^{2int} F_n V_n E_n e^{-int}$$
$$= e^{int} V_n .$$

So
$$U_t A U_t^* = \sum_n e^{int} V_n$$

and
$$\|U_t A U_t^* - A\| = \sup \{ \Sigma \| e^{int} V_n x - V_n x \| ; \ \|x\| = 1 \}$$

If $2\pi t$ is irrational,

$$\{e^{int} ; n \in \mathbb{Z}\}$$

is dense in $\mathbb{T}$, so there is an $n_o$ with

$$|e^{in_o t} - 1| \geq 1 .$$

Let $x \in E_o\mathcal{H}$, $\|x\| = 1$, then

$$\|U_t A U_t^* - A\| \geq |e^{in_o t} - 1| \|x\| \geq 1$$

$$\|\alpha(t)(A) - A\| \not\to 0 \quad \text{as} \quad t \to 0 .$$

Example 2 This example is well known in Quantum Field Theory as the canonical representation of the Heisenberg Commutation relation.

Let $R^2$ have planar Lebesgue measure. For $f \in L^2(R^2)$ ; $r, s, t \in R$, define

$$U_t f(r, s) = f(r, s-t)$$

$$V_t f(r, s) = e^{ist} f(r-t, s)$$

Then clearly $U: t \to U_t$ and $V: t \to V_t$ are strongly continuous unitary representations of $R$ on $L^2(R^2)$ and

$$\begin{aligned} U_t V_{-t} V_{-s} f(p,q) &= V_s U_{-t} V_{-s} f(p, q-t) \\ &= U_{-t} V_{-s} e^{is(q-t)} f(p-s, q-t) \\ &= e^{is(q-t)} V_{-s} f(p-s, q) \\ &= e^{-ist} f(p, q) . \end{aligned}$$

Thus $$U_t V_s = e^{-ist} V_s U_t \quad (s, t \in R) \qquad (**)$$

Consider $\alpha : (s, t) \to \text{ad}\, U_s V_t$

Then by (**), $\text{ad}\, U_s V_t = \text{ad}\, V_t U_s$ ,

so
$$\alpha(s_1, t_1)\, \alpha(s_2, t_2) = \text{ad}\, U_{s_1} V_{t_1} U_{s_2} V_{t_2}$$
$$= \text{ad}\, U_{s_1} U_{s_2} V_{t_1} V_{t_2}$$
$$= \text{ad}\, U_{s_1 s_2} V_{t_1 t_2}$$
$$= \alpha(s_1 s_2, t_1 t_2) .$$

Thus $\alpha$ is a representation of $R^2$ on $\mathcal{B}(L^2(R^2))$. Since U, V are strongly continuous representations, $\alpha$ is also a strongly continuous representation.

Now $\alpha(s, t) = \text{ad}\, U_s V_t$

and $\alpha(t, s) = U_t V_s$ .

I claim that these unitaries do not commute if $s \neq t$, since

$$(U_s V_t)(U_t V_s) = e^{it^2} U_s U_t V_t V_s$$
$$= e^{it^2} U_{s+t} V_{s+t}$$

and

$$(U_t V_s)(U_s V_t) = e^{is^2} U_{s+t} V_{s+t} .$$

Thus $\alpha$ is a strongly continuous representation of the locally compact separable abelian group $R^2$ on $\mathcal{B}(L^2(R^2))$ by unitaries which do not commute. Hence there is no unitary representation of $R^2$ on $L^2(R^2)$. which induces $\alpha$.

# CHAPTER VII

## ERGODIC THEORY AND VON NEUMANN ALGEBRAS

Ergodic theory is the study of groups of transformations of a measure space. We shall show below how this links up with the study of groups of automorphisms of operator algebras. We shall also show how this link up motivates certain definitions and results about groups of automorphisms. Our starting point is the restatement of two well known theorems about abelian operator algebras from the introduction.

<u>7.1 Theorem</u> ([7], Ch.1, §7, p.118, Theorem 1). Let $\mathcal{Z}$ be an abelian von Neumann algebra acting on a Hilbert space $\mathcal{H}$, then there is a locally compact space Z, a positive regular borel measure $\nu$ on Z, with support equal to Z, and a *-isomorphism from $\mathcal{Z}$ onto the algebra $L^\infty(Z,\nu)$ of essentially bounded $\nu$-measurable complex valued functions on Z. If $\mathcal{H}$ is separable, then Z may be chosen to be compact and second countable.

<u>7.2 Theorem</u> ([8], 1.4.1 Theorem, p.9). Let $\mathfrak{A}$ be an abelian C*-algebra, $\Phi_{\mathfrak{A}}$ the carrier space of $\mathfrak{A}$ with the weak* topology. Then $\Phi_{\mathfrak{A}}$ is locally compact and $\mathfrak{A}$ is * isomorphic to the algebra $C_o(\Phi_{\mathfrak{A}})$ of all complex valued continuous functions on $\Phi_{\mathfrak{A}}$, vanishing at infinity.

Suppose $\mathfrak{A}$ is an abelian C*-algebra and G is a group of automorphisms of $\mathfrak{A}$. Let $\rho \in \Phi_{\mathfrak{A}}$, and define $g^*(\rho)$ by

$$g^*(\rho)(A) = \rho(g(A)) \qquad (A \in \mathfrak{A}) .$$

Then

$$g^*: \Phi_{\mathfrak{A}} \to \Phi_{\mathfrak{A}}$$

is a homeomorphism and

$$G^* = \{g^*; \quad g \in G\}$$

is a group of homeomorphisms of $\Phi_{\mathfrak{A}}$.

If $\mathfrak{R}$ is an abelian von Neumann algebra, and G is a group of automorphisms of $\mathfrak{R}$, then by Theorem 7.1, we can regard G as a group of automorphisms of $L^\infty(Z,\nu)$ for some locally compact space Z. Let E be a borel set in Z, then $\chi_E$, the characteristic function of E, is a projection in $\mathfrak{R}$, thus if $g \in G$, $g(\chi_E)$ is also a projection in $\mathfrak{R}$, so there is a borel set F with

$$g(\chi_E) = \chi_F$$

Let

$$g^*: E \to F$$

Then

$$G^* = \{g^*; \quad g \in G\}$$

is a group of transformations of the $\sigma$-ring of borel sets in Z.

The above remarks show that a group of automorphisms of an abelian C*-algebra, or an abelian von Neumann algebra, may be regarded as a group of transformations acting on the $\sigma$-ring of borel sets of some locally compact space. The study of such groups is part of Ergodic theory (see [16], Introduction p.2, ℓℓ.9-25 and Example, p.5). As remarked in [16] (p.61, ℓℓ. 9-14), the fact that two measurable transformations of a measure space X are identified if they differ only on null sets means that for most purposes, we need only consider groups of transformations of the $\sigma$-ring of measurable subsets of X.

In this chapter we shall generalise some concepts and results in Ergodic theory to the case of a group of automorphisms of an operator algebra. To begin with, we shall give some ergodic theoretic definitions and their corresponding generalisations to the operator algebra situation.

Let $(X, \mathcal{B}, \mu)$ be a measure space, where $\mathcal{B}$ is the $\sigma$-ring of $\mu$-measurable subsets of X. A map $T: X \to X$ is a measurable transformation if $E \in \mathcal{B}$ implies $T^{-1}(E) \in \mathcal{B}$ where

$$T^{-1}(E) = \{x; Tx \in E\}$$

T is measure preserving if T is measurable and

$$\mu(T^{-1}(E)) = \mu(E) \quad \text{for all } E \in \mathcal{B}.$$

It is also said in this case that $\mu$ is invariant under T. (See [16], p.6, ℓℓ.1-3). If T is a measurable transformation of X, then T is said to be ergodic if $E \in \mathcal{B}$ and $T^{-1}(E) \setminus E$ is a null set imply that either

$$\mu(E) = 0 \quad \text{or} \quad \mu(X \setminus E) = 0 .$$

If $\mu$ is also invariant under T, $\mu$ is said to be an ergodic measure (relative to T). (See [16], p.25, ℓ.14).

7.3 Definition Let $\mathcal{R}$ be a von Neumann algebra, G a group of automorphisms of $\mathcal{R}$. G acts ergodically on $\mathcal{R}$ if given a projection E in $\mathcal{R}$,

$$g(E) = E \quad \text{for all} \quad g \in G$$

implies either

$$E = 0 \quad \text{or} \quad E = I.$$

In the abelian case, we may regard $\mathcal{R}$ as equal to $L^{\infty}(X, \mu)$ for some locally compact space X, by Theorem 7.1. Let G* be the corresponding group of transformations of $\mathcal{B}$, the $\sigma$-ring of borel subsets of X. If $E \in \mathcal{B}$ and $g^*(E) = E$ for all $g^* \in G^*$, then $\chi_E$ is a projection in $\mathcal{R}$, invariant under G. Hence if G acts ergodically in the sense of Definition 7.3, then either $\chi_E = 0$ or $\chi_E = I$. Thus either $\mu(E) = 0$ or $\mu(X \setminus E) = 0$. This shows that Definition 7.3 is a reasonable generalisation of ergodicity.

<u>7.4 Proposition</u> Let $\mathcal{R}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$; G a group of automorphisms of $\mathcal{R}$. Then G acts ergodically on $\mathcal{R}$ if and only if the following condition is satisfied:

If

$$A \in \mathcal{R}, \quad g(A) = A \quad \text{for all } g \in G$$

then

$$A = \lambda I \quad \text{for some } \lambda \in \mathbb{C} .$$

<u>Proof</u> Suppose condition (1) is satisfied, and let E be a G-invariant projection in $\mathcal{R}$. i.e.

$$g(E) = E \quad (g \in G).$$

Then $E = \lambda I$ for some $\lambda \in \mathbb{C}$. Now

$$\sigma(E) = \{0, 1\}$$

thus $\lambda = 0$ or 1. Hence $E = 0$ or I and G acts ergodically.

Conversely, if G acts ergodically on $\mathcal{R}$, let $A \in \mathcal{R}$ be G-invariant, then $A^*$ is also G-invariant thus if $A = A_1 + iA_2$ is the decomposition of A into real and imaginary parts, then both $A_1$ and $A_2$ are G-invariant. We may thus assume A is self-adjoint. If E is a spectral projection of A, then E is the limit in the ultraweak topology of a sequence $\{P_n\}$ of polynomials in A. Each such polynomial is a G-invariant element of $\mathfrak{R}$. If $g \in G$, then g is ultraweakly continuous, thus $g(E) = \lim g(P_n) = \lim P_n = E$. Hence E is G-invariant. Since G acts ergodically on $\mathfrak{R}$, either $E = 0$ or $E = I$. By spectral theory, A is the norm limit of linear combinations of such spectral projections. Each such linear combination is of the form $\theta I$ for some scalar $\theta$, so $A = \lambda I$ for some $\lambda \in \mathbb{C}$, giving the result.

In view of Proposition 7.4, the following definition is consistent with Definition 7.3.

<u>7.5 Definition</u> Let $\mathfrak{A}$ be a C*-algebra with identity element I, G a group of automorphisms of $\mathfrak{A}$. G <u>acts ergodically</u> on $\mathfrak{A}$ if

$$A \in \mathfrak{A} \text{ and } g(A) = A \qquad (g \in G)$$

imply

$$A = \lambda I \text{ for some scalar } \lambda.$$

If $\mathfrak{A}$ is an abelian C*-algebra, with identity, then by Theorem 7.2, $\mathfrak{A}$ may be identified with C(X) for some compact Hausdorff space X. By the ~~Reitz~~ Riesz representation theorem ([19], Theorem 12.36, p.177), the positive linear functionals of $\mathfrak{A}$ may be regarded as the positive regular borel measures on X. In particular the states of $\mathfrak{A}$ are simply the probability measures on X (i.e. those positive regular borel measures $\mu$ such that $\mu(X) = 1$). If T is a homeomorphism from X onto X, the following result is well known. For completeness we include a proof.

<u>7.6 Proposition</u> Let $\mu$ be a probability measure on X, T a

homeomorphism from X to X. Then $\mu$ is an ergodic measure relative to T if and only if $\mu$ is an extreme point of the set of those probability measures on X which are invariant under T.

