Answers to exercises

Answer to Exercise 1.3.2, page 15

If $a_{-k} = -a_k$ in

$$f(x) = \sum_{k=-\infty}^{\infty} a_k \exp\left[i(2\pi)kx\right],$$

then

$$f(x) = a_0 + \sum_{k=1}^{\infty} a_k \left\{\exp\left[i(2\pi)kx\right] - \exp\left[-i(2\pi)kx\right]\right\} = a_0 + \sum_{k=1}^{\infty} 2ia_k \sin\left[(2\pi)kx\right].$$

In our case, we have $a_k = (-1)^k/[-i(2\pi)k]$, so $2ia_k = (-1)^{k+1}/(\pi k)$.

Answer to Exercise 1.3.3, page 16

We have $a_0 = \int_0^1 f(x)dx = 1/2$, and

$$\begin{aligned} a_k &= \int_0^1 f(x)\exp\left[i(2\pi)kx\right]dx = \int_{1/4}^{3/4} \exp\left[i(2\pi)kx\right]dx \\ &= \frac{\exp\left[i(2\pi)k3/4\right] - \exp\left[i(2\pi)k/4\right]}{(2\pi)k} \\ &= 2\cdot\frac{\exp\left[i(2\pi)k1/2\right]}{(2\pi)k}\cdot\frac{\exp\left[i(2\pi)k/4\right] - \exp\left[-i(2\pi)k/4\right]}{2i} \\ &= \frac{(-1)^k\sin(\pi k/2)}{\pi k}. \end{aligned}$$

As a result $a_{2k} = 0$ for $k \neq 0$, while

$$a_{2k-1} = \frac{(-1)\sin(k\pi - \pi/2)}{(2k-1)\pi} = \frac{(-1)^k}{(2k-1)\pi}.$$

Answer to Exercise 1.3.4, page 27

$$\widetilde{\mathbf{W}}_2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\widetilde{\mathbf{W}}_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\widetilde{\mathbf{W}}_0 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Answer to Exercise 1.3.5, page 28

The columns and rows of all matrices above are normalised by multiplication with $\sqrt{2}$ of the columns that are not canonical vectors, leading to

$$\widetilde{\mathbf{W}}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\widetilde{\mathbf{W}}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\widetilde{\mathbf{W}}_0 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Answer to Exercise 2.1.2, page 36
The matrix $\widetilde{\mathbf{J}}_j^\top$ has 5 rows and 9 columns

$$\widetilde{\mathbf{J}}_j^\top = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The matrix $\widetilde{\mathbf{J}}_j^{o\top}$ has 4 rows and 9 columns

$$\widetilde{\mathbf{J}}_j^{o\top} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Answer to Exercise 2.1.3, page 41
The rescaling matrix is a $5 \times 5 = n_j \times n_j$ diagonal matrix

$$\mathbf{D}_j = \frac{1}{2}\mathbf{I}_{n_j}.$$

The update matrix is a $5 \times 4 = n_j \times (n_{j+1} - n_j)$ matrix, given by

$$\mathbf{U}_j = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The prediction matrix is a $4 \times 5 = (n_{j+1} - n_j) \times n_j$ matrix, given by

$$\mathbf{P}_j = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Answer to Exercise 2.1.3, page 41
Switching the positions of the update and the rescaling leads to an equivalent scheme if the rescaling uses the same diagonal matrix $\mathbf{D}_j$ and if the update uses the matrix $\mathbf{U}'_j = \mathbf{D}_j^{-1}\mathbf{U}_j$. The nonzero elements of this matrix are

$$U'_{j;k,k} = \left(\frac{\Delta_{j+1,2k}}{\Delta_{j,k}}\right)^{-1} \frac{\Delta_{j+1,2k+1}}{\Delta_{j,k}} = \frac{\Delta_{j+1,2k+1}}{\Delta_{j+1,2k}}.$$

Moving the rescaling further to the end of the forward transform leads to a new prediction matrix $\mathbf{P}'_j = \mathbf{D}_j\mathbf{P}_j$, which amounts to

$$P'_{j;k,k} = \frac{\Delta_{j+1,2k}}{\Delta_{j,k}}$$

for the nonzero matrix entries.
Answer to Exercise 2.1.5, page 42
The prediction and update matrices have the same shape and the same elements as those in Exercise 2.1.3. The rescaling matrix is a $4 \times 4 = (n_{j+1} - n_j) \times (n_{j+1} - n_j)$ diagonal matrix

$$\mathbf{D}_j = \frac{1}{2}\mathbf{I}_{n_{j+1}-n_j}.$$

Answer to Exercise 2.1.6, page 47
The function $\varphi_{j,x}(x)$ takes the form $(x - x_{j,k-1})/(x_{j,k} - x_{j,k-1})$ on the interval $[x_{j,k-1}, x_{j,k}])$, and the form $(x_{j,k+1} - x)/(x_{j,k+1} - x_{j,k})$ on the interval $[x_{j,k}, x_{j,k+1}])$, and it is zero elsewhere. The first moment is the integral of the function, which is the area under a triangle,

$$M_{j,k}^{[0]} = (x_{j,k+1} - x_{j,k-1})/2.$$

The second moment is given by

$$
\begin{aligned}
M_{j,k}^{[1]} &= \int_{x_{j,k-1}}^{x_{j,k}} \frac{u - x_{j,k-1}}{x_{j,k} - x_{j,k-1}} u du + \int_{x_{j,k}}^{x_{j,k+1}} \frac{x_{j,k+1} - u}{x_{j,k+1} - x_{j,k}} u du \\
&= \frac{x_{j,k}^3 - x_{j,k-1}^3}{3(x_{j,k} - x_{j,k-1})} - x_{j,k-1} \frac{(x_{j,k}^2 - x_{j,k-1}^2)}{2(x_{j,k} - x_{j,k-1})} \\
&\quad - \frac{x_{j,k+1}^3 - x_{j,k}^3}{3(x_{j,k+1} - x_{j,k})} + x_{j,k+1} \frac{(x_{j,k+1}^2 - x_{j,k}^2)}{2(x_{j,k+1} - x_{j,k})} \\
&= \frac{x_{j,k}^2 + x_{j,k}x_{j,k-1} + x_{j,k-1}^2}{3} - x_{j,k-1} \frac{(x_{j,k} + x_{j,k-1})}{2} \\
&\quad - \frac{x_{j,k}^2 + x_{j,k+1}x_{j,k} + x_{j,k+1}^2}{3} + x_{j,k+1} \frac{(x_{j,k+1} + x_{j,k})}{2} \\
&= \frac{(x_{j,k} - x_{j,k-1})^2}{3} + x_{j,k}x_{j,k-1} - x_{j,k-1} \frac{(x_{j,k} + x_{j,k-1})}{2} \\
&\quad - \frac{(x_{j,k+1} - x_{j,k})^2}{3} - x_{j,k+1}x_{j,k} + x_{j,k+1} \frac{(x_{j,k+1} + x_{j,k})}{2} \\
&= \frac{(x_{j,k} - x_{j,k-1})^2}{3} + x_{j,k-1} \frac{(x_{j,k} - x_{j,k-1})}{2} \\
&\quad - \frac{(x_{j,k+1} - x_{j,k})^2}{3} + x_{j,k+1} \frac{(x_{j,k+1} - x_{j,k})}{2} \\
&= (x_{j,k} - x_{j,k-1})(x_{j,k}/3 + x_{j,k-1}/6) + (x_{j,k+1} - x_{j,k})(x_{j,k}/3 + x_{j,k+1}/6) \\
&= (x_{j,k+1} - x_{j,k-1}) \frac{x_{j,k}}{3} - x_{j,k}(x_{j,k+1} - x_{j,k-1})/6 + (x_{j,k+1}^2 - x_{j,k-1}^2)/6 \\
&= (x_{j,k+1} - x_{j,k-1})(x_{j,k-1} + x_{j,k} + x_{j,k+1})/6
\end{aligned}
$$

Answer to Exercise 2.1.7, page 47

Let $\Delta_j = x_{j,k} - x_{j,k-1}$ be the interknot distance at level j. The prediction coefficients are given by (2.15), which becomes:

$$
\begin{aligned}
P_{j;k,k} &= \frac{\Delta_{j+1}}{\Delta_{j+1} + \Delta_{j+1}} = \frac{1}{2} \\
P_{j;k,k+1} &= \frac{\Delta_{j+1}}{\Delta_{j+1} + \Delta_{j+1}} = \frac{1}{2}
\end{aligned}
$$

