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INFINITE TOPOLOGICALLY RANDOM CHANNEL NETWORKS[1]

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ABSTRACT

Individual channel networks ordinarily are portions of far larger, essentially infinite, networks. The all network is by definition at infinite topologically random network if the populations of subnetworks w it are topologically random. From such a network, the probability of randomly drawing a link, a subnetw or a basin with Strahler order ω is $1/2^{\omega}$; and that of randomly drawing a stream of order ω is $3/4^{\omega}$. The pr bility of drawing a link of magnitude μ, that is, one having μ sources ultimately tributary to it, is equal t probability of a first passage through the origin at step $2\mu - 1$ in a symmetric random walk, a fact w suggests a useful mathematical analogy between random walks and infinite topologically random netw Assuming uniform link length equal to the constant of channel maintenance, which in turn is the recipro drainage density, the probability distributions for links and streams of various orders may be interpret crude geomorphological "laws" analogous to Horton's laws of drainage composition. These distribu predict geometric-series "laws" in which, using Strahler orders, the bifurcation ratio is $\frac{1}{4}$, the link-nu ratio is $\frac{1}{2}$, the length ratio is 2, the cumulative-length ratio is 4, and the basin-area ratio is 4, all in agreement with the observed ratios. They also predict values of $\frac{4}{3}$ and $\frac{2}{3}$, respectively, for the dimensio ratios of total number of Strahler streams to network magnitude and of Strahler stream frequency t square of the drainage density, in agreement with the values of 1.34 and 0.694 found empirically.

INTRODUCTION

Individual channel networks in nature do not ordinarily exist independently, but are portions of far larger networks that for practical purposes are essentially infinite. The hypothesis that in the absence of geologic controls natural networks will be topologically random (Shreve, 1966, p. 27) thus implicitly contains the corollary that in the over-all infinite network, which will be termed an *infinite topologically random channel network*, all topologically distinct subnetworks with the same number of sources occur with equal frequency. The purposes of this paper are to deduce some of the geomorphologically significant probability distributions associated with these infinite topologically random networks, to demonstrate the close connection of such networks with random walks, and to derive theoretically several of the well-known empirical "laws" of geomorphology.

DEFINITIONS

Most of the specialized terms used in this paper—such as *link, fork, source, topologically identical, topologically distinct,* *topologically random*—have been defined previous publication (Shreve, 1966, p. 27). Strahler stream orders (1952, p. 1120 Shreve, 1966, p. 21–22) will be used ex sively. The terms *channel network* or, convenience, simply *network* will refer to idealized concept of channel netw (Shreve, 1966, p. 27) in which one, and o one, path exists between any two points in which at its upstream end each link ei connects to two other links or terminate a source. Stated negatively, and somew less precisely, no lakes with multiple inl no islands of major extent, and no con ences of more than two channels at a sin place are permissible in the idealized chan networks considered in this paper. terms *exterior link* and *interior link* will nify links terminating at their upstr ends in sources and forks, respectively. these terms, Melton's relationships (195 345; Shreve, 1966, p. 27) state that ev idealized network with μ sources will h $2\mu - 1$ links, of which μ are exterior li and $\mu - 1$ are interior links.

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hrase *link drawn at random* will sig- nk selected in such a way that all the specified target population, ay be either real or hypothetical,

are equally likely to be drawn (see Mood and Graybill, 1963, p. 141–142). It is completely equivalent to the phrases *network drawn at random* or *basin drawn at random*, because each link of given magnitude or order or both in a population of networks is the outlet of a single subnetwork of the same magnitude or order or both. Actually drawing links or networks at random from natural populations is not easily accomplished; nevertheless, for meaningful statistical analysis it is an absolute necessity.

CONCEPT OF MAGNITUDE

The *magnitude* of a link in a channel network is herein defined by the following two rules. (1) Each exterior link has magnitude 1. (2) If links or magnitude μ_1 and μ_2 join, then the resultant link downstream has magnitude $\mu_1 + \mu_2$. The application of these rules to a typical channel network is illustrated in figure 1. By induction, the magnitude of a link is equal to the total number of sources ultimately tributary to it. Magnitude is thus a purely topological concept, for it involves only the interconnections and not the lengths, shapes, or orientations of the links comprising the network. Like the systems of stream ordering of Horton (1945, p. 281), Strahler (1952, p. 1120 n.), and, more recently, Scheidegger (1965, p. B188), the concept of magnitude assumes the existence of sources, that is, of objectively describable points that in fact or by definition are the points farthest upstream in a channel network.