Proof  Let $\mathcal{E}$ denote the set of probability measures on X invariant under T. Suppose $\mu$ is an extreme point of $\mathcal{E}$, and let E be a borel subset of X invariant under T. We have

$$0 \leq \mu(E) \leq 1$$

If

$$\mu(E) = \lambda \ , \quad 0 < \lambda < 1 \ ,$$

define borel measures $\mu_1$ and $\mu_2$ by

$$\mu_1(A) = \frac{1}{\lambda} \ \mu(A \cap E)$$

$$\mu_2(A) = \frac{1}{1 - \lambda} \ \mu(A \cap (X \setminus E)) \ .$$

Then $\mu_1$ and $\mu_2$ are probability measures on X, invariant under T and $\mu = \lambda\mu_1 + (1 - \lambda)\mu_2$. Since $\mu$ is extremal, this is a contradiction. Thus $\lambda = 0$ or 1, showing that $\mu$ is ergodic. Conversely, if $\mu$ is an ergodic probability measure on X, suppose

$$\mu = \lambda\mu_1 + (1-\lambda)\mu_2 \quad \text{with } \mu_1, \mu_2 \in \mathcal{E}, \quad 0 < \lambda < 1 \ .$$

Let

$$\nu = \lambda\mu_1 \ ,$$

then $\nu \leq \mu$, thus by the Radon-Nikodym theorem ([19] Theorem 19.23, p.315), there is an $f \in L^1(X, \mu)$ such that

$$\nu(A) = \int_A f d\mu \quad \text{for each borel set } A$$

Denote by $f \circ T$ the map $x \to f(T(x))$. If A is a borel set

$$\int_A f d\mu = \nu(A)$$

$$= \nu(T(A))$$

$$= \int_{T(A)} f d\mu \quad .$$

By [17] (Theorem C, p.163),

$$\int_X \chi_{T(A)} f d\mu = \int_X (\chi_{T(A)} f) \circ T \, d\mu T^{-1}$$

where $\mu T^{-1}$ denotes the measure

$$A \to \mu(T^{-1}(A))$$

Since $\mu$ is invariant,

$$\mu T^{-1} = \mu$$

Hence

$$\int_A f d\mu = \int_X (\chi_{T(A)} \circ T)(f \circ T) d\mu$$

$$= \int_X \chi_A (f \circ T) d\mu$$

$$= \int_A f \circ T \, d\mu$$

This holds for all borel sets A, so by [17], Theorem E, p.105,

$$f = f \circ T \qquad \text{a.e.}(\mu)$$

If A is a borel set,

$$\nu(A) = \int_A f d\mu$$

$$= \int_A \mathrm{Re}\, f \, d\mu + i \int_A \mathfrak{Im}\, f \, d\mu \; .$$

Since $\nu$, $\mu$ are positive measures,

$$\int_A \mathfrak{Im}\, f \, d\mu = 0$$

Thus by [17] (Theorem E, p.105) again , $\mathcal{J}mf = 0$. A similar argument shows that f must take positive values a.e.($\mu$). Let

$$E = \{x \in X;\ f(x) > 1\} .$$

Then E is a borel set, and

$$\nu(E) = \int_E f d\mu > \mu(E) \qquad \text{unless } \mu(E) = 0$$

But $\nu \leq \mu$, thus $\mu(E) = 0$. Hence f is essentially bounded, and $0 \leq f(x) \leq 1$ a.e.($\mu$) on X. Let

$$\begin{aligned} M &= \text{ess. sup. } f \\ &= \inf \{\alpha;\ f(x) \leq \alpha \text{ a.e.}(\mu) \text{ on } X\} \\ m &= \text{ess. inf. } f \\ &= \sup \{\alpha;\ f(x) \geq \alpha \text{ a.e. } (\mu) \text{ on } X\}. \end{aligned}$$

If $m \neq M$, let

$$m < t < M .$$

Then

$$S_1 = \{x \in X;\ f(x) < t\}$$

and

$$S_2 = \{x \in X;\ f(x) > t\}$$

are both borel subsets of X invariant under T, and both $S_1$ and $S_2$ must have positive measure by definition of m and M. Since $\mu$ is an ergodic measure, this implies that

$$\mu(S_1) = \mu(S_2) = 1 .$$

But if $\mu(S_1) = 1$ then $\mu(S_2) = 0$. This contradiction shows that $m = M$, and $f(x) = m$ a.e.($\mu$) on X. Thus

$$\lambda \mu_1 = \nu = m . \mu .$$

But

$$\lambda = \lambda \mu_1(X)$$
$$= m . \mu(X)$$
$$= m$$

so

$$\mu_1 = \mu$$

A similar argument shows that

$$\mu_2 = \mu .$$

Hence $\mu$ is an extreme point of $\mathcal{E}$. This finishes the proof.

The above proposition motivates the following definition:

7.7 Definition Let G be a group of automorphisms of a C*-algebra $\mathfrak{A}$. If f is a state of $\mathfrak{A}$, f is G-invariant if $f(A) = f(g(A))$ for all $A \in \mathfrak{A}$, $g \in G$. f is said to be ergodic if f is an extreme point of the G-invariant states of $\mathfrak{A}$.

7.7 Remark If there is a G-invariant state on $\mathfrak{A}$, then there is an ergodic state on $\mathfrak{A}$, for the set of G-invariant states is a non-void, weak*-closed (hence weak*-compact) convex subset of $E(\mathfrak{A})$, so has an extreme point by the Krein-Mil'man theorem.

The definition of an ergodic state was first given by I.E. Segal in [48].

If G is a group of automorphisms of the C*-algebra $\mathfrak{A}$, and E is a G-invariant projection in $\mathfrak{A}$, the map

$$EAE \rightarrow Eg(A)E$$

is an automorphism of $E\mathfrak{A}E$, since

$$Eg(A)E = g(EAE) .$$

We can consider G in this way as a group of automorphisms of $E\mathfrak{A}E$, and denote this group by $G_E$. If $\pi$ is a faithful representation of $\mathfrak{A}$, then the map

$$\pi g \pi^{-1}: \pi(A) \to \pi(g(A)) \qquad (A \in \mathfrak{A}) ,$$

an automorphism of $\pi(\mathfrak{A})$. We shall denote the group of such automorphisms by $\pi G \pi^{-1}$.

We are now in a position to obtain our first theorem.

<u>7.9 Theorem</u> Let G be a group of automorphisms of the von Neumann algebra $\mathfrak{R}$, f a G-invariant normal state on $\mathfrak{R}$ with support $E_f$. Then $E_f$ is a G-invariant projection in $\mathfrak{R}$, and f is an ergodic state if and only if $G_{E_f}$ acts ergodically on $E_f\,\mathfrak{R}E_f$.

<u>Proof</u> $E_f$ is the unique smallest projection F in $\mathfrak{R}$ such that $f(F) = 1$. Since

$$f(g(E_f)) = f(E_f) \qquad (g \in G) ,$$

we have

$$g(E_f) = E_f \qquad (g \in G) ,$$

so $E_f$ is G-invariant. Let $\pi = \pi_f$ be the representation corresponding to f on the Hilbert space $\mathcal{H}_f$. Suppose first that f is faithful i.e.

$$E_f = I .$$

Then the von Neumann algebra $\pi(\mathfrak{R})$ has a separating-generating vector $x_o$, and $\pi$ is a *-isomorphism.

For $A \in \pi(\mathfrak{R})$, let S denote the map

$$Ax_o \rightarrow A^* x_o$$

Let $\{U_g;\ g \in G\}$ be the Segal unitaries for $\pi$.

By the Tomita-Takesaki theory, S has a minimal closed linear extension, also denoted S, and if

$$S = J\Delta^{\frac{1}{2}}$$

is the polar decomposition of S, then

$$J\pi(\mathfrak{R})J = \pi(\mathfrak{R})' .$$

Now if $B \in \mathfrak{R}$,

$$U_g S\pi(B)x_o = U_g \pi(B^*)x_o$$

$$= \pi(g(B^*))x_o$$

$$= \pi(g(B))^* x_o$$

$$= S\pi(g(B))x_o$$

$$= SU_g \pi(B)x_o .$$

If x lies in the domain of S, then

$$x \in \mathcal{H}_f = \overline{\pi(\mathfrak{R})x_o} ,$$

so there is a sequence $\{B_n\}$ in $\pi(\mathfrak{R})$ with

$$B_n x_o \rightarrow x , \quad B_n^* x_o \rightarrow Sx .$$

Thus

$$SU_g B_n x_o = U_g SB_n x_o \rightarrow U_g Sx .$$

Also

$$U_g B_n x_o \rightarrow U_g x ,$$

so $U_g x$ lies in the domain of S, and

$$S U_g x = U_g S x$$

since S has closed graph. This shows that

$$S U_g = U_g S \qquad (g \in G)$$

Now

$$S = J\Delta^{\frac{1}{2}} ,$$

so

$$S = U_g S U_g^*$$

$$= U_g J U_g^* U_g \Delta^{\frac{1}{2}} U_g^* .$$

By uniqueness of the polar decomposition,

$$U_g J = J U_g \qquad (g \in G) .$$

Suppose

$$A \in \pi(\mathfrak{R}) \cap \{U_g ; g \in G\}',$$

then

$$J A J \in \pi(\mathfrak{R})' ,$$

and

$$J A J U_g = J A U_g J$$

$$= U_g J A J ,$$

thus

$$J A J \in \pi(\mathfrak{R})' \cap \{U_g ; g \in G\}' .$$

Similarly if

$$B \in \pi(\mathfrak{R})' \cap \{U_g ; g \in G\}' ,$$

then

$$B = J A J$$

for some $A \in \pi(\mathfrak{R})$, and

$$J A J U_g = U_g J A J$$

implies that

$$J A U_g J = J U_g A J ,$$

hence

$$A U_g = U_g A$$

since J is a conjugate unitary. This proves that

$$J(\pi(\mathfrak{R}) \cap \{U_g; g \in G\}')J = \pi(\mathfrak{R})' \cap \{U_g; g \in G\}'.$$

We assert that G acts ergodically on $\mathfrak{R}$ if and only if

$$\pi(\mathfrak{R}) \cap \{U_g; g \in G\}' = \mathbb{C} I_{\pi(\mathfrak{R})} . \qquad (*)$$

For suppose (*) holds, and $A \in \mathfrak{R}$,

$$g(A) = A \qquad (g \in G) .$$

Then

$$U_g \pi(A) U_g^* = \pi(g(A)) = \pi(A) \quad (g \in G)$$

thus

$$\pi(A) = \lambda I_{\pi(\mathfrak{R})}$$

for some $\lambda \in \mathbb{C}$. Since $\pi$ is faithful, we have

$$A = \lambda I_{\mathfrak{R}} .$$

and G acts ergodically. The converse argument is similar.

Hence G acts ergodically on $\mathfrak{R}$

$$\Leftrightarrow \pi(\mathfrak{R}) \cap \{U_g; g \in G\}' = \mathbb{C} I_{\pi(\mathfrak{R})}$$

$$\Leftrightarrow J(\pi(\mathfrak{R}) \cap \{U_g; g \in G\}') J = \mathbb{C} I_{\pi(\mathfrak{R})'}$$

$$\Leftrightarrow \pi(\mathfrak{R})' \cap \{U_g;\ g \in G\}' = \mathbb{C}I_{\pi(\mathfrak{R})'}$$

$$\Leftrightarrow f \text{ is ergodic by [45], Theorem 6.3.3.}$$

Suppose now that f is not necessarily faithful, then $f|_{E_f\mathfrak{R}E_f}$ is faithful, so by the above, $G_{E_f}$ acts ergodically on $E_f\mathfrak{R}E_f$ if and only if $f|_{E_f\mathfrak{R}E_f}$ is an extreme point of the $G_{E_f}$-invariant states of $E_f\mathfrak{R}E_f$. The proof is finished by the following proposition.

<u>7.10 Proposition</u> Let G be a group of automorphisms of the von Neumann algebra $\mathfrak{R}$, f a normal G-invariant state on $\mathfrak{R}$, then f is ergodic if and only if $f|_{E_f\mathfrak{R}E_f}$ is an extreme point of the $G_{E_f}$ - invariant states of $E_f\mathfrak{R}E_f$.

<u>Proof</u> ($\Rightarrow$) clear

($\Leftarrow$) Let

$$f = \lambda g_1 + (1-\lambda) g_2$$

with $g_1$ and $g_2$ G-invariant states of $\mathfrak{R}$. Then

$$f - \lambda g_1 = (1-\lambda) g_2 \geq 0 ,$$

thus

$$\frac{1}{\lambda} f \geq g_1 \geq 0$$

Hence if $(E_\alpha)_{\alpha \in A}$ is an orthogonal family of projections in $\mathfrak{R}$, and

$$E = \Sigma E_\alpha ,$$

then for any finite subset J of A,

$$0 \le g_1(E - \sum_{\alpha \in J} E_\alpha)$$

$$\le \frac{1}{\lambda} f(E - \sum_{\alpha \in J} E_\alpha)$$

and the right hand side goes to 0 as J runs through the directed set of finite subsets of A. Thus $g_1$ is completely additive and hence ultraweakly continuous. Now

$$0 \le g_1(I - E_f)$$

$$\le \frac{1}{\lambda} f(I - E_f) = 0 ,$$

so

$$g_1(E_f) = 1 ,$$

and

$$E_{g_1} \le E_f .$$

Since f is an extreme point of the $G_{E_f}$-invariant states of $E_f \mathfrak{R} E_f$,

$$f|_{E_f \mathfrak{R} E_f} = g_1|_{E_f \mathfrak{R} E_f} .$$

If $A \in \mathfrak{R}$, then

$$f(A) = f(E_f A E_f)$$

$$= g_1(E_f A E_f)$$

$$= g_1(E_{g_1} E_f A E_f E_{g_1})$$

$$= g_1(E_{g_1} A E_{g_1})$$

$$= g_1(A) ,$$

so

$$f = g_1 .$$

Similarly

$$g_2 = f ,$$

completing the proof, and thus completing also the proof of Theorem 7.9.

Suppose now that $\mathfrak{A}$ is a C*-algebra acting in its universal representation, f is a state of $\mathfrak{A}$, and G is a group of automorphisms of $\mathfrak{A}$. Denote by $\tilde{f}$ the extension of f to a normal state of $\mathfrak{A}^-$, and by $\tilde{G}$ the extension of G to a group of automorphisms of $\mathfrak{A}^-$.

<u>7.11 Lemma</u> With notation as above, f is an ergodic state of $\mathfrak{A}$ (relative to G) if and only if $\tilde{f}$ is an ergodic state of $\mathfrak{A}^-$ (relative to $\tilde{G}$).

<u>Proof</u> (⇒) If f is G-invariant, clearly $\tilde{f}$ is $\tilde{G}$-invariant. Suppose f is an ergodic state of $\mathfrak{A}$ and

$$\tilde{f} = \lambda g_1 + (1-\lambda)g_2 , \quad 0 < \lambda < 1 ,$$

with $g_1$, $g_2$ $\tilde{G}$-invariant states of $\mathfrak{A}^-$. As in Proposition 7.10, $g_1$ and $g_2$ are normal states. Now

$$f = \tilde{f}|_{\mathfrak{A}}$$

$$= \lambda g_1|_{\mathfrak{A}} + (1-\lambda)g_2|_{\mathfrak{A}} .$$

$g_1|_{\mathfrak{A}}$ and $g_2|_{\mathfrak{A}}$ are G-invariant states of $\mathfrak{A}$ thus

$$f = g_1|_{\mathfrak{A}} = g_2|_{\mathfrak{A}} .$$

Hence

$$\tilde{f} = g_1 = g_2 ,$$

so $\tilde{f}$ is ergodic.

($\Leftarrow$) If $\tilde{f}$ is an ergodic state of $\mathfrak{A}^-$, and

$$f = \lambda g_1 + (1-\lambda) g_2$$

with $g_1$, $g_2$ both G-invariant states of $\mathfrak{A}$, then $\tilde{g}_1$, $\tilde{g}_2$ are $\tilde{G}$-invariant states of $\mathfrak{A}^-$, and

$$\tilde{f} = \lambda\tilde{g}_1 + (1-\lambda)\tilde{g}_2 ,$$

so

$$\tilde{f} = \tilde{g}_1 = \tilde{g}_2 .$$

Thus

$$g_1 = g_2 .$$

showing that f is ergodic.