According to Exercise 2.1.6, the moments are given by $M_{j,k}^{[0]} = \Delta_j$ and $M_{j,k}^{[1]} = \Delta_j x_{j,k}$. The moment conditions become

$$
\begin{aligned}
\Delta_{j+1} &= U_{j,k,k}\Delta_j + U_{j,k+1,k}\Delta_j \\
\Delta_{j+1}x_{j+1,2k+1} &= U_{j,k,k}\Delta_j x_{j,k} + U_{j,k+1,k}\Delta_j x_{j,k+1}.
\end{aligned}
$$

Using the fact that $\Delta_j = 2\Delta_{j+1}$, this is

$$
\begin{aligned}
1 &= 2U_{j,k,k} + 2U_{j,k+1,k} \\
x_{j+1,2k+1} &= 2U_{j,k,k}x_{j,k} + 2U_{j,k+1,k}x_{j,k+1}.
\end{aligned}
$$

Now we use that $x_{j,k} = x_{j+1,2k} = x_{j+1,2k+1} - \Delta_{j+1}$ and $x_{j,k+1} = x_{j+1,2k+1} + \Delta_{j+1}$:

$$\begin{aligned} 1 &= 2U_{j,k,k} + 2U_{j,k+1,k} \\ x_{j+1,2k+1} &= 2U_{j,k,k}(x_{j+1,2k+1} - \Delta_{j+1}) + 2U_{j,k+1,k}(x_{j+1,2k+1} + \Delta_{j+1}). \end{aligned}$$

Subtract $x_{j+1,2k+1}$ times the first equation from the second equation to get

$$\begin{aligned} 1 &= 2U_{j,k,k} + 2U_{j,k+1,k} \\ 0 &= -U_{j,k,k} + U_{j,k+1,k} \end{aligned}$$

from which it follows that $U_{j,k,k} = U_{j,k+1,k} = 1/4$.

Answer to Exercise 2.2.1, page 54

$$\widetilde{\mathbf{H}}_2 = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \text{ and } \widetilde{\mathbf{G}}_2 = \begin{bmatrix} -\frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$\widetilde{\mathbf{H}}_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix} \text{ and } \widetilde{\mathbf{G}}_1 = \begin{bmatrix} -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$\widetilde{\mathbf{H}}_0 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \text{ and } \widetilde{\mathbf{G}}_0 = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

Answer to Exercise 2.2.2, page 55

From (2.15) it follows immediately that $\widetilde{G}_{j;2k+1,k} = 1$, $\widetilde{G}_{j;2k,k} = -P_{j;k,k}$, and $\widetilde{G}_{j;2k+2,k} = -P_{j;k,k+1}$. This corresponds to $\widetilde{\mathbf{G}}_{j,o}^\top = \mathbf{I}$. and $\widetilde{\mathbf{G}}_{j,e}^\top = -\mathbf{P}_j$.

Next, from substitution of (2.15) into (2.16), we see that

$$\begin{aligned} s_{j,k} &= s_{j+1,2k} + U_{j,k,k}d_{j,k} + U_{j,k,k-1}d_{j,k-1} \\ &= s_{j+1,2k} + U_{j,k,k}[s_{j+1,2k+1} - (P_{j;k,k}s_{j+1,2k} + P_{j;k,k+1}s_{j+1,2k+2})] \\ &\quad + U_{j,k,k-1}[s_{j+1,2k-1} - (P_{j;k-1,k-1}s_{j+1,2k-2} + P_{j;k-1,k}s_{j+1,2k})] \\ &= -U_{j,k,k-1}P_{j;k-1,k-1}s_{j+1,2k-2} + U_{j,k,k-1}s_{j+1,2k-1}(1 - U_{j,k,k}P_{j;k,k} - U_{j,k,k-1}P_{j;k-1,k})s_{j+1,2k} \\ &\quad + U_{j,k,k}s_{j+1,2k+1} - U_{j,k,k}P_{j;k,k+1}s_{j+1,2k+2}, \end{aligned}$$

meaning that

$$\begin{aligned} \widetilde{H}_{j;2k-2,k} &= -U_{j,k,k-1}P_{j;k-1,k-1} \\ \widetilde{H}_{j;2k-1,k} &= U_{j,k,k-1} \\ \widetilde{H}_{j;2k,k} &= (1 - U_{j,k,k}P_{j;k,k} - U_{j,k,k-1}P_{j;k-1,k}) \\ \widetilde{H}_{j;2k+1,k} &= U_{j,k,k} \\ \widetilde{H}_{j;2k+2,k} &= -U_{j,k,k}P_{j;k,k+1}. \end{aligned}$$

This corresponds to $\widetilde{\mathbf{H}}_{j,e}^\top = \mathbf{I} - \mathbf{U}_j\mathbf{P}_j$ and $\widetilde{\mathbf{H}}_{j,o}^\top = \mathbf{U}_j$.

Answer to Exercise 2.2.3, page 56
From (2.55) we find $\mathbf{P}_0 = -\widetilde{\mathbf{G}}_0^\top \widetilde{\mathbf{J}}_0 = 1/2$ while (2.56) becomes $\mathbf{U}_0 = 1/2$. This, however, leads to $\widetilde{\mathbf{G}}_{0,o}^\top = \mathbf{I} = 1$ and $\widetilde{\mathbf{G}}_{0,e}^\top = -\mathbf{P}_0 = -1/2$.

When $\widetilde{\mathbf{G}}_0^\top = [\begin{array}{cc} -1 & 1 \end{array}]$ and $\widetilde{\mathbf{H}}_0^\top = [\begin{array}{cc} 1 & 1 \end{array}]/2$, we find $\mathbf{P}_0 = 1$ and $\mathbf{U}_0 = 1/2$, which leads back to the same $\widetilde{\mathbf{G}}_0^\top$ and $\widetilde{\mathbf{H}}_0^\top$.

Answer to Exercise 2.2.4, page 56
As in Exercise 2.2.3, we find from (2.55) that

$$\mathbf{P}_1 = -\widetilde{\mathbf{G}}_1^\top \widetilde{\mathbf{J}}_1 = -\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

From (2.56) it follows that

$$\mathbf{U}_1 = \widetilde{\mathbf{H}}_1^\top \widetilde{\mathbf{J}}_1^o = \tfrac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

From there we find

$$\begin{aligned}
\mathbf{H}_1 &= \widetilde{\mathbf{J}}_1 + \widetilde{\mathbf{J}}_1^o \mathbf{P}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \\
\mathbf{G}_1 &= \widetilde{\mathbf{J}}_1^o(\mathbf{I} - \mathbf{P}_1\mathbf{U}_1) - \widetilde{\mathbf{J}}_1\mathbf{U}_1 \\
&= \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) - \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}.
\end{aligned}$$

Answer to Exercise 2.2.5, page 56
For the reconstruction from the update-first lifting scheme, identification of $\boldsymbol{s}_{j+1,o} = \boldsymbol{d}_j + \mathbf{P}_j \boldsymbol{s}_j$, and $\boldsymbol{s}_{j+1,e} = (\mathbf{I} - \mathbf{U}_j\mathbf{P}_j)\boldsymbol{s}_j - \mathbf{U}_j\boldsymbol{d}_j$ with (2.46), i.e., $\boldsymbol{s}_{j+1,o} = \mathbf{H}_{j,o}\boldsymbol{s}_j + \mathbf{G}_{j,o}\boldsymbol{d}_j$, and $\boldsymbol{s}_{j+1,e} = \mathbf{H}_{j,e}\boldsymbol{s}_j + \mathbf{G}_{j,e}\boldsymbol{d}_j$ yields $\mathbf{H}_{j,o} = \mathbf{P}_j$; $\mathbf{G}_{j,o} = \mathbf{I}$; $\mathbf{H}_{j,e} = \mathbf{I} - \mathbf{U}_j\mathbf{P}_j$; and $\mathbf{G}_{j,e} = \mathbf{U}_j$. Using the identity $\widetilde{\mathbf{J}}_j\widetilde{\mathbf{J}}_j^\top + \widetilde{\mathbf{J}}_j^o\widetilde{\mathbf{J}}_j^{o\top} = \mathbf{I}_{n_{j+1}}$, it follows that

