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ELASTICITY SOLUTION OF AN ADHESIVELY BONDED COVER PLATE OF VARIOUS GEOMETRIES

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ABSTRACT

The plane strain problem of adhesively bonded structures which consist of two different isotropic adherends is considered. By expressing the x-y components of the displacements in terms of Fourier integrals and using the corresponding boundary and continuity conditions, the system of integral equations for the general problem is obtained. Then , these integral equations are solved numerically by applying Gauss-Chebyshev integration scheme.

The shear and the normal stresses in the adhesive are calculated for various geometries and material properties for a stiffened plate under uniaxial tension σ_x. Also the numerical results involving the stress intensity factors and the strain energy release rate are presented. The closed-form expressions for the Fredholm kernels are provided, so that the solution for an arbitrary geometry and material properties can easily be obtained.

The numerical solution of the integral equations indicates that as (h_1/a), (h_4/a) and (h_2/a) decrease the convergence becomes slower and hence computations become costlier. For the general geometry, the contribution of the normal stress is quite significant. For the symmetric geometries, however, the dominant stress is the shear stress. More specifically, the normal stress vanishes if the adherends also happen to be of the same material and the same thickness.

1. INTRODUCTION

In order to optimize performance and fuel consumption, aerospace and marine industries have been turning to the use of advanced (fiber-reinforced organic) composites, more and more in commercial aircraft, military aircraft and marine systems. These materials offer very good strength-to-weight and stiffness-to-weight ratios. However, one major drawback is the strength and fatique penalty introduced by mechanical fasteners at joints. So more sophisticated joining methods are required.

Adhesive bonding, on the other hand, provides a desirable alternative to mechanical fastening because of;

1. Load being carried over a larger area, thus reducing the stress concentration,
2. Higher joint efficiency (relative strength-to-weight of the joint region),
3. No decrease in strength due to fastener holes,
4. Less expensive and simpler fabrication techniques, and
5. Lower maintenance costs.

However, adhesive bonding has its own disadvantages. The load is not carried uniformly over the entire bond area, but instead is confined to a small region along the bond edge. Though not as high as the stresses at a rivet, this highly stressed region, can lead to failure.

The past forty years have witnessed the expenditure of considerable analytic effort in an attempt to describe stress-strain distributions in composite structures formed by the adhesive bonding of materials. The efforts of Goland and Reissner [1], have been extended by the computerised and experimental analyses of numerous investigators.

To gain some insight and to provide criteria for further development of bonding materials and bonding techniques, assumptions have been introduced which are justified only by the analytic tools available to the investigator. Goland and Reissner [1], for example, restrict themselves to adherends of the same material having identical length and thickness, with no stress variation within the adhesive film. With progress in analytic techniques, each succeeding investigator has been able to relax the number of assumptions previously required to obtain a solution.

However, because of the nonhomogeneous nature and of the geometrical complexity of the medium, even for the linearly elastic materials the exact analytical treatment of the problem regarding the stress analysis of the structure is, in general, hopelessly complicated. The existing analytical studies are, therefore, based on certain simplyfying assumptions with regard to the modeling of the adhesive and the adherends. The adherends are usually modeled as an isotropic or orthotropic membrane [2], a plate [1,3,4,5] or an elastic continuum [5,6]. The adhesive on the other hand, is usually treated as a shear spring [2,6], a tension shear-spring [1,7], or is neglected [8]. In this report the adhesively bonded joint problem is considered by assuming both the adhesive and the adherends as elastic layers.

In order to design adhesively bonded structures with high degree of reliability, one needs to recognize that their failure mode is characterized by flow growth and progressive crack propogation.

The energy balance criterian for fracture,based on works of Griffith [9] and Irwin [10] is adopted. It supposes that fracture occurs when sufficient energy is released from the stress field to generate new fracture surfaces at the instant of crack propagation. This strain energy release rate provides a measure of the energy required to extend a crack over a unit area, and is termed the fracture energy. In this report, the fracture energy of an adhesive layer will be determined, since this property has been widely recognized as the appropriate criterion for adhesive failure as in [5,11,12,13,14].

2. FORMULATION OF THE PROBLEM

2.1 Equilibrium Equations

The problem considered is a stiffened plate shown in Figure 1, under the following assumptions;

- The medium is composed of homogeneous, isotropic, elastic layers with different mechanical properties,
- The problem is one of plane strain, that is, the bonded joint is very "wide",
- The only external load acting on the medium is the uniaxial tension, $\sigma_{1x} = \sigma_0$ away from the reinforcement region.

In the plane theory of elasticity the equations of equilibrium in terms of displacements for the isotropic materials can be expressed as;

$$(\lambda + \mu) \frac{\partial e}{\partial x} + \mu \nabla^2 u + X = 0 ,$$

$$(\lambda + \mu) \frac{\partial e}{\partial y} + \mu \nabla^2 v + Y = 0 , \qquad (1a,b)$$

where u,v are the x,y-components of the displacement vector, X-Y are the x-y components of the body force vector,

$$e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \quad ,$$

$$\lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)} \quad ,$$

and μ, ν are the shear modulus and the Poisson's ratio, respectively.

For each of the layers shown in Figure 1, and for no body forces, the equations (1a,b) read as,

$$(\lambda_i + \mu_i)\frac{\partial e}{\partial x} + \mu_i \nabla^2 u = 0 \quad ,$$

$$(\lambda_i + \mu_i)\frac{\partial e}{\partial y} + \mu_i \nabla^2 v = 0 \quad , \; i=1,..,4 \qquad \text{(2a,b)}$$

2.2 Solutions u_i, v_i, σ^i_{yy}, σ^i_{xy}

As it is seen from Figure 1, the medium possesses a geometric symmetry with respect to x=0 plane, so the problem is solved for $x \geqslant 0$. Also note that, the x-y components of the displacements may be expressed as Fourier integrals, since the displacements as well as their derivatives decrease sufficiently rapidly as $|x| \rightarrow \infty$, so that the requirement of absolute integrability is satisfied.

Therefore, assuming the x-y components of the displacements in the i'th layer in the form,

$$u_i(x,y) = \frac{2}{\pi} \int_0^\infty \Phi_i(\alpha,y) \, \mathrm{Sin}(\alpha x) \, d\alpha \quad ,$$

$$v_i(x,y) = \frac{2}{\pi} \int_0^\infty \Psi_i(\alpha,y) \, \mathrm{Cos}(\alpha x) \, d\alpha \quad , \qquad (3a,b)$$

and using the field equations (2a,b), one obtains,

$$u_i(x,y) = \frac{2}{\pi} \int_0^\infty [\, (A_{i1} + A_{i2}y) \, e^{-\alpha y} \; + $$

$$(A_{i3} + A_{i4}y) \, e^{\alpha y} \,] \, \mathrm{Sin}(\alpha x) \, d\alpha \quad ,$$

$$v_i(x,y) = \frac{2}{\pi} \int_0^\infty \left[\left[A_{i1} + (\frac{\kappa_i}{\alpha} + y) A_{i2} \right] e^{-\alpha y} \; + \right.$$

$$\left. \left[-A_{i3} + (\frac{\kappa_i}{\alpha} - y) \, A_{i4} \right] e^{\alpha y} \right] \mathrm{Cos}(\alpha x) \, d\alpha \quad , \qquad (4a,b)$$

where A_{ij}'s are functions of α which will be determined from the continuity and the boundary conditions. Afterwards, the stresses are evaluated by Hooke's Law, and expressed as,

$$\frac{1}{2\mu_i}\sigma^i_{yy} = \frac{2}{\pi}\int_0^\infty \Big[-\big[\alpha(A_{i1} + A_{i2}y) + 2(1-\nu_i)\,A_{i2}\big]\, e^{-\alpha y}$$

$$+ \big[-\alpha(A_{i3} + A_{i4}y) + 2(1-\nu_i)\,A_{i4}\big]\, e^{\alpha y}\Big]\, \mathrm{Cos}(\alpha x)\, d\alpha \quad ,$$

$$\frac{1}{2\mu_i}\tau^i_{yx} = \frac{2}{\pi}\int_0^\infty \Big[-[\,\alpha(A_{i1} + A_{i2}y) + (1-2\nu_i)\,A_{i2}\,]\, e^{-\alpha y}$$

$$+ [\,\alpha(A_{i3} + A_{i4}y) - (1-2\nu_i)\,A_{i4}\,]\, e^{\alpha y}\Big]\, \mathrm{Sin}(\alpha x)\, d\alpha \ . \qquad (5a,b)$$

2.3 The Boundary and the Continuity Conditions

On the boundaries $y=h_1$, $y=-h_4$, the medium possesses the following homogeneous boundary conditions;

$$\sigma^1_{yy}(x,h_1) = 0 \quad , \qquad 0 \leqslant x < \infty \qquad (6a)$$

$$\tau^1_{xy}(x,h_1) = 0 \quad , \quad 0 \leqslant x < \infty \tag{6b}$$

$$\sigma^4_{yy}(x,-h_4) = 0 \quad , \quad 0 \leqslant x < \infty \tag{6c}$$

$$\tau^4_{xy}(x,-h_4) = 0 \quad , \quad 0 \leqslant x < \infty \, . \tag{6d}$$

The continuity conditions require that on the interfaces the stress and the displacement vectors in the adjacent layers be equal, that is,

$$\sigma^1_{yy}(x,h_2) - \sigma^2_{yy}(x,h_2) = 0 \quad , \quad 0 \leqslant x < \infty \tag{6e}$$

$$\tau^1_{xy}(x,h_2) - \tau^2_{xy}(x,h_2) = 0 \quad , \quad 0 \leqslant x < \infty \tag{6f}$$

$$u_1(x,h_2) - u_2(x,h_2) = 0 \quad , \quad 0 \leqslant x < \infty \tag{6g}$$

$$v_1(x,h_2) - v_2(x,h_2) = 0 \quad , \quad 0 \leqslant x < \infty \tag{6h}$$

$$\sigma^4_{yy}(x,-h_3) - \sigma^3_{yy}(x,-h_3) = 0 \quad , \quad 0 \leqslant x < \infty \tag{6i}$$

$$\tau^4_{xy}(x,-h_3) - \tau^3_{xy}(x,-h_3) = 0 \quad , \quad 0 \leqslant x < \infty \tag{6j}$$

$$u_4(x,-h_3) - u_3(x,-h_3) = 0 \quad , \quad 0 \leqslant x < \infty \tag{6k}$$

$$v_4(x,-h_3) - v_3(x,-h_3) = 0 \quad , \quad 0 \leqslant x < \infty \tag{6l}$$

The above conditions (6a-l) provide 12 linear homogeneous algebraic equations in terms of 16 unknowns. So 4 more equations are needed. Those are obtained from the surface which has the crack, that is, at y = 0 ,

$$\sigma^2_{yy}(x,0) - \sigma^3_{yy}(x,0) = 0 \quad , \quad 0 \leqslant x < \infty \qquad (7a)$$

$$\tau^2_{xy}(x,0) - \tau^3_{xy}(x,0) = 0 \quad , \quad 0 \leqslant x < \infty \qquad (7b)$$

$$\sigma^2_{yy}(x,0) = \sigma^3_{yy}(x,0) = g(x) \quad , \quad x \in L \qquad (8a)$$

$$\tau^2_{xy}(x,0) = \tau^3_{xy}(x,0) = f(x) \quad , \quad x \in L \ . \qquad (8b)$$

L is the part of the x-axis without the crack and f(x), g(x) are respectively, shear and normal stresses at the very same region. The mixed boundary conditions at y=0 , and the process of superposition as shown in Figure 2, give rise to the integral equations for the problem. Those are,

$$\lim_{y \to 0} \frac{\partial}{\partial x} \left[u_2(x,y) - u_3(x,y) \right] = \lambda \quad , \quad 0 \leqslant x < \infty$$

$$\lim_{y \to 0} \frac{\partial}{\partial x} \left[v_2(x,y) - v_3(x,y) \right] = 0 \ . \quad 0 \leqslant x < \infty \quad (9a,b)$$

Note that the integral equations have been expressed in terms of the first derivatives of the displacement differences with respect to x. Also note, λ appearing in equation (9a), has the following values depending on the geometry of the medium,

$$\lambda = \frac{\sigma_o}{E_4} \qquad \text{for plane stress}$$

$$\lambda = \frac{\sigma_o\,(1-\nu_4^2)}{E_4} \qquad \text{for plane strain .}$$

2.4 Application of the Boundary and the Continuity Conditions

In equations (8a) and (8b), it has been assumed that

$$\tau_{xy}^2(x,0) = \begin{cases} 0 & , \quad x > a \\ f_1(x) & , \quad x \leqslant a \end{cases}$$

$$\sigma_{yy}^2(x,0) = \begin{cases} 0 & , \quad x > a \\ f_2(x) & , \quad x \leqslant a \end{cases} \qquad (10a,b)$$

where "a" is the bond length as shown in Figure 1. Also note that equations (5a,b) at y=0 gives,

$$\frac{1}{2\mu_2}\sigma_{yy}^2(x,0) = \frac{2}{\pi}\int_0^\infty \Big[-\big[\alpha A_{21} + 2(1-\nu_2)A_{22}\big]$$

$$+\big[-\alpha A_{23} + 2(1-\nu_2)A_{24}\big]\Big] \operatorname{Cos}(\alpha x)\, d\alpha \;, \qquad 0 \leqslant x < \infty$$

$$\frac{1}{2\mu_2}\tau_{xy}^2(x,0) = \frac{2}{\pi}\int_0^\infty \Big[-\big[\alpha A_{21} + (1-2\nu_2)A_{22}\big]$$

$$+\big[\alpha A_{23} - (1-2\nu_2)A_{24}\big]\Big] \operatorname{Sin}(\alpha x)\, d\alpha \;, \qquad 0 \leqslant x < \infty. \qquad (11a,b)$$

The above equations with the conditions stated in (10a,b), read as,

$$\frac{1}{2\mu_2} f_2(x) = \frac{2}{\pi}\int_0^\infty F_2(\alpha) \operatorname{Cos}(\alpha x)\, d\alpha \;, \qquad 0 \leqslant x \leqslant a$$

$$\frac{1}{2\mu_2} f_1(x) = \frac{2}{\pi}\int_0^\infty F_1(\alpha) \operatorname{Sin}(\alpha x)\, d\alpha \;, \qquad 0 \leqslant x \leqslant a \qquad (12a,b)$$

with,

$$F_1(\alpha) = -\alpha A_{21} - 2(1-\nu_2)A_{22} - \alpha A_{23} + 2(1-\nu_2)A_{24}$$

$$F_2(\alpha) = -\alpha A_{21} - (1-2\nu_2)A_{22} + \alpha A_{23} - (1-2\nu_2)A_{24} \,. \qquad (12c,d)$$

Conditons (6), (7) and (12 c,d) provide 16 algebraic equations to be solved for the same number of unknowns in terms of F_1 and F_2. Writing these equations in matrix form,

$$\underset{(16\times16)}{\begin{bmatrix} \text{Coefficient} \\ \text{Matrix} \\ f(\alpha, h_i, \mu_i, \nu_i) \end{bmatrix}} \underset{(16\times1)}{\begin{bmatrix} A_{11} \\ \vdots \\ A_{ij} \\ \vdots \\ A_{44} \end{bmatrix}} = \underset{(16\times1)}{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ F_1(\alpha) \\ F_2(\alpha) \end{bmatrix}} \qquad (13)$$

and multiplying each side with the inverse of the coefficient matrix, gives A_{ij}'s.

$$\begin{bmatrix} A_{ij} \end{bmatrix} = \begin{bmatrix} \text{Coef.} & & \\ \text{Matrix} & Q_{ij} & S_{ij} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ F_1(\alpha) \\ F_2(\alpha) \end{bmatrix} \qquad (14)$$

where Q_{ij}'s and S_{ij}'s (i, j =1,..,4) are the 15'th and 16'th coloumns of the inverse coefficient matrix, respectively. After performing the matrix multiplication,

$$\left[A_{ij}\right] = \left[Q_{ij}F_1 + S_{ij}F_2\right] = \left[Q_{ij}\right] F_1 + \left[S_{ij}\right] F_2 , \quad (15)$$

it may easily be shown that A_{ij}'s can be solved in the following general form,

$$A_{ij}(\alpha) = Q_{ij}(\alpha)F_1 + S_{ij}(\alpha)F_2 . \quad (16)$$

These when substituted in (13) give two systems of equations to solve for Q_{ij}'s and S_{ij}'s as follows,

$$\left[\begin{matrix}\text{Coef.}\\ \text{Matrix}\end{matrix}\right]\left[Q_{ij}\right]F_1(\alpha) + \left[\begin{matrix}\text{Coef.}\\ \text{Matrix}\end{matrix}\right]\left[S_{ij}\right]F_2(\alpha) =$$

$$\begin{bmatrix}0\\ \vdots\\ \vdots\\ 0\\ 1\\ 0\end{bmatrix} F_1(\alpha) + \begin{bmatrix}0\\ \vdots\\ \vdots\\ 0\\ 0\\ 1\end{bmatrix} F_2(\alpha) \quad (17)$$

which leads to;

$$\begin{bmatrix} \text{Coef.} \\ \text{Matrix} \end{bmatrix} \begin{bmatrix} Q_{ij} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{18}$$

$$\begin{bmatrix} \text{Coef.} \\ \text{Matrix} \end{bmatrix} \begin{bmatrix} S_{ij} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{19}$$

(18) and (19) are solved first numerically for every desired value of α . It is definitely much easier and less time consuming process rather than trying to solve them analytically. However, it has its own shortcomings. It has to be kept in mind that certain combination of these Q_{ij}'s and S_{ij}'s (envelope functions) will actually be integrated from zero to infinity at every x and t (see Section 2.5-6). This requires (18) and (19) to be solved at sufficiently many α's. This is obviously a very costly and time consuming job. Especially for the thinner geometries, where the convergence of the envelope functions is very slow,(18) and (19) has to be solved at even more α's, in order to achieve certain

significant figure accuracy in the evaluation of the infinite integrals.

So as a result, (18) and (19) are required to be solved analytically. After rather lengthy manipulations, the closed-form expressions for Q_{ij}'s and S_{ij}'s are found.