<u>7.12 Corollary</u> If an amenable group G has a representation as a group of automorphisms of a C*-algebra $\mathfrak{A}$, then it has a representation as an ergodic group of automorphisms of some von Neumann subalgebra of $\pi(\mathfrak{A})^-$ where $\pi$ is the universal representation of $\mathfrak{A}$.

<u>Proof</u> Let $\alpha$ be a representation of G on $\mathfrak{A}$, then $\pi\alpha(G)\pi^{-1}$ is a group of automorphisms of $\pi(\mathfrak{A})$. Let

$$H = \widetilde{\pi\alpha(G)\pi^{-1}} ,$$

(the extension of $\pi\alpha(G)\pi^{-1}$ to a group of automorphisms of $\pi(\mathfrak{A})^-$). Since G is amenable, G has the weak fixed point property (W.F.P.) (see Appendix B). If f is a state of $\pi(\mathfrak{A})$ let

$$\mathcal{E} = \overline{co}^{w^*}\{f \circ \beta(g); \quad g \in G\}$$

where

$$\beta(g) = \pi\alpha(g)\pi^{-1}$$

and w* denotes closure in the weak* topology. $\mathcal{E}$ is a weak*-closed convex subset of $E(\pi(\mathfrak{A}))$. By the W.F.P. property, there is an $h \in \mathcal{E}$ with

$$h \circ \beta(g) = h \qquad (g \in G),$$

so h is a $\pi\alpha(G)\pi^{-1}$-invariant state on $\pi(\mathfrak{A})$. By Remark 7.8, there is an ergodic state k on $\pi(\mathfrak{A})$, so $\tilde{k}$ is an ergodic state on $\pi(\mathfrak{A})^-$ (relative to H) by Lemma 7.11. Thus by Theorem 7.9, the group $H_{E_{\tilde{k}}}$ acts ergodically on $E_{\tilde{k}}\pi(\mathfrak{A})^- E_{\tilde{k}}$. Let

$$\tilde{\alpha}(g) = \pi\alpha(g)\pi^{-1}$$

and

$$\gamma(g)(E_{\tilde{k}} A E_{\tilde{k}}) = E_{\tilde{k}}\tilde{\alpha}(g)(A)E_{\tilde{k}}$$

for $A \in \pi(\mathfrak{A})^-$, then $\gamma$ is a representation of G as an ergodic group of automorphisms of the von Neumann algebra $E_{\tilde{k}}\pi(\mathfrak{A})^- E_{\tilde{k}}$. This completes the proof.

7.13 Corollary  Let f be a normal state on a von Neumann algebra $\mathfrak{R}$, then f is a pure state if and only if $E_f$ is a minimal projection in $\mathfrak{R}$

Proof  Take $G = \{\iota\}$, where $\iota$ is the identity automorphism of $\mathfrak{R}$. f is a pure state of $\mathfrak{R}$ $\Leftrightarrow$ f is ergodic relative to G, since every state on $\mathfrak{R}$ is G-invariant. Now f is ergodic $\Leftrightarrow$ $G_{E_f}$ acts ergodically on $E_f\mathfrak{R}E_f$ by Theorem 7.9. But $G_{E_f}$ is simply the identity automorphism of $E_f\mathfrak{R}E_f$, and hence acts ergodically if and only if $E_f\mathfrak{R}E_f = \mathbb{C}E_f$ i.e. $E_f$ is a minimal projection in $\mathfrak{R}$. This completes the proof.

If $\mathfrak{R}$ is a von Neumann algebra, a tracial state f of $\mathfrak{R}$ is a state on $\mathfrak{R}$ which is invariant under inn($\mathfrak{R}$), the group of all inner automorphisms of $\mathfrak{R}$. Since the unitaries in $\mathfrak{R}$ span $\mathfrak{R}$ algebraically, this is equivalent to $f(AB) = f(BA)$ for all $A, B \in \mathfrak{R}$.

<u>7.14 Remark</u> $\mathfrak{R}$ is a factor if and only if inn($\mathfrak{R}$) acts ergodically on $\mathfrak{R}$. To see this, let inn($\mathfrak{R}$) act ergodically on $\mathfrak{R}$ and let $A \in \mathcal{Z}$, the centre of $\mathfrak{R}$. Then A is invariant under inn($\mathfrak{R}$), so $A = \lambda I$ for some scalar $\lambda$, showing that $\mathfrak{R}$ is a factor. Conversely, if $A \in \mathfrak{R}$ and $UAU^* = A$ for all unitaries $U \in \mathfrak{R}$, then $BA = AB$ for all $B \in \mathfrak{R}$, since the unitaries in $\mathfrak{R}$ span $\mathfrak{R}$. Thus $A \in \mathcal{Z} = \mathbb{C}I$, so $A = \lambda I$ for some scalar $\lambda$, showing that inn($\mathfrak{R}$) acts ergodically on $\mathfrak{R}$.

<u>7.15 Corollary</u> Let f be a normal tracial state on a von Neumann algebra $\mathfrak{R}$, then f is an extreme point of the tracial states if and only if $E_f \mathfrak{R} E_f$ is a factor.

<u>Proof</u> This is an immediate consequence of Theorem 7.9 and Remark 7.14.

<u>7.16 Corollary</u> Let $\mathfrak{R}$ be a factor. If $\mathfrak{R}$ possesses a normal tracial state f, then f is unique.

<u>Proof</u> If f is a normal tracial state of $\mathfrak{R}$, then $E_f \mathfrak{R} E_f$ is a factor (since $\mathfrak{R}$ is) and so f is an extreme point of the tracial states by Corollary 7.15. Suppose $f_1$ and $f_2$ are normal tracial states on $\mathfrak{R}$, then so also is $\frac{1}{2}(f_1 + f_2)$, which is not an extreme

point. This shows that $f_1 = f_2$ must be unique.

7.17 Remark Corollary 7.13 and Corollary 7.16 are known results in Operator Algebra theory. The proofs we have given are, however, new.

7.18 Definition Let G be a group of automorphisms of a von Neumann algebra $\mathfrak{R}$. $\mathfrak{R}$ is G-finite if the G-invariant normal states of $\mathfrak{R}$ separate the points of $\mathfrak{R}^+$ i.e. if $A \in \mathfrak{R}^+$ and $f(A) = 0$ for all G-invariant normal states f of $\mathfrak{R}$, then $A = 0$.

Let $\mathfrak{R}^G = \{A \in \mathfrak{R};\ g(A) = A \quad (g \in G)\}$. Since each $g \in G$ is ultraweakly continuous, $\mathfrak{R}^G$ is a von Neumann subalgebra of $\mathfrak{R}$. If $A \in \mathfrak{R}$, let

$$\mathcal{K}(A, G) = \overline{co}^{\sigma w}\{g(A);\quad g \in G\}$$

where $\sigma w$ denotes closure in the ultraweak operator topology. We need the following result, obtained by I. Kovacs and J. Szűcs ([28], Theorems 1 and 2).

7.19 Theorem Let $\mathfrak{R}$ be a von Neumann algebra and G a group of automorphisms of $\mathfrak{R}$. Suppose $\mathfrak{R}$ is G-finite, then for each $T \in \mathfrak{R}$,

$\mathcal{K}(T,G) \cap \mathfrak{R}^G$ is a single element, denoted $T^G$.

The G-invariant map

$$T \to T^G$$

is a faithful normal G-invariant projection of norm one from $\mathfrak{R}$ onto $\mathfrak{R}^G$.

We shall give a proof of this result due essentially to

E. Størmer. Before doing this we give some preliminary lemmas.

7.20 Lemma It suffices to prove Theorem 7.19 under the assumption that $\mathfrak{R}$ acts on a Hilbert space $\mathcal{H}$, there is a unitary representation $g \to U_g$ of G by unitaries $U_g$ on $\mathcal{H}$ such that $U_g A U_g^* = g(A)$ $(A \in \mathfrak{R},\ g \in G)$, and we may also assume that every normal G-invariant state of $\mathfrak{R}$ is of the form $\omega_x$ for some $x \in \mathcal{H}$ such that $U_g x = x$ $(g \in G)$.

Proof Let $\pi$ be the representation $\Sigma^{\oplus}\{\pi_f;\ f$ is a normal G-invariant state of $\mathfrak{R}\}$. Since $\mathfrak{R}$ is G-finite, $\pi$ is a faithful representation of $\mathfrak{R}$, and also $\pi(\mathfrak{R})$ is a von Neumann algebra. For each G-invariant normal state f of $\mathfrak{R}$, let $(U_g^f)_{g \in G}$ be the Segal unitaries associated with f (as in Chapter V). If

$$x = \Sigma^{\oplus} x_f \in \mathcal{H}_\pi ,$$

define

$$U_g x = \Sigma^{\oplus} U_g^f x_f .$$

Then $g \to U_g$ is a unitary representation of G on $\mathcal{H}_\pi$, and

$$\begin{aligned} U_g \pi(A) U_g^* (\Sigma^{\oplus} x_f) &= \Sigma^{\oplus} U_g^f \pi_f(A) U_g^{f*} x_f \\ &= \Sigma^{\oplus} \pi_f(g(A)) x_f \\ &= \pi(g(A)) (\Sigma^{\oplus} x_f) \end{aligned}$$

Thus

$$U_g \pi(A) U_g^* = \pi(g(A)) \quad (g \in G,\ A \in \mathfrak{R})$$

Now $\pi$ is a *-isomorphism between the von Neumann algebras $\mathfrak{R}$ and

$\pi(\mathfrak{R})$, so $\pi$ and $\pi^{-1}$ are both ultraweakly continuous. If $g \in G$, then

$$\pi g \pi^{-1} : \pi(A) \to \pi(g(A))$$

is an automorphism of $\pi(\mathfrak{R})$. Let

$$\pi G \pi^{-1} = \{\pi g \pi^{-1}; \quad g \in G\}$$

Then $\pi G \pi^{-1}$ is a group of automorphisms of $\pi(\mathfrak{R})$. If $\varphi$ is a $\pi G \pi^{-1}$-invariant normal state of $\pi(\mathfrak{R})$, then $f = \varphi \circ \pi^{-1}$ is a G-invariant normal state of $\mathfrak{R}$. Thus if $x_f$ is a cyclic vector for the representation $\pi_f$ such that

$$f = \omega_{x_f} \circ \pi_f \ ,$$

define $y \in \mathcal{H}_\pi$ by

$$f = \Sigma^{\oplus} y_g$$

where

$$y_g = \begin{cases} 0 & (g \neq f) \\ x_f & (g = f) \end{cases}$$

Clearly

$$\varphi = \omega_y|_{\pi(\mathfrak{R})} \ .$$

Since

$$U_g^f x_f = x_f \qquad (g \in G) \ ,$$

we have

$$U_g y = y \qquad (g \in G) \ .$$

We may thus replace $\mathfrak{R}$ by $\pi(\mathfrak{R})$ and G by $\pi G \pi^{-1}$, proving the result.

<u>7.21 Lemma</u> Let $\mathcal{U}$ be a group of unitary operators on a Hilbert space $\mathcal{H}$, and $x_o \in \mathcal{H}$. Then there is a unique vector $x \in \mathcal{H}$ such that for each $T \in co(\mathcal{U})$ (the convex hull of $\mathcal{U}$), there exists $T_1 \in co(\mathcal{U})$ with

$$\|T_2T_1Tx_o - x\| < \varepsilon$$

for all $T_2 \in co(\mathcal{U})$.

<u>Proof</u> Let

$$\mathcal{Y} = \{x \in \mathcal{H}; \ Ux = x \ (U \in \mathcal{U})\}$$

and

$$\mathcal{M} = \mathcal{Y}^{\perp}.$$

If $y \in \mathcal{M}$, $\omega \in \mathcal{Y}$ then

$$\begin{aligned}\langle Uy, \omega\rangle &= \langle y, U^{-1}\omega\rangle \\ &= \langle y, \omega\rangle \\ &= 0 \ ,\end{aligned}$$

for all $U \in \mathcal{U}$, so both $\mathcal{Y}$ and $\mathcal{M}$ are invariant under $\mathcal{U}$. Fix $y \in \mathcal{M}$, and let

$$\mathcal{X} = \{Ty; \ T \in co(\mathcal{U})\} \ .$$

Then $\mathcal{X} \subseteq \mathcal{M}$ and $\mathcal{X}$ is convex. If $Ty \in \mathcal{X}$ then

$$\begin{aligned}\|Ty\| &\le \|T\| \ \|y\| \\ &\le \|y\| \ ,\end{aligned}$$

so $\overline{\mathcal{X}}$ is a bounded closed convex subset of $\mathcal{M}$. Let z be the unique vector of minimal norm in $\overline{\mathcal{X}}$. If $U \in \mathcal{U}$, then $Uz \in \overline{\mathcal{X}}$ and

$$\|Uz\| = \|z\| \ .$$

By uniqueness,

$$Uz = z \qquad (U \in \mathcal{U}) .$$

Thus

$$z \in \mathcal{Y} \cap \mathcal{M} ,$$

so

$$z = 0 .$$

Hence if $\epsilon > 0$, there is a $T_1 \in \text{co}(\mathcal{U})$ with $\|T_1 y\| < \epsilon$, and so

$$\|T_2 T_1 y\| < \epsilon \qquad (T_2 \in \text{co}(\mathcal{U})) .$$

If now

$$x_0 = x + y_0 .$$

with $x \in \mathcal{Y}$, $y_0 \in \mathcal{M}$, let $T \in \text{co}(\mathcal{U})$ and $\epsilon > 0$. Applying the above argument to $y = Ty_0 \in \mathcal{M}$, there is a $T_1 \in \text{co}(\mathcal{U})$ with

$$\|T_2 T_1 T(x_0 - x)\| < \epsilon \qquad (T_2 \in \text{co}(\mathcal{U})) .$$

7.22 Lemma  Let $\mathcal{U}$ be a group of unitary operators on a Hilbert space $\mathcal{H}$,

$$\mathcal{Y} = \{x \in \mathcal{H};\ Ux = x \qquad (U \in \mathcal{U})\} .$$

Denote by $E_0$ the orthogonal projection onto $\mathcal{Y}$. Then

$$E \in \text{co}(\mathcal{U})^- ,$$

the strong closure of $\text{co}(\mathcal{U})$.