In analogy with basin order, the magnitude of a channel network or drainage basin is equal to the magnitude of its highest magnitude link. Thus, if two networks are tributary to the same link, the resultant network has magnitude equal to the sum of the magnitudes of the two tributary networks. Networks with equal magnitudes have equal numbers of links, forks, sources, Horton streams, and first-order Strahler streams, and are therefore comparable in topological complexity (Shreve, 1966, p. 27). A reasonable conjecture is that they are comparable in other ways as well.

Despite its apparent complexity, the system of stream ordering proposed by Scheidegger (1965) is simply related to the concept of magnitude. If the Scheidegger order and the magnitude of a link (or network) are denoted by X and μ, respectively, then

$$X = \log_2 2\mu, \tag{1}$$

in which $\log_2$ signifies the logarithm to base 2.

According to Scheidegger (1965, p. B188–B189), if the second-order streams as classified by his system "are treated as first-order

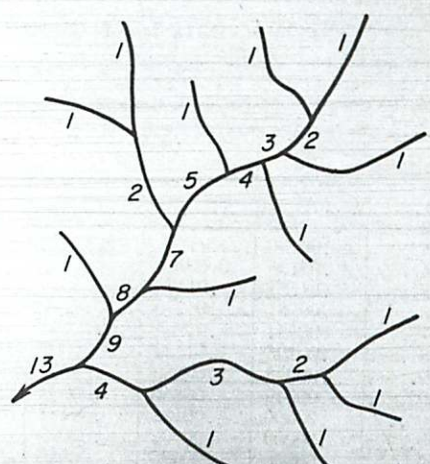

FIG. 1.—Link magnitudes for a typical channel network.

streams, then all orders are simply reduced by one." Equation (1) makes it easy to see, however, that this statement is incorrect, unless the previous first-order streams are treated as zero-order streams that contribute to the order of the higher-order streams even though, as Scheidegger himself states (1965, p. B188), they are in fact non-existent in nature. Thus, one of the main advantages of the Horton and Strahler systems (Shreve, 1966, p. 22–23) is lost in the Scheidegger system.

PROBABILITY DISTRIBUTIONS

The probability $p(\mu, \omega)$ that a link drawn at random from an infinite topologically random network will have specified magnitude μ and Strahler order ω can be computed from the recursive relationship

$$\begin{aligned} p(\mu, \omega) = \tfrac{1}{2} \sum_{\alpha=1}^{\mu-1} [\, p(\alpha, \omega-1) \\ \times p(\mu-\alpha, \omega-1) \\ + 2p(\alpha, \omega) \sum_{\beta=1}^{\omega-1} p(\mu-\alpha, \beta)\,], \end{aligned} \tag{2}$$

$$p(1, 1) = \tfrac{1}{2}, \quad p(\mu, 1) = 0,$$
$$p(1, \omega) = 0, \quad \mu, \omega = 2, 3, \ldots .$$

The factor $\frac{1}{2}$ is the probability of drawing an interior link; and the sum on α accounts for the various ways that pairs of smaller networks can be tributary to this link to give it the specified μ and ω. Recursive relationships for the marginal probabilities

$$u(\omega) = \sum_{\mu=1}^{\infty} p(\mu, \omega)$$

and

$$v(\mu) = \sum_{\omega=1}^{\infty} p(\mu, \omega)$$

can be derived either in the same direct fashion or by summation using (2). Proceeding directly,

$$u(\omega) = \tfrac{1}{2}\Big[u(\omega-1)^2 + 2u(\omega) \sum_{\beta=1}^{\omega-1} u(\beta)\Big], \tag{3}$$
$$u(1) = \tfrac{1}{2}, \quad \omega = 2, 3, \ldots,$$

and

$$v(\mu) = \tfrac{1}{2} \sum_{\alpha=1}^{\mu-1} v(\alpha)\, v(\mu-\alpha), \tag{4}$$
$$v(1) = \tfrac{1}{2}, \quad \mu = 2, 3, \ldots .$$

