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INFLUENCE OF DAMPING AND MASS-STIFFNESS DISCONTINUITIES UPON THE DYNAMIC STABILITY OF A FREE-FREE BEAM UNDER A GIMBALED THRUST OF PERIODICALLY-VARYING MAGNITUDE

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ABSTRACT

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This report considers the dynamic stability of a piecewise continuous beam accelerated through space by a gimbaled, pulsating end thrust and is in essence an investigation into the stability of transverse vibrations induced by the periodically-varying portion of the end thrust. A beam geometry and environment was assumed which could approximate a rocket vehicle. Initial conditions, damping, and longitudinal coupling effects were considered. The analysis was performed on a uniform beam and the method extended to include stepped beams.

The results indicated that parametric instability could develop at arbitrarily small thrust values when damping was neglected. It was determined that damping eliminated instabilities which existed at near zero thrust values and that critical thrust parameters could be obtained. Results indicate that modern rocket vehicles could be unstable for some parameter combinations. author

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NOTATION

a	Longitudinal acceleration
A	Beam cross-sectional area
A_1, A_2	Cross sectional area of beam section (1) and (2) respectively
$\bar{A}$	Nondimensional area ratio A_1/A_2
A_o, A_n, B_n, C_n	Boundary value constants depending upon initial conditions - uniform beam analysis
$(c)_m$	Column matrix m of c's
c	Length of beam section (1)
C	Viscous damping coefficient (transverse vibrations)
$D_{i,j}$	Square array of elements appearing in characteristic determinant
EI	Bending stiffness of uniform beam
f(x)	Displacement function at time $t = 0$
g(x)	Velocity function at time $t = 0$
[F]	Square matrix
F_{jk}	Coefficient in the j^{th} row and k^{th} column of matrix [F]
$\bar{F}_{jj}$	$= \frac{1}{\bar{\Omega}^2} (F_{jj} - \frac{\eta^2}{4})$
g	Longitudinal damping factor
[G]	Square matrix
G_{jk}	Coefficient in the j^{th} row and k^{th} column of matrix [G]
$[H]_s$	Square matrices of H_{jk}
H^s_{jk}	= Coefficient in the j^{th} row and k^{th} column of matrix $[H]_s$, $s = 1,2,\ldots$
i	$\sqrt{-1}$

$H(\alpha)$	Function formed from $\Delta(\alpha)$ to eliminate singularities
$h(x-c)$	$= 1 \quad x \geq c$ a unit step function $= 0 \quad x < c$
$[I]$	Identity matrix
I	Area moment of inertia
I_1, I_2	Area moment of inertia at beam section (1) and (2) respectively
$\bar{I}$	Nondimensional moment of inertia ratio I_1/I_2
K_j	Constants evaluated so as to eliminate singularities of $H(\alpha)$
K_θ	Directional control factor determining thrust vector gimbal angle
ℓ	Length of uniform beam
m	Mass per unit length of beam
m_1, m_2	Mass per unit length of beam section (1) and (2) respectively
M_T	Total mass of the beam
M	Moment distribution in beam; also, range of index m in evaluation of $\Delta_j(\bar{F}_{jj})$
N	Number of bending degrees of freedom assumed in numerical analysis
p	Lateral force on beam arising from component of thrust due to gimbaling
P	Axial force distribution in beam
P^*	Damped axial force distribution in beam
Pr	Product of diagonal elements of $\Delta(\alpha)$
(q)	A column matrix
q_A, q_B	Rigid-body generalized coordinates
q_n	n^{th} bending generalized coordinate

R	Modulus of $z \pm \sqrt{z^2 - 1}$
S	Sum of nondiagonal elements of $\Delta(\alpha)$
t	Real-time variable
T_o	Amplitude of constant thrust
T_1	Amplitude of sinusoidal varying thrust
$\bar{T}_o$	Uniform beam nondimensional thrust parameter = $T_o \ell^2/EI$
$\bar{T}_o$	Stepped beam nondimensional thrust parameter = $T_o \ell^2/EI_2$
(u)	A column matrix
u_k	k^{th} element of matrix u
$u(x,t)$	Longitudinal displacement of particles of beam measured in Lagrangian coordinate system
V	Transverse shear distribution in beam
x	Lagrangian coordinate defining position of particles in unstrained beam relative to one end of the beam
x_G	x-coordinate corresponding to the location of direction-sensing element in the beam
X	Function of x
$y(x,t)$	Lateral displacement of axis of beam from fixed reference line
z	Characteristic value = $\cos 2\pi\alpha$
ζ_j	$= \cos 2\pi \bar{F}_{jj}^{\frac{1}{2}}$
α^2	$= m/AE$
α	Characteristic exponent whose value indicates the stability of a system whose motion is represented by linear differential equations with periodic coefficients
β	Argument of $z \pm \sqrt{z^2 - 1}$
T	$= \frac{\Omega}{\pi} \ln R - \frac{\eta}{2}$

γ	$= T_1/T_o$
$\delta(\xi)$	Dirac delta function
η	Transverse damping factor $= \frac{C}{m\omega_1}$
$\Delta(\alpha)$	Value of the infinite determinant of coefficients obtained from series expansion of $\chi_k(\tau)$
$\Delta_j(\alpha)$	$= \Delta(\alpha) \cdot (-\alpha^2 + \bar{F}_{jj})$
ε	A small quantity
θ	Gimbal angle, equal to rotation of thrust vector from a tangent to the beam-deflection curve
λ_n^4	Uniform beam frequency parameter $= \omega_n^2 \frac{m\ell^4}{EI}$
λ_n^4	Stepped beam frequency parameter $= \omega_n^2 \frac{m_2\ell^4}{EI_2}$
ξ	Nondimensional coordinate $= x/\ell$
ξ_c	Nondimensional coordinate $= c/\ell$
ξ_G	Nondimensional coordinate $= x_G/\ell$
τ	Nondimensional time variable $= \omega_1 t$
ϕ_n	Mode shape of n^{th} vibration mode of uniform free-free beam
$\Phi(\xi)$	A function defining the longitudinal force distribution in a uniform beam arising from the varying thrust component
$\Phi^*(\xi)$	Damped $\Phi(\xi)$
$\psi(x)$	Rotation of the beam element located at x
ψ_G	Rotation of the beam element located at x_G
(χ)	A column matrix
$\chi_k(\tau)$	k^{th} element of (χ) with a periodic variation of $\frac{2\pi}{\bar{\Omega}}$ in τ
ω_L	Fundamental longitudinal frequency of free-free beam
ω_n	Bending frequency of n^{th} mode of free-free beam

$\omega_{(n)}$	Bending frequency of n^{th} mode of free-free beam with end thrust. $n = 1,2,\ldots,N$
$\omega_{(B)}$	Rigid body rotation frequency of free-free beam $B = N + 1$
$\bar{\omega}$	Nondimensional frequency parameter used to represent general form of solution
$\bar{\omega}_L$	Nondimensional longitudinal frequency = ω_L/ω_1
$\bar{\omega}_n$	Nondimensional frequency of free vibration = ω_n/ω_1
$\bar{\omega}_{(n)}$	Nondimensional frequency for a beam under thrust = $\omega_{(n)}/\omega_1$
Ω	Frequency of thrust variation
$\bar{\Omega}$	Nondimensional frequency of thrust variation = Ω/ω_1
$\cdot$	$= \frac{d}{d\tau}$ or $\frac{\partial}{\partial\tau}$
$'$	$= \frac{d}{d\xi}$ or $\frac{\partial}{\partial\xi}$

CHAPTER 1

INTRODUCTION

The theory of dynamic stability of elastic systems has been defined as "the study of vibrations induced by pulsating parametric loading" (ref. 1). Extending this definition, the analysis of the dynamic stability of beams may be defined as the study of transverse motion induced by a pulsating longitudinal force. This investigation considers the dynamic stability of free-free beam subjected to a gimbaled thrust of periodically-varying magnitude including the effects of longitudinal and transverse damping and arbitrary initial conditions.

For the most general model configuration, Fig. 1 describes a slender piecewise continuous beam accelerated through space by a gimbaled thrust, $T_o + T_1 \cos\Omega t$. T_o and T_1 are thrust values associated with the constantly applied thrust and the amplitude of the periodically-varying thrust, respectively. For rotational control, an attitude sensor located at beam station $x = G$, denoted as x_G, controls the thrust gimbal angle, θ, by means of a simple feedback system. Internal and external dissipative forces are assumed to be acting.

The investigation was conducted in two phases. In phase 1 the equations of motion were derived for a uniform beam. In phase 2 the equations of motion were derived for a discontinuous beam which existed as two uniform sections connected by a rigid bulkhead.

In phase 1 the primary objective was to determine the effects of damping and initial conditions upon the dynamic stability of a uniform beam under the imposed boundary conditions. This objective was accomplished with the following considerations:

1. The influence of longitudinal vibrations was considered in deriving the equation of motion.
2. The transient as well as the steady state solution for longitudinal motion of the beam elements was considered.
3. Transverse and longitudinal damping terms were taken into the equation of motion.
4. The partial differential equation of motion was reduced to a system of ordinary differential equations by Galerkin's method.
5. System stability was established.

In phase 2 the primary objectives were twofold; (1) establish a method by which a stepped beam could be analyzed by extending or modifying methods utilized in the uniform beam analysis, and (2) determine the effect of the stepped beam geometry upon the dynamic stability. These objectives were accomplished for the same considerations as those taken in the uniform beam analysis with two exceptions. Initial conditions imposed on the longitudinal displacements were not considered and longitudinal damping was not considered.

Restrictions and assumptions imposed to maintain linearity, simplify analysis, and render a solution were as follows:

1. Motion was restricted to one plane.
2. Shear and rotary inertia effects were neglected.
3. Simple beam theory was assumed applicable.

4. Viscous damping was assumed for lateral or transverse vibration.
5. Structural damping was assumed for longitudinal vibration in the uniform beam analysis.
6. Beam rotation and deformations were restricted to small values.

In brief, a fourth order partial differential equation was derived which governs the transverse mode of vibration. Application of Galerkin's method (ref. 2) reduced the resulting partial differential equation to an infinite set of time dependent second order differential equations which were found to be amenable to the method of solution set forth by T. Beal (ref. 3).

CHAPTER 2

PREVIOUS WORK

Development of fundamentals in the theory of dynamic stability was probably initiated in the early nineteen twenties. However, the subject had more academic interest than widespread practical application. The era of the high-speed aircraft with its minimum weight structure and its affinity for vibration problems sparked renewed interest in the phenomenon of dynamic instability. Therefore it was not until the early nineteen fifties that any extensive treatment was forthcoming.

A recent textbook by V. Bolotin (ref. 1) translated from the Russian by V. Weingarten and others, has compiled and utilized the more significant works in the field of dynamic stability through the nineteen fifties and early sixties. Bolotin approached the subject in a classical sense by developing the theory for elementary problems and then extending the theory to more complex elastic systems. A number of practical applications were included for a variety of structure types.

Significant contributions in the field were made by T. Beal (ref. 3) in the analysis of a free-free beam accelerated by a gimbaled periodically-varying end thrust. The problem itself had very immediate practical application to modern rockets but the most important contribution was the utilization and modification of existing mathematical techniques.

CHAPTER 3

PROBLEM REVIEW

3.1 General

The critical load of any elastic system can be obtained by determining the smallest load at which a disturbance causes a significant departure from the equilibrium position. This criterion is known as the dynamic stability criterion. If the system is also subjected to a periodic loading, and if the amplitude of the load is less than the critical load defined above, the response of the system generally remains bounded. It can be shown, however, that a system becomes unstable for certain relationships between the disturbing frequency and the natural frequencies of a system.

Consider the system described in the introduction, which essentially describes a straight rod compressed by an axial pulsating thrust, $T_o + T_1 \cos\Omega t$. The thrust values T_o and T_1 are associated with the constantly applied thrust and the amplitude of the periodically-varying thrust, respectively.

If damping is neglected, both lateral and longitudinal vibrations are excited by the pulsating thrust component. Ordinary resonance defines instability for forcing frequencies in the vicinity of an natural frequency of longitudinal vibration and is characterized by a rapid unbounded increase in amplitudes of longitudinal vibrations with time. Parametric resonance is characterized by a rapid unbounded response in transverse vibrations.

Contrary to "ordinary resonance" these instabilities may exist in the vicinity of any of the forcing frequencies listed as follows;

$$\text{Type I} \qquad \Omega = \frac{2\omega_{(i)}}{k} \tag{3.1.1}$$

$$\text{Type II} \qquad \Omega = \frac{\omega_{(i)} \pm \omega_{(j)}}{k} \qquad i \neq j \tag{3.1.2}$$

$k = 1,2,\ldots$

i,j an integer

where Ω is the excitation frequency and $\omega_{(n)}$ is the natural frequency of the nth mode of lateral vibration. For purposes of identification and to be consistent with previous investigations, instability associated with Eq. (3.1.1) will be a Type I, kth order instability and instability associated with Eq. (3.1.2) will be a Type II, kth order instability.

The equation of motion for a beam compressed by a pulsating longitudinal force is a fourth order partial differential equation. Under certain boundary conditions such as simple supported ends, the partial differential equation can be reduced to a single second order differential in time with periodic coefficients. This is essentially some form of the Mathieu-Hill equation and permits usage of tabulated results (ref. 4). In general, this is not the case and methods of reduction and subsequent solutions must be sought.

An approximate method using the Galerkin procedure (ref. 2) has been used effectively in reducing the general partial differential equation to a set of dependent linear second order differential equations as given by the following typical matrix equation:

$$(\ddot{q}) + [F](q) + \gamma\Phi(t)\ [G](q) = 0 \qquad (3.1.3)$$

Regardless of the method of reduction used in beam analysis, some form of the equation represented by Eq. (3.1.3) is obtained. The complexity of the solution is dependent on the form of the F and G matrices. For example, if F and G have real distinct eigenvalues and FG = GF, then F and G may be diagonalized simultaneously by the same modal matrix A containing the eigenvectors common to both matrices. Premultiplying the postmultiplying Eq. (3.1.1) by A^{-1} and A, respectively, uncouples the system and yields a set of independent Mathieu or Mathieu-Hill equations which may be solved by classical methods outlined in References 1 , 4 , and 5 .

Previous studies have been oriented toward seeking regions of instability rather than toward a complete solution. Regions of instability are very easily obtained if the system of equations represented by Eq. (3.1.3) has a transformation which will uncouple the equation. Such a transformation was described in the preceding paragraph. For more general and complicated systems, Bolotin (ref. 1) has formulated exact and approximate methods for finding boundaries separating stable and unstable regions.

3.2 Damping

The problem of damping is generally neglected with excellent justification since many problems in vibrations consist of finding natural frequencies in order to determine the spectrum of resonance and determining steady-state amplitudes at frequencies far removed from any resonance frequency. In these two cases, damping has little or no effect on the results.

Damping becomes a significant factor in determining regions of instability for parametrically excited systems. With the inclusion of damping whole regions of instability may be eliminated completely and the more severe regions may be significantly narrowed for relatively small factors. It can also be shown that response to parametric loadings may not be sustained within the regions of stability when arbitrarily small damping terms are taken but will decay exponentially at a rate dependent on the damping factor. This report considers viscous damping of lateral vibrations and structural damping of the longitudinal motion in the analytical formulation with the justification of mathematical compatibility and very limited experimental verification.

The assumption of viscous damping does not define all of the dissipative forces acting on or within a structural member. Structural damping is generally accepted as a function of relative displacement, whereas viscous damping is proportional to the velocity. The problem of accurately describing damping effects analytically is complex and can only be resolved by extensive experimental investigation. At best, damping factors may be determined experimentally for a specified structure to "fit" experimental curves. Experimental investigations may indicate a range of damping factors from which a factor can be chosen which will define the damping influence for certain structural types. Damping factors representing lower bounds attainable by a system may be chosen to construct an envelope curve.

3.3 Initial Condition Considerations

Initial conditions of this investigation are those conditions imposed on the displacement and velocity of the longitudinal elements at $t = 0$. They form the boundary conditions for the transient portion of the general solution for longitudinal motion of an axially forced beam.

In classical forced vibration problems, instability is generally associated with resonance, that is, when the forcing frequency is in the vicinity of a natural frequency. It is conceivable that the natural longitudinal frequency could be a subharmonic of the forcing frequency and not be unstable in the usual sense of resonance. It was assumed in this investigation that a maximum influence would be exerted on the transverse motion at a forcing to longitudinal frequency ratio of 1:n, where n is an integer greater than 1. Only when such a condition existed was a solution possible. More general conditions were unsolvable by the method presented herein.

CHAPTER 4

ANALYTICAL DEVELOPMENT - UNIFORM BEAM

4.1 Equations of Motion

The coordinate system of the beam under analysis, shown in Fig. 2, relates particle or element displacement to a Lagrangian coordinate system. In its initial state, the x coordinate locates particle position while the y coordinate, a function of x & t, measures particle lateral motion relative to a fixed reference line. Particle displacement u(x,t) measures particle motion parallel to the fixed reference line.

An element of the beam is shown in Fig.2(b) in equilibrium at some arbitrary position and time with respect to external and effective forces. The effective forces acting are the longitudinal inertial force $m \frac{\partial^2 u}{\partial x^2}$, the lateral inertial force $m \frac{\partial^2 y}{\partial x^2}$, and the dissipative damping force $C \frac{\partial y}{\partial t}$. This condition of equilibrium is stated as D'Alembert's principle.

The equations of equilibrium are obtained by the summation of forces and moments as

$$\sum F_H = \frac{\partial P}{\partial x} + m \frac{\partial^2 u}{\partial t^2} = 0 \qquad (4.1.1)$$

$$\sum F_V = \frac{\partial V}{\partial x} + m \frac{\partial^2 y}{\partial t^2} + C \frac{\partial y}{\partial t} + p = 0 \qquad (4.1.2)$$

$$\sum M = \frac{\partial M}{\partial x} + P\,\frac{\partial y}{\partial x} - V\left(1 + \frac{\partial u}{\partial x}\right) = 0 \qquad (4.1.3)$$

and the element rotation angle with respect to the horizontal reference line is readily obtained as

$$\psi = \tan^{-1} \frac{\frac{\partial y}{\partial x}}{\left(1 + \frac{\partial u}{\partial x}\right)} \qquad (4.1.4)$$

If rotational stability is maintained by a simple feedback system, the gimbal angle can be expressed as

$$\theta = K_\theta\ \psi_G \qquad (4.1.5)$$

where K_θ is the constant of proportionality between the gimbal angle, θ, and the attitude sensor rotation angle, ψ_G. ψ_G is the angle of rotation at $x = G$ in Eq. (4.1.4).

In simple beam theory, the moment at any cross section may be written as

$$M = EI\,\frac{\partial^2 y}{\partial x^2} \qquad (4.1.6)$$

and the compressive force normal to the beam cross section may be written as

$$P = -\ AE\,\frac{\partial u}{\partial x} \qquad (4.1.7)$$

If the strain term, $\frac{\partial u}{\partial x}$, and the slope, $\frac{\partial y}{\partial x}$, are considered small in comparison to unity, as assumed in deriving Eqs. (4.1.6) and (4.1.7), Eq. (4.1.3) can be reduced to

$$\frac{\partial M}{\partial x} + P\,\frac{\partial y}{\partial x} - V = 0 \qquad (4.1.3a)$$

The shear term, V, can be eliminated between Eqs. (4.1.2) and (4.1.3a)

$$EI\,\frac{\partial^4 y}{\partial x^4} + \frac{\partial}{\partial x}\left[P\,\frac{\partial y}{\partial x}\right] + m\,\frac{\partial^2 y}{\partial t^2} + C\,\frac{\partial y}{\partial t} + p = 0 \qquad (4.1.8)$$

where P(x,t) is to be derived.

The equation of motion for undamped longitudinal motion is,

$$\frac{\partial^2 u}{\partial x^2} = \frac{m}{AE}\,\frac{\partial^2 u}{\partial t^2} \qquad (4.1.9)$$

the familiar wave equation which is obtained by direct substitution of Eq. (4.1.7) into Eq. (4.1.1).

The longitudinal displacement function u is dependent upon the axial component of the thrust. Therefore, if the gimbal angle is restricted to small angles, the boundary conditions existing are at

$$x = \ell; \qquad \frac{\partial u}{\partial x} = -\frac{T_o}{AE} - \frac{T_1}{AE}\cos\Omega t \qquad (4.1.10)$$

and at

$$x = 0; \qquad \frac{\partial u}{\partial x} = 0 \qquad (4.1.11)$$

4.2 Solution to the Wave Equation for Longitudinal Vibrations

4.2.1 Steady State Solution with Zero Damping

The steady state solution of Eq. (4.1.9) can be found in two parts as the displacement resulting from the time varying thrust $T_1 \cos \Omega t$ and the displacement resulting from the constant thrust T_o.

$$u_p = u_1 + u_2 \tag{4.2.1}$$

Let a solution of u_1 be a product of X and T which are functions of x and t, respectively.

$$u_1 = XT \tag{4.2.2}$$

Substitution of Eq. (4.2.2) into Eq. (4.1.9) yields

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{\alpha^2}{T}\frac{d^2T}{dt^2} = \text{Constant} \tag{4.2.3}$$

where $\alpha^2 = \frac{m}{AE}$. Let the constant term in Eq. (4.2.3) equal $-\nu^2$ to achieve a periodic solution. Solutions of Eq. (4.2.3) may be written as

$$X = C_1 \cos \nu x + C_2 \sin \nu x \tag{4.2.4}$$

$$T = C_3 \cos \frac{\nu}{\alpha} t + C_4 \sin \frac{\nu}{\alpha} t \tag{4.2.5}$$

Therefore, by Eq. (4.1.12)

$$u_1 = (C_1 \cos \nu x + C_2 \sin \nu x)(C_3 \cos \frac{\nu}{\alpha} t + C_4 \sin \frac{\nu}{\alpha} t) \tag{4.2.6}$$

$$\frac{\partial u_1}{\partial x} = (-\nu C_1 \sin \nu x + \nu C_2 \cos \nu x)(C_3 \cos \frac{\nu}{\alpha} t + C_4 \sin \frac{\nu}{\alpha} t) \tag{4.2.7}$$

Boundary conditions applicable to a periodic solution are at

$$x = \ell; \qquad \frac{\partial u_1}{\partial x} = - \frac{T_1}{AE} \cos\Omega t \tag{4.2.8}$$

and at

$$x = 0; \qquad \frac{\partial u_1}{\partial x} = 0 \tag{4.2.9}$$

Using Eq. (4.2.7) and the boundary conditions of Eqs. (4.2.8) and (4.2.9)

$$C_2 = 0$$

$$C_4 = 0$$

$$\nu = \alpha\Omega$$

$$C_1 = \frac{T_1}{AE} \frac{1}{\alpha\Omega} \frac{\cos\alpha\Omega x}{\sin\alpha\Omega\ell} \cos\Omega t$$

u_1 can then be written as

$$u_1 = \frac{T_1}{AE} \frac{1}{\alpha\Omega} \frac{\cos\alpha\Omega x}{\sin\alpha\Omega\ell} \cos\alpha\Omega t \tag{4.2.10}$$

A solution of u_2 can be taken as

$$u_2 = X + T \tag{4.2.11}$$

subject to the boundary conditions that at

$$x = \ell; \qquad \frac{\partial u_2}{\partial x} = -\frac{T_o}{AE} \tag{4.2.12}$$

and at

$$x = 0; \qquad \frac{\partial u_2}{\partial x} = 0 \tag{4.2.13}$$

The displacement u_2 can be obtained by substitution of Eq. (4.2.11) into Eq. (4.2.9), which separates variables, then straight integration. Applying the boundary conditions of Eqs. (4.2.12) and (4.2.13) yields

$$u_2 = -\frac{T_o x^2}{2AE\ell} - \frac{T_o t^2}{2AE\ell^2} + \text{constant} \tag{4.2.14}$$

The particular solution of Eq. (4.1.8) can now be written as

$$u_p = u_1 + u_2 + \text{constant}$$

$$u_p = -\frac{T_o x^2}{2AE\ell} - \frac{T_o t^2}{2AE\alpha^2\ell} + \frac{T_1}{AE}\frac{1}{\alpha\Omega}\frac{\cos\alpha\Omega x}{\sin\alpha\Omega\ell}\cos\alpha\Omega t + \text{constant} \tag{4.2.15}$$

4.2.2 <u>Transient Solution</u>

To obtain a general solution, a term u_o must be added to the particular solution u_p such that the sum will satisfy arbitrary initial conditions.

Choose u_o of the form

$$u_o = XT \tag{4.2.16}$$

subject to the boundary conditions

$$u_o(x,0) = f(x) - u_p(x,0) \tag{4.2.17}$$

$f(x)$ = Initial displacement

$$\frac{\partial u_o}{\partial t}(x,0) = g(x) - \frac{\partial u_p}{\partial t}(x,0) \tag{4.2.18}$$

$g(x)$ = Initial velocity

$$\frac{\partial u_o}{\partial x}(0,t) = 0 \tag{4.2.19}$$

$$\frac{\partial u_o}{\partial x}(\ell,t) = 0 \tag{4.2.20}$$

Substitution of Eq. (4.2.16) into the wave equation yields

$$\frac{X''}{X} = \alpha^2 \frac{T''}{T} = - k^2 \tag{4.2.21}$$

$$u_o = (C_1 \cos kx + C_2 \sin kx)(C_3 \cos \frac{k}{\alpha} t + C_4 \sin \frac{k}{\alpha} t) \tag{4.2.22}$$

$$\frac{\partial u_o}{\partial x} = (-C_1 k \sin kx + C_2 k \cos kx)(C_3 \cos \frac{k}{\alpha} t + C_4 \sin \frac{k}{\alpha} t) \tag{4.2.23}$$

From Eq. (4.2.19), Eq. (4.2.20), and Eq. (4.2.23)

$$C_2 = 0$$

$$k = \frac{n\pi}{\ell}$$

The transient solution can then be written as

$$u_o = A_o + \sum_{n=1}^{\infty} [A_n \cos \frac{n\pi t}{\alpha\ell} + B_n \sin \frac{n\pi t}{\alpha\ell}] \cos \frac{n\pi x}{\ell} \quad (4.2.24)$$

$$\frac{\partial u_o}{\partial t} = \sum_{n=1}^{\infty} \frac{n\pi}{\alpha\ell} [-A_n \sin \frac{n\pi t}{\alpha\ell} + B_n \cos \frac{n\pi t}{\alpha\ell}] \cos \frac{n\pi x}{\ell} \quad (4.2.25)$$

Solutions for the constants A_n and B_n may be obtained from Eqs. (4.2.17) and (4.2.18), respectively. A_o does not effect $\frac{\partial u_o}{\partial x}$.

$$f(x) - u_p(x,0) = A_o + \sum_{n=1}^{\infty} A_m \cos \frac{m\pi x}{\ell} \quad (4.2.26)$$

Multiplying by $\cos \frac{n\pi x}{\ell}$ and integrating over the length yields

$$A_n = \frac{2}{\ell} \int_0^{\ell} [f(x) - u_p(x,0)] \cos \frac{n\pi x}{\ell} \, dx \quad (4.2.27)$$

In a similar manner,

$$B_n = \frac{2\alpha}{n\pi} \int_0^{\ell} [g(x) - \frac{\partial u_p}{\partial t} (x,0)] \cos \frac{n\pi x}{\ell} \, dx \quad (4.2.28)$$

The general solution of Eq. (4.1.9) can now be written as

$$u = u_p + u_o = -\frac{T_o x^2}{2AE\ell} - \frac{T_o t^2}{2AE\alpha^2 \ell} + \frac{T_1}{AE}\frac{1}{\alpha\Omega}\frac{\cos\alpha\Omega x}{\sin\alpha\Omega\ell}\cos\Omega t$$

$$+ A_o + \sum_{n=1}^{\infty} [A_n \cos\frac{n\pi}{\alpha\ell} t + B_n \sin\frac{n\pi}{\alpha\ell} t] \cos\frac{n\pi x}{\ell} \tag{4.2.29}$$

where A_n and B_n are defined by Eqs. (4.2.27) and (4.2.28). The normal force P in Eq. (4.1.7) due to the longitudinal vibration is

$$P = -AE\frac{\partial u}{\partial x},$$

then

$$P = \frac{T_o x}{\ell} + T_1 \frac{\sin\alpha\Omega x}{\sin\alpha\Omega\ell}\cos\Omega t$$

$$+ AE\sum_{n=1}^{\infty}\frac{n\pi}{\ell}[A_n \cos\frac{n\pi}{\ell\alpha} t + B_n \sin\frac{n\pi}{\alpha\ell} t] \cdot \sin\frac{n\pi x}{\ell} \tag{4.2.30}$$

or

$$P = P_o + P_1 + P_c \tag{4.2.30a}$$

$$P_o = \frac{T_o x}{\ell} \tag{4.2.31}$$

$$P_1 = T_1 \frac{\sin\alpha\Omega x}{\sin\alpha\Omega\ell}\cos\Omega t \tag{4.2.32}$$

$$P_c = AE\sum_{n=1}^{\infty}\frac{n\pi}{\ell}[A_n \cos\frac{n\pi}{\alpha\ell} t + B_n \sin\frac{n\pi}{\alpha\ell} t] \cdot \sin\frac{n\pi x}{\ell} \tag{4.2.33}$$

4.2.3 Steady State Solution with Longitudinal Damping

In Eq. (4.2.1), u_1 was taken as the displacement resulting from a time varying thrust $T_1\cos\Omega t$. If structural damping is assumed for longitudinal vibrations, the equation of motion, Eq. (4.1.8), will be modified to (ref. 6)

$$\frac{1}{\alpha^2}\frac{\partial^2\bar{u}_1}{\partial t^2} - ig\omega_L^2\bar{u}_1 - \frac{\partial^2\bar{u}_1}{\partial t^2} = 0 \tag{4.2.34}$$

where

$i = \sqrt{-1}$

$\alpha^2 = \frac{m}{AE}$

g = Structural damping factor of longitudinal vibrations

$\omega_L = \frac{\pi}{\alpha \ell} = \frac{\pi}{\ell}\sqrt{\frac{AE}{m}}$ From Eq. (4.2.24)

= Fundamental natural frequency of longitudinal vibrations.

The real portion of $\bar{u}_1$ in Eq. (4.2.34) will represent the desired motion of the damped solution.

As in Section 4.2, a solution is obtained by a separation of variables technique. Let $\bar{u}_1$ be a product of X and T which are functions of x and t, respectively.

$$\bar{u}_1 = XT$$

Substitution into Eq. (4.2.34) yields

$$\frac{1}{\alpha^2}\frac{d^2X}{dx^2}T - ig\omega_L^2XT - X\frac{d^2T}{dt^2} = 0$$

$$\frac{1}{X}\frac{d^2X}{dx^2} - ig\omega_L^2\alpha^2 = \frac{\alpha^2}{T}\frac{d^2T}{dt^2} = -\nu^2 \tag{4.2.35}$$

The boundary conditions imposed for a particular solution are at

$$x = \ell; \qquad \frac{\partial \bar{u}_1}{\partial x} = -\frac{T_1}{AE}\cos\Omega t$$

and at

$$x = 0; \qquad \frac{\partial \bar{u}_1}{\partial x} = 0$$

It follows that

$$\frac{d^2T}{dt^2} + \frac{\nu^2}{\alpha^2}T = 0 \tag{4.2.36}$$

where in order to satisfy the boundary conditions at $x = \ell$ the following relationships must exist:

$$\frac{\nu^2}{\alpha^2} = \Omega^2$$

$$\nu^2 = \Omega^2\alpha^2$$

It then follows that

$$\frac{d^2X}{dx^2} + [\Omega^2\alpha^2 - ig\,(\frac{\pi}{\ell})^2]X = 0 \tag{4.2.37}$$

and

$$X = C_1 \cos\,[\Omega^2\alpha^2 - ig\,(\frac{\pi}{\ell})^2]^{\frac{1}{2}}x$$
$$+ C_2 \sin\,[\Omega^2\alpha^2 - ig\,(\frac{\pi}{\ell})^2]^{\frac{1}{2}}x \tag{4.2.38}$$

Applying the boundary conditions and solving for the constants yields an expression for $\bar{u}_1$ of which the real portion will represent the desired motion. The desired solution for the force distribution is obtained from

$$P = - AE \frac{\partial u}{\partial x}$$

as

$$P_1^* = T_1 \frac{\sin[\Omega^2\alpha^2 - ig\ (\frac{\pi}{\ell})^2]^{\frac{1}{2}}x}{\sin[\Omega^2\alpha^2 - ig\ (\frac{\pi}{\ell})^2]^{\frac{1}{2}}\ell} \cos\Omega t \qquad (4.2.39)$$

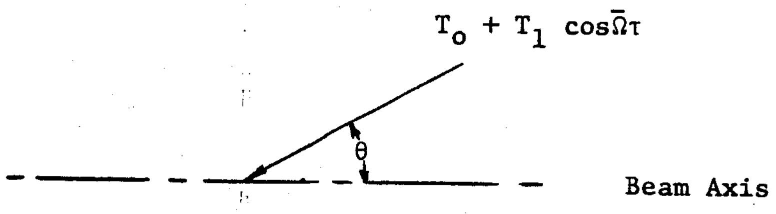
If in Eq. (4.2.39) the damping coefficient equals zero, Eq. (4.2.39) is equivalent to P_1 in Eq. (4.2.32). Therefore, the complex expression for the force distribution with damping is

$$P^* = P_o + P_1^* \qquad (4.2.40)$$

where P_c, the force resulting from satisfying initial conditions, is assumed to damp out with time.

4.3 Lateral Force on the Beam due to Gimbaling

A vectoral sketch of the applied thrust acting on the beam at $x = \ell$ is given below:



For small gimbal angles, $\sin\theta = \theta$, and the normal side load acting at $x = \ell$ can be written as

$$p = (T_o + T_1 \cos\Omega t)\theta\delta(x-\ell) \tag{4.3.1}$$

where $\delta(x-\ell)$ is a dirac delta. The thrust gimbal angle was expressed as a direct function of the beam rotation sensor in Eq. (4.1.5) as $\theta = K_\theta \psi_G$, where

$$\psi_G = \tan^{-1} \frac{\frac{\partial y}{\partial x}(x_G, t)}{(1 + \frac{\partial u}{\partial x})} \tag{4.3.2}$$

If the strain term $\frac{\partial u}{\partial x}$ is assumed to be much smaller than unity, and if the rotation term $\frac{\partial y}{\partial x}$ is restricted to small values, then

$$\theta = K_\theta \frac{\partial y}{\partial x} (x_G, t) \tag{4.3.3}$$

is a good approximation.

The normal side load p is

$$p = (T_o + T_1 \cos\Omega t) K_\theta \frac{\partial y}{\partial x} (x_G, t)\delta(x-\ell) \tag{4.3.4}$$

4.4 Introduction of Dimensionless Variables into the Equations of Motion

The governing equations of motion derived previously are listed below in the dimensional form in which they were derived.

(1) The governing equation of motion for beam vibration.

$$EI \frac{\partial^4 y}{\partial x^4} + \frac{\partial}{\partial x} [P \frac{\partial y}{\partial x}] + m \frac{\partial^2 y}{\partial t^2} + C \frac{\partial y}{\partial t} + p = 0 \qquad (4.4.1)$$

(2) The undamped force distribution.

$$P = \frac{T_o x}{\ell} + T_1 \frac{\sin\alpha\Omega x}{\sin\alpha\Omega\ell} \cos\Omega t + AE \sum_{n=1}^{\infty} \frac{n\pi}{\ell} [A_n \cos \frac{n\pi}{\ell\alpha} t + B_n \sin \frac{n\pi}{\alpha t} t] \sin \frac{n\pi x}{\ell} \qquad (4.4.2)$$

(3) The damped force distribution.

$$P^* = \frac{T_o x}{\ell} + T_1 \frac{\sin[\Omega^2\alpha^2 - ig(\frac{\pi}{\ell})^2]^{\frac{1}{2}} x}{\sin[\Omega^2\alpha^2 - ig(\frac{\pi}{\ell})^2]^{\frac{1}{2}} \ell} \cos\Omega t \qquad (4.4.3)$$

(4) The lateral force p.

$$p = (T_o + T_1 \cos\Omega t) K_\theta \frac{\partial y}{\partial x} (x_G, t) \delta(x - \ell) \qquad (4.4.4)$$

Dimensionless variables may be introduced by the following steps.

Step (1)

The following substitutions are made:

$$x = \xi\ell, \qquad \xi = \frac{x}{\ell} \qquad (4.4.5)$$

$$t = \frac{\tau}{\omega_1}, \qquad \tau = \omega_1 t \qquad (4.4.6)$$

The fundamental frequency of a free-free beam given as ω_1 was chosen as the time normalization constant with the same physical characteristics as the beam under analysis. As shown the ξ term has an obvious definition.

Step (2)

Multiply the equation of motion by $\frac{\ell^4}{EI}$, factor out T_o in the P expressions, and factor out $\frac{T_o}{\ell^2}$ in the p expression.

Step (3)

As a means of grouping terms and rendering a more compact equation, the following substitutions are made into the equations resulting from Steps (1) and (2) above.

$$\bar{\Omega} = \frac{\Omega}{\omega_1} \tag{4.4.7}$$

$$\bar{\sigma} = \frac{\Omega}{\omega_L} \tag{4.4.8}$$

The fundamental longitudinal frequency was found to be

$$\omega_L = \frac{\pi}{\alpha\ell} = \frac{\pi}{\ell}\sqrt{\frac{AE}{m}} \tag{4.4.9}$$

where α is replaced by its original value of

$$\alpha^2 = \frac{m}{AE} \tag{4.4.10}$$

$$\bar{\omega}_L = \frac{\omega_L}{\omega_1} \tag{4.4.11}$$

$$\bar{T}_o = \frac{T_o \ell^2}{EI} \qquad (4.4.12)$$

$$\gamma = \frac{T_1}{T_o} \qquad (4.4.13)$$

$$\eta = \frac{C}{m\omega_1} \qquad (4.4.14)$$

$$\lambda_n^4 = \omega_n^2 \frac{m\ell^4}{EI} \qquad (4.4.15)$$

$$\Phi(\xi) = \frac{\sin\pi\bar{\sigma}\xi}{\sin\pi\bar{\sigma}} \qquad (4.4.16)$$

$$\Phi^*(\xi) = \frac{\sin\pi(\bar{\sigma}^2 - ig)^{\frac{1}{2}}\xi}{\sin\pi(\bar{\sigma}^2 - ig)^{\frac{1}{2}}} \qquad (4.4.17)$$

Performing the indicated steps yields the following equations.

$$\frac{\partial^2 y}{\partial \xi^4} + \frac{\bar{T}_o}{T_o}\frac{\partial}{\partial \xi}\left[P(\xi)\frac{\partial y}{\partial \xi}\right] + \lambda_1^4 \frac{\partial^2 y}{\partial \tau^2} + \eta\lambda_1^4 \frac{\partial y}{\partial \tau}$$
$$+ \bar{T}_o \frac{\ell^2}{T_o} p = 0 \qquad (4.4.18)$$

$$P(\xi) = T_o[\xi + \gamma\Phi(\xi)\cos\bar{\Omega}\tau$$
$$+ \frac{AE}{T_o}\sum_{n=1}^{\infty} \frac{n\pi}{\ell}(A_n\cos\bar{\omega}_L\tau + B_n\sin\bar{\omega}_L\tau)\sin n\pi\xi]$$
$$(4.4.19)$$

$$P^*(\xi) = T_o[\xi + \gamma\Phi^*(\xi)\cos\bar{\Omega}\tau] \qquad (4.4.20)$$

$$p = \frac{T_o}{\ell^2}(1 + \gamma\cos\bar{\Omega}\tau)K_\theta \frac{\partial y}{\partial \xi}(\xi_G, \tau)\delta(\xi - 1) \qquad (4.4.21)$$

which when combined yields the final equation.

$$\frac{\partial^4 y}{\partial \xi^4} + \bar{T}_o \frac{\partial}{\partial \xi} [(\xi + \gamma\Phi(\xi)\cos n\bar{\Omega}\tau) \frac{\partial y}{\partial \xi}$$

$$+ \frac{\partial y}{\partial \xi} \frac{AE}{T_o} \sum_{n=1}^{\infty} \frac{n\pi}{\ell}(A_n \cos n\bar{\omega}_L \tau + B_n \sin n\bar{\omega}_L \tau) \sin n\pi\xi]$$

$$+ \lambda_1^4 \frac{\partial^2 y}{\partial \tau^2} + \eta\lambda_1^4 \frac{\partial y}{\partial \tau}$$

$$+ \bar{T}_o(1 + \gamma\cos\bar{\Omega}\tau)K_\theta \frac{\partial y}{\partial \tau}(\xi_G,\tau)\delta(\xi-1) = 0 \quad (4.4.22)$$

4.5 Reduction of Equation (4.4.22) to a System of Ordinary Differential Equations by Galerkin's Method (ref. 2)

Galerkin's method is normally applied as an approximate method for obtaining a solution to a differential equation under given boundary conditions by taking a function which satisfies the boundary conditions exactly then solving for function modifiers which will render a good approximation. The method is of the same general class as those of Rayleigh and Ritz.

The method of Galerkin will be used in this analysis to reduce the partial differential equation of Eq. (4.4.22) into a set of ordinary differential equations. This process is essentially a separation of time and spatial coordinates where the spatial coordinates are defined by the assumed displacement function and the function modifiers are time variables defined by a set of ordinary differential equations.

A description of the Galerkin method indicates that the solution of Eq. (4.4.22) can be adequately approximated by expressing the deflection

$$y_N(\xi,\tau) = q_A + q_B\xi + \sum_{n=1}^{\infty} q_n(\tau)\ \phi_n(\xi) \tag{4.5.1}$$

as a sum of some translation term $q_A(\tau)$, a rotation term $q_B(\tau)$, and a function $\phi_n(\xi)$ as the n^{th} vibration mode shape of a free-free beam that satisfies the boundary conditions. $q_n(\tau)$ is the generalized coordinate associated with $\phi_n(\xi)$.

Ref. 7 tabulates the characteristic functions and characteristic values for a number of beam configuration including a free-free beam, and include formulas for integrals containing characteristic functions of a vibrating beam.

The approximating function $\phi_n(\xi)$ chosen is a normalized eigenfunction for a uniform beam governed by the well-known differential equation

$$EI\,\frac{\partial^4 y}{\partial x^4} + m\,\frac{\partial^2 y}{\partial t^2} = 0 \tag{4.5.2}$$

and has the mathematical properties

$$\frac{d^4\phi_n}{d\xi^4} = \lambda_n^4\ \phi_n \tag{4.5.3}$$

$$\int_0^1 \phi_n\ \phi_m\ d\xi = \delta_{mn} = 0,\ m \neq n \\ = 1,\ m = n \tag{4.5.4}$$

where λ_n^4 was previously used in Eq. (4.4.15) as a normalization constant.

$$\lambda_n^4 = \omega_n^2 \frac{m\ell^4}{EI}$$

$$\omega_n^2 = \text{natural frequency}$$

The λ_n values are tabulated in ref. 7 as characteristic values of the characteristic functions ϕ_n.

Equation (4.5.4) satisfies the so-called orthogonality relationships within the range of integration and establishes the normalization constant by which ϕ_n is determined and tabulated in ref. 7 .

The chosen function y_N containing a rigid body and a beam displacement function satisfies the dynamic boundary conditions of zero shear and zero moment at the free ends of the beam. That is, at

$$\xi = 0; \quad \frac{\partial^2 y}{\partial \xi^2} = \frac{d^2\phi_n}{d\xi^2} = \frac{d^2 y_N}{d\xi^2} = 0$$

$$\frac{\partial^3 y}{\partial \xi^3} = \frac{d^3\phi_n}{d\xi^3} = \frac{d^3 y_N}{d\xi^3} = 0 \qquad (4.5.5)$$

and at

$$\xi = 1; \quad \frac{\partial^2 y}{\partial \xi^2} = \frac{d^2\phi_n}{d\xi^2} = \frac{d^2 y_N}{d\xi^2} = 0$$

$$\frac{\partial^3 y}{\partial \xi^3} = \frac{d^3\phi_n}{d\xi^3} = \frac{d^3 y_N}{d\xi^3} = 0 \qquad (4.5.6)$$

Galerkin's method requires that the error inherent in the approximate solution given by Eq. (4.5.1) be orthogonal to the weighting function $\frac{\partial y_N}{\partial q_i}$. In equation form,

$$\int_0^1 \S(y_N)\ \frac{\partial y_N}{\partial q_i}\ d\xi = 0 \qquad i = q_A, q_B, q_n \tag{4.5.7}$$

or

$$\int_0^1 \S(y_N) d\xi = 0 \tag{4.5.7a}$$

$$\int_0^1 \Phi(y_N)\xi d\xi = 0 \tag{4.5.7b}$$

$$\int_0^1 \S(y_N)\ \phi_n(\xi) d\xi \qquad n = 1,2,3,\ldots \tag{4.5.7c}$$

where $\S(y_N)$ is the error resulting when Eq. (4.5.1) is substituted into Eq. (4.4.22). The theory behind this process is explained in ref. 2 .

As an aid to integration of the preceding equations a few important formulas are presented below.

$$\int_0^1 \phi_n d\xi = \frac{1}{\lambda_n^4} \int_0^1 \frac{d^4\phi_n}{d\xi^4}\ d\xi = 0 \tag{4.5.8}$$

$$\int_0^1 \xi\phi_n d\xi = -\frac{1}{\lambda_n^4} \int_0^1 \frac{d^3\phi_n}{d\xi^3}\ d\xi = 0 \tag{4.5.9}$$

$$\int_0^1 \frac{d}{d\xi}\ [P\ \frac{dy_N}{d\xi}] d\xi = P(1)\ \frac{dy_N}{d\xi}\ (1) \tag{4.5.10}$$

$$\int_0^1 \frac{d}{d\xi}\ [P\ \frac{dy_N}{d\xi}] d\xi = P(1)\ \frac{dy_N}{d\xi}(1) - \int_0^1 P\ \frac{dy_N}{d\xi}\ d\xi \tag{4.5.11}$$

$$\int_0^1 \phi_n \frac{d}{d\xi}\left[P \frac{dy_N}{d\xi}\right] d\xi = \phi_n(1)\ P(1)\ \frac{dy_N}{d\xi}(1) - \int_0^1 \phi_n' \frac{P dy_N}{d\xi}\, d\xi \qquad (4.5.12)$$

$$\int_0^1 \delta(\xi-1)\, d\xi = 1 \qquad \text{By definition} \qquad (4.5.13)$$

The formulas were obtained by applying the boundary equations for the shear, moment, and force distribution.