Proof  Let $z_1, \ldots, z_n$ be n vectors in $\mathcal{H}$, $\epsilon > 0$ and $\mathcal{M} = \mathcal{Y}^{\perp}$, then

$$z_j = x_j + y_j$$

with $x_j \in \mathcal{Y}$, $y_j \in \mathcal{M}$. By Lemma 7.21, and its proof there is a $T_1 \in co(\mathcal{U})$ with

$$\|T_1(z_1 - x_1)\| < \epsilon$$

Suppose

$$T_1, \ldots, T_k \qquad (1 \le k \le n-1)$$

chosen such that

$$\|T_k T_{k-1} \ldots T_1(z_j - x_j)\| < \epsilon \qquad (j = 1, \ldots, k)$$

By Lemma 7.21, there is a $T_{k+1} \in co(\mathcal{U})$ with

$$\|T_{k+1} T_k \ldots T_1(z_{k+1} - x_{k+1})\| < \epsilon$$

Since

$$\|T_{k+1}\| \le 1$$

and

$$T_{k+1} x_j = x_j \qquad (j = 1, \ldots, k)$$

we have

$$\|T_{k+1} T_k \ldots T_1(z_j - x_j)\| \le \|T_k \ldots T_1 (z_j - x_j)\| < \epsilon$$

Thus $T_1, \ldots, T_n$ can be chosen such that if

$$T = T_n T_{n-1} \ldots T_1 ,$$

then

$$T \in co(\mathcal{U})$$

and

$$\|Tz_j - x_j\| = \|T(z_j - x_j)\|$$

$$< \epsilon \qquad (j = 1, \ldots, n)$$

Hence

$$\|(I - E_o) z_j\| < \epsilon ,$$

and we have shown that

$$E_o \in co(\mathfrak{U})^- .$$

<u>Proof of Theorem 7.19</u> By Lemma 7.20, we may assume $\mathfrak{R}$ acts on a Hilbert space $\mathfrak{H}$, there is a unitary representation $g \to U_g$ of G on $\mathfrak{H}$ with $\alpha(g) = \mathrm{ad}\, U_g$, and that every normal G-invariant state of $\mathfrak{R}$ is a vector state. Let $\mathfrak{U} = \{U_g;\ g \in G\}$, and $\mathfrak{S} = (\mathfrak{R} \cup \mathfrak{U})''$. Denote by $E_o$ the projection onto $\{x \in \mathfrak{H};\ U_g x = x\ \ (g \in G)\}$. By Lemma 7.22,

$$E_o \in co(\mathfrak{U})^- .$$

In particular, $E_o \in \mathfrak{U}''$, so

$$E_o \in (\mathfrak{R}^G)' .$$

Let $\varphi$ be a G-invariant normal state on $\mathfrak{R}$, then by Lemma 7.20, we can assume

$$\varphi = \omega_x|_{\mathfrak{R}} \quad \text{with} \quad E_o x = x$$

But then $C_{E_o}$, the central carrier of $E_o$ in $(\mathfrak{R}^G)'$, is a projection in the centre of $\mathfrak{R}^G$, and

$$\varphi(I - C_{E_o}) = \|x\|^2 - \|C_{E_o} x\|^2$$

Now

$$\begin{aligned} C_{E_o} x &= C_{E_o} E_o x \\ &= E_o x \\ &= x , \end{aligned}$$

so

$$\varphi(I - C_{E_o}) = 0$$

This holds for all such $\varphi$, and $\mathfrak{R}$ is G-finite, so

$$C_{E_o} = I$$

Let

$$\{\sum_i \lambda_i^\alpha U_{g_i^\alpha}\} \quad a \in A$$

be a net in co($\mathfrak{U}$) converging strongly to $E_o$, and let $A \in \mathfrak{R}$, then

$$E_o A E_o = \text{strong} \lim_{\alpha \in A} \sum_i \lambda_i^\alpha U_{g_i^\alpha} A U_{g_i^\alpha}^{-1} E_o$$

since

$$U_{g_i^\alpha}^{-1} E_o = E_o$$

for all i, $\alpha$. The net

$$\{\sum_i \lambda_i^\alpha U_{g_i^\alpha} A U_{g_i^\alpha}^{-1}\}_{\alpha \in A}$$

is bounded by $||A||$, and the ball radius $||A||$ in $\mathfrak{R}$ is weakly compact, so there is a subnet

$$\{\sum_i \lambda_i^\beta U_{g_i^\beta} A U_{g_i^\beta}^{-1}\}_{\beta \in B}$$

converging weakly to $B \in \mathfrak{R}$, say. But the net

$$\{\sum_i \lambda_i^\beta U_{g_i^\beta}\}$$

converges strongly to $E_o$ (being a subnet of a convergent net) so

$$E_o A E_o = B E_o \,.$$

Note now that if $A \in \mathfrak{R}$, then

$$U_g A U_g^{-1} = \alpha(g)(A) \in \mathfrak{R} ,$$

so if $B' \in \mathfrak{R}'$ then

$$U_g B' U_g^{-1} \in \mathfrak{R}' .$$

Also

$$\mathfrak{R}^G = (\mathfrak{R}' \cup \mathfrak{U})'$$

so

$$(\mathfrak{R}^G)' = (\mathfrak{R}' \cup \mathfrak{U})'' .$$

Thus finite sums of the form

$$\sum_j T_j U_{g_j} \qquad (T_j \in \mathfrak{R}', \ g_j \in G)$$

from a *-algebra weakly dense in $(\mathfrak{R}^G)'$. If

$$y \in (\mathfrak{R}^G)' E_o \mathcal{H} ,$$

theyn y is the limit of vectors of the form

$$\sum_{i,j} T_j U_{g_j} E_o x_i \qquad (T_j \in \mathfrak{R}', \ g_j \in G, \ x_i \in \mathcal{H})$$

and

$$\sum_{i,j} T_j U_{g_j} E_o x_i = \sum_{i,j} T_j E_o x_i ,$$

so

$$y \in \mathfrak{R}' E_o \mathcal{H} .$$

The above shows that

$$[(\mathfrak{R}^G)' E_o \mathcal{H}] = [\mathfrak{R}' E_o \mathcal{H}]$$

so

$$\begin{aligned} I &= C_{E_o}((\mathfrak{R}^G)') \\ &= [(\mathfrak{R}^G)' E_o \mathcal{H}] \\ &= [\mathfrak{R}' E_o \mathcal{H}] \end{aligned}$$

Thus if $C \in \mathfrak{R}$ and $C E_o = 0$, then

$$C = CI = C[\mathfrak{R}'.E_o \mathcal{H}]$$

$$= [\mathfrak{R}' C E_o \mathcal{H}]$$

$$= 0 \quad ,$$

showing that B is the unique element of $\mathfrak{R}$ such that

$$E_o A E_o = B E_o$$

Define

$$\Phi(A) = B \ ,$$

then $\Phi$ is a positive linear G-invariant faithful normal projection of norm one from $\mathfrak{R}$ onto $\mathfrak{R}^G$ (since if $B \in \mathfrak{R}^G$,

$$B E_o = E_o B = E_o B E_o \ )$$

By construction

$$\Phi(A) \in \overline{co}^{ow}\{g(A);\ g \in G\} \cap \mathfrak{R}^G \ ,$$

and if

$$B \in \overline{co}^{ow}\{g(A);\ g \in G\} \cap \mathfrak{R}^G \ ,$$

then

$$B E_o = E_o B E_o$$

$$= E_o A E_o \ ,$$

so

$$B = \Phi(A)$$

by uniqueness. This completes the proof.

Suppose now that G is a group of automorphisms of a von Neumann algebra $\mathfrak{R}$, and H is a group of automorphisms of a von Neumann algebra $\mathfrak{S}$, then the direct product G x H can be identified, as in Chapter IV, with a group of automorphisms of

$\mathcal{R} \otimes \mathcal{S}$, via the map

$$(g,h) \to g \otimes h \ ,$$

where $g \otimes h$ denotes the unique automorphism of $\mathcal{R} \otimes \mathcal{S}$ such that

$$(g \otimes h)(A \otimes B) = g(A) \otimes h(B) \qquad (A \in \mathcal{R},\ B \in \mathcal{S}) \ .$$

<u>7.23 Proposition</u> Let $\mathcal{R}$, $\mathcal{S}$, G, H be as above. If $\mathcal{R}$ is G-finite and $\mathcal{S}$ is H-finite then $\mathcal{R} \otimes \mathcal{S}$ is $G \times H$-finite, and

$$(\mathcal{R} \otimes \mathcal{S})^{G \times H} = \mathcal{R}^G \otimes \mathcal{S}^H \ .$$

<u>Proof</u> Let $\mathcal{R}$ act on the Hilbert space $\mathcal{H}$, and $\mathcal{S}$ act on the Hilbert space $\mathcal{K}$. If $\varphi$ is a G-invariant normal state of $\mathcal{R}$ and $\psi$ is an H-invariant normal state of $\mathcal{S}$, then

$$f = \Sigma\, \omega_{x_i} \quad \text{with} \quad x_i \in \mathcal{H} \ ,$$

and

$$g = \Sigma\, \omega_{y_i} , \qquad y_i \in \mathcal{K} \ .$$

Define $\varphi \otimes \psi$ on $\mathcal{R} \otimes \mathcal{S}$ as usual by

$$\varphi \otimes \psi(A) = \sum_{i,j} \omega_{x_i \otimes y_j}(A)$$

Then $\varphi \otimes \psi$ is a normal state of $\mathcal{R} \otimes \mathcal{S}$, and

$$\begin{aligned} \varphi \otimes \psi((g \otimes h)(A \otimes B)) &= \varphi(g(A))\ \psi\ (h(B)) \\ &= \varphi(A)\ \psi\ (B) \\ &= \varphi \otimes \psi(A \otimes B) \end{aligned}$$

Linear combinations of elements of the form $A \otimes B$ are ultra-weakly dense in $\mathcal{R} \otimes \mathcal{S}$, so $\varphi \otimes \psi$ is $G \times H$-invariant.

$$E_\varphi = [\mathcal{R}'\{x_i,\ \ 1 \le i \le \infty\}] \ .$$

$$E_\psi = [\mathcal{S}'\{y_i;\ 1 \le j < \infty\}] \ .$$

and

$$(\varphi \otimes \psi)(E_\varphi \otimes E_\psi) = \varphi(E_\varphi)\ \psi\ (E_\psi) = 1 \ ,$$

thus

$$E_{\varphi \otimes \psi} \le E_\varphi \otimes E_\psi \ .$$

However,

$$E_{\varphi \otimes \psi} = [(\mathcal{R} \otimes \mathcal{S})'\{x_i \otimes y_j,\ 1 \le i,j < \infty\}]$$

$$\ge [\mathcal{R}' \otimes \mathcal{S}'\ \{x_i \otimes y_j;\ 1 \le i,j < \infty\}]$$

$$= E_\varphi \otimes E_\psi \ ,$$

thus

$$E_{\varphi \otimes \psi} = E_\varphi \otimes E_\psi \ .$$

Let

$$E = \vee\{E_\varphi;\ \varphi \text{ is a normal } G \times H\text{-invariant state of } \mathcal{R} \otimes \mathcal{S}\} \ ,$$

$$E_1 = \vee\{E_\varphi;\ \varphi \text{ is a normal } G\text{-invariant state of } \mathcal{R}\} \quad \text{and}$$

$$E_2 = \vee\{E_\varphi;\ \varphi \text{ is a normal } H\text{-invariant state of } \mathcal{S}\} \ .$$

Then

$$E \ge E_1 \otimes E_2$$

by the foregoing. But

$$f(I - E_1) = 0$$

for all normal G-invariant states of $\mathcal{R}$, so

$$E_1 = I$$

since $\mathcal{R}$ is G-finite. Similarly

$$E_2 = I \ ,$$

hence

$$E = I \ ,$$

showing that $\mathcal{R} \otimes \mathcal{S}$ is $G \times H$-finite. Now if $B \in \mathcal{R}$, $C \in \mathcal{S}$, then

$$\Phi_{\mathcal{R}}(B) \in \overline{co}^{ow}\{g(B);\ g \in G\}$$

$$\Phi_{\mathcal{S}}(C) \in \overline{co}^{ow}\{h(C);\ h \in H\}$$

Thus $\Phi_{\mathcal{R}}(B)$ is the ultraweak limit of a net

$$\{\sum_j \lambda_j^{\alpha}\, g_j^{\alpha}\,(B)\}_{\alpha \in A}$$

and $\Phi_{\mathcal{S}}(C)$ is the ultraweak limit of a net

$$\{\sum_k \mu_k^{\beta}\, h_k^{\beta}\,(C)\}_{\beta \in B}$$

Thus $\Phi_{\mathcal{R}}(B) \otimes \Phi_{\mathcal{S}}(C)$ is the ultraweak limit of the net

$$\{\sum_{j,k} \lambda_j^{\alpha}\, \mu_k^{\beta}\, g_j^{\alpha}(B) \otimes h_k^{\beta}(C)\}_{\substack{\alpha \in A \\ \beta \in B}}$$

$$= \{\sum_{j,k} \lambda_j^{\alpha}\, \mu_k^{\beta}\, (g_j^{\alpha} \otimes h_k^{\beta})(B \otimes C)\}_{\substack{\alpha \in A \\ \beta \in B}}$$

Thus

$$\Phi_{\mathcal{R}}(B) \otimes \Phi_{\mathcal{S}}(C) \in \overline{co}^{ow}\{(g \otimes h)(B \otimes C);\ g \in G,\ h \in H\}$$

Since also

$$\Phi_{\mathcal{R}}(B) \otimes \Phi_{\mathcal{S}}(C) \in (\mathcal{R} \otimes \mathcal{S})^{G \times H},$$

we have

$$\Phi_{\mathcal{R} \otimes \mathcal{S}}(B \otimes C) = \Phi_{\mathcal{R}}(B) \otimes \Phi_{\mathcal{S}}(C) \qquad (B \in \mathcal{R},\ C \in \mathcal{S})$$

Now if $A \in (\mathcal{R} \otimes \mathcal{S})^{G \times H}$, then $A \in \mathcal{R} \otimes \mathcal{S}$ so there is a net $B_n \to A$ ultraweakly, with $B_n$ of the form

$$B_n = \sum_i E_i^n \otimes F_i^n \qquad (E_i^n \in \mathcal{R},\ F_i^n \in \mathcal{S}).$$

Thus

$$A = \Phi_{\mathcal{R} \otimes \mathcal{S}}(A)$$

$$= \lim \Phi_{\mathcal{R} \otimes \mathcal{S}}(B_n)$$

$$= \lim \Phi_{\mathcal{R} \otimes \mathcal{S}}(\sum_i E_i^n \otimes F_i^n)$$

$$= \lim \sum_i \Phi_{\mathcal{R}}(E_i^n) \otimes \Phi_{\mathcal{S}}(F_i^n)$$

Thus

$$A \in \mathcal{R}^G \otimes \mathcal{S}^H .$$

This completes the proof.