TABLE 1

INFINITE TOPOLOGICALLY RANDOM CHANNEL NETWORKS

μ	$p(\mu,\omega)$							$v(\mu)$
	$\omega=2$	$\omega=3$	$\omega=4$	$\omega=5$	$\omega=6$	$\omega=7$	$\omega=8$	
1.....	....	....	....	....	....	....	....	0.50000
2.....	0.12500	....	....	....	....	....	....	.12500
3.....	.06250	....	....	....	....	....	....	.06250
4.....	.03125	0.00781	....	....	....	....	....	.03906
5.....	.01562	.01172	....	....	....	....	....	.02734
6.....	.00781	.01270	....	....	....	....	....	.02051
7.....	.00391	.01221	....	....	....	....	....	.01611
8.....	.00195	.01111	0.00003	....	....	....	....	.01309
9.....	.00098	.00983	.00011	....	....	....	....	.01091
10.....	.00049	.00856	.00022	....	....	....	....	.00927
20.....	.00000	.00181	.00141	0.00000	....	....	....	.00322
50.....	.00000	.00002	.00070	.00009	0.00000	....	....	.00080
100.....	.00000	.00000	.00010	.00018	.00000	0.00000	....	.00028
200.....	.00000	.00000	.00000	.00009	.00001	.00000	0.00000	.00010
500.....	.00000	.00000	.00000	.00000	.00002	.00000	.00000	.00002
1000.....	0.00000	0.00000	0.00000	0.00000	0.00001	0.00000	0.00000	0.00001
$u(\omega)$.....	0.25000	0.12500	0.06250	0.03125	0.01562	0.00781	0.00391	....

As before, the factor $\frac{1}{2}$ is the probability of drawing an interior link; and the remaining factor accounts for the various ways that pairs of smaller networks can be tributary to this link to give it the specified μ or ω.

Representative values of $p(\mu, \omega)$, $u(\omega)$, and $v(\mu)$ are presented in table 1. For any particular magnitude the distribution of probabilities with order has a relatively sharp peak, whereas for any particular order the distribution with magnitude has a relatively broad peak; hence, in a topologically random population magnitude more pre-

-ived either in the same direct
-y summation using (2). Proceed-
;

$$\omega-1)^2+2u(\omega)\sum_{\beta=1}^{\omega-1}u(\beta)\Big], \quad (3)$$

$$(1)=\tfrac{1}{2}, \qquad \omega=2,3,\ldots,$$

$$)=\tfrac{1}{2}\sum_{\alpha=1}^{\mu-1}v(\alpha)\,v(\mu-\alpha), \quad (4)$$

$$)=\tfrac{1}{2}, \qquad \mu=2,3,\ldots.$$

NEL NETWORKS

	$\omega=7$	$\omega=8$	$v(\mu)$
...	....	....	0.50000
...	....	....	.12500
...	....	....	.06250
...	....	....	.03906
...	....	....	.02734
...	....	....	.02051
...	....	....	.01611
...	....	....	.01309
...	....	....	.01091
...	....	....	.00927
...	....	....	.00322
00	....	....	.00080
00	0.00000	....	.00028
01	.00000	0.00000	.00010
02	.00000	.00000	.00002
01	0.00000	0.00000	0.00001
62	0.00781	0.00391	....

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-, whereas for any particular order
-ution with magnitude has a rela-
ad peak; hence, in a topologically
-opulation magnitude more pre-

cisely characterizes a network than does order.

A closed expression for $u(\omega)$ can readily be derived from (3). Summation of the first $\omega-1$ terms of $u(\omega)$ gives, using (3),

$$2\sum_{\beta=1}^{\omega-1}u(\beta)=\Big[\sum_{\beta=1}^{\omega-1}u(\beta)\Big]^2 - u(\omega-1)^2+1\,. \quad (5a)$$

Completing the square, taking the square root, and substituting into (3) leads to the relationship

$$2u(\omega)=u(\omega-1)\,, \qquad u(1)=\tfrac{1}{2}\,, \quad \omega=2,3,\ldots, \quad (5b)$$

from which

$$u(\omega)=1/2^{\omega}\,, \qquad \omega=1,2,\ldots, \quad (5c)$$

follows by mathematical induction.