Note that, it is most convenient to retain Q_{22} and Q_{24} as the final two unknowns in equations (6a,b,e,f,g,h) and (12c,d). Equations (12c,d) actually provide two equations for Q_{ij}'s and S_{ij}'s, namely;

$$- \alpha A_{21} - 2(1-\nu_2)A_{22} - \alpha A_{23} + 2(1-\nu_2)A_{24} = F_1(\alpha)$$

$$- \alpha A_{21} - (1-2\nu_2)A_{22} + \alpha A_{23} - (1-2\nu_2)A_{24} = F_2(\alpha)$$

by substituding (16) into above equations we find,

$$- \alpha Q_{21} - 2(1-\nu_2)Q_{22} - \alpha Q_{23} + 2(1-\nu_2)Q_{24} = 1$$

$$- \alpha Q_{21} - (1-2\nu_2)Q_{22} + \alpha Q_{23} - (1-2\nu_2)Q_{24} = 0 \qquad \text{(20a,b)}$$

and

$$- \alpha S_{21} - 2(1-\nu_2)S_{22} - \alpha S_{23} + 2(1-\nu_2)S_{24} = 0$$

$$- \alpha S_{21} - (1-2\nu_2)S_{22} - \alpha S_{23} - (1-2\nu_2)S_{24} = 1 \ . \qquad \text{(21a,b)}$$

Solving (6a,b,e,f,g,h), (20a,b) one gets;

$$Q_{22} = \frac{G_3G_5 - G_2G_6}{G_2G_4 - G_5G_1} \tag{22}$$

$$Q_{24} = \frac{G_1G_6 - G_3G_4}{G_2G_4 - G_5G_1} \tag{23}$$

$$Q_{21} = \frac{1}{2\alpha}\left[-1 - \kappa_2 Q_{22} + Q_{24}\right] \tag{24}$$

$$Q_{23} = \frac{1}{2\alpha}\left[1 - Q_{22} + \kappa_2 Q_{24}\right] \tag{25}$$

$$Q_{14} = \left[\frac{-2\alpha\beta(c-1)}{1+\kappa_1}\right] Q_{21} + \left[\frac{-2\alpha\beta(c-1)(h_2 + \kappa_2/2\alpha)}{1+\kappa_1}\right] Q_{22} + \left[\frac{c + \kappa_2}{1 + \kappa_1}\right] Q_{24} \tag{26}$$

$$Q_{12} = \left[\frac{c + \kappa_2}{1 + \kappa_1}\right] Q_{22} + \left[\frac{-2\alpha(1-c)}{\beta(1 + \kappa_1)}\right] Q_{23} + \left[\frac{-2\alpha(1-c)(h_2 - \kappa_2/2\alpha)}{\beta(1+\kappa_1)}\right] Q_{24} \tag{27}$$

$$Q_{13} = \left[\frac{\kappa_1\beta}{2\alpha}\right] Q_{12} + \left[-h_2 + \frac{\kappa_1}{2\alpha}\right] Q_{14}$$

$$+ \left[\frac{-\kappa_2\beta}{2\alpha}\right] Q_{22} + Q_{23} + \left[h_2 - \frac{\kappa_2}{2\alpha} \right] Q_{24} \tag{28}$$

$$Q_{11} = \left[- h_2 - \frac{\kappa_1}{2\alpha} \right] Q_{12} + \left[\frac{-\kappa_1}{2\alpha\beta}\right] Q_{14}$$

$$+ Q_{21} + \left[h_2 + \frac{\kappa_2}{2\alpha} \right] Q_{22} + \left[\frac{\kappa_2}{2\alpha\beta}\right] Q_{24} \quad . \tag{29}$$

Similarly, retaining Q_{32} and Q_{34} in equations (6c,d,i,j,k,l), (7a,b) is the easiest way to obtain,

$$Q_{32} = \frac{G_{14}G_9 - G_{12}G_{11}}{G_{10}G_{12} - G_{13}G_9} \tag{30}$$

$$Q_{34} = \frac{G_{13}G_{11} - G_{14}G_{10}}{G_{10}G_{12} - G_{13}G_9} \tag{31}$$

$$Q_{31} = - \frac{1}{2\alpha} \left[1 + \kappa_3 Q_{32} - Q_{34}\right] \tag{32}$$

$$Q_{33} = \frac{1}{2\alpha} \left[1 - Q_{32} + \kappa_3 Q_{34}\right] \tag{33}$$

$$Q_{42} = \left[\frac{\kappa_3 + d}{1 + \kappa_4}\right] Q_{32} + \left[\frac{-2\alpha\beta^*(1-d)}{1 + \kappa_4}\right] Q_{33}$$

$$+ \left[\frac{-2\alpha\beta^*(d-1)(h_3 + \kappa_3 /2\alpha)}{1 + \kappa_4}\right] Q_{34} \tag{34}$$

$$Q_{44} = [\frac{-2\alpha(d-1)}{\beta^*(1+\kappa_4)}]\, Q_{31} + [\frac{d+\kappa_3}{1+\kappa_4}]\, Q_{34}$$

$$+ [\frac{-2\alpha(d-1)(-h_3+\kappa_3/2\alpha)}{\beta^*(1+\kappa_4)}]\, Q_{32} \qquad (35)$$

$$Q_{41} = [\, h_3 - \frac{\kappa_4}{2\alpha}]\, Q_{42} + [\frac{-\beta^*\kappa_4}{2\alpha}]\, Q_{44}$$

$$+ Q_{31} + [-h_3 + \frac{\kappa_3}{2\alpha}]\, Q_{32} + [\frac{\beta^*\kappa_3}{2\alpha}]\, Q_{34} \qquad (36)$$

$$Q_{43} = [\frac{\kappa_4}{2\alpha\beta^*}]\, Q_{42} + [h_3 + \frac{\kappa_4}{2\alpha}]\, Q_{44}$$

$$+ [\frac{-\kappa_3}{2\alpha\beta^*}]\, Q_{32} + Q_{33} + [-h_3 - \frac{\kappa_3}{2\alpha}]\, Q_{34}\,, \qquad (37)$$

G_i , (i=1,....,16) , β , β^* are defined in Appendix II.

Similarly using equations (6a,b,c,d,e,f,g,h,i,j,k,l) and (21a,b), one obtains,

$$S_{22} = \frac{P_3P_5 - P_2P_6}{P_2P_4 - P_5P_1} \qquad (38)$$

$$S_{24} = \frac{P_1P_6 - P_3P_4}{P_2P_4 - P_5P_1} \tag{39}$$

$$S_{21} = \frac{1}{2\alpha}[-1 - \kappa_2 S_{22} + S_{24}] \tag{40}$$

$$S_{23} = \frac{1}{2\alpha}[-1 - S_{22} + \kappa_2 S_{24}] \tag{41}$$

$$S_{14} = [\frac{-2\alpha\beta(c-1)}{1 + \kappa_1}] S_{21} + [\frac{-2\alpha\beta(c-1)(h_2 + \kappa_2/2\alpha)}{1 + \kappa_1}] S_{22} + [\frac{c + \kappa_2}{1 + \kappa_1}] S_{24} \tag{42}$$

$$S_{12} = [\frac{\kappa_2 + c}{1 + \kappa_1}] S_{22} + [\frac{-2\alpha(1-c)}{\beta(1 + \kappa_1)}] S_{23} + [\frac{-2\alpha(1-c)(h_2 - \kappa_2/2\alpha)}{\beta(1 + \kappa_1)}] S_{24} \tag{43}$$

$$S_{13} = [\frac{\beta\kappa_1}{2\alpha}] S_{12} + [-h_2 + \frac{\kappa_1}{2\alpha}] S_{14} + [\frac{-\beta\kappa_2}{2\alpha}] S_{22} + S_{23} + [h_2 - \frac{\kappa_2}{2\alpha}] S_{24} \tag{44}$$

$$S_{11} = [-(h_2 + \frac{\kappa_1}{2\alpha})] S_{12} + [\frac{-\kappa_1}{2\alpha\beta}] S_{14}$$

$$+ S_{21} + [\, h_2 + \frac{\kappa_2}{2\alpha}]\, S_{22} + [\frac{\kappa_2}{2\alpha\beta}]\, S_{24} \qquad (45)$$

$$S_{32} = \frac{P_{13}P_{11} - P_{14}P_{10}}{P_{10}P_{12} - P_{13}P_9} \qquad (46)$$

$$S_{34} = \frac{P_{14}P_9 - P_{12}P_{11}}{P_{10}P_{12} - P_{13}P_9} \qquad (47)$$

$$S_{31} = - \frac{1}{2\alpha} [\ 1 + \kappa_3 S_{32} - S_{34}\ ] \qquad (48)$$

$$S_{33} = \frac{1}{2\alpha} [\ -1 - S_{32} + \kappa_3 S_{34}\ ] \qquad (49)$$

$$S_{42} = [\frac{\kappa_3 + d}{1 + \kappa_4}]\, S_{32} + [\frac{-2\alpha\beta^*(1-d)}{1 + \kappa_4}]\, S_{33}$$

$$+ [\frac{-2\alpha\beta^*(d-1)(h_3 + \kappa_3 /2\alpha)}{1 + \kappa_4}]\, S_{34} \qquad (50)$$

$$S_{44} = [\frac{-2\alpha(d-1)}{\beta^*(1 + \kappa_4)}]\, S_{31} + [\frac{d + \kappa_3}{1 + \kappa_4}]\, S_{34}$$

$$+ [\frac{-2\alpha(d-1)(-h_3 + \kappa_3 /2\alpha)}{\beta^*\ (1 + \kappa_4)}]\, S_{32} \qquad (51)$$

$$S_{41} = [\, h_3 - \frac{\kappa_4}{2\alpha}]\, S_{42} + [-\frac{\beta^*\kappa_4}{2\alpha}]\, S_{44}$$

$$+ S_{31} + [-h_3 + \frac{\kappa_3}{2\alpha}]\, S_{32} + [\frac{\beta^*\kappa_4}{2\alpha}]\, S_{34} \qquad (52)$$

$$S_{43} = [\frac{\kappa_4}{2\alpha\beta^*}]\, S_{42} + [\, h_3 + \frac{\kappa_4}{2\alpha}]\, S_{44}$$

$$+ [\frac{-\kappa_3}{2\alpha\beta^*}]\, S_{32} + S_{33} + [-h_3 - \frac{\kappa_3}{2\alpha}]\, S_{34} \qquad (53)$$

$P_i(\alpha)$, $(i=1,\ldots\ldots,16)$, β , β^*, c, d are defined in Appendix III .

2.5 Integral Equations

Integral equations for the problem are derived from (9a, b), that is,

$$\lim_{y\to 0} \frac{\partial}{\partial x} \left[u_2(x,y) - u_3(x,y) \right] = \lambda \quad , \qquad 0 \leqslant x < \infty$$

$$\lim_{y\to 0} \frac{\partial}{\partial x} \left[v_2(x,y) - v_3(x,y) \right] = 0 \quad , \qquad 0 \leqslant x < \infty$$

where, one might recall

$$\lambda = \frac{\sigma_o}{E_4} \qquad \text{for plane stress}$$

$$\lambda = \frac{\sigma_o(1-\nu_4^2)}{E_4} \qquad \text{for plane strain .}$$

Substituding (4a) into above equations, and then replacing the corresponding A_{ij}'s with (16), the following equations are obtained,

$$\lim_{y\to o} \frac{2}{\pi}\int_o^\infty \alpha\Big[\left[(Q_{21}(\alpha) + Q_{22}(\alpha)y)\ e^{-\alpha y} + (Q_{23}(\alpha) + Q_{24}(\alpha)y)\ e^{\alpha y}\right]$$

$$- \left[(Q_{31}(\alpha) + Q_{32}(\alpha)y)\ e^{-\alpha y} + (Q_{33}(\alpha) + Q_{34}(\alpha)y)\ e^{\alpha y}\right]\Big]$$

$$\cdot\ F_1(\alpha)\ \mathrm{Cos}(\alpha x)\ d\alpha$$

$$+ \lim_{y\to o} \frac{2}{\pi}\int_o^\infty \alpha\Big[\left[(S_{21}(\alpha) + S_{22}(\alpha)y)\ e^{-\alpha y} + (S_{23}(\alpha) + S_{24}(\alpha)y)\ e^{\alpha y}\right]$$

$$- \left[(S_{31}(\alpha) + S_{32}(\alpha)y)\ e^{-\alpha y} + (S_{33}(\alpha) + S_{34}(\alpha)y)\ e^{\alpha y}\right]\Big]$$

$$\cdot\ F_2(\alpha)\ \mathrm{Cos}(\alpha x)\ d\alpha = \lambda \qquad (54)$$

$$\lim_{y\to o}\frac{2}{\pi}\int_{o}^{\infty}-\alpha\Bigg[\Bigg[\Big(Q_{21}(\alpha)+Q_{22}(\kappa_2/\alpha+y)\Big)e^{-\alpha y}$$

$$+\Big(-Q_{23}(\alpha)+Q_{24}(\alpha)(\kappa_2/\alpha-y)\Big)e^{\alpha y}\Bigg]$$

$$-\Bigg[\Big(Q_{31}(\alpha)+Q_{32}(\alpha)(\kappa_3/\alpha+y)\Big)e^{-\alpha y}$$

$$+\Big(-Q_{33}(\alpha)+Q_{34}(\alpha)(\kappa_3/\alpha-y)\Big)e^{\alpha y}\Bigg]\Bigg].\ F_1(\alpha)\ \mathrm{Sin}(\alpha x)\ d\alpha$$

$$+\lim_{y\to o}\frac{2}{\pi}\int_{o}^{\infty}-\alpha\Bigg[\Bigg[\Big(S_{21}(\alpha)+S_{22}(\alpha)(\kappa_2/\alpha+y)\Big)e^{-\alpha y}$$

$$+\Big(-S_{23}(\alpha)+S_{24}(\alpha)(\kappa_2/\alpha-y)\Big)e^{\alpha y}\Bigg]$$

$$-\Bigg[\Big(S_{31}(\alpha)+S_{32}(\alpha)(\kappa_3/\alpha+y)\Big)e^{-\alpha y}$$

$$+\Big(-S_{33}(\alpha)+S_{34}(\alpha)(\kappa_3/\alpha-y)\Big)e^{\alpha y}\Bigg]\Bigg].F_2(\alpha)\ \mathrm{Sin}(\alpha x)\ d\alpha = 0\ . \tag{55}$$

F_1, F_2 must be replaced by Fourier inversion of (12a,b)

$$F_1(\alpha) = \int_0^\infty \frac{f_1(t)}{2\mu_2} \, \text{Sin}(\alpha t) \, dt$$

$$F_2(\alpha) = \int_0^\infty \frac{f_2(t)}{2\mu_2} \, \text{Cos}(\alpha t) \, dt \quad . \qquad (56a,b)$$

Noting that,

$$f_1(t) = f_2(t) = 0 \quad , \quad t \in L' \quad , \quad y = 0 \qquad (57)$$

where L' is the part of the x-axis containing the crack. Therefore, (55a,b) with (57) would read as follows ,

$$F_1(\alpha) = \int_0^a \frac{f_1(t)}{2\mu_2} \, \text{Sin}(\alpha t) \, dt$$

$$F_2(\alpha) = \int_0^a \frac{f_2(t)}{2\mu_2} \, \text{Cos}(\alpha t) \, dt \quad . \qquad (58a,b)$$

Replacing $F_1(\alpha)$ and $F_2(\alpha)$ in (54), (55) with (57a,b) give two equations of the form ;

$$\lim_{y \to 0} \frac{2}{\pi} \int_0^a \sum_{j=1}^{2} h_{1j}(x,y,t) \, f_j(t) \, dt = 2\mu_2 \lambda$$

$$\lim_{y\to 0} \frac{2}{\pi}\int_0^a \sum_{j=1}^{2} h_{2j}(x,y,t)\ f_j(t)\ dt = 0 . \qquad (59a,b)$$

Here,

$$h_{11}(x,y,t) = \int_0^\infty RE1(\alpha,y)\ Cos(\alpha x)\ Sin(\alpha t)\ d\alpha \quad , \qquad (60a)$$

$$h_{12}(x,y,t) = \int_0^\infty TE1(\alpha,y)\ Cos(\alpha x)\ Cos(\alpha t)\ d\alpha \quad , \qquad (60b)$$

$$h_{21}(x,y,t) = \int_0^\infty RE2(\alpha,y)\ Sin(\alpha x)\ Sin(\alpha t)\ d\alpha \quad , \qquad (60c)$$

$$h_{22}(x,y,t) = \int_0^\infty TE2(\alpha,y)\ Sin(\alpha x)\ Cos(\alpha t)\ d\alpha \quad , \qquad (60d)$$

where,

$$RE1(\alpha,y) = \alpha\Big[\big[(Q_{21}(\alpha) + Q_{22}(\alpha)y)\ e^{-\alpha y} + (Q_{23}(\alpha) + Q_{24}(\alpha)y)\ e^{\alpha y}\big] - \big[(Q_{31}(\alpha) + Q_{32}(\alpha)y)\ e^{-\alpha y} + (Q_{33}(\alpha) + Q_{34}(\alpha)y)\ e^{\alpha y}\big]\Big],$$

$$TE1(\alpha,y) = \alpha\Big[\big[(S_{21}(\alpha) + S_{22}(\alpha)y)\ e^{-\alpha y} + (S_{23}(\alpha) + S_{24}(\alpha)y)\ e^{\alpha y}\big] - \big[(S_{31}(\alpha) + S_{32}(\alpha)y)\ e^{-\alpha y} + (S_{33}(\alpha) + S_{34}(\alpha)y)\ e^{\alpha y}\big]\Big],$$

$$RE2(\alpha,y) = -\alpha\Big[\Big[\big[Q_{21}(\alpha) + Q_{22}(\alpha)(\kappa_2/\alpha + y)\big]\ e^{-\alpha y}$$

$$+\left[-Q_{23}(\alpha) + Q_{24}(\alpha)(\kappa_2/\alpha - y)\right] e^{\alpha y}\Bigg]$$

$$-\left[\left[Q_{31}(\alpha) + Q_{32}(\alpha)(\kappa_2/\alpha + y)\right] e^{-\alpha y}\right.$$

$$\left.+\left[-Q_{33}(\alpha) + Q_{34}(\alpha)(\kappa_3/\alpha - y)\right] e^{\alpha y}\right]\Bigg],$$

$$TE2(\alpha,y) = -\alpha\Bigg[\Bigg[\left[S_{21}(\alpha) + S_{22}(\alpha)(\kappa_2/\alpha + y)\right] e^{-\alpha y}$$

$$+\left[-S_{23}(\alpha) + S_{24}(\alpha)(\kappa_2/\alpha - y)\right] e^{\alpha y}\Bigg]$$

$$-\left[\left[S_{31}(\alpha) + S_{32}(\alpha)(\kappa_3/\alpha + y)\right] e^{-\alpha y}\right.$$

$$\left.+\left[-S_{33}(\alpha) + S_{34}(\alpha)(\kappa_3/\alpha - y)\right] e^{\alpha y}\right]\Bigg]. \qquad (61a,b,c,d)$$

For the physical problem under consideration, the displacement differences on the bond surface are known and the stresses, $f_i(t)$ are unknown, which may be determined from the integral equations given by (59a,b). After determination of $f_i(t)$, all the desired quantities, like the stress intensity factor, the strain energy release rate and stresses can easily be evaluated.

Here it should be clearly noted that in deriving (59a,b), the derivatives of the displacements, rather than displacements themselves, have been used. Also note that the integral equations should be solved under the following equilibrium conditions

$$\int_{-a}^{a} f_1(t)\, dt = 0$$

$$\int_{-a}^{a} f_2(t)\, dt = 0 \quad . \qquad (62a,b)$$

The infinite integrals giving h_{11} and h_{22} can be expressed as a sum of two integrals, the integrands of which, respectively, are of $O(e^{-\alpha y})$ and $O(e^{-2\alpha h_2}, e^{-2\alpha h_3})$ for $\alpha \longrightarrow \infty$. The first leads to a Cauchy Kernel and the second to a Fredholm Kernel. On the other hand, the integrands of the infinite integrals giving h_{12} and h_{21} are of $O(e^{-2\alpha h_2}, e^{-2\alpha h_3})$ for $\alpha \longrightarrow \infty$, hence leading to Fredholm Kernels only (see Section 2.6) .