<u>7.24 Theorem</u> Let G (resp. H) be a group of automorphisms of a von Neumann algebra $\mathcal{R}$ (resp. $\mathcal{S}$). Let $\varphi$ be a normal G-invariant state of $\mathcal{R}$, $\psi$ a normal H-invariant state of $\mathcal{S}$. If $\varphi$ is an ergodic state relative to G and $\psi$ is an ergodic state relative to H, then $\varphi \otimes \psi$ is an ergodic state relative to $G \times H$.

<u>Proof</u> If $\varphi$ is an ergodic state of $\mathcal{R}$ then $G_{E_\varphi}$ acts ergodically on $E_\varphi \mathcal{R} E_\varphi$ by Theorem 7.9. Similarly, the group $H_{E_\psi}$ acts ergodically on $E_\psi \mathcal{S} E_\psi$, thus

$$(E_\varphi \mathcal{R} E_\varphi)^{G_{E_\varphi}} = \mathbb{C}I$$

$$= (E_\psi \mathcal{S} E_\psi)^{H_{E_\psi}}.$$

Since $E_\varphi \mathcal{R} E_\varphi$ is $G_{E_\varphi}$- finite, and $E_\psi \mathcal{S} E_\psi$ is $H_{E_\psi}$-finite, it follows by Proposition 7.23 that

$$(E_\varphi \mathcal{R} E_\varphi \otimes E_\psi \mathcal{S} E_\psi)^{G_{E_\varphi} \times H_{E_\psi}} = \mathbb{C}I$$

i.e.

$$[(E_\varphi \otimes E_\psi)(\mathfrak{R} \otimes \mathfrak{S})(E_\varphi \otimes E_\psi)]^{(G \times H)_{E_\varphi \otimes E_\psi}} = \mathbf{C}I ,$$

But

$$E_\varphi \otimes E_\psi = E_{\varphi \otimes \psi} ,$$

so $(G \times H)_{E_{\varphi \otimes \psi}}$ acts ergodically on $E_{\varphi \otimes \psi}(\mathfrak{R} \otimes \mathfrak{S})E_{\varphi \otimes \psi}$, thus $\varphi \otimes \psi$ is an ergodic state relative to $G \times H$, by Theorem 7.9. This completes the proof.

We turn now to the question of the existence of normal states of a von Neumann algebra which are invariant under the action of a group of automorphisms of the algebra. In [15], A. Hadjian and S. Kakutani defined the concept of a weakly wandering set. If $(X, \mathfrak{B}\ \mu)$ is a finite measure space (i.e. $\mu(x) < \infty$), where $\mathfrak{B}$ is the $\sigma$-ring of $\mu$-measurable subsets of X, and $T: X \to X$ is a bijective map such that T and $T^{-1}$ are both measurable transformations, then a weakly wandering set is a subset S of X such that for some sequence $(n_k)$ of integers, the measurable sets

$$\{T^{n_k}(S)\}_{k=1}^{\infty}$$

are mutually disjoint. It was shown in [15] that a necessary and sufficient condition for the existence of a finite measure $\nu$ equivalent to $\mu$ (written $\mu \sim \nu$. This means that $\nu$ and $\mu$ have the same null sets) such that

$$\nu(T(E)) = \nu(E)$$

for each $E \in \mathfrak{B}$ is that there are no weakly wandering subsets of X, of positive measure.

Suppose now $\mathfrak{R}$ is an abelian von Neumann algebra. As mentioned in Chapter I, $\mathfrak{R}$ is isomorphic to C(X), all continuous complex-valued functions on X, where X (the carrier space of $\mathfrak{R}$) is a hyperstonean space, and the normal measures on X correspond precisely to the elements of $\mathfrak{R}_*$. By Proposition 1 of [9], a positive (regular borel) measure $\mu$ on X is normal if and only if it annihilates each nowhere dense subset of X. It follows that if $\nu$ is another positive measure and $\nu \sim \mu$, then $\nu$ is also normal. As in the remarks following Theorem 7.1, if

$$\rho \in X = \Phi_{\mathfrak{R}} \quad ,$$

and $\theta$ is an automorphism of $\mathfrak{R}$, then

$$\rho \circ \theta \in X \quad ,$$

and if

$$\theta^* : \rho \to \rho \circ \theta \quad ,$$

then $\theta^*$ is a homeomorphism of X. If we identify $\mathfrak{R}$ with C(X), then

$$\theta(f)(x) = f(\theta^*(x)) \qquad (f \in C(X),\ x \in X) \ .$$

Let $\varphi$ be a faithful normal state of $\mathfrak{R}$, and $\mu$ the corresponding positive normal measure on X. Since $\varphi$ is faithful, the support of $\mu$ is X. By the Hadjian-Kakutani result, there is a measure $\nu \sim \mu$ with

$$\nu(\theta^*(E)) = \nu(E)$$

for each borel set E in X if and only if there are no weakly wandering subsets of X relative to the transformation $\theta^*$. i.e. if and only if there are no borel subsets E of X with positive measure such that for some sequence $(n_k)$ of integers, the borel sets

$$(\theta^*)^{n_k}(E)$$

By regularity of $\mu$, can assume E closed.

are mutually disjoint. Since $\mu$ is normal,

$$\mu(E) = \mu(F)$$

where F is the closure of the interior of E, by [9], Corollary to Proposition 6. Now F is an open-closed subset of X since X is stonean, thus $\chi_F$ is a non-zero projection in C(X). Let

$$F_{ij} = (\theta^*)^{n_i}(F) \cap (\theta^*)^{n_j}(F)$$

Then

$$F_{ij} \subset (\theta^*)^{n_i}(F) \setminus (\theta^*)^{n_i}(E) \ ,$$

Since $(\theta^*)^{n_i}$ is a homeomorphism, $(\theta^*)^{n_i}(F)$ is the closure of the interior of $(\theta^*)^{n_i}(E)$, thus

$$\mu(F_{ij}) = 0$$

since $\mu$ is normal. Hence the projections $(\theta^*)^{n_i}(F)$ are orthogonal. In operator theory language then, we see that there is a faithful normal $\theta$-invariant state on $\mathfrak{R}$ if and only if there are no non-zero projections E in $\mathfrak{R}$ such that for some sequence $(n_k)$ of integers, the projections $\theta^{n_k}(E)$ are mutually orthogonal.

If $G = \{\theta^n ; n \in Z\}$, then G is an abelian group, hence amenable. We shall show that the above result generalises to the situation of an amenable group of automorphisms of a countably decomposable von Neumann algebra, with a new definition of weakly wandering projection. We also show that this definition coincides with the concept of a weakly wandering set when the algebra is abelian.

<u>7.25 Definition</u> Let G be a group of automorphisms of a von Neumann algebra $\mathfrak{R}$. E is a <u>weakly wandering projection</u> if there is a sequence $(g_n)$ in G with $g_n(E) \to 0$ in the weak operator topology. Let f be a faithful normal state of $\mathfrak{R}$. G is said to be <u>poorly mixing</u> for f is given $\epsilon > 0$, there is a $\delta(\epsilon) > 0$ such that if E is a projection in $\mathfrak{R}$ and $f(E) \geq \epsilon$, then $\inf_{g \in G} f(g(E)) \geq \delta(\epsilon)$.

<u>7.26 Theorem</u> Let $\mathfrak{R}$ be a countably decomposable von Neumann algebra, G an amenable group of automorphisms of $\mathfrak{R}$. The following are equivalent

(1) G is poorly mixing for each faithful normal state f of $\mathfrak{R}$.

(2) There is a faithful normal G-invariant state on $\mathfrak{R}$.

(3) There are no non-zero weakly wandering projections in $\mathfrak{R}$.

<u>Proof</u> <u>(1) ⇒ (2)</u>: Let f be a faithful normal state, and $\mathcal{E} = \overline{co}^{\sigma}\{f \circ g;\ g \in G\}$ where $\sigma$ denotes the $\sigma(\mathfrak{R}_*, \mathfrak{R})$ topology on $\mathfrak{R}_*$. We assert that $\mathcal{E}$ is a $\sigma$-compact. For suppose not, then by [2], Theorem II.2, there is an orthogonal sequence $(p_n)$ of projections in $\mathfrak{R}$, an $\epsilon > 0$, and a sequence $(g_n)$ in G with $f(g_n(p_n)) \geq \epsilon$ for all n. Then $f(p_n) \geq \delta(\epsilon)$ for all n, yet $p_n \to 0$ weakly. This is a contradiction so the assertion is proved. Now applying the Ryll-Nardzewski fixed point theorem to $\mathcal{E}$ (see Appendix A), there is an $h \in \mathcal{E}$ with $h \circ g = h$ for all $g \in G$. Thus h is a G-invariant normal state on $\mathfrak{R}$. If $E_h$ is the support of h, then $E_h$ is a G-invariant projection in $\mathfrak{R}$, so $f(E_h) = h(E_h) = 1$. Thus $I = E_f \leq E_h$. Hence $E_h = I$, showing that h is faithful.

(2) ⇒ (3) Let E be a projection in $\mathfrak{R}$ and $(g_n)$ a sequence in G with $g_n(E) \to 0$ weakly. Let h be a faithful normal G-invariant state on $\mathfrak{R}$, then $h(E) = h(g_n(E))$ for all n, thus $h(E) = 0$ and $E = 0$.

(3) ⇒ (2) Let f be a faithful normal state on $\mathfrak{R}$, and let

$$S = \overline{co}^{W^*}\{f \circ g;\ g \in G\}$$

S is a weak*-closed convex subset of the states of $\mathfrak{R}$, so S is weak*-compact. Since G is amenable, there is an $h \in S$ with $h \circ G = h$ (see Appendix B). Thus h is a G-invariant state on $\mathfrak{R}$. There is a unique decomposition $h = h_n + h_s$ with $h_n$ a normal positive linear functional on $\mathfrak{R}$, and $h_s$ a singular positive linear functional on $\mathfrak{R}$ ([54], Theorem 3). By uniqueness of this decomposition, $h_n$ and $h_s$ are both G-invariant. If $h_n \neq 0$, let E be the support of $h_n$. Suppose $E \neq I$, then $h_n(I - E) = 0$. Let F be a projection in $\mathfrak{R}$ with $0 < F < I-E$ and $h_s(F) = 0$. This exists by [55], Theorem 1. We have $h(F) = h_n(F) + h_s(F) = 0$. Let

$$\lambda = \inf_{g \in G} f(g(F))$$

then

$$f(g(F)) \geq \lambda \geq 0 \qquad (g \in G)$$

so

$$\varphi(F) \geq \lambda \qquad (\varphi \in S)$$

In particular, $h(F) \geq \lambda$. Hence $\lambda = 0$. Let $(g_n)$ be a sequence in G with $f(g_n(F)) \to 0$, then $g_n(F) \to 0$ weakly since f is faithful ([7], Chapter I, §4, Proposition 4), so F is a weakly wandering projection in $\mathfrak{R}$. Thus $F = 0$ and $h_n$ is faithful. If

$h_n = 0$, then h is singular and by [55], Theorem 1, there is a non-zero projection E in $\mathfrak{R}$ with $h(E) = 0$. A similar argument to the foregoing shows that E is a weakly wandering projection in $\mathfrak{R}$, hence $E = 0$. This contradiction proves that $h_n$ is a faithful normal G-invariant positive linear functional on $\mathfrak{R}$, and gives the result.

(2) ⇒ (1) Let $\varphi$ be a faithful normal G-invariant state on $\mathfrak{R}$, and suppose (1) is not true, then there is a sequence $(E_n)$ of projections in $\mathfrak{R}$ and an $\epsilon > 0$ with $f(E_n) \geq \epsilon$ for all n, and $\inf_{g \in G} f(g(E_n)) < \frac{1}{n}$.

For each n, choose $g_n \in G$ with $f(g_n(E_n)) < \frac{1}{n}$, then $g_n(E_n) \to 0$ ultraweakly, by [7], Chapter I, §4, Proposition 4, thus $\varphi(E_n) = \varphi(g_n(E_n)) \to 0$, so $E_n \to 0$ ultraweakly (by [7], Chapter I, §4, Proposition 4 again). Hence $f(E_n) \to 0$, a contradiction. This completes the proof.

Theorem 7.27 Let $\mathfrak{R}$ be an abelian countably decomposable von Neumann algebra acting on a Hilbert space $\mathcal{H}$, G a group of automorphisms of $\mathfrak{R}$, then the following are equivalent:

(1) There is a projection $E \neq 0$ in $\mathfrak{R}$ and a sequence $(g_n)$ in G with $g_n(E) \to 0$ weakly.

(2) There is a projection $F \neq 0$ in $\mathfrak{R}$ and a sequence $(h_n)$ in G such that the projections $h_n(F)$ are mutually orthogonal.