A closed expression for $v(\mu)$ can be derived from (4) by means of the generating function $R(r)$, where

$$R(r)=\sum_{\mu=1}^{\infty}v(\mu)\,r^{\mu}\,. \quad (6a)$$

Squaring both sides of (6a) and collecting the coefficients of like powers of r gives

$$R^2=\sum_{\mu=2}^{\infty}r^{\mu}\sum_{\alpha=1}^{\mu-1}v(\alpha)\,v(\mu-\alpha)\,; \quad (6b)$$

hence, using (4),

$$R^2=2R-r\,, \qquad R=1-(1-r)^{1/2}\,, \quad (6c)$$

from which

$$v(\mu)=\frac{2^{-(2\mu-1)}}{2\mu-1}\binom{2\mu-1}{\mu}, \qquad \mu=1,2,\ldots, \quad (6d)$$

follows by the binomial expansion. The minus sign is chosen in the quadratic formula because the power-series expansion in (6a) represents the branch of the curve of R versus r that passes through the origin.

The distribution $v(\mu)$ can also be derived from the probability $w(\mu;\mathrm{M})$ of drawing a link of magnitude μ at random from a topologically random population of networks of magnitude M. This probability is

$$w(\mu;\mathrm{M}) = \frac{(\mathrm{M}-\mu+1)N(\mathrm{M}-\mu+1)N(\mu)}{(2\mathrm{M}-1)N(\mathrm{M})}, \quad (7a)$$

where $N(\mu)$ is the number of topologically distinct networks of magnitude μ. The denominator is equal to the total number of links of all magnitudes in the population of $N(\mathrm{M})$ networks of magnitude M; and the numerator is therefore the number of links of magnitude μ in this population, which in turn is the number of ways that the $N(\mu)$ networks of magnitude μ can be attached to the $\mathrm{M}-\mu+1$ sources of each of the $N(\mathrm{M}-\mu+1)$ networks of magnitude $\mathrm{M}-\mu+1$ to form networks of magnitude M.

Substitution of Cayley's closed expression for $N(\mu)$,

$$N(\mu)=\frac{1}{2\mu-1}\binom{2\mu-1}{\mu} \quad (7b)$$

(Shreve, 1966, p. 29), gives

$$w(\mu;\mathrm{M}) = \frac{1}{2\mu-1}\binom{2\mu}{\mu}\binom{2(\mathrm{M}-\mu)}{\mathrm{M}-\mu}\Big/\binom{2\mathrm{M}}{\mathrm{M}}, \quad (7c)$$

from which (6d) follows by taking the limit as M approaches infinity.

Representative values of $w(\mu;\mathrm{M})$ are presented in table 2.

For any network magnitude M the greatest (or modal) probability is that of drawing a link of magnitude 1, inasmuch as

$$w(1;\mathrm{M})=\mathrm{M}/(2\mathrm{M}-1)\geq\tfrac{1}{2}\,. \quad (8)$$

At the other end of the distribution, however, for $\mathrm{M}>3$ the probability of drawing a link of magnitude M is greater than that of drawing one of magnitude $\mathrm{M}-1$. This is because networks of magnitude M always have one link of that magnitude, whereas they very often have no links of the next smaller magnitude and very rarely have two, which is the maximum possible. Setting the ratio of two successive terms equal to 1 and solving for μ shows that the minimum in the distribution is located approximately at $\mu = 3(M + 1)/4$.

The probability $s(\omega)$ of drawing a Strahler stream of order ω at random from the streams (not links) comprising an infinite topologically random network can be derived from (5c) and (6d) by noting that, because in any network the links whose tributaries are both of order $\omega - 1$ are in one-to-

TABLE 2

POPULATIONS OF FINITE TOPOLOGICALLY RANDOM CHANNEL NETWORKS

M	$w(\mu;M)$							
	$\mu=1$	$\mu=2$	$\mu=3$	$\mu=4$	$\mu=5$	$\mu=6$	$\mu=7$	$\mu=8$
1.....	1.00000	....	....	....	....	....	....	....
2.....	0.66667	0.33333	....	....	....	....	....	....
3.....	0.60000	.20000	0.20000	....	....	....	....	....
4.....	0.57143	.17143	.11429	0.14286	....	....	....	....
5.....	0.55556	.15873	.09524	.07936	0.11111	....	....	....
6.....	0.54545	.15152	.08658	.06494	.06061	0.09091	....	....
7.....	0.53846	.14685	.08158	.05828	.04895	.04895	0.07692	....
8.....	0.53333	.14359	.07832	.05439	.04351	.03916	.04103	0.06667
9.....	0.52941	.14118	.07602	.05183	.04031	.03455	.03258	.03529
10.....	0.52632	.13932	.07430	.05001	.03819	.03183	.02858	.02786
20.....	0.51282	.13167	.06772	.04360	.03151	.02445	.01992	.01683
50.....	0.50505	.12756	.06445	.04072	.02882	.02185	.01737	.01428
100.....	0.50251	.12627	.06346	.03987	.02805	.02115	.01671	.01365
200.....	0.50125	.12563	.06297	.03946	.02769	.02082	.01640	.01336
500.....	0.50050	.12525	.06269	.03922	.02748	.02063	.01623	.01320
1000.....	0.50025	.12512	.06259	.03914	.02741	.02057	.01617	.01314
∞.......	0.50000	0.12500	0.06250	0.03906	0.02734	0.02051	0.01611	0.01309

