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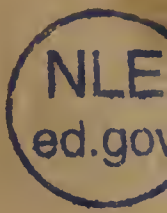
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James Pinkard





A

COURSE ✓

OF

MATHEMATICS,

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IN TWO VOLUMES:

COMPOSED, AND MORE ESPECIALLY DESIGNED,  
FOR THE USE OF THE GENTLEMEN CADETS  
IN THE ROYAL MILITARY ACADEMY AT WOOLWICH.

BY

CHARLES HUTTON, LL.D., F.R.S.

AND PROFESSOR OF MATHEMATICS IN THE SAID ACADEMY.

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THE THIRD EDITION,  
ENLARGED AND CORRECTED.

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VOL. II.

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LONDON:  
PRINTED FOR THE AUTHOR,  
AND FOR G. G. AND J. ROBINSON, PATERNOSTER-ROW,  
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# PLANE TRIGONOMETRY.

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## DEFINITIONS.

1. **P**LANE TRIGONOMETRY treats of the relations and calculations of the sides and angles of plane triangles.

2. The circumference of every circle (as before observed in Geom. Def. 56) is supposed to be divided into 360 equal parts, called Degrees; also each degree into 60 Minutes, each minute into 60 Seconds, and so on.

Hence a semicircle contains 180 degrees, and a quadrant 90 degrees.

3. The Measure of any angle (Def. 57, Geom.) is an arc of any circle contained between the two lines which form that angle, the angular point being the centre; and it is estimated by the number of degrees contained in that arc.

Hence, a right angle, being measured by a quadrant, or quarter of the circle, is an angle of 90 degrees; and the sum of the three angles of every triangle, or two right angles, is equal to 180 degrees. Therefore, in a right-angled triangle, taking one of the acute angles from 90 degrees, leaves the other acute angle; and the sum of two angles, in any triangle, taken from 180 degrees, leaves the third angle; or one angle being taken from 180 degrees, leaves the sum of the other two angles.

4. Degrees are marked at the top of the figure with a small  $^{\circ}$ , minutes with  $'$ , seconds with  $''$ , and so on. Thus,  $57^{\circ} 30' 12''$ , denote 57 degrees 30 minutes and 12 seconds.

5. The Complement of an arc, is what it wants of a quadrant or  $90^\circ$ . Thus, if AD be a quadrant, then BD is the complement of the arc AB; and, reciprocally, AB is the complement of BD. So that, if AB be an arc of  $50^\circ$ , then its complement BD will be  $40^\circ$ .

6. The Supplement of an arc, is what it wants of a semicircle, or  $180^\circ$ .

Thus, if ADE be a semicircle, then BDE is the supplement of the arc AB; and reciprocally, AB is the supplement of the arc BDE. So that, if AB be an arc of  $50^\circ$ , then its supplement BDE will be  $130^\circ$ .

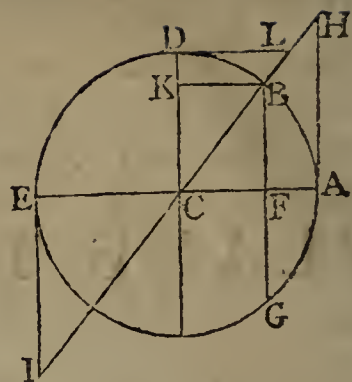
7. The Sine, or Right Sine, of an arc, is the line drawn from one extremity of the arc, perpendicular to the diameter which passes through the other extremity. Thus, BF is the sine of the arc AB, or of the arc BDE.

*Corol.* Hence the sine (BF) is half the chord (BG) of the double arc (BAG).

8. The Versed Sine of an arc, is the part of the diameter intercepted between the arc and its sine. So, AF is the versed sine of the arc AB, and EF the versed sine of the arc EDB.

9. The Tangent of an arc, is a line touching the circle in one extremity of that arc, continued from thence to meet a line drawn from the centre through the other extremity: which last line is called the Secant of the same arc. Thus, AH is the tangent, and CH the secant, of the arc AB. Also, EI is the tangent, and CI the secant, of the supplemental arc BDE. And this latter tangent and secant are equal to the former, but are accounted negative, as being drawn in an opposite or contrary direction to the former.

10. The Cosine, Cotangent, and Cosecant, of an arc, are the sine, tangent, and secant of the complement of that arc, the Co being only a contraction of the word complement. Thus, the arcs AB, BD being the complements of each other, the sine, tangent or secant of the one of these, is the cosine, cotangent or cosecant of the other. So, BF, the sine of AB, is the cosine of BD; and BK, the sine of BD, is the cosine of AB: in like manner, AH, the tangent of AB, is the cotangent of BD; and DL, the tangent of DB, is the cotangent of AB: also, CH, the secant of AB, is the cosecant of BD; and CL, the secant of BD, is the cosecant of AB.



*Corol.* Hence several remarkable properties easily follow from these definitions; as,

1<sup>st</sup>. That an arc and its supplement have the same sine, tangent, and secant; but the two latter, the tangent and secant, are accounted negative when the arc is greater than a quadrant or 90 degrees.

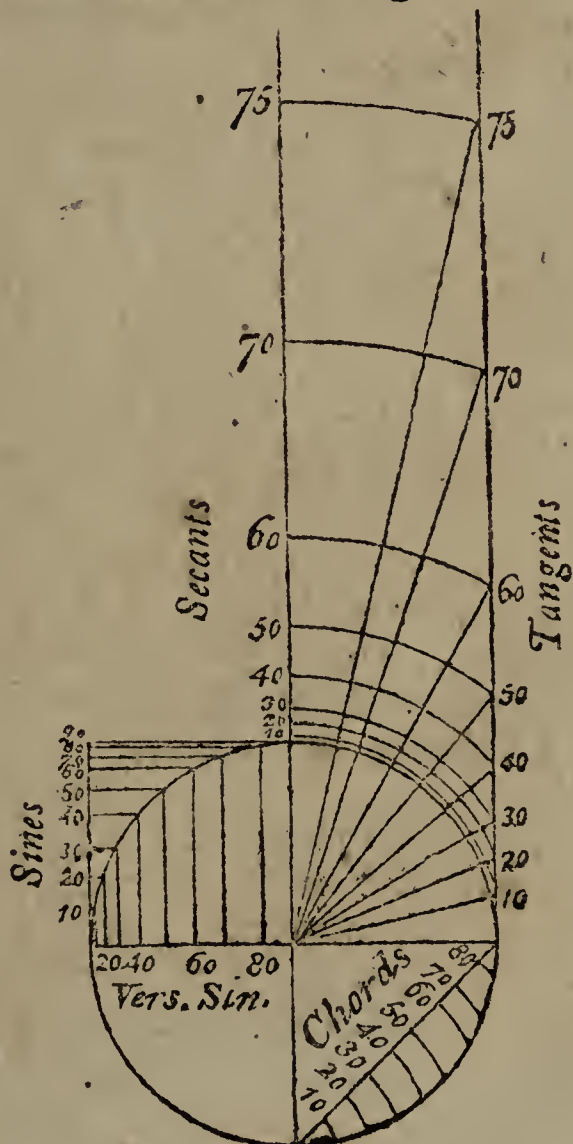
2<sup>d</sup>. When the arc is 0, or nothing, the sine and tangent are nothing, but the secant is then the radius CA.—But when the arc is a quadrant AD, then the sine is the greatest it can be, being the radius CD of the circle; and both the tangent and secant are infinite.

3<sup>d</sup>. Of any arc AB, the versed sine AF, and cosine BK, or CF, together make up the radius CA of the circle.—The radius CA, tangent AH, and secant CH, form a right angled triangle CAH. So also do the radius, sine, and cosine, form another right-angled triangle CBF or CBK. As also the radius, cotangent and cosecant, another right-angled triangle CDL. And all these right-angled triangles are similar to each other.

11. The sine, tangent, or secant of an angle, is the sine, tangent, or secant of the arc by which the angle is measured, or of the degrees, &c. in the same arc or angle.

12. The method of constructing the scales of chords, sines, tangents, and secants, usually engraven on instruments, for practice, is exhibited in the annexed figure.

13. A Trigonometrical Canon, is a table exhibiting the length of the sine, tangent, and secant, to every degree and minute of the quadrant, with respect to the radius, which is expressed by unity or 1, and conceived to be divided into 10000000 or more decimal parts. And farther, the logarithms of these sines, tangents, and secants are also ranged in the tables; which are most commonly used, as they perform the calculations by only addition and subtraction, instead of the multiplication



and division by the natural sines, &c, according to the nature of logarithms.

Upon this table depends the numeral solution of the several cases in trigonometry. It will therefore be proper to begin with the mode of constructing it, which may be done in the following manner :

PROBLEM I.

*To find the Sine and Cosine of a Given Arc.*

THIS problem is resolved after various ways. One of these is as follows, viz. by means of the ratio between the diameter and circumference of a circle, together with the known series for the sine and cosine, hereafter demonstrated. Thus, the semicircumference of the circle, whose radius is 1, being 3.141592653589793 &c, the proportion will therefore be, as the number of degrees or minutes in the semicircle, is to the degrees or minutes in the proposed arc, so is 3.14159265 &c, to the length of the said arc.

This length of the arc being denoted by the letter  $a$ ; and its sine and cosine by  $s$  and  $c$ ; then will these two be expressed by the two following series, viz.

$$\begin{aligned} s &= a - \frac{a^3}{2.3} + \frac{a^5}{2.3.4.5} - \frac{a^7}{2.3.4.5.6.7} + \&c. \\ &= a - \frac{a^3}{6} + \frac{a^5}{120} - \frac{a^7}{5040} + \&c. \\ c &= 1 - \frac{a^2}{2} + \frac{a^4}{2.3.4} - \frac{a^6}{2.3.4.5.6} + \&c. \\ &= 1 - \frac{a^2}{2} + \frac{a^4}{24} - \frac{a^6}{720} + \&c. \end{aligned}$$

*Example I.* If it be required to find the sine and cosine of one minute. Then, the number of minutes in  $180^\circ$  being 10800, it will be first, as  $10800 : 1 :: 3.14159265 \&c. : .000290888208665$  = the length of an arc of one minute. Therefore, in this case,

$$\begin{aligned} a &= .0002908882 \\ \text{and } \frac{1}{6}a^3 &= .000000000004 \&c, \\ \text{the diff. is } s &= .0002908882 \text{ the sine of 1 minute.} \\ \text{Also, from } 1. & \\ \text{take } \frac{1}{2}a^2 &= 0.0000000423079 \&c, \\ \text{leaves } c &= .9999999577 \text{ the cosine of 1 minute.} \end{aligned}$$

*Exam-*

*Example 2.* For the sine and cosine of 5 degrees.  
 Here, as  $180^\circ : 5^\circ :: 3.14159265 \text{ \&c.} : .08726646 = a$  the length of 5 degrees. Hence  $a = .08726646$

$$- \frac{1}{6}a^3 = - .00011076$$

$$+ \frac{1}{120}a^5 = .00000004$$

these collected give  $s = .08715574$  the sine of  $5^\circ$ .

And, for the cosine,  $1 = 1$ .

$$- \frac{1}{2}a^2 = - .00380771$$

$$+ \frac{1}{24}a^4 = .00000241$$

these collected give  $c = .99619470$  the cosine of  $5^\circ$ .

After the same manner, the sine and cosine of any other arc may be computed. But the greater the arc is, the slower the series will converge, in which case a greater number of terms must be taken to bring out the conclusion to the same degree of exactness.

Or, having found the sine, the cosine will be found from it, by the property of the right-angled triangle CBF, viz. the cosine  $CF = \sqrt{CB^2 - BF^2}$ , or  $c = \sqrt{1 - s^2}$ .

There are also other methods of constructing the canon of sines and cosines, which, for brevity's sake, are here omitted.

PROBLEM II.

*To compute the Tangents and Secants.*

THE sines and cosines being known, or found by the foregoing problem; the tangents and secants will be easily found, from the principle of similar triangles, in the following manner;

In the first figure, where, of the arc AB, BF is the sine, CF or BK the cosine, AH the tangent, CH the secant, DL the cotangent, and CL the cosecant, the radius being CA or CB or CD; the three similar triangles CFB, CAH, CDL give the following proportions:

1<sup>st</sup>,  $CF : FB :: CA : AH$ ; whence the tangent is known, being a fourth proportional to the cosine, sine, and radius.

2<sup>d</sup>,  $CF : CB :: CA : CH$ ; whence the secant is known, being a third proportional to the cosine and radius.

3d, -  $BF : FC :: CD : DL$ ; whence the cotangent is known, being a fourth proportional to the sine, cosine, and radius.

4th,  $BF : BC :: CD : CL$ ; whence the cosecant is known, being a third proportional to the sine and radius.

Having given an idea of the calculation of sines, tangents and secants, we may now proceed to resolve the several cases of Trigonometry; previous to which, however, it may be proper to add a few preparatory notes and observations, as below.

*Note 1.* There are usually three methods of resolving triangles, or the cases of trigonometry; namely, Geometrical Construction, Arithmetical Computation, and Instrumental Operation.

*In the First Method,* The triangle is constructed, by making the parts of the given magnitudes, namely the sides from a scale of equal parts, and the angles from a scale of chords, or by some other instrument. Then, measuring the unknown parts, by the same scales or instruments, the solution will be obtained near the truth.

*In the Second Method,* Having stated the terms of the proportion according to the proper rule or theorem, resolve it like any other proportion, in which a fourth term is to be found from three given terms, by multiplying the second and third together, and dividing the product by the first, in working with the natural numbers: or, in working with the logarithms, add the logs. of the second and third terms together, and from the sum take the log. of the first term; then the natural number answering to the remainder is the fourth term sought.

*In the Third Method,* Or Instrumentally, as suppose by the log. lines on one side of the common two-foot scales; Extend the Compasses from the first term, to the second or third, which happens to be of the same kind with it; then that extent will reach from the other term to the fourth term, as required, taking both extents towards the same end of the scale.

*Note 2.* Every triangle has six parts, viz. three sides and three angles. And in every triangle, or case in trigonometry, there must be given three parts, to find the other three. Also, of the three parts that are given, one of them at least must be a side; because, with the same angles the sides may be greater or less in any proportion,

*Note 3.*

*Note 3.* All the cases in trigonometry, may be comprised in three varieties only; viz.

- 1<sup>st</sup>, When a side and its opposite angle are given.
- 2<sup>d</sup>, When two sides and the contained angle are given.
- 3<sup>d</sup>, When the three sides are given.

For there cannot possibly be more than these three varieties of cases; for each of which it will therefore be proper to give a separate theorem, as follows:

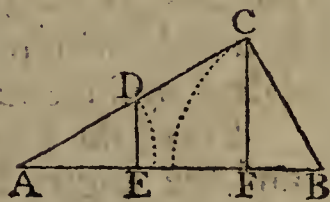
## THEOREM I.

*When a Side and its Opposite Angle, are two of the Given Parts.*

THEN the sides of the triangle have the same proportion to each other, as the sines of their opposite angles have.

That is, As any one side,  
Is to the sine of its opposite angle;  
So is any other side,  
To the sine of its opposite angle.

*Demonstr.* For, let ABC be the proposed triangle, having AB the greatest side, and BC the least. Take AD = BC, considering it as a radius; and let fall the perpendiculars DE, CF, which will evidently be the sines of the angles A and B, to the radius AD or BC.



But the triangles ADE, ACF are equiangular, and therefore have their like sides proportional, namely, AC : CF :: AD or BC : DE; that is, AC is to the sine of its opposite angle B, as BC to the sine of its opposite angle A.

*Note 1.* In practice, to find an angle, begin the proportion with a side opposite a given angle. And to find a side, begin with an angle opposite a given side.

*Note 2.* An angle found by this rule is ambiguous, or uncertain whether it be acute or obtuse, unless it be a right angle, or unless its magnitude be such as to prevent the ambiguity; because the sine answers to two angles, which are supplements to each other: and accordingly the geometrical construction forms two triangles with the same parts that are given, as in the example below; and when there is no restriction or limitation included in the question, either of them may be taken. The degrees in the table, answering to the sine, is the acute angle; but if the angle be obtuse, subtract

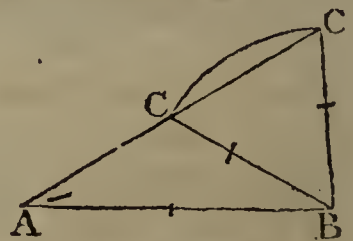
subtract those degrees from  $180^\circ$ , and the remainder will be the obtuse angle. When a given angle is obtuse, or a right one, there can be no ambiguity; for then neither of the other angles can be obtuse, and the geometrical construction will form only one triangle.

EXAMPLE I.

In the plane triangle ABC,

Given  $\begin{cases} AB & 345 \text{ yards} \\ BC & 232 \text{ yards} \\ \angle A & 37^\circ 20' \end{cases}$

Required the other parts.



1. Geometrically.

Draw an indefinite line, upon which set off  $AB = 345$ , from some convenient scale of equal parts.—Make the angle  $A = 37^\circ \frac{1}{3}$ .—With a radius of 232, taken from the same scale of equal parts, and centre B, cross AC in the two points C, C'.—Lastly, join BC, BC', and the figure is constructed, which gives two triangles, and shewing that the case is ambiguous.

Then, the sides AC measured by the scale of equal parts, and the angles B and C measured by the line of chords, or other instrument, will be found to be nearly as below; viz.