Proof (2) ⇒ (1): obvious since an orthogonal family of projections must converge weakly to zero.

(1) ⇒ (2) Let $E \in \mathfrak{R}$ be a non-zero projection and $h_n \in G$ be such that $h_n(E) \to 0$ weakly. Let $\epsilon > 0$ and let $\epsilon_n = \frac{\epsilon}{n \cdot 2^n}$.

Let $\omega$ be a faithful normal state on $\mathcal{R}$ and choose $g_1$ with $\omega(g_1(E)) < \epsilon_1$. Suppose $g_1, \ldots, g_{n-1}$ chosen. The map $A \rightarrow \sum_{j=1}^{n-1} \omega(g_j^{-1}(A))$ is ultraweakly continuous, so there is an ultraweakly open neighbourhood $\mathcal{U}(\epsilon_n)$ of 0 in $\mathcal{R}$ such that $A \in \mathcal{U}(\epsilon_n)$ implies $\omega(g_j^{-1}(A)) < \epsilon_n$ $(1 \leq j \leq n-1)$. By [7], p.32, $\mathcal{U}(\epsilon_n)$ contains a set of the form $\{A; \omega_{x_\ell}(A) < \delta \ (1 \leq \ell \leq k)\}$ where $x_\ell \in \mathcal{H}$. Let $\Theta = \sum_{\ell=1}^{k} \omega_{x_\ell}$, then $\Theta \in \mathcal{R}_*$, and $\Theta(A) < \delta$ implies $\omega(g_j^{-1}(A)) < \epsilon_n$, $(1 \leq j \leq n-1)$. Now $\Theta(h_r(E)) \rightarrow 0$ as $r \rightarrow \infty$, so there is a $g_n$ with $\Theta(g_n(E)) < \delta$. Thus

$$\omega(g_j^{-1} g_n(E)) < \epsilon_n \qquad (1 \leq j \leq n-1) \qquad (*)$$

The proof now follows that of [15], Lemma 4. By induction there is a sequence $(g_n)$ in G such that (*) holds for all n. Let

$$E' = \bigvee_{i=1}^{\infty} \bigvee_{j=1}^{i-1} g_j^{-1} g_i(E)E. \quad \text{Then } E' \text{ is a projection in } \mathcal{R},$$

and

$$\omega(E') \leq \sum_{i=1}^{\infty} \sum_{j=1}^{i-1} \omega(g_j^{-1} g_i(E))$$

$$\leq \sum_{i=1}^{\infty} \sum_{j=1}^{i-1} \epsilon_i \leq \epsilon .$$

Let $F = E - E'$, then $\omega(F) > 0$ if $\epsilon$ is sufficiently small, so F is a non-zero projection in $\mathcal{R}$. Also,

$$F g_j^{-1} g_i(F) \leq (E - E') \, g_j^{-1} g_i(F)$$

$$\leq (E - E') \, E' = 0 .$$

Thus $g_j(F) \, g_i(F) = 0$ $(i \neq j)$. Hence $\{g_n(F)\}$ is a sequence of orthogonal projections in $\mathcal{R}$. This completes the proof.

I would like to thank Professor M. Takesaki for a most stimulating conversation on the above topic.

<u>7.28 Remark</u> It has been pointed out to me that the definition of a poorly mixing group, and the statement (1) ⇔ (2) of Theorem 7.26 are similar to a result of E. Størmer ([51], Corollary ). This was, however, unknown to me at the time.

## APPENDIX A

### The Ryll-Nardzewski Fixed Point Theorem

Let $(E,\tau)$ be a locally convex Hausdorff linear topological space, $\mathcal{K}$ a non-void $\sigma(E^*,E)$-compact convex subset of E, $\mathcal{S}$ a semigroup of $\sigma(E^*,E)$-continuous maps from K to K which satisfy the following properties.

(i) If $x,y \in K$, $x \neq y$, then

$$0 \notin \{Tx - Ty; \; T \in \mathcal{S}\}^-$$

where closure is in the $\tau$-topology. (we say $\mathcal{S}$ is non-contracting if (1) holds)

(ii) If $T \in \mathcal{S}$, then T is affine i.e.

$$T(\lambda x + (1 - \lambda)y) = \lambda Tx + (1 - \lambda)Ty$$

$(x,y \in K, \; 0 \leq \lambda \leq 1)$.

We shall prove that $\mathcal{S}$ has a fixed point in K i.e. there is an $x \in K$ such that

$$Tx = x \qquad (T \in \mathcal{S})$$

We need first some preliminary results. Recall first that a topological space X is said to be of first category if X can be expressed as a countable union of nowhere dense subsets. X is of second category if it is not of first category.

Lemma 1 A non-void compact Hausdorff space of second category.

Proof Let $E \neq \emptyset$ be a compact Hausdorff space and suppose $E = \bigcup_{n=1}^{\infty} E_n$ with $E_n$ closed and $\text{int}(E_n) = \emptyset$. Let

$$G_1 = E \setminus E_1$$

$G_1$ is open and non-void. Since E is a normal space, there is a non-void open set $H_2$ with $\overline{H}_2 \subset G_1$. Let

$$G_2 = H_2 \setminus E_2$$

then $G_2$ is open and non-void. By induction we obtain a sequence $\{G_n\}$ of non-void open sets with

$$\text{(a)} \quad \overline{G}_{n+1} \subset G_n$$

$$\text{(b)} \quad G_n \cap E_n = \emptyset$$

for all n.

The sets $\overline{G}_n$ are compact, and by (b),

$$\bigcap_{n=1}^{\infty} G_n = \emptyset$$

Thus by (a),

$$\bigcap \overline{G}_n = \emptyset$$

Hence $\overline{G}_n = \emptyset$ for some n, giving a contradiction.

<u>Lemma 2</u> Suppose K is $\tau$-separable. Let p be a continuous seminorm on E, and $\epsilon > 0$. Then there is a closed convex set $C \subset K$ such that

$$C \neq K$$

$$p\text{-diam}(K \setminus C) \leq \epsilon$$

i.e.

$$\sup\{p(x - y);\ x, y \in K \setminus C\} \leq \epsilon \ .$$

<u>Proof</u> Let

$$S = \{x;\ p(x) \leq \epsilon/4\} \ .$$

S is closed, convex and $\text{int}(S) \neq \emptyset$. Since S is $\tau$-closed it is also $\sigma(E^*,E)$-closed. Let

$$\text{ext}(K) = \text{set of extreme points of } K .$$

ext(K) is non-void by the Krein-Mil'man theorem. Let

$$D = \sigma(E^*,E)\text{-closure of ext}(K)$$

D is a non-void $\sigma(E^*,E)$-compact Hausdorff space, so by Lemma 1 D is of second category. Since K is $\tau$-separable there is a sequence $\{k_n\}$ of elements of K with

$$K = \bigcup_{n=1}^{\infty} (k_n + S) \quad .$$

Now

$$D \subset \bigcup_{n=1}^{\infty} (k_n + S) \cap D$$

and each $(k_n + S) \cap D$ is closed in the relative weak topology, so at least one of the sets $(k_n + S) \cap D$ contains a non-void relatively weakly open subset. Hence there is a $k \in K$ and a $\sigma(E^*,E)$-open set W with

$$(k + S) \cap D \supset W \cap D \neq \varphi$$

Let

$$K_1 = \overline{co}(D \setminus W)$$

$$K_2 = \overline{co}(W \cap D)$$

(closure in $\tau$-topology). $K_1$ and $K_2$ are closed and convex, hence weakly closed subsets of K. Hence both are weakly compact convex sets. If $K = K_1$ then

$$\text{ext}(K) \subset D \setminus W$$

by the converse of the Krein-Mil'man theorem ([27], Theorem 15.2, p.132). But this implies that

$$D = \text{weak closure } (\text{ext}(K)) \subset D \setminus W ,$$

contradicting the fact that $D \cap W \neq \emptyset$ . Hence

$$K \neq K_1 \tag{1}$$

since

$$W \cap D \subset k + S$$

and $k + S$ is convex and weakly closed, we have

$$K_2 \subset k + S$$

and so

$$\text{p-diam } (K_2) \leq \epsilon/2 . \tag{2}$$

Also

$$K = \text{co } (K_1 \cup K_2) \tag{3}$$

for $(K_1 \cup K_2)$ is convex and weakly compact, and contains ext(K), hence contains K by the Krein-Mil'man theorem.

Let $r \in R$, $0 < r \leq 1$, and define

$$f_r : K_1 \times K_2 \times [r,1] \to K$$

by

$$f_r(x_1, x_2, \lambda) = \lambda x_1 + (1 - \lambda) x_2 .$$

Let $C_r$ be the range of $f_r$. $C_r$ is weakly compact. We assert that $C_r$ is convex, for

$$\alpha(\lambda x_1 + (1 - \lambda)x_2) + (1 - \alpha)(\mu y_1 + (1 - \mu)y_2)$$

$$= \alpha\lambda + (1 - \alpha)\mu \{\frac{\alpha\lambda}{\alpha\lambda + (1 - \alpha)\mu} x_1 + \frac{(1 - \alpha)\mu y_1}{\alpha\lambda + (1 - \alpha)\mu}$$

$$+ \alpha(1 - \lambda) + (1 - \alpha)(1 - \mu) \left\{ \frac{\alpha(1 - \lambda)}{\alpha(1 - \lambda) + (1 - \alpha)(1 - \mu)} x_2 + \frac{(1 - \alpha)(1 - \mu) y_2}{\alpha(1 - \lambda) + (1 - \alpha)(1 - \mu)} \right\}$$

$$= [\alpha\lambda + (1 - \alpha)\mu]z_1 + [\alpha(1 - \lambda) + (1 - \alpha)(1 - \mu)]z_2$$

with $z_1 \in K_1$ , $z_2 \in K_2$

$$= \theta z_1 + (1 - \theta) z_2$$

where $\theta = \alpha\lambda + (1 - \alpha)\mu$ .

Also, $C_r \neq K$

for if $C_r = K$ then every $z \in \text{ext}(K)$ is of the form

$$z = \lambda x_1 + (1 - \lambda)x_2$$

with $x_j \in K_j \quad (j = 1,2)$

and $\lambda \in [r, 1]$ .

Since $K_j \subset K$, this would imply that

$$z = x_1 = x_2$$

or

$$\lambda = 1 \quad \text{and} \quad z = x_1$$

In both cases

$$z \in K_1 ,$$

and

$$\text{ext}(K) \subset K_1$$

so

$$K \subset K_1$$

by the Krein-Mil'man theorem, contradicting (1). Hence

$$C_r \neq K \quad \text{for any } r > 0 . \tag{4}$$

If $y \in K \setminus C_r$, then by (3)

$$y = \lambda x_1 + (1 - \lambda) x_2$$

$$(x_j \in K_j , \quad \lambda \in [0, r[)$$

so

$$y - x_2 = \lambda(x_1 - x_2)$$

$$p(y - x_2) = \lambda p(x_1 - x_2) \leq rd$$

where $d = p\text{-diam}(K)$. Note that $K$, being $\sigma(E^*, E)$-compact is $\tau$-bounded by [27], Theorem 17.5, p.155, so $d < \infty$.

By (2)

$$p\text{-diam}(K_2) \leq \epsilon/2$$

so if

$$y_1, y_2 \in K \setminus C_r ,$$

then there are $x_1$, $x_2$ in $K_2$ such that

$$p(y_1 - x_1) \leq rd ,$$

$$p(y_2 - x_2) \leq rd .$$

Thus

$$p(y_1 - y_2) \leq p(y_1 - x_1) + p(x_1 - x_2) + p(y_2 - x_2)$$

$$\leq 2rd + \epsilon/2 .$$

So

$$p\text{-diam}(K \setminus C_r) \leq 2rd + \epsilon/2 .$$

Take $C = C_r$, and $r = \epsilon/4d$. $C$ is convex, weakly compact, and

$$p\text{-diam}(K \setminus C) \leq \epsilon .$$

Theorem (Ryll-Nardzewski) Let K be a non-void weakly compact convex subset of a linear topological space E. Let $\mathcal{S}$ be a non-contracting semigroup of weakly continuous affine maps from K to K. Then there is an $x \in K$ such that

$$Tx = x \qquad (T \in \mathcal{S})$$

Proof If $T \in \mathcal{S}$, let

$$\mathcal{F}(T) = \{x \in K;\ Tx = x\} \ .$$

We assert that if $T_1, \ldots, T_n \in \mathcal{S}$, then

$$\mathcal{F}(\tfrac{1}{n}(T_1 + \ldots + T_n)) = \bigcap_{k=1}^{n} \mathcal{F}(T_k) \ .$$

This suffices to give the result for $\frac{1}{n}(T_1 + \ldots + T_n)$ is a weakly continuous affine map from K to K, so by [ ], Ch.V, §10, Theorem 6, the sets $\mathcal{F}(T)$ are non-void, weakly compact, and have the finite intersection property. Thus

$$\cap \{\mathcal{F}(T);\ T \in \mathcal{S}\}$$

is non-void, giving the result. It suffices to prove (1). Clearly the right hand side is contained in the left hand side and (1) is true for $n = 1$.

Suppose (1) is false and let $r \geq 2$ be the least positive integer for which it fails for some $T_1, \ldots, T_r$. Let

$$T_o = \frac{1}{r}(T_1 + \ldots + T_r) \ .$$

then there is an $x_o \in K$ with

$$T_o x_o = x_o \ ,$$

but $T_k x_o \neq x_o$ for some $k$, $1 \leq k \leq r$. It follows that

$$T_k x_o \neq x_o \quad (1 \le k \le r) \qquad (2)$$

for suppose $T_r x_o = x_o$ say, then

$$r x_o = r T_o x_o$$

$$= T_1 x_o + \dots + T_{r-1} x_o + x_o$$

Thus

$$(r-1) x_o = T_1 x_o + \dots + T_{r-1} x_o$$

so

$$x_o \in \mathcal{F}\left(\frac{1}{r-1}(T_1 + \dots + T_{r-1})\right) .$$

By minimality of $r$ ,

$$x_o \in \bigcap_{j=1}^{r-1} \mathcal{F}(T_j)$$

thus

$$T_j x_o = x_o \quad (1 \le j \le r-1) ,$$

a contradiction proving (2).