2.6 Cauchy and Fredholm Kernels

After performing asymptotic expansion as $\alpha \rightarrow \infty$, G_i's and P_i's (Appendix II, Appendix III) are found to be ;

$G_1 = 0$,	$G_{11} = -b_2$,	$P_1 = 0$,	$P_{11} = -b_2$,
$G_2 = -a_3$,	$G_{12} = b_3$,	$P_2 = -a_3$,	$P_{12} = b_3$,
$G_3 = 0$,	$G_{13} = 0$,	$P_3 = 0$,	$P_{13} = 0$,
$G_4 = -a_2$,	$G_{14} = 0$,	$P_4 = -a_2$,	$P_{14} = 0$,
$G_5 = 2\alpha a_4$,	$G_{15} = b_2 b_3$,	$P_5 = 2\alpha a_4$,	$P_{15} = b_2 b_3$,

$G_6 = a_2$, $G_{16} = 0$, $P_6 = -a_2$, $P_{16} = 0$,

$G_7 = 0$, $P_7 = 0$,

$G_8 = a_3 a_2$, $P_8 = -a_3 a_2$,

$G_9 = -2\alpha b_4$, $P_9 = -2\alpha b_4$,

$G_{10} = b_2$, $P_{10} = b_2$,

where (a_i, $i=1,..,4$), (b_i, $i=1,..,4$) are given in Appendix I. Using them in (22),(23),(24),(25),(30),(31),(32),(33) and (38),(39),(40),(41), (46),(47),(48),(49) gives the following results as $\alpha \rightarrow \infty$,

$Q_{21}(\alpha) = -(1 + \kappa_2) / 2\alpha$ $\quad$ $Q_{31}(\alpha) = 0.0$

$Q_{22}(\alpha) = 1.0$ $\quad$ $Q_{32}(\alpha) = 0.0$

$Q_{23}(\alpha) = 0.0$ $\quad$ $Q_{33}(\alpha) = (1 + \kappa_3) / 2\alpha$

$Q_{24}(\alpha) = 0.0$ $\quad$ $Q_{34}(\alpha) = 1.0$,

(63a,b,c,d,e,f,g,h)

and

$S_{21}(\alpha) = (-1 + \kappa_2) / 2\alpha$ $\quad$ $S_{31}(\alpha) = 0.0$

$S_{22}(\alpha) = -1.0$ $\quad$ $S_{32}(\alpha) = 0.0$

$S_{23}(\alpha) = 0.0$ $\quad$ $S_{33}(\alpha) = (-1 + \kappa_3) / 2\alpha$

$S_{24}(\alpha) = 0.0$ $\qquad$ $S_{34}(\alpha) = 1.0$

(64a,b,c,d,e,f,g,h)

Substituting them into equations (64a,b,c,d), and noting $\kappa_3=\kappa_2$, give ,

$RE1(\alpha \longrightarrow \infty, y^+) = RE1\infty$

$$= \alpha \left[\frac{-(1+\kappa_2)}{2\alpha} + y - \frac{(1+\kappa_3)}{2\alpha} + y \right] e^{-\alpha y}$$

$$RE1\infty = [\ -(1+\kappa_2) + 2\alpha y\ ]\ e^{-\alpha y} \quad , \qquad (65a)$$

$TE1(\alpha \longrightarrow \infty, y^+) = TE1\infty$

$$= \alpha \left[\frac{-(1+\kappa_2)}{2\alpha} - y - \frac{-1+\kappa_3}{2\alpha} + y \right] e^{-\alpha y}$$

$$TE1\infty = 0.0 \qquad (65b)$$

$RE2(\alpha \longrightarrow \infty, y^+) = RE2\infty$

$$= -\alpha \left[\frac{-(1+\kappa_2)}{2\alpha} + \frac{\kappa_2}{\alpha} + y + \frac{(1+\kappa_3)}{2\alpha} - \frac{\kappa_3}{\alpha} - y \right] e^{-\alpha y}$$

$$RE2\infty = 0.0 \qquad (65c)$$

$TE2(\alpha \longrightarrow \infty, y^+) = TE2$

$$= -\alpha\left[\frac{-1+\kappa_2}{2\alpha} - \frac{\kappa_2}{\alpha} - y + \frac{-1+\kappa_3}{2\alpha} - \frac{\kappa_3}{\alpha} - y\right] e^{-\alpha y}$$

$$TE2\infty = [\ (1+\kappa_2) + 2\alpha y\ ]\ \ e^{-\alpha y} \quad . \qquad (65d)$$

It is obvious that if the infinite parts are substracted from the integrands, the integrals will be uniformly convergent and give bounded kernels, that is, Fredholm kernels, so $\lim_{y\to o}$, can be put under the integration sign. We then obtain,

$$R1(\alpha,0) = \lim_{y\to o}\left[RE1(\alpha,y) - RE1\infty(\alpha,y)\right]$$

$$= \alpha\ [\ Q_{21}(\alpha) + Q_{23}(\alpha) - Q_{31}(\alpha) - Q_{33}(\alpha)\ ] + (1+\kappa_2)\ ,$$

$$T1(\alpha,0) = \lim_{y\to o}\left[TE1(\alpha,y) - TE1\infty(\alpha,y)\right]$$

$$= \alpha\ [\ S_{21}(\alpha) + S_{23}(\alpha) - S_{31}(\alpha) - S_{33}(\alpha)\ ] \quad , \qquad (66a,b)$$

$$R2(\alpha,0) = \lim_{y\to o}\left[RE2(\alpha,y) - RE2\infty(\alpha,y)\right]$$

$$= -\alpha\ [\ Q_{21}(\alpha) + \frac{\kappa_2}{\alpha}Q_{22}(\alpha) - Q_{23}(\alpha) + \frac{\kappa_2}{\alpha}Q_{24}(\alpha)$$

$$-Q_{31}(\alpha) - \frac{\kappa_3}{\alpha} Q_{32}(\alpha) + Q_{33}(\alpha) - \frac{\kappa_3}{\alpha} Q_{34}(\alpha) \,] \quad ,$$

$$T2(\alpha,0) = \lim_{y \to o} \left[TE2(\alpha,y) - TE2_{\infty}(\alpha,y) \right]$$

$$= -\alpha \,[\, S_{21}(\alpha) + \frac{\kappa_2}{\alpha} S_{22}(\alpha) - S_{23}(\alpha) + \frac{\kappa_2}{\alpha} S_{24}(\alpha)$$

$$- S_{31}(\alpha) - \frac{\kappa_3}{\alpha} S_{32}(\alpha) + S_{33}(\alpha) - \frac{\kappa_3}{\alpha} S_{34}(\alpha) \,]$$

$$- (1 + \kappa_2) \quad . \qquad (66c,d)$$

Integral equations (59a,b) with (66a,b,c,d) would then become ;

$$\lim_{y \to o^+} \frac{2}{\pi} \int_0^a f_1(t) \int_0^\infty RE1_\infty(\alpha,y) \, Cos(\alpha x) \, Sin(\alpha t) \, d\alpha \, dt$$

$$+ \frac{2}{\pi} \int_0^a f_1(t) \int_0^\infty R1(\alpha) \, Cos(\alpha x) \, Sin(\alpha t) \, d\alpha \, dt$$

$$+ \frac{2}{\pi} \int_0^a f_2(t) \int_0^\infty T1(\alpha) \, Cos(\alpha x) \, Cos(\alpha t) \, d\alpha \, dt = 2\mu_2 \lambda \qquad (67a)$$

$$\lim_{y\to 0^+} \frac{2}{\pi}\int_0^a f_2(t) \int_0^\infty TE2\infty(\alpha,y) \ Sin(\alpha x) \ Cos(\alpha t) \ d\alpha \ dt$$

$$+\frac{2}{\pi}\int_0^a f_2(t) \int_0^\infty T2(\alpha) \ Sin(\alpha x) \ Cos(\alpha t) \ d\alpha \ dt$$

$$+\frac{2}{\pi}\int_0^a f_1(t) \int_0^\infty R2(\alpha) \ Sin(\alpha x) \ Sin(\alpha t) \ d\alpha \ dt = 0.0 \quad . \qquad (67b)$$

In (67a,b) the kernels in the first terms are Cauchy type whereas the remaining kernels are bounded in the closed interval $0 \leqslant (x,t) \leqslant a$. Next we change the integration limits from (zero-to-a), to (-a to +a). From the physics of the problem, we note that $\tau(x,y)$ is odd, $\sigma(x,y)$ is even, therefore,

$$f_1(t) = -f_1(-t) \qquad -a < t < +a$$

$$f_2(t) = f_2(-t) \qquad -a < t < +a \quad . \qquad (68a,b)$$

and the integral equations (67a,b) become

$$\lim_{y \to 0^+} \frac{1}{\pi} \int_{-a}^{a} f_1(t) \int_{0}^{\infty} RE1\ (\alpha, y)\ Cos(\alpha x)\ Sin(\alpha t)\ d\alpha\ dt$$

$$+ \int_{-a}^{a} f_1(t)\ K_{11}(x,t)\ dt + \int_{-a}^{a} f_2(t)\ K_{12}(x,t)\ dt = 2\mu_2 \lambda$$

$$\lim_{y \to 0^+} \frac{1}{\pi} \int_{-a}^{a} f_2(t) \int_{0}^{\infty} TE2\ (\alpha, y)\ Sin(\alpha x)\ Cos(\alpha t)\ d\alpha\ dt$$

$$+ \int_{-a}^{a} f_1(t)\ K_{21}(x,t)\ dt + \int_{-a}^{a} f_2(t)\ K_{22}(x,t)\ dt = 0.0 \quad . \tag{69a,b}$$

where the fredholm kernels are given by,

$$K_{11}(x,t) = \frac{1}{\pi} \int_{0}^{\infty} R1(\alpha)\ Cos(\alpha x)\ Sin(\alpha t)\ d\alpha \quad ,$$

$$K_{12}(x,t) = \frac{1}{\pi} \int_{0}^{\infty} T1(\alpha)\ Cos(\alpha x)\ Cos(\alpha t)\ d\alpha \quad ,$$

$$K_{21}(x,t) = \frac{1}{\pi} \int_{0}^{\infty} R2(\alpha)\ Sin(\alpha x)\ Sin(\alpha t)\ d\alpha \quad ,$$

$$K_{22}(x,t) = \frac{1}{\pi}\int_0^{\infty} T2(\alpha)\ \mathrm{Sin}(\alpha x)\ \mathrm{Cos}(\alpha t)\ d\alpha \quad . \qquad (70a,b,c,d)$$

The first parts of (69a,b) are reduced to Cauchy integrals as follows ;

$$\lim_{y\to o^+} \frac{-(1+\kappa_2)}{\pi}\int_{-a}^{+a} f_1(t)\int_0^{\infty} e^{-\alpha y}\ \mathrm{Sin}\alpha(t-x)\ d\alpha\ dt$$

$$= \frac{-(1+\kappa_2)}{\pi}\ \lim_{y\to o^+}\int_{-a}^{+a} f_1(t)\ \frac{(t-x)}{(t-x)^2+y^2}\ dt$$

$$= \frac{-(1+\kappa_2)}{\pi}\int_{-a}^{+a} \frac{f_1(t)\ dt}{(t-x)} \qquad (71a)$$

and

$$\lim_{y\to o^+} \frac{(1+\kappa_2)}{\pi}\int_{-a}^{+a} f_2(t)\int_0^{\infty} e^{-\alpha y}\ (-\mathrm{Sin}\alpha(t-x))\ d\alpha\ dt$$

$$= \frac{-(1+\kappa_2)}{\pi}\ \lim_{y\to o^+}\int_{-a}^{+a} f_2(t)\ \frac{(t-x)}{(t-x)^2+y^2}\ dt \quad .$$

$$= \frac{-(1 + \kappa_2)}{\pi} \int_{-a}^{+a} \frac{f_2(t)\, dt}{(t-x)} \quad . \tag{71b}$$

2.7 Normalization of the Integral Equations

To solve the integral equations, it is convenient to define the following dimensionless variables and functions ;

$$r = x/a \quad , \quad -a \leqslant x \leqslant +a \quad , \quad -1 \leqslant r \leqslant +1$$

$$s = t/a \quad , \quad -a \leqslant t \leqslant +a \quad , \quad -1 \leqslant s \leqslant +1$$

$$\delta = \alpha a \quad , \quad 0 \leqslant \alpha < \infty \quad , \quad 0 \leqslant \delta < \infty$$

$$g_1(s) = f_1(s.a) / \sigma_o$$

$$g_2(s) = f_2(s.a) / \sigma_o \ . \tag{72}$$

With (72), (69) and (62) can be written as,

$$\frac{1}{\pi} \int_{-1}^{+1} \frac{g_1(s)\, ds}{(s-r)} + \int_{-1}^{+1} g_1(s)\, K_{11}(r,s)\, ds + \int_{-1}^{+1} g_2(s)\, K_{12}(r,s)\, ds = P_1 \tag{73a}$$

with,

$$\int_{-1}^{+1} g_1(s)\, ds = 0.0 \quad , \tag{73b}$$

and

$$\frac{1}{\pi}\int_{-1}^{+1} \frac{g_2(s)\, ds}{(s-r)} + \int_{-1}^{+1} g_2(s)\, K_{22}(r,s)\, ds$$

$$+ \int_{-1}^{+1} g_1(s)\, K_{21}(r,s)\, ds = 0.0 \tag{73c}$$

with,

$$\int_{-1}^{+1} g_2(s)\, ds = 0.0 \quad . \tag{73d}$$

Here,

$$K_{11}(r,s) = -\frac{1}{\pi(1+\kappa_2)}\int_0^{\infty} R1(\delta/a)\, \mathrm{Cos}(\delta r)\, \mathrm{Sin}(\delta s)\, d\delta \quad ,$$

$$K_{12}(r,s) = -\frac{1}{\pi(1+\kappa_2)}\int_0^{\infty} T1(\delta/a)\, \mathrm{Cos}(\delta r)\, \mathrm{Cos}(\delta s)\, d\delta \quad ,$$

$$K_{21}(r,s) = -\frac{1}{\pi(1+\kappa_2)}\int_0^{\infty} R2(\delta/a)\, \mathrm{Sin}(\delta r)\, \mathrm{Sin}(\delta s)\, d\delta \quad ,$$

$$K_{22}(r,s) = -\frac{1}{\pi(1+\kappa_2)}\int_0^{\infty} T2(\delta/a)\, \mathrm{Sin}(\delta r)\, \mathrm{Cos}(\delta s)\, d\delta \quad ,$$

(74a, b, c, d)

and,

$$P_1 = - \frac{2\mu_2(1 - \nu_4^2)}{E_4(1 + \kappa_2)} \quad .$$

2.8 Evaluation of the Infinite Integrals in the Fredholm Kernels

The integrands of the infinite integrals (74b,d) go to infinity as $\delta \to 0$, that is, they have a pole at $\delta = o$. Hence, those integrals, if treated separately, will be divergent and their evaluation requires special care.

As δ approaches to zero, $\text{Cos}(\delta s)$ goes to unity, therefore the integrands become independent of "s" thus, because of the single valuedness condition, (73d), the coefficient of the unbounded integrals would vanish. Equations (74b,d) could then be replaced by ;

$$K_{12}(r,s) = - \frac{1}{\pi(1 + \kappa_2)} \int_o^\infty T1(\delta/a)\ \text{Cos}(\delta r)\ (\text{Cos}(\delta s) - 1.0)\ d\delta \quad ,$$

$$K_{22}(r,s) = - \frac{1}{\pi(1 + \kappa_2)} \int_o^\infty T2(\delta/a)\ \text{Sin}(\delta r)\ (\text{Cos}(\delta s) - 1.0)\ d\delta \quad .$$

(75a,b)

3. SOLUTION OF THE SINGULAR INTEGRAL EQUATIONS

3.1 Solution of the Integral Equations

The solution of the singular integral equation

$$\int_{-1}^{+1}\left[\frac{1}{\pi}\,\frac{1}{t-x}+k(x,t)\right]\Phi(t)\,dt = g(x)\ , \qquad -1 \leqslant x \leqslant +1 \tag{76}$$

subject to the single valuedness condition

$$\int_{-1}^{+1}\Phi(t)\,dt = 0 \qquad , \tag{77}$$

is given in [15]. The method has been summarized in Appendix IV .

However, the singular integral equations (73a,c), that appear in this report, have two unknown functions g_1 and g_2. So the solution method that has been described in Appendix IV, should be modified accordingly. Also note that, since $g_i(s)$ has a power singularity 1/2 at the end points, the solution will be sought in the form ;

$$g_i(s) = (1 - s^2)^{-1/2}\,\Phi_i(s). \tag{78}$$

where $\Phi_i(s)$ is a function defined in the interval $-1 \leqslant s \leqslant +1$ and the indices of the singular integral equations are +1.

Following the procedure described in [13], the integral equations may be expressed as

$$\frac{\pi}{N-1}\left[\frac{1}{2}k^*_{11}(r_k,s_1)\,\Phi_1(s_1)+\sum_{i=2}^{N-1}k^*_{11}(r_k,s_i)\,\Phi_1(s_i)+\frac{1}{2}k^*_{11}(r_k,s_N)\,\Phi_1(s_N)\right]+$$

$$\frac{\pi}{M-1}\left[\frac{1}{2}k^*_{12}(r_k,s'_1)\,\Phi_2(s'_1)+\sum_{i=2}^{M-1}k^*_{12}(r_k,s'_i)\,\Phi_2(s'_i)+\frac{1}{2}k^*_{12}(r_k,s'_M)\,\Phi_2(s'_M)\right]=P_1 \quad ,$$

$$k=1,\ldots,(N-1) \qquad (79a)$$

with

$$\frac{1}{2}\Phi_1(s_1)+\sum_{i=2}^{N-1}\Phi_1(s_i)+\frac{1}{2}\Phi_1(s_N)=0 \quad , \qquad (79b)$$

and

$$\frac{\pi}{N-1}\left[\frac{1}{2}k^{*}_{21}(r_j,s_1)\ \Phi_1(s_1)\ +\sum_{i=2}^{N-1}k^{*}_{21}(r_j,s_i)\ \Phi_1(s_i)\right.$$

$$\left.+\ \frac{1}{2}\ k^{*}_{21}(r_j,s_N)\ \Phi_1(s_N)\right]+$$

$$\frac{\pi}{M-1}\left[\frac{1}{2}k^{*}_{22}(r_j,s'_1)\ \Phi_2(s'_1)\ +\sum_{i=2}^{M-1}k^{*}_{22}(r_j,s'_i)\ \Phi_2(s'_i)\right.$$

$$\left.+\ \frac{1}{2}\ k^{*}_{22}(r_j,s'_M)\ \Phi_2(s'_M)\right]=0\quad ;$$

$$j=1,\ldots,(M-1) \qquad (80a)$$

with

$$\frac{1}{2}\ \Phi_2(s'_1)\ +\sum_{i=2}^{M-1}\Phi_2(s'_i)\ +\frac{1}{2}\ \Phi_2(s'_M)\ =\ 0\quad , \qquad (80b)$$

where

$$k^{*}_{11}(r,s)\ =\frac{1}{\pi}\ \ \frac{1}{s-r}+K_{11}(r,s)\quad ,$$

$$k^{*}_{12}(r,s)\ =\ K_{12}(r,s)\qquad ,$$

$$k^{*}_{21}(r,s) = K_{21}(r,s) \quad ,$$

$$k^{*}_{22}(r,s) = \frac{1}{\pi} \frac{1}{s-r} + K_{22}(r,s) \quad . \qquad (81a,b,c,d)$$

The kernels $K_{11}(r,s)$, $K_{12}(r,s)$, $K_{21}(r,s)$, $K_{22}(r,s)$ are defined by (74a, 75a, 74c and 75b), respectively, and,

$$s_i = \mathrm{Cos}\left[\frac{i-1}{N-1} \pi\right] \quad , \quad i=1,\ldots.,N$$

$$r_k = \mathrm{Cos}\left[\frac{2k-1}{2N-2} \pi\right] \quad , \quad k=1,\ldots.,(N-1)$$

$$s'_i = \mathrm{Cos}\left[\frac{i-1}{M-1} \pi\right] \quad , \quad i=1,\ldots.,M$$

$$r_j = \mathrm{Cos}\left[\frac{2j-1}{2M-2} \pi\right] \quad , \quad j=1,\ldots.,(M-1) \quad . \qquad (82a,b,c,d)$$

Equations (79a,b) and (80a,b) would give N+M unknowns

$\Phi_1(s_i)$, $i=1,\ldots.,N$ and

$\Phi_2(s'_i)$, $i=1,\ldots.,M$.

3.2 The Stress Intensity Factors

In Section (2.4) , it has been assumed that

$$\tau_{xy}(x,0) = f_1(x) \qquad -a \leqslant x \leqslant +a$$

$$\sigma_{yy}(x,0) = f_2(x) \quad . \qquad -a \leqslant x \leqslant +a$$

They have been non-dimensionalized in Section (2.7), by defining the following variables and functions,

$$r = x/a \qquad , \qquad -1 \leqslant x \leqslant +1$$

$$g_1(r) = f_1(r.a) \; / \; \sigma_o \quad ,$$

$$g_2(r) = f_2(r.a) \; / \; \sigma_o \quad ,$$

where g_i's themselves are defined in the following form (see section 3.1), for the solution of the integral equations;

$$g_1(r) = (1-r^2)^{-1/2} \; \Phi_1(r) \quad ,$$

$$g_2(r) = (1-r^2)^{-1/2}\ \Phi_2(r) \quad .$$

In plane strain crack problems, the symmetric and antisymmetric components of the stress intensity factors may be defined as,

$$k_1 = \lim_{x \to a} \sqrt{2(a-x)}\ \sigma_{yy}(x,0) \quad , \tag{83}$$

$$k_2 = \lim_{x \to a} \sqrt{2(a-x)}\ \sigma_{xy}(x,0) \quad . \tag{84}$$

Substituting the above definitions in (83) and (84), the constants k_1, k_2 may be related to the functions Φ_1, Φ_2 as follows:

$$k_1 = \lim_{r \to 1} \sqrt{2a(1-r)}\ \ f_2(a.r) \quad ,$$

$$= \lim_{r \to 1} \sqrt{2a(1-r)}\ \sigma_o\ g_2(r) \quad ,$$

$$= \lim_{r \to 1} \sqrt{2a(1-r)}\ \sigma_o \frac{\Phi_2(r)}{\sqrt{1-r^2}} \quad ,$$

$$= \lim_{r \to 1} \sqrt{2a}\ \sigma_o \frac{\Phi_2(r)}{\sqrt{1+r}} \ ,$$

$$k_1 = \sigma_o \sqrt{a} \;\; \Phi_2(r=1) \quad . \qquad (85a)$$

similarly,

$$k_2 = \lim_{r \to 1} \sqrt{2a(1-r)} \;\; f_1(a.r) \quad ,$$

$$= \lim_{r \to 1} \sqrt{2a(1-r)} \;\; \sigma_o \; g_1(r) \quad ,$$

$$= \lim_{r \to 1} \sqrt{2a(1-r)} \;\; \sigma_o \frac{\Phi_1(r)}{\sqrt{1-r^2}} \; ,$$

$$= \lim_{r \to 1} \sqrt{2a} \;\; \sigma_o \frac{\Phi_1(r)}{\sqrt{1+r}} \quad ,$$

$$k_2 = \sigma_o \sqrt{a} \;\; \Phi_1(r=1) \quad . \qquad (85b)$$

So the normalized stress intensity factors are ;

$$k_1 = \frac{k_1}{\sigma_o \sqrt{a}} = \Phi_2(r=1)$$

$$k_2 = \frac{k_2}{\sigma_o \sqrt{a}} = \Phi_1(r=1) \qquad (86a,b)$$

where, $\Phi_1(r=1)$, $\Phi_2(r=1)$ are

$$\Phi_1 \; (s_i, \;\; i=1 \longrightarrow s_1=1 \;) \quad ,$$

$$\Phi_2 \; (s'_i, \; i=1 \longrightarrow s'_1=1) \quad , \qquad (87)$$

can be obtained from the solution of integral equations (79) and (80) .