$$\boldsymbol{s}_{j+1} = \widetilde{\mathbf{J}}_j \boldsymbol{s}_{j+1,e} + \widetilde{\mathbf{J}}_j^o \boldsymbol{s}_{j+1,o} = \left[\widetilde{\mathbf{J}}_j(\mathbf{I} - \mathbf{U}_j\mathbf{P}_j) + \widetilde{\mathbf{J}}_j^o\mathbf{P}_j\right]\boldsymbol{s}_j + \left[\widetilde{\mathbf{J}}_j^o - \widetilde{\mathbf{J}}_j\mathbf{U}_j\right]\boldsymbol{d}_j,$$

meaning that

$$\mathbf{H}_j = \widetilde{\mathbf{J}}_j(\mathbf{I} - \mathbf{U}_j\mathbf{P}_j) + \widetilde{\mathbf{J}}_j^o\mathbf{P}_j$$

and

$$\mathbf{G}_j = \widetilde{\mathbf{J}}_j^o - \widetilde{\mathbf{J}}_j\mathbf{U}_j.$$

The forward transform matrices follow from (2.35)

$$\boldsymbol{s}_j = \boldsymbol{s}_{j+1,e} + \mathbf{U}_j \boldsymbol{s}_{j+1,o} = \left(\widetilde{\mathbf{J}}_j^\top + \mathbf{U}_j \widetilde{\mathbf{J}}_j^{o\top}\right) \boldsymbol{s}_{j+1},$$

meaning that

$$\widetilde{\mathbf{H}}_j^\top = \widetilde{\mathbf{J}}_j^\top + \mathbf{U}_j \widetilde{\mathbf{J}}_j^{o\top},$$

and from (2.36)

$$\boldsymbol{d}_j = \boldsymbol{s}_{j+1,o} - \mathbf{P}_j \boldsymbol{s}_j = \left[\widetilde{\mathbf{J}}_j^{o\top} - \mathbf{P}_j \left(\widetilde{\mathbf{J}}_j^\top + \mathbf{U}_j \widetilde{\mathbf{J}}_j^{o\top}\right)\right] \boldsymbol{s}_{j+1},$$

meaning that

$$\widetilde{\mathbf{G}}_j^\top = (\mathbf{I} - \mathbf{P}_j \mathbf{U}_j) \widetilde{\mathbf{J}}_j^{o\top} - \mathbf{P}_j \widetilde{\mathbf{J}}_j^\top.$$

Answer to Exercise 2.2.6, page 58
The dual matrices are found by

$$\begin{bmatrix} \widetilde{\mathbf{H}}_j^\top \\ \widetilde{\mathbf{G}}_j^\top \end{bmatrix} = [\mathbf{H}_j \; \mathbf{G}_j]^{-1}.$$

Answer to Exercise 2.2.7, page 58
The perfect reconstruction condition (2.61) is also satisfied when replacing $\mathbf{G}_j$ by $\mathbf{G}_j^{[1]}$. The dual detail matrix should satisfy (2.63), which becomes

$$\widetilde{\mathbf{G}}_j^{[1]\top} \mathbf{G}_j^{[1]} = \mathbf{I}_{j+1,o} \Leftrightarrow \widetilde{\mathbf{G}}_j^{[1]\top} \mathbf{G}_j \mathbf{A}_j = \mathbf{I}_{j+1,o}.$$

This is fulfilled if

$$\widetilde{\mathbf{G}}_j^{[1]\top} = \mathbf{A}_j^{-1} \widetilde{\mathbf{G}}_j^\top \Leftrightarrow \widetilde{\mathbf{G}}_j^{[1]} = \widetilde{\mathbf{G}}_j \mathbf{A}_j^{-\top}.$$

Answer to Exercise 2.2.8, page 58
It is straightforward to check that

$$\begin{aligned} \varphi_{j,k} &= \chi_{[x_{j,k}, x_{j,k+3})}(x) = \chi_{[x_{j+1,2k}, x_{j+1,2k+6})}(x) \\ &= \chi_{[x_{j+1,2k}, x_{j+1,2k+3})}(x) + \chi_{[x_{j+1,2k+3}, x_{j+1,2k+6})}(x) = \varphi_{j+1,2k}(x) + \varphi_{j+1,2k+3}(x). \end{aligned}$$

Answer to Exercise 2.2.9, page 58
This amounts to taking a different basis of the same detail space. The new basis is a linear combination of the basis $\Psi_j(x)$, as indeed

$$\Psi_j(x)^{[1]} = \Phi_{j+1}(x) \mathbf{G}_j^{[1]} = \Phi_{j+1}(x) \mathbf{G}_j \mathbf{A}_j = \Psi_j(x) \mathbf{A}_j.$$

Answer to Exercise 2.2.10, page 59
We have $\boldsymbol{\varphi}_0 = 1$, $\boldsymbol{\varphi}_1 = (1, 0, 0, 1)$, $\boldsymbol{\varphi}_2 = (1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1)$, and so on. The vector does not converge to a vector of function values of $\varphi_{0,0}(x_{j,k})$.

Answer to Exercise 2.2.11, page 59

1. The forward dual transform has matrices $\mathbf{H}_j^\top$ and $\mathbf{G}_j^\top$ in the filter-bank. Together, this is

$$\begin{bmatrix} \mathbf{H}_j^\top \\ \mathbf{G}_j^\top \end{bmatrix} = \mathbf{W}_j^\top = \widetilde{\mathbf{W}}_j^{-\top},$$

and similarly we find that the inverse of the dual transform is given by the matrices $\widetilde{\mathbf{W}}_j^\top = \mathbf{W}_j^{-\top}$. As a conclusion, the dual transform matrices are given by the inverse transposes of the primal transform.

2. The forward dual transform has matrices $\mathbf{H}_j^\top$ and $\mathbf{G}_j^\top$. In a prediction first scheme, these matrices follow from (2.57):

$$\begin{aligned} \mathbf{H}_j^\top &= \widetilde{\mathbf{J}}_j^\top + \mathbf{P}_j^\top \widetilde{\mathbf{J}}_j^{o\top} \\ \mathbf{G}_j^\top &= (\mathbf{I} - \mathbf{U}_j^\top \mathbf{P}_j^\top)\widetilde{\mathbf{J}}_j^{o\top} - \mathbf{U}_j^\top \widetilde{\mathbf{J}}_j^\top. \end{aligned}$$

Careful comparison of the results above with the expressions from Exercise 2.2.5 reveals that the dual transform of a prediction first lifting scheme is an update first scheme with update $\mathbf{P}_j^\top$ and prediction $\mathbf{U}_j^\top$.

Answer to Exercise 2.3.1, page 62

We proceed by top-down induction, for $j = J^*, J^* - 1, \ldots$, that is. Assume that

$$\int_0^1 \Phi_{j+1}^\top(x)\widetilde{\Phi}_{j+1}(x)dx = \mathbf{I}_{j+1} = \int_0^1 \widetilde{\Phi}_{j+1}^\top(x)\Phi_{j+1}(x)dx,$$

then we know that $\Phi_j(x) = \Phi_{j+1}(x)\mathbf{H}_j$ and $\widetilde{\Phi}_j(x) = \widetilde{\Phi}_{j+1}(x)\widetilde{\mathbf{H}}_j$. As a result

$$\int_0^1 \Phi_j(x)^\top \widetilde{\Phi}_j(x)dx = \int_0^1 \mathbf{H}_j^\top \Phi_{j+1}^\top(x)\widetilde{\Phi}_{j+1}(x)\widetilde{\mathbf{H}}_j dx = \mathbf{H}_j^\top \mathbf{I}_{j+1}\widetilde{\mathbf{H}}_j = \mathbf{I}_j,$$

because of the perfect reconstruction in (2.60). The other results follow in a similar way.

Answer to Exercise 2.3.3, page 64

Construct a tridiagonal update for three primal vanishing moments in a Haar detail basis, i.e., using a Haar scaling basis, find $\mathbf{G}_j$ through a tridiagonal update so that the wavelet basis $\Psi_j(x)$ has three vanishing moments. Then switch the roles of primal and dual bases by taking the dual wavelet transform as above.