AC 174             $\angle B$   $27^\circ$              $\angle C$   $115^\circ \frac{1}{2}$   
 or  $374 \frac{1}{2}$             or  $78 \frac{1}{4}$             or  $64 \frac{1}{2}$

2. Arithmetically.

First, to find the angles at C.

As side	BC	232	-	-	log.	2.3654880
To sin. op.	$\angle A$	$37^\circ 20'$	-	-	-	9.7827958
So side	AB	345	-	-	-	2.5378191
To sin. op.	$\angle C$	$115^\circ 36'$ or $64^\circ 24'$	-	-	-	9.9551269
add	$\angle A$	37 20	37 20			<hr/>
		<hr/>	<hr/>			
the sum		152 56	or 101 44			
taken from		180 00	180 00			
		<hr/>	<hr/>			
leaves	$\angle B$	27 04	or 78 16			
		<hr/>	<hr/>			

Then,



Then, to find the side AC.

As sine	$\angle A$	$37^{\circ} 20'$	-	-	log.	9.7827958
To op. side	BC	232	-	-		2.3654880
So sin.	$\angle B$	$\left\{ \begin{array}{l} 27^{\circ} 04' \\ 78 \quad 16 \end{array} \right.$	-	-		9.6580371
To op. side	AC	$174^{\circ} 07'$	-	-		9.9908291
	or	$374^{\circ} 56'$	-	-		2.2467293
						2.5735213

---

3. *Instrumentally.*

In the first proportion.—Extend the compasses from 232 to 345 upon the line of numbers; then that extent will reach, on the sines, from  $37^{\circ}\frac{1}{3}$  to  $64^{\circ}\frac{1}{2}$ , the angle C.

In the second proportion.—Extend the compasses from  $37^{\circ}\frac{1}{3}$  to  $27^{\circ}$  or  $78^{\circ}\frac{1}{4}$ , on the sines; then that extent will reach, on the line of numbers, from 232 to 174 or  $374\frac{1}{2}$ , the two values of the side AC.

EXAMPLE II.

In the plane triangle ABC,

Given	$\left\{ \begin{array}{l} AB \ 365 \text{ poles} \\ \angle A \ 57^{\circ} 12' \\ \angle B \ 24 \ 45 \end{array} \right.$	Anf.	$\left\{ \begin{array}{l} \angle C \ 98^{\circ} 3' \\ AC \ 154.33 \\ BC \ 309.86 \end{array} \right.$
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Required the other parts.

EXAMPLE III.

In the plane triangle ABC,

Given	$\left\{ \begin{array}{l} AC \ 120 \text{ feet} \\ BC \ 112 \text{ feet} \\ \angle A \ 57^{\circ} 27' \end{array} \right.$	Anf.	$\left\{ \begin{array}{l} \angle B \ 64^{\circ} 34' 21'' \\ \text{or } 115 \ 25 \ 39 \\ \angle C \ 57 \ 58 \ 39 \\ \text{or } 7 \ 7 \ 21 \\ AB \ 112.65 \text{ feet} \\ \text{or } 16.47 \text{ feet.} \end{array} \right.$
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Required the other parts.

THEOREM II.

*When two Sides and their Contained Angle are given.*

THEN it will be,

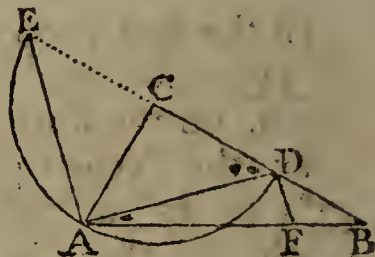
As the sum of those two sides,  
Is to the difference of the same sides;  
So is the tang. of half the sum of their op. angles,  
To the tang. of half the diff. of the same angles.

Hence,

Hence, because it has been shewn (Alg. p. 243), that the half sum of any two quantities increased by their half difference, gives the greater, and diminished by it gives the less, if the half difference of the angles, so found, be added to their half sum, it will give the greater angle, and subtracting it will leave the less angle.

Then all the angles being now known, the unknown side will be found by the former theorem.

*Demonst.* Let  $ABC$  be the proposed triangle, having the two given sides  $AC$ ,  $BC$ , including the given angle  $C$ . With the centre  $C$ , and radius  $CA$ , the less of these two sides, describe a semicircle, meeting the other side  $BC$  produced in  $D$  and  $E$ . Join  $AE$ ,  $AD$ , and draw  $DF$  parallel to  $AE$ .



Then,  $BE$  is the sum, and  $BD$  the diff. of the two given sides  $CB$ ,  $CA$ . Also, the sum of the two angles  $CAB$ ,  $CBA$ , is equal to the sum of the two  $CAD$ ,  $CDA$ , these sums being each the supplement of the vertical angle  $C$  to two right angles: but the two latter  $CAD$ ,  $CDA$  are equal to each other, being opposite to the two equal sides  $CA$ ,  $CD$ : hence either of them, as  $CDA$ , is equal to half the sum of the two unknown angles  $CAB$ ,  $CBA$ . Again, the exterior angle  $CDA$  is equal to the two interior angles  $B$  and  $DAB$ ; therefore the angle  $DAB$  is equal to the diff. between  $CDA$  and  $B$ , or between  $CAD$  and  $B$ : consequently the same angle  $DAB$  is equal to half the difference of the unknown angles  $B$  and  $CAB$ ; of which it has been shewn that  $CDA$  is the half sum.

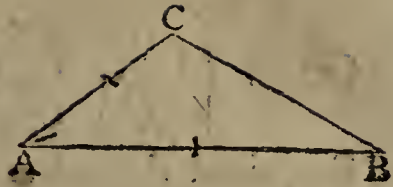
Now the angle  $DAE$ , in a semicircle, is a right angle, or  $AE$  is perpendicular to  $AD$ ; and  $DF$ , parallel to  $AE$ , is also perpendicular to  $AD$ : consequently  $AE$  is the tangent of  $CDA$  the half sum, and  $DF$  the tangent of  $DAB$  the half difference of the angles, to the same radius  $AD$ , by the definition of a tangent. But the tangents  $AE$ ,  $DF$  being parallel, it will be as  $BE : BD :: AE : DF$ ; that is, as the sum of the sides is to the difference of the sides, so is the tangent of half the sum of the opposite angles, to the tangent of half their difference.

*Note.* The sum of the unknown angles is found, by taking the given angle from  $180^\circ$ .

EXAMPLE I.

In the plane triangle ABC,

Given  $\left\{ \begin{array}{l} AB \text{ 345 yards} \\ AC \text{ 174} \cdot 07 \text{ yards} \\ \angle A \text{ } 37^\circ 20' \end{array} \right.$



Required the other parts.

1. Geometrically.

Draw  $AB = 345$  from a scale of equal parts. Make the angle  $A = 37^\circ 20'$ . Set off  $AC = 174$  by the scale of equal parts. Join  $BC$ , and it is done.

Then the other parts being measured, they are found to be nearly as follow; viz. the side  $BC$  232 yards, the angle  $B$   $27^\circ$ , and the angle  $C$   $115^\circ \frac{1}{2}$ .

2. Arithmetically.

As sum of sides AB, AC,	- -	519.07	log.	2.7152259
To diff. of sides AB, AC,	- -	170.93	-	2.2328183
So tang. half sum $\angle$ s C and B	-	71.20	-	10.4712979
Totang. half diff. $\angle$ s C and B	-	44 16	-	9.9888903
<hr style="width: 100%;"/>				
their sum gives $\angle$ C		115 36		
their diff. gives $\angle$ B		27 4		
<hr style="width: 100%;"/>				

Then, by the former theorem,

As fin. $\angle$ C 115.36' or 64.24'	-	log.	9.9551259
To its op. side AB 345	- - -		2.5378191
So fin. $\angle$ A 37.20'	- - -		9.7827958
To its op. side BC 232	- - -		2.3654890
<hr style="width: 100%;"/>			

3. Instrumentally.

In the first proportion.—Extend the compasses from 519 to 171, on the line of numbers; then that extent will reach, on the tangents, from  $71^\circ \frac{1}{3}$  (the contrary way, because the tangents are set back again from  $45^\circ$ ) a little beyond 45, which being set so far back from 45, falls upon  $44^\circ \frac{1}{4}$ , the fourth term.

In the second proportion.—Extend from  $64^\circ \frac{1}{3}$  to  $37^\circ \frac{1}{3}$ , on the fines; then that extent will reach, on the numbers, from 345 to 232, the fourth term sought.

## EXAMPLE II.

In the plane triangle ABC,

$$\text{Given } \begin{cases} AB & 365 \text{ poles} \\ AC & 154 \cdot 33 \\ \angle A & 57^\circ 12' \end{cases}$$

$$\text{Ans. } \begin{cases} BC & 309 \cdot 86 \\ \angle B & 24^\circ 45' \\ \angle C & 98 \quad 3 \end{cases}$$

Required the other parts.

## EXAMPLE III.

In the plane triangle ABC,

$$\text{Given } \begin{cases} AC & 120 \text{ yards} \\ BC & 112 \text{ yards} \\ \angle C & 57^\circ 58' 39'' \end{cases}$$

$$\text{Ans. } \begin{cases} AB & 112 \cdot 65 \\ \angle A & 57^\circ 27' 0'' \\ \angle B & 64 \quad 34 \quad 21 \end{cases}$$

Required the other parts.

## THEOREM III.

*When the Three Sides of the Triangle are given.*

THEN, having let fall a perpendicular from the greatest angle upon the opposite side, or base, dividing it into two segments, and the whole triangle into two right-angled triangles; it will be,

As the base, or sum of the segments,  
Is to the sum of the other two sides;  
So is the difference of those sides,  
To the diff. of the segments of the base.

Then half the difference of the segments being added to the half sum, or the half base, gives the greater segment; and the same subtracted gives the less segment.

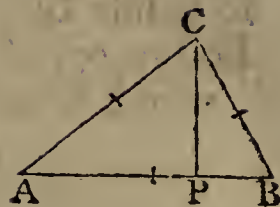
Hence, in each of the two right-angled triangles, there will be known two sides, and the angle opposite to one of them; consequently the other angles will be found by the first theorem.

*Demonstr.* By cor. to theor. 35, Geom. the rectangle under the sum and difference of the two sides, is equal to the rectangle under the sum and difference of the two segments. Therefore, by forming the sides of these rectangles into a proportion, it will appear that the sums and differences are proportional as in this theorem, by theor. 76, Geometry.

EXAMPLE I.

In the plane triangle ABC,

Given the sides  $\begin{cases} AB & 345 \text{ yards} \\ AC & 232 \\ BC & 174.07 \end{cases}$



To find the angles.

1. Geometrically.

Draw the base  $AB = 345$  by a scale of equal parts. With radius 232, and centre A, describe an arc; and with radius 174, and centre B, describe another arc, cutting the former in C. Join AC, BC, and it is done.

Then, by measuring the angles, they will be found to be nearly as follows, viz.

$$\angle A \ 27^\circ, \ \angle B \ 37^\circ\frac{1}{3}, \ \text{and} \ \angle C \ 115^\circ\frac{1}{2}.$$

2. Arithmetically.

Having let fall the perpendicular CP, it will be,  
As the base  $AB : AC + BC :: AC - BC : AP - BP$ ,  
that is, as  $345 : 406.07 :: 57.93 : 68.18 = AP - BP$ ,  
its half is  $\quad \quad \quad 34.09$   
the half base is  $\quad \quad \quad 172.50$

$$\begin{array}{r} \text{the sum of these is } 206.59 = AP \\ \text{and their diff. } \quad \quad 138.41 = BP \end{array}$$

Then, in the triangle APC, right-angled at P,

As the side AC	-	-	232	-	log.	2.3654880
To fin. op. $\angle P$	-	-	$90^\circ$	-	-	10.0000000
So is side AP	-	-	206.59	-	-	2.3151093
To fin. op. $\angle ACP$	-	-	$62^\circ 56'$	-	-	9.9496213
which taken from	-	-	$90^\circ 00'$	-	-	<u>                    </u>

leaves the  $\angle A \ 27^\circ 04'$

Again,

Again, in the triangle BPC, right-angled at P,

As the side BC	-	-	174.07	-	log.	2.2407239
To fin. op. $\angle P$	-	-	90°	-	-	10.0000000
So is side BP	-	-	138.41	-	-	2.1411675
To fin. op. $\angle BCP$	-	-	52° 40'	-	-	9.9004436
which taken from	-	-	90 00	-	-	<u>          </u>

leaves the  $\angle B$  37 20

Also, the  $\angle ACP$  62° 56'  
 added to  $\angle BCP$  52 40  
 gives the whole  $\angle ACB$  115 36

So that all the three angles are as follow, viz.

the  $\angle A$  27° 4'; the  $\angle B$  37° 20'; the  $\angle C$  115° 36'.

### 3. Instrumentally.

In the first proportion.—Extend the compasses from 345 to 406, on the line of numbers; then that extent will reach, on the same line, from 58 to 68.2 nearly, which is the difference of the segments of the base.

In the second proportion.—Extend from 232 to 206½, on the line of numbers; then that extent will reach, on the sines, from 90° to 63°.

In the third proportion.—Extend from 174 to 138½; then that extent will reach from 90° to 52°⅓ on the sines.

### EXAMPLE II.

In the plane triangle ABC,

Given the sides	{	AB	365 poles	Anf.	{	$\angle A$	57° 12'
		AC	154.33			$\angle B$	24 45
		BC	309.86			$\angle C$	98 3

To find the angles.

In the plane triangle ABC,

Given the sides	{	AB	120	Anf.	{	$\angle A$	57° 27' 00''
		AC	112.6			$\angle B$	57 58 39
		BC	112			$\angle C$	64 34 21

To find the angles.

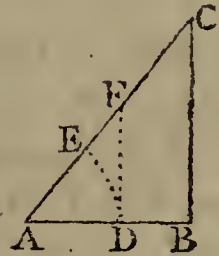
The three foregoing theorems include all the cases of plane triangles, both right-angled and oblique; besides which, there are other theorems suited to some particular forms of triangles, which are sometimes more expeditious in their use than the general ones; one of which, as the case for which it serves so frequently occurs, may be here taken, as follows:

## THEOREM IV.

*When, in a Right-angled Triangle, there are given one Leg and the Angles; to find the other Leg or the Hypothenuse; it will be,*

As radius, (i. e. sin. of  $90^\circ$  or tang. of  $45^\circ$ )  
Is to the given leg,  
So is tang. of its adjacent angle  
To the other leg;  
And so is secant of the same angle  
To the hypothenuse.

*Demonstr.* AB being the given leg, in the right-angled triangle ABC; with the centre A, and any assumed radius AD, describe an arc DE, and draw DF perpendicular to AB, or parallel to BC. Now it is evident, from the definitions, that DF is the tangent, and AF the secant, of the arc DE, or of the angle A which is measured by that arc, to the radius AD. Then, because of the parallels BC, DF, it will be, - - as AD : AB :: DF : BC :: AF : AC, which is the same as the theorem is in words.



## EXAMPLE I.

In the right-angled triangle ABC,

Given  $\left\{ \begin{array}{l} \text{the leg AB } 162 \\ \angle A \ 53^\circ 7' 48'' \end{array} \right\}$  To find AC and BC.

## 1. Geometrically.

Make AB = 162 equal parts, and the angle A =  $53^\circ 7' 48''$ ; then raise the perpendicular BC, meeting AC in C. So shall AC measure 270, and BC 216.

## 2. Arith-

2. *Arithmetically.*

As radius	-	tang. $45^\circ$	-	-	log. $10^{\cdot}0000000$
To leg AB	-	162	-	-	$2^{\cdot}2095150$
So tang. $\angle A$	-	$53^\circ 7' 48''$	-	-	$10^{\cdot}1249371$
To leg BC	-	216	-	-	$2^{\cdot}3344521$
So secant $\angle A$	-	$53^\circ 7' 48''$	-	-	$10^{\cdot}2218477$
To hyp. AC	-	270	-	-	$2^{\cdot}4313627$

3. *Instrumentally.*

Extend the compasses from  $45^\circ$  to  $53^\circ \frac{1}{8}$ , on the tangents. Then that extent will reach from 162 to 216 on the line of numbers.

## EXAMPLE II.

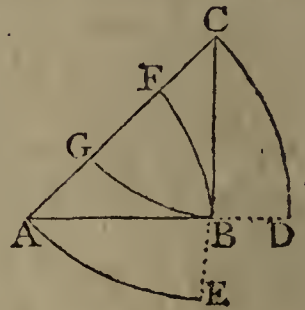
In the right-angled triangle ABC,

$$\text{Given } \begin{cases} \text{the leg AB } 180 \\ \text{the } \angle A \ 62^\circ 40' \end{cases} \quad \text{Anf. } \begin{cases} \text{AC } 392^{\cdot}0146 \\ \text{BC } 348^{\cdot}2464 \end{cases}$$

To find the other two sides.

*Note.* There is sometimes given another method for right-angled triangles, which is this:

ABC being such a triangle, make one leg AB radius, that is, with centre A, and distance AB, describe an arc BF. Then it is evident that the other leg BC represents the tangent, and the hypotenuse AC the secant, of the arc BF, or of the angle A.



In like manner, if the leg BC be made radius; then the other leg AB will represent the tangent, and the hypotenuse AC the secant, of the arc BG or angle C.