one correspondence with the streams of order ω,

$$s(2)/s(1) = v(2)/v(1) = \tfrac{1}{4} \qquad (9a)$$

and

$$s(\omega + 1)/s(\omega) = \tfrac{1}{2}u(\omega)^2/\tfrac{1}{2}u(\omega - 1)^2 = \tfrac{1}{4}, \quad \omega = 2, 3, \ldots. \qquad (9b)$$

These equations define a geometric series with ratio $\frac{1}{4}$, from which

$$s(\omega) = 3/4^{\omega}, \quad \omega = 1, 2, \ldots, \qquad (9c)$$

follows by mathematical induction.

The distributions (5c), (6d), and (9c) could readily be tested against natural popu- lations of channel networks by means of th goodness-of-fit test (see, e.g., Siegel, 1956, p 42–52, 59–60, and Mood and Graybill, 1963 p. 308–309).

CONNECTION WITH RANDOM WALKS

A symbolic representation of any chann network can be constructed as follow Start at the outlet and traverse the networ always turning left at forks and reversi direction at sources, until the outlet is aga reached. During the traverse, generate a sequence of I's and E's by recording an I the first time a given interior link is traversed and an E the first time a given exterior link is traversed. Each link will be traversed twice but recorded only once.

If a right turn instead of a left turn is made at each fork, a different sequence will result, which is the sequence for the mirror-image network. The new sequence will not be the reverse of the original, however; and, in general, symmetry in the network will not be evident in the sequence, and vice versa. The reverse sequence is generated by turn-

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CTION WITH RANDOM WALKS

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e outlet and traverse the network,
ning left at forks and reversing
t sources, until the outlet is again

M CHANNEL NETWORKS

	$\mu=6$	$\mu=7$	$\mu=8$
	....	....	....
	....	....	....
	....	....	....
11	....	....	....
51	0.09091	....	....
95	.04895	0.07692	....
51	.03916	.04103	0.06667
31	.03455	.03258	.03529
19	.03183	.02858	.02786
51	.02445	.01992	.01683
32	.02185	.01737	.01428
95	.02115	.01671	.01365
59	.02082	.01640	.01336
18	.02063	.01623	.01320
11	.02057	.01617	.01314
54	0.02051	0.01611	0.01309

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I's and E's by recording an I the
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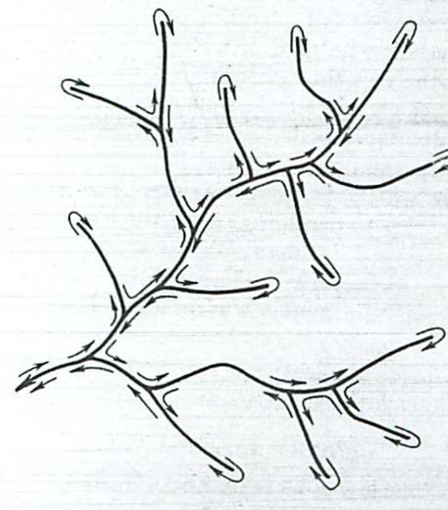
work. The new sequence will not
rse of the original, however; and,
symmetry in the network will not
in the sequence, and vice versa.
e sequence is generated by turn-

ing right and recording the I's and E's the second time given links are traversed rather than the first time. The mirror-image and reverse sequences, although necessary for certain types of investigation, will not be used in this paper.

Topologically identical networks will have identical sequences, and topologically distinct networks will have different unique sequences. Obviously, some possible sequences, such as all I's or all E's, cannot correspond to channel networks. Sequences corresponding to networks of more than one link, for instance, always begin with I and end with two successive E's. More generally, because of the fact that in any network or subnetwork the number of interior links is always exactly one less than the number of exterior links, only those sequences are possible in which, as they are recorded, the number of E's never exceeds the number of I's except at the terminal E. Put another way, if the I's count $+1$ and the E's count -1, then the partial sums can never be negative, except for the last, which will be -1. Thus, on a graph the curve of partial sums may fall to the level of the origin, but it does not drop below it until the terminal E.