3.3 The Strain Energy Release Rate

The strain energy release rate is defined as,

$$G = \frac{K_1^2 + K_2^2}{E} \quad , \qquad (88)$$

where,

$$E \;=\; E \qquad , \text{ for plane stress}$$

$$E \;=\; E \;/\; (1 - \nu^2) \quad , \text{ for plane strain}$$

$$K = \sqrt{\pi} \;\; k \quad .$$

So for our problem, which is a plane strain case, we have

$$G = \frac{\pi(1 - \nu_2^2)}{E_2} (k_1^2 + k_2^2) \quad ,$$

where k_i's are defined by (85a,b), one gets

$$G = \frac{1 + \kappa_2}{8\mu_2} \pi a \, \sigma_o^2 \, (\Phi_1{}^2(1) + \Phi_2^2(1)) \quad . \tag{89}$$

Note that Φ_i's are given by equations (87).

4. RESULTS AND DISCUSSION

The system of integral equations, with the additional conditions (73 a,b,c,d) is solved by the technique described in Appendix IV and Section (3.1). The main problem encountered is the evaluation of the infinite integrals rather than the solution of the integral equations. Unless these integrals, which are to be used later to set up the algebraic system, calculated with sufficiently high accuracy, the solution of the integral equations will not obviously produce dependable results beyond certain number of significant digits. So the numerical integration scheme has a vital importance.

Before going into any integration technique, one should first be able to define the envelope functions, RE1(α), TE1(α), RE2(α), TE2(α), (61 a,b,c,d). Previously, these functions (as mentioned in Section 2.4) were unavailable in closed form. All of their desired properties, including the asymptotic behaviours for α approaching infinity or zero, were accurately determined by plotting and using certain tricks, such as multiplying with the powers of alpha in order to find out the singular behaviour around zero. Since the analytical expressions for the envelope functions were not available, they had to be calculated numerically, by solving sixteen algebraic equations each time, say "n" times, for every α_i,

$i=1,\ldots,n$.

The number "n", which has been mentioned in the previous paragraph, is actually the number of points needed for the numerical integration scheme to give sufficient accuracy. So, obviously "n" depends on the convergence of the integrands in (74 a,c) and (75 a,b). More specifically, slower the convergence, larger will be the "n", or vice versa. A closer look, however, reveals that (74c) and (75a) do not in themselves, have a convergence problem. The convergence of (74a) and (75b), on the other hand, is greatly affected by the adhesive thickness. As it can be seen from Table 1,2 , convergence is very slow for thin geometries. So, as a result, we end up with a very costly procedure, trying to solve sixteen equations "n" times. Because of this major drawback, equation systems (18) and (19) are solved analytically in order to obtain closed-form solutions for the envelope functions.

After having defined the envelope functions, one can then proceed with the selection of the appropriate integration scheme for each of the infinite integrals (74 a,c) and (75 a,b). As mentioned in the previous paragraph, the integrands of the equations (74c) and (75a) do not possess a convergence problem. So, they are integrated from zero to infinity by using Laguerre polynomials in one step. However, the other two, that is, (74a) and (75b) needed special treatment, because of the behaviours of their integrands.

After a closer look, one realizes that the envelope functions of these integrands (74a, 75b) can be handled more easily and accurately, if they are examined in three intervals, instead of just one going from zero to infinity. First, from zero to A , where the functions are very steep and relatively large in magnitude. Then from A to B , where the functions are smoothly decreasing in magnitude, and finally from B to infinity where the envelope functions have been replaced by their asymptotic expressions for large alpha for computational reasons. The constants A and B , depend on the geometry and the material properties of the medium, however, in general A lies between 1 and 5 , and B is around 500 . So the equations (74a), (75b) are integrated in three steps using Filon's integration scheme, first from zero to A, then A to B, and finally from B to infinity.

Finally, the system of integral equations (73 a,c) is solved using Gauss-Chebyshev integration scheme as discussed in Appendix IV. It should be pointed out that, in order to build up the algebraic system for the solution, the kernels, $K_{11}(r,s)$, $K_{12}(r,s)$, $K_{21}(r,s)$, $K_{22}(r,s)$, (74 a,c and 75 a,b), that is, the infinite integrals, have to be evaluated (N x (N-1)), (M x (N-1)), (N x (M-1)) and (M x (M-1)) times at corresponding "r" and "s", respectively. N and M are the number of points for which the unknown functions $\Phi_1(s_i)$, $i=1,\ldots,N$, $\Phi_2(s_j)$, $j=1,\ldots,M$,are evaluated (see Section 3.1). However, taking into consideration the symmetry and the anti-symmetry of the kernels and the unknown functions Φ_1 and Φ_2, these numbers can be reduced to 1/4'th of their original values which means a great deal of saving in computational time and money. This concludes almost all of the numerical considerations regarding the solution of the problem. The results obtained are discussed in the following paragraphs.

The material constants used in the calculations, unless otherwise is specified, are as follows,

Upper and Lower Adherends : Aluminum

$$E_1 = E_4 = 10.5 \times 10^6 \text{ psi} ,$$
$$\mu_1 = \mu_4 = 3.9474 \times 10^6 \text{ psi} ,$$
$$\nu_1 = \nu_4 = 0.33 ,$$

Adhesive : Epoxy

$$E_2 = E_3 = 0.28 \times 10^6 \text{ psi} ,$$
$$\mu_2 = \mu_3 = 1.0 \times 10^5 \text{ psi} ,$$
$$\nu_2 = \nu_3 = 0.40 .$$

The results for the specimen with equal thickness adherends which has the same material properties have been tabulated in Tables

3-10. Corresponding graphs are Figures 3-6 for Tables 4,5,7 and 8 respectively. Since the only applied load is uniaxial tension, the specimen is free to bend. Consequently, in the case of identical adherends the normal stress in the adhesive is found to be zero, which agrees with [4]. Also, it was observed that the adhesive shear stress, the corresponding stress intensity factor and the strain energy release rate increase as the adhesive thickness decreases. Table 5 shows the effect of adherend ticknesses on the adhesive shear stress.

A special geometry is studied in Tables 9 and 10, for comparision with [4]. If the stresses are to be calculated at specific distances away from the right end, rather then at specific values of the non-dimensional variable (x/a), the similarity will become apparent. So we may conclude that the stresses are independent of bond length, hence, the strain energy release rate turn out to be constant(Figure 7). The similar result is found in [4] by using tne plate theory.

Tables 11-20 and Figures 8-10 give the results for the specimen having similar adherends with different thicknesses. In Table 14, upper plate is less stiff than the lower plate, while in Table 11 the relative stiffness is reversed. This is accomplished simply by varying the adherend thicknesses. The peak normal stress changes from tension in Table 14 to compression in Table 11, while its magnitude remains almost the same. The shear stress is compressive in both cases as expected. Tables 13 and 15 shows the effect of (h_1/a) and (h_4/a) ratios on the adhesive stresses, respectively. Again for $h_1 > h_4$ (Table 13) there is compressive normal stress, for $h_1 < h_4$ (Table 15) there is tensile normal stress. Also it has been observed that the adhesive stresses, the stress intensity factors and the strain energy release rate increase as the adherend thickness increases. The same behaviour can also be seen in Tables 5 and 8. This trend has been noted in [3], [6].

Tables 21-28 give the results for the specimen having dissimilar adherends. The adhesive shear stress increases as the shear modulus of the upper plate, μ_1 increases relative to the shear modulus of the lower plate μ_4, no matter what their thickness ratio is. However, the peak normal stress is compressive and increases with increases with increasing μ_1 for $h_1 > h_4$ (Table 21), tensile and decreases with increasing μ_1 for $h_1 < h_4$ (Table 22), and is tensile for $\mu_1 < \mu_4$, zero for $\mu_1 = \mu_4$, compressive for $\mu_1 > \mu_4$ for equal thickness adherends (Tables 23,24). Same trend is observed for the corresponding stress intensity factors (Tables 25,26,27,28 and Figures 11-13). The strain energy release rate, however is as consistent as the shear stress, since shear is the dominant stress. Therefore G increases as μ_1 increases in all three cases, $h_1 > h_4$, $h_1 = h_4$ and $h_1 < h_4$.

The evaluation of the infinite integrals and consequently the numerical solutions of the integral equations as mentioned before, becomes very difficult as the thicknesses decrease, resulting in costlier computations. For these thin geometries convergence becomes very slow, hence no results are given. However, it should be mentioned that in this case the stresses tend to oscillate near the crack tip, which happens to be in agreement with [16].

T A B L E S

Table 1. The effect of adhesive thickness on the envelope function R1(α) (equation 66a).

$(h_1/a)=(h_4/a)=1.0$, a=1.0 in.

α	R1(α)		
	$(h_2/a)=0.0025$	$(h_2/a)=0.005$	$(h_2/a)=0.05$
5.878	2.272879	2.213900	1.279204
10.755	2.223983	2.116758	.763813
20.510	2.126674	1.926407	.430288
30.265	2.030539	1.744085	.348293
40.020	1.935993	1.572448	.265924
54.653	1.797995	1.339239	.138605
64.408	1.709048	1.201911	.078680
74.163	1.622896	1.078744	.041523
83.918	1.539796	.970431	.020802
93.673	1.459961	.876033	.010035
103.429	1.383560	.794602	.004705
113.184	1.310719	.724990	.002158
122.939	1.241522	.665952	.000973
132.694	1.176016	.616218	.000432
142.449	1.114207	.574549	.000189
152.204	1.056074	.539778	.000082
161.959	1.001560	.510833	.000035
171.714	.950589	.486748	.000015
181.469	.903059	.466669	.000006
191.224	.858851	.449849	.000003
200.980	.817834	.435643	.000001
210.735	.779864	.423500	.0
220.490	.744790	.412957	.0
230.245	.712455	.403622	.0
240.000	.682701	.395173	.0

Table 2. The effect of adhesive thickness on the envelope function T2(α) (equation 66d).

$(h_1/a)=(h_4/a)=1.0$, a=1.0 in.

α	T2(α)		
	$(h_2/a)=0.0025$	$(h_2/a)=0.005$	$(h_2/a)=0.05$
5.878	-2.322568	-2.313114	-2.133485
10.755	-2.314808	-2.297474	-1.938180
20.510	-2.299089	-2.265769	-1.486075
30.265	-2.283301	-2.233475	-1.035337
40.020	-2.267406	-2.200334	-.654195
54.653	-2.243282	-2.148607	-.277256
64.408	-2.226963	-2.112594	-.143009
74.163	-2.210420	-2.075291	-.070119
83.918	-2.193628	-2.036698	-.033151
93.673	-2.176563	-1.996856	-.015263
103.429	-2.159209	-1.955842	-.006888
113.184	-2.141551	-1.913753	-.003060
122.939	-2.123578	-1.870701	-.001342
132.694	-2.105284	-1.826807	-.000582
142.449	-2.086666	-1.782191	-.000250
152.204	-2.067723	-1.736974	-.000107
161.959	-2.048458	-1.691273	-.000045
171.714	-2.028877	-1.645198	-.000019
181.469	-2.008986	-1.598856	-.000008
191.224	-1.988795	-1.552346	-.000003
200.980	-1.968316	-1.505762	-.000001
210.735	-1.947559	-1.459192	-.000001
220.490	-1.926539	-1.412722	.0
230.245	-1.905270	-1.366429	.0
240.000	-1.883765	-1.320391	.0

Table 3. The effect of (h_2/a) ratio on the adhesive stresses σ and τ with equal thickness adherends.

$(h_1/a)=(h_4/a)=0.35$, a=1.0 in.

h_2/a →	0.0025	0.004	0.005	0.006
x/a	$-\tau/\sigma_0$	$-\tau/\sigma_0$	$-\tau/\sigma_0$	$-\tau/\sigma_0$
.99997	1.799	1.782	1.775	1.767
.99883	.363	.335	.324	.316
.99533	.263	.222	.209	.200
.98951	.246	.194	.174	.162
.98137	.223	.188	.167	.151
.97094	.203	.176	.161	.146
.95825	.188	.162	.151	.140
.94331	.175	.148	.140	.132
.92617	.164	.135	.128	.122
.90687	.151	.123	.116	.111
.88546	.138	.112	.105	.101
.86197	.124	.102	.0952	.0916
.83647	.108	.0922	.0859	.0826
.80902	.0928	.0827	.0772	.0741
.77967	.0775	.0734	.0690	.0663
.74851	.0631	.0644	.0613	.0590
.71560	.0500	.0558	.0539	.0522
.68102	.0385	.0476	.0470	.0459
.64484	.0290	.0400	.0406	.0401
.60716	.0212	.0331	.0347	.0348
.56806	.0152	.0270	.0293	.0299
.52764	.0106	.0217	.0245	.0255
.48598	.00716	.0171	.0203	.0216
.44319	.00477	.0133	.0166	.0181
.35460	.00192	.00772	.0107	.0123
.30902	.00118	.00574	.00842	.00995
.26271	.00073	.00421	.00652	.00790
.16836	.00021	.00207	.00356	.00453
.07243	.0	.00076	.00139	.00182
.02416	.0	.00025	.00046	.00060

Table 3. (cont.)

$h_2/a \rightarrow$	0.007	0.008	0.009	for all h_2/a ratios
x/a	$-\tau/\sigma_o$	$-\tau/\sigma_o$	$-\tau/\sigma_o$	σ/σ_o
.99997	1.759	1.751	1.743	0.0
.99883	.310	.305	.301	0.0
.99533	.192	.186	.180	0.0
.98951	.154	.148	.143	0.0
.98137	.140	.132	.125	0.0
.97094	.135	.126	.118	0.0
.95825	.130	.121	.114	0.0
.94331	.123	.116	.109	0.0
.92617	.116	.110	.104	0.0
.90687	.107	.102	.0977	0.0
.88546	.0982	.0948	.0912	0.0
.86197	.0892	.0868	.0842	0.0
.83647	.0806	.0789	.0771	0.0
.80902	.0724	.0712	.0700	0.0
.77967	.0648	.0638	.0630	0.0
.74851	.0577	.0570	.0565	0.0
.71560	.0512	.0506	.0503	0.0
.68102	.0451	.0447	.0446	0.0
.64484	.0396	.0393	.0393	0.0
.60716	.0346	.0344	.0345	0.0
.56806	.0300	.0300	.0301	0.0
.52764	.0258	.0259	.0261	0.0
.48598	.0221	.0223	.0225	0.0
.44319	.0187	.0190	.0193	0.0
.35460	.0131	.0135	.0138	0.0
.30902	.0107	.0111	.0114	0.0
.26271	.00863	.00902	.00929	0.0
.16836	.00506	.00536	.00555	0.0
.07243	.00207	.00221	.00229	0.0
.02416	.00068	.00073	.00076	0.0

Table 4. The effect of (h_2/a) ratio on the adhesive stresses σ and τ with equal thickness adherends.

$(h_1/a)=(h_4/a)=0.25$, a=1.0 in.

h_2/a →	0.0025	0.004	0.005	0.006
x/a	$-\tau/\sigma_o$	$-\tau/\sigma_o$	$-\tau/\sigma_o$	$-\tau/\sigma_o$
.99997	1.515	1.496	1.488	1.480
.99883	.305	.281	.272	.265
.99533	.221	.185	.175	.167
.98951	.205	.162	.145	.135
.98137	.185	.156	.138	.125
.97094	.167	.145	.132	.120
.95825	.153	.132	.123	.114
.94331	.141	.119	.112	.106
.92617	.130	.107	.101	.0969
.90687	.118	.0960	.0908	.0875
.88546	.105	.0860	.0809	.0782
.86197	.0909	.0765	.0718	.0694
.83647	.0768	.0675	.0634	.0613
.80902	.0629	.0589	.0556	.0538
.77967	.0497	.0506	.0483	.0469
.74851	.0379	.0428	.0416	.0406
.71560	.0278	.0356	.0355	.0350
.68102	.0196	.0291	.0299	.0298
.64484	.0134	.0233	.0249	.0252
.60716	.00872	.0183	.0204	.0212
.56806	.00550	.0142	.0166	.0176
.52764	.00334	.0107	.0133	.0145
.48598	.00193	.0796	.0105	.0118
.44319	.00110	.00581	.00822	.00957
.35460	.00029	.00293	.00482	.00605
.30902	.00014	.00203	.00362	.00473
.26271	.0	.00138	.00268	.00363
.16836	.0	.00059	.00135	.00197
.07243	.0	.00020	.00050	.00077
.02416	.0	.0	.00016	.00025

Table 4. (cont.)

h_2/a →	0.007	0.008	0.009	for all h_2/a ratios
x/a	$-\tau/\sigma_o$	$-\tau/\sigma_o$	$-\tau/\sigma_o$	σ/σ_o
.99997	1.471	1.463	1.455	0.0
.99883	.259	.255	.251	0.0
.99533	.160	.155	.150	0.0
.98951	.128	.123	.119	0.0
.98137	.116	.109	.103	0.0
.97094	.111	.103	.0970	0.0
.95825	.106	.0985	.0926	0.0
.94331	.0995	.0933	.0879	0.0
.92617	.0922	.0874	.0827	0.0
.90687	.0842	.0807	.0770	0.0
.88546	.0760	.0735	.0709	0.0
.86197	.0678	.0662	.0644	0.0
.83647	.0601	.0591	.0579	0.0
.80902	.0528	.0522	.0515	0.0
.77967	.0462	.0458	.0455	0.0
.74851	.0401	.0399	.0398	0.0
.71560	.0346	.0346	.0346	0.0
.68102	.0297	.0298	.0300	0.0
.64484	.0254	.0255	.0258	0.0
.60716	.0215	.0217	.0220	0.0
.56806	.0181	.0184	.0187	0.0
.52764	.0151	.0155	.0158	0.0
.48598	.0125	.0130	.0133	0.0
.44319	.0103	.0108	.0111	0.0
.35460	.00679	.00724	.00758	0.0
.30902	.00541	.00584	.00615	0.0
.26271	.00424	.00462	.00490	0.0
.16836	.00238	.00265	.00283	0.0
.07243	.00095	.00107	.00115	0.0
.02416	.00031	.00035	.00038	0.0

Table 5. The effect of (h_1/a , h_4/a) ratios on the adhesive stresses σ and τ with equal thickness adherends.