In the succeeding derivations, the substitution

$$C_n \cos(n\omega_L\tau + \psi_n) = \frac{AEn\pi}{T_o\ell}\left[A_n \cos n\omega_L\tau + B_n \sin n\omega_L\tau\right] \qquad (4.5.14)$$

has been made, where

$$A_n = \frac{T_o\ell}{AEn\pi} C_n \sin \psi_n \qquad (4.5.14a)$$

$$B_n = \frac{T_o\ell}{AEn\pi} C_n \cos \psi_n \qquad (4.5.14b)$$

$$C_n = \frac{AEn\pi}{T_o\ell} \sqrt{A_n^2 + B_n^2} \qquad (4.5.14c)$$

Eqs. (4.5.7a, 4.5.7b, and 4.5.7c) are integrated with the aid of Eqs. (4.5.8) through (4.5.13) to obtain

$$\lambda_1^4 \ddot{q}_A + \eta\lambda_1^4\dot{q}_A + \frac{1}{2}\lambda_1^4\dot{q}_B + \frac{\eta\lambda_1^4}{2}\dot{q}_B$$

$$+ \bar{T}_o (1 + \gamma\cos\bar{\Omega}\tau) \left[q_B + \sum_{n=1}^{N} q_n\phi_n'(1)\right]$$

$$+ \bar{T}_o (1 + \gamma\cos\bar{\Omega}\tau)K_\theta \left[q_B + \sum_{n=1}^{N} q_n\phi_n'(\xi_G)\right] = 0 \qquad (4.5.15)$$

$$\frac{\lambda_1^4}{2}\ddot{q}_A + \frac{\eta}{2}\lambda_1^4\dot{q}_A + \frac{\lambda_1^4}{3}\ddot{q}_B + \frac{\eta}{3}\lambda_1^4\dot{q}_B$$

$$+ \bar{T}_o (1 + \gamma\cos\bar{\Omega}\tau) \left[q_B + \sum_{n=1}^{N} q_n\phi_n'(1)\right]$$

$$- \frac{\bar{T}_o}{2} q_B - T_o\gamma\cos\bar{\Omega}\tau \int_0^1 \phi d\xi$$

$$- \bar{T}_o (1 + \gamma\cos\bar{\Omega}\tau) \sum_{n=1}^{N} q_n\phi_n(1)$$

$$+ \bar{T}_o \gamma\cos\bar{\Omega}\tau \sum_{n=1}^{N} q_n \int_0^1 \phi_n\phi' d\xi$$

$$+ \bar{T}_o (1 + \gamma\cos\bar{\Omega}\tau)K_\theta \left[q_B + \sum_{n=1}^{N} q_n\phi_n'(\xi_G)\right]$$

$$+ \bar{T}_o q_B \sum_{s=1}^{\infty} \frac{C_s}{s\pi} [(-1)^s - 1] \cos(s\bar{\omega}_L\tau + \psi_s)$$

$$+ \bar{T}_o \sum_{n=1}^{N} q_n \sum_{s=1}^{\infty} \pi s\, C_s \int_0^1 \phi_n(\xi) \cos(s\pi\xi) d\xi$$

$$\cdot \cos(s\bar{\omega}_L\tau + \psi_s) = 0 \qquad (4.5.16)$$

$$\lambda_1^4 \ddot{q}_k + \eta\lambda_1^4 \dot{q}_k + \lambda_k^4 q_k$$

$$+ \bar{T}_o (1 + \gamma\cos\bar{\Omega}\tau) \left[q_B + \sum_{n=1}^{N} q_n\phi_n'(1)\right] \phi_k(1)$$

$$- \bar{T}_o \phi_k(1) q_B - \bar{T}_o\gamma\cos\bar{\Omega}\tau \left[\phi_k(1) - \int_o^1 \phi_k\Phi' d\xi\right] q_B$$

$$- \bar{T}_o \sum_{n=1}^{N} q_n \int_o^1 \xi\phi_n' \phi_k' d\xi - \bar{T}_o\gamma\cos\bar{\Omega}\tau \sum_{n=1}^{N} q_n \int_o^1 \Phi\phi_n' \phi_k' d\xi$$

$$+ \bar{T}_o (1 + \gamma\cos\bar{\Omega}\tau)K_\theta \left[q_B + \sum_{n=1}^{N} q_n\phi_n'(\xi_G)\right] \phi_k(1)$$

$$- \bar{T}_o q_B \sum_{s=1}^{\infty} C_s \cos (s\bar{\omega}_L\tau + \psi_s) \int_o^1 \phi_k' \sin (s\pi\xi) d\xi$$

$$- \bar{T}_o \sum_{n=1}^{N} q_n \sum_{s=1}^{\infty} C_s \cos (s\bar{\omega}_L + \psi_s) \int_o^1 \phi_n' \phi_k' \sin (s\pi\xi) d\xi = 0$$

$$k = 1,2,\ldots N \qquad (4.5.17)$$

Eq. (4.4.22) has now been reduced to a set of ordinary linear differential equations as given by Eqs. (4.5.15), (4.5.16), and (4.5.17). The coordinate q_A can be eliminated between Eqs. (4.5.15) and (4.5.16) without losing any important modes since translation is uncontrolled. The resultant set of equations may be represented in matrix form as

$$(\ddot{q}) + (\dot{q}) + [F] (q) + \gamma\cos\bar{\Omega}\tau \; [G] (q)$$

$$+ \sum_{s=1}^{\infty} [H]_s \cos (s\omega_L\tau + \psi_s) (q) = (0) \qquad (4.5.18)$$

where

$$(q) = \begin{Bmatrix} q_1 \\ q_2 \\ \cdot \\ \cdot \\ \cdot \\ q_N \\ q_B \end{Bmatrix} \qquad B = N + 1$$

and [F], [G], and [H] are matrices of order N + 1. The elements of these matrices are defined as follows:

$$F_{jk} = (\lambda_j/\lambda_1)^4 \ \delta_{jk} + \frac{\bar{T}_o}{\lambda_1^4} \ [\phi_j(1) \ \phi_k'(1) - \int_o^1 \xi\phi_j' \ \phi_k' \ d\xi$$

$$+ K_\theta\phi_j(1) \ \phi_k'(\xi_G)] \qquad j = 1,2,\ldots N \qquad (4.5.19)$$

$$k = 1,2,\ldots N$$

$$\delta_{jk} = 1 \qquad j = k$$

$$\delta_{jk} = 0 \qquad j \neq k$$

$$F_{j,N+1} = \frac{\bar{T}_o}{\lambda_1^4} K_\theta\phi_j(1) \qquad j = 1,2,\ldots N \qquad (4.5.20)$$

$$F_{N+1,k} = 12 \ \frac{\bar{T}_o}{\lambda_1^4} \left[\frac{1}{2} \ \phi_k'(1) - \phi_k(1) + \frac{1}{2} \ K_\theta\phi_k'(\xi_G) \right] \qquad (4.5.21)$$

$$k = 1,2,\ldots N$$

$$F_{N+1,N+1} = \frac{6\bar{T}_oK_\theta}{\lambda_1^4} \qquad (4.5.22)$$

$$G_{jk} = \frac{\bar{T}_o}{\lambda^4}\left[\phi_j(1)\ \phi_k(1) - \int_o^1 \Phi\phi_j'\ \phi_k' d\xi + K_\theta \phi_j(1)\phi_k'(\xi_G)\right]$$

$j = 1,2,\ldots N$ (4.5.23)

$k = 1,2,\ldots N$

$$G_{j,N+1} = \frac{\bar{T}_o}{\lambda_1^4}\left[\int_o^1 \phi_j \Phi' d\xi + K_\theta \phi_j(1)\right] \tag{4.5.24}$$

$j = 1,2,\ldots N$

$$G_{N+1,k} = 12\frac{\bar{T}_o}{\lambda_1^4}\left[\frac{1}{2}\phi_k'(1) - \phi_k(1) + \int_o^1 \phi_k \Phi' d\xi\right.$$

$$\left. + \frac{1}{2} K_\theta \phi_k'(\xi_G)\right] \quad k = 1,2,\ldots N \tag{4.5.25}$$

$$G_{N+1,N+1} = 12\frac{\bar{T}_o}{\lambda_1^4}\left[\frac{1}{2} - \int_o^1 \Phi d\xi + \frac{1}{2} K_\theta\right] \tag{4.5.26}$$

$$H^i_{jk} = -\frac{\bar{T}_o}{\lambda_1^4} C_s \int_o^1 \phi_k'\ \phi_j'\ \sin(s\pi\xi) d\xi \tag{4.5.27}$$

$j = 1,2,\ldots N$

$k = 1,2,\ldots N$

$s = 1,2,\ldots$

$$H^i_{j,N+1} = -\frac{\bar{T}_o}{\lambda_1^4} C_s \int_o^1 \phi_j'\ \sin(s\pi\xi) d\xi \tag{4.5.28}$$

$j = 1,2,\ldots N$

$s = 1,2,\ldots$

$$H^{i}_{N+1,k} = -12 \frac{\bar{T}_o}{\lambda_1^4} C_s \int_0^1 \phi_k' \sin(s\pi\xi)d\xi \tag{4.5.29}$$

$k = 1,2,\ldots N$

$s = 1,2,\ldots$

$$H^{i}_{N+1,N+1} = 12 \frac{\bar{T}_o}{\lambda_1^4} [(-1)^s - 1] \frac{1}{\pi s} \tag{4.5.30}$$

$s = 1,2,\ldots$

The undefined integrals given in the equations above are reduced to a form compatible with Fortran IV computer language as follows:

Ref. 7 gave the formula for ϕ_n with tables as

$$\phi_n = \cosh \lambda_n\xi + \cos \lambda_n\xi - \alpha_n(\sinh \lambda_n\xi - \sin \lambda_n\xi) \tag{4.5.31}$$

Then

$$\frac{d\phi_n}{d\xi} = \phi_n' = \lambda_n[\sinh \lambda_n\xi - \sin \lambda_n\xi - \alpha_n(\cosh \lambda_n\xi - \cos \lambda_n\xi)] \tag{4.5.32}$$

which can be expressed in exponential form as

$$\phi_n' = \frac{\lambda_n}{2} [e^{\lambda_n\xi} - e^{-\lambda_n\xi} + ie^{i\lambda_n\xi} - ie^{i\lambda_n\xi} - \alpha_n e^{\lambda_n\xi} - \alpha_n e^{\lambda_n\xi} - \alpha_n e^{i\lambda_n\xi} - \alpha_n e^{i\lambda_n\xi}] \tag{4.5.33}$$

Eq. (4.5.33) can be expressed as

$$\phi_n' = \frac{\lambda_n}{2} \sum_{\ell=o}^{3} [(i)^\ell + \alpha_n] \, e^{(i)^\ell \lambda_n \xi} \tag{4.5.34}$$

$$i = \sqrt{-1}$$

which is adaptable to programming.

The most difficult integral to evaluate appears in Eq. (4.5.23) as

$$\int_o^1 \Phi \phi_j' \, \phi_k' d\xi \tag{4.5.35}$$

Substitution of Eqs. (4.4.16) and (4.5.35) for Φ and ϕ_j', respectively, yields

$$\frac{1}{4} \frac{\lambda_j \lambda_k}{\sin \pi \bar{\sigma}} \sum_{\ell=o}^{3} \sum_{m=o}^{3} [(i)^\ell + \alpha_j][(i)^m + \alpha_k] \int_o^1 \sin \pi \sigma \xi^{[(i)^\ell + (i)^m] \lambda_n \xi} d\xi$$

(4.5.36)

which is easily integratable in its complex form from any table of integrals. The real part of the result is the desired integral value. A check of accuracy can be made by comparing the imaginary part to zero.

The remaining integrals are obtained by employing Eq. (4.5.34) in a similar manner.

In the absence of a complex computer routine, the integral formulas of (ref. 7) can be employed. However, for the particular integral of Eq. (4.5.36), a recurrence formula given for integrals of that type proved laborious to employ.

The matrix elements of Eqs. (4.5.19) through (4.5.30) are equally valid for longitudinally damped conditions by making the following substitutions,

Φ^* for Φ, See Eq. (4.4.17)

$H^i_{jk} = 0$ for all j and k

and retaining only the real parts.

4.6 Initial Conditions

The condition of zero displacement and zero velocity along the longitudinal axis is assumed to exist prior to application of the loading. For such a case, $f(x) = 0$ and $g(x) = 0$ in Eqs. (4.2.17) and (4.2.18), respectively.

The unknown constant A_n can be determined from

$$A_n = \frac{2}{\ell} \int_0^{\ell} [0 - u_p(x,0)] \cos \frac{n\pi x}{\ell} \, dx \qquad (4.6.1)$$

where $f(x) = 0$ has been subsituted into Eq. (4.2.27). A solution to $u_p(x,0)$ is obtained by solving Eq. (4.2.15) at $t = 0$ to obtain

$$u_p(x,0) = -\frac{T_o x^2}{2AE\ell} + \frac{T_1}{AE}\frac{1}{\alpha\Omega}\frac{\cos\alpha\Omega x}{\sin\alpha\Omega\ell} \tag{4.6.2}$$

Integration of Eq. (6.1.1) yields

$$A_n = \frac{T_o\ell}{AE}\left(\frac{2(-1)^n}{n^2}\left[\frac{1}{n^2} + \frac{\gamma}{\bar{\sigma}^2-n^2}\right]\right) \tag{4.6.3}$$

In like fashion, B_n can be determined from Eq. (4.2.28) with $g(x) = 0$ as

$$B_n = \frac{2}{n\pi}\int_0^\ell [0 - \frac{\partial u}{\partial t} p\ (x,0)] \cos\frac{n\pi x}{\ell}\, dx \tag{4.6.4}$$

Therefore, since $\frac{\partial u_p}{\partial t}(x,0) = 0$,

$$B_n = 0 \tag{4.6.5}$$

The constant C_n is evaluated from Eq. (4.5.14) as

$$C_n = \frac{AEn\pi}{T_o\ell}(A_n)$$

$$C_n = \frac{2\pi(-1)^n}{n}\left[\frac{1}{n^2} + \frac{\gamma}{\bar{\sigma}^2-n^2}\right] \tag{4.6.6}$$

Since $B_n = 0$, the phase angle, ψ_n, of Eq. (4.5.14) is equal to zero by Eq. (4.5.14b).

CHAPTER 5

ANALYTICAL DEVELOPMENT - STEPPED BEAM

5.1 Equation of Motion of a Stepped Beam

The coordinate system of the stepped beam, shown in Figure No. 2(a), relates particle or element displacement to a Lagrangian coordinate system. In its initial state, the x coordinate locates particle position while the y coordinate, a function of x and t, measures particle position normal to a fixed reference line. Particle displacement parallel to the fixed reference line is given by $u(x,t)$. The coordinate system and loading configuration is identical to that chosen in the uniform beam analysis of Section 4.

In Fig. No. 1(b) the beam coordinate constants are defined for continuous sections. Only one discontinuity will be considered in the analysis. Beam parameters will be designated by subscripts 1 or 2 to indicate the beam section to which the parameter applies. The discontinuity in mass, area, and stiffness are reflected in equation form as

$$m(x) = m_1 + (m_2 - m_1)\, h(x-c) \tag{5.1.1}$$

$$A(x) = A_1 + (A_2 - A_1)\, h(x-c) \tag{5.1.2}$$

$$EI(x) = EI_1 + E(I_2 - I_1)\, h(x-c) \tag{5.1.3}$$

The term, $h(x-c)$, is a unit step function with the following definition:

$$h(x-c) = 1; \qquad x \geq c$$
$$\quad = 0. \qquad x < c$$

The analysis assumes that both sections have identical material properties, therefore, Eq. (5.1.1) can be written in terms of the cross-sectional area as

$$\begin{aligned} m(x) &= \rho A(x) \\ &= \rho[A_1 + (A_2 - A_1)\, h(x-c)] \end{aligned} \tag{5.1.4}$$

The equilibrium equations derived for a uniform beam from Fig. 4-1 are valid for the stepped beam in this analysis, if the mass, area, and stiffness terms are considered functions of x. The element rotation angle, ψ, given by Eq. (4.1.4) is applicable in both cases. The resultant equations of motion can be derived as was done in the uniform beam analysis to yield the following equation forms:

$$\begin{aligned} \frac{\partial^2}{\partial x^2}\left[EI(x)\, \frac{\partial^2 y}{\partial x^2}\right] + \frac{\partial}{\partial x}\left[P(x,t)\, \frac{\partial y}{\partial x}\right] + \rho A(x)\, \frac{\partial^2 y}{\partial t^2} \\ + C(x)\, \frac{\partial y}{\partial x} + p = 0 \end{aligned} \tag{5.1.5}$$

$$\frac{\partial}{\partial x}\left[EA(x)\, \frac{\partial u}{\partial x}\right] - \rho A(x)\, \frac{\partial^2 u}{\partial t^2} = 0 \tag{5.1.6}$$

The force distribution, $P(x,t)$, will be derived through Eq. (5.1.6) from the relation $P = -EA(x)\, \frac{\partial u}{\partial x}$. The lateral force component, p, was derived in the uniform beam analysis and is directly applicable as

$$p = (T_o + T_1 \cos\Omega x)\ K_\theta \frac{\partial y}{\partial x}(x_G, t)\ \delta(x-\ell) \qquad (5.1.7)$$

5.2 Steady State Force Distribution, P(x,t)

In this analysis only the particular solution of Eq. (5.1.6) for the applied load, $T_o + T_1 \cos\Omega t$, will be sought. The force distribution is functionally related to the displacement by

$$P(x,t) = -\ EA(x) \frac{\partial u}{\partial x} \qquad (5.2.1)$$

The boundary conditions are at

$$x = 0; \qquad P(0,t) = 0 \qquad (5.2.2)$$

and at

$$x = \ell; \qquad P(\ell,t) = T_o + T_1 \cos\Omega t \qquad (5.2.3)$$

The constant thrust, T_o, imparts a constant acceleration to the beam. The force, P_o, at any station, x, due to the constant thrust, T_o, is equal to the mass accelerated multiplied by the resultant acceleration, that is,

$$P_o = \frac{T_o \rho}{M_T} [A_1 x + (A_2 - A_1)(x-c)\ h(x-c)] \qquad (5.2.4)$$

Note that the resultant acceleration due to T_o is given by T_o/M_T. M_T is the total mass of the beam.

The force distribution due to the periodically varying thrust component, $T_1 \cos\Omega t$, can be derived from Eq. (5.1.6) by applying Eq. (5.2.1) and the boundary conditions. As a first step, assume that the displacement, u, in Eq. (5.1.6) can be expressed as

$$u = XT = X \cos\Omega t \tag{5.2.5}$$

where X and T are functions of x and t, respectively. $u = X \cos\Omega t$ since the steady state solution is required. Substitution of Eq. (5.2.5) into Eq. (5.1.6) yields

$$\frac{d}{dx}\left[EA(x)\frac{dX}{dx}\right] + \Omega^2 \rho A(x)X = 0 \tag{5.2.6}$$

Let the force distribution due to the periodically-varying thrust be denoted by P_1.

$$P_1 = - EA(x)\frac{\partial u}{\partial x} = P_x \cos\Omega t \tag{5.2.7}$$

$$P_x = - EA(x)\frac{dX}{dx} \tag{5.2.8}$$

It is convenient to take the derivative of each term in Eq. (5.2.6) and to express the results in terms of P_x. The indicated transformation is accomplished as follows:

$$\frac{d^2}{dx^2}\left[EA(x)\frac{dX}{dx}\right] + \Omega^2 \rho A'(x)X + \Omega^2 \rho A(x)\frac{dX}{dx} = 0 \tag{5.2.9}$$

$$A(x) = A_1 + (A_2 - A_1)\ h(x-c)$$

$$A'(x) = \frac{dA}{dx} = (A_2 - A_1)\ \delta(x-c)$$

Eq. (5.2.9) can now be expressed as

$$\frac{d^2P_x}{dx^2} + \Omega^2 \frac{\rho}{E} P_x = \Omega^2 \rho(A_2 - A_1) X(c) \delta(x-c) \tag{5.2.10}$$

Note that X(c) is the value of X at x = c and is a constant. This result follows from the identity, $X(x) \delta(x-c) = X(c) \delta(x-c)$, given in ref. (7).

A solution, P_x, in Eq. (5.2.10) can be obtained by application of Laplace transforms. The Laplace transform of each term in Eq. (5.2.10) yields

$$-P_x'(0) - sP_x(0) + s^2P(s) + \frac{\Omega^2\rho}{E} P(s)$$

$$= \rho\Omega^2(A_2 - A_1) X(c) e^{-sc} \tag{5.2.11}$$

s = Transform variable

$P_x(0) = 0$

Rearrangement of terms in Eq. (5.2.11) with the substitution

$$\frac{\Omega^2\rho}{E} = (\frac{\pi}{\ell} \bar{\sigma})^2 = (\nu^*)^2 \tag{5.2.12}$$

$$\bar{\sigma} = \frac{\Omega}{\omega_L}$$

$$\omega_L = \frac{\pi}{\ell} \sqrt{\frac{E}{\rho}} \tag{5.2.13}$$

yields

$$P_x(s) = \frac{P_x'(0)}{s^2+(\nu^*)^2} + \frac{\Omega^2\rho(A_2-A_1)X(c)e^{-sc}}{s^2+(\nu^*)^2} \quad (5.2.14)$$

The inverse Laplace transform of each term in Eq. (5.2.14) can be taken to yield

$$P_x(x) = \frac{P_x'(0)}{\nu^*} \sin\nu^*x + \frac{\Omega^2\rho}{\nu^*}(A_2-A_1) \sin\nu^*(x-c)\, h(x-c) \quad (5.2.15)$$

$$P_x(\ell) = T_1 \text{ from Eq. } (5.2.3)$$

Therefore,

$$P_x(x) = T_1 \frac{\sin\nu^*x}{\sin\nu^*\ell} - \frac{\Omega^2\rho}{\nu^*}(A_2-A_1)X(c)\frac{\sin\nu^*(\ell-c)}{\sin\nu^*\ell}\sin\nu^*x$$

$$+ \frac{\Omega^2\rho}{\nu^*}(A_2-A_1)X(c)\sin\nu^*(x-c)h(x-c) \quad (5.2.16)$$

$$\nu^* = \frac{\pi}{\ell}\bar{\sigma}$$

The unknown constant X(c) can be obtained by solving Eq. (5.1.6) directly for each continuous section of the discontinuous beam and then matching boundary conditions existing at the discontinuity. It can be verified by performing the indicated operations that X(c) can be written as

$$X(c) = \frac{T\cos\nu^*c}{\nu^*A_2E} \quad \frac{1}{\cos\nu^*c\,\sin\nu^*(\ell-c) + \frac{A_1}{A_2}\sin\nu^*c\,\cos\nu^*(\ell-c)} \quad (5.2.17)$$

Substitution of Eq. (5.2.17) into Eq. (5.2.16) yields the final equation form of P_x. Let β_e denote the expression within the brackets of Eq. (5.2.17), then P_x can be written as

$$P_x = T_1 \left[\frac{\sin\nu^*x}{\sin\nu^*\ell} - (1 - \frac{A_1}{A_2}) \beta_e \cos\nu^*c \sin\nu^*(\ell-c) \frac{\sin\nu^*x}{\sin\nu^*\ell} \right.$$

$$\left. + (1 - \frac{A_1}{A_2}) \beta_e \cos\nu^*c \sin\nu^*(x-c) \; h(x-c) \right] \qquad (5.2.18)$$

5.3 Introduction of Dimensionless Variables into the Equations of Motion

The governing equation of motion is given below in dimensional form for ready reference.

$$\frac{\partial^2}{\partial x^2} [EI(x) \frac{\partial^2 y}{\partial x^2}] + \frac{\partial}{\partial x} [P(x,t) \frac{\partial y}{\partial x}] + \rho A(x) \frac{\partial^2 y}{\partial t^2}$$

$$+ C(x) \frac{\partial y}{\partial t} + p(x,t) = 0 \qquad (5.3.1)$$

Dimensionless variables may be introduced by a change of variables given by

$$x = \xi\ell; \qquad \xi = \frac{x}{\ell}, \quad \text{and} \quad \xi_c = \frac{c}{\ell} \qquad (5.3.2)$$

and

$$t = \frac{\tau}{\omega_1}; \qquad \tau = \omega_1 t \qquad (5.3.3)$$

The constant ω_1 was taken as the natural frequency of a uniform free-free beam having a length ℓ with the material and cross-sectional properties of Section 2 of the stepped beam and is related to the beam geometry and material properties by

$$\lambda_n^4 = \omega_n^2 \frac{m_2 \ell^4}{EI_2} \tag{5.3.4}$$

By performing the indicated change of variable and then multiplying through by ℓ^4/EI_2, Eq. (5.3.1) can be written as

$$\frac{\partial^2}{\partial \xi^2}\left[\frac{I(x)}{I_2}\frac{\partial^2 y}{\partial \xi^2}\right] + \frac{\ell^2}{EI_2}\frac{\partial}{\partial \xi}\left[P(\xi\ell, \frac{\tau}{\omega_1})\right] + \frac{\ell^4}{EI_2} p(\xi\ell, \frac{\tau}{\omega_1})$$

$$+ \omega_1^2 \frac{m_2 \ell^4}{EI_2} \frac{A(\xi)}{A_2} \frac{\partial^2 y}{\partial \tau^2} + \frac{C_2}{m_2\omega_1} \omega_1^2 \frac{m_2 \ell^4}{EI_2} \frac{A(\xi)}{A_2} \frac{\partial y}{\partial \tau} = 0 \tag{5.3.5}$$

It was assumed that $C(x) \propto m(x)$ which essentially states that $\frac{C_1}{m_1} = \frac{C_2}{m_2}$ = constant. C_1 and C_2 are the viscous damping coefficients of section (1) and (2) respectively. As a result of this proportionality one can write

$$C(x) = \rho A(x) \cdot \text{constant} \tag{5.3.6}$$

The constant term is given by the term $C_2/m_2\omega_1$ in Eq. (5.3.5). This constant is called the transverse damping factor and is given by

$$\eta = \frac{C_2}{m_2\omega_1} \tag{5.3.7}$$

The final equation form can be written as

$$\frac{\partial^2}{\partial \xi^2}\left[\frac{I(x)}{I_2}\frac{\partial^2 y}{\partial \xi^2}\right] + \frac{\ell^2}{EI_2}\frac{\partial}{\partial \xi}\left[P(\xi,\tau)\frac{\partial y}{\partial \xi}\right] + \frac{\ell^2}{EI_2} p(\xi,\tau)$$

$$+ \lambda_1^4 \frac{A(\xi)}{A_2}\frac{\partial^2 y}{\partial \tau^2} + \eta\lambda_1^2 \frac{A(\xi)}{A_2}\frac{\partial y}{\partial \tau} = 0 \tag{5.3.8}$$

where

$$P(\xi,\tau) = P_o(\xi) + P_x(\xi)\cos\bar{\Omega}\tau \tag{5.3.9}$$

$$P_o(\xi) = T_o \frac{m_2\ell}{M_T}[\bar{A}\xi + (1-\bar{A})(\xi-\xi_c)\, h(\xi-c)] \tag{5.3.10}$$

$$\frac{M_T}{m_2\ell} = 1 - \xi_c(1-\bar{A}) \tag{5.3.11}$$

$$P_x(\xi) = T_1 \left[\frac{\sin\pi\bar{\sigma}\xi}{\sin\pi\bar{\sigma}} - (1-\bar{A})\,\beta_e \cos\pi\bar{\sigma}\xi_c \right.$$

$$\left. \cdot\; [\sin\pi\bar{\sigma}(1-\xi_c)\frac{\sin\pi\bar{\sigma}\xi}{\sin\pi\bar{\sigma}} - \sin\pi\bar{\sigma}(\xi-\xi_c)h(\xi-c)]\right] \tag{5.3.12}$$

$$\beta_e = \frac{1}{[\cos\pi\bar{\sigma}\xi_c \;\sin\pi\bar{\sigma}(1-\xi_c) + \bar{A}\sin\pi\bar{\sigma}\xi_c \;\cos\pi\bar{\sigma}(1-\xi_c)]} \tag{5.3.13}$$

$$p(\xi,\tau) = T_1(1+\gamma\cos\bar{\Omega}\tau)\,K_\theta \frac{\partial y}{\partial\xi}(\xi_G,\tau)\,\delta(\xi-1) \tag{5.3.14}$$

$$\gamma = \frac{T_1}{T_o} \tag{5.3.15}$$

$$\bar{A} = \frac{A_1}{A_2} \tag{5.3.16}$$

$$\bar{\Omega} = \frac{\Omega}{\omega_1} \tag{5.3.17}$$

$$\bar{\sigma} = \frac{\Omega}{\omega_L} = \frac{\ell}{\pi}\nu^* \tag{5.3.18}$$

$$\omega_L = \frac{\pi}{\ell}\sqrt{\frac{E}{\rho}} \tag{5.3.19}$$

5.4 Application of Galerkin's Method

There is some question as to the applicability of Galerkin's method in the reduction of Eq. (5.3.8) to a system of ordinary differential equations, since the coefficients are not continuous (ref. 1). An extension of the method to piecewise continuous intervals has not been substantiated.

Although the accuracy of the result is questionable in the application of Galerkin's method to Eq. (5.3.8), it is hoped that the results will be of qualitative value in analyzing the dynamic stability of a discontinuous beam.

A discussion of Galerkin's method is presented in Section 4.5 as applied to a uniform beam. The approximating function employed for the uniform beam has been taken as the approximating function of the discontinuous beam, therefore, Eqs. (4.5.1) through (4.5.7c) are applicable to this analysis. Pertinent equations are rewritten below for ready reference. All derivations, definition of terms, and discussions are given in Section 4.5.

(1) The approximating function, Eq. (4.5.1)

$$y_N(\xi,\tau) = q_A + q_B\xi + \sum_{n=1}^{\infty} q_n(\tau)\ \phi_n(\xi) \tag{5.4.1}$$

(2) Resultant system of equations, Eq. (4.5.7a) through (4.5.7c)

$$\int_0^1 \S(y_N)d\xi = 0 \tag{5.4.2a}$$

$$\int_0^1 \S(y_N)\xi d\xi = 0 \tag{5.4.2b}$$

$$\int_0^1 \S(y_N)\ \phi_n(\xi)d\xi = 0 \qquad n = 1,2,3,\cdots \tag{5.4.2c}$$

where $\S(y_N)$ is the error resulting in substituting Eq. (5.3.1) into Eq. (5.3.8).

Equations (5.4.2a) through (5.4.2c) can be represented in matrix form as

$$[A](\ddot{q}) + \eta[A](\dot{q}) + [B](q) + [C]\cos\bar{\Omega}\tau(q) = (0) \tag{5.4.3}$$

where

[ ] = a square matrix of order N + 2

and

$$(q) = \begin{Bmatrix} q_1 \\ q_2 \\ \cdot \\ \cdot \\ \cdot \\ q_N \\ q_{N+1} \\ q_{N+2} \end{Bmatrix} \qquad \begin{array}{l} q_{N+1} = q_B \\ q_{N+2} = q_A \end{array}$$

The elements of the coefficient matrices are obtained by expanding Eqs. (5.4.2a), (5.4.2b), and (5.4.2c) then grouping coefficients of $(\ddot{q})$, $(\dot{q})$, (q), and $\cos\bar{\Omega}\tau(q)$. The expanded equations are lengthy but the expansion and grouping of terms is routine. The resultant expressions for the elements of the coefficient matrices are given in integral form below. It is not necessary to assign equation numbers to each element since the equations are readily identified by their subscripts.

$$\begin{aligned}
A_{j,k} &= \lambda_1^4 \int_0^1 \frac{A(\xi)}{A_2} \phi_j \; \phi_k d\xi \\
A_{j,N+1} &= \lambda_1^4 \int_0^1 \frac{A(\xi)}{A_2} \phi_j \; \xi d\xi \\
A_{j,N+2} &= \lambda_1^4 \int_0^1 \frac{A(\xi)}{A_2} \phi_j \; d\xi \\
A_{N+1,N+2} &= \lambda_1^4 \int_0^1 \frac{A(\xi)}{A_2} \xi d\xi \\
A_{N+1,N+1} &= \lambda_1^4 \int_0^1 \frac{A(\xi)}{A_2} \xi^2 d\xi
\end{aligned} \tag{5.4.3}$$

$$A_{N+2,N+2} = \lambda_1^4 \int_0^1 \frac{A(\xi)}{A_2}\, d\xi$$

Matrix A was found to be symmetric therefore it is only necessary to define elements of A on and above the diagonal.

$$B_{j,k} = \int_0^1 \frac{d^2}{d\xi^2} \left[\frac{I(\xi)}{I_2}\, \phi_k''\right] \phi_j d\xi$$

$$+ \frac{\ell^2}{EI_2} \int_0^1 \frac{d}{d\xi} \left[P_o\ \phi_k'\right] \phi_j d\xi$$

$$+ \frac{T_o \ell^2}{EI_2} K_\theta\ \phi_k'(\xi_G) \int_0^1 \phi_j\ \delta(\xi-1) d\xi \qquad (5.4.4)$$

$$B_{j,N+1} = \frac{\ell^2}{EI_2} \int_0^1 \frac{d}{d\xi} \left[P_o\right] \phi_j d\xi$$

$$+ \frac{T_o \ell^2}{EI_2} K_\theta \int_0^1 \phi_j\ \delta(\xi-1) d\xi$$

$$B_{j,N+2} = 0$$

$$B_{N+1,k} = \int_0^1 \frac{d^2}{d\xi^2} \left[\frac{I(\xi)}{I_2}\, \phi_k''\right] \xi d\xi$$

$$+ \frac{\ell^2}{EI_2} \int_0^1 \frac{d}{d\xi} \left[P_o\ \phi_k'\right] \xi d\xi$$

$$+ \frac{T_o \ell^2}{EI_2} K_\theta\ \phi_k'(\xi_G) \int_0^1 \xi \delta(\xi-1) d\xi$$

$$B_{N+1,N+1} = \frac{\ell^2}{EI_2} \int_0^1 \frac{d}{d\xi} \; [P_o] \; \xi d\xi$$

$$+ \frac{T_o \ell^2}{EI_2} K_\theta \int_0^1 \xi \delta(\xi-1) d\xi$$

$$B_{N+1,N+2} = 0$$

$$B_{N+2,k} = \frac{\ell^2}{EI_2} \int_0^1 \frac{d}{d\xi} \; [P_o \; \phi_k'] d\xi$$

$$+ \frac{T_o \ell^2}{EI_2} K_\theta' \; \phi_k'(\xi_G) \int_0^1 \delta(\xi-1) d\xi$$

$$B_{N+2,N+1} = \frac{\ell^2}{EI_2} \int_0^1 \frac{dP_o}{d\xi} \, d\xi$$

$$B_{N+2,N+2} = 0$$

$$C_{j,k} = \frac{\ell^2}{EI_2} \int_0^1 \frac{d}{d\xi} \; [P_x \; \phi_k'] \; \phi_j d\xi \tag{5.4.5}$$

$$+ \frac{T_1 \ell^2}{EI_2} K_\theta \phi_k'(\xi_G) \int_0^1 \phi_j \; \delta(\xi-1) d\xi$$

$$C_{j,N+1} = \frac{\ell^2}{EI_2} \int_0^1 \frac{d}{d\xi} \; [P_x] \; \phi_j d\xi$$

$$+ \frac{T_1 \ell^2}{EI_2} K_\theta \int_0^1 \phi_j \delta(\xi-1) d\xi$$

$$C_{j,N+2} = 0$$

$$C_{N+1,k} = \frac{\ell^2}{EI_2} \int_o^1 \frac{d}{d\xi} [P_x \phi_k'] \xi d\xi$$

$$C_{N+1,N+1} = \frac{T_1 \ell^2}{EI_2} K_\theta \phi_k'(\xi_G) \int_o^1 \xi \delta(\xi - 1) d\xi$$

$$C_{N+1,N+2} = 0$$

$$C_{N+2,k} = \frac{\ell^2}{EI_2} \int_o^1 \frac{d}{d\xi} [P_x \phi_k'] d\xi$$
$$+ \frac{T_1 \ell^2}{EI_2} K_\theta \phi_k'(\xi_G) \int_o^1 \delta(\xi - 1) d\xi$$

$$C_{N+2,N+1} = \frac{\ell^2}{EI_2} \int_o^1 \frac{d}{d\xi} [P_x] d\xi$$

$$C_{N+2,N+2} = 0$$

One must proceed with caution when manipulating or evaluating integrals containing discontinuous functions. To illustrate, several of the representative integrals in the matrix elements given above are taken as examples. The first integral chosen as an example is in element $A_{j,k}$. Let

$$\int_0^1 \frac{A(\xi)}{A_2} \phi_j \phi_k \, d\xi = INT_1$$

then

$$INT_1 = \frac{1}{A_2} \int_0^1 [A_1 + (A_2 - A_1) h(\xi - \xi_c)] \phi_j \phi_k \, d\xi$$

$$INT_1 = \bar{A} \int_0^1 \phi_j \phi_k \, d\xi + (1-\bar{A}) \int_{\xi_c}^1 \phi_j \phi_k \, d\xi .$$

The following integrals are taken from element $B_{j,k}$,

$$\int_0^1 \frac{d}{d\xi^2} [\frac{I(\xi)}{I_2} \phi_k''] \phi_j \, d\xi = INT_2$$

Integrating by parts yields

$$INT_2 = \frac{d}{d\xi} [\frac{I(\xi)}{I_2} \phi_k''] \phi_j \Big|_0^1 - \int_0^1 \frac{d}{d\xi} [\frac{I(\xi)}{I_2} \phi_k''] \phi_j' \, d\xi .$$

The first term is zero since the shear is zero at $\xi = 0$ and $\xi = 1$.

Integration of the second term then yields

$$INT_2 = - [\frac{I(\xi)}{I_2} \phi_k''] \phi_j \Big|_0^1 + \int_0^1 \frac{I(\xi)}{I_2} \phi_k'' \phi_j'' \, d\xi$$

The first term is zero since the moment is zero at $\xi = 0$ and $\xi = 1$. The resultant integral can be written as

$$INT_2 = \bar{I} \int_0^1 \phi_k'' \phi_j'' \, d\xi + (1-\bar{I}) \int_{\xi_c}^1 \phi_k'' \phi_j'' \, d\xi$$

where $\bar{I} = \frac{I_1}{I_2}$.

Let

$$\frac{\ell^2}{EI_2}\int_0^1 \frac{d}{d\xi}\,[P_o\ \phi_k']\ \phi_j\ d\xi = INT_3$$

Integrating by parts yields

$$INT_3 = \frac{\ell^2}{EI_2}\,[P_o\ \phi_k']\phi_j\Big|_0^1 - \frac{\ell^2}{EI_2}\int_0^1 P_o\ \phi_k'\ \phi_j'\ d\xi\ .$$

By applying the boundary condition that $P_o = 0$ at $\xi = 0$ and $P_o = T_o$ at $\xi = 1$, INT_3 can be written as

$$INT_3 = \frac{T_o\ell^2}{EI_2}\ \phi_k'(1)\ \ \phi_j'(1) - \frac{\ell^2}{EI_2}\int_0^1 P_o\ \phi_k'\ \phi_j'\ d\xi.$$

As the final integral type let

$$\int_0^1 \phi_j\delta(\xi-1)d\xi = INT_4$$

$$INT_4 = \int_0^1 \phi_j(1)\ \ \delta(\xi-1)d\xi = \phi_j(1)$$

The remaining integrals given in the matrix elements may be transformed to a standard form in a similar manner. The final integral forms do not contain derivatives of discontinuous functions, and the dynamic boundary conditions on shear and moment are satisfied at the beam ends. The resultant integrals may be evaluated from tables of integrals of characteristic functions given in ref. 11. The recurrence relation for ϕ_n', given by Eq. (4.5.34), can be employed as was done for the uniform beam in Section 4.5.

Several dimensionless variables were introduced in Section 5.3. Upon substitution of P_o and P_x into the integrals in which they appear two dimensionless factors emerge which are denoted by $\bar{T}_o$ and γ

$$\bar{T}_o = \frac{T_o \ell^2}{EI_2} \qquad (5.4.6)$$

$$\gamma = \frac{T_1}{T_o} \qquad (5.4.7)$$

Note that all the dimensionless products are taken with respect to beam section (2).

Upon assigning values to the dimensionless variables and evaluating the matrix elements the only unknowns in Eq. (5.4.3) are the generalized coordinates, q_n. Eq. (5.4.3) is given by

$$[A](\ddot{q}) + \eta[A](\dot{q}) + [B](q) + \cos\bar{\Omega}\tau \; [C](q) = (0) \qquad (5.4.8)$$

which is equivalent to

$$(\ddot{q}) + \eta(\dot{q}) + [A]^{-1} [B](q) + \gamma\cos\bar{\Omega}\tau \; \frac{[A]^{-1} [C]}{\gamma} (q) = (0).$$

As the final equation form one can then write

$$(\ddot{q}) + \eta(\dot{q}) + [F](q) + \gamma\cos\bar{\Omega}\tau \; [G] = (0) \qquad (5.4.9)$$

which is identical to the form obtained in the uniform beam analysis when the initial conditions are neglected.

Note that each element of column N + 2 in Matrix B and Matrix C is equal to zero, therefore, it is only necessary to consider F and G matrices of order N + 1 in the solution of Eq. (6.1.1). The result is expected since $q_A = q_{N+2}$ is the generalized coordinate of rigid body translation. Since translation is uncontrolled, the natural frequency of translation is zero and the translation term q_A is arbitrary.

The resultant matrix equation has the same form as the equation derived in the uniform beam analysis, therefore, identical methods of solution may be employed. Note that Eq. (6.1.1) of the following section includes an extra term which accounts for the initial conditions chosen in the uniform beam analysis. Consider this term zero in the stepped beam analysis.

CHAPTER 6

METHOD OF SOLUTION

6.1 Form of the Equations

The system of equations represented by Eq. (4.5.18) can be written as

$$(\ddot{q}) + \eta(\dot{q}) + [F](q) + \gamma\cos\bar{\Omega}\tau\ [G](q) + \sum_{s=1}^{\infty} [H]_s \cos s\bar{\omega}_L\tau(q) = 0 \qquad (6.1.1)$$

for the initial conditions assumed in Section 4.6.

In the preceding derivations the damping factor η was assumed to be the same in all modes of vibration. This is by no means an unreasonable assumption as many systems decay at approximately the same rate in all modes of vibration. Furthermore the assumption greatly simplifies subsequent analysis and should be adequate in determining damping effects.

The velocity term can be eliminated from Eq. (6.1.1) by the transformation

$$q_k = e^{-\frac{\eta}{2}\tau} u_k \qquad (6.1.2)$$

suggested in (ref. 7). Performing the transformation yields

$$(\ddot{u}) + \left[[F] - \frac{\eta^2}{4}[I]\right](u) + \gamma\cos\bar{\Omega}\tau\ [G](u)$$

$$+ \sum_{s=1}^{\infty} [H]_s \cos s\bar{\omega}_L \tau (u) = 0 \qquad (6.1.3)$$

In the case of zero γ in Eq. (6.1.3) and for $t >> 0$ (Eq. (6.1.3)) reduces to

$$(\ddot{u}) + \left[[F] - \frac{\eta^2}{4}[I]\right](u) = 0 \qquad (6.1.4)$$

For the system considered, $[F] - \frac{\eta^2}{4}[I]$ is non-symmetric. A solution for the eigenvalues for such a matrix may be found in (ref. 8). Eq. (6.1.4) yields the damped natural frequencies of a beam subjected to a non-conservative constant thrust which is gimbaled to achieve rotational stability.

6.2 Method of Solution

In accordance with (ref. 1), a solution of Eq. (6.1.3) can be taken as

$$u_k = e^{i\alpha\bar{\Omega}\tau} \chi_k(\tau) \qquad k = 1,2,\ldots N+1 \qquad (6.2.1)$$

where χ_k is a complex Fourier series given by

$$(\chi) = \sum_{m=-\infty}^{\infty} (c)_m e^{im\bar{\Omega}\tau} \qquad (6.2.2)$$

with a period of $2\pi/\bar{\Omega}$.

Eq. (6.2.1) then becomes

$$(u) = \sum_{m=-\infty}^{\infty} (c)_m \, e^{i(\alpha+m)\bar{\Omega}\tau} \quad . \tag{6.2.3}$$

A constant α has been introduced which can be real, imaginary, or complex where its type determines the stability of the system as shown in Section 5.3.

The forcing frequency and the longitudinal natural frequency can be restricted to a ratio of

$$\frac{\Omega}{\omega_L} = \frac{1}{n} \tag{6.2.4}$$

where n is taken as an integer greater than 1 to facilitate a solution.

Substitution of Eqs. (6.2.7) and (6.2.8) into Eqs. (6.1.3) yields

$$\begin{aligned} &- \bar{\Omega}^2 \sum_{m=-\infty}^{\infty} (c)_m (\alpha + m)^2 e^{i(\alpha+m)\bar{\Omega}\tau} \\ &+ \left[[F] - \frac{n^2}{4} [I] \right] \sum_{m=-\infty}^{\infty} (c)_m \, e^{i(\alpha+m)\bar{\Omega}\tau} \\ &+ \frac{\gamma}{2} [G] \sum_{m=-\infty}^{\infty} (c)_m \, e^{i(\alpha+m+1)\bar{\Omega}\tau} \\ &+ \frac{\gamma}{2} [G] \sum_{m=-\infty}^{\infty} (c)_m \, e^{i(\alpha+m-1)\bar{\Omega}\tau} \\ &+ \frac{1}{2} \sum_{s=1}^{\infty} [H]_s \sum_{m=-\infty}^{\infty} (c)_m \, e^{i(\alpha+m+sn)\bar{\Omega}\tau} \\ &+ \frac{1}{2} \sum_{s=1}^{\infty} [H]_s \sum_{m=-\infty}^{\infty} (c)_m \, e^{i(\alpha+m-sn)\bar{\Omega}\tau} \end{aligned} \tag{6.2.5}$$

Equation (6.2.5) is satisfied for all α if collected coefficients of like exponentials are equal to zero or, in equation form,

$$
\begin{aligned}
&- (\alpha + m)^2 \, (c)_m + \frac{1}{\bar{\Omega}^2} \left[[F] - \frac{n^2}{4} [I] \right] (c)_m \\
&+ \frac{\gamma}{2\bar{\Omega}^2} [G] \, (c)_{m-p} + \frac{\gamma}{2\omega^2} [G] \, (c)_{m+p} \\
&+ \frac{1}{2\bar{\Omega}^2} \sum_{s=1}^{\infty} [H]_s \, (c)_{m-sn} \\
&+ \frac{1}{2\bar{\Omega}^2} \sum_{s=1}^{\infty} [H]_s \, (c)_{m+sn} = 0 \qquad (6.2.6)
\end{aligned}
$$

$m = \ldots -3, -2, -1, 0, 1, 2, 3 \ldots$

$j = 1, 2, 3, \ldots N+1$

$k = 1, 2, 3, \ldots N+1$

The system of equations represented by Eq. (6.2.6) can be expanded into a single matrix equation as shown on the following page. To ensure that the determinant of the matrix of coefficients is absolutely convergent, Eq. (6.2.6) is first divided by the factor

$$\frac{1}{\bar{\Omega}^2} \left(F_{kk} - \frac{n^2}{4}\right) - (\alpha + m)^2 \qquad (6.2.7)$$

The importance of absolute convergence becomes evident as the development of the method of solution proceeds.

Eq. (6.2.8) below is the matrix equivalent of the system of equations given by Eq. (6.2.6). As shown, the index m ranged from -1 through +1. There is no upper limit except computer capacity but a finite value of m can be chosen, say M, which will give adequate convergence.

$$\begin{bmatrix} & \vdots & \vdots & \vdots & \\ \cdots & D_{-1,-1} & \frac{1}{\Omega^2} D_{-1,0} & \frac{1}{\Omega^2} D_{-1,1} & \cdots \\ \cdots & \frac{1}{\Omega^2} D_{0,-1} & D_{0;0} & \frac{1}{\Omega^2} D_{0.1} & \cdots \\ \cdots & \frac{1}{\Omega^2} D_{1,-1} & \frac{1}{\Omega^2} D_{1,0} & D_{1,1} & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix} \begin{bmatrix} \vdots \\ (c)_{-1} \\ (c)_0 \\ (c)_1 \\ \vdots \end{bmatrix} = (0) \quad (6.2.8)$$

The determinant of the matrix of coefficients is here defined as $\Delta(\alpha)$, and $D_{j,k}$ are the square arrays of elements within $\Delta(\alpha)$. The arrays $D_{j,k}$ are defined by the following equation:

$$D_{m,m+1} = \begin{vmatrix} \dfrac{G_{11}(\gamma/2)}{\bar{F}_{11}-(\alpha+m)^2} & \dfrac{G_{12}(\gamma/2)}{\bar{F}_{11}-(\alpha+m)^2} & \cdots & \dfrac{G_{1,N+1}(\gamma/2)}{\bar{F}_{11}-(\alpha+m)^2} \\ \dfrac{G_{21}(\gamma/2)}{\bar{F}_{22}-(\alpha+m)^2} & \dfrac{G_{22}(\gamma/2)}{\bar{F}_{22}-(\alpha+m)^2} & \cdots & \dfrac{G_{2,N+1}(\gamma/2)}{\bar{F}_{22}-(\alpha+m)^2} \\ \vdots & \vdots & & \vdots \\ \dfrac{G_{N+1,1}(\gamma/2)}{\bar{F}_{33}-(\alpha+m)^2} & \dfrac{G_{N+1,2}(\gamma/2)}{\bar{F}_{33}-(\alpha+m)^2} & \cdots & \dfrac{G_{N+1,N+1}(\gamma/2)}{\bar{F}_{33}-(\alpha+m)^2} \end{vmatrix}$$

$$D_{m,m} = \begin{vmatrix} 1 & \dfrac{F_{12}}{\bar{F}_{11}-(\alpha+m)^2} & \cdots & \dfrac{F_{1,N+1}}{\bar{F}_{11}-(\alpha+m)^2} \\ \dfrac{F_{21}}{\bar{F}_{22}-(\alpha+m)^2} & 1 & \cdots & \dfrac{F_{2,N+1}}{\bar{F}_{22}-(\alpha+m)^2} \\ \vdots & \vdots & & \vdots \\ \dfrac{F_{N+1,1}}{\bar{F}_{33}-(\alpha+m)^2} & \dfrac{F_{N+1,2}}{\bar{F}_{33}-(\alpha+m)^2} & \cdots & 1 \end{vmatrix} \tag{6.2.9}$$

$$D_{m,m\pm sn} = \begin{vmatrix} \frac{H^s_{11}(1/2)}{\bar{F}_{11}-(\alpha+m)^2} & \frac{H^s_{12}(1/2)}{\bar{F}_{11}-(\alpha+m)^2} & \cdots & \frac{H^s_{1,N+1}(1/2)}{\bar{F}_{11}-(\alpha+m)^2} \\ \frac{H^s_{21}(1/2)}{\bar{F}_{22}-(\alpha+m)^2} & \frac{H^s_{22}(1/2)}{\bar{F}_{22}-(\alpha+m)^2} & \cdots & \frac{H^s_{N+1,2}(1/2)}{\bar{F}_{22}-(\alpha+m)^2} \\ \vdots & \vdots & & \vdots \\ \frac{H^s_{N+1,1}(1/2)}{\bar{F}_{33}-(\alpha+m)^2} & \frac{H^s_{N+1,2}(1/2)}{\bar{F}_{33}-(\alpha+m)^2} & \cdots & \frac{H^s_{N+1,N+1}(1/2)}{\bar{F}_{33}-(\alpha+m)^2} \end{vmatrix}$$

$$s = 1,2,3,\ldots$$

where

$$\bar{F}_{jj} = \frac{1}{\bar{\Omega}^2}\left(F_{jj} - \frac{n^2}{4}\right) \qquad (6.2.10)$$

j corresponding to the row within each array.