But if the hypotenuse be made radius; then each leg will represent the sine of its opposite angle; namely, the leg AB the sine of the arc AE or angle C, and the leg BC the sine of the arc CD or angle A.

And then the general rule for all these cases, is this, namely, that the sides of the triangle bear to each other the same proportion as the parts which they represent.

And this is called, Making every side radius.



OF HEIGHTS AND DISTANCES, &c.

BY the mensuration and protraction of lines and angles, are determined the lengths, heights, depths, and distances of bodies or objects.

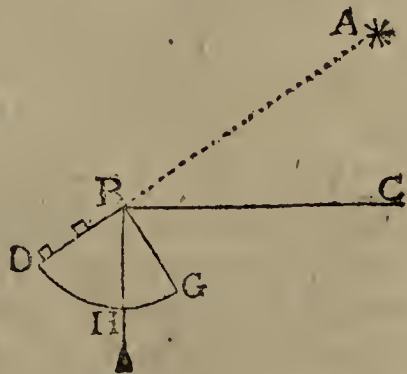
Accessible lines are measured by applying to them some certain measure a number of times, as an inch, or foot, or yard. But inaccessible lines must be measured by taking angles, or by such-like method, drawn from the principles of geometry.

When instruments are used for taking the magnitude of the angles in degrees, the lines are then calculated by trigonometry: in the other methods, the lines are calculated from the principle of similar triangles, without regard to the measure of the angles.

Angles of elevation, or of depression, are usually taken either with a theodolite, or with a quadrant, divided into degrees and minutes, and furnished with a plummet suspended from the centre, and two sights fixed on one of the radii, or else with telescopic sights.

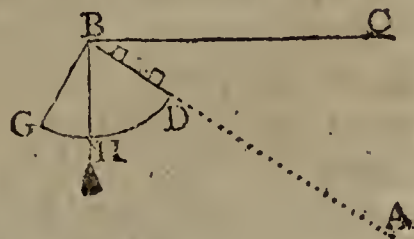
*To take an Angle of Altitude and Depression with the Quadrant.*

Let A be any object, as the sun, moon, or a star, or the top of a tower, or hill, or other eminence: and let it be required to find the measure of the angle ABC, which a line drawn from the object makes above the horizontal line BC.



Fix the centre of the quadrant in the angular point, and move it round there as a centre till, with one eye at D, the other being shut, you perceive the object A through the sights: then will the arc GH of the quadrant, cut off by the plumb line BH, be the measure of the angle ABC as required.

The angle ABC of depression of any object A, below the horizontal line BC, is taken in the same manner; except that here the eye is applied to the centre, and the measure of the angle is the arc GH, on the other side of the plumb line.



The following examples are to be constructed and calculated by the foregoing methods, treated of in Trigonometry.

## EXAMPLE I.

Having measured a distance of 200 feet, in a direct horizontal line, from the bottom of a steeple, the angle of elevation of its top, taken at that distance, was found to be  $47^{\circ} 30'$ : from hence it is required to find the height of the steeple.

*Construction.*

Draw an indefinite line, upon which set off  $AC = 200$  equal parts, for the measured distance. Erect the indefinite perpendicular  $AB$ ; and draw  $CB$  so as to make the angle  $C = 47^{\circ} 30'$ , the angle of elevation; and it is done. Then  $AB$ , measured on the scale of equal parts, is nearly  $218\frac{1}{4}$ .

*Calculation.*

As radius	-	-	10'0000000
To AC 200	-	-	2'3010300
So tang. $\angle C$ $47^{\circ} 30'$			10'0379475
To AB 218.26 required			2'3389775



## EXAMPLE II.

What was the perpendicular height of a cloud, or of a balloon, when its angles of elevation were  $35^{\circ}$  and  $64^{\circ}$ , as taken by two observers, at the same time, both on the same side of it, and in the same vertical plane; the distance between them being half a mile or 880 yards. And what was its distance from the said two observers?

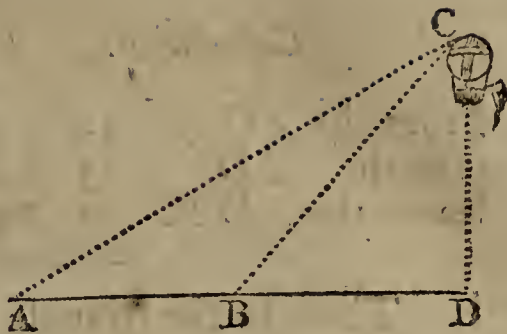
*Construction.*

Draw an indefinite ground line, upon which set off the given distance  $AB = 880$ ; then  $A$  and  $B$  are the places of the observers. Make the angle  $A = 35^{\circ}$ , and the angle  $B = 64^{\circ}$ ; and the intersection of the lines at  $C$  will be the place of the balloon: from whence the perpendicular  $CD$ , being let fall, will be its perpendicular height. Then, by measurement, are found the distances and height nearly as follows, viz.  $AC$  1631,  $BC$  1041,  $DC$  936.

*Cal-*

*Calculation.*

First, from  $\angle B$   $64^\circ$   
 take  $\angle A$   $35$   
 leaves  $\angle ACB$   $29$



Then, in the triangle ABC,

As sin. $\angle ACB$	$29^\circ$	-	-	-	9.6855712
To op. side AB	880	-	-	-	2.9444827
So sin. $\angle A$	$35^\circ$	-	-	-	9.7585913
To op. side BC	1041.125	-	-	-	3.0175028

As sin. $\angle ACB$	$29^\circ$	-	-	-	9.6855712
To op. side AB	880	-	-	-	2.9444827
So sin. $\angle B$	$116^\circ$ or $64^\circ$	-	-	-	9.9536602
To op. side AC	1631.442	-	-	-	3.2125717

And, in the triangle BCD,

As sin. $\angle D$	$90^\circ$	-	-	-	10.0000000
To op. side BC	1041.125	-	-	-	3.0175028
So sin. $\angle B$	$64^\circ$	-	-	-	9.9536602
To op. side CD	935.757	-	-	-	2.9711630

EXAMPLE III.

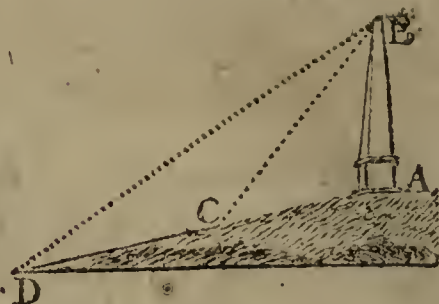
Having to find the height of an obelisk standing on the top of a declivity, I first measured from its bottom a distance of 40 feet, and there found the angle; formed by the oblique plane and a line imagined to go to the top of the obelisk,  $41^\circ$ ; but, after measuring on in the same direction 60 feet further, the like angle was only  $23^\circ 45'$ . What then was the height of the obelisk?

*Construction.*

Draw an indefinite line for the sloping plane or declivity, in which assume any point A for the bottom of the obelisk, from whence set off the distance  $AC = 40$ , and again  $CD = 60$  equal parts. Then make the angle  $C = 41^\circ$ , and the angle  $D = 23^\circ 45'$ ; and the point B where the two lines meet will be the top of the obelisk. Therefore AB, joined, will be its height.

*Calculation.*

From the  $\angle C$   $41^\circ 00'$   
 take the  $\angle D$   $23 \quad 45$   
 leaves the  $\angle DBC$   $17 \quad 15$



Then, in the triangle DBC,

As fin. $\angle DBC$	$17^\circ 15'$	-	-	-	9.4720856
To op. side DC	60	-	-	-	1.7781513
So fin. $\angle D$	$23 \quad 45$	-	-	-	9.6050320
To op. side CB	$81.488$	-	-	-	<u>1.9110977</u>

And, in the triangle ABC,

As sum of sides CB, CA	$121.488$	-	-	-	2.0845334
To diff. of sides CB, CA	$41.488$	-	-	-	1.6179225
So tang. half sum $\angle$ s A, B	$69^\circ 30'$	-	-	-	10.4272623
To tang. half diff. $\angle$ s A, B	$42 \quad 24\frac{1}{2}$	-	-	-	<u>9.9606514</u>

the diff. of these is  $\angle CBA$   $27 \quad 5\frac{1}{2}$

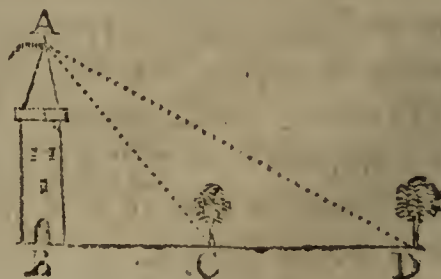
Lastly, as fin. $\angle CBA$	$27^\circ 5\frac{1}{2}$	-	-	-	9.6582842
To op. side CA	40	-	-	-	1.6020600
So fin. $\angle C$	$41^\circ 0'$	-	-	-	9.8169429
To op. side AB	$57.623$	-	-	-	<u>1.7607187</u>

## EXAMPLE IV.

Wanting to know the distance between two inaccessible trees, or other objects, from the top of a tower, 120 feet high, which lay in the same right line with the two objects, I took the angles formed by the perpendicular wall and lines conceived to be drawn from the top of the tower to the bottom of each tree, and found them to be  $33^\circ$  and  $64^\circ\frac{1}{2}$ . What then may be the distance between the two objects?

*Construction.*

Draw the indefinite ground line BD, and perpendicular to it BA = 120 equal parts. Then draw the two lines AC, AD making the two angles BAC, BAD equal to the given



given angles  $33^\circ$  and  $64^\circ\frac{1}{2}$ . So shall C and D be the places of the two objects.

*Calculation.*

First, In the right-angled triangle ABC,

As radius	-	-	-	-	10.0000000
To AB	-	120	-	-	2.0791812
So tang. $\angle$ BAC		$33^\circ$	-	-	9.8125174
To BC	-	77.929	-	-	1.8916986

And, in the right-angled triangle ABD,

As radius	-	-	-	-	10.0000000
To AB	-	120	-	-	2.0791812
So tang. $\angle$ BAD		$64^\circ\frac{1}{2}$	-	-	10.3215039
To BD	-	251.585	-	-	2.4006851

from which take BC 77.929  
leaves the dist. CD 173.656 as required.

EXAMPLE V.

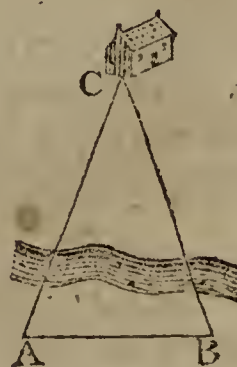
Being on the side of a river, and wanting to know the distance to a house which was seen on the other side, I measured 200 yards in a straight line by the side of the river; and then at each end of this line of distance took the horizontal angle formed between the house and the other end of the line; which angles were, the one of them  $68^\circ 2'$ , and the other  $73^\circ 15'$ . What then were the distances from each end to the house?

*Construction.*

Draw the line AB = 200 equal parts. Then draw AC so as to make the angle A =  $68^\circ 2'$ , and BC to make the angle B =  $73^\circ 15'$ . So shall the point C be the place of the house required.

*Calculation.*

To the given $\angle$ A	$68^\circ$	$2'$
add the given $\angle$ B	73	15
then their sum	141	17
being taken from	180	0
leaves the third $\angle$ C	38	43



Hence

Hence, As fin. $\angle C$	$38^{\circ} 43'$	-	9.7962062
To op. side AB	200	-	2.3010300
So fin. $\angle A$	$68^{\circ} 2'$	-	9.9672679
To op. side BC	296.54	-	2.4720917

And, As fin. $\angle C$	$38^{\circ} 43'$	-	9.7962062
To op. side AB	200	-	2.3010300
So fin. $\angle B$	$73^{\circ} 15'$	-	9.9811711
To op. side AC	306.19	-	2.4859949

EXAM. VI. From the edge of a ditch, of 36 feet wide, surrounding a fort, having taken the angle of elevation of the top of the wall, it was found to be  $62^{\circ} 40'$ : required the height of the wall, and the length of a ladder to reach from my station to the top of it?

Ans.  $\left\{ \begin{array}{l} \text{height of wall } 69.64, \\ \text{ladder } 78.4 \text{ feet.} \end{array} \right.$

EXAM. VII. Required the length of a shoar, which, being to strut 11 feet from the upright of a building, will support a jamb 23 feet 10 inches from the ground?

Ans. 26 feet 3 inches.

EXAM. VIII. A ladder, 40 feet long, can be so planted, that it shall reach a window 33 feet from the ground, on one side of the street; and by turning it over, without moving the foot out of its place, it will do the same by a window 21 feet high, on the other side: required the breadth of the street?

Ans. 56.649 feet.

EXAM. IX. A Maypole, whose top was broken off by a blast of wind, struck the ground at 15 feet distance from the foot of the pole: what was the height of the whole maypole, supposing the broken piece to measure 39 feet in length?

Ans. 75 feet.

EXAM. X. At 170 feet distance from the bottom of a tower, the angle of its elevation was found to be  $52^{\circ} 30'$ : required the altitude of the tower?

Ans. 221 feet.

EXAM. XI. From the top of a tower, by the sea-side, of 143 feet height, it was observed that the angle of depression of a ship's bottom, then at anchor, measured  $35^{\circ}$ ; what then was the ship's distance from the bottom of the wall?

Ans. 204.22 feet.

EXAM. XII. What is the perpendicular height of a hill; its angle of elevation, taken at the bottom of it, being  $46^{\circ}$ ,  
and

and 200 yards farther off, on a level with the bottom of it, the angle was  $31^{\circ}$  ?

Anf. 286.28 yards.

EXAM. XIII. Wanting to know the height of an inaccessible tower; at the least distance from it, on the same horizontal plane, I took its angle of elevation equal to  $58^{\circ}$ ; then going 300 feet directly from it, found the angle there to be only  $32^{\circ}$ : required its height, and my distance from it at the first station ?

Anf.  $\left\{ \begin{array}{l} \text{height } 307.53 \\ \text{distance } 192.15 \end{array} \right.$

EXAM. XIV. Being on a horizontal plane, and wanting to know the height of a tower placed on the top of an inaccessible hill; I took the angle of elevation of the top of the hill equal  $40^{\circ}$ , and of the top of the tower equal  $51^{\circ}$ ; then measuring in a line directly from it to the distance of 200 feet farther, I found the angle to the top of the tower to be  $33^{\circ} 45'$ . What then is the height of the tower ?

Anf. 93.33148 feet.

EXAM. XV. From a window near the bottom of a house, which seemed to be on a level with the bottom of a steeple, I took the angle of elevation of the top of the steeple equal  $40^{\circ}$ ; then from another window, 18 feet directly above the former, the like angle was  $37^{\circ} 30'$ : what then is the height and distance of the steeple ?

Anf.  $\left\{ \begin{array}{l} \text{height } 210.44 \\ \text{distance } 250.79 \end{array} \right.$

EXAM. XVI. Wanting to know the height of, and my distance from, an object on the other side of a river, which seemed to be on a level with the place where I stood, close by the side of the river; and not having room to measure backward, on the same plane, because of the immediate rise of the bank, I placed a mark where I stood, and measured in a direction from the object, up the ascending ground to the distance of 264 feet, where it was evident that I was above the level of the top of the object; there the angles of depression were found to be, viz. of the mark left at the river's side  $42^{\circ}$ , of the bottom of the object  $27^{\circ}$ , and of its top  $19^{\circ}$ . Required then the height of the object, and the distance of the mark from its bottom ?

Anf.  $\left\{ \begin{array}{l} \text{height } 57.26 \\ \text{distance } 150.50 \end{array} \right.$

EXAM. XVII. If the height of the mountain called the Pike of Teneriff be 3 miles, and the angle taken at the top of it, as formed between a plumb line and a line conceived

to touch the earth in the horizon, or farthest visible point, be  $87^{\circ} 46' 33''$ ; it is required from hence to determine the magnitude of the whole earth, and the utmost distance that can be seen on its surface from the top of the mountain, supposing the form of the earth to be perfectly round?

$$\text{Ans. } \left\{ \begin{array}{l} \text{dist. } 154.539 \\ \text{diam. } 7957.75 \end{array} \right\} \text{miles.}$$

EXAM. XVIII. Two ships of war, intending to cannonade a fort, are, by the shallowness of the water, kept so far from it, that they suspect their guns cannot reach it with effect. In order, therefore, to measure the distance, they separate from each other a quarter of a mile, or 440 yards; then each ship observes and measures the angle which the other ship and the fort subtends, which angle were  $83^{\circ} 45'$  and  $85^{\circ} 15'$ . What then was the distance between each ship and the fort?

$$\text{Ans. } \left\{ \begin{array}{l} 2292.26 \text{ yards.} \\ 2298.05 \end{array} \right.$$

EXAM. XIX. Being on the side of a river, and wanting to know the distance to a house which was seen at a distance on the other side; I measured out for a base 400 yards in a right line by the side of the river, and found that the two angles, one at each end of this line, subtended by the other end and the house, were  $68^{\circ} 2'$  and  $73^{\circ} 15'$ . What then was the distance between each station and the house?

$$\text{Ans. } \left\{ \begin{array}{l} 593.08 \text{ yards.} \\ 612.38 \end{array} \right.$$

EXAM. XX. Wanting to know the breadth of a river, I measured a base of 500 yards in a straight line close by one side of it; and at each end of this line I found the angles subtended by the other end and a tree close on the bank on the other side of the river, to be  $53^{\circ}$  and  $79^{\circ} 12'$ . What then was the perpendicular breadth of the river?

$$\text{Ans. } 529.48 \text{ yards.}$$

EXAM. XXI. Wanting to know the extent of a piece of water, or distance between two headlands; I measured from each of them to a certain point inland, and found the two distances to be 735 yards and 846 yards; also, the horizontal angle subtended between these two lines was  $55^{\circ} 40'$ . What then was the distance required?

$$\text{Ans. } 741.2 \text{ yards.}$$

EXAM. XXII. A point of land was observed, by a ship at sea, to bear east-by-south; and after sailing north-east 12 miles, it was found to bear south-east-by-east. It is required to



to determine the place of that headland, and the ship's distance from it at the last observation.      Ans. 26.0728 miles.