$(h_2/a)=0.0025$, $h_1=h_4$, $a=1.0$ in.

h_1/a →	0.25	0.30	0.35	0.50	1.0	for all h_1/a ratios
x/a	$-\tau/\sigma_o$	$-\tau/\sigma_o$	$-\tau/\sigma_o$	$-\tau/\sigma_o$	$-\tau/\sigma_o$	σ/σ_o
.99997	1.515	1.663	1.799	2.157	3.059	0.0
.99883	.305	.336	.363	.436	.619	0.0
.99533	.221	.243	.263	.317	.451	0.0
.98951	.205	.226	.246	.297	.424	0.0
.98137	.185	.205	.223	.271	.389	0.0
.97094	.167	.186	.203	.248	.358	0.0
.95825	.153	.171	.188	.231	.337	0.0
.94331	.141	.159	.175	.217	.321	0.0
.92617	.130	.147	.164	.205	.307	0.0
.90687	.118	.135	.151	.193	.293	0.0
.88546	.105	.122	.138	.179	.278	0.0
.86197	.0909	.108	.124	.164	.261	0.0
.83647	.0768	.0932	.108	.148	.243	0.0
.80902	.0629	.0783	.0928	.131	.223	0.0
.77967	.0497	.0639	.0775	.114	.203	0.0
.74851	.0379	.0506	.0631	.0973	.182	0.0
.71560	.0278	.0388	.0500	.0817	.163	0.0
.68102	.0196	.0289	.0385	.0674	.144	0.0
.64484	.0134	.0208	.0290	.0547	.127	0.0
.60716	.00872	.0145	.0212	.0437	.111	0.0
.56806	.00550	.00982	.0152	.0345	.0964	0.0
.52764	.00334	.00646	.0106	.0268	.0836	0.0
.48598	.00193	.00408	.00716	.0205	.0720	0.0
.44319	.00110	.00254	.00477	.0155	.0618	0.0
.35460	.00029	.00086	.00192	.00845	.0442	0.0
.30902	.00014	.00048	.00118	.00611	.0367	0.0
.26271	.0	.00029	.00073	.00437	.0300	0.0
.16836	.0	.0	.00021	.00200	.0179	0.0
.07243	.0	.0	.0	.00074	.00749	0.0
.02416	.0	.0	.0	.00025	.00250	0.0

Table 6. The effect of (h_2/a) ratio on the normalized stress intensity factors $(k_1/k_o\ ,\ k_2/k_o)$ and the strain energy release rate, G/G_o, with equal thickness adherends.

$(h_1/a)=(h_4/a)=0.35\ ,\ k_o=\sigma_o\sqrt{a}$

$G_o=\sigma_o^2 a/E_2\ ,\ a=1.0$ in.

h_2/a	0.0025	0.004	0.005	0.006
k_1/k_o	0.0	0.0	0.0	0.0
k_2/k_o	-.01394	-.01380	-.01375	-.01369
G/G_o	.513E-3	.503E-3	.499E-3	.495E-3

h_2/a	0.007	0.008	0.009
k_1/k_o	0.0	0.0	0.0
k_2/k_o	-.01363	-.01356	-.01350
G/G_o	.490E-3	.485E-3	.481E-3

Table 7. The effect of (h_2/a) ratio on the normalized stress intensity factors $(k_1/k_o$, $k_2/k_o)$ and the strain energy release rate, G/G_o, with equal thickness adherends.

$(h_1/a)=(h_4/a)=0.25$, $k_o=\sigma_o\sqrt{a}$

$G_o=\sigma_o^2 a/E_2$, $a=1.0$ in.

h_2/a	0.0025	0.004	0.005	0.006
k_1/k_o	0.0	0.0	0.0	0.0
k_2/k_o	-.01173	-.01159	-.01153	-.01146
G/G_o	.363E-3	.354E-3	.351E-3	.347E-3

h_2/a	0.007	0.008	0.009
k_1/k_o	0.0	0.0	0.0
k_2/k_o	-.01140	-.01133	-.01127
G/G_o	.343E-3	.339E-3	.335E-3

Table 8. The effect of (h_1/a , h_4/a) ratios on the normalized stress intensity factors (k_1/k_o , k_2/k_o) and the strain energy release rate, G/G_o, with equal thickness adherends.

$(h_2/a)=0.0025$, $k_o=\sigma_o\sqrt{a}$, $h_1=h_4$

$G_o=\sigma_o^2 a/E_2$, $a=1.0$ in.

h_1/a	0.25	0.30	0.35	0.50	1.0
k_1/k_o	0.0	0.0	0.0	0.0	0.0
k_2/k_o	-.01173	-.01288	-.01394	-.01671	-.02369
G/G_o	.363E-3	.438E-3	.513E-3	.737E-3	.148E-2

Table 9. The effect of N , the number of unknown functions selected in the solution of the system of integral equations (Section 3.1) , on the stress intensity factor and the strain energy release rate for thin geometries and the comparision of (G/G_o) with the plate solution in Ref.[4] .
$h_1=h_4=0.125$ in., $h_2=0.0025$ in.

$k_o=\sigma_o\sqrt{a}$, $G_o=\sigma_o^2 a/E_2$.

$G_{plate}=0.2581$, $\sigma_o=2E+4$ lb/in^2 .

					Plate Soln.
		N=26	N=29	N=33	
a=1.0	k_1/k_o	0.0	0.0	0.0	0.0
	k_2/k_o	-.00816	-.00821	-.00818	
	G/G_o	.176E-3	.178E-3	.177E-3	.181E-3
		N=37	N=40	N=45	
a=2.0	k_1/k_o	0.0	0.0	0.0	0.0
	k_2/k_o	-.00578	-.00581	-.00579	
	G/G_o	.882E-4	.891E-4	.885E-4	.903E-4
		N=41	N=45		
a=3.0	k_1/k_o	0.0	0.0		0.0
	k_2/k_o	-.00465	-.00471		
	G/G_o	.571E-4	.585E-4		.602E-4

Table 10. The effect of bond length (a), on the adhesive stresses σ and τ, with equal thickness adherends, for thin geometries.

$h_1=h_4=0.125$ in., $h_2=0.0025$ in.

a ⟶	1.0	2.0	3.0	1.0,2.0,3.0
x/a	$-\tau/\sigma_o$	$-\tau/\sigma_o$	$-\tau/\sigma_o$	σ/σ_o
.99997	1.052	.748	.609	0.0
.99938	.265	.208	.189	0.0
.99751	.172	.151	.144	0.0
.99440	.144	.137	.124	0.0
.99005	.138	.121	.110	0.0
.98447	.131	.106	.0990	0.0
.97766	.121	.0943	.0861	0.0
.96963	.109	.0832	.0704	0.0
.96039	.0979	.0718	.0526	0.0
.94996	.0872	.0600	.0350	0.0
.93834	.0774	.0479	.0202	0.0
.92556	.0684	.0362	.00993	0.0
.91162	.0600	.0258	.00408	0.0
.89655	.0522	.0170	.00139	0.0
.88036	.0448	.0104	.00038	0.0
.86307	.0380	.00578	.0	0.0
.84471	.0317	.00293	.0	0.0
.82529	.0261	.00135	.0	0.0
.80485	.0210	.00054	.0	0.0
.78340	.0167	.00020	.0	0.0
.76098	.0129	.0	.0	0.0
.73761	.00983	.0	.0	0.0
.71332	.00731	.0	.0	0.0
.66211	.00378	.0	.0	0.0
.60759	.00178	.0	.0	0.0
.55006	.00076	.0	.0	0.0
.48978	.00029	.0	.0	0.0
.42706	.00010	.0	.0	0.0
.39488	.0	.0	.0	0.0
.01765	.0	.0	.0	0.0

Table 11. The effect of (h_2/a) ratio on the adhesive stresses σ and τ with different thickness adherends.

$(h_1/a)=1.0$, $(h_4/a)=0.5$, a=1.0 in.

	$(h_2/a)=0.03$		$(h_2/a)=0.04$	
x/a	$-\tau/\sigma_o$	σ/σ_o	$-\tau/\sigma_o$	σ/σ_o
.99997	2.174	-.523	2.023	-.464
.99883	.356	-.0891	.330	-.0778
.99533	.190	-.0527	.173	-.0441
.98951	.139	-.0444	.124	-.0354
.98137	.116	-.0426	.101	-.0327
.97094	.101	-.0420	.0879	-.0318
.95825	.0912	-.0409	.0786	-.0312
.94331	.0844	-.0389	.0716	-.0301
.92617	.0801	-.0363	.0666	-.0284
.90687	.0771	-.0334	.0631	-.0264
.88546	.0745	-.0301	.0606	-.0240
.86197	.0717	-.0267	.0583	-.0215
.83647	.0689	-.0232	.0561	-.0189
.80902	.0659	-.0197	.0538	-.0162
.77967	.0629	-.0162	.0514	-.0134
.74851	.0597	-.0128	.0489	-.0107
.71560	.0565	-.00950	.0463	-.00794
.68102	.0532	-.00632	.0437	-.00531
.64484	.0499	-.00326	.0411	-.00276
.60716	.0466	-.00034	.0384	-.00033
.56806	.0432	.00243	.0356	.00197
.52764	.0398	.00505	.0329	.00412
.48598	.0364	.00750	.0301	.00612
.44319	.0330	.00977	.0273	.00795
.35460	.0261	.0137	.0216	.0111
.30902	.0226	.0154	.0188	.0124
.26271	.0191	.0168	.0159	.0135
.16836	.0122	.0190	.0102	.0152
.07243	.00523	.0203	.00436	.0162
.02416	.00174	.0205	.00145	.0163

Table 11. (cont.)

x/a	$(h_2/a)=0.05$		$(h_2/a)=0.06$	
	$-\tau/\sigma_o$	σ/σ_o	$-\tau/\sigma_o$	σ/σ_o
.99997	1.905	-.419	1.806	-.385
.99883	.309	-.0695	.292	-.0635
.99533	.160	-.0384	.151	-.0344
.98951	.113	-.0297	.105	-.0260
.98137	.0912	-.0266	.0838	-.0227
.97094	.0786	-.0255	.0716	-.0214
.95825	.0700	-.0249	.0635	-.0207
.94331	.0635	-.0242	.0574	-.0201
.92617	.0583	-.0231	.0525	-.0193
.90687	.0545	-.0216	.0485	-.0182
.88546	.0516	-.0199	.0455	-.0169
.86197	.0494	-.0180	.0431	-.0153
.83647	.0475	-.0160	.0412	-.0137
.80902	.0455	-.0138	.0394	-.0119
.77967	.0436	-.0116	.0377	-.0101
.74851	.0415	-.00940	.0360	-.00822
.71560	.0394	-.00716	.0342	-.00633
.68102	.0372	-.00496	.0323	-.00445
.64484	.0350	-.00280	.0304	-.00260
.60716	.0327	-.00072	.0285	-.00080
.56806	.0304	.00126	.0265	.00092
.52764	.0281	.00314	.0245	.00256
.48498	.0257	.00490	.0224	.00410
.44319	.0233	.00652	.0204	.00553
.35460	.0185	.00933	.0162	.00800
.30902	.0160	.0105	.0141	.00903
.26271	.0136	.0115	.0119	.00993
.16836	.00866	.0130	.00762	.0113
.07243	.00372	.0139	.00327	.0121
.02416	.00124	.0141	.00109	.0122

Table 11. (cont.)

x/a	$(h_2/a)=0.07$ $-\tau/\sigma_o$	σ/σ_o	$(h_2/a)=0.1$ $-\tau/\sigma_o$	σ/σ_o
.99997	1.723	-.357	1.535	-.299
.99883	.278	-.0586	.247	-.0486
.99533	.143	-.0314	.126	-.0254
.98951	.0990	-.0233	.0859	-.0182
.98137	.0781	-.0199	.0666	-.0149
.97094	.0662	-.0184	.0555	-.0132
.95825	.0585	-.0176	.0483	-.0123
.94331	.0527	-.0171	.0432	-.0117
.92617	.0481	-.0164	.0392	-.0112
.90687	.0443	-.0156	.0359	-.0106
.88546	.0411	-.0146	.0330	-.0100
.86197	.0386	-.0133	.0305	-.00927
.83647	.0366	-.0120	.0284	-.00842
.80902	.0349	-.0105	.0266	-.00746
.77967	.0333	-.00895	.0250	-.00644
.74851	.0318	-.00736	.0237	-.00537
.71560	.0302	-.00575	.0224	-.00426
.68102	.0286	-.00412	.0211	-.00313
.64484	.0269	-.00251	.0199	-.00199
.60716	.0252	-.00093	.0187	-.00087
.56806	.0235	.00060	.0174	.00024
.52764	.0217	.00206	.0161	.00131
.48598	.0199	.00345	.0147	.00233
.44319	.0180	.00474	.0134	.00330
.35460	.0143	.00700	.0107	.00501
.30902	.0124	.00796	.00926	.00575
.26271	.0105	.00879	.00785	.00639
.16836	.00673	.0101	.00501	.00737
.07243	.00289	.0108	.00215	.00793
.02416	.00096	.0109	.00072	.00804

Table 12. The effect of bond length (a), on the adhesive stresses σ and τ, with different thickness adherends.

h_1=1.0 in., h_4=0.5 in., h_2=0.1 in.

x/a	a=2.0 $-\tau/\sigma_o$	a=2.0 σ/σ_o	a=3.0 $-\tau/\sigma_o$	a=3.0 σ/σ_o	a=4.0 $-\tau/\sigma_o$	a=4.0 σ/σ_o
.99997	1.511	-.414	1.333	-.375	1.176	-.333
.99883	.245	-.0686	.218	-.0632	.193	-.0570
.99533	.127	-.0377	.115	-.0362	.103	-.0340
.98951	.0894	-.0289	.0827	-.0294	.0761	-.0288
.98137	.0718	-.0256	.0679	-.0271	.0631	-.0271
.97094	.0616	-.0242	.0588	-.0259	.0547	-.0254
.95825	.0547	-.0233	.0522	-.0243	.0487	-.0229
.94331	.0493	-.0222	.0474	-.0221	.0449	-.0197
.92617	.0451	-.0207	.0441	-.0194	.0423	-.0160
.90687	.0419	-.0189	.0417	-.0163	.0400	-.0122
.88546	.0395	-.0168	.0397	-.0131	.0377	-.00834
.86197	.0376	-.0145	.0378	-.00979	.0354	-.00472
.83647	.0359	-.0121	.0358	-.00660	.0331	-.00147
.80902	.0343	-.00961	.0338	-.00361	.0308	.00129
.77967	.0326	-.00720	.0317	-.00092	.0285	.00348
.74851	.0310	-.00486	.0297	.00139	.0262	.00507
.71560	.0292	-.00265	.0277	.00329	.0240	.00609
.68102	.0275	-.00061	.0257	.00476	.0218	.00658
.64484	.0257	.00122	.0237	.00579	.0197	.00661
.60716	.0240	.00282	.0217	.00643	.0177	.00628
.56806	.0222	.00418	.0198	.00670	.0157	.00570
.52764	.0204	.00530	.0179	.00669	.0139	.00498
.48598	.0186	.00619	.0161	.00643	.0122	.00420
.44319	.0168	.00686	.0143	.00602	.0106	.00344
.35460	.0133	.00766	.0110	.00492	.00771	.00220
.30902	.0115	.00784	.00935	.00435	.00645	.00177
.26271	.00971	.00793	.00780	.00381	.00528	.00145
.16836	.00617	.00792	.00486	.00295	.00319	.00112
.07243	.00264	.00784	.00205	.00244	.00133	.00101
.02416	.00088	.00782	.00068	.00234	.000439	.00099

Table 13. The effect of (h_1/a) ratio on the adhesive stresses σ and τ with different thickness adherends.

(h_4/a)=0.5 , (h_2/a)=0.1 , a=1.0 in.

x/a	(h_1/a)=1.0		(h_1/a)=2.0		(h_1/a)=3.0	
	$-\tau/\sigma_o$	σ/σ_o	$-\tau/\sigma_o$	σ/σ_o	$-\tau/\sigma_o$	σ/σ_o
.99997	1.535	-.299	1.605	-.391	1.620	-.405
.99883	.247	-.0486	.259	-.0637	.261	-.0659
.99533	.126	-.0254	.131	-.0333	.133	-.0344
.98951	.0859	-.0182	.0899	-.0238	.0907	-.0246
.98137	.0666	-.0149	.0698	-.0195	.0704	-.0202
.97094	.0555	-.0132	.0582	-.0173	.0587	-.0179
.95825	.0483	-.0123	.0506	-.0160	.0511	-.0166
.94331	.0432	-.0117	.0453	-.0152	.0457	-.0158
.92617	.0392	-.0112	.0412	-.0146	.0415	-.0151
.90687	.0359	-.0106	.0377	-.0139	.0380	-.0144
.88546	.0330	-.0100	.0347	-.0131	.0350	-.0135
.86197	.0305	-.00927	.0322	-.0121	.0324	-.0125
.83647	.0284	-.00842	.0299	-.0110	.0302	-.0114
.80902	.0266	-.00746	.0281	-.00971	.0283	-.0101
.77967	.0250	-.00644	.0264	-.00837	.0267	-.00868
.74851	.0237	-.00537	.0250	-.00696	.0253	-.00723
.71560	.0224	-.00426	.0237	-.00551	.0239	-.00573
.68102	.0211	-.00313	.0224	-.00403	.0226	-.00420
.64484	.0199	-.00199	.0211	-.00255	.0213	-.00267
.60716	.0187	-.00087	.0198	-.00108	.0200	-.00115
.56806	.0174	.00024	.0185	.00035	.0187	.00034
.52764	.0161	.00131	.0171	.00174	.0173	.00178
.48598	.0147	.00233	.0157	.00307	.0159	.00316
.44319	.0134	.00330	.0143	.00432	.0145	.00446
.35460	.0107	.00501	.0113	.00653	.0115	.00677
.30902	.00926	.00575	.00980	.00748	.0100	.00776
.26271	.00785	.00639	.00830	.00831	.00850	.00862
.21578	.00644	.00693	.00681	.00901	.00698	.00936
.16836	.00501	.00737	.00531	.00958	.00544	.00995
.07243	.00215	.00793	.00229	.0103	.00234	.0107
.02416	.00071	.00804	.00076	.0104	.00078	.0109

Table 14. The effect of (h_2/a) ratio on the adhesive stresses σ and τ with different thickness adherents.

$(h_1/a)=0.5$, $(h_4/a)=1.0$, a=1.0 in.

	$(h_2/a)=0.03$		$(h_2/a)=0.04$		$(h_2/a)=0.05$	
x/a	$-\tau/\sigma_o$	σ/σ_o	$-\tau/\sigma_o$	σ/σ_o	$-\tau/\sigma_o$	σ/σ_o
.99997	2.173	.525	2.023	.464	1.904	.419
.99883	.356	.0895	.330	.0778	.309	.0696
.99533	.190	.0529	.173	.0441	.160	.0385
.98951	.139	.0446	.124	.0354	.113	.0298
.98137	.115	.0427	.101	.0323	.0911	.0267
.97094	.101	.0421	.0879	.0318	.0785	.0255
.95825	.0911	.0410	.0786	.0312	.0700	.0249
.94331	.0843	.0390	.0716	.0301	.0634	.0242
.92617	.0801	.0364	.0666	.0284	.0583	.0231
.90687	.0771	.0334	.0631	.0264	.0544	.0216
.88546	.0744	.0301	.0606	.0240	.0516	.0199
.86197	.0717	.0266	.0583	.0215	.0494	.0180
.83647	.0688	.0231	.0561	.0189	.0474	.0159
.80902	.0659	.0195	.0538	.0162	.0455	.0138
.77967	.0628	.0160	.0514	.0134	.0435	.0116
.74851	.0596	.0126	.0489	.0107	.0415	.00931
.71560	.0564	.00931	.0463	.00794	.0393	.00707
.68102	.0532	.00612	.0437	.00531	.0372	.00486
.64484	.0499	.00306	.0411	.00276	.0350	.00271
.60716	.0465	.00014	.0384	.00033	.0327	.00064
.56806	.0432	-.00262	.0356	-.00197	.0304	-.00134
.52764	.0398	-.00521	.0329	-.00412	.0281	-.00321
.48598	.0364	-.00763	.0301	-.00612	.0257	-.00495
.44319	.0330	-.00985	.0273	-.00795	.0233	-.00655
.35460	.0261	-.0137	.0216	-.0111	.0185	-.00931
.30902	.0227	-.0153	.0188	-.0124	.0161	-.0105
.26271	.0192	-.0167	.0159	-.0135	.0137	-.0114
.16836	.0123	-.0188	.0102	-.0152	.00872	-.0129
.07243	.00526	-.0201	.00436	-.0162	.00374	-.0138
.02416	.00175	-.0203	.00145	-.0163	.00125	-.0140

Table 14. (cont.)

x/a	$(h_2/a)=0.06$		$(h_2/a)=0.07$		$(h_2/a)=0.1$	
	$-\tau/\sigma_o$	σ/σ_o	$-\tau/\sigma_o$	σ/σ_o	$-\tau/\sigma_o$	σ/σ_o
.99997	1.806	.385	1.723	.358	1.536	.299
.99795	.222	.0489	.211	.0450	.187	.0371
.99179	.117	.0281	.110	.0253	.0962	.0200
.98156	.0841	.0228	.0784	.0200	.0670	.0150
.96729	.0688	.0211	.0635	.0181	.0530	.0129
.94906	.0595	.0203	.0547	.0173	.0449	.0119
.92692	.0527	.0193	.0483	.0165	.0394	.0112
.90097	.0476	.0178	.0433	.0153	.0351	.0105
.87132	.0440	.0160	.0395	.0138	.0315	.00956
.83809	.0413	.0138	.0367	.0120	.0285	.00845
.80141	.0390	.0115	.0345	.0100	.0262	.00718
.76145	.0367	.00900	.0324	.00797	.0242	.00578
.71835	.0343	.00650	.0303	.00583	.0225	.00432
.67230	.0319	.00401	.0282	.00367	.0209	.00282
.62349	.0293	.00159	.0259	.00155	.0192	.00132
.57212	.0267	-.00073	.0236	-.00049	.0175	-.00016
.51839	.0240	-.00290	.0213	-.00242	.0158	-.00157
.46254	.0213	-.00489	.0189	-.00419	.0140	-.00289
.40478	.0186	-.00667	.0164	-.00579	.0122	-.00409
.34537	.0158	-.00822	.0140	-.00719	.0103	-.00516
.28453	.0130	-.00953	.0115	-.00838	.00842	-.00608
.22252	.0101	-.0106	.00894	-.00935	.00656	-.00683
.15960	.00723	-.0114	.00640	-.0101	.00470	-.00741
.09602	.00434	-.0120	.00384	-.0106	.00283	-.00780
.03205	.00145	-.0122	.00128	-.0108	.00094	-.00799

Table 15. The effect of (h_4/a) ratio on the adhesive stresses σ and τ with different thickness adherends.