The system of equations represented by the matrix Eq. (6.2.8) can have a non-trivial solution only if the determinant of the matrix of coefficients equals zero, that is,

$$\Delta(\alpha) = 0 \qquad (6.2.11)$$

where only the values of α which satisfy Eq. (6.2.11) are permitted. The problem then is to solve Eq. (6.2.11) for α.

To ensure the validity of the solution the infinite determinant represented by $\Delta(\alpha)$ must converge absolutely. It has been proven in (ref. 5) that an infinite determinant converges if the product of the diagonal elements and the sum of the non-diagonal elements are absolutely convergent.

The product of the diagonal elements is identically equal to 1 since all elements along the diagonal are equal to 1. This condition was deliberately obtained by dividing Eq. (6.2.6) by Eq. (6.2.7).

The sum of the non-diagonal elements of $\Delta(\alpha)$ is

$$S = \frac{1}{\bar{\Omega}^2} \sum_{\substack{j=1 \\ j\neq k}}^{N+1} \sum_{k=1}^{N+1} \sum_{m=-\infty}^{\infty} \frac{\bar{\Omega}^2 F_{jk} + \gamma/2\ G_{jk} + H^s_{jk}(1/2)}{\bar{F}_{jj} - (\alpha+m)^2}$$

$$+ \frac{1}{\bar{\Omega}^2} \sum_{j=1}^{N+1} \sum_{m=-\infty}^{\infty} \frac{(\gamma/2)\ G_{jj} + H^s_{jj}(1/2)}{\bar{F}_{jj} - (\alpha+m)^2} \qquad (6.2.12)$$

which is convergent by comparison with the series $\sum_{n=-\infty}^{\infty} \frac{1}{n^2}$. It can therefore be concluded that $\Delta\alpha$ is convergent for all α except where the factor $\bar{F}_{jj} - (\alpha+m)^2$ is equal to zero.

In Eq. (6.2.12) convergence is assured for all values of α. A criterion must be established to determine the minimum value for m, denoted by M, which will give adequate convergence. A finite M can be chosen such that the denominator in Eq. (6.2.12) becomes larger in absolute value for an increase to M+1. Since M is squared, the elements of $D_{j,k}$ rapidly grow smaller. From the criterion established, M satisfying

$$(\alpha + M)^2 > \bar{F}_{max}$$

should be sufficient. $\bar{F}_{max}$ is the maximum value of the $\bar{F}_{jj}$'s in Eq. (6.2.10).

6.3 Determination of the Characteristic Values

The characteristic determinant $\Delta(\alpha)$ has been expressed in arrays, $D_{j,k}$, with each array defined in terms of its corresponding elements in Eq. (6.2.9). The postulates of Liouville's theorem may be employed to establish a solution for the characteristic values, α, from $\Delta(\alpha)$ as follows:

1. Writing $-\alpha$ for α and $-m$ for m does not alter $(\alpha+m)^2$. Transposition of the elements containing $+m$ and $-m$ leaves $\Delta(\alpha)$ unaltered since m assumes all integral values from $-\infty$ to ∞. Therefore, $\Delta(\alpha) = \Delta(-\alpha)$ so that $\Delta(\alpha)$ is an even function of α.

2. $[(\alpha+1) + m]^2 = [\alpha + (m+1)]^2$, therefore replacing m+1 by m gives $(\alpha+m)^2$. Hence $\Delta(\alpha) = \Delta(\alpha+1)$ proving that $\Delta\alpha$ has a period of 1.

3. Singularities exist only when α satisfies $(\alpha+m)^2 - \bar{F}_{jj} = 0$. That is when $\alpha = \pm (\bar{F}_{jj})^{\frac{1}{2}} -m$. The possibility that $(\bar{F}_{jj})^{\frac{1}{2}} = m$ is excluded. Therefore, there is no singularity

at $\alpha = 0$. A function can be defined, $\rho_j(\alpha) = \dfrac{1}{\cos 2\alpha\pi - \cos\pi\bar{F}_{jj}^{\frac{1}{2}}}$, that has the singularities and period of $\Delta(\alpha)$.

From 3 it follows that

$$H(\alpha) = \Delta(\alpha) - \sum_{j=1}^{N+1} K_j \rho_j(\alpha) \qquad (6.3.1)$$

will have no singularities if the constants K_j are suitably chosen. $H(\alpha)$ must be a constant by Liouville's theorem. By allowing $\alpha \to i\infty$, $H(\alpha)$ is evaluated as

$$H(\alpha) = \lim_{\alpha\to i\infty} [\Delta(\alpha) - \sum_{j=1}^{N+1} K_j \rho_j(\alpha)] = 1 \qquad (6.3.2)$$

since the limit of the summation term goes to zero and all the elements off the diagonal of $\Delta(\alpha)$ tend to zero as α approaches $i\infty$.

The constants $K_1, K_2, K_3, \ldots K_{n+1}$ are evaluated by allowing α to approach $(\bar{F}_{11})^{\frac{1}{2}}$, $(\bar{F}_{22})^{\frac{1}{2}}$, ... $(\bar{F}_{n+1,N+1})^{\frac{1}{2}}$, respectively. Letting $\alpha = (\bar{F}_{jj})^{\frac{1}{2}} + \varepsilon$,

$$K_j = -2\pi \sin 2\pi(\bar{F}_{jj})^{\frac{1}{2}} \lim_{\varepsilon\to 0} \varepsilon\Delta\, (\bar{F}_{jj}^{\frac{1}{2}} + \varepsilon). \qquad (6.3.3)$$

In order to evaluate Eq. (6.3.3) it becomes necessary to define a new determinant

$$\Delta_j(\alpha) = (-\alpha^2 + \bar{F}_{jj})\Delta(\alpha) \qquad (6.3.4)$$

which does not have a singularity at $\alpha = (\bar{F}_{jj})^{\frac{1}{2}}$. Substitution of Eq. (6.3.3) into Eq. (6.3.4) and taking the limit yields

$$K_j = \frac{\pi \sin 2\pi(\bar{F}_{jj})^{\frac{1}{2}}}{\bar{F}_{jj}} \Delta_j(\bar{F}_{jj}{}^{\frac{1}{2}}) \qquad (6.3.5)$$

The characteristic equation can now be written as

$$1 - \sum_{j=1}^{N+1} \frac{K_j}{\zeta_j - z} = 0 \qquad (6.3.6)$$

where

$$\zeta_j = \cos 2\pi(\bar{F}_{jj})^{\frac{1}{2}} \qquad (6.3.7)$$

and

$$z = \cos 2\pi\alpha \qquad (6.3.8)$$

6.4 Stability of Solution

The characteristic values α of the solutions

$$u_k = e^{i\alpha\omega\tau} \chi_k \qquad k = 1,2,\ldots N+1$$

are evaluated from Eq. (6.3.8) as

$$\alpha = -\frac{i}{2\pi} \ln (z \pm \sqrt{z^2 - 1}) \qquad (6.4.1)$$

Solutions to the original differential equation can now be obtained from the relationship

$$(q) = e^{-\frac{1}{2}\eta\tau} (u)$$

of Eq. (6.1.2). q_k can now be written in terms of the characteristic values of u_k.

$$(q) = e^{-\frac{1}{2}\eta\tau} \cdot e^{i\alpha\omega\tau}\ (\chi) \tag{6.4.2}$$

$$(q) = e^{\tau(i\alpha\omega-\frac{1}{2}\eta)}\ (\chi) \tag{6.4.3}$$

The stability criteria governing the solutions of Eq. (6.3.3) are established as follows:

1. A solution is defined as <u>unstable</u> if q_k tends to $\pm\infty$ as τ approaches $+\infty$.

2. A solution is defined to be <u>stable</u> if q_k tends to zero or remains bounded as τ approaches $+\infty$.

3. A solution with period 2π is neutral but q_k may be regarded as a special case of a stable solution.

The elements in the column matrix χ_k have a period $\frac{2\pi}{\bar{\Omega}}$ which is neutral. Therefore stability depends upon $e^{\tau(i\alpha\omega-\frac{1}{2}\eta)}$ or, more specifically, upon the relative values of η and α. The term $(i\alpha\omega-\frac{1}{2}\eta)$ may in general have any real, imaginary, or complex value depending upon α and η.

α is determined as a function of the computed z value from Eq. (6.4.1) where z may be real or complex. For convenience Eq. (6.4.1) is written as

$$\alpha = -\frac{i}{2\pi}\ln\ (Re^{i\beta}) \tag{6.4.4}$$

$$\alpha = -\frac{i}{2\pi}\ln R + \frac{\beta}{2\pi} \qquad (6.4.5)$$

where R is the modulus of $z \pm \sqrt{z^2 - 1}$ and β the argument. Then

$$(q) = e^{\tau\left(\frac{\bar{\Omega}}{2\pi}\ln R - \frac{\eta}{2}\right)} \cdot e^{\left(\frac{i\beta}{2\pi}\right)} (X) \qquad (6.4.6)$$

From the criteria established, a stable solution requires that

$$\eta \geq \frac{\bar{\Omega}}{\pi}\ln R \qquad (6.4.7)$$

or for zero damping that

$$\frac{\bar{\Omega}}{2\pi}\ln R \leq 0 \qquad (6.4.8)$$

CHAPTER 7

RESULTS

7.1 Definition of Terms and Symbols

Section 6.4 established the criterion upon which the solutions of Eq. (6.1.1) became unstable. In Eq. (6.4.6) a stable solution required that $\eta \geq \frac{\bar{\Omega}}{\pi} \ln R$ or, conversely, an unstable solution required that $\frac{\bar{\Omega}}{\pi} \ln R - \eta > 0$. The numerical amount by which $(\frac{\bar{\Omega}}{\pi} \ln R - \eta)$ exceeds zero is a measure of the growth rate of the amplitudes of response and is indicative of the severity of the instability. The forcing frequency at which the term becomes zero is a boundary point separating regions of instability from stable regions.

For purposes of identification and of presenting data, the term governing stability is defined as the characteristic argument and is given the symbol T where

$$T = \frac{\bar{\Omega}}{\pi} \ln R - \eta \tag{7.1.1}$$

and

$$R = \left| z \pm \sqrt{z^2 - 1} \right| \tag{7.1.2}$$

The system is stable if

$T \leq 0$ <u>Stable</u>

and unstable if

$T > 0.$ <u>Unstable</u>

The z term can be considered a characteristic value associated with each degree of freedom chosen as a solution in Eq. (4.5.1). The form of z clearly determines the magnitude of R in Eq. (7.1.2) and as such defines the stability of the system through T. Since instability results if only one solution is unstable, it is necessary to compute a maximum T value possible from a set of characteristic values.

It is seen from Eq. (7.1.1) that instability can occur only for $R > 1$. In turn $R > 1$ can occur only when z is complex or greater than unity in absolute value. Section 3.1 classified forcing frequencies at which instabilities could develop by types. A real z value exceeding unity in absolute value relates to the Type I, k^{th} order instability given by

$$\Omega = \frac{2\omega(i)}{k} \tag{7.1.3}$$

Complex z values exist as complex conjugates and relate to the Type II, k^{th} order instability given by

$$\Omega = \frac{\omega(i) \pm \omega(j)}{k} \tag{7.1.4}$$

The characteristic argument provided a value which could be calculated for any set of beam parameters. When plotted against the forcing frequency (see for example Fig. 8), the resultant plot could be used to determine such pertinent information as unstable regions, boundaries separating stable and unstable regions, relative severity of an unstable region, damping effects, and longitudinal coupling effects.

7.2 Natural Frequencies of a Free-Free Beam under a Gimbaled Thrust

The computation of natural frequencies was necessary in determining the forcing frequencies at which parametric resonance could occur. This was not, however, a primary objective of this investigation and the results are given as a by-product.

Natural frequencies are plotted for a number of beam configurations as a function of the applied thrust in Figs. 3 through 7. The frequencies may be converted from the normalized values plotted to the actual values by methods outlined in the example problems of Appendix A.

It was established in reference 3 that an unstable condition could exist due to a static loading from the reduction of a natural frequency to zero or from a coalescence of adjacent natural frequencies, whichever occurred first. As an illustration, consider Fig. 3 and Fig. 4 which are plotted for a uniform beam. An instability is seen to exist at $\bar{T}_o = 25.67$ due to the reduction of the rotational frequency, $\bar{\omega}_{(B)}$, to zero. This thrust also defines the critical load for inertial buckling. Fig. 4 illustrates frequency coalescence where an instability develops at approximately $\bar{T}_o = 15$ due to the coalescence of the rotational mode and the first bending mode.

Natural frequencies are plotted in Figs. 3 through 7 to illustrate the effects of a variation in a single parameter. The norm chosen in each figure is that of a uniform beam under the influence of the following parameters:

$N = 2$, Two bending degrees of freedom

ξ_G = 0.5, Sensor location at midspan

K_θ = 1.0, Control constant

η = 0.00, Zero damping.

Fig. 3 illustrates the effects of the control constant, K_θ, upon the natural frequencies of a uniform beam. The curves do not reflect a general trend but do indicate that the control factor has a significant effect upon the natural frequencies.

In Figure 4 the natural frequencies are seen to be sensitive to the location of the attitude sensor and for the given choice of parameters is unstable at a thrust lower than the thrust at inertial buckling.

Figures 5, 6, and 7 illustrate the effect of beam geometry upon the natural frequencies. It must be noted that the frequency normalization factor, ω_1, for the stepped beam is based upon the mass distribution, stiffness, and material properties of beam section (2) and the length of the total beam. The effect of a variation in sensor location or control factor is expected to be similar to the effects noted in Figs. 3 and 7 for a uniform beam.

It was found that damping decreased the natural frequencies slightly but the variation was very small. For all practical purposes damped frequencies may be considered equal to undamped frequencies for small damping factors. Although damping did not appreciably alter the natural frequencies, the minute shift in frequency could alter the critical thrust determined by a reduction of a natural frequency to zero or frequency coalescence. This phenomenon was not investigated.

Two important conclusions may be drawn which will effect the presentation of data in the following stability investigation.

(1) Damping does not appreciably alter natural frequencies for small damping factors. This indicates that parametric resonance would occur at nearly identical forcing frequencies with or without damping.

(2) A control system limits the thrust parameter which can be applied to 25.67 for the uniform beam. It is not necessary to investigate the stability of the system at thrust levels above this value since the system is unstable for thrust ranges greater than 25.67.

7.3 Stability

The stability investigation is limited to forcing frequencies in the vicinity of predicted first order instabilities, i.e., k = 1 in Eqs. (3.1.2) and (3.1.3). Let it be clearly understood that the system under analysis could and often does develop higher order instabilities. In general these second order instabilities, if they exist, are not as severe as the corresponding first order instability. If the 1st order instability exhibits a severe nature, then it would be necessary to investigate the 2nd order instability also.

Figures 8 through 16 form a sequence of graphs valid for a uniform beam which may be used to determine critical thrust parameters. Critical thrust parameters define the combination of constant thrust T_o and thrust ratio T_1/T_o at which a parametric instability could develop with some assumed transverse damping factor. Figures 17 through 25 form a similar sequence which is valid for a single geometry of a stepped beam. Each change in geometry would require an additional set of graphs.

Tables 1 and 2 tabulate a range of forcing frequencies in the vicinity of $\bar{\Omega} = \bar{\omega}_{(1)} \pm \bar{\omega}_{(j)}$ for a uniform beam and a stepped beam, respectively. The characteristic argument, T, defined in Section 7.1 is tabulated for each forcing frequency chosen for stability investigation. The damped values were not tabulated since it was found that the damped and undamped T differed by the chosen damping factor.

In Figures 8 and 17 the characteristic arguments tabulated in Tables 1 and 2 were plotted against the corresponding forcing frequencies. The frequency scale is broken between regions of 1st order instability to facilitate greater resolution and to provide a more direct comparison of the unstable regions. As plotted, instability exists at forcing frequencies for which $T > 0$. Damped values of T are plotted as dashed lines on the same graphs for a damping factor of $\eta = 0.01$ and are seen to be shifted an amount equal to the chosen damping factor.

At unstable forcing frequencies, the T curves plotted as half circles with a radius center at $\bar{\Omega} = \bar{\omega}_{(i)} \pm \bar{\omega}_{(j)}$. An exception is seen to exist in the vicinity of $2\bar{\omega}_{(B)}$ due to the development of higher order instabilities within the unstable regions of $2\bar{\omega}_{(B)}$. Another exception developed for regions of instability existing near resonance (see Figure 26).

Although the reason T plotted as a circular function is not clearly understood, the fact can be used to advantage. A close approximation can be made to the boundary frequency at T = 0 by

$$\bar{\Omega}_b = [\bar{\omega}_{(i)} \pm \bar{\omega}_{(j)}] \pm T_x \tag{7.3.1}$$

where

$$T_x = \text{Value of } T \text{ at } \bar{\Omega} = \bar{\omega}_{(i)} \pm \bar{\omega}_{(j)}$$

and

$$\bar{\Omega}_b = \text{Boundary frequency} - \bar{\Omega} \text{ at } T = 0.$$

The resultant boundaries separating stable and unstable frequencies may be plotted as $\bar{\Omega}$ versus $\bar{T}_o$ with T_1/T_o held constant, or as $\bar{\Omega}$ versus Υ with $\bar{T}_o$ held constant. The choice of boundary plots are arbitrary. Both types are included under this investigation. Figures 9 and 10 and Figures 18 and 19 are examples of boundary plots of uniform and of stepped beams, respectively. The curves of Figure 10 and Figure 19 indicate that the boundaries are linear with the ratio, T_1/T_o.

7.3.1 Transverse Damping

Figure 8 and Figure 17 clearly depict the effects of damping upon regions of instability. The damped T values are shifted along the T axis by the value of the chosen damping factor. The boundaries, $\bar{\Omega}_b$, are not appreciably altered in the larger regions of instability when damping is considered. The magnitude of the characteristic argument, T, was significantly reduced, however, which indicates that the severity of the instability is reduced by damping. It is also seen that T is less than zero in stable regions, indicating that transverse vibrations damp out with time. Inspection of the transformation of Eq. (6.1.2) substantiates this result.

T_x very closely approximates the maximum value of T within an unstable region except near a longitudinal resonance frequency (see Fig. 26). This property, coupled with the fact that damping shifts the T curves by an amount equal to the chosen damping factor, provides a method of calculating critical $\bar{T}_o$ and γ values when a forcing frequency is in the immediate vicinity of $\bar{\Omega} = \bar{\omega}_{(i)} \pm \bar{\omega}_{(j)}$.

A plot of T_x versus γ for a range of T_o values is shown in the top half of Fig. 11 through Fig. 14. A line is passed through each thrust curve at $T_x = \eta$. These points define values of constant thrust and γ at which instability is possible for the chosen η value. An envelope curve is constructed in the lower half of Fig. 11 through Fig. 14 plotting critical thrust and γ combinations. Any combination of $\bar{T}_o$ and γ below this curve is considered stable for the assumed damping factor. Similar curves are given for a stepped beam in Fig. 20 through Fig. 23.

Boundary plots of the damped systems are given in Figs. 16 and 17 for the uniform beam and in Figs. 24 and 25 for the stepped beam. The critical thrust values at each region of instability were located by the curves of Figs. 11 through 14 and Figs. 20 through 24. It can be seen that transverse damping has a significant effect upon the regions of instability at small thrust values and that instabilities will not develop until sufficiently large $\bar{T}_o$ and T_1/T_o values are attained.

7.3.2 Longitudinal Compliance and Longitudinal Damping

Resonance, as used in this study is the term applied when the forcing frequency is in the immediate vicinity of a natural longitudinal frequency. T. Beal (ref. 3) established that longitudinal compliance or coupling had a particular strong influence upon the dynamic stability of a system when resonance occurred at or near $\frac{\omega_{(i)} \pm \omega_{(j)}}{k}$. If damping is neglected the force distribution becomes infinite at resonance.

Figure 8 is a plot of the characteristic argument, T, of a beam which is assumed to have an infinitely high longitudinal frequency, i.e., $\bar{\omega}_L = \infty$ and $\bar{\omega}_L >> \bar{\Omega}$. The T plot remains constant with a decrease in $\bar{\omega}_L$ until $\bar{\omega}_L$ is in the vicinity of a parametric resonance frequency. Figure 26 illustrates the effect of longitudinal frequency upon the characteristic argument T_x at parametric resonance. T_x was defined in Section 7.3. For longitudinal frequencies removed from resonance, $\bar{\omega}_L > \bar{\Omega}$, T_x approached a constant, however, at resonance, $\bar{\omega}_L = \bar{\Omega}$, T_x became infinite. This is good reason to avoid all rocket vehicle designs in which the longitudinal frequency is near the predicted forcing frequency. The expressions containing the periodically-varying force distribution appear in the matrix elements of the G matrix of Eq. (4.5.18) in the form of an integral. Evaluation of these integrals indicates the relative contribution of longitudinal compliance. It is interesting to investigate these integrals near resonant regions and to determine the effects of longitudinal damping.

The integrals were evaluated for Ω/ω_L from 0 to 2, with resonance occurring at $\Omega/\omega_L = 1,2$. It was found that the value of integrals on the diagonal of [G] approached infinity for $\Omega/\omega_L = 1$ and off-diagonal integrals approached infinity for $\Omega/\omega_L = 2$.

A curve is presented in Fig. 27 evaluating the integral of the matrix element $G_{N+1,N+1}$ for variations of Ω/ω_L.

For reference, the integral values are taken as the real part of $\int_0^1 \Phi^* d\xi$ where Φ^* has been defined as

$$\phi^* = \frac{\sin\pi(\bar{\sigma}^2 - ig)^{\frac{1}{2}}\xi}{\sin\pi(\bar{\sigma}^2 - ig)^{\frac{1}{2}}}$$

by Eq. (4.4.17). The curve was plotted for damping factors of g = 0.00 and g = 0.04. Damping is seen to have a negligible effect at forcing frequencies removed from the immediate vicinity of resonance. At resonance, however, integral values become finite with damping and actually pass through zero since the integral values change in sign at resonance.

A curve of $\bar{\Omega}$ versus T is plotted in Fig. 28 for forcing frequencies in the vicinity of $2\bar{\omega}_{(1)}$ and resonance. Curves are plotted for damping factors of g = 0.00, 0.04, and 0.08. The amplitudes of T become infinite for a damping factor of g = 0.04 but the region of instability is not appreciably altered. A narrow stable region develops at $\bar{\Omega} = 2.0$ for g = 0.04. This paradoxical result can be explained by observing that the integral values illustrated in Fig. 27 pass through zero at resonance. The regions of instability are seen to be divided into two unstable regions with a damping factor of 0.04. A damping factor of g = 0.08 further reduced the amplitudes of T and eliminated the region of instability existing above $\bar{\Omega} = 2.0$.

Fig. 28 is a graph of boundaries of instability for a longitudinal frequency of $\bar{\omega}_L = 2.0$ with zero transverse damping. The region of instability at $\bar{\Omega} = 2\bar{\omega}_{(1)}$ is seen to be extremely broad. For comparison a similar plot is given in Fig. 8 with $\bar{\omega}_L = \infty$. Longitudinal compliance exerts a maximum influence when resonance occurs near $\bar{\omega}_{(i)} \pm \bar{\omega}_{(j)}$. An additional instability developed at $\bar{\Omega} = 4$ which is not predicted in the usual theory of dynamic stability where longitudinal coupling is neglected.

A damping factor of 0.04 did not appreciably alter the boundaries of instability at $\bar{\Omega} = 2\bar{\omega}_{(1)}$ as seen in Fig. 28. Fig. 29 plots boundaries of instability for a longitudinal damping factor of g = 0.08. Overdamped regions are crosshatched. The instability at $\bar{\Omega} = 4.0$ damps out and the width of the unstable region at $\bar{\Omega} = 2\bar{\omega}_{(1)}$ is narrowed slightly while other boundaries are unchanged.

7.3.3 Effect of Initial Conditions

Conditions for solution were highly restrictive upon inclusion of initial conditions. It was assumed in this investigation that initial conditions would exert a maximum influence on the transverse motion at a forcing to longitudinal frequency ratio of 1:2.

The initial conditions chosen for analysis were those of zero velocity and zero displacement as taken in Section 4.6. Solutions for the characteristic values were taken at $\bar{\Omega} = \bar{\omega}_{(i)} \pm \bar{\omega}_{(j)}$ for a longitudinal frequency of $2\bar{\Omega}$. A tabulation of results is given in Table 2 in terms of T computed at the predicted unstable forcing frequencies with and without the inclusion of initial conditions.

The results indicate that zero initial conditions do not appreciably alter the dynamic stability characteristics of the system. Furthermore, the results do not substantiate a conclusion as to whether initial conditions reinforce the response in unstable regions. It is noted that for a couple of unstable frequencies, initial conditions actually tend to stabilize the system.

CHAPTER 8

CONCLUSIONS

8.1 General Conclusions

A restatement of conclusions obtained in preceeding investigations and substantiated in this report are given below.

1. Unstable solutions occur for frequencies of variation of the thrust in the vicinity of twice one of the natural frequencies of the bending modes, or the sum or difference of two of these frequencies (see, for example, Figs. 8, 9, and 10).

2. The width of the unstable regions stated above is approximately linear with a ratio of $\gamma = T_1/T_0$ (see for example, Fig. 10).

3. With finite longitudinal compliance instabilities also occur for frequencies of the thrust variation in the vicinity of the longitudinal natural frequencies (see, for example, Fig. 29).

4. Longitudinal compliance has a negligible effect at forcing frequencies removed from the longitudinal natural frequencies.

8.2 Effects of a Stepped Beam Configuration Upon the Dynamic Stability

A beam composed of two uniform sections joined by a rigid bulkhead was analyzed (see Section 5) and the following results were established.

1. The natural frequencies shifted as expected with any change in mass, moment of inertia, or location of step (see, for example Figs. 5, 6, and 7).

2. As a result of a shift in frequencies the regions of parametric instability shifted accordingly. The width and severity of the instabilities were altered but no trends were established in this analysis (compare, for example, Figs. 8 and 17).

8.3 Effects of Transverse Damping

Transverse damping was found to be a significant factor in determining regions of parametric instability for the system investigated. The following salient results were established:

1. A pulsating end thrust will not sustain transverse vibrations unless the system is unstable.

2. Some regions existing as very weak instabilities are completely eliminated with damping.

3. Instabilities may result for sufficiently large thrust and T_1/T_o ratios; conversely, damping prohibits the formation of a parametric instability at lower thrust values (see, for example, Figs. 7.6, 7.7, 7.8, and 7.9).

8.4 Effects of Longitudinal Damping

Regions of instability developed at resonant frequencies which are not predicted in the usual theory of dynamic stability. In addition, critical regions within the vicinity of a resonance frequency were many orders of magnitude more severe than the same critical region away from resonance. Inclusion of longitudinal damping produced the following effects:

1. Longitudinal damping had no effect on unstable regions removed from resonance.

2. Unstable regions which existed near resonance were narrowed slightly and reduced in severity but the instability remained quite severe.

3. Instabilities which developed at resonance were eliminated with longitudinal damping.

Critical regions near a resonance frequency develop very severe instabilities even with large longitudinal damping factors.

8.5 Effects of Zero Initial Conditions upon Regions of Instability

The effect of initial conditions on critical regions was inconclusive. Results (see, for example, Table 3) indicate that the effects may be considered slight.

Limited experimental work (ref. 1) for beams subjected to a pulsating loading indicate that unstable conditions do not in general precipitate violent response conditions. It was found that the response curves increased exponentially for a period of time then stabilized at some higher amplitude response.

Since the transient solution damps out completely with time, the contribution of initial conditions would damp out as the amplitudes of response increase under unstable conditions. The transient solution could not sustain or appreciably reinforce amplitudes of response over a period of time.

The investigation substantiates the position that initial conditions may be neglected in the analysis of parametrically-excited systems.

8.6 Suggestions for Further Study

The modern launch vehicle is a highly discontinuous structure. Due to fuel consumption during flight the mass properties are under continuous change. It is possible that parametric instability could exist during some time interval. In order to determine if parametric instability occurs during some portion of a missile flight it is necessary that quantitative results be available. Such quantitative results are not available by existing methods. In order to adequately approximate mass and stiffness distributions in the investigation of the dynamic stability of modern space vehicles, it may be desirable to employ representative lumped-mass systems. Such a simplification has been used effectively in calculating natural frequencies.

It is proposed that a method of analysis be devised for investigating the dynamic stability of highly discontinuous structures. The analysis would be specifically oriented toward existing space vehicles. The primary objectives of the proposed investigation would be twofold; (1) establish a simplified analytical model which would adequately describe the mass and stiffness distribution, and (2) determine forcing frequencies at which instability is possible due to parametric resonance. Such quantitative information could be directly applied to an actual vehicle.

TABLE 1

CHARACTERISTIC VALUES - UNIFORM BEAM

$\xi_G = 0.50$ | $K_\theta = 0.5$ | $\bar{\omega}_L = \infty$ | $\bar{T}_o = 20.0$ | $\gamma = .10$ | $\eta = 0.0$ | $g = 0.0$

	$\bar{\Omega}$	T		$\cos 2\pi\alpha$	
	.371	.05799	REAL IMAG	-.87719 0	-.04559 0	-1.12299 0
	.374	.05630		-.84207 0	-.37565 0	-1.11392 0
	.375	.02830		-.96003 +.08570	-.96003 -.08570	-.47932 0
	.376	.03233		-.57301 0	-.87226 0	-1.03672 0
$2\bar{\omega}_{(B)}$	.383	.04094		-.69751 0	-.97429 0	-1.05706 0
	.388	.03922		-.95577 0	-.55875 0	-1.05098 0
	.398	.02834		-.23829 0	-.25696 0	-1.02518 0
	1.064	0		.49510 0	.54281 0	-.59028 0
	1.066	.00689		.51588 -.01740	.51588 +.01740	-.56769 0
	1.070	.01724		.50970 -.04357	.50970 +.04357	-.52150 0
	1.080	.02812		.49371 -.07126	.49371 +.07126	-.40105 0
$\bar{\omega}_{(B)}+\bar{\omega}_{(1)}$	1.100	.03404		.46111 -.08649	.46111 +.08649	-.14765 0
	1.812	0		.78824 0	-.73128 0	-.99998 0

TABLE 1-Continued

CHARACTERISTIC VALUES - UNIFORM BEAM

$\xi_G = 0.50$	$K_\theta = 0.5$	$\bar{\omega}_L = \infty$	$\bar{T}_o = 20.0$	$\gamma = .10$	$\eta = 0.0$	$g = 0.0$
	$\bar{\Omega}$	T		$\cos 2\pi\alpha$		
	1.814	.00114	REAL IMAG	.78858 0	-.72643 0	-1.00000 0
$2\bar{\omega}_{(1)}$	1.818	.00493		.78958 0	-.71173 0	-1.00004 0
	1.822	.00280		.79058 0	-.69678 0	-1.00001 0
	1.824	0		.79091 0	-.69174 0	-.99999 0
	2.665	0		.90581 0	.92058 0	-.54161 0
	2.672	.01002		.91052 +.00487	.91052 -.00487	-.53693 0
	2.679	.01847		.90775 +.00910	.90775 -.00910	-.53225 0
$\bar{\omega}_{(B)}+\bar{\omega}_{(2)}$	2.693	.02311		.90193 +.01167	.90193 +.01167	-.52292 0
	2.707	.01851		.89576 +.00956	.89576 -.00956	-.51363 0
	2.714	.01012		.89256 +.00528	.89256 -.00528	-.50899 0
	2.721	0		.89734 0	.88120 0	-.50436 0
$\bar{\omega}_{(2)}+\bar{\omega}_{(1)}$	3.411	0		.93834 0	-.10222 0	-.10526 0
	4.971	0		.97089 0	.40950 0	-.99994 0
	4.975	.01161		.97098 0	.41117 0	-1.00003 0
$2\bar{\omega}_{(2)}$	5.003	.02659		.97126 0	.41617 0	-1.00014 0
	5.027	.01155		.97153 0	.42111 0	-1.00003 0
	5.035	0		.97162 0	.42274 0	-.99994 0

TABLE 2

CHARACTERISTIC VALUE - STEPPED BEAM

$\xi_G = 0.50$	$K_\theta = 1.0$	$\bar{\omega}_L = \infty$	$\bar{T}_o = 20.0$	$\gamma = 0.10$	$\eta = 0.00$
		$\xi_c = 0.6$	$\bar{A} = 0.06$	$\bar{I} = 0.06$	

	$\bar{\Omega}$	I		$\cos 2\pi\alpha$		
	.422	0	REAL IMAG	+.40135 —	-.44838 —	-.89637 —
	.434	0.00923		+.71929 —	-.97760 -.01473	-.97760 +.01473
	.446	0.00541		+.92008 —	-.65398 —	-1.00072 —
$2\bar{\omega}_{(B)}$	.496	0.04110		+.63676 —	+.48731 —	-1.03410 —
	.520	0.02421		+.14714 —	-.80233 —	-1.01070 —
	.533	0		-.10557 —	-.99109 —	-.99941 —
	1.111	0		+.39890 —	+.55177 —	+.23434 —
	1.125	0.02212		+.66801 —	+.29486 -.05907	+.29486 +.05907
	1.139	0.03857		+.79968 —	+.27358 -.10256	+.27358 +.10256
$\bar{\omega}_{(1)} + \bar{\omega}_{(B)}$	1.168	0.04753		+.94892 —	+.23086 -.12482	+.23086 +.12482
	1.196	0.03743		+.99990 —	+.18849 -.09672	+.18849 -.09672
	1.210	0.01892		+.99090 —	+.16771 -.04842	+.16771 +.04842
	1.225	0		+.96117 —	+.23124 —	+.06336 —

TABLE 2 - Continued

CHARACTERISTIC VALUES - STEPPED BEAM

ξ_G = 0.50	K_θ = 1.0	$\bar{\omega}_I = \infty$	$\bar{I}_o$ = 20.0	γ = 0.10	η = 0.00	
		ξ_c = 0.6	$\bar{A}$ = 0.6	$\bar{I}$ = 0.6		

	$\bar{\Omega}$	T		$\cos 2\pi\alpha$		
	1.461	0	REAL IMAG	.47731 —	-.65166 —	-.67122 —
	1.464	0.00710		.47932 —	-.67480 -.01124	-.67480 +.01124
	1.468	0.01049		.48145 —	-.68786 -.01630	-.68786 +.01630
$\bar{\omega}_{(2)} - \bar{\omega}_{(1)}$	1.475	0.01218		.48586 —	-.71303 -.01819	-.71303 +.01819
	1.483	0.00893		.49036 —	-.73692 -.01279	-.73692 +.01279
	1.486	0.00235		.49262 —	-.74839 -.00329	-.74839 +.00329
	1.490	0		.49488 —	-.74613 —	-.77291 —
	1.830	0		.65754 —	-.36334 —	-.99994 —
	1.832	0.00227		.65843 —	-.35299 —	-1.00001 —
	1.835	0.00618		.65932 —	-.34263 —	-1.00006 —
$2\bar{\omega}_{(1)}$	1.839	0.00823		.66108 —	-.32187 —	-1.00010 —
	1.844	0.00696		.66283 —	-.30106 —	-1.00007 —
	1.847	0.00451		.66370 —	-.29065 —	-1.00003 —
	1.849	0		.66457 —	-.28023 —	-.99997 —

TABLE 2 - Continued

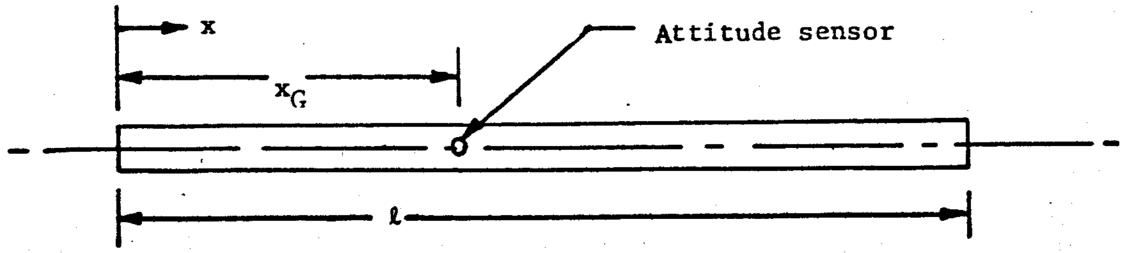
CHARACTERISTIC VALUES - STEPPED BEAM

$\xi_G = 0.50$	$K_\theta = 1.0$	$\bar{\omega}_L = \infty$	$\bar{T}_o = 20.0$	$\gamma = 0.10$	$\eta = 0.00$	
		$\xi_c = 0.6$	$\bar{A} = 0.6$	$\bar{I} = 0.6$		
	$\bar{\Omega}$	T			$\cos 2\pi\alpha$	
	2.615	0	REAL IMAG	.85513 —	.83505 —	−.59649 —
	2.622	0.00953		.84166 −.00617	.84166 +.00617	−.59181 —
	2.629	0.01821		.83815 −.01187	.83815 +.01187	−.58715 —
$\bar{\omega}_{(2)} + \bar{\omega}_{(B)}$	2.643	0.02300		.83090 −.01520	.83090 −.01520	−.57783 —
	2.657	0.01853		.82336 −.01244	.82336 +.01244	−.56854 —
	2.664	0.01043		.81949 −.00705	.81949 +.00705	−.56390 —
	2.671	0		.82566 —	.80544 —	−.55926 —
	4.740	0		.94641 —	.34451 —	−.99984 —
	4.753	0.01830		.94669 —	.34751 —	−1.00007 —
	4.765	0.03329		.94697 —	.35049 —	−1.00024 —
$2\bar{\omega}_{(2)}$	4.790	0.04152		.94752 —	.35639 —	−1.00037 —
	4.815	0.03312		.94805 —	.36221 —	−1.00023 —
	4.827	0.01785		.94832 —	.36510 —	−1.00007 —
	4.840	0		.94858 —	.36796 —	−.99984 —

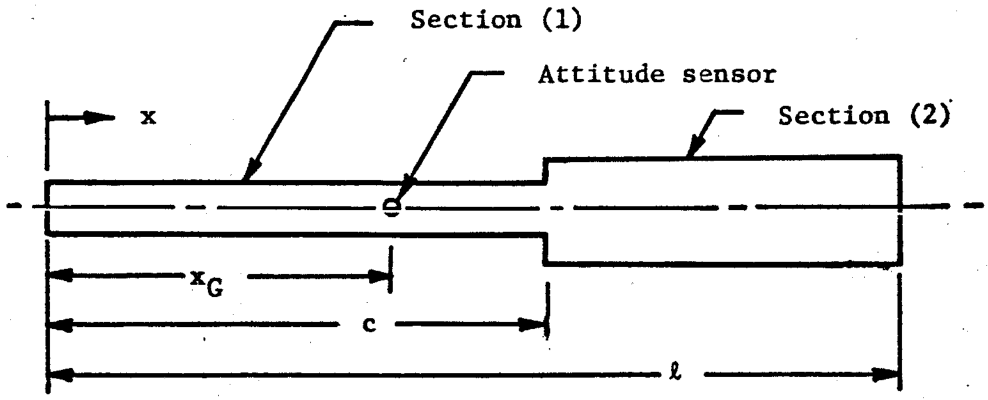
TABLE 3

EFFECTS OF INITIAL CONDITIONS UPON STABILITY

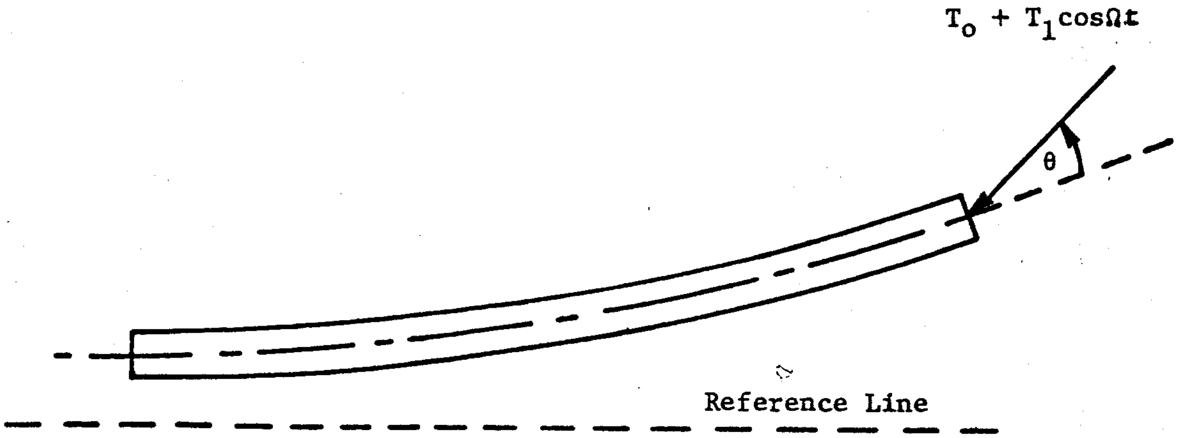
$\xi_G = 0.50$	$K_\theta = 0.50$	$T_o = 20$	$\gamma = 0.10$	$\eta = 0$	$g = 0$
	$\bar{\Omega}$	$\bar{\omega}_L$	T		
			WITHOUT INITIAL CONDITIONS	WITH INITIAL CONDITIONS	
$2\omega_{(B)}$	0.383	0.765	0.05141	0	
$2\omega_{(1)}$	1.818	3.636	0.01509	0.01724	
$2\omega_{(2)}$	5.003	10.007	0.03482	0.03495	
$\omega_{(1)}+\omega_{(2)}$	1.100	2.201	0.03803	0.03408	
$\omega_{(2)}+\omega_{(B)}$	2.693	5.386	0.02475	0.02727	
$\omega_{(2)}+\omega_{(1)}$	3.411	6.821	0	0	
$\omega_{(1)}-\omega_{(B)}$	0.718	1.435	0	0	
$\omega_{(2)}-\omega_{(B)}$	2.310	4.621	0	0	
$\omega_{(2)}-\omega_{(1)}$	1.593	3.185	0.00117	0	



(a) Uniform Beam - Beam Geometry in Undeformed State

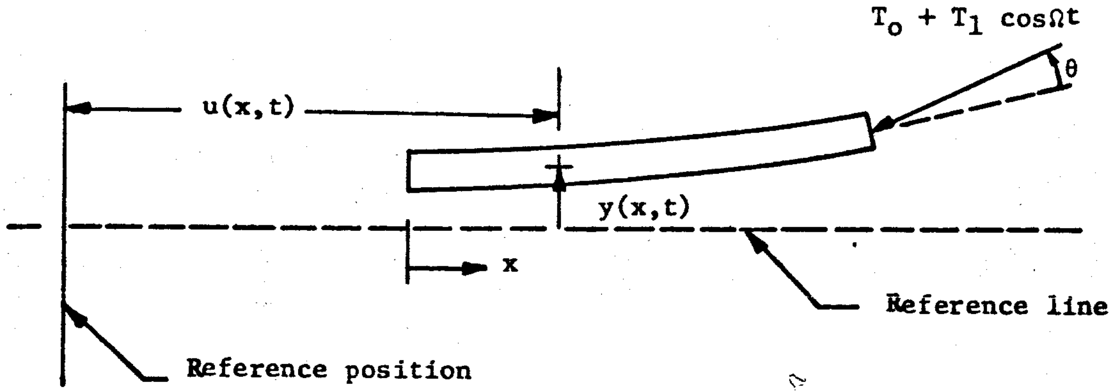


(b) Stepped Beam - Beam Geometry in Undeformed State

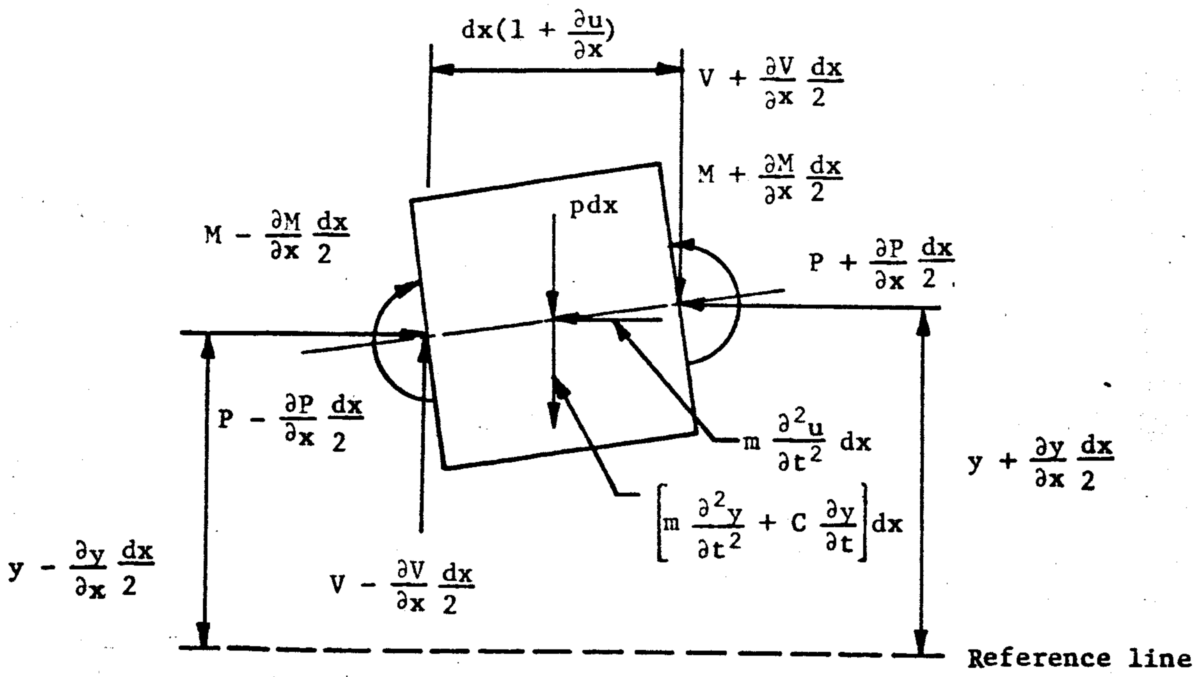


(c) Deformed Beam of Arbitrary Cross Section with Applied Loading

FIGURE 1. FREE-FREE BEAM



(a) Displacements in Lagrangian Coordinate System



(b) Forces on Beam Elements

FIGURE 2. CONTROLLED BEAM WITH THRUST OF PERIODICALLY-VARYING MAGNITUDE

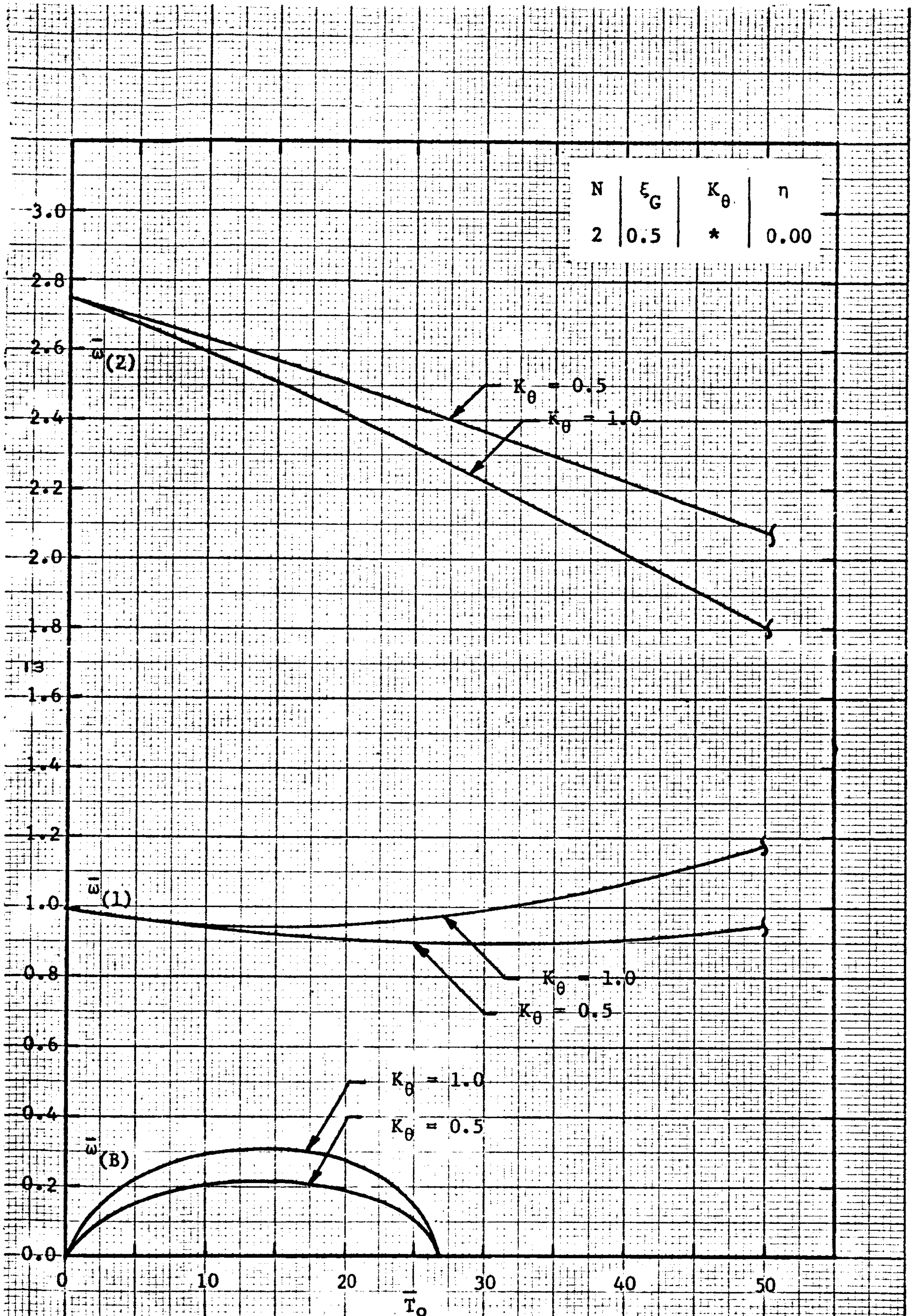


FIGURE 3. NATURAL FREQUENCY OF A UNIFORM BEAM UNDER THRUST (EFFECT OF CONTROL FACTOR, K_θ)