EXAM. XXIII. Wanting to know the distance between a house and a mill, which were seen at a distance on the other side of a river, I measured a base line along the side, where I was, of 600 yards, and at each end of it took the angles subtended by the other end and the house and mill, which were as follow, viz. at one end the angles were  $58^{\circ} 20'$  and  $95^{\circ} 20'$ , and at the other end the like angles were  $53^{\circ} 30'$  and  $98^{\circ} 45'$ . What then was the distance between the house and mill?

Ans. 959.5866 yards.

EXAM. XXIV. Wanting to know my distance from an inaccessible object O, on the other side of a river; and having no instrument for taking angles, but only a chain or cord for measuring distances; from each of two stations, A and B, which were taken at 500 yards asunder, I measured in a direct line from the object O 100 yards, viz. AC and BD each equal to 100 yards; also the diagonal AD measured 550 yards, and the diagonal BC 560. What then was the distance of the object O from each station A and B?

Ans.  $\begin{cases} AO & 536.25 \\ BO & 500.09 \end{cases}$

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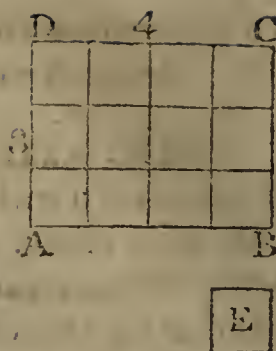
## MENSURATION OF PLANES.

THE Area of any plane figure, is the measure of the space contained within its extremes or bounds; without any regard to thickness.

This area, or the content of the plane figure, is estimated by the number of little squares that may be contained in it; the side of those little measuring squares being an inch, a foot, a yard, or any other fixed quantity. - And hence, the area or content is said to be so many square inches, or square feet, or square yards, &c,

Thus,

Thus, if the figure to be measured be the rectangle ABCD, and the little square E, whose side is one inch, be the measuring unit proposed: then, as often as the said little square is contained in the rectangle, so many square inches the rectangle is said to contain, which in the present case is 12.



## PROBLEM I.

To find the Area of any Parallelogram, whether it be a Square, a Rectangle, a Rhombus, or a Rhomboid.

MULTIPLY the length by the perpendicular breadth, or height, and the product will be the area\*.

## EXAMPLES.

Ex. 1. To find the area of a parallelogram, whose length is 12.25, and height 8.5.

$$\begin{array}{r}
 12.25 \text{ length} \\
 8.5 \text{ breadth} \\
 \hline
 6125 \\
 9800 \\
 \hline
 104.125 \text{ area} \\
 \hline
 \end{array}$$

\* The truth of this rule is proved in the Geom. theor. 81, cor. 2.

The same is otherwise proved thus: Let the foregoing rectangle be the figure proposed; and let the length and breadth be divided into equal parts, each equal to the linear measuring unit, being here 4 for the length, and 3 for the breadth; and let the opposite points of division be connected by right lines.—Then, it is evident that these lines divide the rectangle into a number of little squares, each equal to the square measuring unit E; and farther, that the number of these little squares, or the area of the figure, is equal to the number of linear measuring units in the length, repeated as often as there are linear measuring units in the breadth, or height; that is, equal to the length drawn into the height; which here is  $4 \times 3$  or 12.

And it is proved (Geom. theor. 25, cor. 2), that a rectangle is equal to any oblique parallelogram, of equal length and perpendicular breadth. Therefore the rule is general for all parallelograms whatever.

Ex. 2.

Ex. 2. To find the area of a square, whose side is 35·25 chains.  
 Anf. 124 acres, 1 rood, 1 perch.

Ex. 3. To find the area of a rectangular board, whose length is  $12\frac{1}{2}$  feet, and breadth 9 inches. Anf.  $9\frac{3}{8}$  feet.

Ex. 4. To find the content of a piece of land, in form of a rhombus, its length being 6·20 chains, and perpendicular height 5·45.  
 Anf. 3 acres, 1 rood, 20 perches.

Ex. 5. To find the number of square yards of painting in a rhomboid, whose length is 37 feet, and breadth 5 feet 3 inches.  
 Anf.  $21\frac{7}{2}$  square yards.

## PROBLEM II.

*To find the Area of a Triangle.*

RULE I. MULTIPLY the base by the perpendicular height, and take half the product for the area\*. Or, multiply the one of these dimensions by half the other.

## EXAMPLES.

Ex. 1. To find the area of a triangle, whose base is 625, and perpendicular height 520 links?

Here  $625 \times 260 = 162500$  square links,  
 or equal 1 acre, 2 roods, 20 perches, the answer.

Ex. 2. How many square yards contains the triangle, whose base is 40, and perpendicular 30 feet?

Anf.  $66\frac{2}{3}$  square yards.

Ex. 3. To find the number of square yards in a triangle, whose base is 49 feet, and height  $25\frac{1}{4}$  feet?

Anf.  $68\frac{5}{2}$ , or 68·7361.

Ex. 4. To find the area of a triangle, whose base is 18 feet 4 inches, and height 11 feet 10 inches?

Anf. 108 feet,  $5\frac{2}{3}$  inches.

\* The truth of this rule is evident, because any triangle is the half of a parallelogram of equal base and altitude, by Geom. theor. 26.

**RULE II.** When two sides and their contained angle are given: Multiply the two given sides together, and take half their product: Then say, as radius is to the sine of the given angle, so is that half product, to the area of the triangle.

Or, multiply that half product by the sine of the said angle\*.

**Ex. I.** What is the area of a triangle, whose two sides are 30 and 40, and their contained angle  $28^{\circ} 57' 18''$ ?

Here  $\frac{1}{2} \times 40 \times 30 = 600$ ,  
therefore  $1 : .4841226 \text{ nat. sin. } 28^{\circ} 57' 18''$   
 $600$

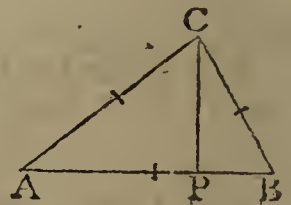
290.47356 the answer.

**Ex. 2.** How many square yards contains the triangle, of which one angle is  $45^{\circ}$ , and its containing sides 25 and  $21\frac{1}{4}$  feet? Ans. 20.86947.

**RULE III.** When the three sides are given: Add all the three sides together, and take half that sum. Next, subtract each side severally from the said half sum, obtaining three remainders. Then multiply the said half sum and those three remainders all together, and, lastly, extract the square root of the last product, for the area of the triangle †.

**Ex. I.**

\* For, let AB, AC be the two given sides, including the given angle A. Now  $\frac{1}{2}AB \times CP$  is the area, by the first rule, CP being perpendicular. But, by trigon. as  $\text{fin. } \angle P$ , or radius : AC ::  $\text{fin. } \angle A$  : CP =  $\text{fin. } \angle A \times AC$ , taking radius = 1. Therefore the area  $\frac{1}{2}AB \times CP$  is =  $\frac{1}{2}AB \times AC \times \text{fin. } \angle A$ , to radius 1; or as radius :  $\text{fin. } \angle A$  ::  $\frac{1}{2}AB \times AC$  : the area.



† For, let  $b$  denote the base AB of the triangle ABC (see the last fig.), also  $a$  the side AC, and  $c$  the side BC. Then, by th. 3, Trigon. as  $b : a + c :: a - c : \frac{aa - cc}{b} = AP - PB$  the diff. of the segments;

theref.

Ex. 1. To find the area of the triangle whose three sides are 20, 30, 40.

20	45	45	45
30	20	30	40
40	—	—	—
—	25 1st rem.	15 2d rem.	5 3d rem.
2) 90	—	—	—
45 half sum			
—			

Then  $45 \times 25 \times 15 \times 5 = 84375$ .

The root of which is 290.4737, the area.

Ex. 2. How many square yards of plastering are in a triangle, whose sides are 30, 40, 50 feet? Ans.  $66\frac{2}{3}$ .

Ex. 3. How many acres, &c. contains the triangle, whose sides are 2569, 4900, 5025 links?

Ans. 61 acres, 1 rood, 39 perches.

PROBLEM III.

*To find the Area of a Trapezoid.*

ADD together the two parallel sides; then multiply their sum by the perpendicular breadth, or the distance between them; and take half the product for the area. By Geom. theor. 29.

theref.  $\frac{1}{2}b + \frac{an - cc}{2b} = \frac{bb + aa - cc}{2b}$  = the segment AP;

hence  $\sqrt{AC^2 - AP^2}$  = the perp. CP, that is,

$$\sqrt{\left( aa - \left( \frac{bb + aa - cc}{2b} \right)^2 \right)} = \dots$$

$$\sqrt{\frac{2a^2b^2 - a^4 + 2b^2c^2 - b^4 + 2a^2c^2 - c^4}{4bb}} = CP.$$

But  $\frac{1}{2}AB \times CP$  is the area, that is,

$$\frac{1}{2}b \times CP = \sqrt{\frac{2a^2b^2 - a^4 + 2b^2c^2 - b^4 + 2a^2c^2 - c^4}{16}}$$

$$= \sqrt{\left( \frac{-aa + bb + cc + 2bc}{4} \times \frac{aa - bb - cc + 2bc}{4} \right)}$$

$$= \sqrt{\left( \frac{a + b + c}{2} \times \frac{-a + b + c}{2} \times \frac{a - b + c}{2} \times \frac{a + b - c}{2} \right)}$$

=  $\sqrt{(s \times s - a \times s - b \times s - c)}$ , which is the rule, where  $s$  denotes half the sum of the three sides.

Ex. 1.

Ex. 1. In a trapezoid, the parallel sides are 750 and 1225, and the perpendicular distance between them 1540 links: to find the area.

$$\begin{array}{r} 1225 \\ 750 \\ \hline \end{array}$$

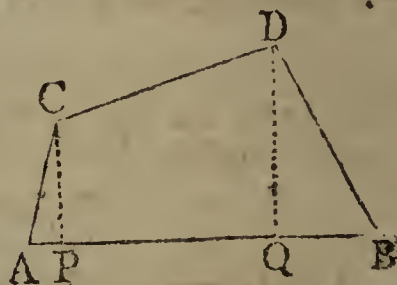
$$1975 \times 770 = 152075 \text{ square links} = 15 \text{ acr. } 33 \text{ perches.}$$

Ex. 2. How many square feet are contained in the plank, whose length is 12 feet 6 inches, the breadth at the greater end 15 inches, and at the less end 11 inches?

Anf.  $13\frac{3}{4}$  feet.

Ex. 3. In measuring along one side AB of a quadrangular field, that side and the two perpendiculars let fall on it from the two opposite corners, measured as below: required the content.

$$\begin{array}{l} AP = 110 \text{ links.} \\ AQ = 745 \\ AB = 1110 \\ CP = 352 \\ DQ = 595 \end{array}$$



Anf. 4 acres, 1 rood, 5.792 perches.

#### PROBLEM IV.

*To find the Area of any Trapezium.*

DIVIDE the trapezium into two triangles by a diagonal; then find the areas of these triangles, and add them together.

*Note.* If two perpendiculars be let fall on the diagonal, from the other two opposite angles, the sum of these perpendiculars being multiplied by the diagonal, half the product will be the area of the trapezium.

Ex. 1. To find the area of the trapezium, whose diagonal is 42, and the two perpendiculars on it 16 and 18.

$$\text{Here } 16 + 18 = 34, \text{ its half is } 17.$$

$$\text{Then } 42 \times 17 = 714 \text{ the area.}$$

Ex. 2. How many square yards of paving are in the trapezium, whose diagonal is 65 feet, and the two perpendiculars let fall on it 28 and  $33\frac{1}{2}$  feet? Anf.  $222\frac{1}{2}$  yards.

Ex. 3.

Ex. 3. In the quadrangular field ABCD, on account of obstructions there could only be taken the following measures, viz. the two sides BC 265 and AD 220 yards, the diagonal AC 378, and the two distances of the perpendiculars from the ends of the diagonal, namely, AE 100, and CF 70 yards. Required the area in acres, when 4840 square yards make an acre? Anf. 17 acres, 2 roods, 21 perches.

## PROBLEM V.

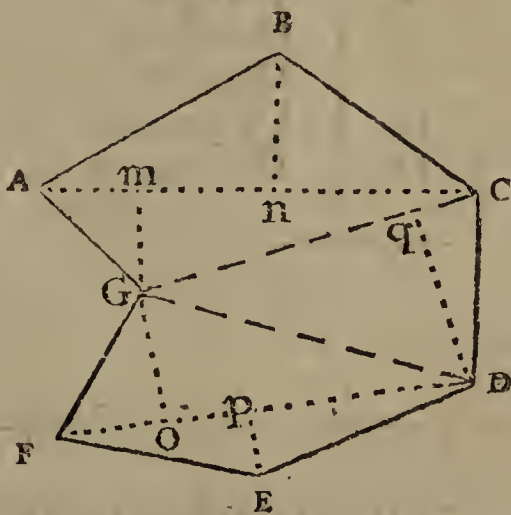
*To find the Area of an Irregular Polygon.*

DRAW diagonals dividing the proposed polygon into trapeziums and triangles. Then find the areas of all these separately, and add them together for the content of the whole polygon.

EXAM. To find the content of the irregular figure ABCDEFGA, in which are given the following diagonals and perpendiculars: namely,

AC 55  
 FD 52  
 GC 44  
 Gm 13  
 Bn 18  
 Go 12  
 Ep 8  
 Dq 23

Anf.  $1878\frac{1}{2}$



## PROBLEM VI.

*To find the Area of a Regular Polygon.*

RULE I. MULTIPLY the perimeter of the polygon, or sum of its sides, by the perpendicular drawn from its centre on one of its sides, and take half the product for the area\*.

\* This is only in effect resolving the polygon into as many equal triangles as it has sides, by drawing lines from the centre to all the angles; then finding their areas, and adding them all together.

EX. I.

Ex. 1. To find the area of the regular pentagon, each side being 25 feet, and the perpendicular from the centre on each side is 17.2047737.

Here  $25 \times 5 = 125$  is the perimeter.

And  $17.2047737 \times 125 = 2150.5967125$ .

Its half 1075.298356 is the area sought.

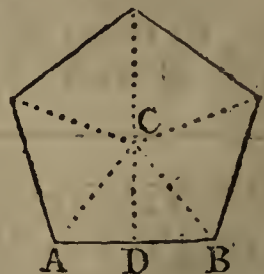
RULE II. Square the side of the polygon; then multiply that square by the area or multiplier set against its name in the following table, and the product will be the area\*.

No. of Sides.	Names.	Areas, or Multipliers.
3	Trigon, or triangle	0.4330127
4	Tetragon, or square	1.0000000
5	Pentagon	1.7204774
6	Hexagon	2.5980762
7	Heptagon	3.6339124
8	Octagon	4.8284271
9	Nonagon	6.1818242
10	Decagon	7.6942088
11	Undecagon	9.3656399
12	Dodecagon	11.1961524

EXAM.

\* This rule is founded on the property, that like polygons, being similar figures, are to one another as the squares of their like sides; which is proved in the Geom. theor. 89. Now, the multipliers in the table, are the areas of the respective polygons to the side 1. Whence the rule is manifest.

*Note.* The areas in the table, to each side 1, may be computed in the following manner: From the centre C of the polygon draw lines to every angle, dividing the whole figure into as many equal triangles as the polygon has sides; and let ABC be one of those triangles, the perpendicular of which is CD. Divide 360 degrees by the number of sides in the polygon, the quotient gives the angle at the centre ACB. The half



of



EXAM. Taking here the same example as before, namely a pentagon, whose side is 25 feet.

Then  $25^2$  being = 625,

And the tabular area 1.7204774;

Theref.  $1.7204774 \times 625 = 1075.298375$ , as before.

Ex. 2. To find the area of the trigon, or equilateral triangle, whose side is 20. Ans. 173.20508.

Ex. 3. To find the area of the hexagon, whose side is 20. Ans. 1039.23048.

Ex. 4. To find the area of an octagon, whose side is 20. Ans. 1931.37084.

Ex. 5. To find the area of a decagon, whose side is 20. Ans. 3077.68352.

PROBLEM VII.

*To find the Diameter and Circumference of any Circle, the one from the other.*

THIS may be done nearly by either of the two following proportions, viz.

As 7 is to 22, so is the diameter to the circumference.