Answer to Exercise 3.1.1, page 68

The functions in $\Psi_L(x)$ do not overlap, so

$$\int_0^1 \Psi_L(x)^\top \Psi_L(x)dx = \mathbf{W}_L,$$

where $\mathbf{W}_L$ is a diagonal matrix with elements

$$\begin{aligned}
W_{L;k,k} &= \int_0^1 [\varphi_{L+1,2k+1}(x)]^2\,dx \\
&= \int_{x_{L+1,2k}}^{x_{L+1,2k+1}} \left(\frac{x - x_{L+1,2k}}{x_{L+1,2k+1} - x_{L+1,2k}}\right)^2 dx \\
&\quad + \int_{x_{L+1,2k+1}}^{x_{L+1,2k+2}} \left(\frac{x_{L+1,2k+2} - x}{x_{L+1,2k+2} - x_{L+1,2k+1}}\right)^2 dx \\
&= (x_{L+1,2k+2} - x_{L+1,2k})/3.
\end{aligned}$$

The matrix defined by

$$\mathbf{V}_L = \int_0^1 \Psi_L(x)^\top \Phi_L(x) dx$$

is bidiagonal, as indeed

$$V_{L;k,l} = \int_0^1 \psi_{L,k}(x)\varphi_{L,l}(x)dx = \int_0^1 \varphi_{L+1,2k+1}(x)\varphi_{L,l}(x)dx$$

which is nonzero if $l \in \{k, k+1\}$. We have

$$\begin{aligned}
V_{L;k,k} &= \int_{x_{L+1,2k}}^{x_{L+1,2k+1}} \frac{x - x_{L+1,2k}}{x_{L+1,2k+1} - x_{L+1,2k}} \cdot \frac{x_{L+1,2k+2} - x}{x_{L+1,2k+2} - x_{L+1,2k}} dx \\
&\quad + \int_{x_{L+1,2k+1}}^{x_{L+1,2k+2}} \frac{x_{L+1,2k+2} - x}{x_{L+1,2k+2} - x_{L+1,2k+1}} \cdot \frac{x_{L+1,2k+2} - x}{x_{L+1,2k+2} - x_{L+1,2k}} dx \\
&= \frac{1}{6}(2x_{L+1,2k+2} - x_{L+1,2k+1} - x_{L+1,2k})
\end{aligned}$$

and

$$\begin{aligned}
V_{L;k,k+1} &= \int_{x_{L+1,2k}}^{x_{L+1,2k+1}} \frac{x - x_{L+1,2k}}{x_{L+1,2k+1} - x_{L+1,2k}} \cdot \frac{x - x_{L+1,2k+2}}{x_{L+1,2k+2} - x_{L+1,2k}} dx \\
&\quad + \int_{x_{L+1,2k+1}}^{x_{L+1,2k+2}} \frac{x_{L+1,2k+2} - x}{x_{L+1,2k+2} - x_{L+1,2k+1}} \cdot \frac{x - x_{L+1,2k+2}}{x_{L+1,2k+2} - x_{L+1,2k}} dx \\
&= \frac{1}{6}(x_{L+1,2k+2} + x_{L+1,2k+1} - 2x_{L+1,2k}).
\end{aligned}$$

The detail coefficients are given by

$$d_{L,k} = -\frac{V_{L;k,k}}{W_{L;k,k}} s_{L,k} - \frac{V_{L;k,k+1}}{W_{L;k,k}} s_{L,k+1}.$$

Answer to Exercise 3.1.2, page 69

1. This is a straightforward rewriting of the refinement equation in (2.28):

$$\begin{aligned}\Phi_j(x) &= \Phi_{j+1,e}(x) + \Phi_{j+1,o}(x)\mathbf{P}_j \\ &= \Phi_{j+1}(x)\widetilde{\mathbf{J}}_j + \Phi_{j+1}(x)\widetilde{\mathbf{J}}_j^o\mathbf{P}_j \\ &= \Phi_{j+1}(x)\left[\widetilde{\mathbf{J}}_j + \widetilde{\mathbf{J}}_j^o\mathbf{P}_j\right].\end{aligned}$$

2. With $\mathbf{G}_j = \widetilde{\mathbf{J}}_j^o\mathbf{A}_j$ we have $\Psi_j(x) = \Phi_{j+1}(x)\widetilde{\mathbf{J}}_j^o\mathbf{A}_j = \Phi_{j+1,o}\mathbf{A}_j$. The qth moments, as defined in (2.30) and (2.31) are then given by $\boldsymbol{O}_j^{(q)} = \mathbf{A}_j^\top\boldsymbol{M}_{j+1,o}^{(q)}$. With $\mathbf{A}_j$ a non-singular matrix, the only vector $\boldsymbol{M}_{j+1,o}^{(q)}$ in the null space is the zero vector.

   As the multiplication with $\mathbf{A}_j$ amounts to a basis transform, all decompositions in the oiginal hierarchical basis, including the nontrivial decompositions of the zero function, transform into decompositions in the transformed basis.

Answer to Exercise 3.1.3, page 71

The two-scale equation in (2.28) does not depend on $\mathbf{U}_j$. This is confirmed by the Expression (2.57) from which $\mathbf{H}_j$ can be identified to be $\mathbf{H}_j = \widetilde{\mathbf{J}}_j + \widetilde{\mathbf{J}}_j^o\mathbf{P}_j$. When the update comes before the prediction in the forward transform, the refinement becomes (2.39), which does depend on the update.

Answer to Exercise 3.1.4, page 71

We have $\mathrm{var}(s_{j,k}) = (1-u)^2 + 2u^2 - 2(u/2)^2$, which takes a minimum at $u = 2/7$. The value is different from the $u = 1/4$ for vanishing moment updates. A compromise with just one vanishing moment is impossible, because, on an equidistant grid, the first vanishing moment implies the second, for reasons of symmetry.

Answer to Exercise 3.1.6, page 75

Let $j = J-1 = L$, then $\boldsymbol{d}_j = \mathbf{0}$, so $\widehat{s}_{j+1,2k} = s_{j,k} = (1-u)s_{j+1,2k} + us_{j+1,2k-1} + us_{j+1,2k+1} - (u/2)s_{j+1,2k-2} - (u/2)s_{j+1,2k+2}$, while

$$\begin{aligned}\widehat{s}_{j+1,2k+1} &= \frac{1}{2}(\widehat{s}_{j+1,2k} + \widehat{s}_{j+1,2k+2}). \\ &= \frac{1}{2}[(1-3u/2)s_{j+1,2k} + (1-3u/2)s_{j+1,2k+2} \\ &\quad +us_{j+1,2k-1} + 2us_{j+1,2k+1} + us_{j+1,2k+3} \\ &\quad -(u/2)s_{j+1,2k-2} - (u/2)s_{j+1,2k+4}].\end{aligned}$$

The variance is then

$$\mathrm{var}(\widehat{s}_{j+1,2k+1}) = [2(1-3u/2)^2 + (u^2+4u^2+u^2) + 2(u/2)^2]/4.$$

This is minimised at $u = 3/11$.

Answer to Exercise 3.1.7, page 75

The multiscale variance propagation depends on the projections onto the coarse scaling spaces, not on the detail offsets, so not on the choice of $\mathbf{G}_j^{[1]} = \mathbf{G}_j\mathbf{A}_j$.