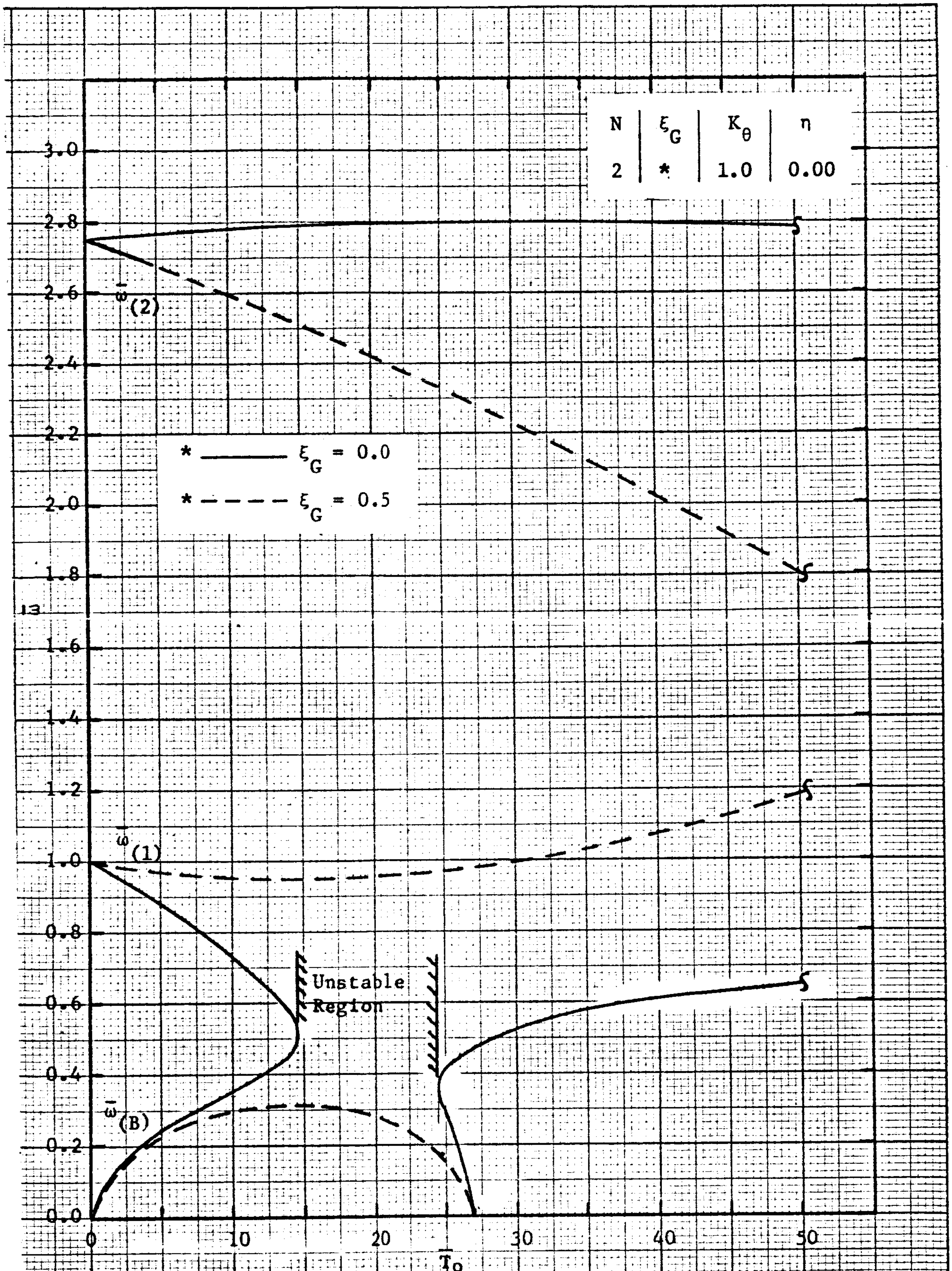


FIGURE 4. NATURAL FREQUENCY OF A UNIFORM BEAM UNDER THRUST (EFFECT OF SENSOR LOCATION, ξ_G)

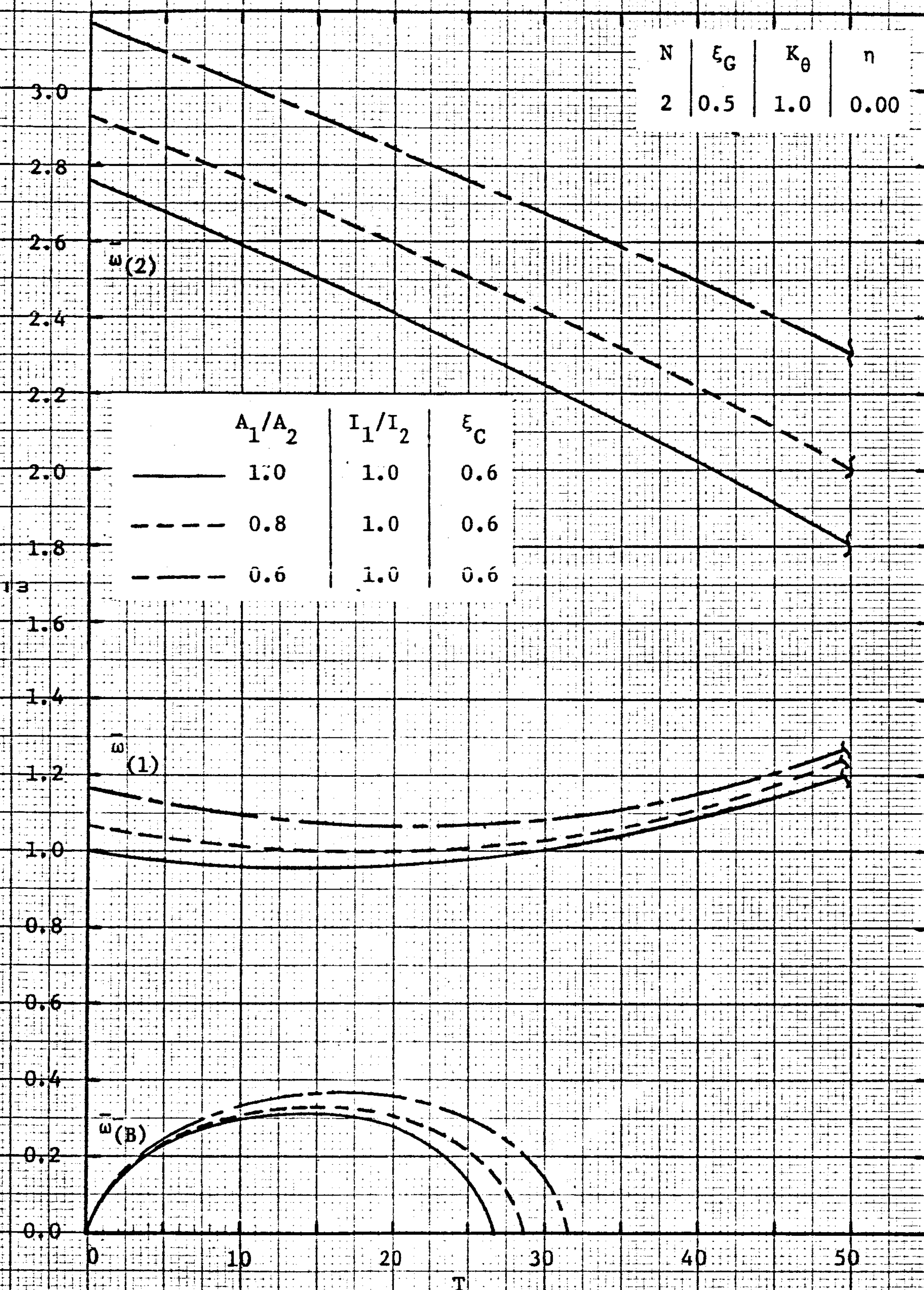


FIGURE 5 . NATURAL FREQUENCY OF A STEPPED BEAM UNDER THRUST (EFFECT OF AREA RATIO, A_1/A_2)

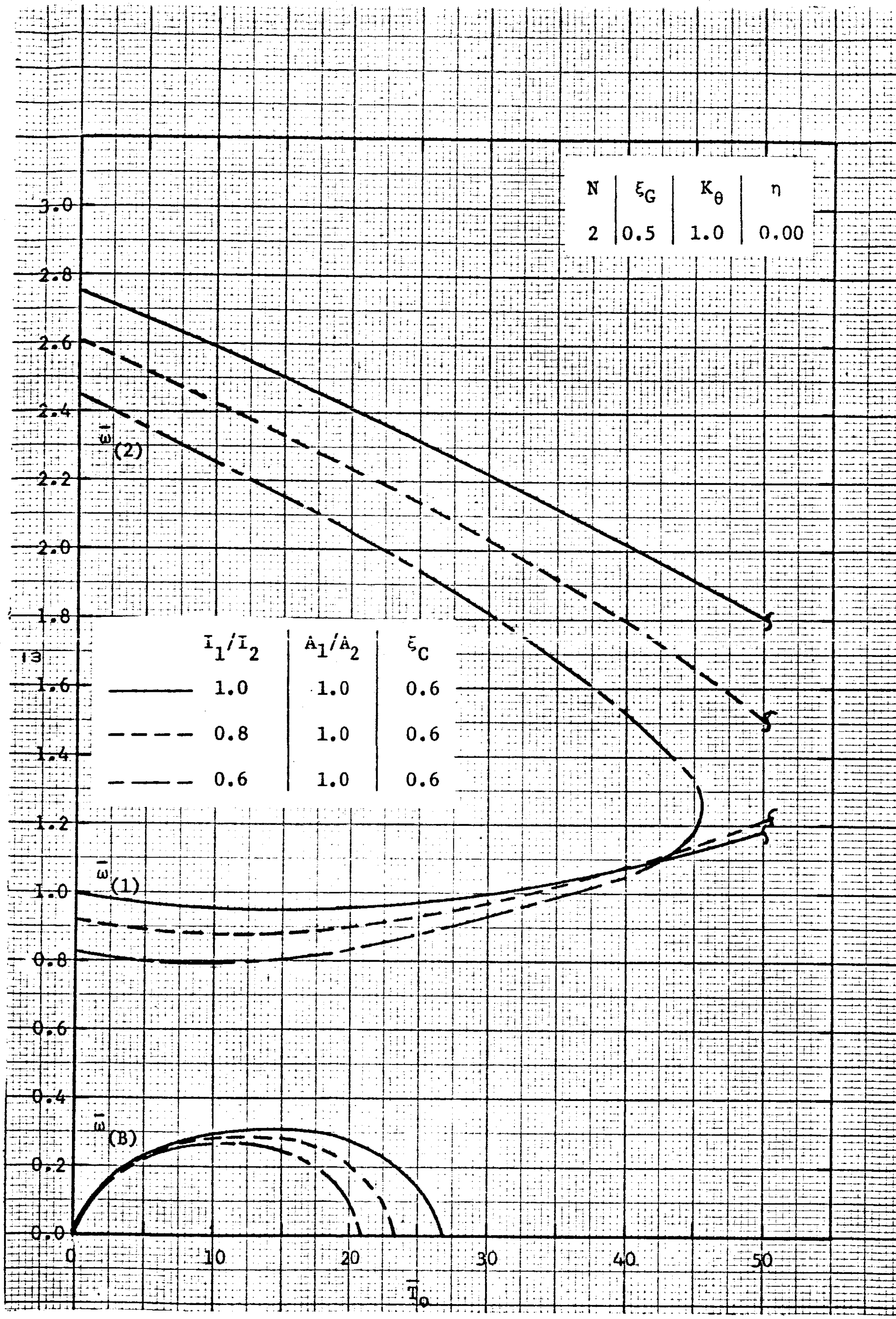


FIGURE 6. NATURAL FREQUENCY OF A STEPPED BEAM UNDER THRUST (EFFECT OF STIFFNESS RATIO, I_1/I_2)

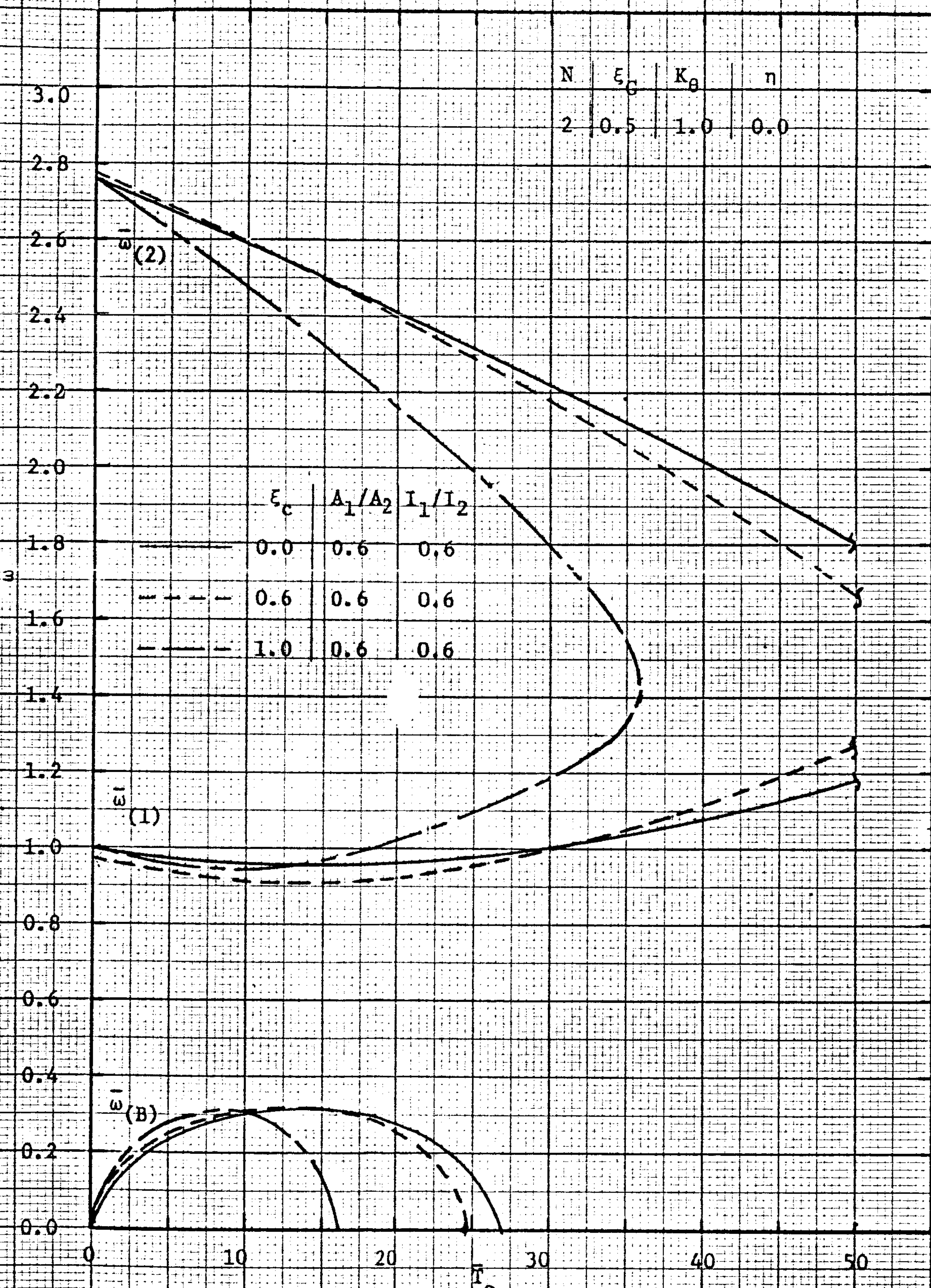


FIGURE 7. NATURAL FREQUENCY OF A STEPPED BEAM UNDER THRUST (EFFECT OF DISCONTINUITY LOCATION, ξ_c)

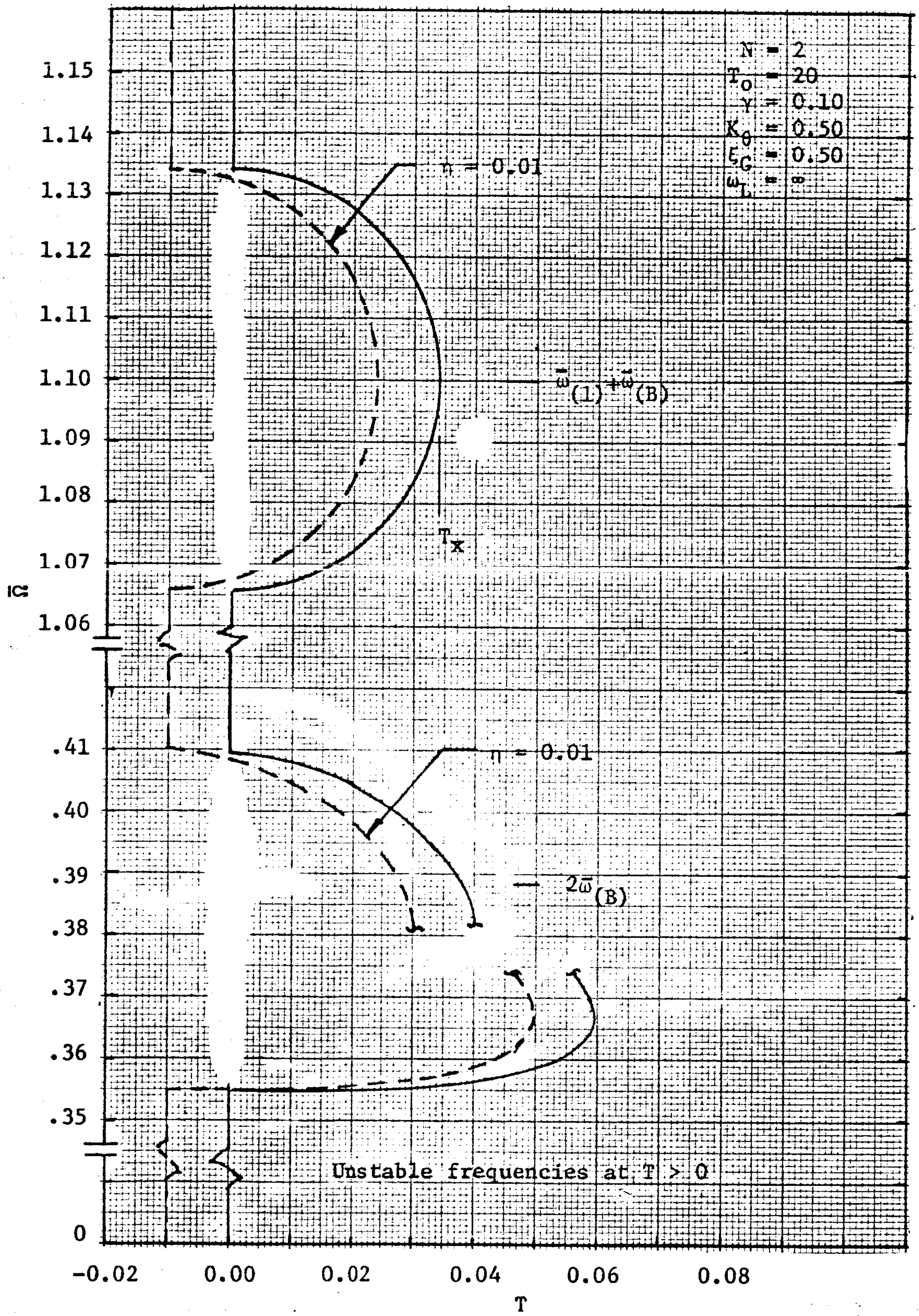


FIGURE 8. $\bar{\Omega}$ VERSUS T, PLOT OF CHARACTERISTIC ARGUMENT - UNIFORM BEAM

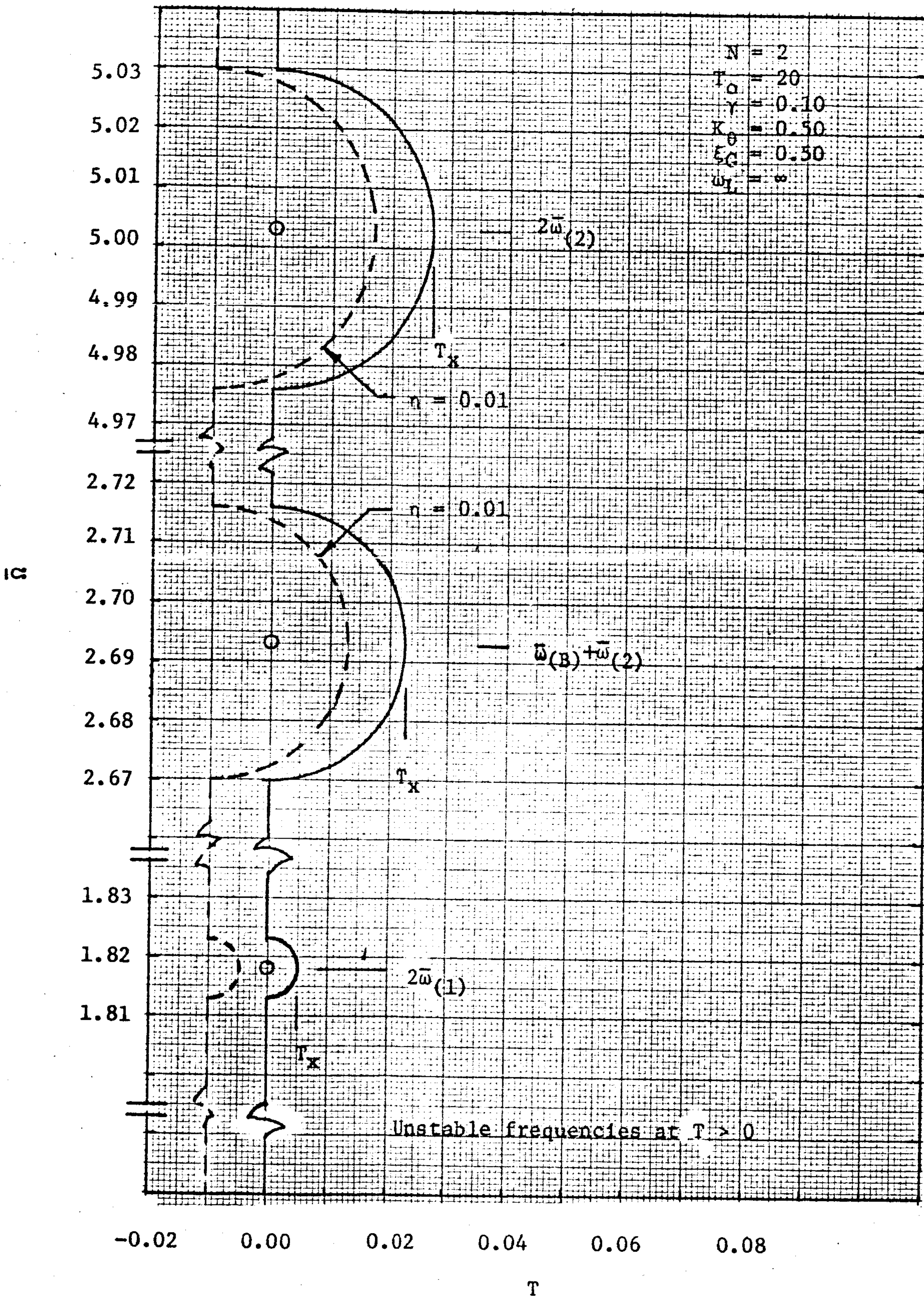


FIGURE 8 - CONTINUED

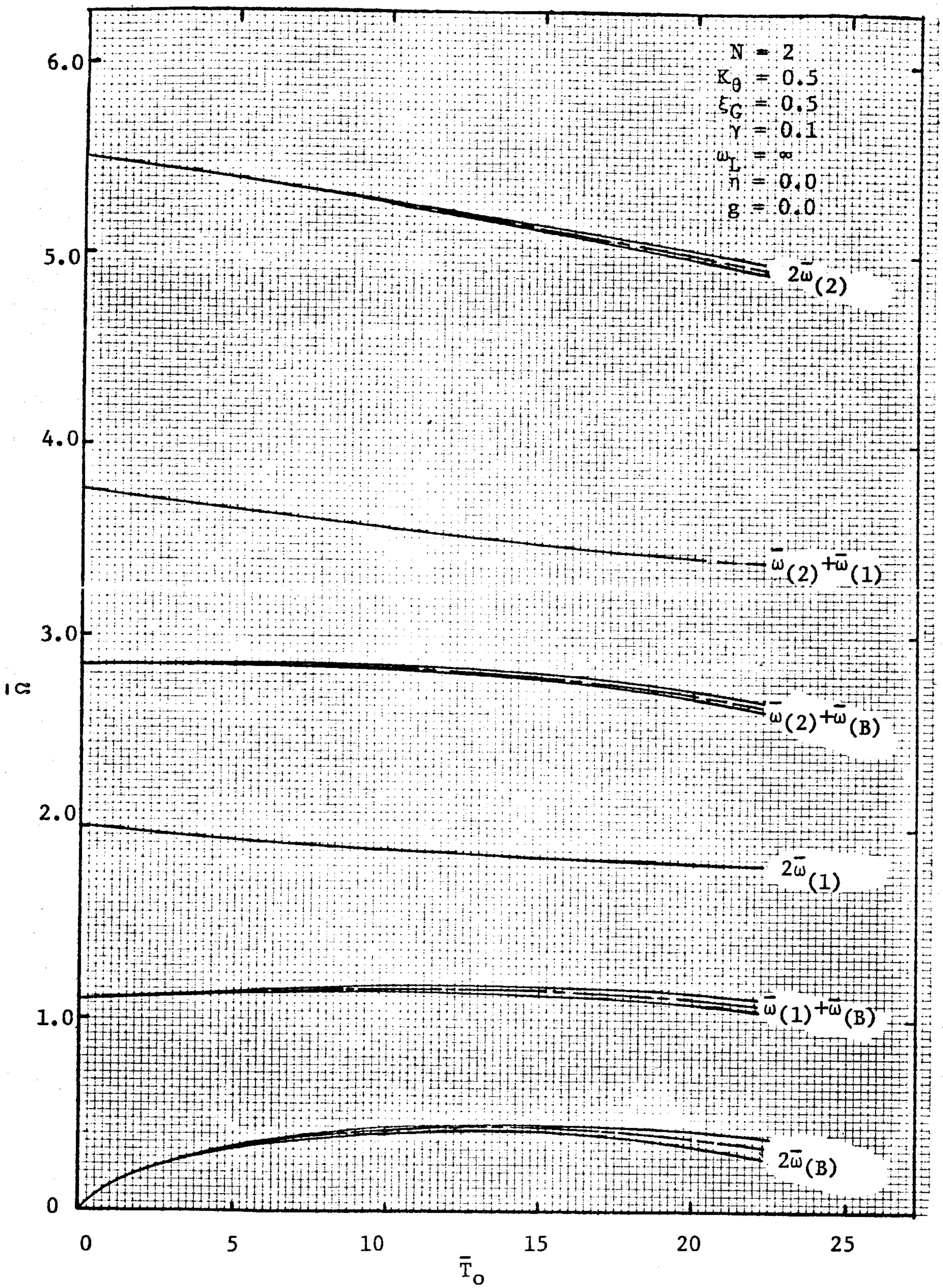


FIGURE 9. UNDAMPED BOUNDARIES SEPARATING STABLE AND UNSTABLE FORCING FREQUENCIES, $\gamma = T_1/T_o$ HELD CONSTANT - UNIFORM BEAM

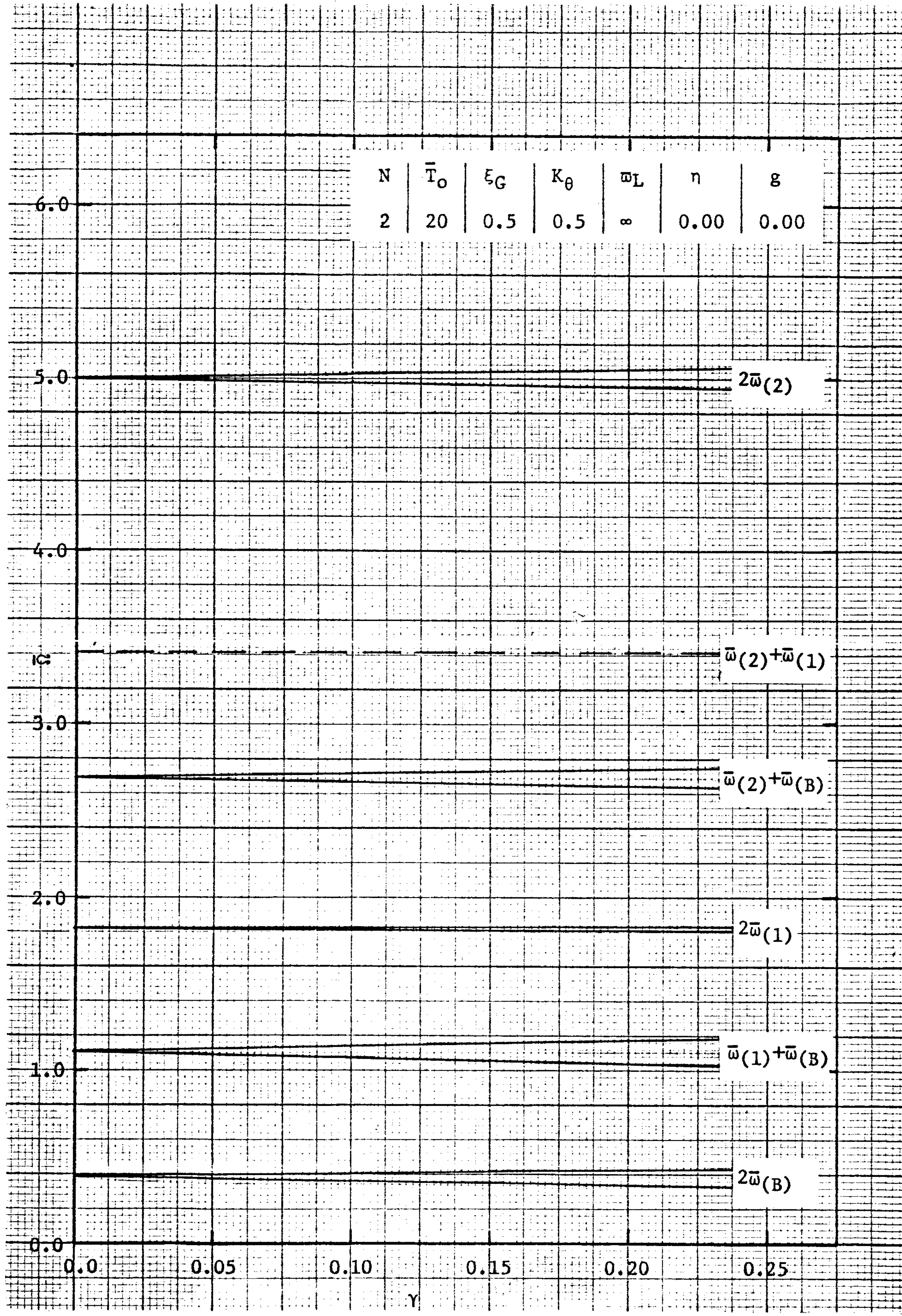


FIGURE 10. UNDAMPED BOUNDARIES SEPARATING STABLE AND UNSTABLE FORCING FREQUENCIES, $\bar{T}_o$ HELD CONSTANT - UNIFORM BEAM

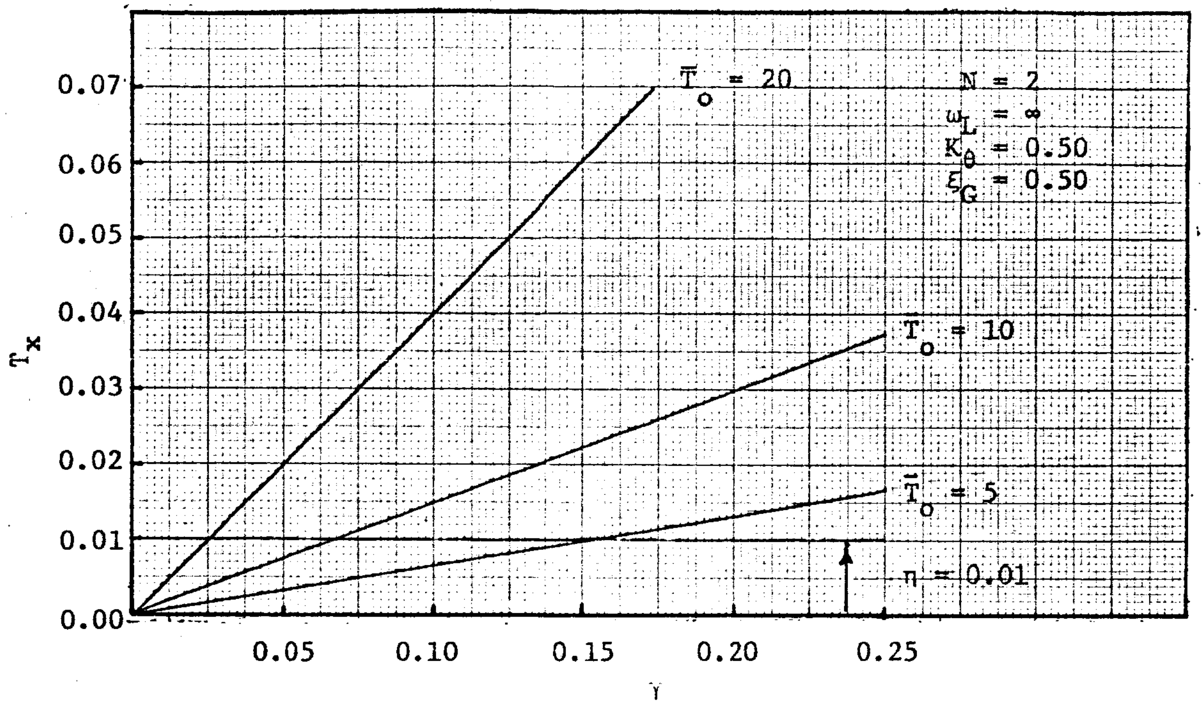


FIGURE 11.1 T_x VERSUS γ AT $\bar{\Omega} = 2\bar{\omega}_{(B)}$

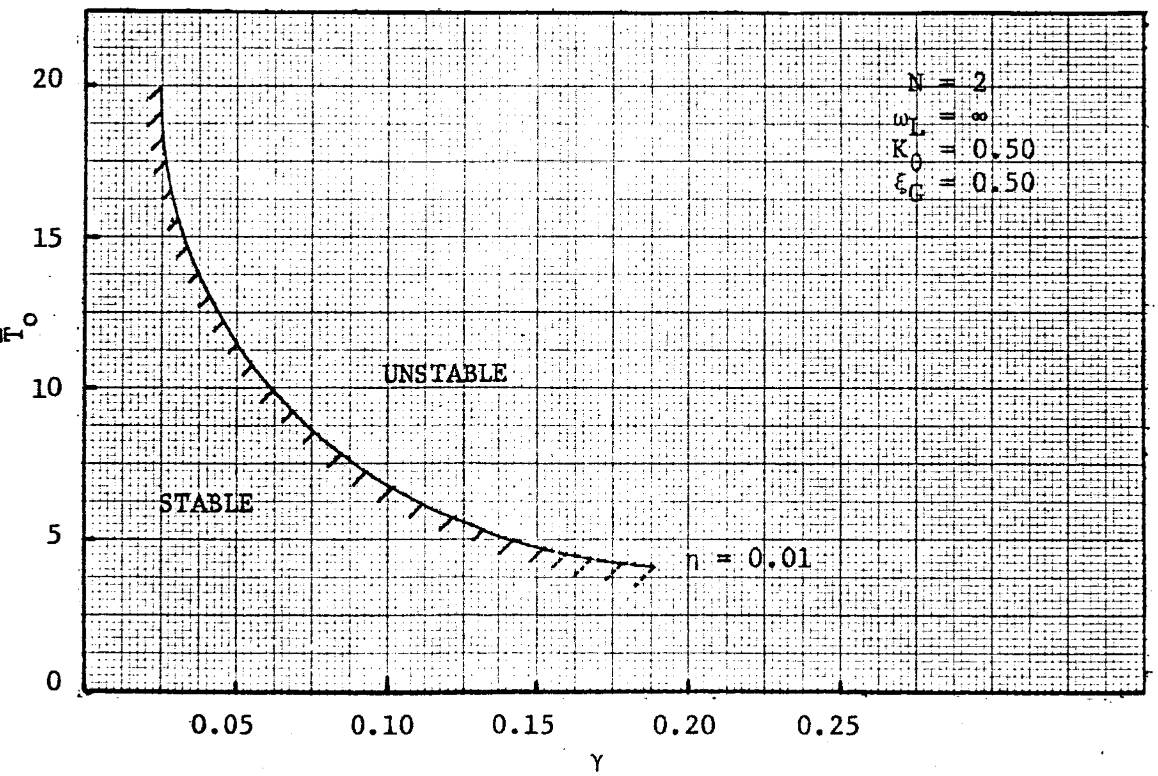


FIGURE 11.2 CRITICAL $\bar{T}_o$ AND γ AT $\bar{\Omega} = 2\bar{\omega}_{(B)}$

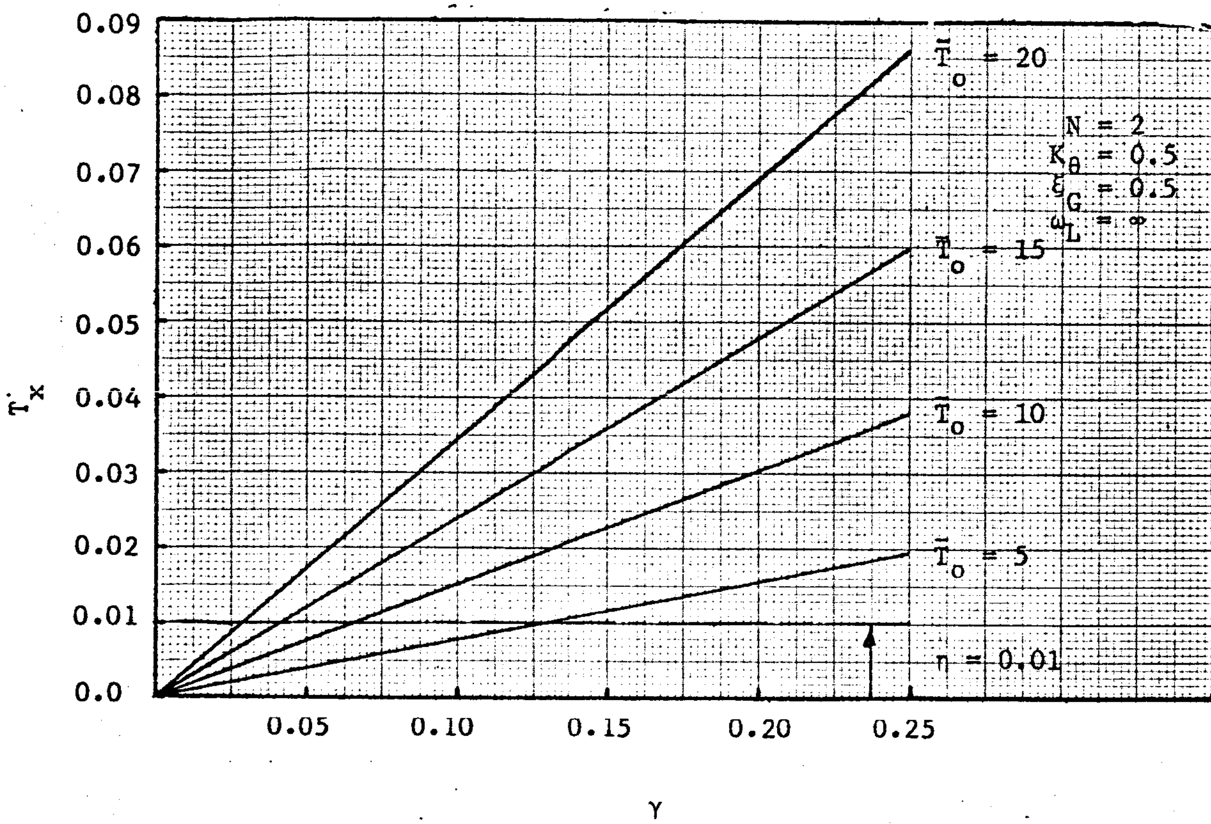


FIGURE 12.1. T_x VERSUS γ AT $\bar{\Omega} = \bar{\omega}_{(1)} + \bar{\omega}_{(B)}$

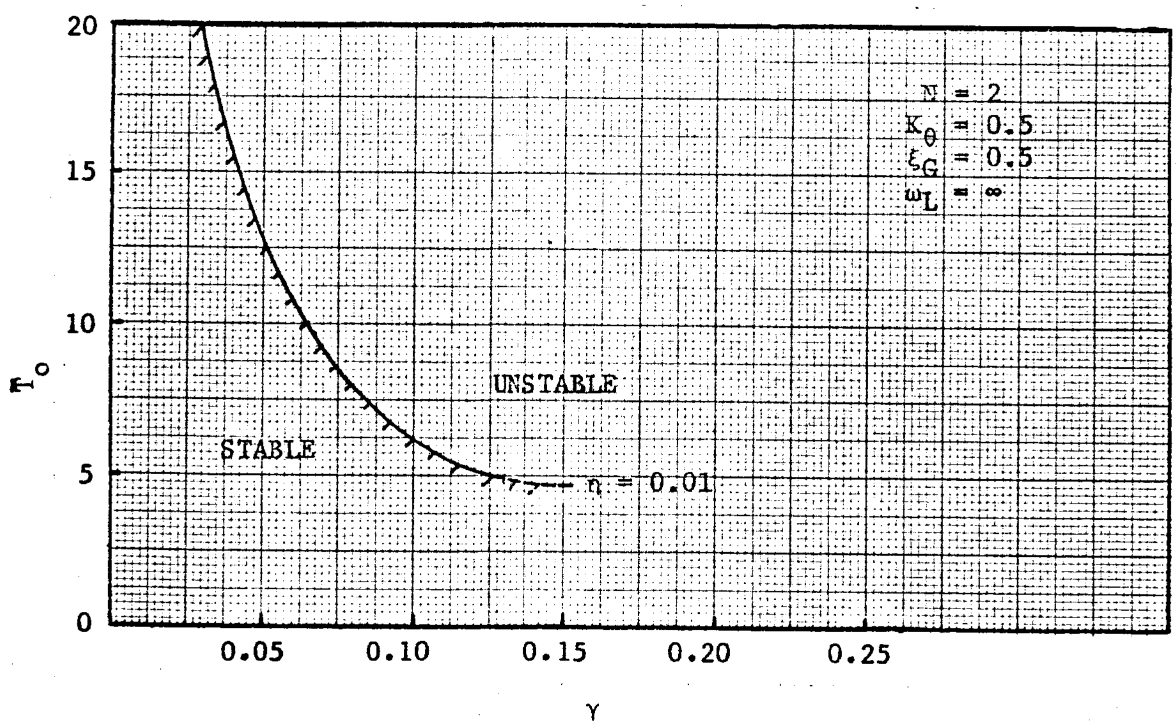


FIGURE 12.2. CRITICAL $\bar{T}_o$ AND γ AT $\bar{\Omega} = \bar{\omega}_{(1)} + \bar{\omega}_{(B)}$