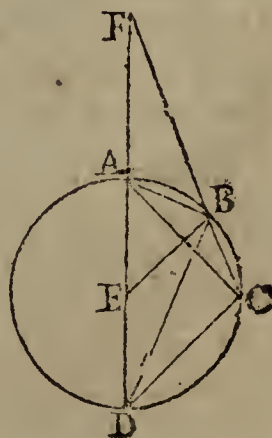
Or, As 1 is to 3.1416, so is the diameter to the circumference\*.

Ex. 1.

of this gives the angle ACD; and this taken from  $90^\circ$ , leaves the angle CAD. Then, as radius is to AD, so is tang. angle CAD, to the perpendicular CD. This multiplied by AD, gives the area of the triangle ABC; which being multiplied by the number of the triangles, or of the sides of the polygon, gives its whole area, as in the table.

\* For, let ABCD be any circle, whose centre is E, and let AB, BC be any two equal arcs. Draw the several chords as in the figure, and join BE; also draw the diameter DA, which produce to F, till BF be equal to the chord BD.

Then the two isosceles triangles DEB, DBF are equiangular, because they have the angle at D common; consequently  $DE:DB::DB:DF$ . But the two triangles AFB, DCB are identical, or equal in all respects, because they have the angle F = the angle BDC, being each equal to the angle ADB, these being subtended by the equal



Ex. 1. To find the circumference of the circle whose diameter is 20.

By the first rule, as  $7 : 22 :: 20 : 62\frac{6}{7}$ , the answer.

Ex. 2.

arcs AB, BC; also the exterior angle FAB of the quadrangle ABCD, is equal to the opposite interior angle at C; and the two triangles have also the side BF = the side BD; therefore the side AF is also equal to the side DC. Hence the proportion above, viz. DE : DB :: DB : DF = DA + AF, becomes DE : DB :: DB : 2DE + DC. Then, by taking the rectangles of the extremes and means, it is  $DB^2 = 2DE^2 + DE \cdot DC$ .

Now, if the radius DE be taken = 1, this expression becomes  $DB^2 = 2 + DC$ , and hence the root  $DB = \sqrt{2 + DC}$ . That is, If the measure of the supplemental chord of any arc be increased by the number 2, the square root of the sum will be the supplemental chord of half that arc.

Now, to apply this to the calculation of the circumference of the circle, let the arc AC be taken equal to  $\frac{1}{6}$  of the circumference, and be successively bisected by the above theorem: thus, the chord AC, of  $\frac{1}{6}$  of the circumference, is the side of the inscribed regular hexagon, and is therefore equal the radius AE or 1: hence, in the right-angle triangle ACD, it will be  $DC = \sqrt{AD^2 - AC^2} = \sqrt{2^2 - 1^2} = \sqrt{3} = 1.7320508076$ , the supplemental chord of  $\frac{1}{6}$  of the periphery.

Then, by the foregoing theorem, by always bisecting the arcs, and adding 2 to the last square root, there will be found the supplemental chords of the 12th, the 24th, the 48th, the 96th, &c. parts of the periphery; thus,

$\sqrt{3.7320508076} = 1.9318516525$	}	for the supplemental chord of	}	$\frac{1}{12}$	}	of the periphery.
$\sqrt{3.9318516525} = 1.9828897227$				$\frac{1}{24}$		
$\sqrt{3.9828897227} = 1.9957178465$				$\frac{1}{48}$		
$\sqrt{3.9957178465} = 1.9989291743$				$\frac{1}{96}$		
$\sqrt{3.9989291743} = 1.9997322757$				$\frac{1}{192}$		
$\sqrt{3.9997322757} = 1.9999330678$				$\frac{1}{384}$		
$\sqrt{3.9999330678} = 1.9999832669$				$\frac{1}{768}$		
$\sqrt{3.9999832669} = - - - -$				$\frac{1}{1536}$		

Since then it is found that 3.9999832669 is the square of the supplemental chord of the 1536th part of the periphery, let this number be taken from 4, which is the square of the diameter, and the remainder 0.0000167331 will be the square of the chord of the said 1536th part of the periphery, and consequently the root  $\sqrt{0.0000167331} = 0.0040906112$  is the length of that chord; this number then being multiplied by 1536, gives 6.2831788 for the perimeter of a regular polygon of 1536 sides inscribed in the circle; which, as the

sides

EX. 2. If the circumference of the earth be 25000 miles, what is its diameter?

By the 2d rule, as  $3.1416 : 1 :: 25000 : 7957\frac{3}{4}$  nearly the diameter.

PROBLEM VIII.

*To find the Length of any Arc of a Circle.*

MULTIPLY the degrees in the given arc by the radius of the circle, and the product again by the decimal  $.01745$ , for the length of the arc \*.

EX. I.

sides of the polygon nearly coincide with the circumference of the circle, must also express the length of the circumference itself, very nearly.

But now, to shew how near this determination is to the truth, let  $AQP = 0.0040906112$  represent one side of such a regular polygon of 1536 sides, and  $SRT$  a side of another similar polygon described about the circle; and from the centre  $E$  let the perpendicular  $EQR$  be drawn, bisecting  $AP$  and  $ST$  in  $Q$  and  $R$ . Then, since  $AQ$  is  $= \frac{1}{2}AP = 0.0020453056$ , and  $EA = 1$ , therefore  $EQ^2 = EA^2 - AQ^2 = .9999958167$ , and consequently its root gives  $EQ = .9999979084$ ; then, because of the parallels  $AP, ST$ , it is  $EQ : ER :: AP : ST ::$  as the whole inscribed perimeter : to the circumscribed one, that is, as  $.9999979084 : 1 :: 6.2831788 : 6.2831920$  the perimeter of the circumscribed polygon. Now, the circumference of the circle being greater than the perimeter of the inner polygon, but less than that of the outer, it must consequently be greater than  $6.2831788$ , but less than  $6.2831920$ , and must therefore be nearly equal  $\frac{1}{2}$  their sum, or  $6.2831854$ , which in fact is true to the last figure, which should be a 3 instead of the 4.



Hence, the circumference being  $6.2831854$  when the diameter is 2, it will be the half of that, or  $3.1415927$ , when the diameter is 1, to which the ratio in the rule, viz. 1 to  $3.1416$  is very near. Also, the other ratio in the rule 7 to 22 or 1 to  $3\frac{1}{7} = 3.1428$  &c. is another near approximation.

\* It having been found, in the demonstration of the foregoing problem, that when the radius of a circle is 1, the length of the whole circumference is  $6.2831854$ , which consists of 360 degrees; therefore as  $360^\circ : 6.2831854 :: 1^\circ : .01745$  &c. the length of the arc of 1 degree. Hence, the number  $.01745$  multiplied by any number of degrees, will give the length of the arc of those degrees.

Ex. 1. To find the length of an arc of 30 degrees, the radius being 9 feet. Anf. 4.7115.

Ex. 2. To find the length of an arc of  $12^{\circ} 10'$ , or  $12^{\circ}\frac{1}{6}$ , the radius being 10 feet. Anf. 2.1231.

## PROBLEM IX.

To find the Area of a Circle\*.

RULE I. MULTIPLY half the circumference by half the diameter. Or multiply the whole circumference by the whole diameter, and take  $\frac{1}{4}$  of the product.

RULE II. Square the diameter, and multiply that square by the decimal .7854, for the area.

RULE III. Square the circumference, and multiply that square by the decimal .07958.

Ex. 1. To find the area of a circle whose diameter is 10, and its circumference 31.416.

By Rule 1.	By Rule 2.	By Rule 3.
31.416	.7854	31.416
10	$10^2 = 100$	31.416
4 ) 314.16	78.54	986.965
78.54	78.54	.07958
78.54	78.54	78.54

So that the area is 78.54 by all the rules.

And, because the circumferences, and arcs, are as the diameters, or as the radii of the circles, therefore as the radius 1 is to any other radius  $r$ , so is the length of the arc above mentioned, to  $r \times .01745 \times$  degrees in the arc, which is the length of that arc, as in the rule.

\* The first rule is proved in the Geom. theor. 94.

And the 2d and 3d rules are deduced from the first rule, in this manner.—By that rule,  $dc \div 4$  is the area, when  $d$  denotes the diameter, and  $c$  the circumference. But, by prob. 7,  $c$  is  $= 3.1416d$ ; therefore the said area  $dc \div 4$ , becomes  $d \times 3.1416d \div 4 = .7854d^2$ , which gives the 2d rule.—Also, by the same prob. 7,  $d$  is  $= c \div 3.1416$ ; therefore again the same first area  $dc \div 4$ , becomes  $c \div 3.1416 \times c \div 4 = c^2 \div 12.5664$ , which is  $= c^2 \times .07958$ , by taking the reciprocal of 12.5664, or changing that divisor into the multiplier .07958; which gives the 3d rule.

Corol Hence, the areas of different circles are in proportion to one another, as the square of their diameters, or as the square of their circumferences; as before proved in the Geom. theor. 93.

Ex. 2. To find the area of a circle, whose diameter is 7, and circumference 22. Ans.  $38\frac{1}{2}$ .

Ex. 3. How many square yards are in a circle, whose diameter is  $3\frac{1}{2}$  feet. Ans. 1.069.

Ex. 4. To find the area of a circle whose circumference is 12 feet. Ans. 11.4595.

## PROBLEM X.

*To find the Area of a Circular Ring, or Space included between two Concentric Circles.*

TAKE the difference between the areas of the two circles, as found by the last problem, for the area of the ring.—Or, which is the same thing, subtract the square of the less diameter from the square of the greater, and multiply their difference by .7854.—Or lastly, multiply the sum of the diameters by the difference of the same, and that product by .7854; which is still the same thing, because the product of the sum and difference of any two quantities, is equal to the difference of their squares.

Ex. 1. The diameters of two concentric circles being 10 and 6, required the area of the ring contained between their circumferences.

Here  $10 + 6 = 16$  the sum, and  $10 - 6 = 4$  the diff.  
Therefore  $.7854 \times 16 \times 4 = .7854 \times 64 = 50.2656$ ,  
the area.

Ex. 2. What is the area of the ring, the diameters of whose bounding circles are 10 and 20? Ans. 235.62.

## PROBLEM XI.

*To find the Area of the Sector of a Circle.*

RULE I. MULTIPLY the radius, or half the diameter, by half the arc of the sector, for the area. Or, multiply the whole diameter by the whole arc of the sector, and take  $\frac{1}{4}$  of the product. The reason of which is the same as for the first rule to problem 9.

RULE II. As 360 is to the degrees in the arc of the sector, so is the area of the whole circle, to the area of the sector.

This is evident, because the sector is proportional to the length of the arc, or to the degrees contained in it.

Ex. 1. To find the area of a circular sector, whose arc contains 18 degrees; the diameter being 3 feet?

1. By

1. By the 1st Rule.

First,  $3 \cdot 1416 \times 3 = 9 \cdot 4248$ , the circumference.

And  $360 : 18 :: 9 \cdot 4248 : \cdot 47124$ , the length of the arc.

Then  $\cdot 47124 \times 3 \div 4 = \cdot 11781 \times 3 = \cdot 35343$ , the area.

2. By the 2d Rule.

First,  $\cdot 7854 \times 3^2 = 7 \cdot 0686$ , the area of the whole circle.

Then, as  $360 : 18 :: 7 \cdot 0686 : \cdot 35343$ , the area of the sector.

Ex. 2. To find the area of a sector, whose radius is 10, and arc 20. Ans. 100.

Ex. 3. Required the area of a sector, whose radius is 25, and its arc containing  $147^\circ 29'$ . Ans.  $804 \cdot 3986$ .

### PROBLEM XII.

*To find the Area of a Segment of a Circle.*

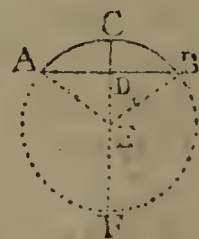
**RULE I.** FIND the area of the sector having the same arc with the segment, by the last problem.

Find also the area of the triangle, formed by the chord of the segment and the two radii of the sector.

Then take the sum of these two for the answer, when the segment is greater than a semicircle: or take their difference for the answer, when it is less than a semicircle.—As is evident by inspection.

Ex. 1. To find the area of the segment ACBDA, its chord AB being 12, and the radius AE or CE 10.

First, As  $AE : AD :: \sin. \angle D90^\circ : \sin. 36^\circ 52\frac{1}{2}' = 36 \cdot 87$  degrees, the degrees in the  $\angle AEC$  or arc AC. Their double,  $73 \cdot 74$ , are the degrees in the whole arc ACB.



Now  $\cdot 7854 \times 400 = 314 \cdot 16$ , the area of the whole circle,

Therefore  $360^\circ : 73 \cdot 74 :: 314 \cdot 16 : 64 \cdot 3504$ , area of the sector ACBE.

Again,  $\sqrt{AE^2 - AD^2} = \sqrt{100 - 36} = \sqrt{64} = 8 = DE$ .  
 Theref.  $AD \times DE = 6 \times 8 = 48$ , the area of the triangle AEB.

Hence, sector ACBA — triangle AEB =  $16 \cdot 3504$ , area of seg. ACBDA.

**RULE I.**

**RULE II.** Divide the height of the segment by the diameter; and find the quotient in the column of heights in the following tablet:—Take out the corresponding area in the next column on the right hand; and multiply it by the square of the circle's diameter, for the area of the segment\*.

*Note.* When the quotient is not found exactly in the table, proportion may be made between the next less and greater area, in the same manner as is done for logarithms, or any other table.

*Table of the Areas of Circular Segments.*

Height.	Area of the Segm.	Height.	Area of the Segm.	Height.	Area of the Segm.	Height.	Area of the Segm.	Height.	Area of the Segm.
·01	·00133	·11	·04701	·21	·11990	·31	·20738	·41	·30819
·02	·00375	·12	·05339	·22	·12811	·32	·21667	·42	·31304
·03	·00687	·13	·06000	·23	·13646	·33	·22603	·43	·32293
·04	·01054	·14	·06683	·24	·14494	·34	·23547	·44	·33284
·05	·01468	·15	·07387	·25	·15354	·35	·24498	·45	·34278
·06	·01924	·16	·08111	·26	·16226	·36	·25455	·46	·35274
·07	·02417	·17	·08853	·27	·17109	·37	·26418	·47	·36272
·08	·02944	·18	·09613	·28	·18002	·38	·27386	·48	·37270
·09	·03502	·19	·10390	·29	·18905	·39	·28359	·49	·38270
·10	·04088	·20	·11182	·30	·19817	·40	·29337	·50	·39270

Ex. 2. Taking the same example as before, in which are given the chord AB 12, and the radius 10, or diameter 20.

And having found, as above,  $DE = 8$ ; then  $CE = DE = CD = 10 - 8 = 2$ . Hence, by the rule,  $CD \div CF = 2 \div 20 = \cdot 1$  the tabular height. This being found in the first column of the table, the corresponding tabular area is  $\cdot 04088$ . Then  $\cdot 04088 \times 20^2 = \cdot 04088 \times 400 = 16\cdot 352$ , the area, nearly the same as before.

\* The truth of this rule depends on the principle of similar plane figures, which are to one another as the square of their like linear dimensions. The segments in the table are those of a circle whose diameter is 1; and the first column contains the corresponding heights or versed sines divided by the diameter. Thus then, the area of the similar segment, taken from the table, and multiplied by the square of the diameter, gives the area of the segment to this diameter.

Ex. 3.

Ex. 3. What is the area of the segment, whose height is 18, and diameter of the circle 50? Anf. 636.375.

Ex. 4. Required the area of the segment whose chord is 16, the diameter being 20? Anf. 44.728.

## PROBLEM XIII.

*To measure long Irregular Figures.*

\* TAKE or measure the breadth in several places, at equal distances. Then add all these breadths together, and divide the sum by the number of them, for the mean breadth; which multiply by the length, for the area.

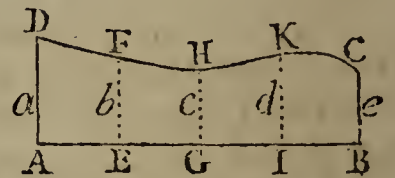
*Note 1.* Take half the sum of the extreme breadths for one of the said breadths; which makes the number to divide by the same as the number of parts in the base.

*Note 2.* If the perpendiculars or breadths be not at equal distances, compute all the parts separately, as so many trapezoids, and add them all together for the whole area.

Or else, add all the perpendicular breadths together, and divide their sum by the number of them for the mean breadth, to multiply by the length; which will give the whole area, not far from the truth.

\* This rule is made out as follows:—

Let ABCD be the irregular piece; having the several breadths AD, EF, GH, IK, BC, at the equal distances AE, EG, GI, IB. Let the several breadths in order be denoted by the corresponding letters  $a, b, c, d, e$ , and the whole length AB by  $l$ ; then compute the areas of the parts into which the figure is divided by the perpendiculars, as so many trapezoids, by prob. 3, and add them all together. Thus, the sum of the parts is,



$$\frac{a+b}{2} \times AE + \frac{b+c}{2} \times EG + \frac{c+d}{2} \times GI + \frac{d+e}{2} \times IB$$

$$= \frac{a+b}{2} \times \frac{1}{4}l + \frac{b+c}{2} \times \frac{1}{4}l + \frac{c+d}{2} \times \frac{1}{4}l + \frac{d+e}{2} \times \frac{1}{4}l$$

$$= \left(\frac{1}{2}a + b + c + d + \frac{1}{2}e\right) \times \frac{1}{4}l = (m + b + c + d) \frac{1}{4}l,$$

which is the whole area, agreeing with the rule;  $m$  being the arithmetic mean between the extremes, or half the sum of them both, and 4 the number of the parts. And the same for any other number of parts.