Since $\mathcal{S}$ is non-contracting, it follows from (2) that there is a continuous seminorm $p$ on $(E,\tau)$ and an $\varepsilon > 0$ with

$$p(T T_k x_o - T x_o) > \varepsilon \quad (T \in \mathcal{S} ,\ 1 \le k \le r) \qquad (3)$$

Let $\mathcal{S}_o$ be the semigroup generated by $T_1, \dots, T_r$, and let $K_o$ be the weakly closed convex hull of $\{T x_o ; T \in \mathcal{S}_o\}$. $K_o$ is a weakly compact, non-void separable convex subset of $K$. By Lemma 2, there is a closed convex subset $C$ of $K_o$ with $C \neq K_o$, and

$$p\text{-diam}\,(K_o \setminus C) \le \varepsilon .$$

Since $C \neq K_o$, there is an $S \in \mathcal{S}_o$ such that

$$S x_o \in K_o \setminus C .$$

Since $T_o x_o = x_o$, and $S$ is affine, we have

$$S x_o = S T_o x_o = \frac{1}{r} (S T_1 x_o + \ldots + S T_r x_o)$$

Thus

$$S T_j x_o \in K_o \setminus C$$

for some j (since otherwise $S x_o \in C$).

Since

$$S T_j x_o \in K_o \setminus C ,$$

$$S x_o \in K_o \setminus C ,$$

and

$$p\text{-diam} (K_o \setminus C) \leq \varepsilon$$

we have

$$p(S T_j x_o - S x_o) \leq \varepsilon$$

contradicting (3). This contradiction proves (1) and finishes the proof of the theorem.

The above proof is due to I. Namioka and E. Asplund ([40]). The following corollary is the form most useful to us:

Corollary Let K be a non-void weakly compact convex subset of a Banach space E. Then there is an $x \in K$ such that

$$T x = x$$

for all isometric linear maps $T : E \to E$ such that $T(K) \subseteq K$.

Proof Let $\mathcal{S}$ be the set of all isometric $T : X \to X$ such that $T(Q) \subseteq Q$. Clearly $\mathcal{S}$ is a non-contradicting semigroup of linear maps. A basic weakly open neighbourhood of 0 is of the form

$$\mathcal{U} = \{x;\ |f_j(x)| < 1 \quad (1 \leq j \leq n)\}$$

with $f_j \in E^*$. Let

$$\mathcal{V} = \{x;\ |T^*(f_j)(x)| < 1 \quad (1 \leq j \leq n)\} .$$

Then $\mathcal{V}$ is a weak neighbourhood of 0 and

$$T(\mathcal{V}) \subset \mathcal{U} ,$$

thus each T is weakly continuous. The result follows from the theorem above.

# APPENDIX B

## The Weak Fixed Point Property for Amenable Groups

Let G be a group, B(G) the set of all complex valued bounded functions on G. If $h, g \in G$, $f \in B(G)$, define

$$\lambda g(f)(h) = f(gh)$$

$$\rho g(f)(h) = f(hg)$$

Then

$$\{\lambda g \ ; \ g \in G\}$$

and

$$\{\rho g \ ; \ g \in G\}$$

are groups of transformations of B(G). Each $\lambda g$ (resp. $\rho g$) is called a left (resp. right) translation. If B(G) is endowed with the supremum norm, it becomes a Banach space. An invariant mean on G is a map $\Omega \in B(G)^*$ such that

(i) $\Omega$ is translation invariant i.e.

$$\Omega(\lambda g(f)) = \Omega(\rho g(f)) = \Omega(f) \qquad (f \in B(G)\ , \quad g \in G)$$

(ii) $|\Omega(f)| \leq \sup\{|f(h)| \ ; h \in G\}$ .

(iii) If $\underline{1}$ denotes the function taking the value 1 everywhere on G, then

$$\Omega(\underline{1}) = 1 \ .$$

(iv) If $f \geq 0$ then

$$\Omega(f) \geq 0 \ ,$$

so $\Omega$ takes real values on real valued functions. Hence

$$\operatorname{Re} \Omega(f) = \Omega(\operatorname{Re} f) \qquad (f \in B(G))$$

If there is an invariant mean on B(G), we say that G is an amenable group.

<u>Theorem</u> Let E be a Banach space, G an amenable group of isometric linear maps from E to E. Let $f \in E^*$ and define

$$\mathcal{K} = \overline{co}^{w^*} \{f \circ g \; ; \; g \in G\} \; .$$

(closure in weak*-topology). Then there is a $\varphi \in \mathcal{K}$ such that

$$\varphi \circ g = \varphi \quad (g \in G) \; .$$

<u>Proof</u> For $a \in E$, define

$$\psi_a(g) = f(g(a)) \quad (g \in G) \; .$$

$$\begin{aligned} |\psi_a(g)| &\le \|f\| \|g(a)\| \\ &= \|f\| \; \|a\| \end{aligned}$$

so $\psi_a \in B(G)$ for each $a \in E$. Let $\Omega$ be an invariant mean on B(G), and define

$$\varphi(a) = \Omega(\psi_a) \qquad (a \in E) \; .$$

Then if $a, b \in E$, $\lambda, \mu \in C$ ,

$$\begin{aligned} \varphi(\lambda a + \mu b) &= \Omega(\psi_{\lambda a + \mu b}) \\ &= \Omega(\lambda\psi_a + \mu\psi_b) \\ &= \lambda\varphi(a) + \mu\varphi(b) \; . \end{aligned}$$

so $\varphi$ is linear. Also if $a \in E$ ,

$$\begin{aligned} |\varphi(a)| &= |\Omega(\psi_a)| \\ &\le \sup \{|\psi_a(g)| \; ; \; g \in G\} \end{aligned}$$

$$= \sup \{|f(g(a))| \ ; g \in G\}$$

$$\leq \|f\| \, \|a\| \ .$$

Thus

$$\varphi \in E^* .$$

Now

$$\psi_{g(a)}(h) = f(h(g(a)))$$

$$= f(hg(a))$$

$$= \psi_a(hg)$$

$$= \rho g(\psi_a)(h)$$

Thus

$$\psi_{g(a)} = \rho g(\psi_a) \quad (g \in G, \ a \in E) \ .$$

so if $g \in G$, $a \in E$,

$$\varphi(g(a)) = \Omega(\psi_{g(a)})$$

$$= \Omega(\rho g(\psi_a))$$

$$= \Omega(\psi_a)$$

$$= \varphi(a)$$

Showing that

$$\varphi \circ g = \varphi \qquad (g \in G) \ .$$

It remains to prove that $\varphi \in K$.

Suppose $\varphi \notin K$, then by the Hahn-Banach theorem there is a weak* continuous linear functional $\gamma$ on $E^*$, $t \in R$ and an $\varepsilon > 0$ such that

$$\operatorname{Re} \gamma(\varphi) \leq t < t + \varepsilon \leq \operatorname{Re} \gamma(k)$$

for all $k \in K$.

Since $\gamma$ is weak* continuous, $\gamma$ corresponds to an element of E, by [13] (Chapter V, §3, Theorem 9, p.421). Hence there is an $x \in E$ such that

$$\text{Re } \varphi(x) \leq t \leq t + \varepsilon \leq \text{Re } f(g(x))$$

for all $g \in G$. Now

$$\text{Re } f(g(x)) = (\text{Re } \psi_x)(g)$$

Thus

$$(\text{Re } \psi_x)(g) \geq t + \varepsilon \qquad \text{for all } g \in G.$$

Hence

$$\Omega(\text{Re } \psi_x) \geq t + \varepsilon \quad .$$

But

$$\begin{aligned} \Omega(\text{Re } \psi_x) &= \text{Re } \Omega(\psi_x) \\ &= \text{Re } \varphi(x) \\ &\leq t \quad . \end{aligned}$$

This contradiction gives the required conclusion.

# APPENDIX C

In this Appendix we shall prove that if $\varphi$ is a continuous surjection from a compact totally disconnected Hausdorff space K onto a Stonean space S, there is a continuous map $f : S \to K$ such that $\varphi \circ f$ is the identity map on S. The result is due to Gleason, but the proof we give here is due to A. Connes and can be found in [54], Theorem 48.

Lemma Let $\varphi$ be a continuous map from a compact totally disconnected Hausdorff space K into a Stonean space S. $\varphi$ is an open mapping if and only if for each non-void open set E in K, $\varphi(E)$ is not a nowhere dense set.

Proof If $\varphi$ is an open map, and E is a non-void open set in K, then $\varphi(E)$ is a non-void open set in S, so $\varphi(E)$ cannot be nowhere dense. Conversely, since K is a compact totally disconnected Hausdorff space, its topology has a basis consisting of open-closed sets ([49], Theorem C, p.150). Thus it suffices to show that if V is an open-closed set in K, then $\varphi(V)$ is open in S. Now V is compact, so $\varphi(V)$ is a compact set. Note also that the interior, int $\varphi(V)$, of $\varphi(V)$ is open in $\varphi(V)$ and if $W = \overline{\text{int}\,\varphi(V)}$ then $W \subset \varphi(V)$. Since S is Stonean, W is open, thus $W \subset \text{int}\,\varphi(V)$. This shows that int $\varphi(V)$ is an open-closed set. Hence $\varphi^{-1}(\text{int}\,\varphi(V))$ is open-closed in K. Now $Y = \varphi(V)\setminus\text{int}\,\varphi(V)$ is closed, and has void interior, so Y is a nowhere dense set. However,

$$Y = \varphi(V\setminus\varphi^{-1}(\text{int}\,\varphi(V)))$$

and $V\setminus\varphi^{-1}$ (int $\varphi(V)$) is open, thus $V\setminus\varphi^{-1}$ (int $\varphi(V)$) is void, showing that

$$\varphi(V) = \text{int } \varphi(V) \quad .$$

So $\varphi$ is an open mapping.

Theorem Let $\varphi$ be a continuous surjection from a totally disconnected compact Hausdorff space K onto a Stonean space S. Then there is a continuous map $f : S \to K$ such that $\varphi \circ f$ is the identity map on S.

Proof Let $\mathcal{K}$ denote the set of all compact subsets $K'$ of K such that $\varphi(K') = S$. $K \in \mathcal{K}$, so $\mathcal{K}$ is non-void. $\mathcal{K}$ is a partially ordered set under set inclusion. Let $(K_\alpha)_{\alpha \in A}$ be a chain in $\mathcal{K}$. By indexing the sets $K_\alpha$ by themselves, we can assume that A is a directed set. Let

$$K_1 = \bigcap_{\alpha \in A} K_\alpha$$

Each $K_\alpha$ is compact and K is compact, so K is a non-void compact set.

Let $s \in S$. For each $\alpha \in A$, there is a $k_\alpha \in K_\alpha$ with $\varphi(k_\alpha) = s$. Then $\{k_\alpha\}_{\alpha \in A}$ is a net in K, so has a cofinal convergent subnet $\{k_\beta\}_{\beta \in B}$. Let $k_\beta \to k$, then

$$\begin{aligned} \varphi(k) &= \lim \varphi(k_\beta) \\ &= s \quad . \end{aligned}$$

We claim that $k \in K_1$, for suppose not. Then for some $\beta_0$, $k \notin K_\beta$ $(\beta \geq \beta_0)$ so for some $\beta_1$, $\beta \geq \beta_1$ implies $k_\beta \notin K_\gamma$ $(\gamma \geq \beta_0)$. But if $\beta \geq \beta_1$ and $\beta \geq \beta_0$ then $k_\beta \in K_\beta$ - a contradiction. Thus

$k \in K_1$, and $\varphi(K_1) = S$. This shows that $\mathcal{K}$ is inductively ordered. By Zorn's Lemma, $\mathcal{K}$ contains a minimal element $K_o$.

(1) <u>$\varphi|K_o$ is an open mapping</u>: Let V be an open non-void set in $K_o$. Since $\varphi|K_o$ is surjective,

$$S\setminus\varphi(V) \subset \varphi(K_o\setminus V) ,$$

so

$$\overline{S\setminus\varphi(V)} \subset \varphi(K_o\setminus V)$$

since $\varphi(K_o\setminus V)$ is compact. Suppose $\varphi(V)$ is nowhere dense, then

$$\begin{aligned} S &= S\setminus\operatorname{int}(\overline{\varphi(V)}) \\ &= \overline{S\setminus\overline{\varphi(V)}} \\ &\subseteq \overline{S\setminus\varphi(V)} \\ &\subseteq \varphi(K_o\setminus V) \end{aligned}$$

Thus $K_o\setminus V \in \mathcal{K}$. By minimality of $K_o$, $\varphi(V)$ is not nowhere dense, thus by the lemma, $\varphi|K_o$ is an open map.

(2) <u>$\varphi|K_o$ is injective</u>: Suppose there are elements $x_1$, $x_2$ in $K_o$ with $x_1 \neq x_2$ and $\varphi(x_1) = \varphi(x_2)$. Since K has a base consisting of open-closed sets ([49], Theorem C, p.150), there is an open-closed set V containing $x_1$ but not $x_2$. Now $\varphi(V)$ is open-closed since $\varphi|K_o$ is an open mapping, and

$$V' = \varphi^{-1}(\varphi(V))\setminus V$$

is an open neighbourhood of $x_2$ with

$$\varphi(V') \subseteq \varphi(V) .$$

If $x \notin \varphi(K_o\setminus V')$, then $x \in \varphi(V') \subset \varphi(V)$ so $S\setminus\varphi(V) \subset \varphi(K_o\setminus V')$.
If $x \in V$, then $x \notin V'$, thus

$$\varphi(K_o \setminus V') \supset \varphi(V) \cup S \setminus \varphi(V)$$

$$= S$$

Thus $K_o \setminus V' \in \mathcal{K}$, contradicting the minimality of $K_o$. This shows that $\varphi|K_o$ is bijective, open and continuous. Thus if $f = (\varphi|K_o)^{-1}$, then $f : S \to K$ is a continuous map, and for $s \in S$,

$$(\varphi \circ f)(s) = \varphi((\varphi|_{K_o})^{-1}(s))$$

$$= \varphi|_{K_o} \circ (\varphi|_{K_o})^{-1}(s)$$

$$= s .$$

This completes the proof.