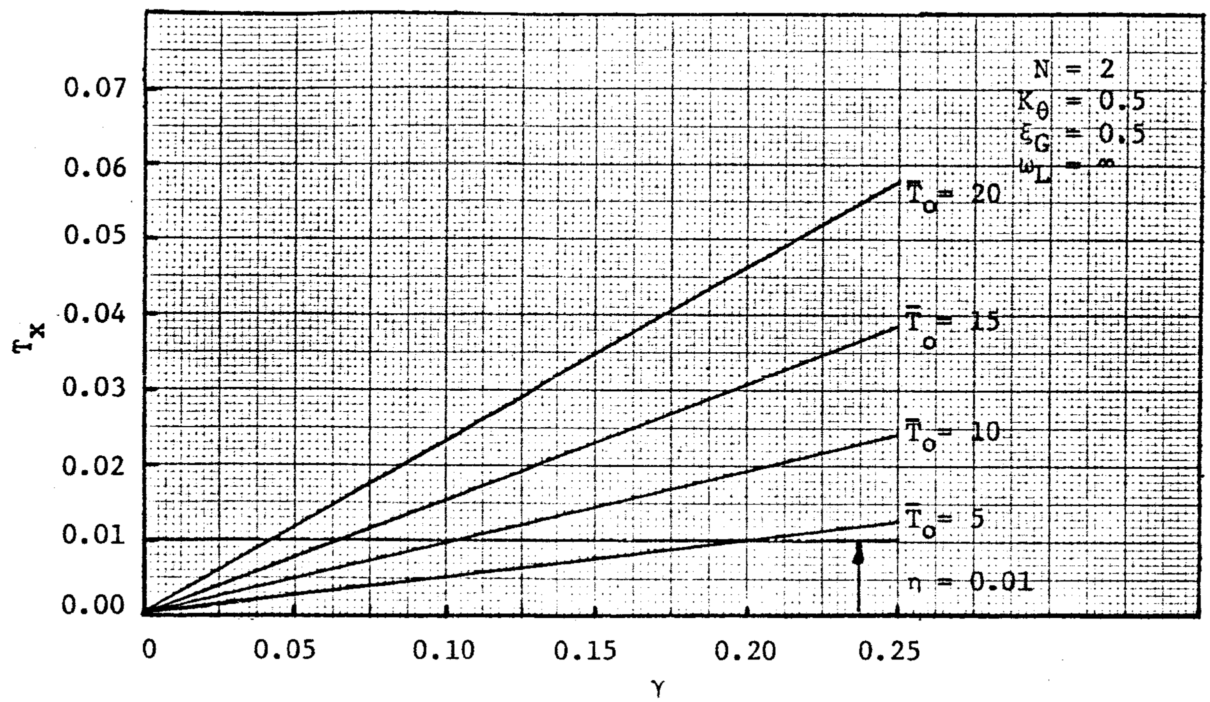


FIGURE 13.1. τ_x VERSUS γ AT $\bar{\Omega} = \bar{\omega}_{(2)} + \bar{\omega}_{(B)}$

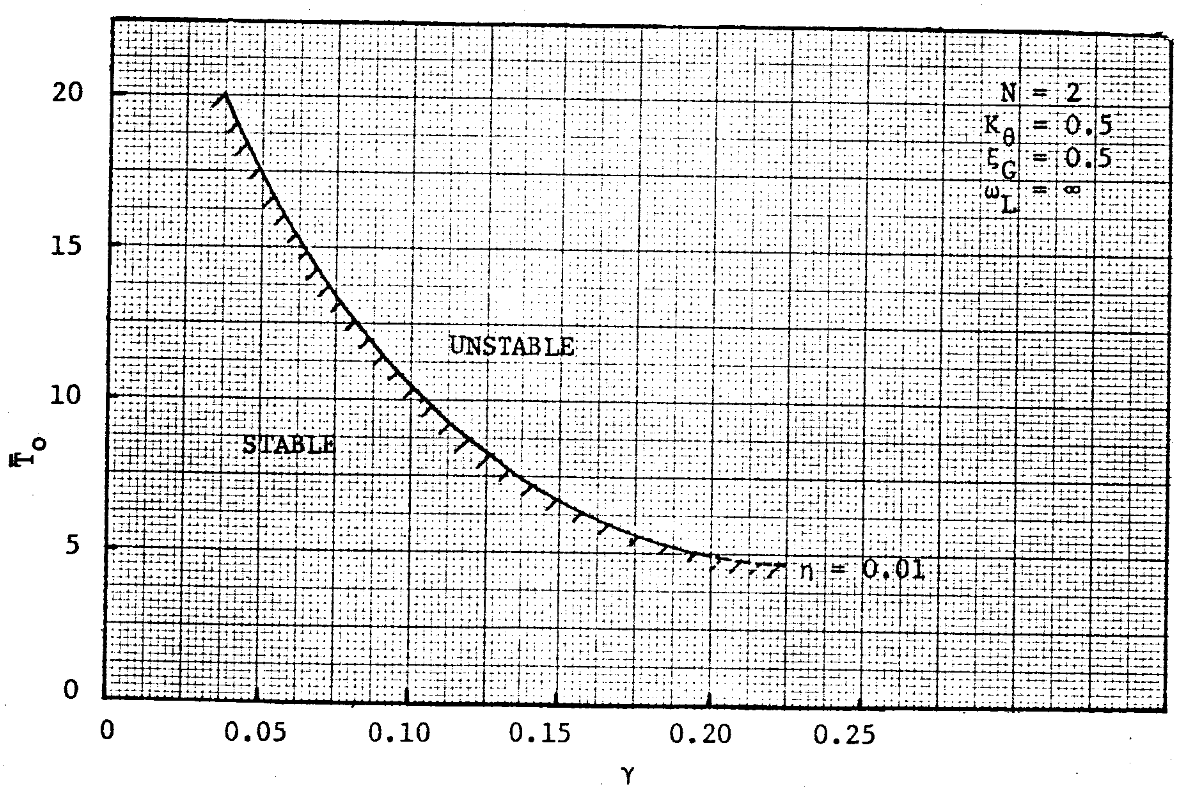


FIGURE 13.1. CRITICAL $\bar{T}_o$ AND γ AT $\bar{\Omega} = \bar{\omega}_{(2)} + \bar{\omega}_{(B)}$

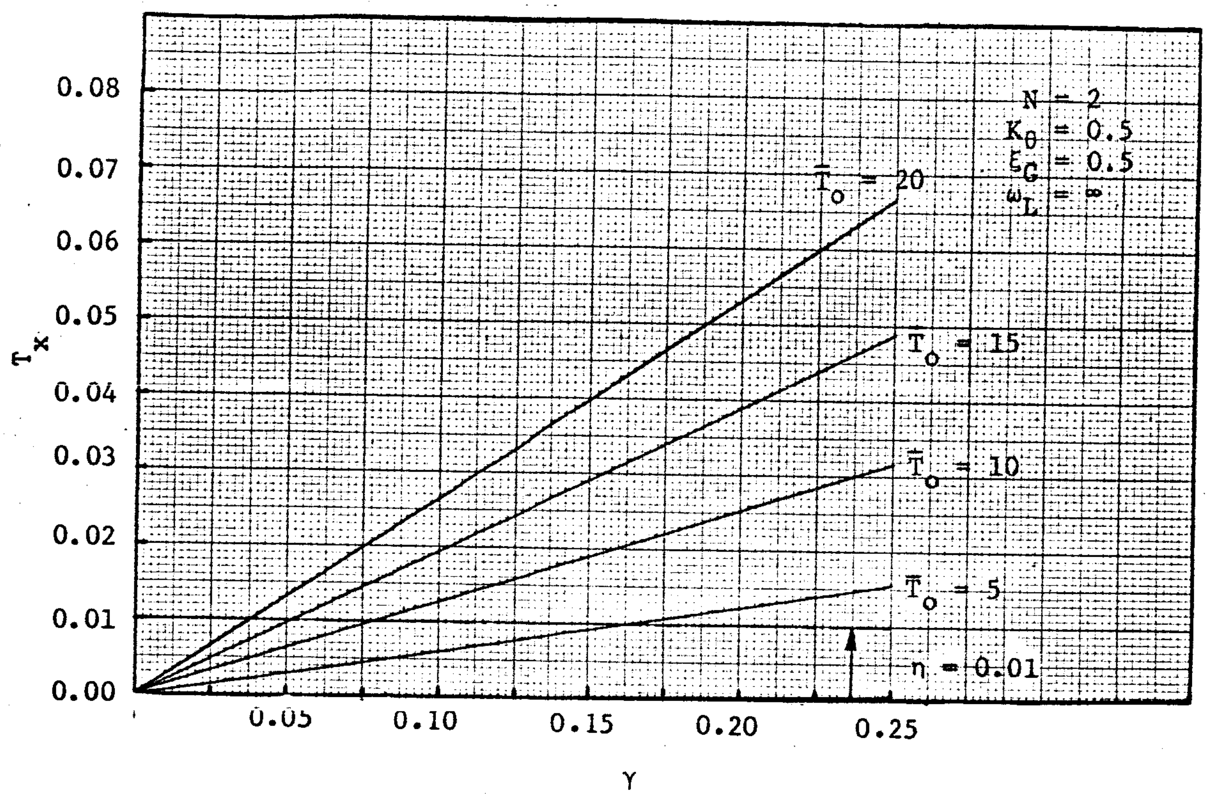


FIGURE 14.1. T_x VERSUS γ AT $\bar{\Omega} = 2\,\bar{\omega}_{(2)}$

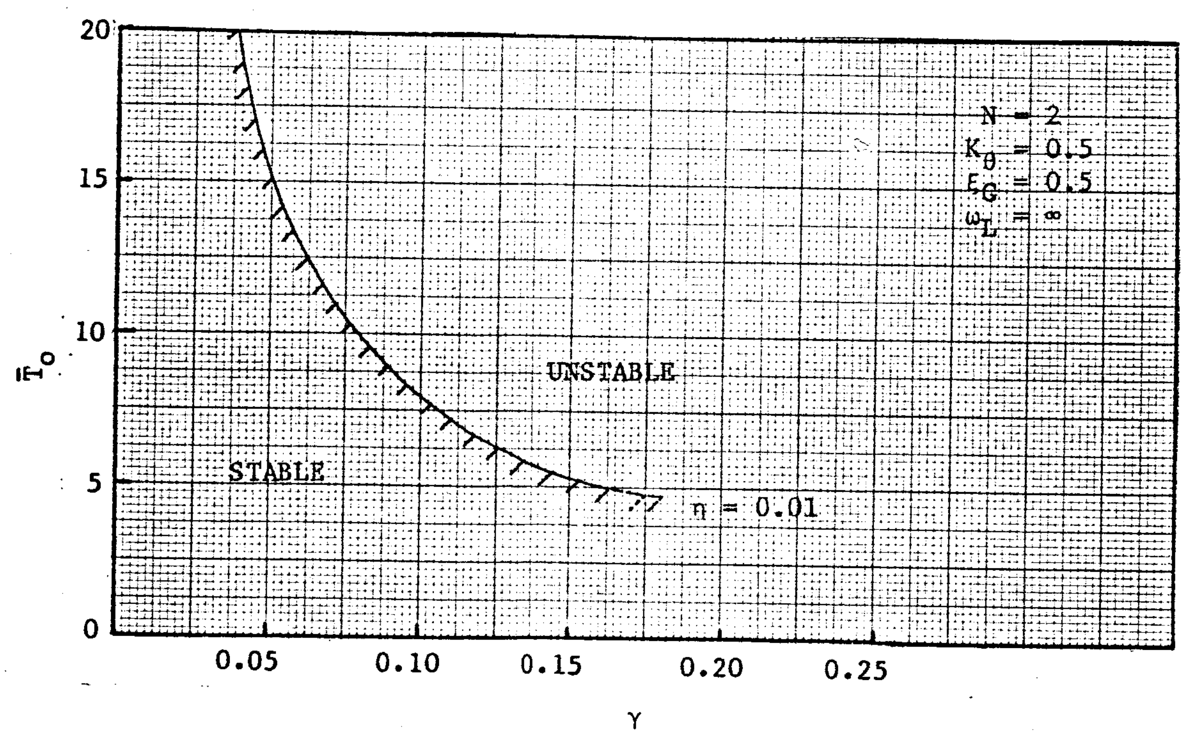


FIGURE 14.2. CRITICAL $\bar{T}_o$ AND γ AT $\bar{\Omega} = 2\bar{\omega}_{(2)}$

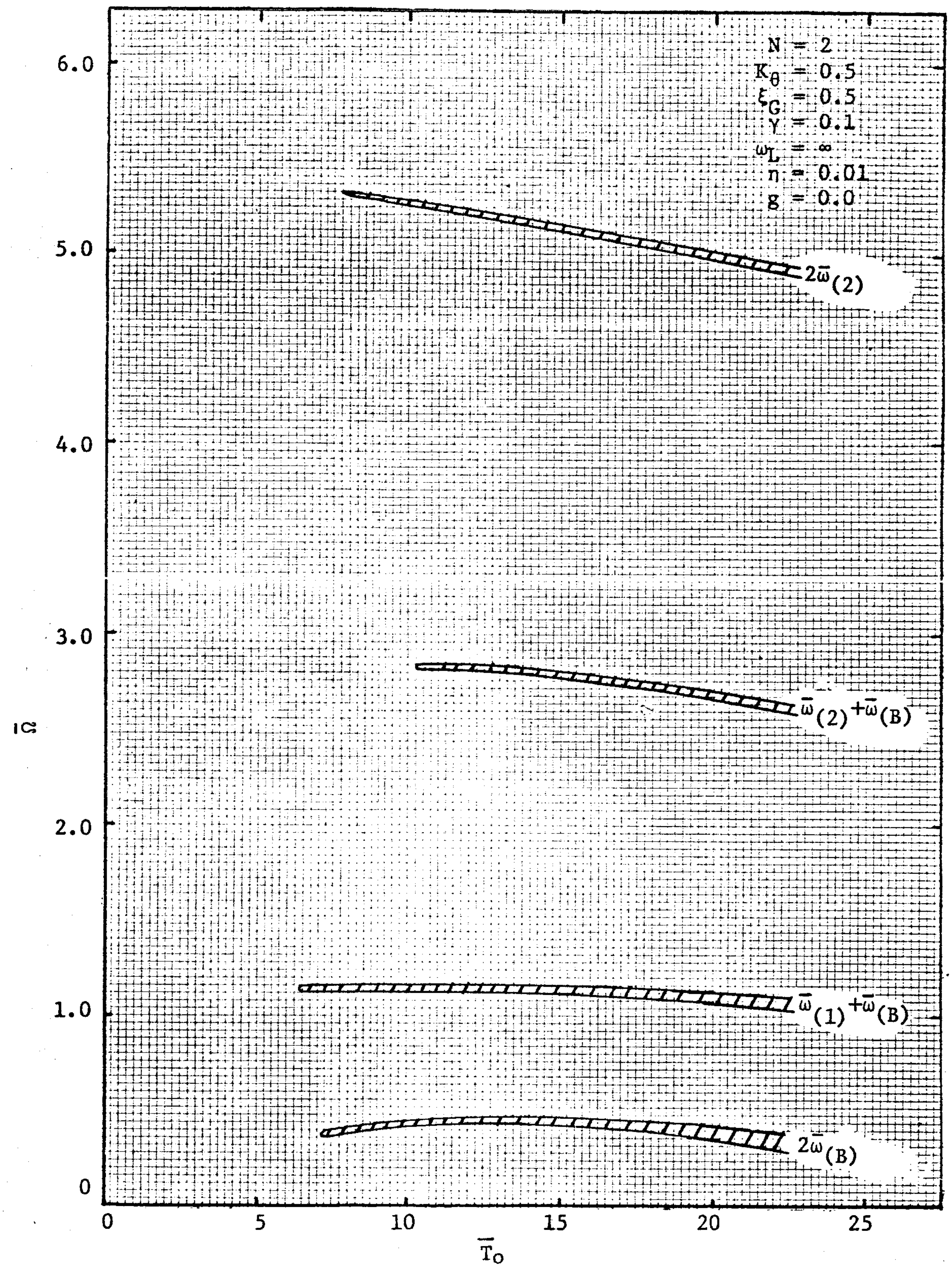


FIGURE 15. UNSTABLE REGIONS EXISTING WITH DAMPING (SEE FIGURE 8 FOR PLOT OF BOUNDARIES WITHOUT DAMPING) $\gamma = T_1/T_o$ HELD CONSTANT - UNIFORM BEAM

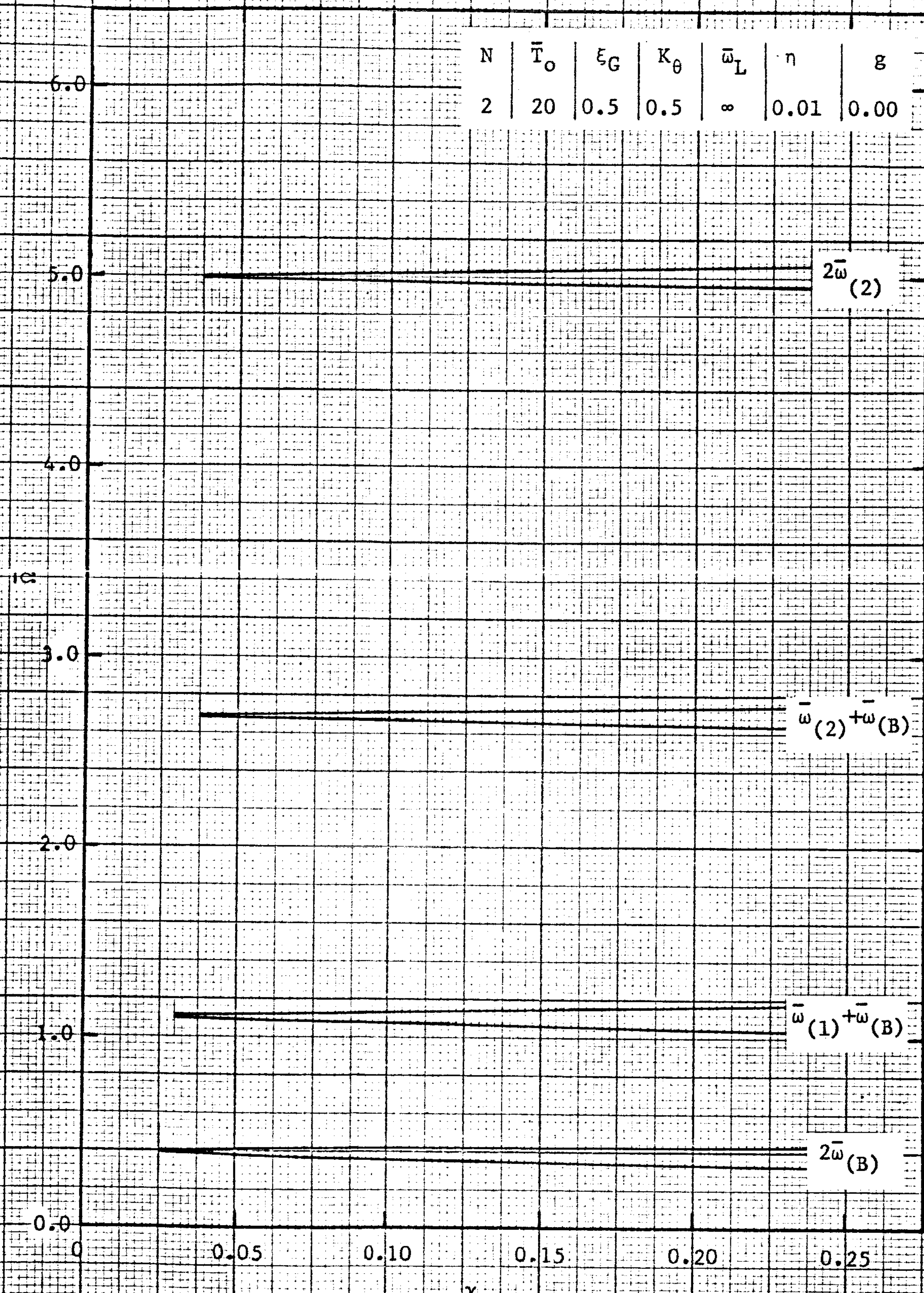


FIGURE 16. UNSTABLE REGIONS EXISTING WITH DAMPING (SEE FIGURE 10 FOR PLOT OF BOUNDARIES WITHOUT DAMPING) $\bar{T}_o$ HELD CONSTANT - UNIFORM BEAM

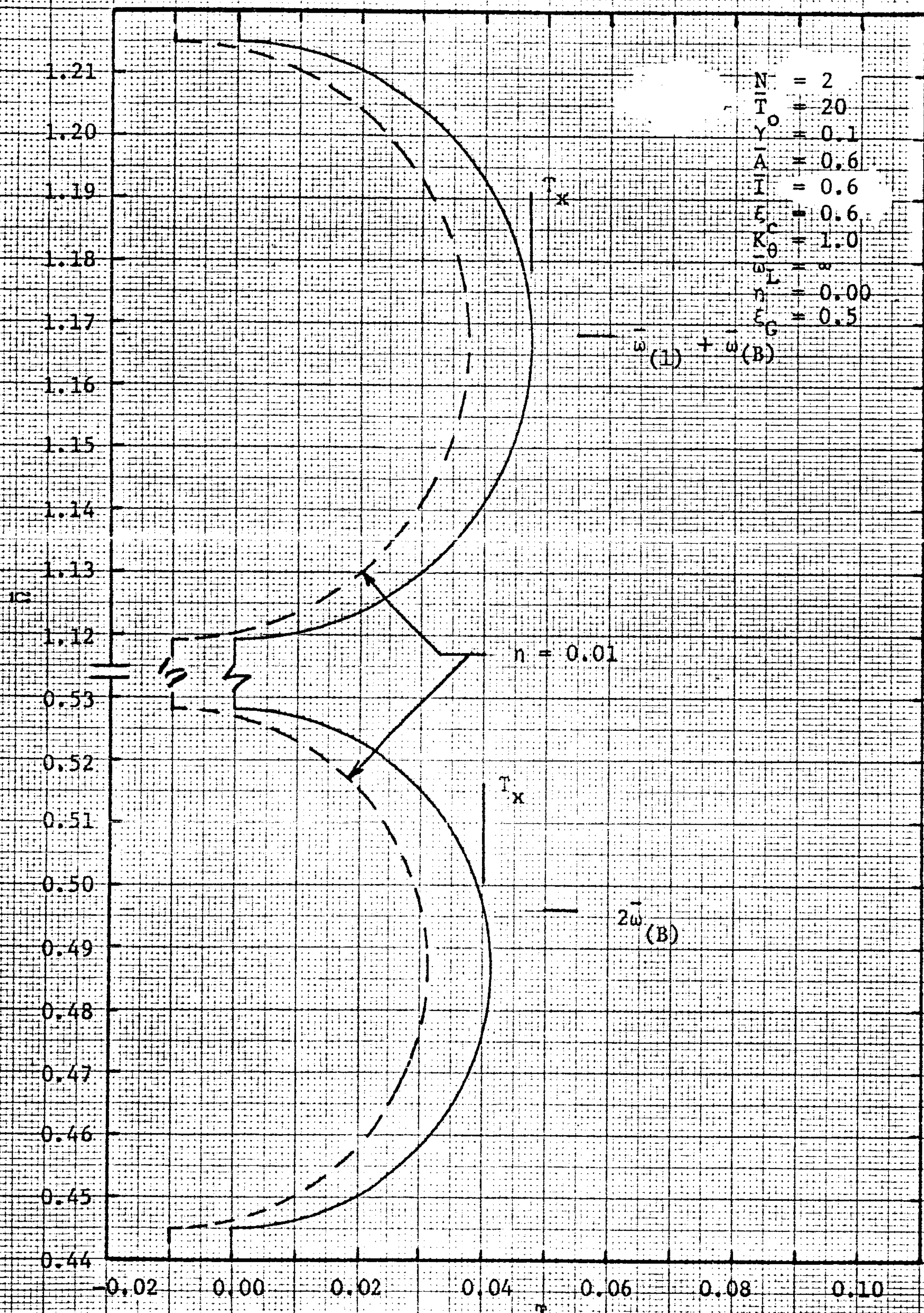


FIGURE 17. $\bar{\Omega}$ VERSUS T, PLOT OF CHARACTERISTIC ARGUMENT - STEPPED BEAM

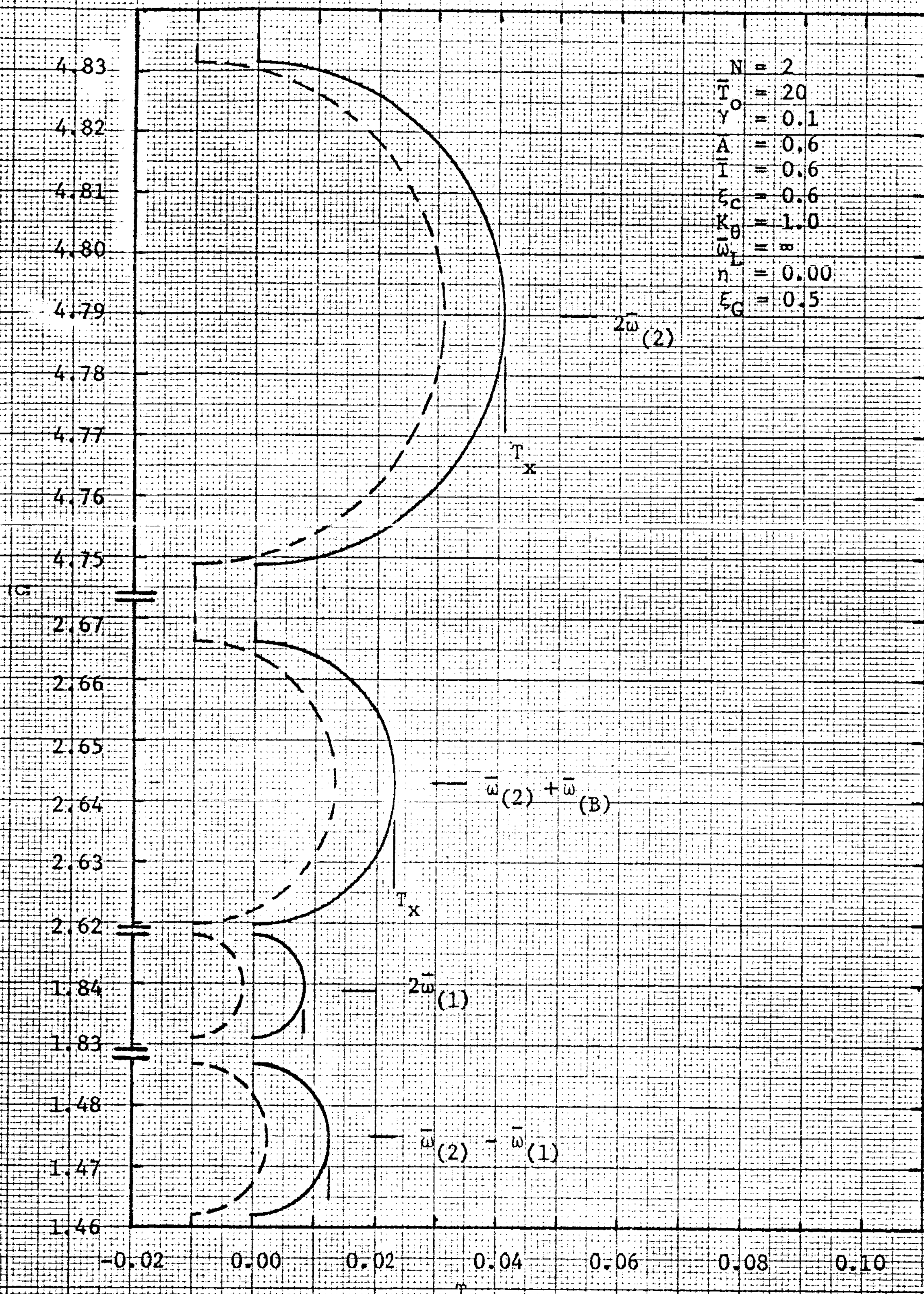


FIGURE 17. - CONTINUED

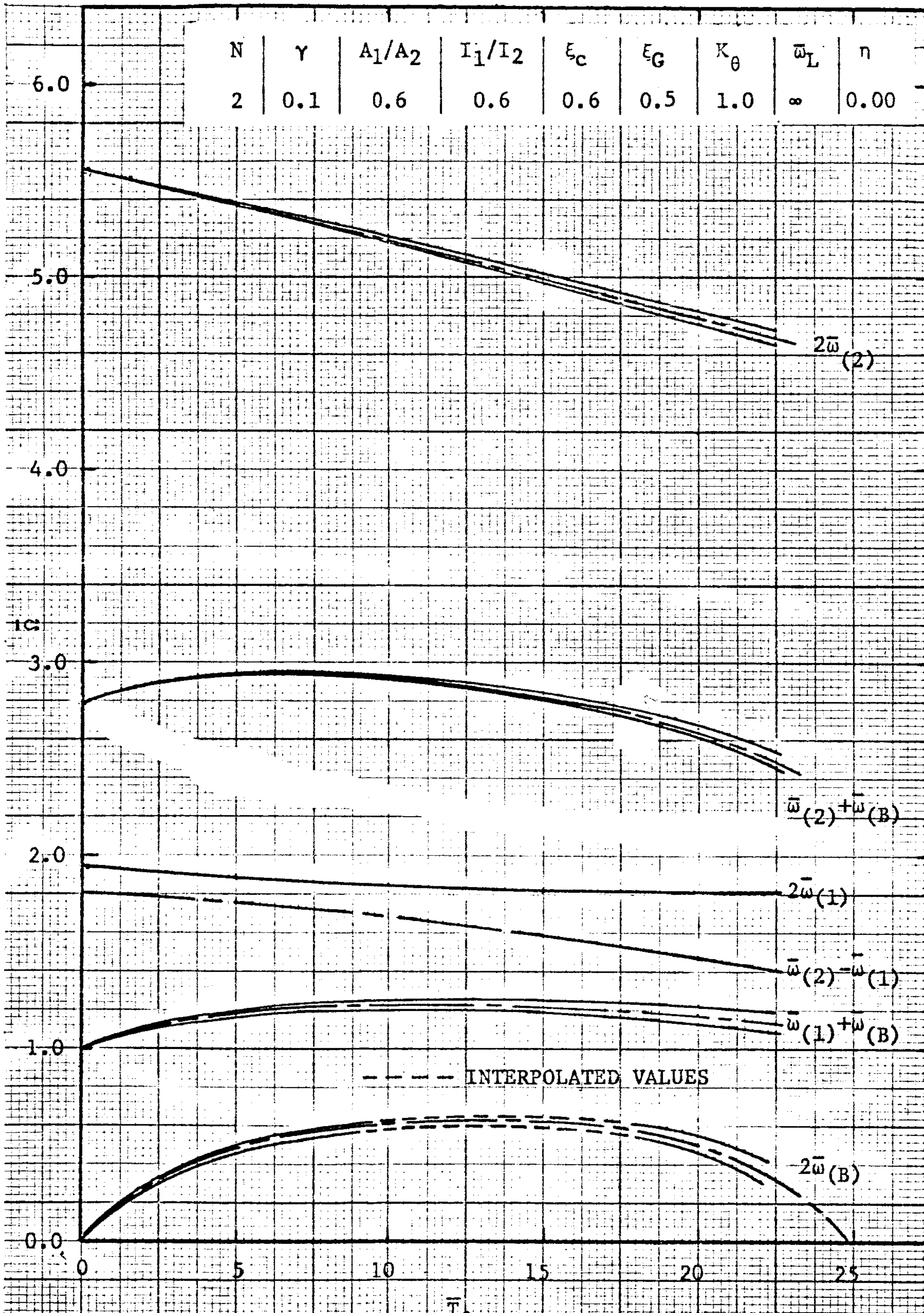


FIGURE 18. UNDAMPED BOUNDARIES SEPARATING STABLE AND UNSTABLE FORCING FREQUENCIES, $\gamma = T_1/T_o$ HELD CONSTANT - STEPPED BEAM

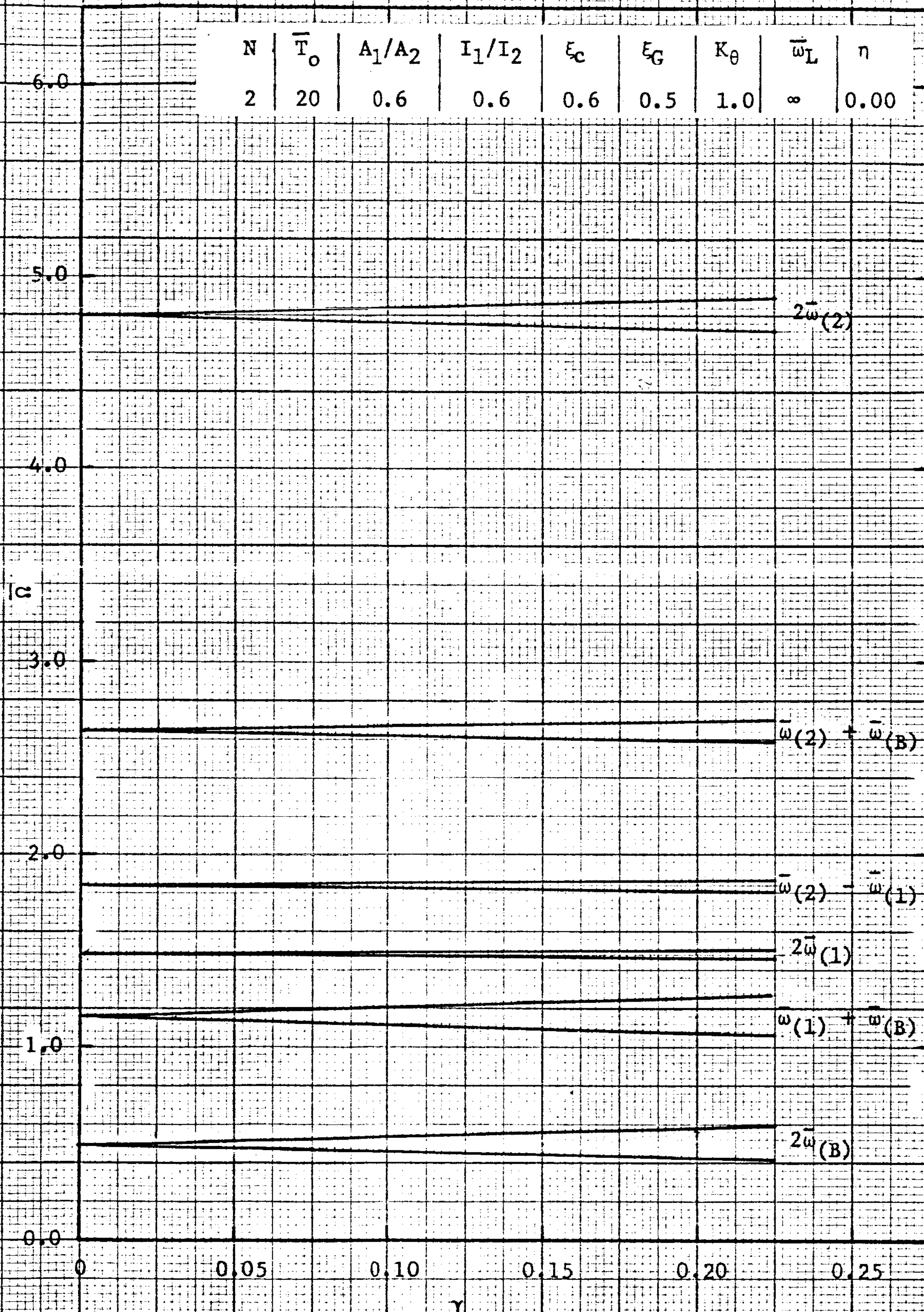


FIGURE 19. UNDAMPED BOUNDARIES SEPARATING STABLE AND UNSTABLE FORCING FREQUENCIES, T_o HELD CONSTANT - STEPPED BEAM

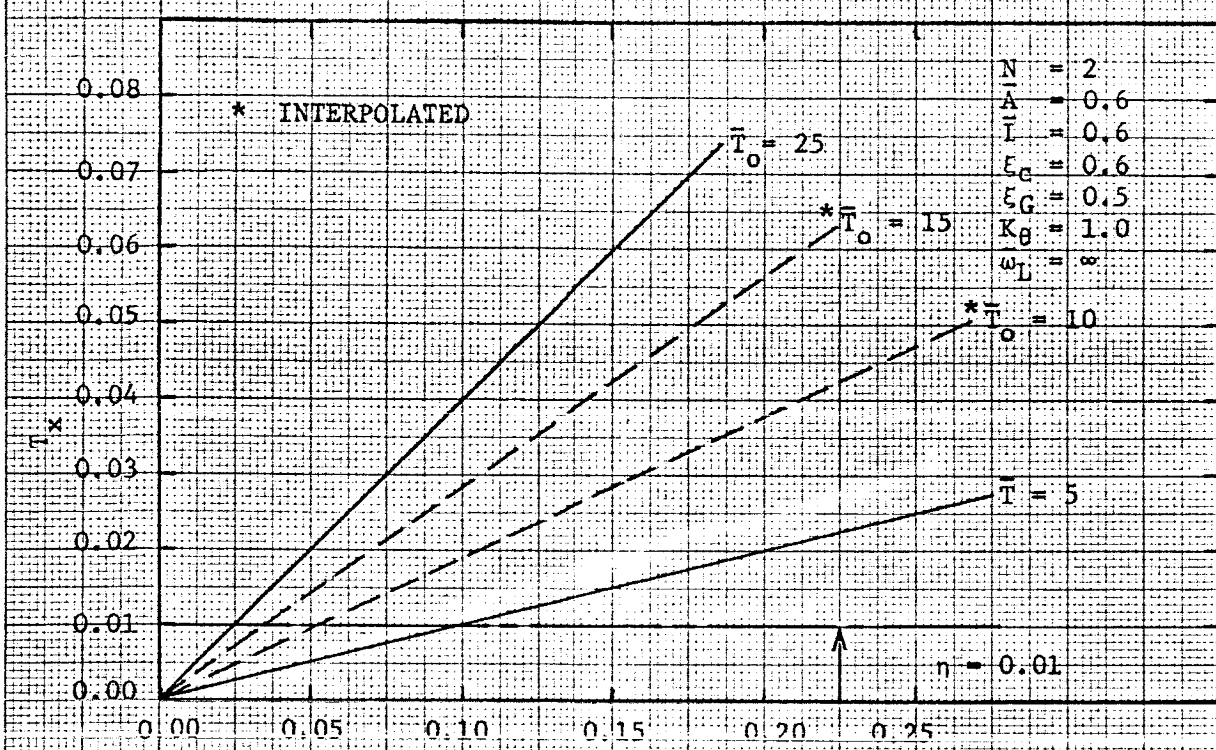


FIGURE 20.1. T_x VERSUS γ AT $\bar{\Omega} = 2\bar{\omega}_{(B)}$

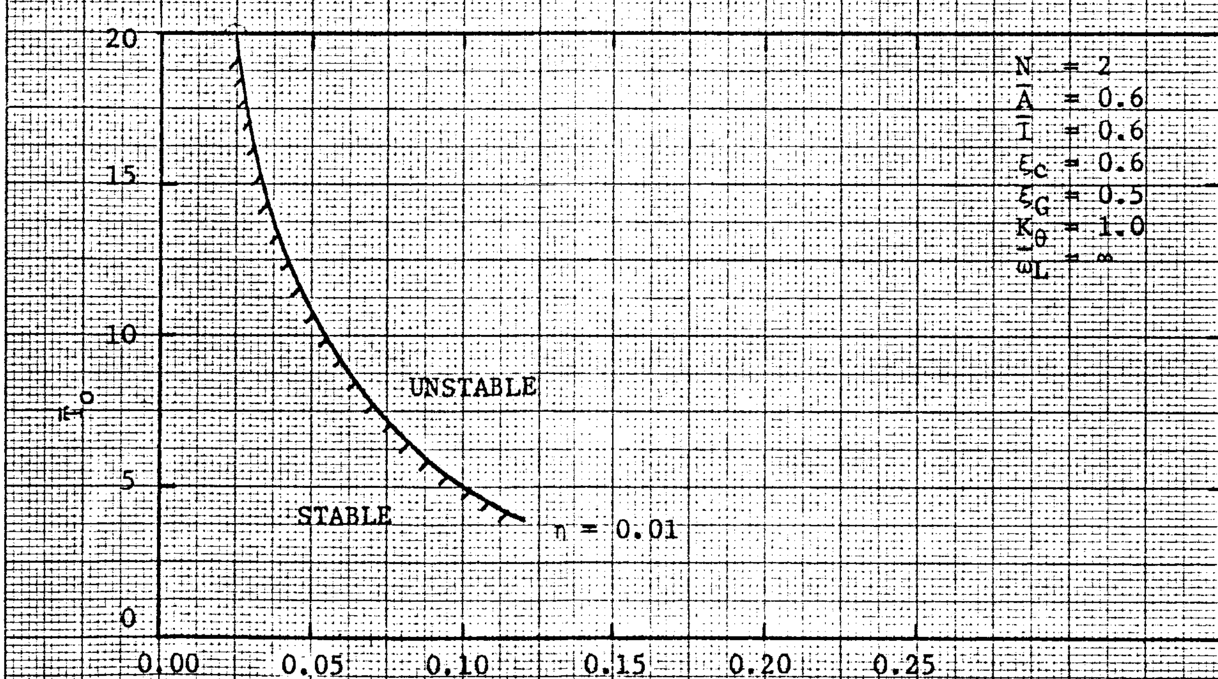


FIGURE 20.2. CRITICAL $\bar{T}_o$ AND γ AT $\bar{\Omega} = 2\bar{\omega}_{(B)}$

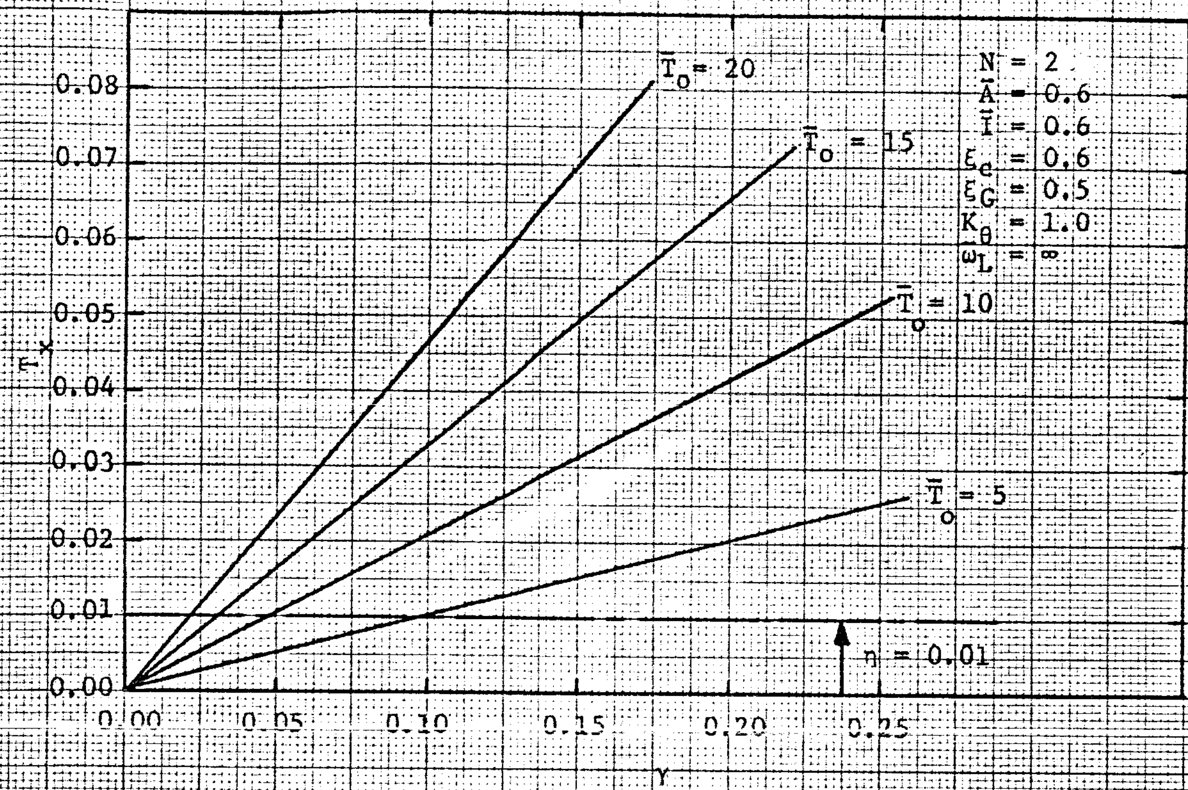


FIGURE 21.1. T_x VERSUS γ AT $\bar{\Omega} = \bar{\omega}_{(1)} + \bar{\omega}_{(B)}$

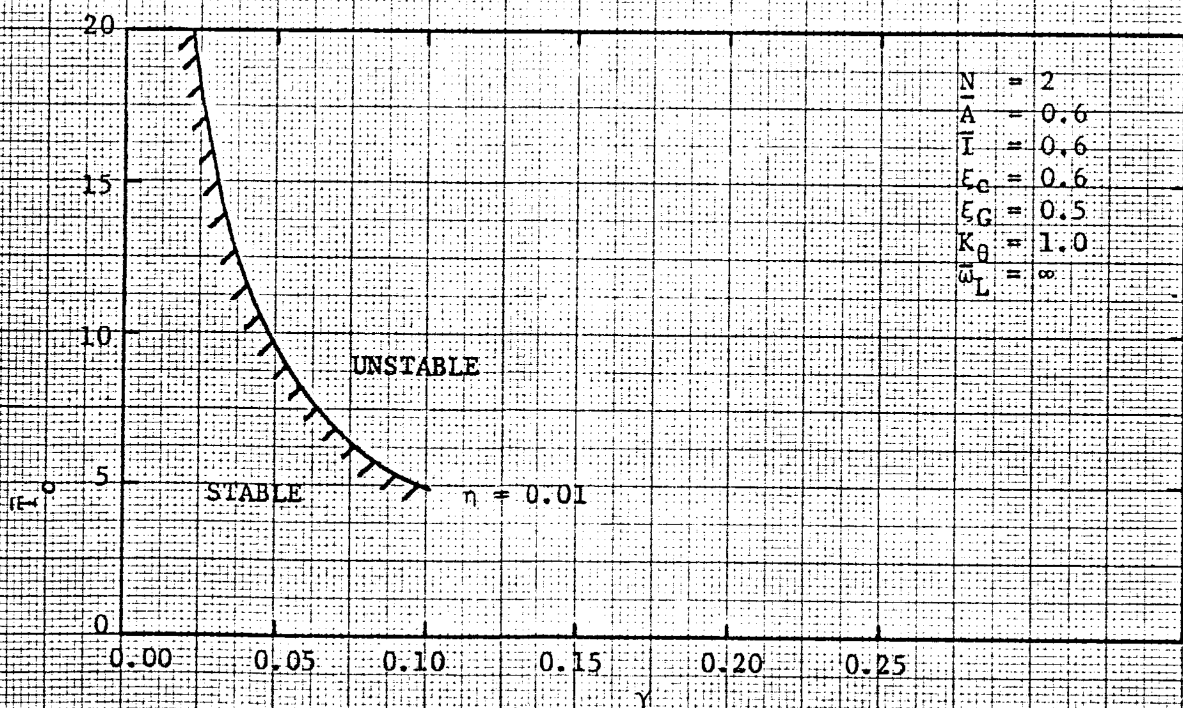


FIGURE 21.2. CRITICAL $\bar{T}_o$ AND γ AT $\bar{\Omega} = \bar{\omega}_{(1)} + \bar{\omega}_{(B)}$