Ex. I.



Ex. 1. The breadths of an irregular figure, at five equidistant places, being 8.2, 7.4, 9.2, 10.2, 8.6; and the whole length 39; required the area?

First,  $(8.2 + 8.6) \div 2 = 8.4$ , the mean of the two extremes.

Then  $8.4 + 7.4 + 9.2 + 10.2 = 35.2$ , sum of breadths.

And  $35.2 \div 4 = 8.8$ , the mean breadth.

Hence  $8.8 \times 39 = 343.2$ , the answer.

Ex. 2. The length of an irregular figure being 84, and the breadths at six equidistant places 17.4, 20.6, 14.2, 16.5, 20.1, 24.4; what is the area? Ans. 1550.64.

## MENSURATION OF SOLIDS.

By the Mensuration of Solids are determined the spaces included by contiguous surfaces, and the sum of the measures of these including surfaces, is the whole surface or superficies of the body.

The measure of a solid, is called its solidity, capacity, or content.

Solids are measured by cubes, whose sides are inches, or feet, or yards, &c. And hence the solidity of a body is said to be so many cubic inches, feet, yards, &c. as will fill its capacity or space, or another of equal magnitude.

The least solid measure is the cubic inch, other cubes being taken from it according to the proportion in the following table.

*Table of Cubic or Solid Measures.*

1728	cubic inches make	1 cubic foot
27	cubic feet -	1 cubic yard
166 $\frac{3}{8}$	cubic yards -	1 cubic pole
64000	cubic poles -	1 cubic furlong
512	cubic furlongs	1 cubic mile.

## PROBLEM I.

*To find the Superficies of a Prism.*

MULTIPLY the perimeter of one end of the prism by the length or height of the solid, and the product will be the surface of all its sides. To which add also the area of the two ends of the prism, when required\*.

Or, compute the areas of all the sides and ends separately, and add them all together.

Ex. 1. To find the surface of a cube, the length of each side being 20 feet. Ans. 2400 feet.

Ex. 2. To find the whole surface of a triangular prism, whose length is 20 feet, and each side of its end or base 18 inches. Ans. 91.948 feet.

Ex. 3. To find the convex surface of a round prism, or cylinder, whose length is 20 feet, and diameter of its base is 2 feet. Ans. 125.664.

Ex. 4. What must be paid for lining a rectangular cistern with lead, at 2d. a pound weight, the thickness of the lead being such as to weigh 7lb. for each square foot of surface; the inside dimensions of the cistern being as follow, viz. the length 3 feet 2 inches, the breadth 2 feet 8 inches, and depth 2 feet 6 inches. Ans. 2l. 3s. 10½d.

## PROBLEM II.

*To find the Surface of a Pyramid or Cone.*

MULTIPLY the perimeter of the base by the slant height, or length of the side, and half the product will evidently be the surface of the sides, or the sum of the areas of all the triangles which form it. To which add the area of the end or base, if requisite.

\* The truth of this will easily appear, by considering that the sides of any prism are parallelograms, whose common length is the same as the length of the solid, and their breadths taken all together make up the perimeter of the ends of the same.

And the rule is evidently the same for the surface of a cylinder.

Ex. I.

EX. 1. What is the upright surface of a triangular pyramid, the slant height being 20 feet, and each side of the base 3 feet?  
Anf. 90 feet.

EX. 2. Required the convex surface of a cone, or circular pyramid, the slant height being 50 feet, and the diameter of its base  $8\frac{1}{2}$  feet.  
Anf.  $667\cdot59$ .

## PROBLEM III.

*To find the Surface of the Frustum of a Pyramid or Cone; being the lower part, when the top is cut off by a plane parallel to the base.*

ADD together the perimeters of the two ends, and multiply their sum by the slant height, taking half the product for the answer.—As is evident, because the sides of the solid are trapezoids, having the opposite sides parallel.

EX. 1. How many square feet are in the surface of the frustum of a square pyramid, whose slant height is 10 feet; also, each side of the base or greater end being 3 feet 4 inches, and each side of the less end 2 feet 2 inches? Anf. 110 feet.

EX. 2. To find the convex surface of the frustum of a cone, the slant height of the frustum being  $12\frac{1}{2}$  feet, and the circumferences of the two ends 6 and  $8\cdot4$  feet. Anf. 90 feet.

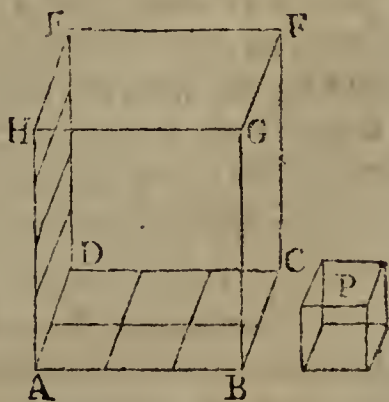
## PROBLEM IV.

*To find the Solid Content of any Prism or Cylinder.*

\* FIND the area of the base, or end, whatever the figure of it may be; and multiply it by the length of the prism or cylinder, for the solid content.

*Note.* For a cube, take the cube of its side; and for a parallelepipedon, multiply the length, breadth and depth all together, for the content.

\* This rule appears from the Geom. theor. 110, cor. 2, and is more particularly shewn as follows: Let the annexed rectangular parallelepipedon be the solid to be measured, and the cube P the solid measuring unit, its side being 1 inch, or 1 foot, &c.; also, let the length and breadth of the base ABCD, and the height AH, be divided into spaces equal to the length of the base of the cube P, namely here 3 in the length and 2 in the breadth, making 3 times 2  
or



Ex. 1. To find the solid content of a cube, whose side is 24 inches. Anf. 13824.

Ex. 2. How many cubic feet are in a block of marble, its length being 3 feet 2 inches, breadth 2 feet 8 inches, and thickness 2 feet 6 inches? Anf.  $21\frac{1}{8}$ .

Ex. 3. How many gallons of water will the cistern contain, whose dimensions are the same as in the last example, when 282 cubic inches are contained in one gallon? Anf.  $129\frac{7}{77}$ .

Ex. 4. Required the solidity of a triangular prism, whose length is 10 feet, and the three sides of its triangular end or base, are 3, 4, 5 feet. Anf. 60.

Ex. 5. Required the content of a round pillar, or cylinder, whose length is 20 feet, and circumference 5 feet 6 inches. Anf.  $48\cdot1459$ .

#### PROBLEM V.

*To find the Content of any Pyramid or Cone.*

\* FIND the area of the base, and multiply that area by the perpendicular height; then take  $\frac{1}{3}$  of the product for the content.

Ex. 1. Required the solidity of the square pyramid, each side of its base being 30, and its slant height 25. Anf. 6000.

Ex. 2. To find the content of a triangular pyramid, whose perpendicular height is 30, and each side of the base 3. Anf.  $38\cdot97117$ .

or 6 squares in the base AC, each equal to the base of the cube P. Hence it is manifest that the parallelepipedon will contain the cube P, as many times as the base AC contains the base of the cube, repeated as often as the height AH contains the height of the cube. That is, the content of any parallelepipedon is found, by multiplying the area of the base by the altitude of that solid. And, because all prisms and cylinders are equal to parallelepipedons of equal bases and altitudes, by Geom. theor. 108, it follows that the rule is general for all such solids, whatever the figure of the base may be.

\* This rule follows from that of the prism, because any pyramid is  $\frac{1}{3}$  of a prism of equal base and altitude; by Geom. theor. 115, cor. 1 and 2.

Ex. 3.

Ex. 3. To find the content of a triangular pyramid, its height being 14 feet 6 inches, and the three sides of its base 5, 6, 7 feet.

Anf. 71.0352.

Ex. 4. What is the content of a pentagonal pyramid, its height being 12 feet, and each side of its base 2 feet?

Anf. 27.5276.

Ex. 5. What is the content of the hexagonal pyramid, whose height is 6.4 feet, and each side of its base 6 inches?

Anf. 1.38564 feet.

Ex. 6. Required the content of a cone, its height being  $10\frac{1}{2}$  feet, and the circumference of its base 9 feet.

Anf. 22.56093.

### PROBLEM VI.

*To find the Solidity of the Frustum of a Cone or Pyramid.*

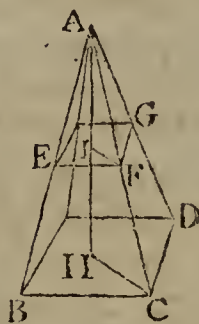
\* ADD into one sum, the areas of the two ends, and the mean proportional between them, or the square root of their product; and  $\frac{1}{3}$  of that sum will be a mean area; which being multiplied by the perpendicular height or length of the frustum, will give its content.

Ex. 1. To find the number of solid feet in a piece of timber, whose bases are squares, each side of the greater end

\* Let ABCD be any pyramid, of which BCDGFE is a frustum. And put  $a^2$  for the area of the base BCD,  $b^2$  the area of the top EFG,  $b$  the height IH of the frustum, and  $c$  the height AI of the top part above it. Then  $c + b = AH$  is the height of the whole pyramid.

Hence, by the last prob.  $\frac{1}{3}a^2(c + b)$  is the content of the whole pyramid ABCD, and  $\frac{1}{3}b^2c$  the content of the top part AEF; therefore the difference  $\frac{1}{3}a^2(c + b) - \frac{1}{3}b^2c$  is the content of the frustum BCDGFE. But, by Geom. theor. 112,  $a^2 : b^2 :: (c + b)^2 : c^2$ , or  $a : b :: c + b : c$ , and  $a - b : b :: b : c$ ; hence  $c = \frac{bb}{a - b}$ , and  $c + b = \frac{ab}{a - b}$ ; then

these values of  $c$  and  $c + b$  being substituted for them in the expression for the content of the frustum, gives that content  $= \frac{1}{3}a^2 \times \frac{ab}{a - b} - \frac{1}{3}b^2 \times \frac{bb}{a - b} = \frac{1}{3}b \times \frac{a^3 - b^3}{a - b} = \frac{1}{3}b \times (a^2 + ab + b^2)$ ; which is the rule above given.



being

being 15 inches, and each side of the less end 6 inches; also, the length or perpendicular altitude 24 feet? Anf.  $19\frac{1}{2}$ .

Ex. 2. Required the content of a pentagonal frustum, whose height is 5 feet, each side of the base 18 inches, and each side of the top or less end 6 inches. Anf.  $9\cdot31925$  feet.

Ex. 3. To find the content of a conic frustum, the altitude being 18, the greatest diameter 8, and the least diameter 4. Anf.  $527\cdot7888$ .

Ex. 4. What is the solidity of the frustum of a cone, the altitude being 25, also the circumference at the greater end being 20, and at the less end 10? Anf.  $464\cdot205$ .

Ex. 5. If a cask, which is two equal conic frustums joined together at the bases, have its bung diameter 28 inches, the head diameter 20 inches, and length 40 inches; how many gallons of wine will it hold? Anf.  $79\cdot0613$ .

#### PROBLEM VII.

*To find the Surface of a Sphere, or any Segment.*

\* RULE I. MULTIPLY the circumference of the sphere by its diameter, and the product will be the whole surface of it.

RULE II.

\* These rules come from the following theorems for the surface of a sphere, viz. That the said surface is equal to the curve surface of its circumscribing cylinder; or that it is equal to 4 great circles of the same sphere, or of the same diameter; which are thus proved.

Let ABCD be a cylinder, circumscribing the sphere EFGH, the former generated by the rotation of the rectangle FBCH about the axis or diameter FH; and the latter by the rotation of the semicircle FGH about the same diameter FH. Draw two lines KL, MN perpendicular to the axis, intercepting the parts LN, OP of the cylinder and sphere; then shall the ring or cylindric surface generated by the rotation of LN, be equal to the ring or spherical surface generated by the arc OP. For, first, suppose the parallels KL and MN to be indefinitely near together; drawing IO, and also OQ parallel to LN. Then, the two triangles IKO, OQP being equiangular, it



is,

**RULE II.** Multiply the square of the diameter by 3·1416, and the product will be the surface.

*Note.* For the surface of a segment or frustum, multiply the whole circumference of the sphere by the height of the part required.

**Ex. 1.** Required the convex superficies of a sphere, whose diameter is 7, and circumference 22. Anf. 154.

**Ex. 2.** Required the superficies of a globe, whose diameter is 24 inches. Anf. 1809·5616.

**Ex. 3.** Required the area of the whole surface of the earth, its diameter being  $7957\frac{1}{4}$  miles, and its circumference 25000 miles. Anf. 198943750 sq. miles.

**Ex. 4.** The axis of a sphere being 42 inches, what is the convex superficies of the segment whose height is 9 inches? Anf. 1187·5248 inches.

**Ex. 5.** Required the convex surface of a spherical zone, whose breadth or height is 2 feet, and cut from a sphere of  $12\frac{1}{2}$  feet diameter. Anf. 78·54 feet.

is, as  $OP : OQ$  or  $LN :: IO$  or  $KL : KO ::$  circumference described by  $KL : \text{circumf. described by } KO$ ; therefore the rectangle  $OP \times \text{circumf. of } KO$  is equal to the rectangle  $LN \times \text{circumf. of } KL$ ; that is, the ring described by  $OP$  on the sphere, is equal to the ring described by  $LN$  on the cylinder.

And as this is every where the case, therefore the sums of any corresponding number of these are also equal; that is, the whole surface of the sphere, generated by the whole semicircle  $FGH$ , is equal to the whole curve surface of the cylinder, generated by the height  $BC$ ; as well as the surface of any segment generated by  $FO$ , equal to the surface of the corresponding segment generated by  $BL$ .

*Corol. 1.* Hence the surface of the sphere is equal to 4 of its great circles, or equal to the circumference  $EF GH$ , or of  $DC$ , multiplied by the height  $BC$ , or the diameter  $FH$ .

*Corol. 2.* Hence also, the surface of any such part as a segment, or frustum, or zone, is equal to the same circumference of the sphere, multiplied by the height of the said part. And consequently such spherical curve surfaces are to one another, in the same proportion as their altitudes.

## PROBLEM VIII.

*To find the Solidity of a Sphere or Globe.*

\* Rule I. Multiply the surface by the diameter, and take  $\frac{1}{6}$  of the product for the content.

RULE II. Take the cube of the diameter, and multiply it by the decimal  $\cdot 5236$ , for the content.

Ex. 1. To find the content of a sphere whose axis is 12.  
Anf. 904.7808.

Ex. 2. To find the solid content of the globe of the earth, supposing its circumference to be 25000 miles.  
Anf. 263858149120 miles.

## PROBLEM IX.

*To find the Solid Content of a Spherical Segment.*

† RULE I. From 3 times the diameter of the sphere, take double the height of the segment; then multiply the remainder

\* For, put  $d =$  the diameter,  $c =$  the circumference, and  $s =$  the surface of the sphere, or of its circumscribing cylinder; also,  $a =$  the number 3.1416.

Then,  $\frac{1}{4}c$  is the base of the cylinder, or one great circle of the sphere; and  $d$  is the height of the cylinder; therefore  $\frac{1}{4}ds$  is the content of the cylinder. But  $\frac{2}{3}$  of the cylinder is the sphere, by th. 117, Geom. that is,  $\frac{2}{3}$  of  $\frac{1}{4}ds$ , or  $\frac{1}{6}ds$  is the sphere; which is the first rule.

Again, because the surface  $s$  is  $= ad^2$ ; therefore  $\frac{1}{6}ds = \frac{1}{6}ad^3 = \cdot 5236d^3$ , the content, as in the 2d rule.

† By corol. 3, of theor. 117, Geom. it appears that the spheric segment PFN, is equal to the difference between the cylinder ABLO, and the conic frustum ABMQ.

But, putting  $d = AB$  or  $FH$  the diameter of the sphere or cylinder,  $b = FK$  the height of the segment,  $r = PK$  the radius of its base, and  $a = 3.1416$ ; then the content of the cone ABI is  $= \frac{1}{4}ad^2 \times \frac{1}{3}FI = \frac{1}{24}ad^3$ ; and by the similar cones ABI, QMI, as

$FI^3 : KI^3 :: \frac{1}{24}ad^3 : \frac{1}{24}ad^3 \times \left(\frac{\frac{1}{2}d - b}{\frac{1}{2}d}\right)^3 =$  the cone QMI;

therefore





remainder by the square of the height, and the product by the decimal  $\cdot 5236$ , for the content.

**RULE II.** To three times the square of the radius of the segment's base, add the square of its height; then multiply the sum by the height, and the product by  $\cdot 5236$ , for the content.

**Ex. 1.** To find the content of a spherical segment, of 2 feet in height, cut from a sphere of 8 feet diameter.

Anf. 41.888.

**Ex. 2.** What is the solidity of the segment of a sphere, its height being 9, and the diameter of its base 20?

Anf. 1795.4244.

*Note.* The general rules for measuring all sorts of figures having been now delivered, we may next proceed to apply them to the several practical uses in life, as follows.

therefore the cone  $ABI$  — the cone  $QMI = \frac{1}{24}ad^3 - \frac{1}{24}ad^3 \times (\frac{\frac{1}{2}d - b}{\frac{1}{2}d})^3 = \frac{1}{4}ad^2b - \frac{1}{2}adb^2 + \frac{1}{3}ab^3$  is = the conic frustum  $ABMQ$ .