The steps from network to sequence to graph are illustrated in figure 2.

Graphs like that of figure 2 occur widely in the theory of random walks; and it is worthwhile to compare their properties with those of topologically random networks. The graphs dropping below the axis for the first time at step $2\mu - 1$ correspond to the topologically distinct networks with $2\mu - 1$ links; hence, the number of such graphs is $N(\mu)$, as shown directly by Feller (1957, p. 71; see Shreve, 1966, p. 29, for derivation in terms of networks). In random-walk terminology $N(\mu)$ different one-dimensional paths make a first passage of the origin at step $2\mu - 1$. Following the analogy further, if the graphs are generated by unbiased coin tossing, counting heads as I and tails as E, then the probability that the graph will drop below the axis, that is, that a first passage will occur, at step $2\mu - 1$ is equal to $v(\mu)$ (Feller, 1957, p. 73–75). Thus, the probability that a link drawn at random from an infinite topologically random network will have magnitude μ, thereby defining a subnetwork with $2\mu - 1$ links, is exactly the same as the probability that in a symmetric random walk the first passage of the origin will occur at step $2\mu - 1$ (Feller, 1957, p. 76).



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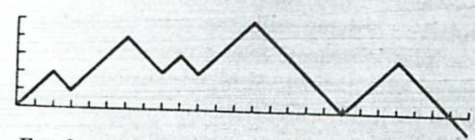

FIG. 2.—Symbolic and graphical representations of a typical channel network.

A symmetric random walk can be regarded as the outcome of a sequence of Bernoulli trials with probability of "success" equal to $\frac{1}{2}$ (Feller, 1957, p. 135, 311); hence, in the infinite sequence of I's and E's corresponding to an infinite topologically random network, E appears at any specified position with probability $\frac{1}{2}$ regardless of the pattern anywhere else in the sequence. All possible subsequences are equally likely to appear, corresponding to the definition of topologically random networks (for proof of the zero probability of occurrence of "pathological" sequences, such as all E's, see Feller, 1957, p. 189–197).

In the sequence corresponding to an infinite network that is not topologically random, on the other hand, although the two letters appear in equal numbers as required by Melton's relationships, the probability that E appears at a specified position is not necessarily $\frac{1}{2}$. Instead, it depends upon the preceding pattern of letters, corresponding to the fact that certain network topologies, hence certain letter sequences, are preferentially developed.

As random walks and Bernoulli trials occur widely in fields ranging from nuclear physics to gambling, their theory is highly developed in many different guises (see, e.g., Feller, 1957, chaps. iii, xi, xiii, xiv); hence, the analogy with infinite topologically random channel networks makes available for application a vast body of theorems and solutions.

GEOMORPHOLOGICAL "LAWS"

The law of large numbers (Feller, 1957, p. 141–142, 189–191) in conjunction with (9c) implies that, in random samples from an infinite topologically random network, the average number of Strahler streams of successive orders will, as the number of samples increases, tend toward a geometric series with ratio $\frac{1}{4}$. Similarly, in conjunction with (5c), it implies that the average number of links of successive orders will tend toward a geometric series with ratio $\frac{1}{2}$. In analogy with Horton's laws of drainage composition, these relationships might be termed the "law of stream numbers" and the "law of link numbers," respectively, although, like Horton's laws, they are not true statistical laws in the sense proposed in my paper on the law of stream numbers (Shreve, 1966, p. 17). Combining these two "laws" leads to the further relationship that the average number of links per stream increases with order as a geometric series with ratio 2.

Derivation of a "law of stream lengths" requires a hypothesis concerning the lengths of individual links. The simplest, and one that does not appear to be too far from the truth, is that all of the links have the same length. In this case, the "law of stream lengths" corresponding to the previous crude "laws" would state that the average length of streams increases with order as a geometric series with ratio 2. In like manner the average total length of streams in basins of given order would increase with successive orders approximately as a geometric series with ratio 4, in good agreement with observation (e.g., Schumm, 1956, p. 604–605).