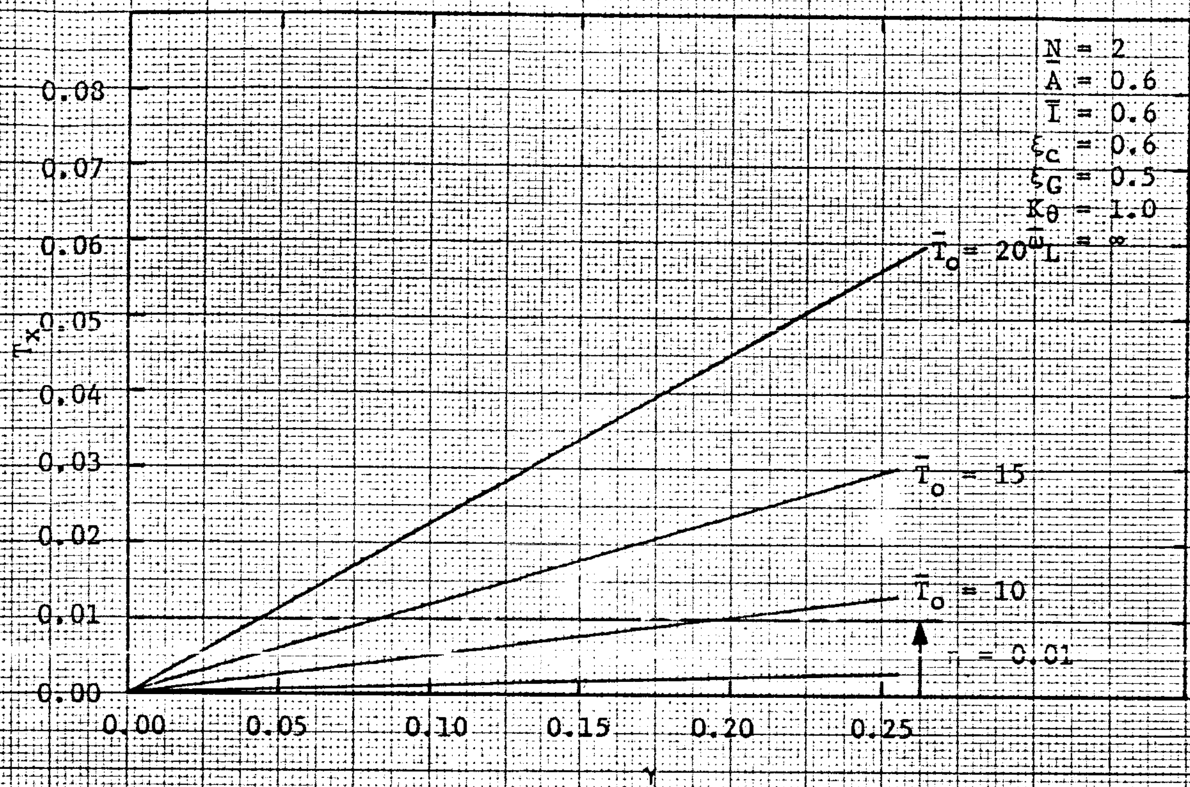


FIGURE 22.1. T_x VERSUS γ AT $\bar{\Omega} = \bar{\omega}_{(2)} + \bar{\omega}_{(B)}$

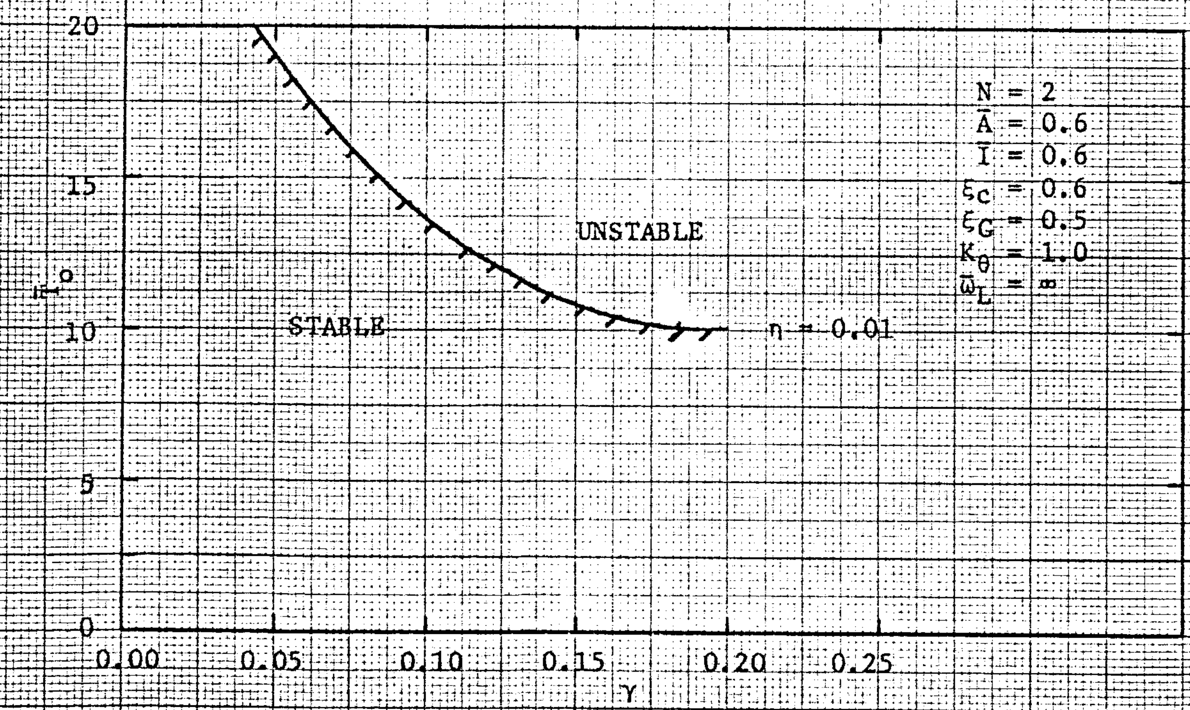


FIGURE 22.2. CRITICAL $\bar{T}_o$ AND γ AT $\bar{\Omega} = \bar{\omega}_{(2)} + \bar{\omega}_{(B)}$

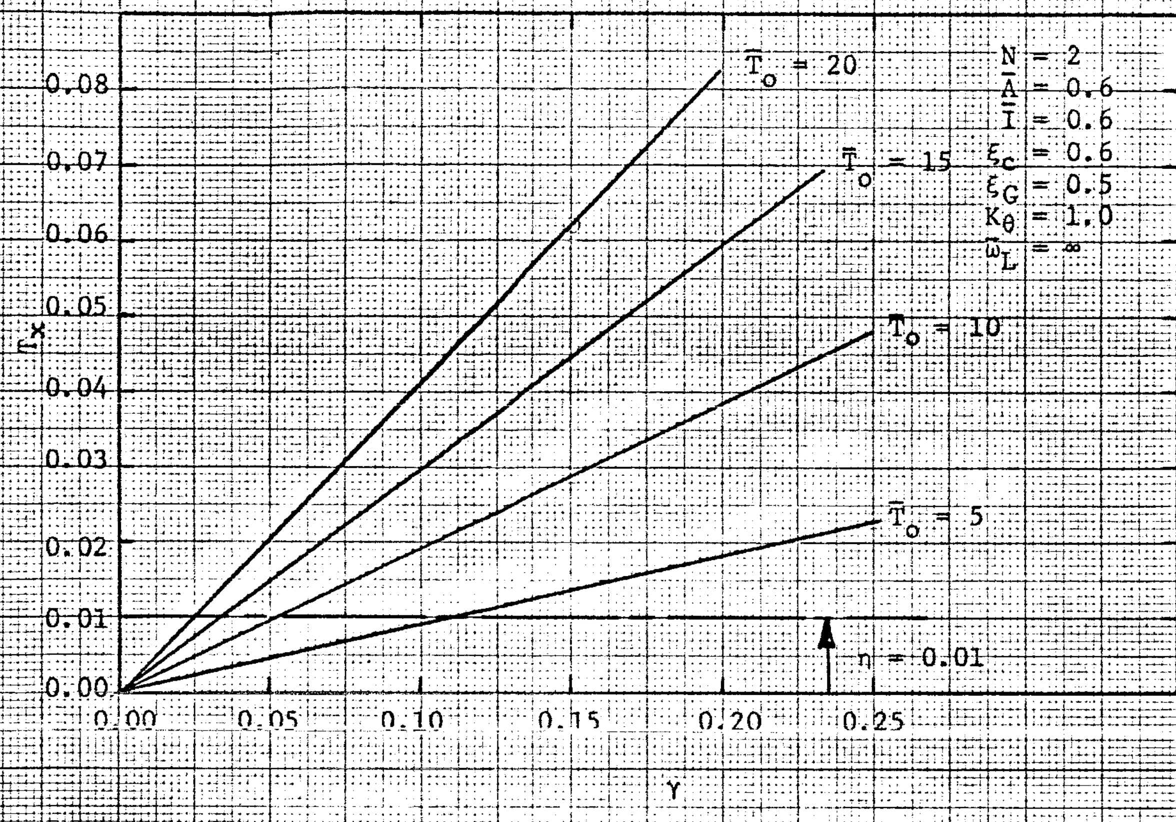


FIGURE 23.1. T_x VERSUS γ AT $\overline{\Omega} = 2\overline{\omega}_{(2)}$

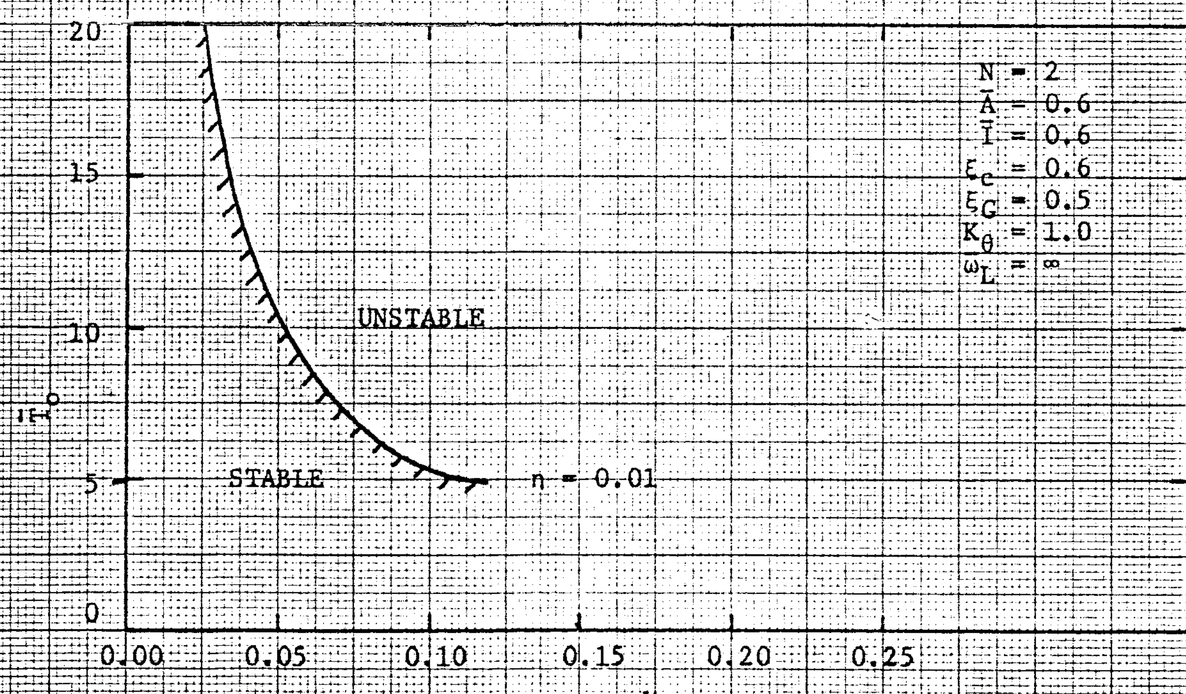


FIGURE 23.2. CRITICAL $\overline{T}_o$ AND γ AT $\overline{\Omega} = 2\overline{\omega}_{(2)}$

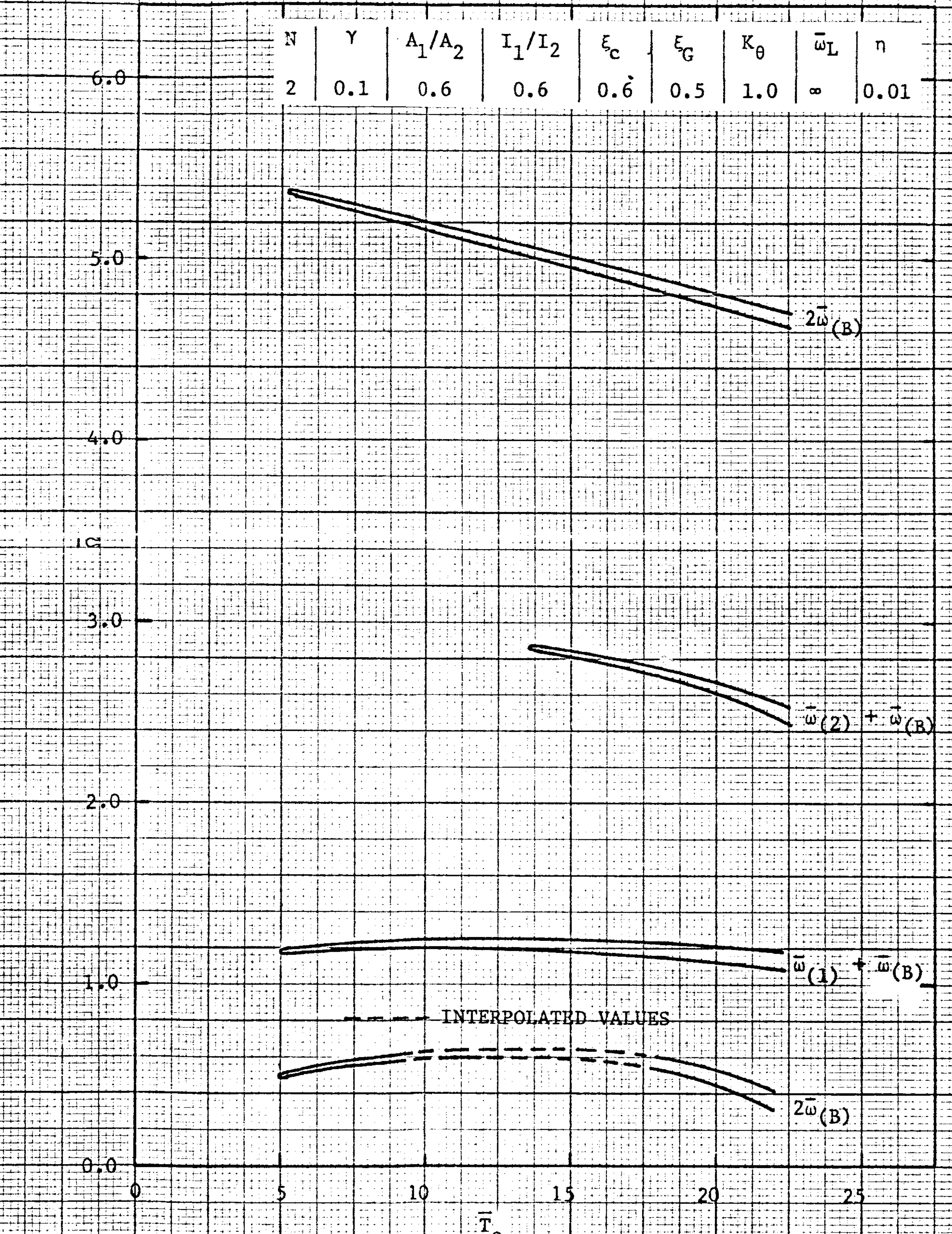


FIGURE 24. UNSTABLE REGIONS EXISTING WITH DAMPING (SEE FIGURE 17 FOR PLOT OF BOUNDARIES WITHOUT DAMPING) $\gamma = T_1/T_o$ HELD CONSTANT - STEPPED BEAM

N	$\bar{T}_o$	A_1/A_2	I_1/I_2	ξ_c	ξ_G	K_θ	$\bar{\omega}_L$	η
2	20	0.6	0.6	0.6	0.5	1.0	∞	0.01

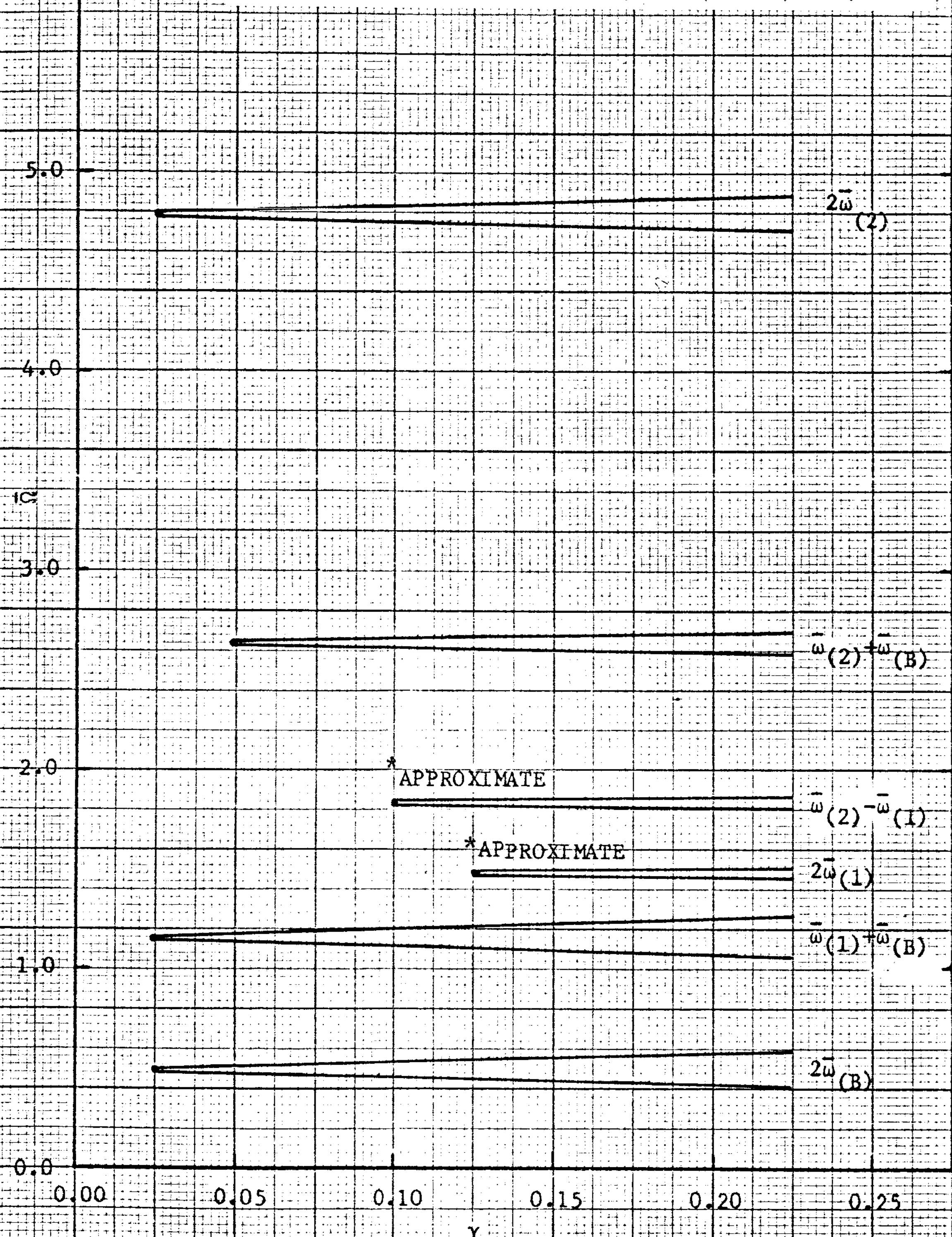


FIGURE 25. UNSTABLE REGIONS EXISTING WITH DAMPING (SEE FIGURE 18 FOR PLOT OF BOUNDARIES WITHOUT DAMPING) $\bar{T}_o$ HELD CONSTANT

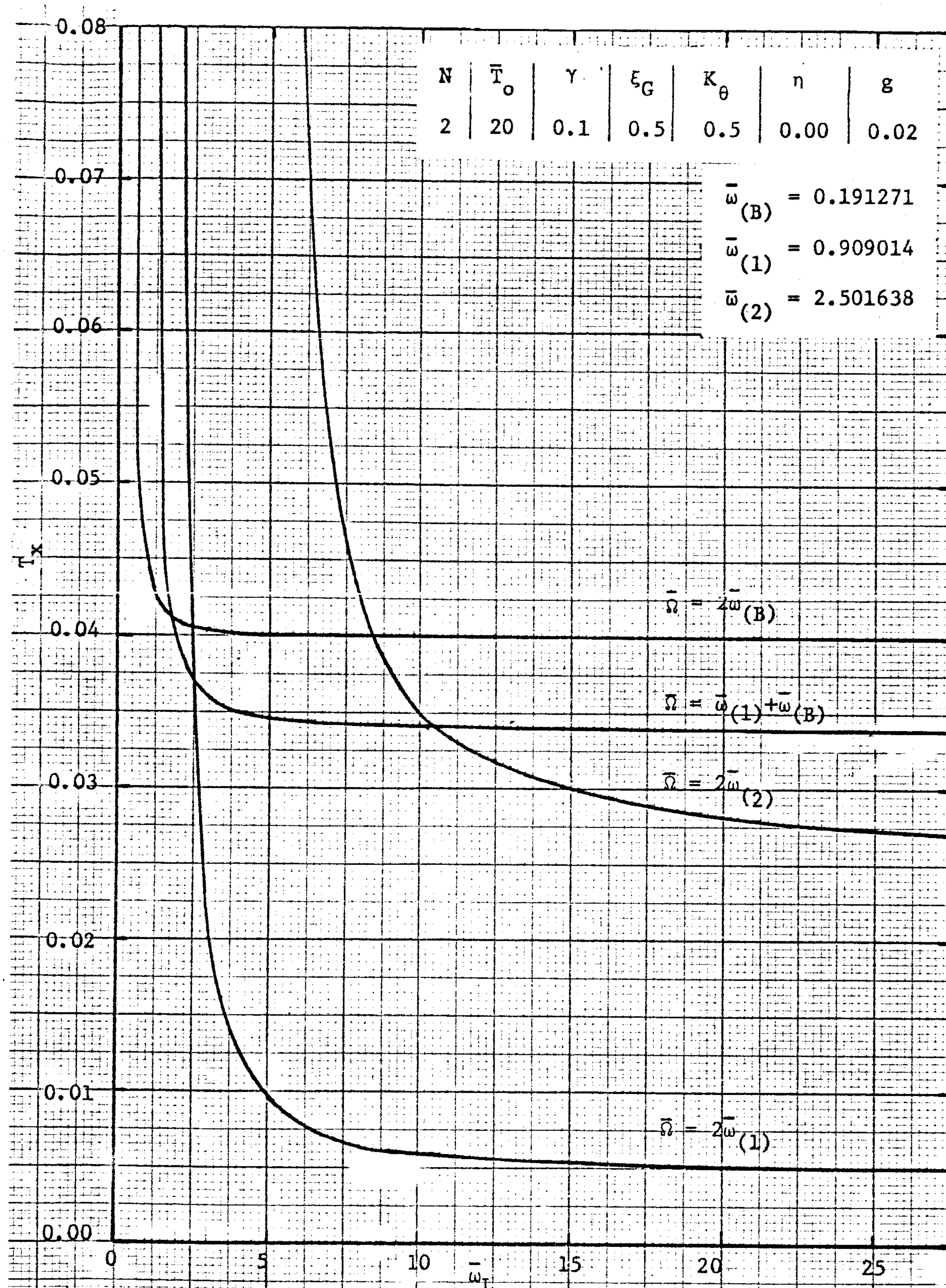


FIGURE 26. T_x VERSUS $\bar{\omega}_L$, EFFECT OF LONGITUDINAL COMPLIANCE UPON THE CHARACTERISTIC ARGUMENT FOR FORCING FREQUENCIES AT PARAMETRIC RESONANCE - UNIFORM BEAM

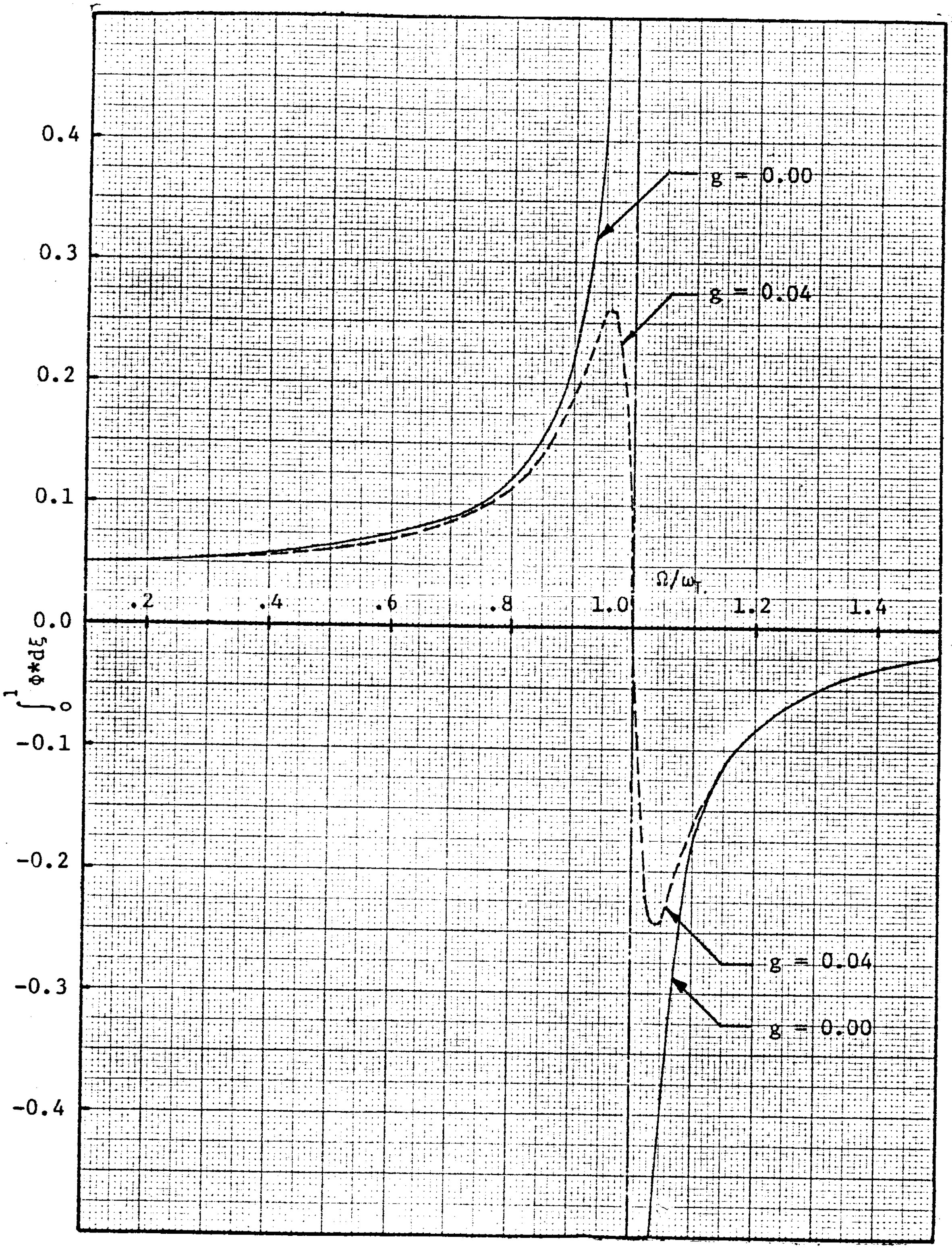


FIGURE 27. EFFECT OF LONGITUDINAL DAMPING UPON LONGITUDINAL COMPLIANCE NEAR RESONANCE - UNIFORM BEAM

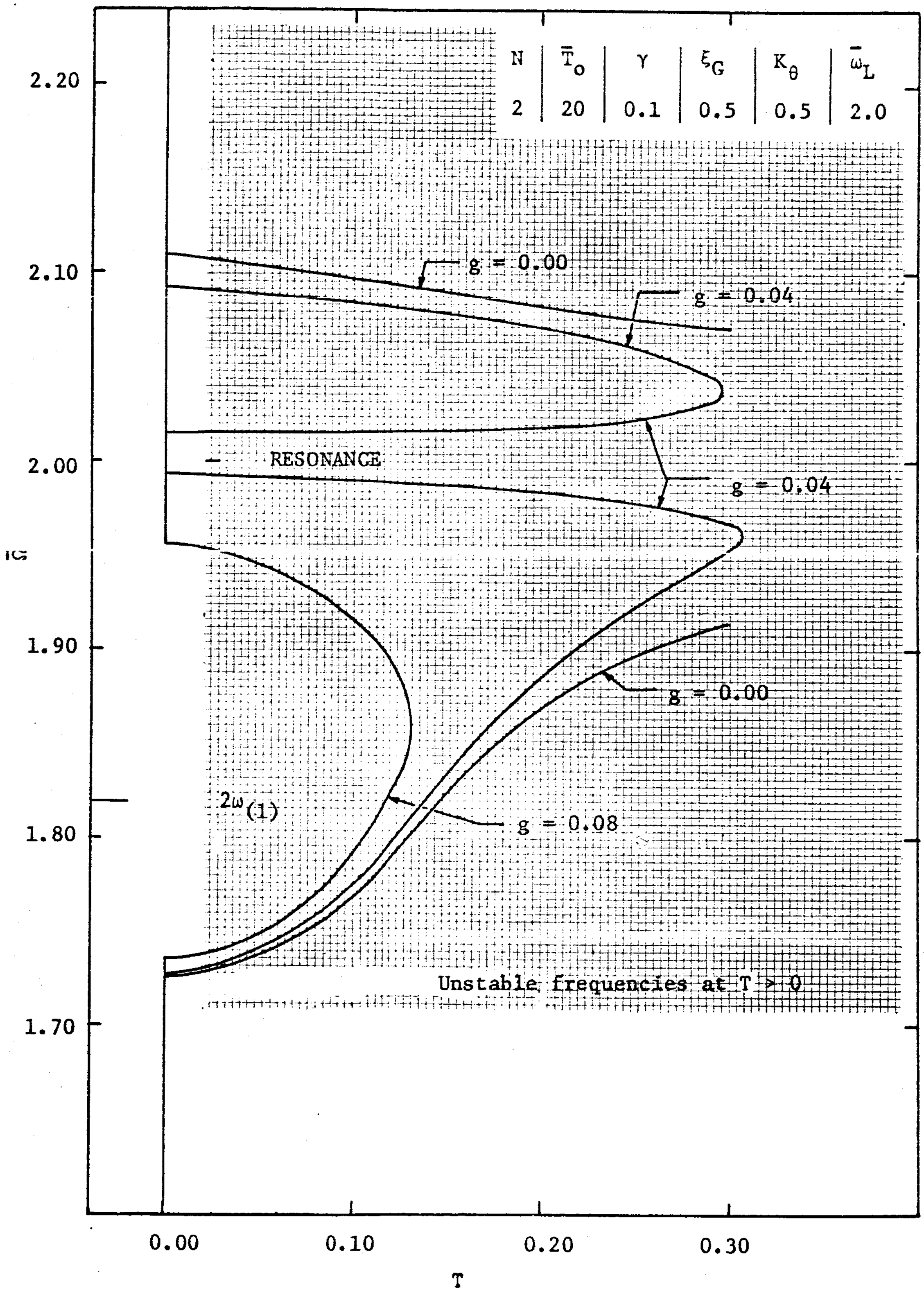


FIGURE 28. $\bar{\Omega}$ VERSUS T, PLOT OF CHARACTERISTIC ARGUMENT NEAR RESONANCE - UNIFORM BEAM

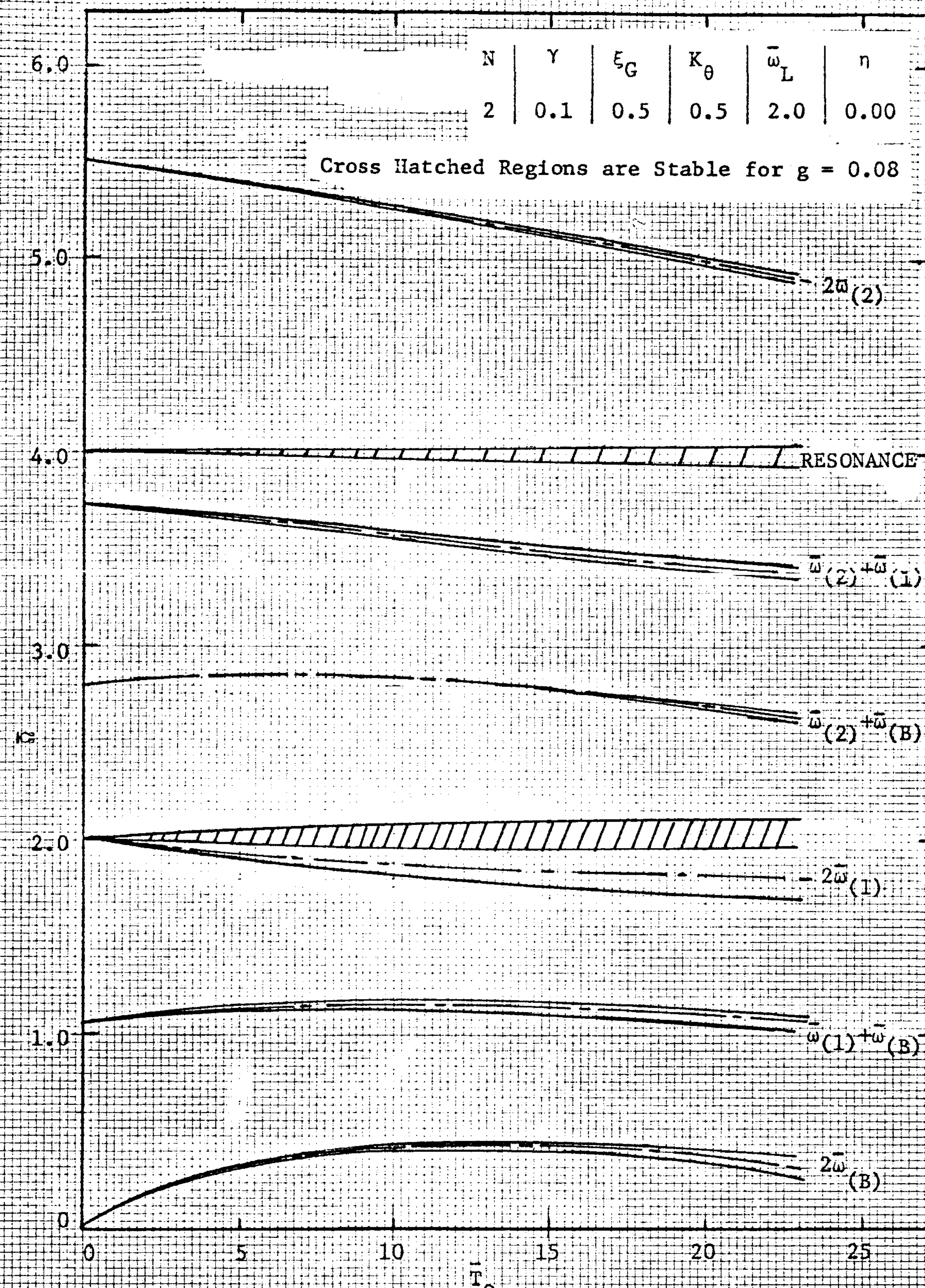


FIGURE 29. EFFECT OF LONGITUDINAL DAMPING UPON REGIONS OF INSTABILITY - UNIFORM BEAM

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5. Whittaker, E. T., and Watson, G. N., A Course of Modern Analysis. 4th ed., reprinted. London: Cambridge University Press, 1962.

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APPENDIX A

PROBLEMS

Two examples are presented which convert normalized parameters used in the presentation of results into terms and values that are familiar to the reader.

A very serious limitation exists in the direct application of the results of the analysis. A rocket vehicle is a complex structure whose structural weight is only a small percentage of the total. The assumption of an equivalent cylindrical beam accelerating through space due to some thrust could lead to incorrect conclusions if the mass of the propellant was not considered in evaluating natural frequencies.

In order to avoid calculation of a natural frequency by assuming an equivalent mass distribution which will compensate for the mass of the propellant, a natural frequency will be chosen which approximates the natural frequency of an actual rocket vehicle under zero thrust. These natural frequencies could be available from experimental investigations.

An equivalent stepped beam would be a better approximation than an equivalent uniform beam, however, certain assumptions must then be made regarding the ratio of the masses and the ratio of the moment of inertias between uniform sections. The location of the discontinuity must also be assumed. At best this must be an "educated guess" approximation.

Examples have been chosen within the range of parameters chosen in the illustration of results. The stepped beam stability evaluation considered only one beam geometry. Any other configuration would require computer utilization.

Consider a rocket vehicle under the same limitations and environment as the analytical model chosen. A typical fundamental frequency of a large rocket vehicle is approximately 2π rad/sec. The overall length of this vehicle is 4800 in. and the acceleration attained is 5g (g is the acceleration due to gravity). The rocket engine pulses at some arbitrary frequency, Ω, and the amplitude of the periodically-varying portion of the thrust is 10 percent of the constant thrust value, T_o, that is,

$$T = T_o + T_1 \cos\Omega t$$

and

$$\frac{T_1}{T_o} = 0.10.$$

Example 1: Uniform Beam Approximation

Assume that the mass and stiffness distribution of the vehicle described above can be approximated by a uniform beam. A reasonable thrust value can be established from the equation

$$T_o = m\ell a$$

where a is the longitudinal acceleration attained. From Eq. (4.4.15)

$$\lambda_n^4 = \omega_n^2 \frac{m\ell^4}{EI} .$$

$$\lambda_1^4 = 500.6, \quad \text{from tabulated results in (ref. 7).}$$

From Eq. (4.4.12)

$$\bar{T}_o = \frac{T_o \ell^2}{EI}$$

which can be written in terms of the acceleration as

$$\bar{T}_o = m\ell a \frac{\ell^2}{EI} = \frac{\lambda_1^4}{\omega_1^2} \frac{a}{\ell} \quad .$$

The thrust parameter can now be evaluated as

$$\bar{T}_o = \frac{(500.6)}{(2\pi)^2} \frac{(5g)}{(4800)} \; .$$

$$g = 386 \text{ in/sec}^2$$

$$\bar{T}_o = 5.1$$

The natural frequencies of the rocket under thrust can be obtained from the relationship

$$\omega_{(n)} = \bar{\omega}_{(n)} \, \omega_1$$

where $\bar{\omega}_{(n)}$ is obtained from the curves of Fig. 3 or Fig. 4, (something must be known about the sensor location, ξ_G, and the value of the control factor, K_θ).

A stability evaluation with damping can be made in Fig. 15. It is seen that no instabilities may exist for $\bar{T}_o = 5.1$, for any forcing frequency.

Example 2: Stepped Beam Approximation

Assume that the mass and stiffness distribution of the vehicle is best approximated as two uniform sections (see Fig. 1(b)). Some assumption must also be made regarding the change in mass and the change in moment of inertia between the assumed uniform sections. Therefore let it be assumed that

$$\frac{m_1}{m_2} = 0.6$$

$$\frac{I_1}{I_2} = \bar{I} = 0.6$$

but

$$m(x) \propto A(x) \qquad \text{from Eq. (5.1.4)}$$

therefore

$$\frac{m_1}{m_2} = \frac{A_1}{A_2} = \bar{A} = 0.6 \ .$$

The discontinuity is assumed to be located at

$$\xi_c = 0.6 .$$

The fundamental bending frequency of the vehicle was taken as 2π rad/sec. Unlike the uniform beam approximation this frequency can not be applied directly in determining $\bar{T}_o$. Figure 7 plots normalized natural frequencies versus the applied thrust parameter $\bar{T}_o$. At $\bar{T}_o = 0$ the normalized natural frequency of first bending is plotted as

$$\bar{\omega}_{(1)} = 0.974$$

but

$$\bar{\omega}_{(1)} = \frac{\omega_{(1)}}{\omega_1}$$

therefore,

$$\omega_1 = \frac{\omega_{(1)}}{\bar{\omega}_{(1)}} .$$

Since $\omega_{(1)} = 2\pi$ is the true natural frequency of the rocket at zero thrust the normalization constant ω_1 can then be calculated as

$$\omega_1 = \frac{2\pi}{0.974} = 6.55.$$

The thrust required to impart the desired acceleration can determined from the equation

$$T_o = M_T\ a$$

where

M_T = total mass of the vehicle

a = longitudinal acceleration.

An expression for $\bar{T}_o$ in terms of T_o is given by Eq. (5.4.6) as

$$\bar{T}_o = \frac{T_o \ell^2}{EI_2}$$

then

$$\bar{T}_o = M_T\ a \frac{\ell^2}{EI_2} = \frac{M_T a}{m_2 \omega_1^2 \ell^2}\ \omega_1^2 \frac{m_2 \ell^4}{EI_2} .$$

$$\omega_1^2 \frac{m_2 \ell^4}{EI_2} = \lambda_1^4 = 500.6 \quad \text{(ref. 7)}$$

$$\frac{M_T}{m_2 \ell} = \frac{m_1 \ell + (m_2 - m_1)(\ell - \ell_1)}{m_2 \ell} = 1 - \xi_c (1 - \bar{A}).$$

Therefore,

$$\bar{T}_o = [1 - \xi_c(1-\bar{A})] a \frac{\lambda_1^4}{\omega_1^2 \ \ell}$$

$$= [1 - 0.6 \ (1-0.6)] \ \frac{(500.6)(5)(386)}{(6.55)^2(4800)}$$

$$= 3.56 .$$

A stability evaluation with damping can now be made from Fig. 24.

APPENDIX B

COMPUTER PROGRAMS

B-1 General

This appendix presents two FORTRAN IV Computer Programs that are operationally compatible with the FORTRAN processors of the IBM 7094 and the UNIVAC 1107 digital computers.

Independent computer programs were written for the uniform and the stepped beam analyses. The stepped beam computer program is adaptable to the uniform beam analysis but does not have the capabilities of handling longitudinal damping or initial condition considerations.

The computers capacity to perform complex algebra internally was a great asset in evaluating integrals with complex arguments. Complex arguments result in application of Eq. (4.5.34) to integral evaluation with or without the inclusion of longitudinal damping. The real part of the result could be retained as a solution. It was found that a high degree of accuracy was attainable where the results could be checked by formulas.

Computer capacity became a problem as more degrees of freedom were used in the assumed solution. As presently programmed, only two bending modes and a rotational mode, that is, three degrees of freedom are taken. A choice of more degrees of freedom could exceed the capacity of the computer for stability analysis of forcing frequencies near zero.

This is evident by examining the convergence criteria established for the infinite determinant in Section 6.3. A maximum of four degrees of freedom could be handled with slight modification to the program (see B-5).

B-2 Program Variables

B-2.1 Input Variables Used Only in the Stepped Beam Program

Input Variable	Program Symbol	Increment To	Increment By	Range of Variable MIN. MAX.
ξ_G	XG	XGMAX	DELXG	0.0, $\overset{\rightarrow}{0.1}$, 1.0
ξ_c	RC	RCMIN	DELRC	0.0, $\overset{\leftarrow}{0.1}$, 1.0
$\bar{A}$	RA	RAMIN	DELRA	0.0, $\leftarrow$, ——
$\bar{I}$	RI	RIMIN	DELRI	0.0, $\leftarrow$, ——

B-2.2 Input Variables Used Only in the Uniform Beam Program

Input Variable	Program Symbol	Increment To	Increment By	Range of Variable
g	GA	GMAX	DELG	0.0, $\rightarrow$, ——

B-2.3 Input Variables Common to the Stepped Beam Program and the Uniform Beam Program

Input Variable	Program Symbol	Increment To	Increment By	Range of Variable
$\bar{T}_o$	TO	TOMAX	DELTO	0,0, $\rightarrow$, ——

Input Variable	Program Symbol	Increment To	Increment By	Range of Variable
ω_L	OMEGL	OLMAX	DELOL	0.0, →, ——
η	ETA	ETAMAX	DELETA	0.0, →, ——
K_θ	SPC	SPCMAX	DELSPC	0.0, →, ——
γ	GAMMA	GAMAX	DELGA	0.0, →, ——

B-2.4 Output Variables

Program Symbol	Equivalent	Definition
FREQ(1)	$\bar{\omega}_{(B)}$	Rotational frequency
FREQ(2)	$\bar{\omega}_{(1)}$	1st bending frequency
FREQ(3)	$\bar{\omega}_{(2)}$	2nd bending frequency
ROOTR	Real z	See Eq. (5.3.8)
ROOTI	Imag z	See Eq. (5.3.8)
STAB1	T	Characteristic argument see Eq. (7.1.1)
OMEGB	$\bar{\Omega}$	Forcing frequency

B-2.5 Program Control Variables

Program control symbols establish program branch routes to be used, output formats written, and the reading of additional data cards which change the preceding input variables. The control symbols and their function are given below.

Program Symbol	Numerical Code	Function
ITR	0	STAB is called from DRIVER
	1	SWITCH is called from DRIVER
NAT	1	Computes Natural Frequencies only
	0	Stability evaluation
ICOUNT	0	READ additional data cards
	1	Read additional data cards after location 401
	2	Stop at completion of original set of input variables.

B-3 Usage

The preceding section lists all program variables which may be incremented within the program DRIVER (see B-5 and B-6). The range over which a variable can be incremented is given by the columns under min and max signifying the minimum and maximum values permissible. The direction arrow over a numerical value in the column between the min and max columns is the permissible increment and the direction of incrementation; i.e., from a low to high (→) or from a high to low (←) value. A dash (-) indicates that any positive numerical value is acceptable.

For example, consider the variable which locates the sensor location. This parameter is denoted by the symbol XG. If it is necessary to hold XG constant at, say XG = 0.6 throughout a computer run then the data card corresponding to XG must be filled out according to the format allocated to XG, XGMAX, and DELXG. In this case

XG = 0.6

XGMAX = 0.6

and DELXG = 0.1. (Note: DEL--- is greater than 0.001)

If, instead it is required that XG be incremented during a run then the following procedure would be followed. Suppose that it is necessary that XG be run at XG = 0.6 and XG = 0.8, then the data card corresponding to XG must be filled out as follows:

XG = 0.6

XGMAX = 0.80

DELXG = 0.2 .

The program variables RC, RA, and RI are incremented downward. When RC, RA, and RI are followed by MIN, i.e., RCMIN, this represents the value to which the parameter is incremented. When preceded by DEL, the symbol refers to the amount by which the parameter is incremented. If for example, it is required that RC be taken at 1.0, 0.8, and 0.6 then

RC = 1.0

RCMIN = 0.60

DELRC = 0.2.

The program has three output formats written into the program (see, for example, B-5).

Format (1) tabulates three natural frequencies which exist for some input $\bar{T}_o$ value. The thrust and resultant frequencies are tabulated in columns under the appropriate heading. Pertinent data constants are also included in the heading. To obtain this type of table set NAT = 1 and fill out the data card for TO according to the increments of TO desired.

Format (2) is a tabulation of the stability evaluation for a forcing frequency existing at parametric resonance. This format is obtained by setting NAT = 0 and ITR = 0.

Format (3) is a tabulation of the stability evaluation for a range of forcing frequencies existing at and around parametric resonance. This format is used in conjunction with SUBROUTINE SWITCH, a routine that automatically increments OMEGB if the initial value of OMEGB is unstable. This format is obtained by setting NAT = 0 and ITR = 1.

The control variable ICOUNT may be employed for extended parameter investigations. For example, it becomes necessary to redefine program input variables for an extended computer run. In this case set ICOUNT = 0 and furnish the appropriate READ and data cards to the source deck. If the input variables do not change set ICOUNT = 1.

B-4 Restrictions and Recommendations

As presently programmed only two bending degrees of freedom (N = 2) may be used without modification of the programs. SUBROUTINE STAB contains the formulation necessary in expanding Eq. (6.3.6) into a form suitable for the polynomial root solver SUBROUTINE MULLER. Additional formulation would be required for $N \neq 2$.

Longitudinal coupling effects were found to be negligible for OMEGL > 50.0. The program is most efficient for OMEGL > 50.0 since a number of computations are eliminated.

The stability evaluation for small forcing frequencies (OMEGB < 1.0) may be in error due to truncation of terms in the evaluation of the infinite determinant in Eq. (6.3.5). A warning will be given in this case and the program will continue to operate. The computational time required at smaller forcing frequencies (OMEGB < 1.0) is also much greater.

Several library routines were used in matrix and determinant operations. These routines are listed below with the operations performed.

SUBROUTINE	PURPOSE
HESSEN	Evaluation of the eigenvalues of a non-symmetric matrix
GASDET	Evaluation of a real determinant
MULLER	Polynomial root finder
GASINV	Inverse of a real matrix

Any routines which perform the same functions can be substituted into the program.

B-5 Uniform Beam Computer Program

The computer program used in the analysis of a uniform beam is listed in this appendix. Library subroutines are listed by name with variable array dimensions without a complete listing. Program input data is listed at the end of the program with the resultant output listed immediately following.