Answer to Exercise 3.1.12, page 80
The proof inserts the transpose of the perfect reconstruction of (2.58),

$$\widetilde{\mathbf{H}}_j\mathbf{H}_j^\top + \widetilde{\mathbf{G}}_j\mathbf{G}_j^\top = \mathbf{I},$$

leading to

$$\begin{aligned}\Psi_j(x) &= \Phi_{j+1}(x)\mathbf{G}_j = \widetilde{\Phi}_{j+1}(x)\mathbf{\Pi}_{j+1}\mathbf{G}_j \\ &= \widetilde{\Phi}_{j+1}(x)(\widetilde{\mathbf{H}}_j\mathbf{H}_j^\top + \widetilde{\mathbf{G}}_j\mathbf{G}_j^\top)\mathbf{\Pi}_{j+1}\mathbf{G}_j \\ &= \widetilde{\Phi}_{j+1}(x)\widetilde{\mathbf{H}}_j \cdot \mathbf{H}_j^\top\mathbf{\Pi}_{j+1}\mathbf{G}_j + \widetilde{\Phi}_{j+1}(x)\widetilde{\mathbf{G}}_j \cdot \mathbf{G}_j^\top\mathbf{\Pi}_{j+1}\mathbf{G}_j \\ &= \widetilde{\Phi}_j(x)\mathbf{\Upsilon}_j + \widetilde{\Psi}_j(x)\mathbf{\Xi}_j = \mathbf{0} + \widetilde{\Psi}_j(x)\mathbf{\Xi}_j.\end{aligned}$$

Answer to Exercise 3.2.3, page 90
Since the scaling basis is interpolating, the associated wavelet transform consists of one prediction step and one update step. Hence, following (2.23), $\boldsymbol{s}_j = \boldsymbol{s}_{j+1,e} + \mathbf{U}_j\boldsymbol{d}_j$. If $\boldsymbol{d}_j = \mathbf{0}$ or if $\mathbf{U}_j$ is the zero matrix, then $\boldsymbol{s}_j = \boldsymbol{s}_{j+1,e} = \boldsymbol{f}_{j+1,e} = \boldsymbol{f}_j$.
Answer to Exercise 4.1.1, page 101
Since

$$[\mathbf{H}_j\ \mathbf{G}_j] = [\mathbf{H}_j^{[0]}\ \mathbf{G}_j^{[0]}]\begin{bmatrix}\mathbf{I}_{j+1,e} & -\mathbf{U}_j \\ \mathbf{0}_{j+1,o,e} & \mathbf{I}_{j+1,o}\end{bmatrix},$$

we have

$$\begin{aligned}[\widetilde{\mathbf{H}}_j\ \widetilde{\mathbf{G}}_j] &= \begin{bmatrix}\widetilde{\mathbf{H}}_j^\top \\ \widetilde{\mathbf{G}}_j^\top\end{bmatrix}^\top = [\mathbf{H}_j\ \mathbf{G}_j]^{-\top} \\ &= [\mathbf{H}_j^{[0]}\ \mathbf{G}_j^{[0]}]^{-\top}\begin{bmatrix}\mathbf{I}_{j+1,e} & -\mathbf{U}_j \\ \mathbf{0}_{j+1,o,e} & \mathbf{I}_{j+1,o}\end{bmatrix}^{-\top} \\ &= \begin{bmatrix}\widetilde{\mathbf{H}}_j^{[0]\top} \\ \widetilde{\mathbf{G}}_j^{[0]\top}\end{bmatrix}^\top\begin{bmatrix}\mathbf{I}_{j+1,e} & +\mathbf{U}_j \\ \mathbf{0}_{j+1,o,e} & \mathbf{I}_{j+1,o}\end{bmatrix}^\top \\ &= [\widetilde{\mathbf{H}}_j^{[0]}\ \widetilde{\mathbf{G}}_j^{[0]}]\begin{bmatrix}\mathbf{I}_{j+1,e} & \mathbf{0}_{j+1,e,o} \\ \mathbf{U}_j^\top & \mathbf{I}_{j+1,o}\end{bmatrix}\end{aligned}$$

We used the facts that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ and $(\mathbf{AB})^\top = \mathbf{B}^\top\mathbf{A}^\top$, so $(\mathbf{AB})^{-\top} = \mathbf{A}^{-\top}\mathbf{B}^{-\top}$.
The dual wavelet transform is lifted by a prediction at the coarse scale side, using $\mathbf{U}_j^\top$ as prediction matrix in

$$\widetilde{\mathbf{H}}_j = \widetilde{\mathbf{H}}_j^{[0]} + \widetilde{\mathbf{G}}_j^{[0]}\mathbf{U}_j^\top.$$

This is the same form as (4.14).
Answer to Exercise 4.1.8, page 108
The dual refinement in a semi-orthogonal wavelet transform is given by (3.6) or (3.12). These dual refinements satisfy the perfect reconstruction (2.60). From (4.35), it then follows that $\mathbf{U}_j = \widetilde{\mathbf{J}}_j^\top\mathbf{G}_j^{[0]}$.

Answer to Exercise 4.1.9, page 108
The perfect reconstruction (2.60) is satisfied, as

$$\widetilde{\mathbf{H}}_j^\top \mathbf{H}_j = \widetilde{\mathbf{J}}_j^\top (\widetilde{\mathbf{J}}_j + \widetilde{\mathbf{J}}_j^o \mathbf{P}_j) = \mathbf{I}_{j+1,e}.$$

The choice $\widetilde{\mathbf{H}}_j = \widetilde{\mathbf{J}}_j$ corresponds to a hierarchical, interpolating basis, see Exercise 3.1.2. From (4.35), it then follows that $\mathbf{U}_j = \widetilde{\mathbf{J}}_j^\top \mathbf{G}_j^{[0]}$, meaning that the final update is one half of the detail matrix accumulated in the process of factoring the refinement.
Answer to Exercise 4.1.10, page 108
Let $\widetilde{\mathbf{H}}_j$ and $\widetilde{\mathbf{G}}_j^{[1]}$ be the dual transform matrices of the given multiscale decomposition, defined by

$$[\mathbf{H}_j \, \mathbf{G}_j^{[1]}] \left[\begin{array}{c} \widetilde{\mathbf{H}}_j^\top \\ \widetilde{\mathbf{G}}_j^{[1]\top} \end{array} \right] = \mathbf{I}_{j+1}.$$

With $\mathbf{G}_j^{[0]}$ the detail matrix that follows from the factoring in (4.30), fix $\mathbf{U}_j = \widetilde{\mathbf{H}}_j^\top \mathbf{G}_j^{[0]}$ as in (4.35) and $\mathbf{G}_j = \mathbf{G}_j^{[0]} - \mathbf{H}_j \mathbf{U}_j$. Then

$$\mathbf{H}_j \widetilde{\mathbf{H}}_j^\top + \mathbf{G}_j \widetilde{\mathbf{G}}_j^\top = \mathbf{I}_{j+1} = \mathbf{H}_j \widetilde{\mathbf{H}}_j^\top + \mathbf{G}_j^{[1]} \widetilde{\mathbf{G}}_j^{[1]\top} \Rightarrow \mathbf{G}_j \widetilde{\mathbf{G}}_j^\top = \mathbf{G}_j^{[1]} \widetilde{\mathbf{G}}_j^{[1]\top}.$$

The latter equality can be right multiplied by $\mathbf{G}_j$ and $\mathbf{G}_j^{[1]}$. Perfect reconstruction in the multiscale transform induced by the factoring given multiscale transform then reads $\widetilde{\mathbf{G}}_j^{[1]\top} \mathbf{G}_j^{[1]} = \mathbf{I}_j$, leading to $\mathbf{G}_j^{[1]} = \mathbf{G}_j (\widetilde{\mathbf{G}}_j^\top \mathbf{G}_j^{[1]})$, while perfect reconstruction in the given multiscale transform leads to $\mathbf{G}_j = \mathbf{G}_j^{[1]} (\widetilde{\mathbf{G}}_j^{[1]\top} \mathbf{G}_j)$. This can be understood as $\mathbf{G}_j^{[1]} = \mathbf{G}_j \mathbf{A}_j$ and $\mathbf{G}_j = \mathbf{G}_j^{[1]} \mathbf{A}_j^{-1}$, where $\mathbf{A}_j = \widetilde{\mathbf{G}}_j^\top \mathbf{G}_j^{[1]}$ and $\mathbf{A}_j^{-1} = \widetilde{\mathbf{G}}_j^{[1]\top} \mathbf{G}_j$. The forward transform with $\widetilde{\mathbf{H}}_j$ and $\widetilde{\mathbf{G}}_j^{[1]}$ is realised by the lifting factoring for $\widetilde{\mathbf{H}}_j$ and $\widetilde{\mathbf{G}}_j$ (including the final update), followed by a multiplication with $\mathbf{A}_j^{-1}$ on the detail branch, as in Exercise 2.2.7.