Suggestive as these geomorphological "laws" are, they are based upon averages over an infinite population and so, like Horton's laws (Shreve, 1966, p. 17), do not supply the complete distributions needed for statistical analysis of observations made on natural populations. Moreover, the hypothesis of constant link length amounts to using the average link length. A better hypothesis, for example, might be that the link lengths have a log-normal distribution (M. A. Melton, unpublished analysis; Schumm 1956, p. 607–608) with mean inversely proportional to drainage density, or perhaps with exterior links having one mean and interior links another. Surprisingly, the statistics of natural link lengths, or even of stream lengths has not received much investigation beyond the work of Schumm (1956, p. 607–608) on the badlands at Perth Amboy.

The total length L of channels in a basin of magnitude μ is

$$L = l(2\mu - 1)\,, \qquad (10a)$$

where l is the link length, which is assumed constant. Similarly, the area A of the basin may be written

$$A = \kappa l^2 (2\mu - 1)\,, \qquad (10b)$$

in which the dimensionless coefficient κ is a constant if, as will be assumed, the drainage density

$$D = L/A = 1/(\kappa l) \qquad (10c)$$

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is uniform, as would be the case in mature topography developed in a homogeneous environment. Elimination of l between (10a) and (10c) gives κ in terms of commonly measured network parameters,

$$\kappa = (2\mu - 1)/LD\,. \qquad (10d)$$

For the 11 networks tabulated by Melton which are definitely free of geologic controls (1957, Table 2, facing p. 88; Chinle Badlands N and S, Finley and Adams Canyon I and II, Sycamore Canyon I and II, Whipple Basin, Mesa Gulch, Dory Hill Basin, Peters Dome Basin, and Cerro Pavo Basin), the mean value of κ is 0.96. Thus, from (10b) the average basin area drained by each link is approximately that of a square of side l; and from (10c) the constant of channel maintenance (Schumm, 1956, p. 607) is approximately equal to the mean link length. Assuming $\kappa = 1$, $\mu \gg 1$, and noting from (9c) that the bifurcation ratio is approximately $\frac{1}{4}$, so that (Shreve, 1966, p. 21)

$$\mu \approx 4^{\omega-1}\,, \qquad (10e)$$

where μ and ω are the basin magnitude and order, respectively, and $\approx$ denotes approximate equality, leads to a "law of basin areas,"

$$A \approx (2/D^2)4^{\omega-1}\,. \qquad (10f)$$

I agree!

ie. $R_A = R_B$

The basin-area ratio is thus 4, again in good agreement with observation (e.g., Schumm, 1956, p. 604–605).

Because from (9c) the bifurcation ratio is approximately $\frac{1}{4}$, a simple summation for $\mu \gg 1$ shows that the total number S_S of Strahler streams in networks of magnitude μ will on the average be close to $4\mu/3$; hence, for all networks of sufficient size, the average ratio S_S/μ will be approximately $\frac{4}{3}$, in agreement with the mean observed ratio of 1.34 for the 11 networks free of geologic controls tabulated by Melton, whose magnitudes range from 19 to 111.

Once more assuming $\kappa = 1$, $\mu \gg 1$, defining $F_S = S_S/A$, and using (10b) and (10c), leads to the approximate relationship

$$F_S/D^2 \approx \tfrac{2}{3}\,. \qquad (11a)$$

This may be compared with the empirical equation

$$F_S/D^2 \approx 0.694 \qquad (11b)$$

obtained by Melton (1958, p. 36–37) by analysis of data from 156 drainage basins. For the 11 networks previously considered, the mean value of F_S/D^2 is 0.71.

In terms of links rather than Strahler streams, the relationship is exactly

$$F/D^2 = \kappa\,, \qquad (12)$$

where F is the average number of links per unit area. This equation is tautological, inasmuch as it follows directly from the definitions of F, D, and κ; it does not depend upon constant link lengths, uniform drainage density, or topologically random networks. Similarly, the ratio F_S/D^2 investigated by Melton (1958, p. 37) is exactly equal to κ divided by the number of links per Strahler stream. Part of the scatter in his diagram of F_S versus D is therefore due to variations in network topology, as he recognized (Melton, 1958, p. 37, 38, 43–46); and the remainder is due to fluctuations in κ. Use of the channel link (a theoretical concept unrecognized at the time) rather than the Strahler stream as the basic channel unit would have eliminated the scatter due to topological variations and reduced the problem to investigation of the behavior of the conceptually simple quantity κ.

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