The output formats are listed as format (1), format (2), and format (3). These formats are obtained in one computer run using the program source deck and data deck listed.

```
C       PROGRAM DRIVER
        DIMENSION R(5),B(5),PH(5),DPH(5),DPHG(5)
        DIMENSION F(8,8),G(8,8),H(5,8,8),RTR(8), RTI(8)
        DIMENSION FF(8,8),FG(8,8),FH(5,8,8)
        COMMON R,B,PH,DPH,DPHG
        COMMON N,PI,XG,SPC,TO,OMEGB,GAMMA,ETA,P,Q,S
        COMMON OMEGL,LV,GA
        COMMON F,G,H
        INTEGER S,P,Q
        READ(5,5) N,XG,P1,S1,PI
        READ (5, 6) (R(I),B(I),PH(I),DPH(I), I=1,5)
        READ(5,7)(DPHG(I),I=1,5)
        READ(5,8) FQ,FQMAX,DELFQ
        READ(5,8) OMEGL,OLMAX,DELOL
        READ(5,8) ETA,ETAMAX,DELETA
        READ(5,8)  SPC,SPCMAX,DELSPC
        READ(5,8)  GAMMA,GAMAX,DELGA
        READ(5,8)  TO,TOMAX,DELTO
        READ(5,8) GA,GMAX,DELG
        READ(5,500) ITR,NAT,ICOUNT
  500   FORMAT(3I5)
    5   FORMAT(I3,3E7.0,1E12.0,36X)
    6   FORMAT (4E12.0,24X)
    7   FORMAT(5E12.0,12X)
    8   FORMAT(3E12.0,36X)
  400   CONTINUE
        LV = 1
        Q = FQ + .5
        P = P1
        S = S1 + .5
        FQ1 = FQ
        FETA = ETA
        FTO = TO
        FSPC = SPC
        FGAMMA = GAMMA
        FOMEGL = OMEGL
        FGA = GA
        N1 = N+1
        IF(SPC.EQ.0.0) N1=N
   15   CALL COEF(F,G,H)
        DO 10 J=1,N1
        DO 10 K=1,N1
        FF(J,K) = F(J,K)
        FG(J,K) = G(J,K)
        DO 9 I=1,5
    9   FH(I,J,K) = H(I,J,K)
   10   CONTINUE
  46    CONTINUE
        IF(NAT.EQ.0) GO TO 45
        WRITE(6,501)
        WRITE (6,28)
        WRITE(6,27) GAMMA,N,OMEGL,ETA,SPC,XG,GA
```

```
      WRITE(6,60) (I,I=1,N1)
28    FORMAT(//////20X,29HNATURAL          FREQUENCIES///)
60    FORMAT(15X3HTO ,8X5HFREQ(I1,1H),8X5HFREQ(I1,1H),8X5HFREQ(I1,1H)/)
70    FORMAT(13X,1F5.1,3F15.6//)
 27 FORMAT(8X6HGAMMA=F4.2,4H, N=I1,7H,OMEGL=F6.2,6H, ETA=F4.3,6H, SPC=
   1F4.2,5H, XG=F3.1,5H, GA=F4.3/)
 45 CONTINUE
      TOR4 = TO/R(1)**4
      DO 13 J=1,N1
      DO 13 K=1,N1
      F(J,K) = TOR4*FF(J,K)
      G(J,K) = TOR4*FG(J,K)
      IF(OMEGL.GT.50.0) G(J,K) = F(J,K)
      IF(J.EQ.(N+1)) GO TO 16
      IF(J.EQ.K) F(J,K) = F(J,K) + (R(J)/R(1))**4 - (ETA*ETA)/4.0
 16 CONTINUE
      DO 12 I=1,5
 12 H(I,J,K) = FH(I,J,K)*TOR4
 13 CONTINUE
      CALL HESSEN(F,N1)
      CALL QREIG(F,N1,RTR,RTI,0)
      N1M1 = N1-1
      DO 110 I=1,N1M1
      IP1 = I+1
      DO 110 J=IP1,N1
      IF(RTR(I)-RTR(J)) 110,110,120
120 TEMP = RTR(I)
      RTR(I) = RTR(J)
      RTR(J) = TEMP
110 CONTINUE
      DO 130 I=1,N1
130 RTR(I) = SQRT(ABS(RTR(I)))
      DO 14 J=1,N1
      DO 14 K=1,N1
      F(J,K) = TOR4*FF(J,K)
      IF(J.EQ.(N+1)) GO TO 14
      IF(J.EQ.K)F(J,K) = F(J,K) + (R(J)/R(1))**4 - (ETA*ETA)/4.0
 14 CONTINUE
      IF(NAT.EQ.1) WRITE(6,70) TO,(RTR(I),I=1,N1)
      IF(NAT.EQ.1) GO TO 300
501 FORMAT(1H1)
      WRITE  (6,501)
      DO 200 KZ = 1, N1
      WRITE(6,502) KZ
502 FORMAT(//12X,46HCHARACTERISTIC VALUES IN VICINITY OF 2 X FREQ(I1,1
   1H)//)
      WRITE(6,60) (I,I = 1,N1)
      WRITE(6,70) TO,(RTR(I),I=1,N1)
      OMEGB = 2.0 * RTR(KZ)
      WRITE(6,27) GAMMA,N,OMEGL,ETA,SPC,XG,GA
      IF(ITR.EQ.0) CALL STAB(ITS,STAB2)
      IF(ITR.EQ.1) CALL SWITCH(OMEGB,DELOM)
200 CONTINUE
      WRITE(6,501)
      DO 201 KZ = 1,N1
      DO 201 JZ = 1, N1
```

```
      IF(KZ.EQ.JZ.OR.KZ.GT.JZ) GO TO 201
      WRITE(6,503) JZ,KZ
  503 FORMAT(//12X,42HCHARACTERISTIC VALUES IN VICINITY OF FREQ(I1,1H),9
     1H   + FREQ(I1,1H)//)
      WRITE(6,60) (I,I = 1,N1)
      WRITE(6,70) TO,(RTR(I),I=1,N1)
      WRITE(6,27) GAMMA,N,OMEGL,ETA,SPC,XG,GA
      OMEGB = RTR(JZ)   +RTR(KZ)
      IF(ITR.EQ.0) CALL STAB(ITS,STAB2)
      IF(ITR.EQ.1) CALL SWITCH(OMEGB,DELOM)
  201 CONTINUE
      WRITE(6,501)
      DO 204 KZ = 1, N1
      DO 204 JZ = 1, N1
      IF (KZ.EQ.JZ.OR.KZ.GT.JZ) GO TO 204
      WRITE(6,504) JZ, KZ
  504 FORMAT(//12X,42HCHARACTERISTIC VALUES IN VICINITY OF FREQ(I1,1H),9
     1H   - FREQ(I1,1H)//)
      WRITE(6,60) (I,I = 1,N1)
      WRITE(6,70) TO,(RTR(I),I=1,N1)
      WRITE(6,27) GAMMA,N,OMEGL,ETA,SPC,XG,GA
      OMEGB = RTR(JZ) - RTR(KZ)
      IF(ITR.EQ.1) CALL SWITCH(OMEGB,DELOM)
      IF(ITR.EQ.0) CALL STAB(ITS,STAB2)
  204 CONTINUE
 300  CONTINUE
      TO = TO + DELTO
      IF(TO.LT.(TOMAX+0.001)) GO TO 45
      TO = FTO
      OMEGL = OMEGL + DELOL
      IF(OMEGL.LT.(OLMAX+0.001))GO TO 45
      OMEGL = FOMEGL
      ETA = ETA + DELETA
      IF(ETA.LT.(ETAMAX+0.001))GO TO 46
      ETA = FETA
      SPC = SPC + DELSPC
      IF(SPC.LT.(SPCMAX+0.001))GO TO 15
      SPC = FSPC
      GAMMA = GAMMA + DELGA
      IF(GAMMA.LT.(GAMAX +0.001))GO TO 45
      GAMMA = FGAMMA
      GA = GA + DELG
      IF(GA.LT.(GMAX+0.001)) GO TO 15
      GA = FGA
      IF(ICOUNT.EQ.2) STOP
      IF(ICOUNT.EQ.1) GO TO 401
C     READ IN ANY CHANGE OF PARAMETERS HERE
      READ(5,8)  TO,TOMAX,DELTO
      READ(5,500)ITR,NAT,ICOUNT
      GO TO 400
  401 CONTINUE
      READ(5,500)ITR,NAT,ICOUNT
      GO TO 400
      STOP
      END
```

```
      SUBROUTINE SWITCH(OMEGB,DELOM)
      FOMEGB = OMEGB
      LOC4 = 0
    3 LOC1 = 0
      LOC2 = 0
      LOC3 = 0
    5 CONTINUE
      CALL STAB(ITS,STAB2)
      IF(LOC1.EQ.0) DELOM = STAB2*1.2
      IF(DELOM.GT.0.10) DELOM = 0.04
      LOC1 = LOC1 + 1
      GO TO (1,2),ITS
    1 IF(LOC1.EQ.1) GO TO 100
      LOC2 = LOC2 + 1
      IF(LOC2.EQ.2.AND.LOC3.EQ.1) GO TO 74
      GO TO (11,12,74),LOC2
   11 OMEGB = OMEGB + DELOM/2.0
      GO TO 5
   12 OMEGB = OMEGB + DELOM/4.0
      GO TO 5
    2 IF(LOC2.EQ.0) GO TO 20
      LOC3 = LOC3 + 1
      IF(LOC2.EQ.2.AND.LOC3.EQ.1) GO TO 75
      IF(LOC2.EQ.1.AND.LOC3.EQ.2) GO TO 75
      IF(LOC2.EQ.1.AND.LOC3.EQ.1) GO TO 21
      STOP
   20 OMEGB = OMEGB - DELOM
      GO TO 5
   21 OMEGB = OMEGB - DELOM/4.0
      GO TO 5
   74 IF(ABS(DELOM).LT.0.10) GO TO 90
      LOC5 = 0
      DO 84 IZ = 1,4
      IF(DELOM.GT.0.)OMEGB = OMEGB + .002
      IF(DELOM.LT.0.) OMEGB = OMEGB - .002
      CALL STAB(ITS,STAB2)
      IF(ITS.EQ.2) LOC5 = LOC5+1
      IF(LOC5.EQ.1.AND.ITS.EQ.2) WRITE(6,101)
   84 CONTINUE
      GO TO 90
   75 IF(ABS(DELOM).LT.0.10) GO TO 90
      LOC5 = 0
      DO 85 IZ = 1,4
      IF(DELOM.GT.0.) OMEGB = OMEGB - .002
      IF(DELOM.LT.0.) OMEGB = OMEGB + .002
      CALL STAB(ITS,STAB2)
      IF(ITS.EQ.1) LOC5 = LOC5 + 1
      IF(LOC5.EQ.1.AND.ITS.EQ.1) WRITE(6,101)
   85 CONTINUE
      GO TO 90
   90 IF(LOC4.EQ.1) GO TO 100
      LOC4 = LOC4 + 1
      DELOM =-DELOM
      OMEGB = FOMEGB
```

```
      LOC2 = 0
      LOC3 = 0
      GO TO 2
  100 CONTINUE
  101 FORMAT(5X59HTHERE IS A BOUNDARY POINT BETWEEN THE LAST TWO OMEGB V
     1ALUES)
      RETURN
      END



C     PROGRAM TO CALCULATE STABILITY CONSTANTS OF A UNIFORM BEAM SUB-
C     JECTED TO A PERIODICALLY VARYING THRUST WITH DIRECTIONAL CONTROL
      SUBROUTINE STAB(ITS,STAB2)
      DIMENSION A(7), ROOTR(6), ROOTI(6)
      DIMENSION DT(6),ACHE(6),     Y(6),W(6)
      DIMENSION R(5),B(5),PH(5),DPH(5),DPHG(5)
      DIMENSION RMOD1(6),RMOD2(6)
      COMMON R,B,PH,DPH,DPHG
      COMMON N,PI,XG,SPC,TO,OMEGB,GAMMA,ETA,P,Q,S
      COMMON OMEGL,LV,GA
      COMPLEX CZ,CARG1,CARG2,CSQRT
      INTEGER P,Q,S
      N1 = N + 1
      IF(SPC.EQ.0.0)N1=N
      CALL DETER (DT,W)
      DO 5  I=1,N1
      ACHE(I) = PI*SIN(2.*PI*W(I))*DT(I)/W(I)
      Y(I) = COS(2.*PI*W(I))
    5 CONTINUE
      A(1) = -1.
      A(2) = Y(1)+Y(2)+Y(3) - ACHE(1)-ACHE(2)-ACHE(3)
      A(3)=-(Y(1)*Y(2)+Y(1)*Y(3)+Y(2)*Y(3)-ACHE(1)*(Y(2)+Y(3))
     1-ACHE(2)*(Y(1)+Y(3))-ACHE(3)*(Y(1)+Y(2)))
      A(4) = Y(1)*Y(2)*Y(3) - ACHE(1)*Y(2)*Y(3) - ACHE(2)*Y(1)*Y(3) -
     1ACHE(3)*Y(1)*Y(2)
      CALL MULLER (A,ROOTR,ROOTI,3)
      DO 111 I=1,3
      IF(ABS(ROOTI(I)).LT..1E-4) ROOTI(I) = 0.0
      CZ = CMPLX(ROOTR(I),ROOTI(I))
      CARG1 = CZ    + CSQRT(CZ**2-1.)
      CARG2 = CZ -  CSQRT(CZ**2-1.)
      RMOD1(I) = CABS(CARG1)
  111 RMOD2(I) = CABS(CARG2)
      RMAX =  AMAX1(RMOD1(1),RMOD1(2),RMOD1(3),RMOD2(1),RMOD2(2),
     1RMOD2(3))
      STAB2 = ALOG(RMAX)*OMEGB/PI
      STAB1 = STAB2 - ETA
      IF(STAB1 - .1E-6) 1,1,2
    1 ITS = 1
      WRITE(6,21) OMEGB
      GO TO 3
    2 ITS = 2
      WRITE(6,22) OMEGB
   21 FORMAT(11X6HOMEGB=F6.3,5X6HSTABLE)
```

```
22 FORMAT(11X6HOMEGB=F6.3,5X8HUNSTABLE)
 3 WRITE(6,23)  (ROOTR(I),I=1,3),(ROOTI(I),I=1,3)
   WRITE(6,50)          STAB1
23 FORMAT(18X5HROOTR,12X,3F10.5/18X5HROOTI,12X,3F10.5)
50 FORMAT(                      18X5HSTAB1,12XF10.5)
   RETURN
   END


C      SUBPROGRAM TO EVALUATE AN INFINITE DETERMINANT
       SUBROUTINE DETER (DT,W)
       DIMENSION R(5),B(5),PH(5),DPH(5),DPHG(5)
       DIMENSION A(100,100),DT(6),F(8,8),G(8,8),H(5,8,8)
       DIMENSION FF(8,8),FG(8,8),FH(5,8,8)
       DIMENSION W(6)
       COMMON R,B,PH,DPH,DPHG
       COMMON N,PI,XG,SPC,TO,OMEGB,GAMMA,ETA,P,Q,S
       COMMON OMEGL,LV,GA
       COMMON F,G,H
       INTEGER P,Q,S
       SOMEGB = OMEGB**2
       N1=N+1
       IF (SPC.EQ.0.0) N1 = N
       IF(OMEGL.GT.50.0.AND.GA.EQ.0.0) GO TO 12
    11 CALL COEF(FF,FG,FH)
       TOR4=TO/R(1)**4
       DO 13 J=1,N1
       DO 13 K=1,N1
    13 G(J,K)=FG(J,K)*TOR4
    12 CONTINUE
       DO105 I1=1,N1
       W(I1) = SQRT(ABS(    F(I1,I1)        ))/(OMEGB    )
       M1 = (SQRT(ABS(F(N,N))) + SQRT(ABS(F(I1,I1))))/OMEGB + 1.0
       IF(M1-14) 200,200,201
   201 WRITE(6,202)
   202 FORMAT(5X,35HCONVERGENCE OF DETERMINANT DOUBTFUL)
       M1 = 14
   200 CONTINUE
       MO= 2*M1+1
       L = 0
     2 L = L + 1
       DO100 M=1,MO
       DO100 J=1,N1
       DO100 K=1,N1
       IO = M1-L+1
       E = IO
       DENOM = -(W(I1) - E)**2 + (    F(J,J)          )/(SOMEGB    )
       IF(I1.NE.J) GO TO 4
       IF (IO) 4,3,4
     3 DENOM = 1.0
     4 J1= J+(L-1)*N1
       K1= K+(M-1)*N1
       A(J1,K1) = 0.0
       IF(L-M)10,5,10
```

```
    5 IF(J1-K1)55,50,55
   10 IF  (IABS(L-M)-1) 20,60, 20
   20 IF(Q.EQ.1) GO TO 100
      DO 30 I=1,5
      IF(IABS(L-M).EQ.I*Q) A(J1,K1) = .5*H(I,J,K)/(DENOM*SOMEGB)
   30 CONTINUE
      GO TO 100
   50 A(J1,K1)= 1
      IF(I1.NE.J) GO TO 100
      IF  (10) 100,54,100
   54 A(J1,K1) = 0.0
      GO TO 100
   55 A(J1,K1)= F(J,K)/(DENOM*SOMEGB)
      GO TO 100
   60 A(J1,K1)= .5*GAMMA*G(J,K)/(DENOM*SOMEGB)
  100 CONTINUE
      IF(L.LT.MO) GO TO 2
      MO = N1*MO
      CALL GASDET(A,MO,DET)
      DT(I1)=DET
  105 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE COEF (F,G,H)
      DIMENSION R(5),B(5),PH(5),DPH(5),DPHG(5),A(5)
      DIMENSION F(8,8),G(8,8),H(5,8,8),FJK(8,8),GJK(8,8),HIJK(5,8,8)
      COMMON R,B,PH,DPH,DPHG
      COMMON N,PI,XG,SPC,TO,OMEGB,GAMMA,ETA,P,Q,S
      COMMON OMEGL,LV,GA
      INTEGER S,P,Q
      COMPLEX  CI,Z,CG,CCOS,CSIN,CSQRT
      CI = (0.0,1.0)
      N1=N+1
      SIG = OMEGB/OMEGL
      CALL INTEG(FJK,GJK,HIJK)
      SOMEGB = OMEGB**2
      X = SIG
C     COMPUTE  A ARRAY
      DO 1  I=1,S
      SGNI = (-1.)**I
      XI= I
    1 A(I) =  2.*SGNI*(1.0/XI**2 + GAMMA/(XI**2-X**2))/PI**2
      X = PI*SIG
      DO 15 J=1,N
      DO 15 K=1,N
   10 F(J,K)=          PH(J)*DPH(K)-FJK(J,K)+SPC*PH(J)*DPHG(K)
   11 G(J,K)=          PH(J)*DPH(K)-GJK(J,K)+SPC*PH(J)*DPHG(K)
      DO 12 I=1,S
      XI= I
   12 H(I,J,K) =  A(I)*(-1.0 )*HIJK(I,J,K)
   15 CONTINUE
      DO 20 K=1,N
```

```
      F(N+1,K)= 12.*     (.5*DPH(K)-PH(K)+.5*SPC*DPHG(K))
      G(N+1,K)= 12.*     (.5*DPH(K)-PH(K)+GJK(N+1,K)+.5*SPC*DPHG(K) )
      DO 20 I=1,S
      XI= I
   20 H(I,N+1,K)=       -A(I)*12.*     HIJK(I,N+1,K)
      DO 25 J=1,N
      F(J,N+1)=       SPC*PH(J)
      G(J,N+1)=        GJK(J,N+1)+SPC*PH(J)
      DO 25 I=1,S
      XI=I
   25 H(I,J,N+1)=(-1.0 )* A(I)*HIJK(I,J,N+1)
      F(N+1,N+1)= 6.*SPC
      IF(GA.GT.0.0) GO TO 27
   26 CONTINUE
      G(N+1,N+1)= 12.*     (.5-(1.0-COS(X))/(X*SIN(X))+.5*SPC)
      GO TO 28
   27 CONTINUE
      Z = PI*CSQRT(SIG**2 - CI*GA)
      CG = (1.0 - CCOS(Z))/(Z*CSIN(Z))
      GJK(N1,N1) = REAL(CG)
      G(N+1,N+1) = 12.*(.5 - REAL(CG) + .5*SPC)
   28 CONTINUE
      DO 30 I=1,S
      SGNI = (-1.)**I
      XI= I
   30 H(I,N+1,N+1)= 12.*     (SGNI-1.)* A(I)/(PI*XI)
      RETURN
      END


C     SUBROUTINE TO EVALUATE  INTEGRALS
      SUBROUTINE INTEG (FJK,GJK,HIJK)
      DIMENSION R(5),B(5),PH(5),DPH(5),DPHG(5)
      DIMENSION FJK(8,8),CGJK(8,8),GJK(8,8),CHIJK(5,8,8),HIJK(5,8,8)
      COMMON R,B,PH,DPH,DPHG
      COMMON N,PI,XG,SPC,TO,OMEGB,GAMMA,ETA,P,Q,S
      COMMON OMEGL,LV,GA
      INTEGER P,Q,S
      COMPLEX CGJK,CHIJK,CI,CA,CB,CEXP
      COMPLEX CX1,CX2,CX3
      SIG = OMEGB/OMEGL
      COMPLEX     Z,CG,CCOS,CSIN,CSQRT
      X=PI*SIG
      Z = PI*CSQRT(SIG**2 - CI*GA)
      DO 31 J=1,N
      DO 31 K=1,N
      CGJK(J,K) = (0.0,0.0)
      DO 4 I = 1,5
    4 CHIJK(I,J,K) = (0.0,0.0)
      SGN= (-1.)**(J+K)
      IF(K-J)25,5,10
    5 FJK(J,K)= (B(J)*R(J)/2.0)*(B(J)*R(J)-6.0)
      GO TO 15
   10 FJK(J,K)= (4.*SGN*R(J)*R(K)/(R(K)**4-R(J)**4))*(-B(J)*R(K)**3+B(K)
```

```
  1*R(J)**3)+((16.0*R(J)**4*R(K)**4)/(R(K)**4-R(J)**4)**2)*(SGN-1.0)
15 DO 20 L=1,4
   DO 20 M=1,4
   CI= (0.0,1.0)
   CA= R(J)*CI**(L-1) + R(K)*CI**(M-1)
   CB= (R(J)*R(K)/4.0)*(B(J) + CI**(L-1))*(B(K) + CI**(M-1))
   IF(GA.GT.0.0) GO TO 22
21 CONTINUE
   CGJK(J,K) = CGJK(J,K) + (CB/(CA**2+X**2))*((CA*SIN(X)-X*COS(X))*
  1CEXP(CA) + X)/SIN(X)
   GO TO 23
22 CONTINUE
   CGJK(J,K) = CGJK(J,K) + (CB/(CA**2+Z**2))*((CA*CSIN(Z)-Z*CCOS(Z))*
  1CEXP(CA) + Z)/CSIN(Z)
23 CONTINUE
   DO 20 I=1,S
   SGNI = (-1.)**I
   Y = PI*FLOAT(I)
   CHIJK(I,J,K)=CHIJK(I,J,K)+(CB*Y/(CA**2+Y**2))*(1.0-SGNI*CEXP(CA))
20 HIJK(I,J,K) = REAL (CHIJK(I,J,K))
   GJK(J,K) = REAL(CGJK(J,K))
   GO TO 30
25 FJK(J,K)= FJK(K,J)
   CGJK(J,K)= CGJK(K,J)
   DO 26 I=1,S
   CHIJK(I,J,K)= CHIJK(I,K,J)
26  HIJK(I,J,K)= REAL(CHIJK(I,J,K))
    GJK(J,K)= REAL(CGJK(J,K))
30 IF(SPC.EQ.0.0) GJK(J,K) = FJK(J,K)
31 CONTINUE
   DO35 K=1,N
   XK = 1.0/(R(K)**4-X**4)
   SGNK = (-1.)**K
   IF(GA.GT.0.0) GO TO 33
32 CONTINUE
   GJK( N+1,K)= XK*((X**3)*DPH(K)*(SGNK-COS(X))-(X**4)*PH(K)*SIN(X))/
  1SIN(X)
   GO TO 24
33 CONTINUE
   CG = 1.0/(R(K)**4 - Z**4)
   CG =CG*((Z**3)*DPH(K)*(SGNK-CCOS(Z))-(Z**4)*PH(K)*CSIN(Z))/CSIN(Z)
   GJK(N+1,K) = REAL(CG)
24 CONTINUE
   DO34 I=1,S
   SGNI = (-1.)**I
   Y = PI*FLOAT(I)
   YK = 1.0/(R(K)**4-Y**4)
34 HIJK(I,N+1,K) = YK*(Y**3)*DPH(K)*(SGNI-SGNK)
35 CONTINUE
   DO 40 J=1,N
   XJ = 1.0/(R(J)**4-X**4)
   SGNJ = (-1.)**J
   IF(GA.GT.0.0) GO TO 37
36 CONTINUE
   GJK(J,N+1)= (-XJ)*((X**2)*DPH(J)*SIN(X)+(X**3)*PH(J)*(COS(X)+
  1SGNJ))/SIN(X)
```

```
      GO TO 38
   37 CONTINUE
      CG =  1.0/(R(J)**4 - Z**4)
      CG = -CG*(Z**2*DPH(J)*CSIN(Z)+(Z**3)*PH(J)*(CCOS(Z)+SGNJ))/CSIN(Z)
      GJK(J,N+1) = REAL(CG)
   38 CONTINUE
      DO 39 I=1,S
      SGNI = (-1.)**I
      Y = PI*FLOAT(I)
      YJ = 1.0/(R(J)**4-Y**4)
   39 HIJK(I,J,N+1)=YJ*(Y**3)*DPH(J)*(SGNI-SGNJ)
   40 CONTINUE
      RETURN
      END



      SUBROUTINE GASDET(A,N,DET)                                        IF100
      DIMENSION A(100,100)
      RETURN
      END                                                               IB101.



      SUBROUTINE MULLER(COE,ROOTR,ROOTI,N1)                             IB100
      DIMENSION COE(7),ROOTR(6),ROOTI(6)
      RETURN                                                            IF100
      END                                                               IF100



C     SUBROUTINE TO PUT MATRIX IN UPPER HESSENBERG FORM.                HES
      SUBROUTINE HESSEN(A,M)                                            HESS0
      DIMENSION A(5,5),B(4)
   30 RETURN                                                            HESS0
       END                                                              HESS0



C       PROGRAM TO CALL QR TRANSFORMATION, MAXIMUM ITER IS 50.          QRCN0
      SUBROUTINE QREIG(A,M,ROOTR,ROOTI,IPRNT)
      DIMENSION A(5,5),ROOTR(5),ROOTI(5)
  107 FORMAT(56X  E13.8)                                                QRCN0
       END                                                              QRCN0



      SUBROUTINE QRT(A,N,R,SIG,D)                                       QRT 0
      DIMENSION A(5,5),PSI(3),G(3)
  101 RETURN
       END


  2  +0.5   +1.0  +5.0 +3.141592
4.7300408   -0.9825022    2.0         9.29453
```

```
7.8532046     -1.0007773    -2.0        -15.71858
10.9956078    -0.99996645    2.0        21.99045
14.1371655    -1.00000145   -2.0        -28.27433
17.2787596    -0.99999993    2.0        34.55751
+0.0          +10.80066     +0.0        -20.017144   +0.0
+1.0          +1.0          +1.0                                       FQ
+100.0        +100.0        +10.0                                       OMEG
+0.0          +0.0          +0.01                                      ETA
+0.5          +0.5           +0.5                                      SPC
+0.10         +0.10         +0.15                                      GAMMA
+0.0          +20.0         +2.0                                       TO
+0.00         +0.00         +0.02                                       GA
    1     1     0                                                      CONTR
+20.0         +20.0         +5.0                                       TO
    0     0     1                                                      CONTR
    1     0     2                                                      CONTR
      END OF LISTING
```

FORMAT (1)

NATURAL FREQUENCIES

GAMMA= .10, N=2,OMEGL=100.00, ETA=.000, SPC= .50, XG= .5, GA=.000

TO	FREQ(1)	FREQ(2)	FREQ(3)
.0	.000000	1.000000	2.756538
2.0	.106666	.987895	2.731931
4.0	.146440	.976210	2.707137
6.0	.173364	.965031	2.682152
8.0	.192492	.954452	2.656973
10.0	.205617	.944572	2.631596
12.0	.213472	.935491	2.606020
14.0	.216275	.927310	2.580239
16.0	.213897	.920120	2.554250
18.0	.205875	.914001	2.528051
20.0	.191271	.909014	2.501638

FORMAT (2)

```
CHARACTERISTIC VALUES IN VICINITY OF 2 X FREQ(1)

      TO            FREQ(1)         FREQ(2)         FREQ(3)

     20.0           .191271         .909014        2.501638

GAMMA= .10, N=2,OMEGL=100.00, ETA=.000, SPC= .50, XG= .5, GA=.000

   OMEGR=  .383      UNSTABLE
           ROOTR                     -1.05427   -.69691   -.97445
           ROOTI                       .00000    .00000    .00000
           STAB1                       .03994

CHARACTERISTIC VALUES IN VICINITY OF 2 X FREQ(2)

      TO            FREQ(1)         FREQ(2)         FREQ(3)

     20.0           .191271         .909014        2.501638

GAMMA= .10, N=2,OMEGL=100.00, ETA=.000, SPC= .50, XG= .5, GA=.000

   OMEGR= 1.818      UNSTABLE
           ROOTR                       .78958   -.71173  -1.00004
           ROOTI                       .00000    .00000    .00000
           STAB1                       .00492

CHARACTERISTIC VALUES IN VICINITY OF 2 X FREQ(3)

      TO            FREQ(1)         FREQ(2)         FREQ(3)

     20.0           .191271         .909014        2.501638

GAMMA= .10, N=2,OMEGL=100.00, ETA=.000, SPC= .50, XG= .5, GA=.000

   OMEGR= 5.003      UNSTABLE
           ROOTR                       .97126    .41617  -1.00014
           ROOTI                       .00000    .00000    .00000
           STAB1                       .02653
```

FORMAT (2) (continued)

```
CHARACTERISTIC VALUES IN VICINITY OF FREQ(2)  + FREQ(1)

     TO          FREQ(1)        FREQ(2)        FREQ(3)

    20.0         .191271        .909014       2.501638

GAMMA= .10, N=2,OMEGL=100.00, ETA=.000, SPC= .50, XG= .5, GA=.000

  OMEGB= 1.100      UNSTABLE
         ROOTR                    .46111     .46111    -.14765
         ROOTI                   -.08649     .08649     .00000
         STAB1                    .03404

CHARACTERISTIC VALUES IN VICINITY OF FREQ(3)  + FREQ(1)

     TO          FREQ(1)        FREQ(2)        FREQ(3)

    20.0         .191271        .909014       2.501638

GAMMA= .10, N=2,OMEGL=100.00, ETA=.000, SPC= .50, XG= .5, GA=.000

  OMEGB= 2.693      UNSTABLE
         ROOTR                    .90193     .90193    -.52292
         ROOTI                   -.01166     .01166     .00000
         STAB1                    .02311

CHARACTERISTIC VALUES IN VICINITY OF FREQ(3)  + FREQ(2)

     TO          FREQ(1)        FREQ(2)        FREQ(3)

    20.0         .191271        .909014       2.501638

GAMMA= .10, N=2,OMEGL=100.00, ETA=.000, SPC= .50, XG= .5, GA=.000

  OMEGB= 3.411      STABLE
         ROOTR                    .93634    -.10222    -.10526
         ROOTI                    .00000     .00000     .00000
         STAB1                   -.00000
```

FORMAT (2) (continued)

```
         CHARACTERISTIC VALUES IN VICINITY OF FREQ(2)  - FREQ(1)

            TO          FREQ(1)        FREQ(2)        FREQ(3)

           20.0         .191271        .909014       2.501638

 GAMMA= .10, N=2,OMEGL=100.00, ETA=.000, SPC= .50, XG= .5, GA=.000

    OMEGR=  .718     STABLE
         ROOTR                      -.34505     .09017    -.99605
         ROOTI                       .00000     .00000     .00000
         STAB1                       .00000

         CHARACTERISTIC VALUES IN VICINITY OF FREQ(3)  - FREQ(1)

            TO          FREQ(1)        FREQ(2)        FREQ(3)

           20.0         .191271        .909014       2.501638

 GAMMA= .10, N=2,OMEGL=100.00, ETA=.000, SPC= .50, XG= .5, GA=.000

    OMEGR= 2.310     STABLE
         ROOTR                       .88338     .65206    -.78430
         ROOTI                       .00000     .00000     .00000
         STAB1                       .00000

         CHARACTERISTIC VALUES IN VICINITY OF FREQ(3)  - FREQ(2)

            TO          FREQ(1)        FREQ(2)        FREQ(3)

           20.0         .191271        .909014       2.501638

 GAMMA= .10, N=2,OMEGL=100.00, ETA=.000, SPC= .50, XG= .5, GA=.000

    OMEGR= 1.593     UNSTABLE
         ROOTR                       .72615    -.90256    -.90256
         ROOTI                       .00000    -.00147     .00147
         STAB1                       .00173
```

FORMAT (3)

CHARACTERISTIC VALUES IN VICINITY OF 2 X FREQ(1)

TO	FREQ(1)	FREQ(2)	FREQ(3)
20.0	.191271	.909014	2.501638

GAMMA= .10, N=2,OMEGL=100.00, ETA=.000, SPC= .50, XG= .5, GA=.000

```
OMEGR=  .383     UNSTABLE
         ROOTR               -1.05427    -.69691    -.97445
         ROOTI                 .00000     .00000     .00000
         STAB1                 .03994
CONVERGENCE OF DETERMINANT DOUBTFUL
OMEGR=  .335     STABLE
         ROOTR                -.99018    -.21531    -.93393
         ROOTI                 .00000     .00000     .00000
         STAB1                 .00000
OMEGR=  .359     UNSTABLE
         ROOTR                 .98698    -.92604   -1.12718
         ROOTI                 .00000     .00000     .00000
         STAB1                 .05697
CONVERGENCE OF DETERMINANT DOUBTFUL
OMEGR=  .347     STABLE
         ROOTR                -.73490     .20277    -.96981
         ROOTI                 .00000     .00000     .00000
         STAB1                 .00000
OMEGR=  .430     STABLE
         ROOTR                 .75722     .37538    -.95966
         ROOTI                 .00000     .00000     .00000
         STAB1                 .00000
OMEGR=  .407     STABLE
         ROOTR                 .56913    -.26284    -.39982
         ROOTI                 .00000     .00000     .00000
         STAB1                 .00000
OMEGR=  .395     UNSTABLE
         ROOTR                -.53747    -.33576   -1.03214
         ROOTI                 .00000     .00000     .00000
         STAB1                 .03176
```

CHARACTERISTIC VALUES IN VICINITY OF 2 X FREQ(2)

TO	FREQ(1)	FREQ(2)	FREQ(3)
20.0	.191271	.909014	2.501638

GAMMA= .10, N=2,OMEGL=100.00, ETA=.000, SPC= .50, XG= .5, GA=.000

```
OMEGR= 1.818     UNSTABLE
         ROOTR                 .78958    -.71173   -1.00004
         ROOTI                 .00000     .00000     .00000
```

FORMAT (3) (continued)

```
          STAB1                      .00492
OMEGB= 1.812       STABLE
          ROOTR                      .78824     -.73124     -.99998
          ROOTI                      .00000      .00000      .00000
          STAB1                      .00000
OMEGB= 1.815       UNSTABLE
          ROOTR                      .78891     -.72154    -1.00002
          ROOTI                      .00000      .00000      .00000
          STAB1                      .00365
OMEGB= 1.814       UNSTABLE
          ROOTR                      .78858     -.72640    -1.00000
          ROOTI                      .00000      .00000      .00000
          STAB1                      .00113
OMEGB= 1.824       STABLE
          ROOTR                      .79091     -.69178     -.99999
          ROOTI                      .00000      .00000      .00000
          STAB1                      .00000
OMEGB= 1.821       UNSTABLE
          ROOTR                      .79024     -.70180    -1.00003
          ROOTI                      .00000      .00000      .00000
          STAB1                      .00421
OMEGB= 1.822       UNSTABLE
          ROOTR                      .79058     -.69680    -1.00001
          ROOTI                      .00000      .00000      .00000
          STAB1                      .00280


       CHARACTERISTIC VALUES IN VICINITY OF 2 X FREQ(3)


       TO            FREQ(1)         FREQ(2)          FREQ(3)

      20.0           .191271         .909014          2.501638


GAMMA= .10, N=2,OMEGL=100.00, ETA=.000, SPC= .50, XG= .5, GA=.000

OMEGB= 5.003       UNSTABLE
          ROOTR                      .97126      .41617    -1.00014
          ROOTI                      .00000      .00000      .00000
          STAB1                      .02653
OMEGB= 4.971       STABLE
          ROOTR                      .97089      .40951     -.99994
          ROOTI                      .00000      .00000      .00000
          STAB1                      .00000
OMEGB= 4.987       UNSTABLE
          ROOTR                      .97107      .41285    -1.00009
          ROOTI                      .00000      .00000      .00000
          STAB1                      .02122
OMEGB= 4.979       UNSTABLE
          ROOTR                      .97098      .41119    -1.00003
          ROOTI                      .00000      .00000      .00000
          STAB1                      .01159
OMEGB= 5.035       STABLE
          ROOTR                      .97162      .42272     -.99994
          ROOTI                      .00000      .00000      .00000
          STAB1                      .00000
OMEGB= 5.019       UNSTABLE
```

FORMAT (3) (continued)

```
        ROOTR                 .97144     .41946   -1.00009
        ROOTI                 .00000     .00000     .00000
        STAB1                 .02121
OMEGR=  5.027     UNSTABLE
        ROOTR                 .97153     .42109   -1.00003
        ROOTI                 .00000     .00000     .00000
        STAB1                 .01152
```

FORMAT (3) (continued)

```
   CHARACTERISTIC VALUES IN VICINITY OF FREQ(2)  + FREQ(1)


      TO          FREQ(1)        FREQ(2)        FREQ(3)

     20.0         .191271        .909014       2.501638


GAMMA= .10, N=2,OMEGL=100.00, ETA=.000, SPC= .50, XG= .5, GA=.000

   OMEGR= 1.100      UNSTABLE
          ROOTR                  .46111     .46111    -.14765
          ROOTI                 -.08649     .08649     .00000
          STAR1                  .03404
   OMEGR= 1.059      STABLE
          ROOTR                  .47263     .58000    -.64343
          ROOTI                  .00000     .00000     .00000
          STAR1                  .00000
   OMEGR= 1.080      UNSTABLE
          ROOTR                  .49437     .49437    -.40627
          ROOTI                 -.07053     .07053     .00000
          STAR1                  .02783
   OMEGR= 1.070      UNSTABLE
          ROOTR                  .51069     .51069    -.52892
          ROOTI                 -.04062     .04062     .00000
          STAR1                  .01607
   OMEGR= 1.141      STABLE
          ROOTR                  .33735     .44692     .35428
          ROOTI                  .00000     .00000     .00000
          STAR1                 -.00000
   OMEGR= 1.121      UNSTABLE
          ROOTR                  .42682     .42682     .11189
          ROOTI                 -.06775     .06775     .00000
          STAR1                  .02668
   OMEGR= 1.131      UNSTABLE
          ROOTR                  .40877     .40877     .23669
          ROOTI                 -.04494     .04494     .00000
          STAR1                  .01771


   CHARACTERISTIC VALUES IN VICINITY OF FREQ(3)  + FREQ(1)


      TO          FREQ(1)        FREQ(2)        FREQ(3)

     20.0         .191271        .909014       2.501638


GAMMA= .10, N=2,OMEGL=100.00, ETA=.000, SPC= .50, XG= .5, GA=.000

   OMEGR= 2.693      UNSTABLE
          ROOTR                  .90193     .90193    -.52292
          ROOTI                 -.01166     .01166     .00000
          STAR1                  .02311
   OMEGR= 2.665      STABLE
```

FORMAT (3) (continued)

```
         ROOTR                    .90581     .92058    -.54161
         ROOTI                    .00000     .00000     .00000
         STAB1                    .00000
OMEGR= 2.679      UNSTABLE
         ROOTR                    .90775     .90775    -.53225
         ROOTI                   -.00910     .00910     .00000
         STAB1                    .01647
OMEGR= 2.672      UNSTABLE
         ROOTR                    .91052     .91052    -.53692
         ROOTI                   -.00487     .00487     .00000
         STAB1                    .01002
OMEGR= 2.721      STABLE
         ROOTR                    .89734     .88120    -.50437
         ROOTI                    .00000     .00000     .00000
         STAB1                    .00000
OMEGR= 2.707      UNSTABLE
         ROOTR                    .89577     .89577    -.51363
         ROOTI                    .00956    -.00956     .00000
         STAB1                    .01851
OMEGR= 2.714      UNSTABLE
         ROOTR                    .89256     .89256    -.50899
         ROOTI                    .00528    -.00528     .00000
         STAB1                    .01012
```

CHARACTERISTIC VALUES IN VICINITY OF FREQ(3) + FREQ(2)

TO	FREQ(1)	FREQ(2)	FREQ(3)
20.0	.191271	.909014	2.501638

GAMMA= .10, N=2,OMEGL=100.00, ETA=.000, SPC= .50, XG= .5, GA=.000

```
OMEGR= 3.411      STABLE
         ROOTR                    .93834    -.10222    -.10526
         ROOTI                    .00000     .00000     .00000
         STAB1                   -.00000
```

FORMAT (3) (continued)

```
CHARACTERISTIC VALUES IN VICINITY OF FREQ(2)  - FREQ(1)

   TO          FREQ(1)        FREQ(2)        FREQ(3)

  20.0         .191271        .909014       2.501638

GAMMA= .10, N=2,OMEGL=100.00, ETA=.000, SPC= .50, XG= .5, GA=.000

 OMEGB=  .718       STABLE
         ROOTR                -.34505    .09017   -.99605
         ROOTI                 .00000    .00000    .00000
         STAB1                 .00000

CHARACTERISTIC VALUES IN VICINITY OF FREQ(3)  - FREQ(1)

   TO          FREQ(1)        FREQ(2)        FREQ(3)

  20.0         .191271        .909014       2.501638

GAMMA= .10, N=2,OMEGL=100.00, ETA=.000, SPC= .50, XG= .5, GA=.000

 OMEGB= 2.310       STABLE
         ROOTR                 .88338    .85206   -.78430
         ROOTI                 .00000    .00000    .00000
         STAB1                 .00000

CHARACTERISTIC VALUES IN VICINITY OF FREQ(3)  - FREQ(2)

   TO          FREQ(1)        FREQ(2)        FREQ(3)

  20.0         .191271        .909014       2.501638

GAMMA= .10, N=2,OMEGL=100.00, ETA=.000, SPC= .50, XG= .5, GA=.000

 OMEGB= 1.593       UNSTABLE
         ROOTR                 .72815   -.90256   -.90256
         ROOTI                 .00000   -.00147    .00147
         STAB1                 .00173
 OMEGB= 1.591       STABLE
         ROOTR                 .72747   -.89841   -.89906
         ROOTI                 .00000    .00000    .00000
         STAB1                 .00000
 OMEGB= 1.592       UNSTABLE
         ROOTR                 .72781   -.90066   -.90066
         ROOTI                 .00000   -.00135    .00135
         STAB1                 .00158
 OMEGB= 1.591       UNSTABLE
```

FORMAT (3) (continued)

```
        ROOTR                        .72764   -.89970   -.89970
        ROOTI                        .00000   -.00104    .00104
        STAB1                        .00120
OMEGB= 1.595          STABLE
        ROOTR                        .72683   -.90497   -.90760
        ROOTI                        .00000    .00000    .00000
        STAB1                       -.00000
OMEGB= 1.594          UNSTABLE
        ROOTR                        .72649   -.90443   -.90443
        ROOTI                        .00000   -.00097    .00097
        STAB1                        .00115
OMEGB= 1.594          STABLE
        ROOTR                        .72666   -.90582   -.90490
        ROOTI                        .00000    .00000    .00000
        STAB1                        .00000
```

STOP

B-6 Stepped Beam Computer Program

The computer program used in the analysis of a stepped beam is listed in this appendix. Library subroutines are listed by name with any variable array dimensions without a complete listing. Program input data is listed at the end of the program with the resultant output listed immediately following.