The example of hierarchical bases has $\mathbf{G}_j^{[1]} = \widetilde{\mathbf{J}}_j^o$. The even rows of $\mathbf{G}_j = \mathbf{G}_j^{[1]} \mathbf{A}_j^{-1}$ will therefore contain zero elements.
Answer to Exercise 4.2.7, page 111
We have $\widetilde{p}$ intervals on which $s_{j,k}(x)$ is a piecewise polynomial. That means that we have $\widetilde{p}^2$ coefficients $a_{l,q}$: $l \in \{k - \lfloor \widetilde{p}/2 \rfloor, \ldots, k + \lceil \widetilde{p}/2 \rceil\}$ and $q \in \{0, 1, \ldots, \widetilde{p} - 1$. We have $\widetilde{p} + 1$ knots with $\widetilde{p} - 1$ continuity conditions, stating that left and right limits of the qth derivative at each knot are the same, for $q = 0, 1, \ldots, \widetilde{p} - 2$. All these consitions lead to a homogeneous system of $(\widetilde{p} + 1)(\widetilde{p} - 1) = \widetilde{p}^2 - 1$ independent linear equations. In the leftmost and rightmost knots, this means that the qth derivatives are zero. In the leftmost interval, we find that $s_{j,k}(x) = a_{k - \lfloor \widetilde{p}/2 \rfloor, \widetilde{p} - 1} x^{\widetilde{p} - 1}$. Putting $a_{k - \lfloor \widetilde{p}/2 \rfloor, \widetilde{p} - 1} = 0$ leads to the zero solution.
Answer to Exercise 5.1.2, page 133
The factoring of $\mathbf{H}_j$ defines a dual detail matrix, as in (4.32). The dual detail matrix defined by the factoring need not be exactly the same as the solution

proposed in (5.20). Therefore, the dual detail generated by the factoring can be denoted here by $\widetilde{\mathbf{G}}_j^{[1]}$. Then both $\widetilde{\mathbf{G}}_j$ proposed in (5.20) and $\widetilde{\mathbf{G}}_j^{[1]}$, proposed in (4.32) satisfy the PR condition in (2.62), meaning that the columns of both matrices span the left null space of $\mathbf{H}_j$. As a result, there exists a matrix $\mathbf{A}_j$ so that $\widetilde{\mathbf{G}}_j = \widetilde{\mathbf{G}}_j^{[1]}\mathbf{A}_j$. Let $\widetilde{\mathbf{H}}_j^{[0]}$ be the primitive dual refinement matrix resulting from the factoring in (4.32), then we have

$$\begin{aligned}\left[\begin{array}{c}\widetilde{\mathbf{H}}_j^{[0]^\top}\\ \widetilde{\mathbf{G}}_j^\top\end{array}\right] &= \left[\begin{array}{cc}\mathbf{I}_{n_j} & \mathbf{0}\\ \mathbf{0} & \mathbf{A}_j^\top\end{array}\right]\left[\begin{array}{c}\widetilde{\mathbf{H}}_j^{[0]^\top}\\ \widetilde{\mathbf{G}}_j^{[1]^\top}\end{array}\right]\\ &= \left[\begin{array}{cc}\mathbf{D}_j^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{A}_j^\top\end{array}\right]\left(\prod_{s=0}^{u-1}\left[\begin{array}{cc}\mathbf{I}_{n_j} & \mathbf{0}\\ -\mathbf{P}_j^{[q-s]} & \mathbf{I}_{n'_j}\end{array}\right]\left[\begin{array}{cc}\mathbf{I}_{n_j} & \mathbf{U}_j^{[q-s]}\\ \mathbf{0} & \mathbf{I}_{n'_j}\end{array}\right]\right).\end{aligned}$$

The primitive inverse transform then follows straightforwardly

$$\left[\begin{array}{cc}\mathbf{H}_j & \mathbf{G}_j^{[0]}\end{array}\right] = \left(\prod_{s=1}^{u}\left[\begin{array}{cc}\mathbf{I}_{n_j} & -\mathbf{U}_j^{[s]}\\ \mathbf{0} & \mathbf{I}_{n'_j}\end{array}\right]\left[\begin{array}{cc}\mathbf{I}_{n_j} & \mathbf{0}\\ \mathbf{P}_j^{[s]} & \mathbf{I}_{n'_j}\end{array}\right]\right)\left[\begin{array}{cc}\mathbf{D}_j & \mathbf{0}\\ \mathbf{0} & \mathbf{A}_j^{-\top}\end{array}\right].$$

A final update as in Section 4.1.5 can be added to enrich the wavelet transform with more properties.

Answer to Exercise 5.2.4, page 135

$$\begin{aligned}\psi_{j,l}(x) &= \sum_{k=-\infty}^{\infty} g_{k-2l}\varphi_{j+1,k}(x)\\ &= \sum_{k=-\infty}^{\infty} g_{k-2l}d^{j+1}\varphi\left(2^{j+1}x-k\right)\\ &= d^j c\sum_{m=-\infty}^{\infty} g_m\varphi\left(2(2^jx)-2l-m\right)\\ &= d^j c\sum_{m=-\infty}^{\infty} g_m\varphi\left(2\left(2^jx-l\right)-m\right)\\ &= d^j\psi\left(2^jx-l\right).\end{aligned}$$

Answer to Exercise 5.3.1, page 138

We have $h(x) = \int_0^1 g(x-u)du = \int_{x-1}^x g(v)dv$. The derivative is $h'(x) = g(x)-g(x-1)$, which is a difference between two piecewise polynomials of degree $\widetilde{p}-1$ with common knots. Therefore $h'(x)$ is a spline of degree $\widetilde{p}-1$, order $\widetilde{p}$, ans so $h(x)$ is a spline of order $\widetilde{p}+1$, nonzero on the interval $[0,\widetilde{p}+1]$. As the result in Lemma 4.2.8 on equispaced knots simplifies to the same expression as $h'(x) = g(x)-g(x-1)$, we conclude that $h(x)$ is the B-spline defined on the knots $\{0,1,\ldots,\widetilde{p}+1\}$. Note that this is $N_{j,0}^{[\widetilde{p}]}(x)$, not $\varphi_{j,0}^{[\widetilde{p}]}(x)$.

Answer to Exercise 5.3.2, page 138

The density of a sum of independent random variables is given by the convolution integral of the two densities. Using the result in Exercise 5.3.1 as induction step, repeated convolution of the uniform density of $[0, 1]$ (which is the Haar scaling function, which is the B-spline of order 1 on the knots $\{0, 1\}$) leads to a B-spline basis function on the knots $\{0, 1, \ldots, \widetilde{p}\}$. Rescaling of the random variable corresponds to a mere stretching (dilation) of its density function. The central limit theorem provides a proof for the B-splines of increasing order converging to the Gaussian bell curve.

Answer to Exercise 5.4.9, page 147

We have to prove that $\lim_{n\to\infty} \sup_{x\in D} |R_n(x)| = 0$, where

$$R_n(x) = \log(1 + x) - \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} x^k = 0.$$

Taking the derivative

$$R'_n(x) = \frac{1}{1+x} - \sum_{k=1}^{n} (-x)^{k-1} = \frac{1}{1+x} - \frac{1-(-x)^n}{1+x} = \frac{(-x)^n}{1+x},$$

it is found that $R'_n(0) = 0$, leading to a local and global minimum at the origin (not surprisingly for a Taylor series) while the absolute error increases monotoneously from there, reaching its maximum on D at $x = 1$. As the series converges at $x = 1$, it does so uniformly on the whole interval.

Answer to Exercise 6.1.2, page 165

With $\varphi(x) = \sin(\pi x)/(\pi x)$ and $\Phi(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(x) \exp(-i\omega x) dx$ its Fourier transform, we have

$$\int_{-\infty}^{\infty} \frac{\sin(\pi x)}{\pi x} dx = 2\pi\Phi(0).$$

From Example 6.1.1, we know that $\Phi(\omega) = 1/(2\pi)$, leading immediately to the conclusion that the integral equals one.