And  $\frac{1}{4}ad^2b$  is = the cylinder  $ABLO$ .

Then the difference of these two is  $\frac{1}{2}adb^2 - \frac{1}{3}ab^3 = \frac{1}{6}ab^2 \times (3d - 2b)$ , for the spheric segment  $PFN$ ; which is the first rule.

Again, because  $PK^2 = FK \times KH$ , or  $r^2 = b(d - b)$ , therefore  $d = \frac{r^2}{b} + b$ , and  $3d - 2b = \frac{3r^2}{b} + b = \frac{3r^2 + b^2}{b}$ ; which being substituted in the former rule, it becomes  $\frac{1}{6}ab^2 \times \frac{3r^2 + b^2}{b} = \frac{1}{6}ab \times (3r^2 + b^2)$ , which is the 2d rule.

*Note.* By subtracting a segment from a half sphere, or from another segment, the content of any frustum or zone may be found.

## LAND SURVEYING.

## SECTION I.

## DESCRIPTION AND USE OF THE INSTRUMENTS.

## I. OF THE CHAIN.

LAND is measured with a chain, called Gunter's Chain, from its inventor, of 4 poles or 22 yards, or 66 feet in length. It consists of 100 equal links; and the length of each link is therefore  $\frac{22}{100}$  of a yard, or  $\frac{66}{100}$  of a foot, or 7.92 inches.

Land is estimated in acres, roods, and perches. An acre is equal to 10 square chains, that is, 10 chains in length and 1 chain in breadth. Or it is  $220 \times 22 = 4840$  square yards. Or it is  $40 \times 4 = 160$  square poles. Or it is  $1000 \times 100 = 100000$  square links. These being all the same quantity.

Also, an acre is divided into 4 parts called roods, and a rood into 40 parts called perches, which are square poles, or the square of a pole of  $5\frac{1}{2}$  yards long, or the square of  $\frac{1}{4}$  of a chain, or of 25 links, which is 625 square links. So that the divisions of land measure will be thus :

$$\begin{aligned} 625 \text{ sq. links} &= 1 \text{ pole or perch} \\ 40 \text{ perches} &= 1 \text{ rood} \\ 4 \text{ roods} &= 1 \text{ acre.} \end{aligned}$$

The length of lines, measured with a chain, are best set down in links as integers, every chain in length being 100 links; and not in chains and decimals. Therefore after the content is found, it will be in square links; then cut off five of the figures on the right-hand for decimals, and the rest will be acres. These decimals are then multiplied by 4 for roods, and the decimals of these again by 40 for perches.

EXAM. Suppose the length of a rectangular piece of ground be 792 links, and its breadth 385; to find the area in acres, roods, and perches.

792	3'04920
385	4
3960	·19680
6336	40
2376	7'87200
3'04920	

Anf. 3 acres, 0 roods, 7 perches.

## 2. OF THE PLAIN TABLE.

THIS instrument consists of a plain rectangular board, of any convenient size: the centre of which, when used, is fixed by means of screws to a three-legged stand, having a ball and socket, or other joint, at the top, by means of which, when the legs are fixed on the ground, the table is inclined in any direction.

To the table belong various parts, as follow.

1. A frame of wood, made to fit round its edges, and to be taken off, for the convenience of putting a sheet of paper upon the table. The one side of this frame is usually divided into equal parts, for drawing lines across the table, parallel or perpendicular to the sides; and the other side of the frame is divided into 360 degrees from a centre which is in the middle of the table; by means of which the table is to be used as a theodolite, &c.

2. A needle and compass, either screwed into the side of the table, or fixed beneath its centre, to point out the directions, and to be a check upon the sights.

3. An index, which is a brass two-foot scale, with either a small telescope, or open sights erected perpendicularly upon the ends. These sights and one edge of the index are in the same plane, and that edge is called the fiducial edge of the index.

To use this instrument, take a sheet of paper which will cover it, and wet it to make it expand; then spread it flat on the table, pressing down the frame upon the edges, to stretch it and keep it fixed there; and when the paper is become dry, it will, by contracting again, stretch itself smooth and flat from any cramps and unevenness. On this paper is to be drawn the plan or form of the thing measured.

Thus, begin at any part of the ground the most proper,

and make a point on a convenient part of the paper or table, to represent that point of the ground ; then fix in that point one leg of the compasses, or a fine steel pin, and apply to it the fiducial edge of the index, moving it round till through the sights you perceive some remarkable object, as the corner of a field, &c ; and from the station point draw a line with the point of the compasses along the fiducial edge of the index ; then set another object or corner, and draw its line ; do the same by another, and so on, till as many objects are set as may be thought fit. Then measure from the station, towards as many of the objects as may be necessary, and no more, taking the requisite offsets to corners or crooks in the hedges, laying the measures down on their respective lines on the table. Then, at any convenient place, measured to, fix the table in the same position, and set the objects which appear from thence, &c, as before ; and thus continue till the work is finished, measuring such lines as are necessary, and determining as many as may be, by intersecting lines of direction drawn from different stations.

*Of shifting the Paper on the Plain Table.*

When one paper is full, and you have occasion for more ; draw a line in any manner through the farthest point of the last station line, to which the work can be conveniently laid down ; then take the sheet off the table, and fix another on, drawing a line upon it, in a part the most convenient for the rest of the work ; then fold or cut the old sheet by the line drawn on it, applying the edge to the line on the new sheet, and as they lie in that position, continue the last station line on the new paper, placing on it the rest of the measure, beginning at where the old sheet left off. And so on from sheet to sheet.

When the work is done, and you would fasten all the sheets together into one piece, or rough plan, the aforesaid lines are to be accurately joined together, in the same manner as when the lines were transferred from the old sheets to the new ones.

But it is to be noted, that if the said joining lines, on the old and new sheets, have not the same inclination to the side of the table, the needle will not point to the original degree when the table is rectified ; and if the needle be required to respect still the same degree of the compass, the easiest way of drawing the lines in the same position, is to draw them both parallel to the same sides of the table, by means of the equal divisions marked on the other two sides.

## 3. OF THE THEODOLITE.

THE theodolite is a brazen circular ring, divided into 360 degrees, and having an index with sights, or a telescope, placed on the centre, about which the index is moveable; also a compass fixed to the centre, to point out courses and check the sights; the whole being fixed by the centre on a stand of a convenient height for use.

In using this instrument, an exact account, or field-book, of all measures and things necessary to be remarked in the plan, must be kept, from which to make out the plan on returning home from the ground.

Begin at such part of the ground, and measure in such directions, as you judge most convenient; taking angles or directions to objects, and measuring such distances as appear necessary, under the same restrictions as in the use of the plain table. And it is safest to fix the theodolite in the original position at every station by means of fore and back objects, and the compass, exactly as in using the plain table; registering the number of degrees cut off by the index when directed to each object; and, at any station, placing the index at the same degree as when the direction towards that station was taken from the last preceding one, to fix the theodolite there in the original position.

The best method of laying down the aforesaid lines of direction, is to describe a pretty large circle; then quarter it, and lay on it the several numbers of degrees cut off by the index in each direction, and drawing lines from the centre to all these marked points in the circle. Then, by means of a parallel ruler, draw, from station to station, lines parallel to the aforesaid lines drawn from the centre to the respective points in the circumference.

## 4. OF THE CROSS.

THE cross consists of two pair of sights set at right angles to each other, upon a staff having a sharp point at the bottom to stick in the ground.

The cross is very useful to measure small and crooked pieces of ground. The method is to measure a base or chief line, usually in the longest direction of the piece, from corner to corner; and while measuring it, finding the places where perpendiculars would fall on this line, from the several corners and bends in the boundary of the piece, with the cross, by fixing it, by trials, on such parts of the line, so that through one pair of the sights both ends of the line may appear, and through the other pair you can perceive the

corresponding bends or corners; and then measuring the lengths of the said perpendiculars.

#### R E M A R K S.

Besides the fore-mentioned instruments, which are most commonly used, there are some others; as the circumferentor, which resembles the theodolite in shape and use; and the semicircle, for taking angles, &c.

The perambulator is used for measuring roads, and other great distances on level ground, and by the sides of rivers. It has a wheel of  $8\frac{1}{4}$  feet, or half a pole in circumference, by the turning of which the machine goes forward; and the distance measured, is pointed out by an index, which is moved round by clock work.

Levels, with telescopic or other sights, are used to find the level between place and place, or how much one place is higher or lower than another. And in measuring any sloping or oblique line, either ascending or descending, a small pocket level is useful for showing how many links for each chain are to be deducted, to reduce the line to the true horizontal length.

An offset-staff is a very useful and necessary instrument for measuring the offsets and other short distances. It is 10 links in length, being divided and marked at each of the 10 links.

Ten small arrows, or rods of iron or wood, are used to mark the end of every chain length, in measuring lines. And sometimes pickets, or staves with flags, are set up as marks or objects of direction.

Various scales are also used in protracting and measuring on the plan or paper; such as plane scales, line of chords, protractor, compasses, reducing scale, parallel and perpendicular rules, &c. Of plane scales, there should be several sizes, as a chain in 1 inch, a chain in  $\frac{3}{4}$  of an inch, a chain in  $\frac{1}{2}$  an inch, &c. And of these, the best for use are those that are laid on the very edges of the ivory scale, to mark off distances, without compasses.

#### 5. OF THE FIELD-BOOK.

In surveying with the plane table, a field-book is not used, as every thing is drawn on the table immediately when it is measured. But in surveying with the theodolite, or any other instrument, some sort of a field-book must be used, to write down in it a register or account of all that is done and occurs relative to the survey in hand.

This book every one contrives and rules as he thinks fittest for himself. The following is a specimen of a form which has been formerly used. It is ruled into 3 columns: the middle, or principal column, is for the stations, angles, bearings,

bearings, distances measured, &c; and those on the right and left are for the offsets on the right and left, which are set against their corresponding distances in the middle column; as also for such remarks as may occur, and may be proper to note in drawing the plan, &c.

Here  $\odot 1$  is the first station, where the angle or bearing is  $105^{\circ} 25'$ . On the left, at 73 links in, the distance or principal line, is an offset of 92; and at 610 an offset of 24 to a cross hedge. On the right, at 0, or the beginning, an offset 25 to the corner of the field; at 248 Brown's boundary hedge commences; at 610 an offset 35; and at 954, the end of the first line, the  $\odot$  denotes its terminating in the hedge. And so on for the other stations.

A line is drawn under the work, at the end of every station line, to prevent confusion.

*Form of this Field-Book.*

Offsets and Remarks on the left.	Stations, Bearings, and Distances.	Offsets and Remarks on the right.
<p>92</p> <p>cross a hedge 24</p>	<p><math>\odot 1</math></p> <p><math>105^{\circ} 25'</math></p> <p>00</p> <p>73</p> <p>248</p> <p>610</p> <p>954</p>	<p>25 corner</p> <p>Brown's hedge</p> <p>35</p> <p>00</p>
<p>house corner 51</p> <p>34</p>	<p><math>\odot 2</math></p> <p><math>53^{\circ} 10'</math></p> <p>00</p> <p>25</p> <p>120</p> <p>734</p>	<p>00</p> <p>21</p> <p>29 a tree</p> <p>40 a stile</p>
<p>a brook 30</p> <p>foot-path 16</p> <p>cross hedge 18</p>	<p><math>\odot 3</math></p> <p><math>67^{\circ} 20'</math></p> <p>61</p> <p>248</p> <p>639</p> <p>810</p> <p>973</p>	<p>35</p> <p>16 a spring</p> <p>20 a pond</p>

But some skilful surveyors now make use of a different method for the field-book, namely, beginning at the bottom of the page and writing upwards; by which they sketch a neat boundary on either hand, as they pass along; an example of which will be given further on, in the method of surveying a large estate.

In smaller surveys and measurements, a good way of setting down the work, is, to draw by the eye, on a piece of paper, a figure resembling that which is to be measured; and so writing the dimensions, as they are found, against the corresponding parts of the figure. And this method may be practised to a considerable extent, even in the larger surveys.

## SECTION II.

### THE PRACTICE OF SURVEYING.

THIS part contains the several works proper to be done in the field, or the ways of measuring by all the instruments, and in all situations.

#### PROBLEM I.

##### *To Measure a Line or Distance.*

To measure a line on the ground with the chain: Having provided a chain, with 10 small arrows, or rods, to stick one into the ground, as a mark, at the end of every chain; two persons take hold of the chain, one at each end of it; and all the 10 arrows are taken by one of them, who goes foremost, and is called the leader; the other being called the follower, for distinction's sake.

A picket, or station staff, being set up in the direction of the line to be measured, if there do not appear some marks naturally in that direction; they measure straight towards it, the leader fixing down an arrow at the end of every chain, which the follower always takes up, till all the ten arrows are used. They are then all returned to the leader, to use over again. And thus the arrows are changed from the one to the other at every 10 chains length, till the whole line is finished; then the number of changes of the arrows shews the number of tens, to which the follower adds the arrows  
he



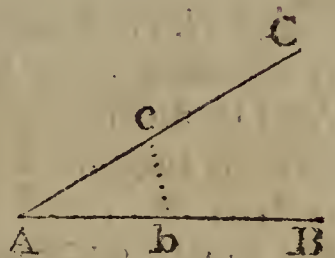
he holds in his hand, and the number of links of another chain over to the mark or end of the line. So, if there have been 3 changes of the arrows, and the follower hold 6 arrows, and the end of the line cut off 45 links more, the whole length of the line is set down in links thus, 3645.

When the ground is not level, but either ascending or descending; at every chain length, lay the offset staff, or link-staff down in the slope of the chain, upon which lay the small pocket level, to shew how many links or parts the slope line is longer than the true level one; then draw the chain forward so many links or parts, which reduces the line to the horizontal direction.

### PROBLEM II.

*To take Angles and Bearings.*

LET B and C be two objects, or two pickets set up perpendicular, and let it be required to take their bearings, or the angle formed between them at any station A.



#### 1. *With the Plain Table.*

The table being covered with a paper, and fixed on its stand: plant it at the station A, and fix a fine pin, or a point of the compasses, in a proper point of the paper, to represent the point A: Close by the side of this pin lay the fiducial edge of the index, and turn it about, still touching the pin, till one object B can be seen through the sights: then by the fiducial edge of the index draw a line. In the very same manner draw another line in the direction of the other object C. And it is done.

#### 2. *With the Theodolite, &c.*

Direct the fixed sights along one of the lines, as AB, by turning the instrument about till the mark B is seen through these sights; and there screw the instrument fast. Then turn the moveable index about till, through its sights, you see the other mark C. Then the degrees cut by the index, upon the graduated limb or ring of the instrument, shew the quantity of the angle.

#### 3. *With the Magnetic Needle and Compass.*

Turn the instrument, or compass, so, that the north end of the needle point to the flower-de-luce. Then direct the  
sights

lights to one mark as B, and note the degrees cut by the needle. Next direct the sights to the other mark C, and note again the degrees cut by the needle. Then their sum or difference, as the case is, will give the quantity of the angle BAC.

4. *By Measurement with the Chain, &c.*

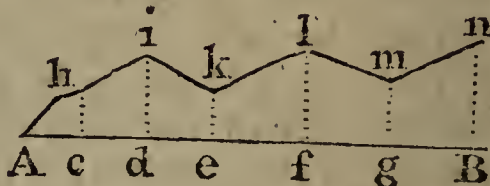
Measure one chain length, or any other length, along both directions, as to b and c. Then measure the distance b c, and it is done.—This is easily transferred to paper, by making a triangle A b c with these three lengths, and then measuring the angle A.

PROBLEM III.

*To measure the Offsets.*

A h i k l m n being a crooked hedge, or river, &c. From A measure in a straight direction along the side of it to B. And in measuring along this line A B, observe when you are directly opposite any bends or corners of the boundary, as at c, d, e, &c; and from thence measure the perpendicular offsets c h, d i, &c, with the offset-staff, if they are not very large, otherwise with the chain itself. And the work is done. The register, or field-book, may be as follows :

Offs. left.		Base line AB.	
	o	⊙ A	
c h	62	45	A c
d i	84	220	A d
e k	70	340	A e
f l	98	510	A f
g m	57	634	A g
B n	91	785	A B

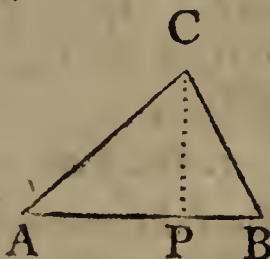


PROBLEM IV.

*To survey a Triangular Field ABC.*

1. *By the Chain.*

AP 794  
 AB 1321  
 PC 826



Having

Having set up marks at the corners, which is to be done in all cases where there are not marks naturally; measure with the chain from A to P, where a perpendicular would fall from the angle C, and set up a mark at P, noting down the distance AP. Then complete the distance AB by measuring from P to B. Having set down this measure, return to P, and measure the perpendicular PC. And thus, having the base and perpendicular, the area from them is easily found. Or having the place P of the perpendicular, the triangle is easily constructed.

Or, measure all the three sides with the chain, and note them down. From which the content is easily found, or the figure constructed.