The output formats are listed as format (1), format (2), and format (3). These formats are obtained in one computer run using the program source deck and data deck listed.

```
C     PROGRAM DRIVER
      DIMENSION F(5,5),G(5,5),RTR(5),RTI(5),FF(5,5)
      DIMENSION R(5),B(5),D0PH(11,5),D1PH(11,5),D2PH(11,5),D3PH(11,5)
      COMMON N,PI,XG,SPC,TO,OMEGB,GAMMA,ETA,OMEGL,N1
      COMMON RA,RC,RI
      COMMON R,B,D0PH,D1PH,D2PH,D3PH
      COMMON F,G,RTR,RTI
      READ(5,5) N,PI
      READ(5,6) (R(I),I=1,5)
      READ(5,6) (B(I),I=1,5)
      READ(5,7) ((D0PH(J,K),J=1,10),K=1,5)
      READ(5,7) ((D1PH(J,K),J=1,10),K=1,5)
      READ(5,7) ((D2PH(J,K),J=1,10),K=1,5)
      READ(5,7) ((D3PH(J,K),J=1,10),K=1,5)
      READ(5,8) XG,XGMAX,DELXG
      READ(5,8) OMEGL,OLMAX,DELOML
      READ(5,8) ETA,ETAMAX,DELETA
      READ(5,8)  SPC,SPCMAX,DELSPC
      READ(5,8)  GAMMA,GAMAX,DELGA
      READ(5,8)  TO,TOMAX,DELTO
      READ(5,9) RC,RCMIN,DELRC,RA,RAMIN,DELRA,RI,RIMIN,DELRI
      READ(5,10) ITR,NAT,ICOUNT
    5 FORMAT(I5,E12.1)
    6 FORMAT(5E12.1)
    7 FORMAT(7E10.1)
    8 FORMAT(3E12.0,36X)
    9 FORMAT (9E6.1)
   10 FORMAT(3I5)
      DO 1 J=1,N
      D0PH(11,J) = (-1.)**(J-1)*D0PH(10,J)
      D1PH(11,J) = (-1.)**J*D1PH(10,J)
      D2PH(11,J) = 0.0
    1 D3PH(11,J) = 0.0
  400 CONTINUE
      FXG = XG
      FOMEGL = OMEGL
      FRC = RC
      FRA = RA
      FRI = RI
      FOMEGB = OMEGB
      FETA = ETA
      FTO = TO
      FSPC = SPC
      FGAMMA = GAMMA
      N1 = N+1
      IF(SPC.EQ.0.0) N1=N
      N2 = N1 + 1
   46 CONTINUE
      IF(NAT.EQ.0) GO TO 45
      WRITE(6,501)
      WRITE (6,28)
      WRITE(6,60) (I,I=1,N1)
      WRITE(6,27) GAMMA,N,OMEGL,ETA,SPC,XG,RC,RA,RI
```

```
28   FORMAT(//////20X,29HNATURAL              FREQUENCIES///)
60   FORMAT(15X3HTO  ,8X5HFREQ(I1,1H),8X5HFREQ(I1,1H),8X5HFREQ(I1,1H)/)
70   FORMAT(13X,1F5.1,3F15.6//)
 27  FORMAT(8X6HGAMMA=F4.2,4H, N=I1,7H,OMEGL=F6.2,6H, ETA=F6.3,6H, SPC=
    1F4.2,5H, XG=F3.1,//8X,4H RC=F4.2,5H, RA=F4.2,5H, RI=F4.2/)
 45  CONTINUE
     CALL COEF(F,G)
     DO 12 J = 1, N1
     DO 12 K = 1,N1
 12  FF(J,K) = F(J,K)
     CALL HESSEN(F,N1)
     CALL QREIG(F,N1,RTR,RTI,0)
     N1M1 = N1 - 1
     DO 110 I = 1, N1M1
     IP1 = I + 1
     DO 110 J = IP1,N1
     IF(RTR(I) - RTR(J))110,110,120
120  TEMP = RTR(I)
     RTR(I) = RTR(J)
     RTR(J) = TEMP
110  CONTINUE
     DO 130 I = 1,N1
130  RTR(I)=SQRT(ABS(RTR(I)))
     DO 2 J = 1,N1
     DO 2 K = 1,N1
  2  F(J,K) = FF(J,K)
     IF(NAT.EQ.1) WRITE(6,70) TO,(RTR(I),I=1,N1)
     IF(NAT.EQ.1) GO TO 300
501  FORMAT(1H1)
     WRITE  (6,501)
     DO 200 KZ = 1, N1
     WRITE(6,502) KZ
502  FORMAT(//12X,46HCHARACTERISTIC VALUES IN VICINITY OF 2 X FREQ(I1,1
    1H)//)
     WRITE(6,60) (I,I = 1,N1)
     WRITE(6,70) TO,(RTR(I),I=1,N1)
     OMEGB = 2.0 * RTR(KZ)
     WRITE(6,27) GAMMA,N,OMEGL,ETA,SPC,XG,RC,RA,RI
     IF(ITR.EQ.0) CALL STAB(ITS,STAB2)
     IF(ITR.EQ.1) CALL SWITCH(OMEGB,DELOM)
200  CONTINUE
     WRITE(6,501)
     DO 201 KZ = 1,N1
     DO 201 JZ = 1, N1
     IF(KZ.EQ.JZ.OR.KZ.GT.JZ) GO TO 201
     WRITE(6,503) JZ,KZ
503  FORMAT(//12X,42HCHARACTERISTIC VALUES IN VICINITY OF FREQ(I1,1H),9
    1H  + FREQ(I1,1H)//)
     WRITE(6,60) (I,I = 1,N1)
     WRITE(6,70) TO,(RTR(I),I=1,N1)
     OMEGB = RTR(JZ)  +RTR(KZ)
     IF(ITR.EQ.0) CALL STAB(ITS,STAB2)
     IF(ITR.EQ.1) CALL SWITCH(OMEGB,DELOM)
201  CONTINUE
     WRITE(6,501)
     DO 204 KZ = 1, N1
```

```
      DO 204 JZ = 1, N1
      IF (KZ.EQ.JZ.OR.KZ.GT.JZ) GO TO 204
      WRITE(6,504) JZ, KZ
  504 FORMAT(//12X,42HCHARACTERISTIC VALUES IN VICINITY OF FREQ(I1,1H),9
     1H  - FREQ(I1,1H)//)
      WRITE(6,60) (I,I = 1,N1)
      WRITE(6,70) TO,(RTR(I),I=1,N1)
      OMEGB = RTR(JZ) - RTR(KZ)
      WRITE(6,27) GAMMA,N,OMEGL,ETA,SPC,XG,RC,RA,RI
      IF(ITR.EQ.1) CALL SWITCH(OMEGB,DELOM)
      IF(ITR.EQ.0) CALL STAB(ITS,STAB2)
  204 CONTINUE
 300  CONTINUE
      TO = TO + DELTO
      IF(TO.LT.(TOMAX+0.001)) GO TO 45
      TO = FTO
      RC = RC - DELRC
      IF(RC.GT.(RCMIN-0.001)) GO TO 46
      RC = FRC
      RA = RA-DELRA
      IF(RA.GT.(RAMIN-0.001)) GO TO 46
      RA = FRA
      RI = RI - DELRI
      IF(RI.GT.(RIMIN-0.001)) GO TO 46
      RI = FRI
      OMEGL = OMEGL + DELOML
      IF(OMEGL.LT.(OLMAX+0.001))GO TO 45
      OMEGL = FOMEGL
      ETA = ETA + DELETA
      IF(ETA.LT.(ETAMAX+0.001))GO TO 46
      ETA = FETA
      SPC = SPC + DELSPC
      IF(SPC.LT.(SPCMAX+0.001))GO TO 46
      SPC = FSPC
      GAMMA = GAMMA + DELGA
      IF(GAMMA.LT.(GAMAX +0.001))GO TO 45
      GAMMA = FGAMMA
      XG = XG + DELXG
      IF(XG.LT.(XGMAX+0.001)) GO TO 46
      XG = FXG
      IF(ICOUNT.EQ.2) STOP
      IF(ICOUNT.EQ.1) GO TO 401
C     READ IN ANY CHANGE OF PARAMETERS HERE
      READ(5,8)  TO,TOMAX,DELTO
      READ(5,10) ITR,NAT,ICOUNT
      GO TO 400
  401 CONTINUE
      READ(5,10) ITR,NAT,ICOUNT
      GO TO 400
      STOP
      END


      SUBROUTINE SWITCH(OMEGB,DELOM)
      FOMEGB = OMEGB
```

```
      LOC4 = 0
    3 LOC1 = 0
      LOC2 = 0
      LOC3 = 0
    5 CONTINUE
      CALL STAB(ITS,STAB2)
      IF(LOC1.EQ.0) DELOM = STAB2*1.2
      IF(DELOM.GT.0.10) DELOM = 0.04
      LOC1 = LOC1 + 1
      GO TO (1,2),ITS
    1 IF(LOC1.EQ.1) GO TO 100
      LOC2 = LOC2 + 1
      IF(LOC2.EQ.2.AND.LOC3.EQ.1) GO TO 74
      GO TO (11,12,74),LOC2
   11 OMEGB = OMEGB + DELOM/2.0
      GO TO 5
   12 OMEGB = OMEGB + DELOM/4.0
      GO TO 5
    2 IF(LOC2.EQ.0) GO TO 20
      LOC3 = LOC3 + 1
      IF(LOC2.EQ.2.AND.LOC3.EQ.1) GO TO 75
      IF(LOC2.EQ.1.AND.LOC3.EQ.2) GO TO 75
      IF(LOC2.EQ.1.AND.LOC3.EQ.1) GO TO 21
      STOP
   20 OMEGB = OMEGB - DELOM
      GO TO 5
   21 OMEGB = OMEGB - DELOM/4.0
      GO TO 5
   74 IF(ABS(DELOM).LT.0.10) GO TO 90
      LOC5 = 0
      DO 84 IZ = 1,4
      IF(DELOM.GT.0.)OMEGB = OMEGB + .002
      IF(DELOM.LT.0.) OMEGB = OMEGB - .002
      CALL STAB(ITS,STAB2)
      IF(ITS.EQ.2) LOC5 = LOC5+1
      IF(LOC5.EQ.1.AND.ITS.EQ.2) WRITE(6,101)
   84 CONTINUE
      GO TO 90
   75 IF(ABS(DELOM).LT.0.10) GO TO 90
      LOC5 = 0
      DO 85 IZ = 1,4
      IF(DELOM.GT.0.) OMEGB = OMEGB - .002
      IF(DELOM.LT.0.) OMEGB = OMEGB + .002
      CALL STAB(ITS,STAB2)
      IF(ITS.EQ.1) LOC5 = LOC5 + 1
      IF(LOC5.EQ.1.AND.ITS.EQ.1) WRITE(6,101)
   85 CONTINUE
      GO TO 90
   90 IF(LOC4.EQ.1) GO TO 100
      LOC4 = LOC4 + 1
      DELOM =-DELOM
      OMEGB = FOMEGB
      LOC2 = 0
      LOC3 = 0
      GO TO 2
  100 CONTINUE
```

```
 101 FORMAT(5X59HTHERE IS A BOUNDARY POINT BETWEEN THE LAST TWO OMEGB V
    1ALUES)
     RETURN
     END

C      SUBPROGRAM TO EVALUATE AN INFINITE DETERMINANT
      SUBROUTINE DETER (DT,W)
      DIMENSION R(5),B(5),D0PH(11,5),D1PH(11,5),D2PH(11,5),D3PH(11,5)
      DIMENSION A(100,100),DT(5),F(5,5),G(5,5),W(5)
      COMMON N,PI,XG,SPC,TO,OMEGB,GAMMA,ETA,OMEGL,N1
      COMMON RA,RC,RI
      COMMON R,B,D0PH,D1PH,D2PH,D3PH
      COMMON F,G,RTR,RTI
      SOMEGB = OMEGB**2
      CALL COEF(F,G)
      DO105 I1=1,N1
      W(I1) = SQRT(ABS(    F(I1,I1)        ))/(OMEGB    )
      M1 = (SQRT(ABS(F(N,N))) + SQRT(ABS(F(I1,I1))))/OMEGB + 1.0
      IF(M1 - 14) 200,200,201
 201  WRITE(6,202)
 202  FORMAT(5X,35HCONVERGENCE OF DETERMINANT DOUBTFUL)
      M1 = 14
 200  CONTINUE
      MO= 2*M1+1
      L = 0
    2 L = L + 1
      DO100 M=1,MO
      DO100 J=1,N1
      DO100 K=1,N1
      IO = M1-L+1
      E = IO
      DENOM = -(W(I1) - E)**2 + (    F(J,J)         )/(SOMEGB    )
      IF(I1.NE.J) GO TO 4
      IF (IO) 4,3,4
    3 DENOM = 1.0
    4 J1= J+(L-1)*N1
      K1= K+(M-1)*N1
      A(J1,K1) = 0.0
      IF(L-M)10,5,10
    5 IF(J1-K1)55,50,55
   10 IF (IABS(L-M)-1) 20,60, 20
   20 GO TO 100
   50 A(J1,K1)= 1
      IF(I1.NE.J) GO TO 100
      IF (IO) 100,54,100
   54 A(J1,K1) = 0.0
      GO TO 100
   55 A(J1,K1)= F(J,K)/(DENOM*SOMEGB)
      GO TO 100
   60 A(J1,K1)= .5*GAMMA*G(J,K)/(DENOM*SOMEGB)
      GO TO 100
  100 CONTINUE
      IF(L.LT.MO) GO TO 2
```

```
      MO = N1*MO
      CALL GASDET(A,MO,DET)
      DT(I1)=DET
  105 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE COEF(F,G)
      DIMENSION F(5,5),G(5,5),RTR(5),RTI(5),A(6,6),C(5,5),D(5,5)
      DIMENSION R(5),B(5),D0PH(11,5),D1PH(11,5),D2PH(11,5),D3PH(11,5)
      DIMENSIONT00CL(5,5),T22CL(5,5),P111(5,5),P211(5,5),P11(5),P21(5)
      COMMON N,PI,XG,SPC,TO,OMEGB,GAMMA,ETA,OMEGL,N1
      COMMON RA,RC,RI
      COMMON R,B,D0PH,D1PH,D2PH,D3PH
      COMMON F,G,RTR,RTI
      REAL MR
      MR = RA + (1.-RA)*(1.-RC)
      LC = (10.*RC) + .5
      IF(RC.LT.0.01) LC = 11
      IG = (10.*XG) +.5
      IF(XG.LT.0.01) IG = 11
      LG = IG
      N2 = N1 + 1
      CALL INTEG (T00CL,T22CL,P111,P211,P11,P21,P1,P2)
      TOR4 = TO/R(1)**4
      DO 15 J=1,N
      DO 15 K=1,N
      A(J,K) = (1.-RA)*T00CL(J,K)
      IF(J.EQ.K) A(J,K) = A(J,K) + RA
      C(J,K) = ((D0PH(10,J)*D1PH(10,K) + SPC*D0PH(10,J)*D1PH(LG,K))*R(K)
     1 - P111(J,K))*TOR4 + (1. - RI)*T22CL(J,K)/R(1)**4
      IF(J.EQ.K) C(J,K) = C(J,K) + (RI*(R(J)/R(1))**4)
      D(J,K) =((D0PH(10,J)*D1PH(10,K) + SPC*D0PH(10,J)*D1PH(LG,K))*R(K)-
     1       P211(J,K)) * TOR4
   15 CONTINUE
      DO 20 K=1,N
      A(N+1,K) = -(1.-RA)*(RC*R(K)*D3PH(LC,K) - D2PH(LC,K))/R(K)**2
      A(N+2,K) = -(1.-RA)*D3PH(LC,K)/R(K)
      C(N+1,K)=((D1PH(10,K) + SPC*D1PH(IG,K))*R(K) - P11(K)) * TOR4
      C(N+2,K) =((D1PH(10,K) + SPC*D1PH(LG,K))*R(K)) * TOR4
      D(N+1,K) =((D1PH(10,K) + SPC*D1PH(IG,K))*R(K) -       P21(K))*TOR4
      D(N+2,K) = C(N+2,K)
   20 CONTINUE
      DO 25 J=1,N
      A(J,N+1) = -(1.-RA)*(RC*R(J)*D3PH(LC,J) - D2PH(LC,J))/R(J)**2
      A(J,N+2) = -(1.-RA)*D3PH(LC,J)/R(J)
      C(J,N+1) = (D0PH(10,J)*(1.0+SPC) - P11(J))* TOR4
      C(J,N+2) = 0.0
      D(J,N+1) = (D0PH(10,J)*(1.0+SPC) -  P21(J))*TOR4
      D(J,N+2) = 0.0
   25 CONTINUE
      A(N+1,N+1) = (RA +(1.-RA)*(1.-RC**3))/3.0
      A(N+1,N+2) = .5* (RA+(1.-RA)*(1.-RC**2))
```

```
      C(N+1,N+1) = (1.0 + SPC - P1) * TOR4
      C(N+1,N+2) = 0.0
      D(N+1,N+1) = (1.0 + SPC -        P2) * TOR4
      D(N+1,N+2) = 0.0
      A(N+2,N+1) = A(N+1,N+2)
      C(N+2,N+1) =(1.0 + SPC) * TOR4
      D(N+2,N+1) = C(N+2,N+1)
      A(N+2,N+2) = MR
      C(N+2,N+2) = 0.0
      D(N+2,N+2) = 0.0
      N2 = N+2
      CALL GASINV (A,N2,DET)
      IF (DET) 35,30,35
   30 WRITE(6,31)
   31 FORMAT(1H ,20HMATRIX A IS SINGULAR)
      STOP
   35 DO 41 I=1,N2
      DO 41 K=1,N2
      F(I,K) = 0.0
      G(I,K) = 0.0
      DO 40 J=1,N2
      F(I,K) = A(I,J) * C(J,K) + F(I,K)
      G(I,K) = A(I,J)*D(J,K)  + G(I,K)
   40 CONTINUE
      IF(I.EQ.K) F(I,K) = F(I,K) - (ETA**2)/4.
   41 CONTINUE
      RETURN
      END


C     SUBROUTINE TO EVALUATE  INTEGRALS
      SUBROUTINE INTEG (T00CL,T22CL,P111,P211,P11,P21,P1,P2)
      DIMENSION R(5),B(5),D0PH(11,5),D1PH(11,5),D2PH(11,5),D3PH(11,5)
      DIMENSION P11(5),P21(5),P111(5,5),P211(5,5)
      DIMENSION T00CL(5,5),T22CL(5,5)
      COMMON N,PI,XG,SPC,TO,OMEGB,GAMMA,ETA,OMEGL,N1
      COMMON RA,RC,RI
      COMMON R,B,D0PH,D1PH,D2PH,D3PH
      REAL MR
      MR = RA + (1.-RA)*(1.-RC)
      COMPLEX CI,CA,CB,CYOL,CYCL,CSUM,RD
      CI = (0.0,1.0)
      LC = (10.*RC) + .5
      IF(RC.LT.0.01) LC = 11
      P1 = .5*(RA + (1.-RA)*(1.-RC)**2)/MR
      IF(OMEGL.GT.50.0) GO TO 1
      SIG = OMEGB/OMEGL
      X = PI*SIG
      BE = 1./(COS(X*RC)*SIN(X*(1.-RC)) + RA*SIN(X*RC)*COS(X*(1.-RC)))
      BC = (1.-RA)*BE*COS(X*RC)
      BD = (1.-(1.-RA)*BE*COS(X*RC)*SIN(X*(1.-RC)))/SIN(X)
      P2 = (BD*(-COS(X)+1.0) + BC*(-COS(X*(1.-RC))+1.0))/X
    1 IF(OMEGL.GT.50.0) P2 = P1
      DO 11 J=1,N
```

```
      P11(J) = (RA*DOPH(10,J) + (1.-RA)*((1.-RC)*DOPH(10,J) + D3PH(LC,J)
     1/R(J)))/MR
      IF(OMEGL.GT.50.0) GO TO 2
      CYOL =(0.0,0.0)
      CYCL =(0.0,0.0)
      DO 10 L=1,4
      CA = R(J)*CI**(L-1)
      CB = B(J) + CI**(L-1)
      RD = CA**2 + X**2
      CYOL = CYOL + CB*(CSUM(1.0,X,CA,1.0)-CSUM(0.0,X,CA,0.0)) /RD
   10 CYCL = CYCL + CB*(CSUM((1.-RC),X,CA,1.0)-CSUM((0.0),X,CA,RC))/RD
      P21(J) = (BD*REAL(CYOL) + BC*REAL(CYCL))*R(J)/2.0
    2 IF(OMEGL.GT.50.0) P21(J) = P11(J)
   11 CONTINUE
      DO 22 J=1,N
      DO 22 K=1,N
      IF(J.EQ.K) GO TO 41
      T00CL(J,K) =-(D0PH(LC,K)*D3PH(LC,J)*R(J)**3 - D0PH(LC,J)*D3PH(LC,K
     1)*R(K)**3 - D1PH(LC,K)*D2PH(LC,J)*R(K)*R(J)**2 + D1PH(LC,J)*D2PH(
     2LC,K)*R(J)*R(K)**2)/(R(J)**4-R(K)**4)
      T22CL(J,K) =-(R(J)*R(K))**2*(D1PH(LC,J)*D2PH(LC,K)*R(J)**3 - D1PH(
     1LC,K)*D2PH(LC,J)*R(K)**3 - D0PH(LC,J)*D3PH(LC,K)*R(K)*R(J)**2 +
     2D0PH(LC,K)*D3PH(LC,J)*R(J)*R(K)**2)/(R(J)**4 - R(K)**4)
      GO TO 42
   41 T00CL(J,J) = .25*(D0PH(10,J)**2) - .25*(3.0*D0PH(LC,J)*D3PH(LC,J)/
     1R(J) + RC*D0PH(LC,J)**2 -2.0*RC*D1PH(LC,J)*D3PH(LC,J) - D1PH(LC,J
     2)*D2PH(LC,J)/R(J) + RC*D2PH(LC,J)**2)
      T22CL(J,J)=(-(3.0*D2PH(LC,J)*D1PH(LC,J) + RC*R(J)*D2PH(LC,J)**2 -
     12.0*RC*R(J)*D1PH(LC,J)*D3PH(LC,J) - D0PH(LC,J)*D3PH(LC,J) + RC*
     2R(J)*D0PH(LC,J)**2) + R(J)*D0PH(10,J)**2)*.25*R(J)**3
   42 CONTINUE
      IF(OMEGL.GT.50.0) GO TO 3
      CYOL = (0.0,0.0)
      CYCL = ( 0.0,0.0)
      DO 20 L=1,4
      DO 20 M=1,4
      CB = (B(J)+CI**(L-1))*(B(K)+CI**(M-1))
      CA =  R(J)*CI**(L-1)+R(K)*CI**(M-1)
      RD = CA**2 + X**2
      CYOL = CYOL + CB*   (CSUM(1.0,X,CA,1.)-CSUM(0.0,X,CA,0.))/RD
   20 CYCL = CYCL + CB*(CSUM((1.-RC),X,CA,1.)-CSUM(0.0,X,CA,RC))/RD
       P211(J,K)= .25*R(J)*R(K)*(BD*REAL(CYOL)+BC*REAL(CYCL))
    3 CONTINUE
      IF(J.EQ.K) GO TO 21
      P111(J,K) =(XJK(10,J,K) - RA*XJK(11,J,K) - (1.-RA)*XJK(LC,J,K)
     1-RC*(1.-RA)*CJK(10,J,K) + RC*(1.-RA)*CJK(LC,J,K))/MR
      IF(OMEGL.GT.50.0) P211(J,K) = P111(J,K)
      GO TO 22
   21 CONTINUE
      P111(J,J) = (XJJ(10,J,K) - RA*XJJ(11,J,K) - (1.-RA)*XJJ(LC,J,K)
     1-RC*(1.-RA)*CJJ(10,J,K) + RC*(1.-RA)*CJJ(LC,J,K))/MR
      IF(OMEGL.GT.50.0) P211(J,K) = P111(J,K)
   22 CONTINUE
      RETURN
      END
```

```
      FUNCTION XJJ(L,J,K)
      DIMENSION R(5),B(5),D0PH(11,5),D1PH(11,5),D2PH(11,5),D3PH(11,5)
      COMMON N,PI,XG,SPC,TO,OMEGB,GAMMA,ETA,OMEGL,N1
      COMMON RA,RC,RI
      COMMON R,B,D0PH,D1PH,D2PH,D3PH
      XL = L
      XL =XL/10.
      IF(L.EQ.11) XL= 0.0
      XJJ = .125*(6.*XL*D0PH(L,J)*D1PH(L,J)*R(J) +(D1PH(L,J)*R(J)*XL)**2
     1- 2.0*D0PH(L,J)*D2PH(L,J)*(XL*R(J))**2-2.*D2PH(L,J)*D3PH(L,J)*XL*
     2R(J) +(D3PH(L,J)*XL*R(J))**2 - 3.0*D0PH(L,J)**2 + D2PH(L,J)**2)
      RETURN
      END


      FUNCTION CJJ(L,J,K)
      DIMENSION R(5),B(5),D0PH(11,5),D1PH(11,5),D2PH(11,5),D3PH(11,5)
      COMMON N,PI,XG,SPC,TO,OMEGB,GAMMA,ETA,OMEGL,N1
      COMMON RA,RC,RI
      COMMON R,B,D0PH,D1PH,D2PH,D3PH
      XL = L
      XL =XL/10.
      IF(L.EQ.11) XL= 0.0
      CJJ =.25*(3.0*D0PH(L,J)*D1PH(L,J)*R(J) + XL*(D1PH(L,J)*R(J))**2
     1-2.*XL*D0PH(L,J)*D2PH(L,J)*R(J)**2- D2PH(L,J)*D3PH(L,J)*R(J) +
     2XL*(D3PH(L,J)*R(J))**2)
      END


      FUNCTION CSUM(Y,X,CA,Z)
      DIMENSION R(5),B(5),D0PH(11,5),D1PH(11,5),D2PH(11,5),D3PH(11,5)
      COMMON N,PI,XG,SPC,TO,OMEGB,GAMMA,ETA,OMEGL,N1
      COMMON RA,RC,RI
      COMMON R,B,D0PH,D1PH,D2PH,D3PH
      COMPLEX CA, CSUM,CEXP
      CSUM= CEXP(Z*CA)*(CA*SIN(X*Y)-X*COS(X*Y))
      RETURN
      END


      FUNCTION XJK(L,J,K)
      DIMENSION R(5),B(5),D0PH(11,5),D1PH(11,5),D2PH(11,5),D3PH(11,5)
      COMMON N,PI,XG,SPC,TO,OMEGB,GAMMA,ETA,OMEGL,N1
      COMMON RA,RC,RI
      COMMON R,B,D0PH,D1PH,D2PH,D3PH
      XL = L
      XL =XL/10.
```

```
      IF(L.EQ.11) XL= 0.0
      RD = R(J)**4 - R(K)**4
      XJK1= (XL*(D0PH(L,J)*D1PH(L,K)*R(K)*R(J)**4 - D0PH(L,K)*D1PH(L,J)*
     1R(J)*R(K)**4 - D2PH(L,K)*D3PH(L,J)*R(K)**2*R(J)**3 + D2PH(L,J)*
     2D3PH(L,K)*R(J)**2*R(K)**3))/RD
      XJK2 = -(D1PH(L,K)*D3PH(L,J)*R(K)*R(J)**3*(R(J)**4+3.0*R(K)**4) +
     1D1PH(L,J)*D3PH(L,K)*R(J)*R(K)**3*(3.0*R(J)**4+R(K)**4) - 2.0*D2PH(
     2L,K)*D2PH(L,J)*(R(J)*R(K))**2*(R(J)**4+R(K)**4)-4.*D0PH(L,J)*D0PH(
     3L,K)*(R(J)*R(K))**4)/RD**2
      XJK = XJK1 + XJK2
      RETURN
      END



      FUNCTION CJK(L,J,K)
      DIMENSION R(5),B(5),D0PH(11,5),D1PH(11,5),D2PH(11,5),D3PH(11,5)
      COMMON N,PI,XG,SPC,TO,OMEGB,GAMMA,ETA,OMEGL,N1
      COMMON RA,RC,RI
      COMMON R,B,D0PH,D1PH,D2PH,D3PH
      CJK = (D0PH(L,J)*D1PH(L,K)*R(K)*R(J)**4 - D0PH(L,K)*D1PH(L,J)*R(J)
     1*R(K)**4 - D2PH(L,K)*D3PH(L,J)*R(K)**2*R(J)**3 + D2PH(L,J)*D3PH(L,
     2K)*R(J)**2*R(K)**3)/(R(J)**4-R(K)**4)
      RETURN
      END



C     PROGRAM TO CALCULATE STABILITY CONSTANTS OF A UNIFORM BEAM SUB-
C     JECTED TO A PERIODICALLY VARYING THRUST WITH DIRECTIONAL CONTROL
      SUBROUTINE STAB(ITS, STAB2)
      DIMENSION A(7), ROOTR(6), ROOTI(6)
      DIMENSION DT(6),ACHE(6),      Y(6),W(6)
      DIMENSION RMOD1(6),RMOD2(6)
      COMMON N,PI,XG,SPC,TO,OMEGB,GAMMA,ETA,OMEGL,N1
      COMMON RA,RC,RI
      COMPLEX CZ,CARG1,CARG2,CSQRT
      CALL DETER (DT,W)
      DO 5  I=1,N1
      ACHE(I) = PI*SIN(2.*PI*W(I))*DT(I)/W(I)
      Y(I) = COS(2.*PI*W(I))
    5 CONTINUE
      A(1) = -1.
      A(2) = Y(1)+Y(2)+Y(3) - ACHE(1)-ACHE(2)-ACHE(3)
      A(3)=-(Y(1)*Y(2)+Y(1)*Y(3)+Y(2)*Y(3)-ACHE(1)*(Y(2)+Y(3))
     1-ACHE(2)*(Y(1)+Y(3))-ACHE(3)*(Y(1)+Y(2)))
      A(4) = Y(1)*Y(2)*Y(3) - ACHE(1)*Y(2)*Y(3) - ACHE(2)*Y(1)*Y(3) -
     1ACHE(3)*Y(1)*Y(2)
      CALL MULLER (A,ROOTR,ROOTI,3)
      DO 111 I=1,3
      IF(ABS(ROOTI(I)).LT..1E-4) ROOTI(I) = 0.0
      CZ = CMPLX(ROOTR(I),ROOTI(I))
      CARG1 = CZ     +CSQRT(CZ**2 - 1.)
      CARG2 = CZ - CSQRT(CZ**2 - 1.)
```

```
      RMOD1(I) = CABS(CARG1)
  111 RMOD2(I) = CABS(CARG2)
      RMAX =  AMAX1(RMOD1(1),RMOD1(2),RMOD1(3),RMOD2(1),RMOD2(2),
     1RMOD2(3))
      STAB2 = ALOG(RMAX)*OMEGB/PI
      STAB1 = STAB2 - ETA
      IF(STAB1 - .1E-6) 1,1,2
    1 ITS = 1
      WRITE(6,21) OMEGB
      GO TO 3
    2 ITS = 2
      WRITE(6,22) OMEGB
   21 FORMAT(11X6HOMEGB=F6.3,5X6HSTABLE )
   22 FORMAT(11X6HOMEGB=F6.3,5X8HUNSTABLE)
    3 WRITE(6,23) (ROOTR(I),I=1,3),(ROOTI(I),I=1,3)
      WRITE(6,50)       STAB1
   23 FORMAT(18X5HROOTR,12X,3F10.5/18X5HROOTI,12X,3F10.5)
   50 FORMAT(                        18X5HSTAB1,12XF10.5)
      RETURN
      END


      SUBROUTINE GASDET(A,N,DET)                                        IF100
      DIMENSION A(100,100)
      RETURN
      END                                                               IB101.


      SUBROUTINE MULLER(COE,ROOTR,ROOTI,N1)                             IB100
      DIMENSION COE(7),ROOTR(6),ROOTI(6)
      RETURN                                                            IF100
      END                                                               IF100


      SUBROUTINE GASINV(A,N,DET)                                        IG000
      DIMENSION A(6,6),IORD(6)
      RETURN                                                            IG000
      END


C     SUBROUTINE TO PUT MATRIX IN UPPER HESSENBERG FORM.                HES
      SUBROUTINE HESSEN(A,M)                                            HESS0
      DIMENSION A(5,5),B(4)
   30 RETURN                                                            HESS0
       END                                                              HESS0


C       PROGRAM TO CALL QR TRANSFORMATION, MAXIMUM ITER IS 50.          QRCN0
      SUBROUTINE QREIG(A,M,ROOTR,ROOTI,IPRNT)
      DIMENSION A(5,5),ROOTR(5),ROOTI(5)
  107 FORMAT(56X  E13.8)                                                QRCN0
```

```
      END                                                              QRCNO

      SUBROUTINE QRT(A,N,R,SIG,D)                                     QRT 0
      DIMENSION A(5,5),PSI(3),G(3)
 101  RETURN
       END

C                   DATA
    2 +3.141592
+4.7300408  +7.8532046  +10.9956078 +14.1371655 +17.2787596          R(I)
-0.9825022  -1.0007773  -0.99996645 -1.00000145 -0.99999994          B(I)
+1.07433   +0.19545   -0.54401   -1.04050   -1.21565   -1.04050   -0.54401
+0.19545   +1.07433   +2.00000   +0.45486   -0.79450   -1.32402   -0.96605
+0.00000   +0.96605   +1.32402   +0.79450   -0.45486   -2.00000   -0.10393
-1.28572   -0.79387   +0.65569   +1.42238   +0.65569   -0.79387   -1.28572
-0.10393   +2.00000   -0.58802   -1.20092   +0.45136   +1.40010   +0.00000
-1.40010   -0.45136   +1.20092   +0.58802   -2.00000   -0.96646   -0.61048
+1.35061   +0.22226   -1.41386   +0.22226   +1.35061   -0.61048   -0.96646
+2.00000
-1.93383   -1.74814   -1.34074   -0.73007   +0.00000   +0.73007   +1.34074
+1.74814   +1.93383   +1.96500   -1.87176   -1.21002   -0.09872   +0.94823
+1.37532   +0.94823   -0.09872   -1.21002   -1.87176   -2.00155 -1.67795
-0.33199   +1.10762   +1.24912   +0.00000   -1.24912   -1.10762   +0.33199
+1.67795   +1.99993   -1.38736   +0.58286   +1.33056   -0.22494   -1.415192
-0.22494   +1.33056   +0.58286   -1.38736   -2.00000   -1.00891   +1.22851
+0.43141   -1.39777   +0.00000   +1.39777   -0.43141   -1.22851   +1.00891
+2.00000
+0.18910   +0.61939   +1.09600   +1.45545   +1.58815   +1.45545   +1.09600
+0.61939   +0.18910   +0.00000   +0.45573   +1.20674   +1.50550   +1.03457
+0.00000   -1.03457   -1.50550   -1.20674   -0.45573   +0.00000   +0.77005
+1.50782   +0.86864   -0.62837   -1.40600   -0.62837   +0.86864   +1.50782
+0.77005   +0.00000   +1.07449   +1.31923   -0.42268   -1.39351   +0.00000
+1.39351   +0.42268   -1.31923   -1.07449   +0.00000   +1.32178   +0.67360
-1.33938   -0.22021   +1.41457   -0.22021   -1.33938   +0.67360   +1.32178
+0.00000
+0.72655   +1.02342   +0.93338   +0.54723   +0.00000   -0.54723   -0.93338
-1.02342   -0.72655   +0.00000   +0.95776   +0.79030   -0.09916   -1.05271
-1.45420   -1.05271   -0.09916   +0.79030   +0.95776   +0.00000   +1.01202
+0.11050   -1.18057   -1.27099   +0.00000   +1.27099   +1.18057   -0.11050
-1.01202   +0.00000   +0.90088   -0.70122   -1.35944   +0.21753   +1.41251
+0.21753   -1.35944   -0.70122   +0.90080   +0.00000   +0.65359   -1.29164
-0.44262   +1.39584   +0.00000   -1.39584   +0.44262   +1.29164   -0.65359
+0.00000
+.50          +.50          +.1                                        XG
+100.0        +100.0        +100.0                                   OMEGL
+0.00         +0.00         +0.01                                     ETA
 +1.0          +1.0         +0.1                                       SPC
+0.10         +0.10         +0.05                                    GAMMA
+0.00         +20.0         +2.0                                     TO
+0.60 +0.60 +0.60 +0.60 +0.60 +0.60 +0.60 +0.60 +0.60             RC RA RI
    1     1     0                                                    CONTR
+20.0         +20.0         +5.0                                     TO
```

```
0     0     1                                                              CONTR
1     0     2                                                              CONTR
  END OF LISTING
```

FORMAT (1)

NATURAL FREQUENCIES

TO	FREQ(1)	FREQ(2)	FREQ(3)
GAMMA= .10, N=2,OMEGL=100.00, ETA= .000, SPC=1.00, XG= .5			
RC= .60, RA= .60, RI= .60			
.0	.000086	.973546	2.778828
2.0	.162000	.957970	2.742831
4.0	.222785	.943613	2.706368
6.0	.263711	.930871	2.669420
8.0	.292027	.920191	2.631964
10.0	.310043	.912042	2.593976
12.0	.318480	.906861	2.555432
14.0	.317333	.904976	2.516302
16.0	.306115	.906547	2.476554
18.0	.283750	.911535	2.436153
20.0	.247899	.919723	2.395058

2

FORMAT (2)

CHARACTERISTIC VALUES IN VICINITY OF 2 X FREQ(1)

TO	FREQ(1)	FREQ(2)	FREQ(3)
20.0	.247899	.919723	2.395058

GAMMA= .10, N=2,OMEGL=100.00, ETA= .000, SPC=1.00, XG= .5

RC= .60, RA= .60, RI= .60

OMEGB= .496	UNSTABLE			
	ROOTR	.63676	.48731	-1.03409
	ROOTI	.00000	.00000	.00000
	STAB1	.04109		

CHARACTERISTIC VALUES IN VICINITY OF 2 X FREQ(2)

TO	FREQ(1)	FREQ(2)	FREQ(3)
20.0	.247899	.919723	2.395058

GAMMA= .10, N=2,OMEGL=100.00, ETA= .000, SPC=1.00, XG= .5

RC= .60, RA= .60, RI= .60

OMEGB= 1.839	UNSTABLE			
	ROOTR	.66108	-.32187	-1.00010
	ROOTI	.00000	.00000	.00000
	STAB1	.00824		

CHARACTERISTIC VALUES IN VICINITY OF 2 X FREQ(3)

TO	FREQ(1)	FREQ(2)	FREQ(3)
20.0	.247899	.919723	2.395058

GAMMA= .10, N=2,OMEGL=100.00, ETA= .000, SPC=1.00, XG= .5

RC= .60, RA= .60, RI= .60

OMEGB= 4.790	UNSTABLE			
	ROOTR	.94752	.35639	-1.00037
	ROOTI	.00000	.00000	.00000
	STAB1	.04145		

2

FORMAT (2) (continued)

```
CHARACTERISTIC VALUES IN VICINITY OF FREQ(2)  + FREQ(1)

     TO          FREQ(1)         FREQ(2)          FREQ(3)

    20.0         .247899         .919723          2.395058

  OMEGB= 1.168      UNSTABLE
         ROOTR                 .94892     .23086     .23086
         ROOTI                 .00000    -.12482     .12482
         STAB1                 .04753

CHARACTERISTIC VALUES IN VICINITY OF FREQ(3)  + FREQ(1)

     TO          FREQ(1)         FREQ(2)          FREQ(3)

    20.0         .247899         .919723          2.395058

  OMEGB= 2.643      UNSTABLE
         ROOTR                 .83091     .83091    -.57783
         ROOTI                -.01520     .01520     .00000
         STAB1                 .02297

CHARACTERISTIC VALUES IN VICINITY OF FREQ(3)  + FREQ(2)

     TO          FREQ(1)         FREQ(2)          FREQ(3)

    20.0         .247899         .919723          2.395058

  OMEGB= 3.315      STABLE
         ROOTR                 .89112    -.16043    -.18359
         ROOTI                 .00000     .00000     .00000
         STAB1                 .00000
```

2

FORMAT (2) (continued)

CHARACTERISTIC VALUES IN VICINITY OF FREQ(2) - FREQ(1)

TO	FREQ(1)	FREQ(2)	FREQ(3)
20.0	.247899	.919723	2.395058

GAMMA= .10, N=2,OMEGL=100.00, ETA= .000, SPC=1.00, XG= .5

RC= .60, RA= .60, RI= .60

OMEGB= .672	STABLE			
	ROOTR	-.91185	-.50980	-.81120
	ROOTI	.00000	.00000	.00000
	STAB1	.00000		

CHARACTERISTIC VALUES IN VICINITY OF FREQ(3) - FREQ(1)

TO	FREQ(1)	FREQ(2)	FREQ(3)
20.0	.247899	.919723	2.395058

GAMMA= .10, N=2,OMEGL=100.00, ETA= .000, SPC=1.00, XG= .5

RC= .60, RA= .60, RI= .60

OMEGB= 2.147	STABLE			
	ROOTR	.77009	.72547	-.90064
	ROOTI	.00000	.00000	.00000
	STAB1	.00000		

CHARACTERISTIC VALUES IN VICINITY OF FREQ(3) - FREQ(2)

TO	FREQ(1)	FREQ(2)	FREQ(3)
20.0	.247899	.919723	2.395058

GAMMA= .10, N=2,OMEGL=100.00, ETA= .000, SPC=1.00, XG= .5

RC= .60, RA= .60, RI= .60

OMEGB= 1.475	UNSTABLE			
	ROOTR	.48586	-.71303	-.71303
	ROOTI	.00000	-.01819	.01819
	STAB1	.01218		

FORMAT (3)

CHARACTERISTIC VALUES IN VICINITY OF 2 X FREQ(1)

TO	FREQ(1)	FREQ(2)	FREQ(3)
20.0	.247899	.919723	2.395058

GAMMA= .10, N=2,OMEGL=100.00, ETA= .000, SPC=1.00, XG= .5

RC= .60, RA= .60, RI= .60

OMEGB=	.496	UNSTABLE			
	ROOTR		.63676	.48731	-1.03409
	ROOTI		.00000	.00000	.00000
	STAB1		.04109		
OMEGB=	.446	UNSTABLE			
	ROOTR		.92018	-.65350	-1.00072
	ROOTI		.00000	.00000	.00000
	STAB1		.00538		
OMEGB=	.397	STABLE			
	ROOTR		.98257	-.43555	-.77440
	ROOTI		.00000	.00000	.00000
	STAB1		.00000		
OMEGB=	.422	STABLE			
	ROOTR		.40175	-.44935	-.89647
	ROOTI		.00000	.00000	.00000
	STAB1		.00000		
OMEGB=	.434	UNSTABLE			
	ROOTR		.71951	-.97765	-.97765
	ROOTI		.00000	-.01460	.01460
	STAB1		.00916		
OMEGB=	.545	STABLE			
	ROOTR		-.96692	-.33979	-.78556
	ROOTI		.00000	.00000	.00000
	STAB1		.00000		
OMEGB=	.520	UNSTABLE			
	ROOTR		-.80219	.14722	-1.01069
	ROOTI		.00000	.00000	.00000
	STAB1		.02420		
OMEGB=	.533	STABLE			
	ROOTR		-.99108	-.10545	-.99942
	ROOTI		.00000	.00000	.00000
	STAB1		.00000		

CHARACTERISTIC VALUES IN VICINITY OF 2 X FREQ(2)

TO	FREQ(1)	FREQ(2)	FREQ(3)
20.0	.247899	.919723	2.395058

GAMMA= .10, N=2,OMEGL=100.00, ETA= .000, SPC=1.00, XG= .5

FORMAT (3) (continued)

RC= .60, RA= .60, RI= .60

OMEGR= 1.839		UNSTABLE			
	ROOTR		.66108	-.32187	-1.00010
	ROOTI		.00000	.00000	.00000
	STAB1		.00624		
OMEGR= 1.830		STABLE			
	ROOTR		.65754	-.36340	-.99994
	ROOTI		.00000	.00000	.00000
	STAB1		-.00000		
OMEGR= 1.834		UNSTABLE			
	ROOTR		.65932	-.34266	-1.00006
	ROOTI		.00000	.00000	.00000
	STAB1		.00620		
OMEGR= 1.832		UNSTABLE			
	ROOTR		.65843	-.35304	-1.00001
	ROOTI		.00000	.00000	.00000
	STAB1		.00228		
OMEGR= 1.849		STABLE			
	ROOTR		.66457	-.28016	-.99997
	ROOTI		.00000	.00000	.00000
	STAB1		-.00000		
OMEGR= 1.844		UNSTABLE			
	ROOTR		.66283	-.30103	-1.00007
	ROOTI		.00000	.00000	.00000
	STAB1		.00697		
OMEGR= 1.847		UNSTABLE			
	ROOTR		.66370	-.29060	-1.00003
	ROOTI		.00000	.00000	.00000
	STAB1		.00452		

CHARACTERISTIC VALUES IN VICINITY OF 2 X FREQ(3)

TO	FREQ(1)	FREQ(2)	FREQ(3)
20.0	.247899	.919723	2.395058

GAMMA= .10, N=2,OMEGL=100.00, ETA= .000, SPC=1.00, XG= .5

RC= .60, RA= .60, RI= .60

OMEGR= 4.790		UNSTABLE			
	ROOTR		.94752	.35639	-1.00037
	ROOTI		.00000	.00000	.00000
	STAB1		.04145		
OMEGR= 4.740		STABLE			
	ROOTR		.94642	.34453	-.99984
	ROOTI		.00000	.00000	.00000
	STAB1		.00000		
OMEGR= 4.765		UNSTABLE			
	ROOTR		.94697	.35050	-1.00024
	ROOTI		.00000	.00000	.00000
	STAB1		.03323		
OMEGR= 4.753		UNSTABLE			

FORMAT (3) (continued)

	ROOTR		.94669	.34752	-1.00007
	ROOTI		.00000	.00000	.00000
	STAB1		.01827		
OMEGB=	4.840	STABLE			
	ROOTR		.94858	.36794	-.99984
	ROOTI		.00000	.00000	.00000
	STAB1		.00000		
OMEGB=	4.815	UNSTABLE			
	ROOTR		.94805	.36220	-1.00023
	ROOTI		.00000	.00000	.00000
	STAB1		.03307		
OMEGB=	4.827	UNSTABLE			
	ROOTR		.94832	.36508	-1.00007
	ROOTI		.00000	.00000	.00000
	STAB1		.01782		

FORMAT (3) (continued)

CHARACTERISTIC VALUES IN VICINITY OF FREQ(2) + FREQ(1)

TO	FREQ(1)	FREQ(2)	FREQ(3)
20.0	.247899	.919723	2.395058

OMEGB=	1.168	UNSTABLE			
	ROOTR		.94892	.23086	.23086
	ROOTI		.00000	-.12482	.12482
	STAB1		.04753		
OMEGB=	1.111	STABLE			
	ROOTR		.39889	.55181	.23435
	ROOTI		.00000	.00000	.00000
	STAB1		.00000		
OMEGB=	1.139	UNSTABLE			
	ROOTR		.79970	.27357	.27357
	ROOTI		.00000	-.10255	.10255
	STAB1		.03857		
OMEGB=	1.125	UNSTABLE			
	ROOTR		.68803	.29486	.29486
	ROOTI		.00000	-.05907	.05907
	STAB1		.02212		
OMEGB=	1.225	STABLE			
	ROOTR		.96118	.23124	.06337
	ROOTI		.00000	.00000	.00000
	STAB1		-.00000		
OMEGB=	1.196	UNSTABLE			
	ROOTR		.99990	.18850	.18850
	ROOTI		.00000	-.09672	.09672
	STAB1		.03743		
OMEGB=	1.210	UNSTABLE			
	ROOTR		.99091	.16772	.16772
	ROOTI		.00000	-.04841	.04841
	STAB1		.01691		

CHARACTERISTIC VALUES IN VICINITY OF FREQ(3) + FREQ(1)

TO	FREQ(1)	FREQ(2)	FREQ(3)
20.0	.247899	.919723	2.395058

OMEGB=	2.643	UNSTABLE			
	ROOTR		.83091	.83091	-.57783
	ROOTI		-.01520	.01520	.00000
	STAB1		.02297		
OMEGB=	2.615	STABLE			
	ROOTR		.85513	.83505	-.59648
	ROOTI		.00000	.00000	.00000
	STAB1		-.00000		
OMEGB=	2.629	UNSTABLE			

FORMAT (3) (continued)

```
        ROOTR                 .83615     .83815    -.58715
        ROOTI                -.01187     .01187     .00000
        STAB1                 .01620
OMEGR= 2.622     UNSTABLE
        ROOTR                 .84166     .84166    -.59181
        ROOTI                -.00617     .00617     .00000
        STAB1                 .00953
OMEGR= 2.671     STABLE
        ROOTR                 .82566     .80544    -.55927
        ROOTI                 .00000     .00000     .00000
        STAB1                 .00000
OMEGR= 2.657     UNSTABLE
        ROOTR                 .82337     .82337    -.56854
        ROOTI                -.01244     .01244     .00000
        STAB1                 .01653
OMEGR= 2.664     UNSTABLE
        ROOTR                 .81949     .81949    -.56390
        ROOTI                -.00705     .00705     .00000
        STAB1                 .01043
```

CHARACTERISTIC VALUES IN VICINITY OF FREQ(3) + FREQ(2)

TO	FREQ(1)	FREQ(2)	FREQ(3)
20.0	.247899	.919723	2.395058

```
OMEGR= 3.315     STABLE
        ROOTR                 .89112    -.16043    -.18359
        ROOTI                 .00000     .00000     .00000
        STAB1                 .00000
```

FORMAT (3) (continued)

```
     CHARACTERISTIC VALUES IN VICINITY OF FREQ(2)  - FREQ(1)

       TO            FREQ(1)          FREQ(2)          FREQ(3)

      20.0           .247899          .919723          2.395058

 GAMMA= .10, N=2,OMEGL=100.00, ETA=  .000, SPC=1.00, XG= .5

  RC= .60, RA= .60, RI= .60

    OMEGB=  .672      STABLE
            ROOTR                 -.91185    -.50980    -.81120
            ROOTI                  .00000     .00000     .00000
            STAB1                  .00000

     CHARACTERISTIC VALUES IN VICINITY OF FREQ(3)  - FREQ(1)

       TO            FREQ(1)          FREQ(2)          FREQ(3)

      20.0           .247899          .919723          2.395058

 GAMMA= .10, N=2,OMEGL=100.00, ETA=  .000, SPC=1.00, XG= .5

  RC= .60, RA= .60, RI= .60

    OMEGB= 2.147      STABLE
            ROOTR                  .77009     .72547    -.90064
            ROOTI                  .00000     .00000     .00000
            STAB1                  .00000

     CHARACTERISTIC VALUES IN VICINITY OF FREQ(3)  - FREQ(2)

       TO            FREQ(1)          FREQ(2)          FREQ(3)

      20.0           .247899          .919723          2.395058

 GAMMA= .10, N=2,OMEGL=100.00, ETA=  .000, SPC=1.00, XG= .5

  RC= .60, RA= .60, RI= .60

    OMEGB= 1.475      UNSTABLE
            ROOTR                  .48586    -.71303    -.71303
            ROOTI                  .00000    -.01819     .01819
            STAB1                  .01218
    OMEGB= 1.461      STABLE
            ROOTR                  .47731    -.65166    -.67122
            ROOTI                  .00000     .00000     .00000
```

FORMAT (3) (continued)

```
          STAB1                    -.00000
OMEGB=  1.468       UNSTABLE
          ROOTR                     .48145     -.68786     -.68786
          ROOTI                     .00000     -.01630      .01630
          STAB1                     .01049
OMEGB=  1.464       UNSTABLE
          ROOTR                     .47932     -.67480     -.67480
          ROOTI                     .00000     -.01124      .01124
          STAB1                     .00710
OMEGB=  1.490       STABLE
          ROOTR                     .49488     -.74613     -.77290
          ROOTI                     .00000      .00000      .00000
          STAB1                     .00000
OMEGB=  1.483       UNSTABLE
          ROOTR                     .49037     -.73692     -.73692
          ROOTI                     .00000     -.01279      .01279
          STAB1                     .00893
OMEGB=  1.486       UNSTABLE
          ROOTR                     .49262     -.74838     -.74838
          ROOTI                     .00000     -.00329      .00329
          STAB1                     .00235

STOP
```