Answer to Exercise 6.1.11, page 168

Plugging in

$$H(\omega) = \sum_{s=-\infty}^{\infty} h_s \exp(-i\omega s)$$

into

$$H(\omega)\widetilde{H}(-\omega) + H(\pi + \omega)\widetilde{H}(\pi - \omega) = 2,$$

we find

$$\sum_{s=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} h_s \widetilde{h}_t \exp[-i\omega(s-t)] + \sum_{s=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} h_s \widetilde{h}_t (-1)^{s+t} \exp[-i\omega(s-t)] = 2.$$

With $r = s - t$, we have $s = r + t$ and $s + t = r + 2t$, so we get

$$\sum_{r=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} h_{r+t} \widetilde{h}_t \exp[-i\omega r] + \sum_{r=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} h_{r+t} \widetilde{h}_t (-1)^r \exp[-i\omega r] = 2,$$

which simplifies to

$$\sum_{r=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} 2h_{2r+t}\widetilde{h}_t \exp[-i\omega 2r] = 2,$$

or, with $k = -r$ and $l = t$,

$$\sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} h_{l-2k}\widetilde{h}_l \exp[i\omega 2k] = 1.$$

The left hand side can be seen as a Fourier series as in (1.2) with $\omega = \pi x$ of the function $f(\omega) = 1$ on the right hand side. The coefficients are here

$$a_k = \sum_{l=-\infty}^{\infty} h_{l-2k}\widetilde{h}_l.$$

By (1.4) These coefficients equal

$$a_k = \int_0^1 1\,[-i(2\pi)kx]\,dx = \delta_k,$$

from which (5.16) follows.

Answer to Exercise 6.2.3, page 173

We find

$$\begin{aligned}
\mathcal{F}\left(f(x-a)\right) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x-a)\exp(-i\omega x)dx \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\exp(-i\omega(u+a))du \\
&= \exp(-i\omega a)F(\omega)
\end{aligned}$$

and

$$\mathcal{F}\left(\exp(ibx)f(x)\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\exp(ibx)\exp(-i\omega x)dx = F(\omega-b).$$

The Fourier transform of $\exp(ibx)f(x-a)$ is then

$$\mathcal{F}\left(\exp(ibx)f(x-a)\right) = \exp[-i(\omega-b)a]F(\omega-b) = \exp(iab)\exp(-i\omega a)F(\omega-b).$$

The effect of a dilation (stretching) is found by

$$\begin{aligned}
\mathcal{F}\left(f(sx)\right) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(sx)\exp(-i\omega x)dx \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\exp(-i\omega u/s)\frac{du}{s} = \frac{1}{s}F\left(\frac{u}{s}\right).
\end{aligned}$$

Dilation and translation combined lead to

$$\begin{aligned}\mathcal{F}\left(f(sx-a)\right) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(sx-a)\exp(-i\omega x)dx \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\exp(-i\omega(u+a)/s)\frac{du}{s} \\ &= \frac{\exp(-i\omega a/s)}{s}F\left(\frac{u}{s}\right).\end{aligned}$$

Note that $f(sx-a)$ is a dilation of a translation of $f(x)$: the translation comes first, then the dilation. That is, we dilate the function $f(x-a)$ by evaluating it in sx. This is in contrast to the function $f(s(x-a))$ which is a translation of a dilation: the dilated function $f(sx)$ is evaluated in $(x-a)$.

Answer to Exercise 6.2.4, page 173

We have $\overline{x}_g = \overline{x}_f + a/s$, which can be seen from

$$\begin{aligned}\overline{x}_g &= \frac{\int_{-\infty}^{\infty}|g(x)|^2 x dx}{\int_{-\infty}^{\infty}|g(x)|^2 dx} \\ &= \frac{\int_{-\infty}^{\infty}|f(sx-a)|^2 x dx}{\int_{-\infty}^{\infty}|f(sx-a)|^2 dx} \\ &= \frac{\int_{-\infty}^{\infty}|f(u)|^2\left(\frac{u+a}{s}\right)\frac{du}{s}}{\int_{-\infty}^{\infty}|f(u)|^2\frac{du}{s}} = \frac{\frac{1}{s}\int_{-\infty}^{\infty}|f(u)|^2(u+a)du}{\int_{-\infty}^{\infty}|f(u)|^2 du} = \frac{\overline{x}_f + a}{s}.\end{aligned}$$

The width of the uncertainty window is

$$\begin{aligned}(\Delta_g x)^2 &= \frac{\int_{-\infty}^{\infty}|g(x)|^2(x-\overline{x}_g)^2 dx}{\int_{-\infty}^{\infty}|g(x)|^2 dx} \\ &= \frac{\int_{-\infty}^{\infty}|f(sx-a)|^2\left(x-\frac{\overline{x}_f+a}{s}\right)^2 dx}{\int_{-\infty}^{\infty}|f(sx-a)|^2 dx} \\ &= \frac{\int_{-\infty}^{\infty}|f(u)|^2\left(\frac{u-\overline{x}_f}{s}\right)^2\frac{du}{s}}{\int_{-\infty}^{\infty}|f(u)|^2\frac{du}{s}} = \frac{(\Delta_f x)^2}{s^2}.\end{aligned}$$

From Exercise 6.2.3, we can see that $|G(\omega)| = (1/s)|F(\omega/s)|$. The factor $1/s$ in front has no impact on $(\Delta_G\omega)^2$, while the scaling by $1/s$ leads, in line with the result above, to $(\Delta_G\omega)^2 = s^2(\Delta_F\omega)^2$. As a result of all this, we conclude that

$$(\Delta_f x)(\Delta_F\omega) = (\Delta_g x)(\Delta_G\omega).$$

Answer to Exercise 6.2.5, page 173
Using partial integration

$$\begin{aligned}
\mathcal{F}\left(f'(x)\right) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f'(x)\exp(-i\omega x)dx \\
&= \frac{1}{2\pi}\left[f(x)\exp(-i\omega x)\right]_{-\infty}^{\infty} - \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)(-i\omega)\exp(-i\omega x)dx \\
&= 0 + i\omega F(\omega).
\end{aligned}$$

Answer to Exercise 6.2.6, page 173
Isolation of a squared sum in the integrand, followed by a substitution $u = x + i\omega\sigma^2$ yields

$$\begin{aligned}
F(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left[-\frac{x^2}{2\sigma^2} - i\omega x\right]dx \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left[-\frac{x^2}{2\sigma^2} - \frac{2i\omega\sigma x}{2\sigma} - \frac{(i\omega\sigma)^2}{2}\right]\exp\left[\frac{(i\omega\sigma)^2}{2}\right]dx \\
&= \frac{1}{2\pi}\exp\left[-(\omega\sigma)^2/2\right]\int_{-\infty}^{\infty}\exp\left[-\left(\frac{x}{\sqrt{2}\sigma} + \frac{i\omega\sigma}{\sqrt{2}}\right)^2\right]dx \\
&= \frac{1}{2\pi}\exp\left[-(\omega\sigma)^2/2\right]\int_{-\infty}^{\infty}\exp(-u^2/2\sigma^2)du = \frac{\sigma}{\sqrt{2\pi}}\exp\left[-(\omega\sigma)^2/2\right].
\end{aligned}$$

Answer to Exercise 6.2.7, page 173
Applying Fubini's theorem, we find

$$\begin{aligned}
H(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} h(x)\exp(-i\omega x)dx \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)g(x-u)du\exp(-i\omega x)dx \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)g(v)du\exp(-i\omega(u+v))dv \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\exp(-i\omega u)du\cdot\int_{-\infty}^{\infty} f(v)\exp(-i\omega v)dv \\
&= (2\pi)F(\omega)G(\omega).
\end{aligned}$$

Let $\varphi(x)$ be the Haar scaling function, which is the characteristic function on $[0, 1]$. Then

$$\begin{aligned}
\Phi(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(x)\exp(-i\omega x)dx \\
&= \frac{1}{2\pi}\int_{0}^{1}\exp(-i\omega x)dx = \frac{1}{2\pi}\left[\frac{\exp(-i\omega)-1}{-i\omega}\right] \\
&= \frac{\exp(-i\omega/2)}{2\pi}\frac{\exp(i\omega/2)-\exp(-i\omega/2)}{2i(\omega/2)} = \frac{\exp(-i\omega/2)}{2\pi}\frac{\sin(\omega/2)}{\omega/2}.
\end{aligned}$$