$(h_1/a)=0.5$, $(h_2/a)=0.07$, a=1.0 in.

	$(h_4/a)=1.0$		$(h_4/a)=2.0$		$(h_4/a)=3.0$	
x/a	$-\tau/\sigma_o$	σ/σ_o	$-\tau/\sigma_o$	σ/σ_o	$-\tau/\sigma_o$	σ/σ_o
.99997	1.723	.358	1.818	.479	1.840	.499
.99795	.211	.0450	.223	.0603	.226	.0628
.99179	.110	.0253	.116	.0339	.118	.0352
.98156	.0784	.0200	.0829	.0268	.0839	.0278
.96729	.0635	.0181	.0672	.0243	.0680	.0252
.94906	.0547	.0173	.0579	.0231	.0586	.0241
.92692	.0483	.0165	.0512	.0220	.0518	.0229
.90097	.0433	.0153	.0460	.0205	.0465	.0213
.87132	.0395	.0138	.0420	.0185	.0426	.0192
.83809	.0367	.0120	.0391	.0161	.0396	.0167
.80141	.0345	.0100	.0368	.0134	.0372	.0140
.76145	.0324	.00797	.0346	.0106	.0351	.0110
.71835	.0303	.00583	.0325	.00778	.0329	.00806
.67230	.0282	.00367	.0302	.00489	.0306	.00506
.62349	.0259	.00155	.0279	.00206	.0282	.00212
.57212	.0236	-.00049	.0254	-.00066	.0258	-.00071
.51839	.0213	-.00242	.0230	-.00323	.0233	-.00337
.46254	.0189	-.00419	.0204	-.00560	.0207	-.00583
.40478	.0164	-.00579	.0178	-.00774	.0180	-.00804
.34537	.0140	-.00719	.0151	-.00961	.0154	-.00999
.28453	.0115	-.00838	.0124	-.0112	.0126	-.0116
.22252	.00894	-.00935	.00972	-.0125	.00986	-.0130
.15960	.00640	-.0101	.00696	-.0135	.00707	-.0140
.09602	.00384	-.0106	.00419	-.0141	.00425	-.0147
.03205	.00128	-.0108	.00140	-.0145	.00142	-.0150

Table 16. The effect of (h_2/a) ratio on the normalized stress intensity factors $(k_1/k_o,\ k_2/k_o)$ and the strain energy release rate, G/G_o, with different thickness adherends.

$(h_1/a)=1.0$, $(h_4/a)=0.5$, $k_o=\sigma_o\sqrt{a}$

$G_o=\sigma_o^2 a/E_2$, $a=1.0$ in.

h_2/a	0.03	0.04	0.05	0.06	0.07	0.1
k_1/k_o	-.00405	-.00359	-.00324	-.00298	-.00277	-.00232
k_2/k_o	-.0168	-.0157	-.0148	-.0140	-.0133	-.0119
G/G_o	.745E-3	.650E-3	.578E-3	.517E-3	.467E-3	.374E-3

Table 17. The effect of bond length (a), on the normalized stress intensity factors (k_1/k_o, k_2/k_o) and the strain energy release rate, G/G_o, with different thickness adherends.

h_1=1.0 in., h_2=0.1 in., h_4=0.5 in., $k_o=\sigma_o\sqrt{a}$,

$G_o=\sigma_o^2 a/E_2$.

	a=2.0	a=3.0	a=4.0
k_1/k_o	-.00321	-.00291	-.00258
k_2/k_o	-.0117	-.0103	-.00911
G/G_o	.361E-3	.280E-3	.219E-3

Table 18. The effect of (h_1/a) ratio on the normalized stress intensity factors (k_1/k_o, k_2/k_o) and the strain energy release rate, G/G_o, with different thickness adherends.

(h_4/a)=0.5 , (h_2/a)=0.1 , $k_o=\sigma_o$ a

$G_o=\sigma_o^2 a/E_2$, a=1.0 in.

h_1/a	1.0	2.0	3.0
k_1/k_o	-.00232	-.00303	-.00314
k_2/k_o	-.0119	-.0124	-.0125
G/G_o	.374E-3	.406E-3	.412E-3

Table 19. The effect of (h_2/a) ratio on the normalized stress intensity factors $(k_1/k_o, k_2/k_o)$ and the strain energy release rate, G/G_o, with different thickness adherends.

$(h_1/a)=0.5$, $(h_4/a)=1.0$, $k_o=\sigma_o\sqrt{a}$

$G_o=\sigma_o^2 a/E_2$, $a=1.0$ in.

h_2/a	0.03	0.04	0.05	0.06	0.07	0.1
k_1/k_o	.00407	.00360	.00325	.00298	.00277	.00232
k_2/k_o	-.0168	-.0157	-.0148	-.0140	-.0133	-.0119
G/G_o	.789E-3	.685E-3	.606E-3	.541E-3	.487E-3	.388E-3

Table 20. The effect of (h_4/a) ratio on the normalized stress intensity factors $(k_1/k_o, k_2/k_o)$ and the strain energy release rate, G/G_o, with different thickness adherends.

$(h_1/a)=0.5$, $(h_2/a)=0.07$, $k_o=\sigma_o\sqrt{a}$

$G_o=\sigma_o^2 a/E_2$, $a=1.0$ in.

h_4/a	1.0	2.0	3.0
k_1/k_o	.00277	.00371	.00386
k_2/k_o	-.0138	-.0141	-.0143
G/G_o	.523E-3	.561E-3	.579E-3

Table 21. The effect of the material properties on the adhesive stresses σ and τ with different thickness adherends.

$(h_1/a)=1.0$, $(h_4/a)=0.5$,

$(h_2/a)=0.07$, $a=1.0$ in.

	$\mu_1=\mu_4/2$		$\mu_1=\mu_4$		$\mu_1=2\mu_4$	
x/a	$-\tau/\sigma_o$	σ/σ_o	$-\tau/\sigma_o$	σ/σ_o	$-\tau/\sigma_o$	σ/σ_o
.99997	1.582	-.205	1.723	-.358	1.813	-.459
.99795	.194	-.0258	.211	-.0450	.222	-.0579
.99179	.101	-.0145	.110	-.0253	.116	-.0326
.98156	.0716	-.0114	.0784	-.0200	.0828	-.0258
.96729	.0578	-.0104	.0635	-.0181	.0672	-.0234
.94906	.0496	-.00990	.0547	-.0173	.0580	-.0224
.92692	.0436	-.00946	.0483	-.0165	.0513	-.0213
.90097	.0389	-.00880	.0433	-.0153	.0461	-.0199
.87132	.0354	-.00793	.0395	-.0138	.0422	-.0180
.83809	.0327	-.00689	.0367	-.0120	.0393	-.0157
.80141	.0306	-.00574	.0345	-.0100	.0370	-.0132
.76145	.0286	-.00454	.0324	-.00797	.0349	-.0105
.71835	.0267	-.00331	.0303	-.00583	.0327	-.00771
.67230	.0247	-.00207	.0282	-.00367	.0305	-.00491
.62349	.0226	-.00086	.0259	-.00155	.0281	-.00214
.57212	.0206	.00030	.0236	.00049	.0257	.00054
.51839	.0184	.00140	.0213	.00242	.0232	.00307
.46254	.0163	.00241	.0189	.00419	.0206	.00540
.40478	.0141	.00332	.0164	.00579	.0180	.00751
.34537	.0120	.00411	.0140	.00719	.0153	.00937
.28453	.00981	.00479	.0115	.00838	.0126	.0109
.22252	.00763	.00534	.00894	.00935	.00982	.0122
.15960	.00545	.00576	.00640	.0101	.00704	.0132
.09602	.00327	.00604	.00384	.0106	.00423	.0139
.03205	.00109	.00618	.00128	.0108	.00141	.0142

Table 22. The effect of the material properties on the adhesive stresses σ and τ with different thickness adherends.

$(h_1/a)=0.5$, $(h_4/a)=1.0$,

$(h_2/a)=0.07$, a=1.0 in.

x/a	$\nu_1=\nu_4/2$		$\nu_1=\nu_4$		$\nu_1=2\nu_4$	
	$-\tau/\sigma_o$	σ/σ_o	$-\tau/\sigma_o$	σ/σ_o	$-\tau/\sigma_o$	σ/σ_o
.99997	1.570	.534	1.723	.358	1.839	.172
.99795	.192	.0670	.211	.0450	.226	.0216
.99179	.100	.0374	.110	.0253	.118	.0122
.98156	.0712	.0293	.0784	.0200	.0840	.00971
.96729	.0575	.0263	.0635	.0181	.0682	.00890
.94906	.0493	.0247	.0547	.0173	.0588	.00858
.92692	.0435	.0233	.0483	.0165	.0520	.00828
.90097	.0389	.0213	.0433	.0153	.0467	.00780
.87132	.0354	.0189	.0395	.0138	.0428	.00712
.83809	.0329	.0161	.0367	.0120	.0398	.00628
.80141	.0308	.0132	.0345	.0100	.0374	.00534
.76145	.0289	.0102	.0324	.00797	.0352	.00433
.71835	.0270	.00708	.0303	.00583	.0330	.00326
.67230	.0251	.00408	.0282	.00367	.0307	.00216
.62349	.0231	.00120	.0259	.00155	.0283	.00107
.57212	.0210	-.00150	.0236	-.00049	.0259	-.00001
.51839	.0189	-.00399	.0213	-.00242	.0233	-.00104
.46254	.0168	-.00622	.0189	-.00419	.0207	-.00201
.40478	.0146	-.00818	.0164	-.00579	.0180	-.00290
.34537	.0124	-.00986	.0140	-.00719	.0153	-.00370
.28453	.0102	-.0113	.0115	-.00838	.0126	-.00438
.22252	.00797	-.0124	.00894	-.00935	.00981	-.00494
.15960	.00571	-.0132	.00640	-.0101	.00702	-.00537
.09602	.00343	-.0138	.00384	-.0106	.00422	-.00567
.03205	.00115	-.0141	.00128	-.0108	.00141	-.00580

Table 23. The effect of the material properties on the adhesive stresses σ and τ with equal thickness adherends.

$(h_1/a)=(h_4/a)=0.5$, $(h_2/a)=0.07$, a=1.0 in.

	$\mu_1=\mu_4/2$		$\mu_1=\mu_4$		$\mu_1=2\mu_4$	
x/a	$-\tau/\sigma_o$	σ/σ_o	$-\tau/\sigma_o$	σ/σ_o	$-\tau/\sigma_o$	σ/σ_o
.99997	1.368	.204	1.529	0.0	1.669	-.198
.99883	.221	.0335	.247	0.0	.270	-.0324
.99533	.113	.0178	.127	0.0	.138	-.0174
.98951	.0781	.0131	.0876	0.0	.0959	-.0129
.98137	.0614	.0111	.0691	0.0	.0757	-.0110
.97094	.0518	.0102	.0584	0.0	.0642	-.0102
.95825	.0455	.00959	.0515	0.0	.0566	-.00973
.94331	.0408	.00915	.0463	0.0	.0511	-.00940
.92617	.0370	.00867	.0421	0.0	.0466	-.00903
.90687	.0338	.00808	.0386	0.0	.0429	-.00855
.88546	.0312	.00739	.0357	0.0	.0398	-.00795
.86197	.0291	.00663	.0334	0.0	.0374	-.00727
.83647	.0273	.00581	.0316	0.0	.0354	-.00650
.80902	.0259	.00496	.0300	0.0	.0337	-.00569
.77967	.0245	.00409	.0285	0.0	.0321	-.00484
.74851	.0232	.00322	.0271	0.0	.0306	-.00395
.71560	.0219	.00235	.0256	0.0	.0290	-.00306
.68102	.0205	.00151	.0241	0.0	.0274	-.00217
.64484	.0192	.00070	.0226	0.0	.0258	-.00128
.60716	.0178	-.00008	.0211	0.0	.0241	-.00042
.56806	.0164	-.00080	.0195	0.0	.0224	.00041
.52764	.0151	-.00147	.0179	0.0	.0207	.00121
.48598	.0137	-.00208	.0164	0.0	.0189	.00195
.44319	.0124	-.00264	.0148	0.0	.0171	.00264
.35460	.00968	-.00356	.0116	0.0	.0136	.00384
.30902	.00836	-.00393	.0101	0.0	.0118	.00434
.26271	.00705	-.00424	.00851	0.0	.00996	.00477
.16836	.00446	-.00469	.00540	0.0	.00634	.00542
.07243	.00191	-.00494	.00231	0.0	.00272	.00578
.02416	.00064	-.00499	.00077	0.0	.00091	.0058

Table 24. The effect of the material properties on the adhesive stresses σ and τ with equal thickness adherends.

$(h_1/a)=(h_4/a)=1.0$, $(h_2/a)=0.07$, $a=1.0$ in.

	$\nu_1=\nu_4/2$		$\nu_1=\nu_4$		$\nu_1=2\nu_4$	
x/a	$-\tau/\sigma_o$	σ/σ_o	$-\tau/\sigma_o$	σ/σ_o	$-\tau/\sigma_o$	σ/σ_o
.99997	1.716	.146	1.852	0.0	1.938	-.103
.99883	.277	.0239	.299	0.0	.313	-.0169
.99533	.142	.0128	.153	0.0	.161	-.00908
.98951	.0984	.00940	.107	0.0	.112	-.00674
.98137	.0776	.00798	.0841	0.0	.0883	-.00577
.97094	.0657	.00729	.0714	0.0	.0750	-.00532
.95825	.0579	.00690	.0631	0.0	.0663	-.00508
.94331	.0522	.00659	.0570	0.0	.0600	-.00490
.92617	.0475	.00625	.0520	0.0	.0549	-.00470
.90687	.0437	.00585	.0480	0.0	.0506	-.00445
.88546	.0406	.00538	.0446	0.0	.0472	-.00415
.86197	.0381	.00487	.0420	0.0	.0445	-.00380
.83647	.0361	.00432	.0399	0.0	.0423	-.00341
.80902	.0344	.00374	.0380	0.0	.0404	-.00300
.77967	.0328	.00315	.0364	0.0	.0387	-.00257
.74851	.0312	.00255	.0347	0.0	.0370	-.00213
.71560	.0297	.00194	.0330	0.0	.0352	-.00167
.68102	.0281	.00135	.0313	0.0	.0334	-.00121
.64484	.0264	.00076	.0295	0.0	.0315	-.00075
.60716	.0247	.00019	.0277	0.0	.0296	-.00030
.56806	.0230	-.00035	.0258	0.0	.0276	.00013
.52764	.0212	-.00087	.0239	0.0	.0256	.00056
.48598	.0195	-.00136	.0219	0.0	.0235	.00096
.44319	.0176	-.00181	.0199	0.0	.0214	.00133
.35460	.0140	-.00260	.0158	0.0	.0170	.00200
.30902	.0122	-.00293	.0137	0.0	.0148	.00229
.26271	.0103	-.00322	.0117	0.0	.0126	.00254
.16836	.00657	-.00366	.00744	0.0	.00803	.00292
.07243	.00282	-.00391	.00320	0.0	.00345	.00314
.02416	.00094	-.00396	.00107	0.0	.00115	.00318

Table 25. The effect of the material properties on the normalized stress intensity factors (k_1/k_o, k_2/k_o) and the strain energy release rate, G/G_o, with different thickness adherends.

$(h_1/a)=1.0$, $(h_4/a)=0.5$, $(h_2/a)=0.07$,

$k_o=\sigma_o\sqrt{a}$, $G_o=\sigma_o^2 a/E_2$, a=1.0 in.

	$\nu_1=\nu_4/2$	$\nu_1=\nu_4$	$\nu_1=2\nu_4$
k_1/k_o	-.00159	-.00277	-.00356
k_2/k_o	-.0123	-.0133	-.0140
G/G_o	.399E-3	.467E-3	.517E-3

Table 26. The effect of the material properties on the normalized stress intensity factors (k_1/k_o, k_2/k_o) and the strain energy release rate, G/G_o, with different thickness adherends.

$(h_1/a)=0.5$, $(h_4/a)=1.0$, $(h_2/a)=0.07$,

$k_o=\sigma_o\sqrt{a}$, $G_o=\sigma_o^2 a/E_2$, a=1.0 in.

	$\nu_1=\nu_4/2$	$\nu_1=\nu_4$	$\nu_1=2\nu_4$
k_1/k_o	.00413	.00277	.00133
k_2/k_o	-.0122	-.0133	-.0142
G/G_o	.438E-3	.487E-3	.537E-3

Table 27. The effect of the material properties on the normalized stress intensity factors (k_1/k_o, k_2/k_o) and the strain energy release rate, G/G_o, with equal thickness adherends.

$(h_1/a)=(h_4/a)=0.5$, $(h_2/a)=0.07$, $k_o=\sigma_o\sqrt{a}$

$G_o=\sigma_o^2 a/E_2$, $a=1.0$ in.

	$\nu_1=\nu_4/2$	$\nu_1=\nu_4$	$\nu_1=2\nu_4$
k_1/k_o	.00158	.0	-.00129
k_2/k_o	-.0106	-.0118	-.0150
G/G_o	.303E-3	.367E-3	.594E-3

Table 28. The effect of the material properties on the normalized stress intensity factors (k_1/k_o, k_2/k_o) and the strain energy release rate, G/G_o, with equal thickness adherends.

$(h_1/a)=(h_4/a)=1.0$, $(h_2/a)=0.07$, $k_o=\sigma_o\sqrt{a}$

$G_o=\sigma_o^2 a/E_2$, $a=1.0$ in.

	$\nu_1=\nu_4/2$	$\nu_1=\nu_4$	$\nu_1=2\nu_4$
k_1/k_o	.00113	.0	-.0008
k_2/k_o	-.0133	-.0143	-.0150
G/G_o	.470E-3	.540E-3	.594E-3

F I G U R E S

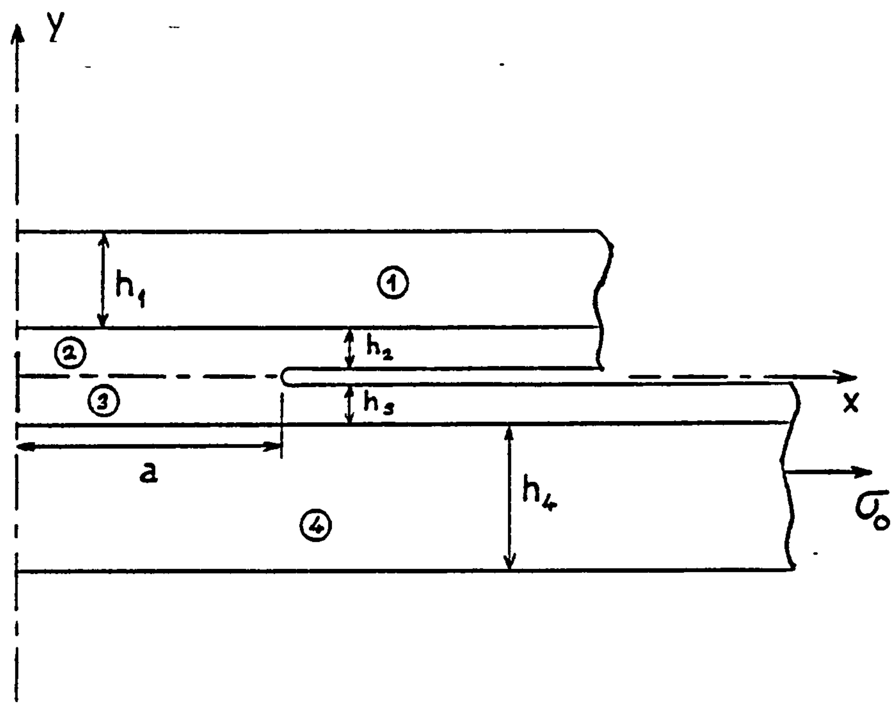


Figure 1. Geometry and notation for an adhesively bonded joint.