## APPENDIX D

A topological space X is said to be <u>locally arcwise connected</u> if every neighbourhood of a point in X contains an arcwise connected neighbourhood of that point. X is <u>simply connected</u> if every loop in X is homotopic to a point. If G and G′ are topological groups with identities e and e′ respectively, a <u>local homomorphism</u> from G to G′ is a continuous map f from a neighbourhood $\mathcal{U}$ of e in G to a neighbourhood $\mathcal{V}$ of e′ in G′ such that

if a, b and ab lie in $\mathcal{U}$, then

$$f(a)\, f(b) \in \mathcal{V}\,. \text{ and}$$

$$f(a)\, f(b) = f(ab)$$

<u>Theorem</u> ([42], Theorem 80, p.366) Let G and G′ be arcwise connected topological groups with identities e and e′ respectively. Suppose that G is also locally arcwise connected and simply connected. Let f be a local homomorphism from G to G′. Then f extends uniquely to a continuous homomorphism from G into G′ i.e. there is a continuous homomorphism $\varphi : G \to G'$ such that on some neighbourhood $\mathcal{W}$ of e in G,

$$\varphi|_{\mathcal{W}} = f|_{\mathcal{W}}$$

<u>Proof</u> Note firstly that if $\varphi$ is any homomorphism from G into G′ extending f, then $\varphi$ is continuous at e, hence continuous everywhere on G. We show next that the extension is unique. Suppose $\varphi$ and $\varphi'$ are two homomorphisms from G to G′ extending f. Let $g \in G$, and suppose $\mathcal{W}$ is a neighbourhood of e in G such that

$$\varphi|_{\mathcal{W}} = \varphi'|_{\mathcal{W}} = f$$

Since G is connected, there are elements $g_1, \ldots, g_n \in \mathcal{W}$ with

$$g = g_1 \ldots g_n$$

by [42], Theorem 14, p.129. Then

$$\begin{aligned}\varphi(g) &= \varphi(g_1) \ldots \varphi(g_n) \\ &= f(g_1) \ldots f(g_n) \\ &= \varphi'(g_1) \ldots \varphi'(g_n) \\ &= \varphi'(g)\end{aligned}$$

Thus $$\varphi = \varphi'$$

We now construct the extension $\varphi$. Let $P$ be a path in G with initial point e, terminal point g. We shall show that there is a unique corresponding path $P'$ in $G'$ such that

(a) $P'(0) = e'$

(b) If $f$ is defined on the open neighbourhood $U$ of $e$, then there is an $\varepsilon > 0$ such that

$$|t_1 - t_2| < \varepsilon \text{ implies}$$

$$P(t_1)^{-1} P(t_2) \in U \text{ and}$$

$$f(P(t_1)^{-1}P(t_2)) = P'(t_1)^{-1} P'(t_2)\ .$$

We prove firstly that the path $P'$ is unique. The initial point of $P'$ is determined by condition (a). Moreover if $P'$ is uniquely determined for $t < t_o$, then it is uniquely determined for $t \leq t_o$ by continuity of $f$. Suppose now that $t_o < t < t_o + \varepsilon$, then

$$P(t_o)^{-1} P(t) \in U \quad \text{and}$$

$$f(P(t_o)^{-1} P(t)) = P'(t_o)^{-1} P'(t)$$

Thus

$$P'(t) = P'(t_o) f(P(t_o)^{-1} P(t))$$

This shows that $P'$ is uniquely determined for all $t$, $0 < t < t_o + \epsilon$. Hence $P'$ is uniquely determined for all $t \in [0, 1]$ by conditions (a) and (b).

We now show that such a path $P'$ exists. Since the map $(g, h) \to g^{-1}h$ is continuous from $G \times G$ to $G$, there is a neighbourhood $V$ of $e$ with $V^{-1}V \subset U$. The map $(t, s) \to P(t)^{-1} P(s)$ is continuous on $[0, 1]^2$, thus there is a positive integer $n$ such that

$$|t_1 - t_2| < \frac{1}{n} \quad \text{implies} \quad P(t_1)^{-1} P(t_2) \in V.$$

Let $\epsilon = \frac{1}{n}$. If $m$ is an integer with $0 \leq m < n$, suppose $P'$ has already been defined for all $t$, with $0 \leq t \leq m\epsilon$, in such a way that conditions (a) and (b) are satisfied. We shall extend the domain of $P'$ to the interval $0 \leq t \leq (m+1)\epsilon$. Since the case $m = 0$ is covered by the condition $P'(0) = e'$, it follows by induction that $P'$ can be defined on all of $[0, 1]$ as required.

Let $0 \leq h \leq \epsilon$. Define

$$P'(m\epsilon + h) = P'(m\epsilon) f(P(m\epsilon)^{-1} P(m\epsilon + h)) \qquad (*)$$

Clearly (a) holds for the extended path $P'$. To show that (b) still holds let $h'$ be a real number with $|h'| \leq \epsilon$. If $h' \geq 0$,

$$P'(m\epsilon + h') = P'(m\epsilon) f(P(m\epsilon)^{-1} P(m\epsilon + h)) \qquad (**)$$

by (*). If $h < 0$, then $m\epsilon + h' < m\epsilon$, and $|(m\epsilon + h') - m\epsilon| \leq \epsilon$, so by inductive hypothesis, $P(m\epsilon + h')^{-1}(P(m\epsilon)) \in V$ and (**) holds by condition (b). Hence (**) holds for all real $h'$ with $|h'| \leq \epsilon$. It follows that

$$P'(m\epsilon + h)^{-1} P'(m\epsilon + h') = [P'(m\epsilon) f(P(m\epsilon)^{-1} P(m\epsilon + h))]^{-1} \times [P'(m\epsilon) f(P(m\epsilon)^{-1} P(m\epsilon + h'))]$$

Note that since $0 \leq h \leq \epsilon$,

$$P(m\epsilon)^{-1} P(m\epsilon + h) \in V,$$

hence
$$P(m\epsilon + h)^{-1} P(m\epsilon) \in V^{-1} \subset V,$$

so
$$f(P(m\epsilon + h)^{-1} P(m\epsilon) \cdot P(m\epsilon)^{-1} P(m\epsilon + h)) = f(e) = e'.$$

showing that

$$[f(P(m\epsilon)^{-1} P(m\epsilon + h)]^{-1} = f(P(m\epsilon + h)^{-1} \cdot P(m\epsilon))$$

Thus

$$\begin{aligned} P'(m\epsilon + h)^{-1} P'(m\epsilon + h') &= f(P(m\epsilon + h)^{-1} P(m\epsilon)) \cdot P'(m\epsilon)^{-1} \\ &\quad \times P'(m\epsilon) \cdot f(P(m\epsilon)^{-1} P(m\epsilon + h')) \\ &= f(P(m\epsilon + h)^{-1} P(m\epsilon)) \cdot f(P(m\epsilon)^{-1} P(m\epsilon + h')). \end{aligned}$$

Now
$$P(m\epsilon + h)^{-1} P(m\epsilon) \in V \subset U$$

and
$$P(m\epsilon)^{-1} P(m\epsilon + h') \in V^{-1} \subset U.$$

Thus
$$P(m\epsilon + h)^{-1} P(m\epsilon) P(m\epsilon)^{-1} P(m\epsilon + h') \in V^{-1} V \subset U.$$

Hence
$$P'(m\epsilon + h)^{-1} P'(m\epsilon + h')^{-1} = f(P(m\epsilon + h)^{-1} P(m\epsilon + h'))$$

This shows that (b) is satisfied, and proves the existence of the path $P'$. Suppose now that $0 \leq t_1 < t_2 \leq 1$ and $|t_1 - t_2| \leq \epsilon$. Let P be subjected to a continuous deformation leaving fixed all points except those in the parameter range $t_1 < t < t_2$. Call such a deformation of P a small deformation. By (b), if $0 \leq t \leq t_1$, $P'(t)$ depends only on the behavious of $P(t)$ in the range $0 \leq t \leq t_1$. If $t = t_2$, then

$$P'(t_2) = P'(t_1)\, f(P(t_2)^{-1}\, P(t_1)) .$$

Thus $P'(t_2)$ is determined by $P'(t_1)$. If $1 \geq t \geq t_2$, $P'(t)$ depends only on $P'(t_2)$ and the behaviour of $P(t)$ in the range $1 \geq t \geq t_2$ by (b). Hence if P is subjected to a small deformation, the end points of $P'$ remain fixed. Suppose now that Q is another path in G from e to g. Let $W$ be a path in G from $Q(\epsilon/2)$ to $P(\epsilon)$ and define

$$\begin{aligned} F(s) &= Q(s) \qquad (0 \leq s \leq \epsilon/2) \\ &= W(\tfrac{2}{\epsilon}(s - \epsilon/2)) \qquad (\epsilon/2 \leq s \leq \epsilon) \\ &= P(s) \qquad (1 \geq s \geq \epsilon) . \end{aligned}$$

Then F is homotopic to P since G is simply connected, and F is obtained from P by a small deformation. Thus the corresponding paths $F'$ and $P'$ have the same endpoints. Also F coincides with Q on the interval $[0, \epsilon/2]$. Continuing in this manner, we can deform P to Q by a sequence of small deformations. Hence the paths $P'$ and $Q'$ have the same endpoints. It follows that the terminal point $g'$ of $P'$ depends only on g and not on the path P. In this way, to each element $g \in G$ we can associate a uniquely determined $g' \in G'$ defining a map $\varphi : G \to G'$ by $\varphi(g) = g'$.

We now show that there is a neighbourhood $\mathcal{W} \subset U$ of e on which $\varphi$ equals f. Let $\mathcal{W}$ be an arcwise connected neighbourhood of e with $\mathcal{W}^{-1}\mathcal{W} \subset U$. Let $g \in \mathcal{W}$, and P a path from e to g in $\mathcal{W}$. The path $P' = f \circ P$ in G′ satisfies (a) and (b) since

$$P'(e) = f(e) = e'$$

and if $P(t) \in \mathcal{W}$, then $P(t)^{-1} \in \mathcal{W}^{-1}$ so if $0 \leq t_1, t_2 \leq 1$, then

$$P(t_1)^{-1} P(t_2) \in \mathcal{W}^{-1}\mathcal{W} \subset U .$$

Hence
$$f(P(t_1)^{-1}P(t_2)) = f(P(t_1))^{-1}f(P(t_2))$$
$$= P'(t_1)^{-1} P'(t_2)$$

Thus, by uniqueness, the terminal point of P′ is $\varphi(g)$. But $P'(1) = f(P(1)) = f(g)$, so $\varphi = f$ on $\mathcal{W}$.

We prove finally that $\varphi$ is a homomorphism. Let $g, h \in G$, and P, Q be paths in G from e to g and e to h respectively. Denote by P′, Q′ the corresponding paths in G′. If $g' = P'(1)$, $h' = Q'(1)$, then $g' = \varphi(g)$, $h' = \varphi(h)$.

Let
$$R(t) = P(2t) \qquad (0 \leq t \leq \tfrac{1}{2})$$
$$= g\,Q(2t - 1) \qquad (\tfrac{1}{2} \leq t \leq 1)$$

Then R is a path from e to gh,

Let
$$R'(t) = P'(2t) \qquad (0 \leq t \leq \tfrac{1}{2})$$
$$= g'\,Q'(2t - 1) \qquad (\tfrac{1}{2} \leq t \leq 1)$$

R′ is a path in G′ from e′ to g′h′. We claim that R′ is the path in G′ corresponding to R satisfying conditions (a) and (b). Clearly $R'(0) = e'$. We have to show that (b) is satisfied.

If $0 \leq t_1 \leq t_2 \leq \frac{1}{2}$ and $t_1$ is close to $t_2$, then

$$R'(t_1)^{-1} R'(t_2) = P'(2t_1)^{-1} P'(2t_2)$$
$$= f(P(2t_1)^{-1} P(2t_2))$$
$$= f(R'(2t_1)^{-1} R'(2t_2))$$

Similarly if $\frac{1}{2} \leq t_1 \leq t_2 \leq 1$ and $t_1$ is close to $t_2$, then $2t_1 - 1$ is close to $2t_2 - 1$, so

$$R'(t_1)^{-1} R'(t_2) = Q'(2t_1 - 1)^{-1} Q'(2t_2 - 1)$$
$$= f(Q(2t_1 - 1)^{-1} Q(2t_2 - 1))$$
$$= f(R(t_1)^{-1} R(t_2)) .$$

Now if $0 \leq t_1 \leq \frac{1}{2} \leq t_2 \leq 1$ and $t_1$ and $t_2$ are close, then $t_1$ and $t_2$ are both close to $\frac{1}{2}$, so $2t_1$ is close to 1 and $2t_2 - 1$ is close to 0. Thus since $P(1) = g$ and $P'(1) = g'$ ,

$$P'(2t_1)^{-1}g' = f(P(2t_1)^{-1}g)$$

and, since $Q(0) = e$, $Q'(0) = e'$ ,

$$Q'(2t_2 - 1)^{-1} = f(Q(2t_2 - 1)^{-1})$$

Hence $\quad R'(t_1)^{-1}R'(t_2) = P'(2t_1)^{-1}g'Q'(2t_2 - 1)$ lies in U

if $t_1$, $t_2$ sufficiently close, and

$$R'(t_1)^{-1} R'(t_2) = P'(2t_1)^{-1}g'Q'(2t_2 - 1)$$
$$= f(P(2t_1)^{-1}g) f(Q(2t_2 - 1))$$
$$= f(P(2t_1)^{-1}g Q(2t_2 - 1))$$
$$= f(R(t_1)^{-1} R(t_2)) .$$

Hence $R'$ satisfies (a) and (b) relative to $R$. It follows that the terminal point of $R'$ is $\varphi(gh)$, so

$$\begin{aligned} \varphi(gh) &= R'(1) \\ &= g' Q'(1) \\ &= g'h' \\ &= \varphi(g)\,\varphi(h) \end{aligned}$$

showing that $\varphi$ is a homomorphism. This completes the proof.

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