Answer to Exercise 6.2.8, page 174
The denominator is

$$\int_{-\infty}^{\infty}|f(x)|^2dx = 2\int_0^1(1-x)^2dx = \frac{2}{3}.$$

Since $\overline{x}_f = 0$, the numerator is

$$\int_{-\infty}^{\infty}|f(x)|^2x^2dx = 2\int_0^1(1-x)^2x^2dx = \frac{1}{15}.$$

Hence $(\Delta_f x)^2 = \frac{1}{20}$.
Answer to Exercise 6.2.9, page 174
The denominator is the same as in Exercise 6.2.8. The numerator becomes

$$\int_{-\infty}^{\infty}|f'(x)|^2dx = \int_{-1}^{0}1^2dx + \int_0^1(-1)^2dx = 2.$$

All together we find

$$(\Delta_f x)(\Delta_F\omega) = \frac{\sqrt{(1/15)\cdot 2}}{(2/3)} = 0.5477 \geq \frac{1}{2}.$$

Answer to Exercise 6.2.10, page 174
We have

$$|F(\omega)|^2\omega^2 = \left|\frac{\sin(\omega/2)}{\omega/2}\right|^2\omega^2 = 4\sin^2(\omega/2),$$

which has an infinite integral on the $\mathbb{R}$.
Answer to Exercise 6.2.11, page 174
The denominators of $(\Delta_g x)^2$ and $(\Delta_G\omega)^2$ are given by

$$\int_{\infty}^{\infty}|g(x)|^2dx = 2\left(\int_0^1 x^2dx + \int_1^3(3-x)^2/4dx\right) = 2\left(\frac{1}{3}+\frac{1}{4}\cdot\frac{8}{3}\right) = 2.$$

The numerator of $(\Delta_g x)^2$ becomes

$$\int_{\infty}^{\infty}|g(x)|^2x^2dx = 2\left(\int_0^1 x^4dx + \int_1^3(3-x)^2x^2/4dx\right) = \frac{18}{5}.$$

The numerator of $(\Delta_G\omega)^2$ is

$$\int_{\infty}^{\infty} |g'(x)|^2 dx = \int_{-3}^{-1} \left(-\frac{1}{2}\right)^2 dx + \int_{-1}^{1} 1^2 dx + \int_{1}^{3} \left(-\frac{1}{2}\right)^2 dx = 3,$$

leading to $(\Delta_g x)^2 = \frac{9}{5}$, $(\Delta_G\omega)^2 = \frac{3}{2}$, and $(\Delta_g x)(\Delta_G\omega) = 1.643$. Somehow suprisingly, the wavelet function has a much less precise time-frequency window than the accompanying scaling function.

Answer to Exercise 6.2.17, page 177

Let $S_\Delta(\omega) = \sum_{k\in\mathbb{Z}} F\left(\frac{\omega+2\pi k}{\Delta}\right)$, then $S_\Delta(\omega+2\pi) = S_\Delta(\omega)$. Hence, by (1.9), we have $S_\Delta(\omega) = \sum_{s=-\infty}^{\infty} a_s \exp(-i\omega s)$, where

$$\begin{aligned}
a_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} S_\Delta(\omega)\exp(i\omega k)d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{s\in\mathbb{Z}} F\left(\frac{\omega+2\pi s}{\Delta}\right)\exp(i\omega k)d\omega \\
&= \frac{1}{2\pi}\sum_{s\in\mathbb{Z}} \int_{-\pi}^{\pi} F\left(\frac{\omega+2\pi s}{\Delta}\right)\exp(i\omega k)d\omega \\
&= \frac{1}{2\pi}\sum_{s\in\mathbb{Z}} \int_{-\pi+2\pi s}^{\pi+2\pi s} F\left(\frac{w}{\Delta}\right)\exp(iwk)\exp(-i2\pi sk)dw \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} F\left(\frac{w}{\Delta}\right)\exp(iwk)dw \\
&= \frac{\Delta}{2\pi}\int_{-\infty}^{\infty} F(v)\exp(i\Delta vk)dv = \frac{\Delta}{2\pi} f(k\Delta).
\end{aligned}$$

Answer to Exercise 9.1.1, page 221

From (9.7), taking $\alpha = 1/2$, and using the primal version of (9.5), we find

$$\begin{aligned}
\boldsymbol{s}_{j+1}^{[q]} &= \frac{1}{2}\left[\mathbf{H}_j^{[2q]}\boldsymbol{s}_j^{[2q]} + \mathbf{G}_j^{[2q]}\boldsymbol{d}_j^{[2q+1]} + \mathbf{H}_j^{[2q+1]}\boldsymbol{s}_j^{[2q+1]} + \mathbf{G}_j^{[2q+1]}\boldsymbol{d}_j^{[2q]}\right] \\
&= \frac{1}{2}\left[\boldsymbol{\mathcal{H}}_j^{[q]}\widetilde{\mathbf{J}}_j\widetilde{\mathbf{J}}_j^\top\boldsymbol{s}_j^{[2q,2q+1]} + \boldsymbol{\mathcal{G}}_j^{[q]}\widetilde{\mathbf{J}}_j^{[1]}\widetilde{\mathbf{J}}_j^{[1]\top}\boldsymbol{d}_j^{[2q,2q+1]}\right. \\
&\quad \left. + \boldsymbol{\mathcal{H}}_j^{[q]}\widetilde{\mathbf{J}}_j^{[1]}\widetilde{\mathbf{J}}_j^{[1]\top}\boldsymbol{s}_j^{[2q,2q+1]} + \boldsymbol{\mathcal{G}}_j^{[q]}\widetilde{\mathbf{J}}_j\widetilde{\mathbf{J}}_j^\top\boldsymbol{d}_j^{[2q,2q+1]}\right].
\end{aligned}$$

Now, (2.54), which reads here as

$$\widetilde{\mathbf{J}}_j\widetilde{\mathbf{J}}_j^\top + \widetilde{\mathbf{J}}_j^{[1]}\widetilde{\mathbf{J}}_j^{[1]\top} = \mathbf{I}_{n_{j+1}},$$

leads straightforwardly to (9.8).

Answer to Exercise 9.1.1, page 221

We look for the vector $\boldsymbol{s}_{j+1}$ that minimises

$$\begin{aligned}
r(\boldsymbol{s}_{j+1}) &= \|\widetilde{\mathbf{H}}_j^{[2q]\top}\boldsymbol{s}_{j+1} - \boldsymbol{s}_j^{[2q]}\|_2^2 + \|\widetilde{\mathbf{G}}_j^{[2q]\top}\boldsymbol{s}_{j+1} - \boldsymbol{d}_j^{[2q]}\|_2^2 \\
&\quad + \|\widetilde{\mathbf{H}}_j^{[2q+1]\top}\boldsymbol{s}_{j+1} - \boldsymbol{s}_j^{[2q+1]}\|_2^2 + \|\widetilde{\mathbf{G}}_j^{[2q+1]\top}\boldsymbol{s}_{j+1} - \boldsymbol{d}_j^{[2q+1]}\|_2^2.
\end{aligned}$$

Because of the orthogonality, this can be written as

$$r(\boldsymbol{s}_{j+1}) = \|\boldsymbol{s}_{j+1} - \boldsymbol{s}_{j+1}^{[2q]}\|_2^2 + \|\boldsymbol{s}_{j+1} - \boldsymbol{s}_{j+1}^{[2q+1]}\|_2^2,$$

where $\boldsymbol{s}_{j+1}^{[2q]} = \mathbf{H}_j^{[2q]}\boldsymbol{s}_{j,e}^{[2q,2q+1]} + \mathbf{G}_j^{[2q]}\boldsymbol{d}_{j,o}^{[2q,2q+1]}$ and $\boldsymbol{s}_{j+1}^{[2q+1]} = \mathbf{H}_j^{[2q+1]}\boldsymbol{s}_{j,o}^{[2q,2q+1]} + \mathbf{G}_j^{[2q+1]}\boldsymbol{d}_{j,e}^{[2q,2q+1]}$. The minimisation of $r(\boldsymbol{s}_{j+1})$ proceeds componentwise. The kth component is

$$r(s_{j+1,k}) = (s_{j+1,k} - s_{j+1,k}^{[2q]})^2 + (s_{j+1,k} - s_{j+1,k}^{[2q+1]})^2.$$

In general $r(x) = (x - x_0)^2 + (x - x_1)^2$ takes a minimum at $x = (x_0 + x_1)/2$.