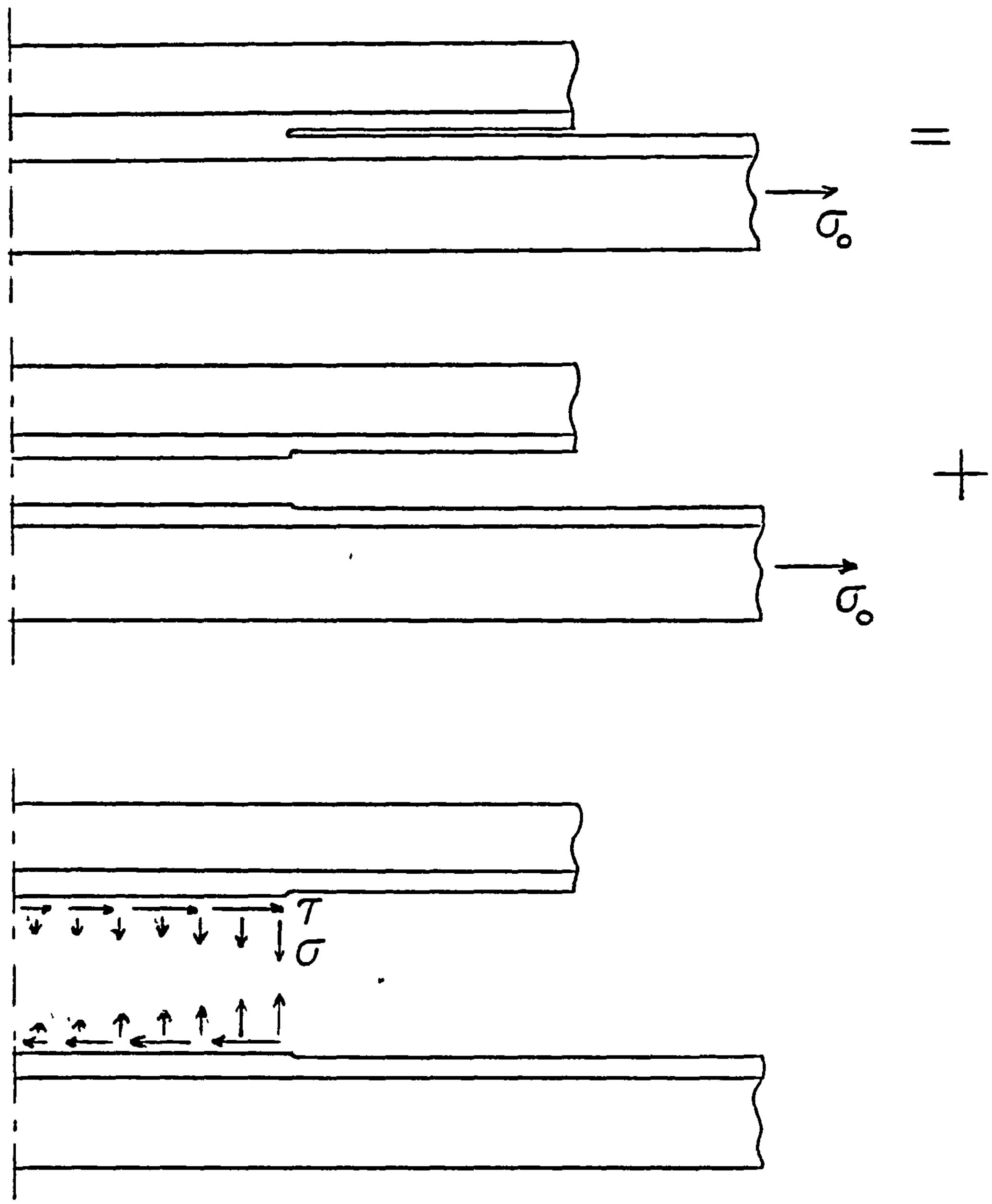
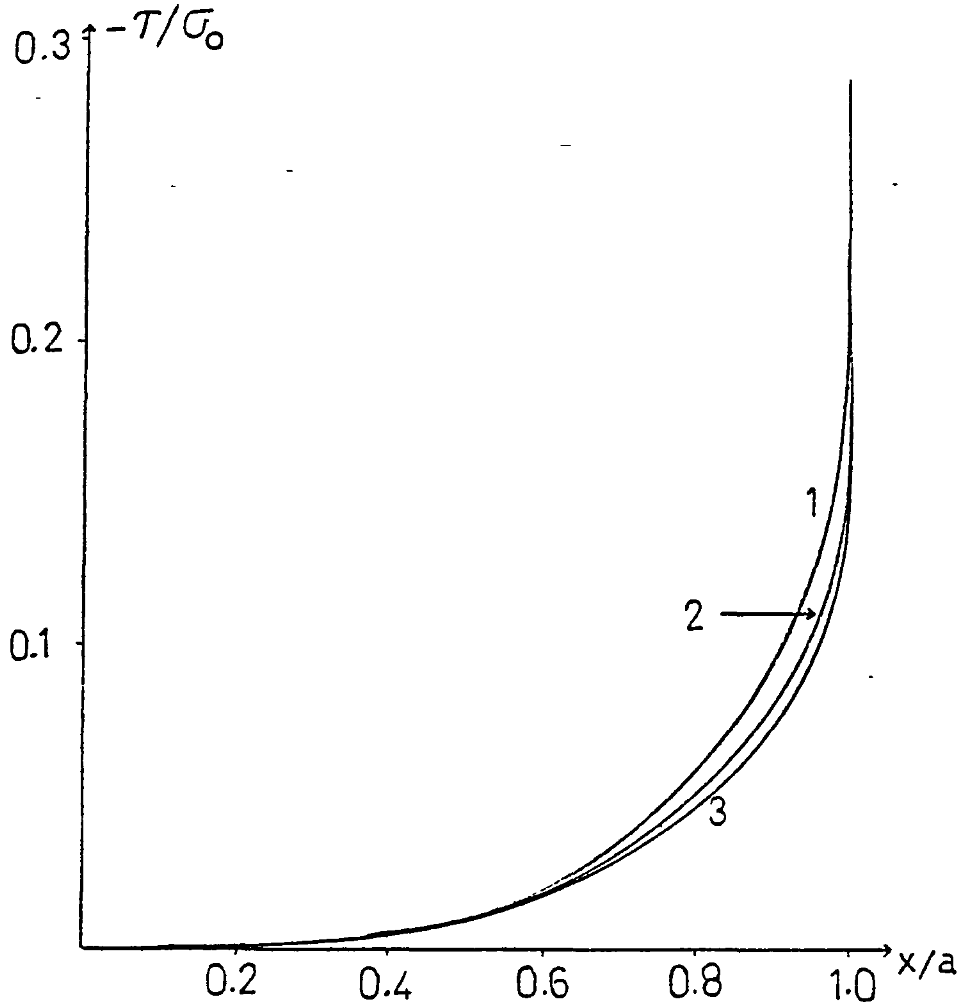


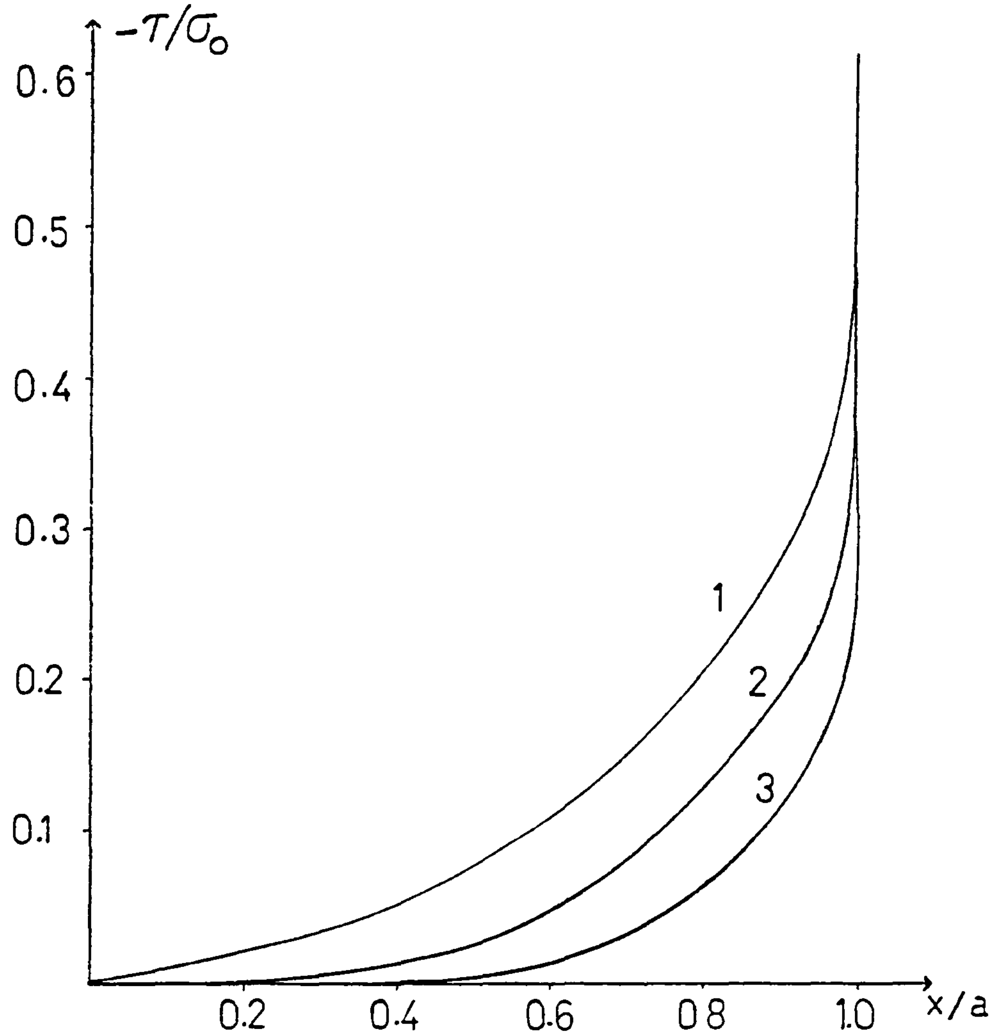
Figure 2. Superposition technique used in the solution of the problem.



1: h_2/a= 0.004
2: h_2/a= 0.006
3: h_2/a= 0.009

Figure 3. The adhesive shear stress for different (h_2/a) ratios with equal thickness adherends.

$(h_1/a)=(h_4/a)=0.25$, a=1.0 in.



1: $h_1/a=1.00$
2: $h_1/a=0.50$
3: $h_1/a=0.25$

Figure 4. The adhesive shear stress for different (h_1/a , h_4/a) ratios with equal thickness adherends.

($h_2/a=0.0025$, $h_1=h_4$, a=1.0 in.

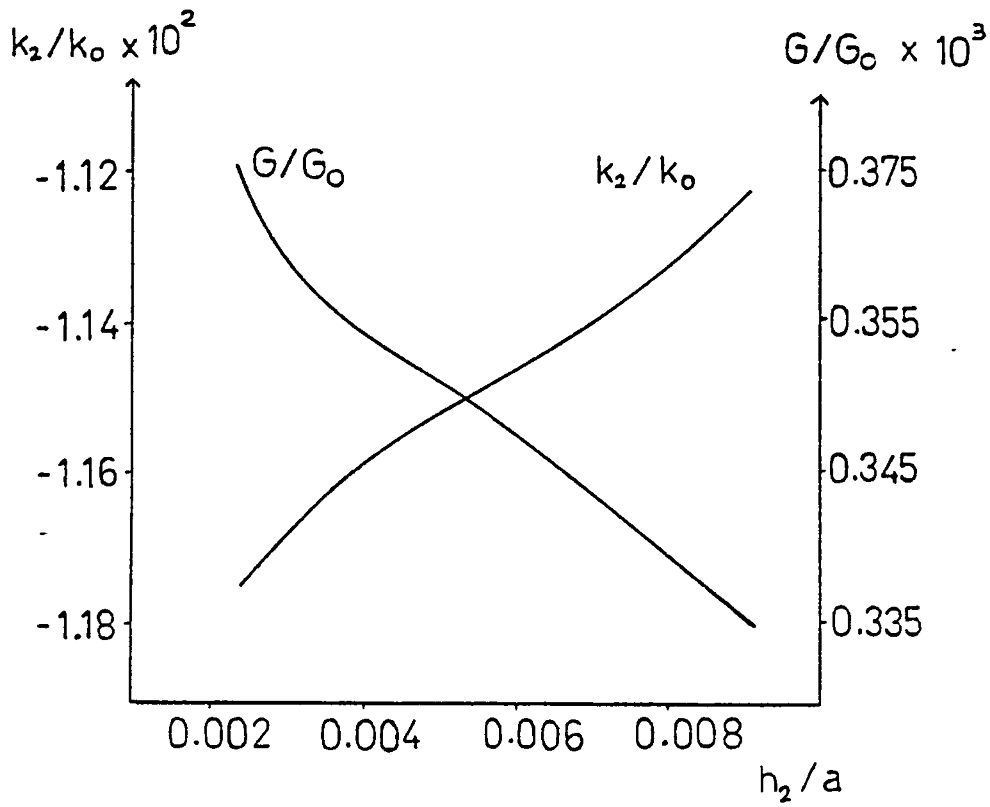


Figure 5. The normalized stress intensity factor and the strain energy release rate versus (h_2/a) ratio with equal thickness adherends.

$(h_1/a)=(h_4/a)=0.25$, a=1.0 in.

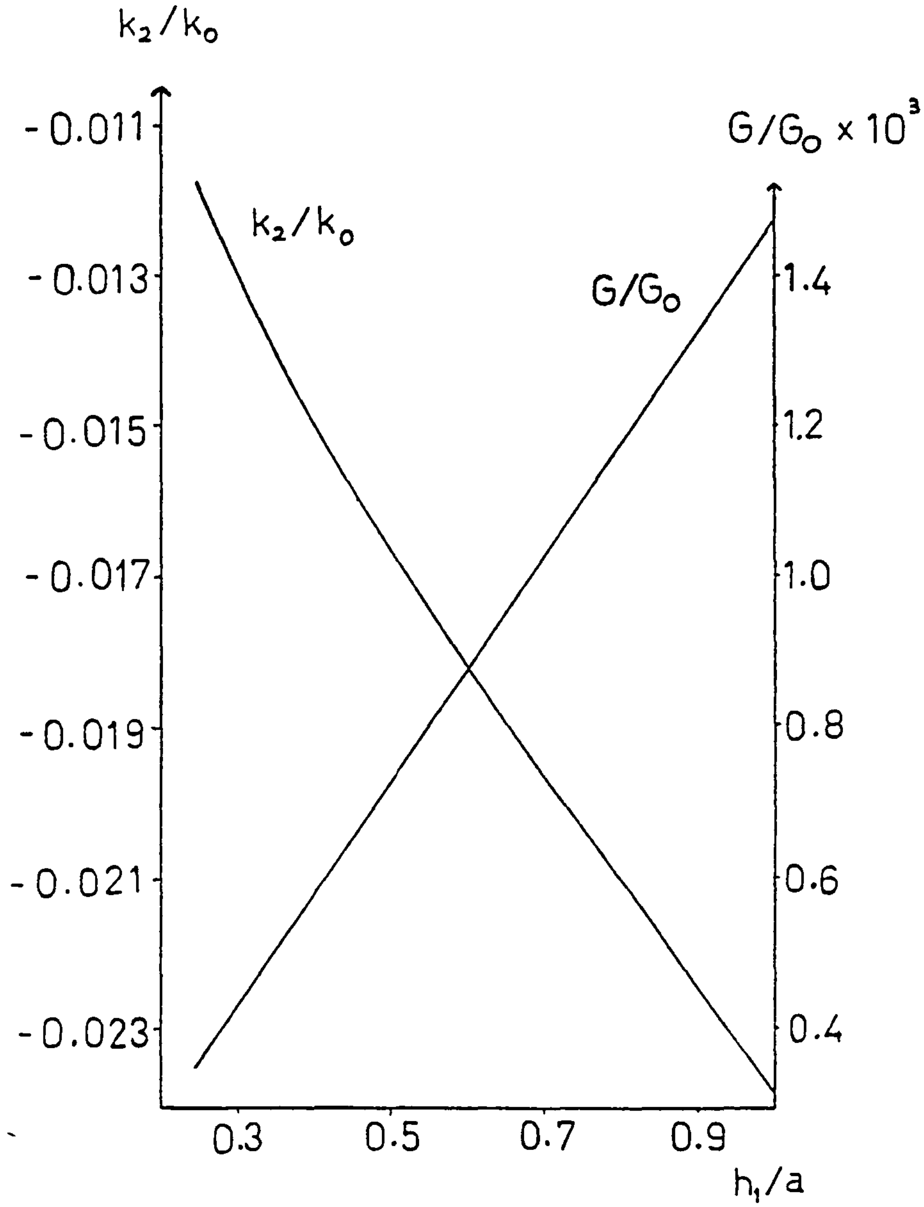


Figure 6. The normalized stress intensity factor and the strain energy release rate versus (h_1/a) ratio with equal thickness adherends.

(h_2/a)=0.0025 , a=1.0 in.

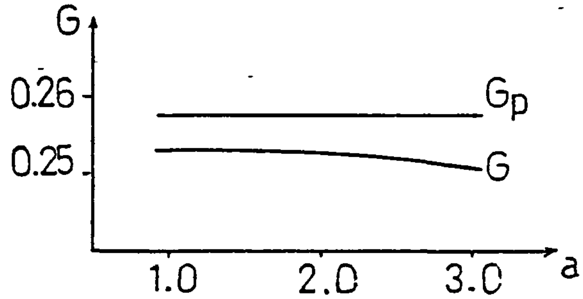


Figure 7. Comparison of the strain energy release rate calculated in this study (G) with the plate solutions (G_p) for the specific geometry discussed in Tbl. 9 .

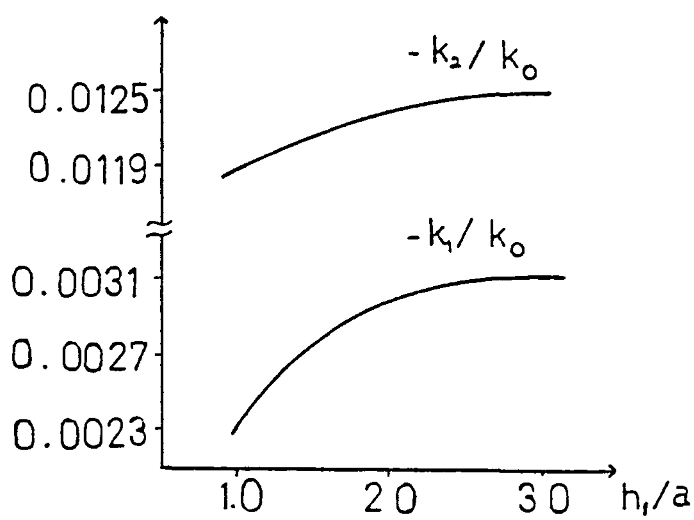


Figure 8. The effect of (h_1/a) ratio on the normalized stress intensity factors with different thickness adherends. (h_4/a)=0.5 , (h_2/a)=0.1 , a=1.0 in.

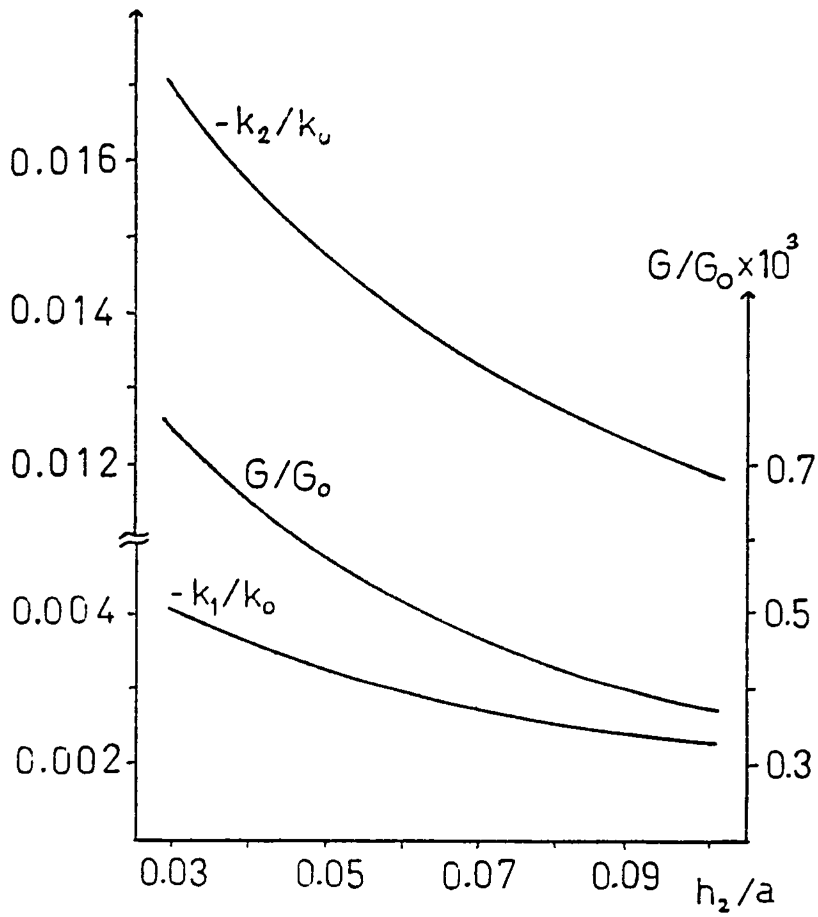


Figure 9. The effect of (h_2/a) ratio on the normalized stress intensity factors and the strain energy release rate with different thickness adherends.

(h_1/a)=1.0 , (h_4/a)=0.5 , a=1.0 in.

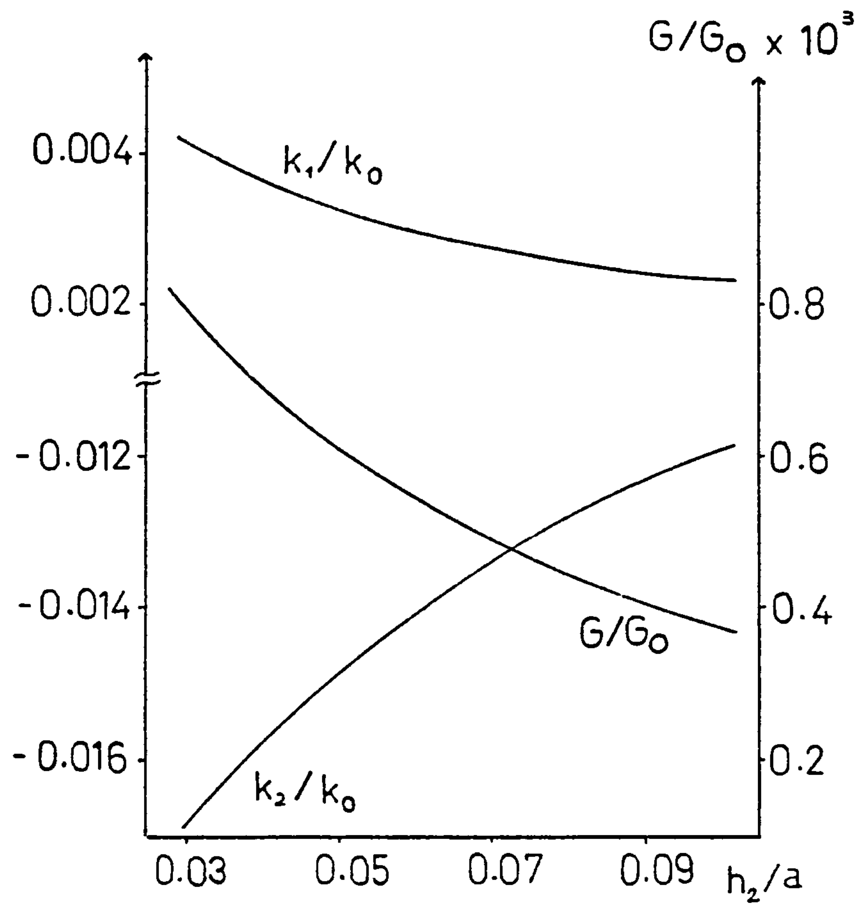


Figure 10. The effect of (h_2/a) ratio on the normalized stress intensity factors and the strain energy release rate with different thickness adherends.

(h_1/a)=0.5 , (h_4/a)=1.0 , a=1.0 in.

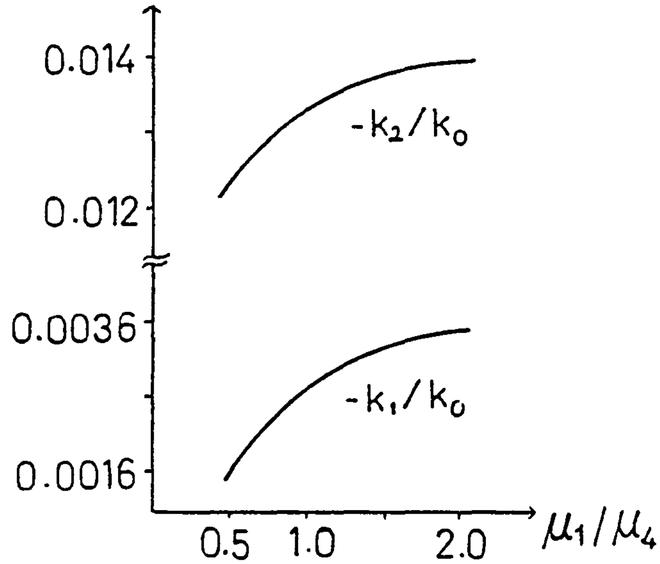


Figure 11. Effect of material properties on normalized stress intensity factors with different thickness adherends. $(h_1/a)=1.0$, $(h_4/a)=0.5$, $(h_2/a)=0.02$, a=1.0 in.

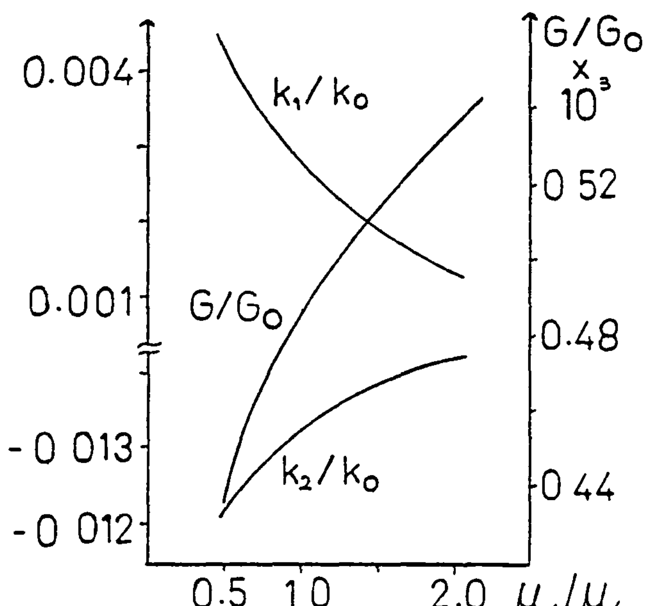
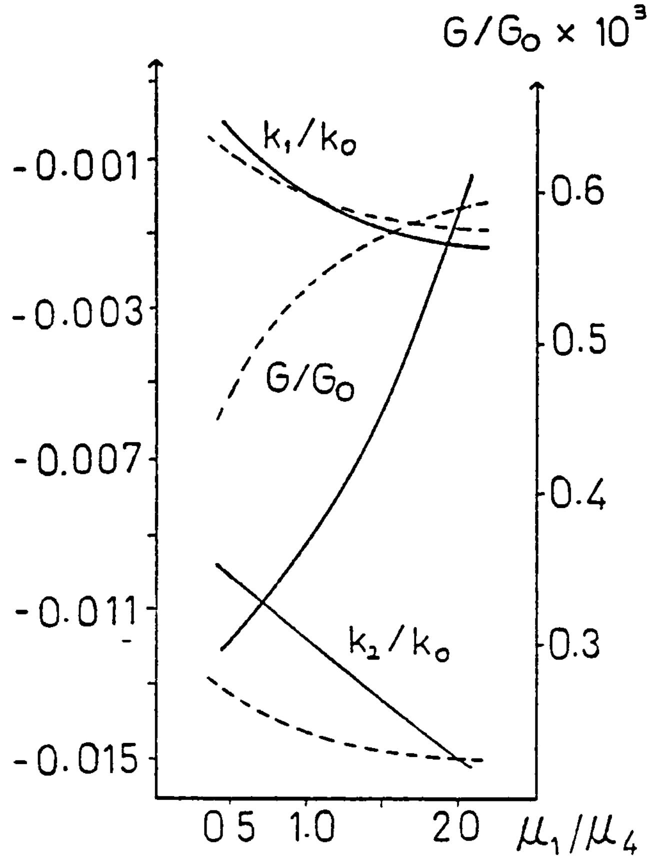


Figure 12. Same as Figure 11 with $(h_1/a)=0.5$, $(h_4/a)=1.0$, $(h_2/a)=0.07$, a=1.0 in.



——— $(h_1/a)=(h_4/a)=0.5$, $(h_2/a)=0.07$, a=1.0 in.
- - - - - - $(h_1/a)=(h_4/a)=1.0$, $(h_2/a)=0.07$, a=1.0 in.

Figure 13. Effect of material properties on normalized stress intensity factors and strain energy release rate with equal thickness adherends.

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APPENDIX I

$$c = \mu_2 / \mu_1 \quad , \quad d = \mu_3 / \mu_4$$

$$a_1 = (\kappa_2 - \kappa_1 c) / (c - 1) \quad ,$$

$$a_2 = (\kappa_1 c + 1) / (c - 1) \quad ,$$

$$a_3 = (\kappa_2 + c) / (c - 1) \quad ,$$

$$a_4 = (a_2 h_2 + a_3 h_1) \quad ,$$

$$b_1 = (\kappa_3 - \kappa_4 d) / (d - 1) \quad ,$$

$$b_2 = (\kappa_4 d + 1) / (d - 1) \quad ,$$

$$b_3 = (\kappa_3 + d) / (d - 1) \quad ,$$

$$b_4 = (b_2 h_3 + b_3 h_4) \quad ,$$

where μ_i's, h_i's, κ_i's are defined in Figure 1 .

APPENDIX II

$$\beta = e^{-2\alpha h_2} \quad ,$$

$$\beta^* = e^{-2\alpha h_3} \quad ,$$

$$\gamma = e^{-2\alpha(h_1 + h_2)} \quad ,$$

$$\gamma^* = e^{-2\alpha(h_4 + h_3)} \quad ,$$

$$\xi = e^{-2\alpha h_1} \quad ,$$

$$\xi^* = e^{-2\alpha h_4} \quad ,$$

$$G_1(\alpha) = 2\alpha\gamma a_4 + 2\alpha(h_2\beta - h_1\xi) \quad ,$$

$$G_2(\alpha) = a_2\gamma + \beta + a_1\xi + 4\alpha^2 h_1 h_2 \xi - a_3 \quad ,$$

$$G_3(\alpha) = - a_2\gamma - \beta + 2\alpha h_1 \xi \quad ,$$

$$G_4(\alpha) = - 4\alpha^2 h_1 h_2 \beta - a_2 - \xi - a_1\beta + a_3\gamma \quad ,$$

$$G_5(\alpha) = 2\alpha a_4 + 2\alpha(h_2\xi - h_1\beta) \quad ,$$

$$G_6(\alpha) = 2\alpha h_1 \beta + \xi + a_2 \quad ,$$

$$G_7(\alpha) = G_1(\alpha)G_6(\alpha) - G_3(\alpha)G_4(\alpha) \quad ,$$

$$G_8(\alpha) = G_3(\alpha)G_5(\alpha) - G_2(\alpha)G_6(\alpha) \quad ,$$

$$G_9(\alpha) = -2\alpha b_4 - 2\alpha(h_3\xi^* - h_4\beta^*) \quad ,$$

$$G_{10}(\alpha) = b_2 + \xi^* + 4\alpha^2 h_3 h_4 \beta^* + b_1\beta^* - b_3\gamma^* \quad ,$$

$$G_{11}(\alpha) = -b_2 - \xi^* - 2\alpha h_4\beta^* \quad ,$$

$$G_{12}(\alpha) = -b_1\xi^* + b_3 - 4\alpha^2 h_3 h_4 \xi^* - \beta^* - b_2\gamma^* \quad ,$$

$$G_{13}(\alpha) = -2\alpha\gamma^* b_4 - 2\alpha(h_3\beta^* - h_4\xi^*) \quad ,$$

$$G_{14}(\alpha) = b_2\gamma^* - 2\alpha h_4\xi^* + \beta^* \quad ,$$

$$G_{15}(\alpha) = G_{14}(\alpha)G_9(\alpha) - G_{12}(\alpha)G_{11}(\alpha) \quad ,$$

$$G_{16}(\alpha) = G_{13}(\alpha)G_{11}(\alpha) - G_{14}(\alpha)G_{10}(\alpha) \quad ,$$

where h_i's, a_i's, b_i's are defined in Appendix I .

APPENDIX III

$$P_1(\alpha) = G_1(\alpha) \quad ,$$

$$P_2(\alpha) = G_2(\alpha) \quad ,$$

$$P_3(\alpha) = -a_2\gamma - \beta - 2\alpha h_1\xi \quad ,$$

$$P_4(\alpha) = G_4(\alpha) \quad ,$$

$$P_5(\alpha) = G_5(\alpha) \quad ,$$

$$P_6(\alpha) = 2\alpha h_1\beta - \xi - a_2 \quad ,$$

$$P_7(\alpha) = P_1(\alpha)P_6(\alpha) - P_3(\alpha)P_4(\alpha) \quad ,$$

$$P_8(\alpha) = P_3(\alpha)P_5(\alpha) - P_2(\alpha)P_6(\alpha) \quad ,$$

$$P_9(\alpha) = G_9(\alpha) \quad ,$$

$$P_{10}(\alpha) = G_{10}(\alpha) \quad ,$$

$$P_{11}(\alpha) = - b_2 - \xi^* + 2\alpha h_4\beta^* \quad ,$$

$$P_{12}(\alpha) = G_{12}(\alpha) \quad ,$$

$$P_{13}(\alpha) = G_{13}(\alpha) \quad ,$$

$$P_{14}(\alpha) = - b_2\gamma^* - 2\alpha h_4\xi^* - \beta^* \quad ,$$

$$P_{15}(\alpha) = P_{14}(\alpha)P_9(\alpha) - P_{12}(\alpha)P_{11}(\alpha) \quad ,$$

$$P_{16}(\alpha) = P_{13}(\alpha)P_{11}(\alpha) - P_{14}(\alpha)P_{10}(\alpha) \quad ,$$

where (h_i's, a_i's, b_i's) and (β, β^*, γ, γ^*, ξ, ξ^*, G_i's) are defined in Figure 1 and Appendix II, respectively.

APPENDIX IV

Solution of the Singular Integral Equation

In [15] a quadrature formula of closed type is derived for the principal part of the singular integral equation of the form ;

$$a\,\Phi(t) + \frac{b}{\pi}\int_{-1}^{1} \frac{\Phi(t)}{t-x}\,dt + \int_{-1}^{1} \Phi(t)\,k(x,t)\,dt = g(x) \quad ,$$

$$-1 < x < +1 \quad .$$

The solution will be sought in the form

$$\Phi(t) = \Psi(t)\,W(t) \quad ,$$

where $\Psi(t)$ is a bounded function, and $W(t)$ is given by

$$W(t) = (1-t)^{\alpha}\,(1+t)^{\beta} \quad ,$$

$$\alpha = \frac{1}{2\pi i}\log\left[\frac{a - ib}{a + ib}\right] + N \quad ,$$

$$\beta = -\frac{1}{2\pi i}\log\left[\frac{a - ib}{a + ib}\right] + M \quad .$$

N and M are arbitrary integers determined from the physics of the problem. The index of the singular integral equation is defined by

$$\kappa = -(\alpha + \beta) = -(N + M) .$$

Defining,

$$k^*(x,t) = \frac{b}{\pi(t-x)} + k(x,t) \quad ,$$

the approximate solution to the singular integral equation is determined from

$$(1+\alpha)\, H_1\, k^*(x_k,t_1)\, \psi(t_1) + \sum_{i=2}^{n-1} H_i\, k^*(x_k,t_i)\, \psi(t_i)$$

$$+ (1+\beta)\, H_n\, k^*(x_k,t_n)\, \psi(t_n) = g(x_k) \quad , \quad k=1,..,-1$$

and

$$(1+\alpha)\, H_1\, \psi(t_1) + \sum_{i=2}^{n-1} H_i\, \psi(t_i) + (1+\beta)\, H_n\, \psi(t_n) = C \quad ,$$

where the last equation is the approximation to the extra condition,

(when $\kappa=1$)

$$\int_{-1}^{1} \Phi(t)\, dt = C \ .$$

Here t_i, x_k and H_i are defined as follows ,

$$(1 - t_i^2)\, P_{n-2}^{(1+\alpha,\ 1+\beta)}\ (t_i) = 0 \ , \quad t_1 > t_2 > \dots > t_n$$

$$P_{n-1}^{(-1-\alpha,\ -1-\beta)}(x_k) = 0 \quad , \ k=1,2,\dots,(n-1)$$

and

$$H_i = \frac{\Gamma(n+\alpha)\ \Gamma(n+\beta)}{(n-1)\ \left[\Gamma(n) P_{n-1}^{(\alpha,\beta)}\ (t_i)\right]^2} \qquad ,$$

where $P_n^{(\alpha,\beta)}$ (x) is the Jacobi polynomial of degree "n" .

In the analysis of the report the singular integral equations are of the first kind and $\alpha = \beta = -1/2$ giving $\kappa=1$. In this case Jacobi polynomials reduce to the Chebyshev polynomials and n unknowns $\psi(t_i)$ can be determined from

$$\frac{1}{2} k^*(x_k, t_1)\ \psi(t_1) + \sum_{i=2}^{n-1} k^*(x_k, t_i)\ \psi(t_i)$$

$$+ \frac{1}{2} k^*(x_k, t_n)\ \psi(t_n) = \frac{n-1}{\pi} g(x_k) \quad ,$$

and,

$$\frac{1}{2} \psi(t_1) + \sum_{i=2}^{n-1} \psi(t_i) + \frac{1}{2} \psi(t_n) = \frac{n-1}{\pi} C \quad ,$$

where

$$t_i = \mathrm{Cos}\left[\frac{i-1}{n-1}\ \pi\right] \quad , \quad i=1,\ldots.,n$$

$$x_k = \mathrm{Cos}\left[\frac{2k-1}{2n-2}\ \pi\right] \quad , \quad k=1,\ldots.,(n-1) \quad .$$

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15 Supplementary Notes

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16 Abstract

The plane strain problem of adhesively bonded structures which consist of two different isotropic adherends is considered. By expressing the x-y components of the displacements in terms of Fourier integrals and using the corresponding boundary and continuity conditions, the system of integral equations for the general problem is obtained. Then, these integral equations are solved numerically by applying Gauss-Chebyshev integration scheme.

The shear and the normal stresses in the adhesive are calculated for various geometries and material properties for a stiffened plate under uniaxial tension σ_x. Also, the numerical results involving the stress intensity factors and the strain energy release rate are presented. The closed-form expressions for the Fredholm kernels are provided, so that the solution for an arbitrary geometry and material properties can easily be obtained.

The numerical solution of the integral equations indicates that as (h_1/a), (h_4/a) and (h_2/a) decrease the convergence becomes slower and hence, computations become costlier. For the general geometry of a cover plate, the contribution of the normal stress is quite significant. For the symmetric geometries, however, the dominant stress is the shear stress. More specifically, the normal stress varies if the adherends also happen to be of the same material and the same thickness.

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