2. *By taking some of the Angles.*

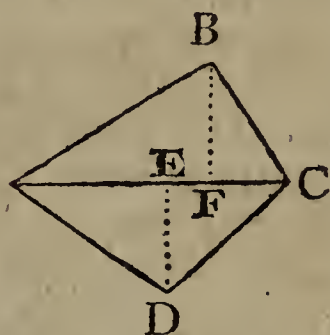
Measure two sides AB, AC, and the angle A between them. Or measure one side AB, and the two adjacent angles A and B. From either of these ways the figure is easily planned; then by measuring the perpendicular CP on the plan, and multiplying it by half AB, you have the content.

PROBLEM V.

*To measure a Four-sided Field.*

1. *By the Chain.*

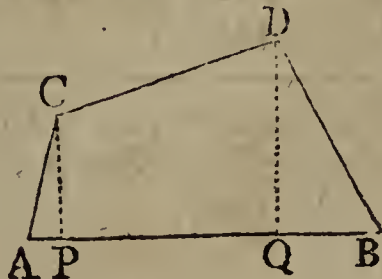
AE	214		210	DE
AF	362		306	BF
AC	592			



Measure along either of the diagonals, as AC; and either the two perpendiculars DE, BF, as in the last problem; or else the sides AB, BC, CD, DA. From either of which the figure may be planned and computed as before directed.

*Otherwise, by the Chain.*

AP	110		352	PC
AQ	745		595	QD
AB	1110			



Measure on the longest side, the distances AP, AQ, AB; and the perpendicular PC, QD.

2. By

2. *By taking some of the Angles.*

Measure the diagonal AC (see the last fig. but one), and the angles CAB, CAD, ACB, ACD.—Or measure the four sides, and any one of the angles, as BAD.

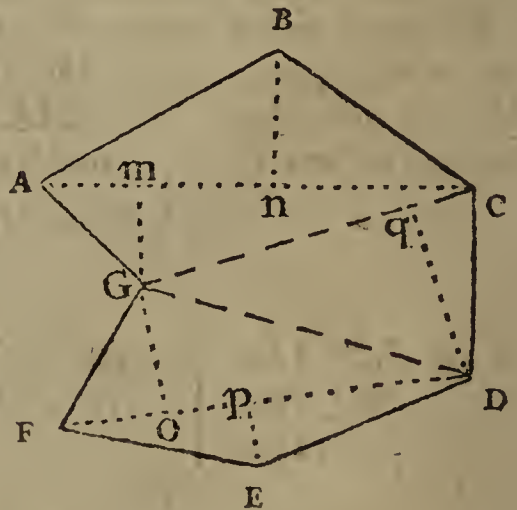
Thus		Or thus
AC 591		AB 486
CAB 37° 20'		BC 394
CAD 41 15		CD 410
ACB 72 25		DA 462
ACD 54 40		BAD 78° 35'

PROBLEM VI.

*To survey any Field by the Chain only.*

HAVING set up marks at the corners, where necessary, of the proposed field ABCDEFG, walk over the ground, and consider how it can best be divided in triangles and trapeziums; and measure them separately as in the last two problems. Thus, the following figure is divided into the two trapeziums ABCG, GDEF, and the triangle GCD. Then, in the first trapezium, beginning at A, measure the diagonal AC, and the two perpendiculars G m, B n. Then the base GC, and the perpendicular D q. Lastly, the diagonal DF, and the two perpendiculars p E, o G. All which measures write against the corresponding parts of a rough figure drawn to resemble the figure to be surveyed, or set them down in any other form you choose.

Thus		Thus
A m 135		130 m G
A n 410		180 n B
A c 550		
C q 152		230 q D
CG 440		
Fo 206		120 o G
F p 288		80 p E
FD 520		



Or

*Or thus.*

Measure all the sides AB, BC, CD, DE, EF, FG, and GA; and the diagonals AC, CG, GD, DF.

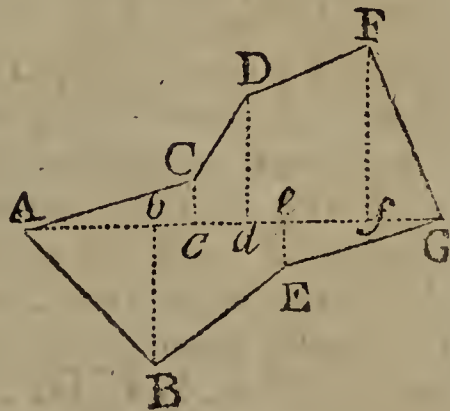
*Otherwise.*

Many pieces of land may be very well surveyed, by measuring any base line, either within or without them, together with the perpendiculars let fall upon it from every corner of them. For they are by those means divided into several triangles and trapezoids, all whose parallel sides are perpendicular to the base line; and the sum of these triangles and trapeziums will be equal to the figure proposed if the base line fall within it; if not, the sum of the parts which are without being taken from the sum of the whole which are both within and without, will leave the area of the figure proposed.

In pieces that are not very large, it will be sufficiently exact to find the points, in the base line, where the several perpendiculars will fall, by means of the *cross*, and from thence measuring to the corners for the lengths of the perpendiculars.—And it will be most convenient to draw the line so as that all the perpendiculars may fall within the figure.

Thus, in the following figure, beginning at A, and measuring along the line AG, the distances and perpendiculars on the right and left, are as below.

A b	315	350	b B
A c	440	70	c C
A d	585	320	d D
A e	610	50	e E
A f	990	470	f F
AG	1020	0	

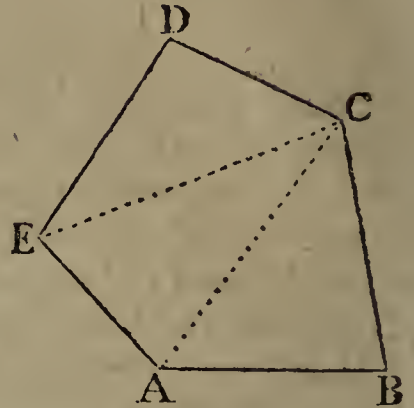


## PROBLEM VII.

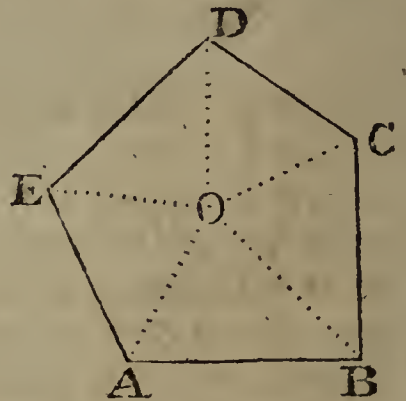
*To survey any Field with the Plain Table.*

1. *From one Station.*

PLANT the table at any angle as C, from whence all the other angles, or marks set up, can be seen; turn the table about till the needle point to the flower-de-luce; and there screw it fast. Make a point for C on the paper on the table, and lay the edge of the index to C, turning it about C till through the sights you see the mark D; and by the edge of the index draw a dry or obscure line: then measure the distance CD, and lay that distance down on the line CD. Then turn the index about the point C, till the mark E be seen through the sights, by which draw a line, and measure the distance to E, laying it on the line from C to E. In like manner determine the positions of CA and CB, by turning the sights successively to A and B; and lay the lengths of those lines down. Then connect the points with the boundaries of the field, by drawing the black lines CD, DE, EA, AB, BC.

2. *From a Station within the Field.*

When all the other parts cannot be seen from one angle, choose some place O within; or even without, if more convenient, from whence the other parts can be seen. Plant the table at O, then fix it with the needle north, and mark the point O on it. Apply the index successively to O, turning it round with the sights to each angle A, B, C, D, E, drawing dry lines to them by the edge of the index; then measuring the distances OA, OB, &c, and laying them down upon those lines. Lastly, draw the boundaries AB, BC, CD, DE, EA.

3. *By going round the Figure.*

When the figure is a wood, or water, or from some other obstruction you cannot measure lines across it; begin at any point

point A, and measure round it, either within or without the figure, and draw the directions of all the sides thus: Plant the table at A, turn it with the needle to the north or flower-de-luce; fix it, and mark the point A. Apply the index to A, turning it till you can see the point E, and there draw a line; then the point B, and there draw a line: then measure these lines, and lay them down from A to E and B. Next, move the table to B, lay the index along the line AB, and turn the table about until you can see the mark A, and screw fast the table; in which position also the needle will again point to the flower-de-luce, as it will do indeed at every station when the table is in the right position. Here turn the index about B till through the sights you see the mark C; there draw a line, measure BC, and lay the distance upon that line after you have set down the table at C. Turn it then again into its proper position, and in like manner find the next line CD. And so on quite round by E to A again. Then the proof of the work will be the joining at A: for if the work is all right, the last direction EA on the ground, will pass exactly through the point A on the paper; and the measured distance will also reach exactly to A. If these do not coincide, or nearly so, some error has been committed, and the work must be examined over again.

#### PROBLEM VIII.

*To survey a Field with the Theodolite, &c.*

##### 1. *From one Point or Station.*

WHEN all the angles can be seen from one point, as the angle C (first fig. to last prob.) place the instrument at C, and turn it about till, through the fixed sights, you see the mark B, and there fix it. Then turn the moveable index about till the mark A is seen through the sights, and note the degrees cut on the instrument. Next turn the index successively to E and D, noting the degrees cut off at each; which gives all the angles BCA, BCE, BCD. Lastly, measure the lines CB, CA, CE, CD; and enter the measures in a field-book, or rather against the corresponding parts of a rough figure drawn by guess to resemble the field.

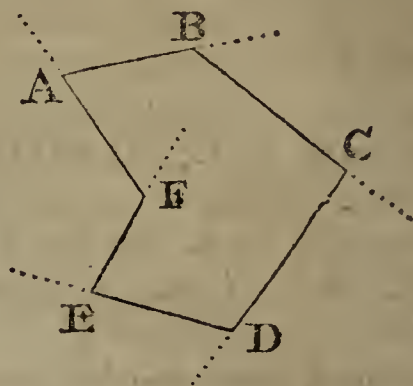
##### 2. *From a Point within or without.*

Plant the instrument at O (last fig.), and turn it about till the fixed sights point to any object, as A; and there screw it fast. Then turn the moveable index round till the  
sights

lights point successively to the other points E, D, C, B, noting the degrees cut off at each of them; which gives all the angles round the point O. Lastly, measure the distances OA, OB, OC, OD, OE, noting them down as before, and the work is done.

### 3. *By going round the Field.*

By measuring round, either within or without the field, proceed thus. Having set up marks at B, C, &c, near the corners as usual, plant the instrument at any point A, and turn it till the fixed index be in the direction AB, and there screw it fast: then turn the moveable index to the direction AF; and the degrees cut off will be the angle A. Measure the line AB, and plant the instrument at B, and there in the same manner observe the angle B. Then measure BC, and observe the angle C. Then measure the distance CD, and take the angle D. Then measure DE, and take the angle E. Then measure EF, and take the angle F. And lastly measure the distance FA.



To prove the work; add all the inward angles A, B, C, &c, together, for when the work is right, their sum will be equal to twice as many right angles as the figure has sides, wanting 4 right angles. But when there is an angle, as F, that bends inwards, and you measure the external angle, which is less than two right angles, subtract it from four right angles, or 360 degrees, to give the internal angle greater than a semicircle or 180 degrees.

#### *Otherwise.*

Instead of observing the internal angles, you may take the external angles, formed without the figure by producing the sides further out. And in this case, when the work is right, their sum altogether will be equal to 360 degrees. But when one of them, as F, runs inwards, subtract it from the sum of the rest, to leave 360 degrees.

#### PROBLEM IX.

##### *To survey a Field with Crooked Hedges, &c.*

WITH any of the instruments, measure the lengths and positions of imaginary lines running as near the sides of the field  
as

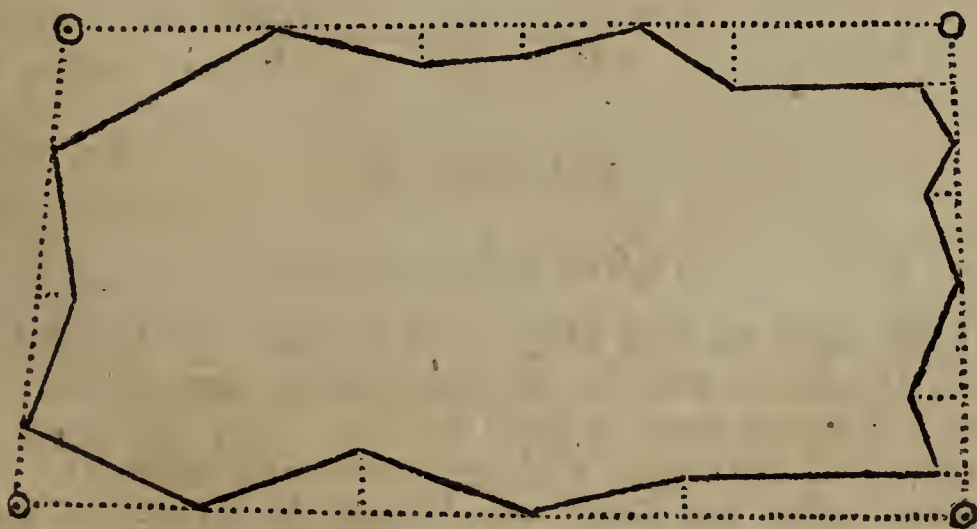


as you can: and, in going along them, measure the offsets in the manner before taught; then you will have the plan on the paper in using the plain table, drawing the crooked hedges through the ends of the offsets; but in surveying with the theodolite, or other instrument, set down the measures properly in a field-book, or memorandum-book, and plan them after returning from the field, by laying down all the lines and angles.



So, in surveying the piece ABCDE, set up marks a, b, c, d, dividing it into as few sides as may be. Then begin at any station a, and measure the lines ab, bc, cd, da, taking their positions, or the angles a, b, c, d; and, in going along the lines, measure all the offsets, as at m, n, o, p, &c, along every station line.

And this is done either within the field, or without, as may be most convenient. When there are obstructions within, as wood, water, hills, &c, then measure without, as in the figure here below.



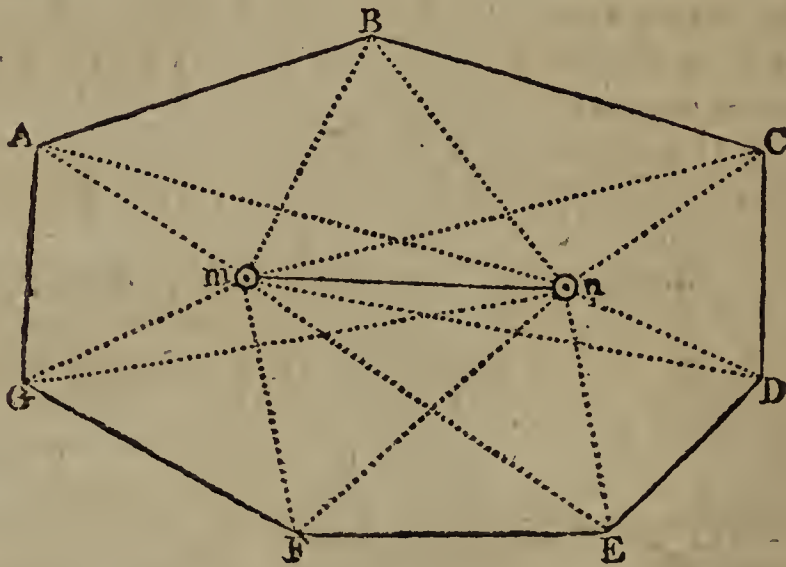
## PROBLEM X.

*To survey a Field, or any other Thing, by two Stations.*

THIS is performed by choosing two stations, from whence all the marks and objects can be seen; then measuring the distance between the stations, and at each station, taking the angles formed by every object, from the station line or distance.

The two stations may be taken either within the bounds, or in one of the sides, or in the direction of two of the objects, or quite at a distance and without the bounds of the objects, or part to be surveyed.

In this manner, not only grounds may be surveyed, without even entering them, but a map may be taken of the principal parts of a county, or the chief places of a town, or any part of a river or coast surveyed, or any other inaccessible objects; by taking two stations, on two towers, or two hills, or such like.



## PROBLEM XI.

*To survey a Large Estate.*

IF the estate be very large, and contain a great number of fields, it cannot well be done by surveying all the fields singly, and then putting them together; nor can it be done by taking all the angles and boundaries that inclose it. For in these cases, any small errors will be so multiplied, as to render it very much distorted.

1. Walk

1. Walk over the estate two or three times, in order to get a perfect idea of it, and till you can carry the map of it tolerably well in your head. And to help your memory, draw an eye draught of it on paper, or at least of the principal parts of it, to guide you; setting the names within the fields in that draught.

2. Choose two or more eminent places in the estate, for stations, from whence all the principal parts of it can be seen: and let these stations be as far distant from one another as possible.

3. Take such angles, between the stations, as you think necessary, and measure the distances from station to station, always in a right line: these things must be done, till you get as many angles and lines as are sufficient for determining all the points of station. And in measuring any of these station distances, mark accurately where these lines meet with any hedges, ditches, roads, lanes, paths, rivulets, &c; and where any remarkable object is placed, by measuring its distance from the station line; and where a perpendicular from it cuts that line. And thus as you go along any main station line, take offsets to the ends of all hedges, and to any pond, house, mill, bridge, &c, omitting nothing that is remarkable, and noting every thing down.

4. As to the inner parts of the estate, they must be determined in like manner, by new station lines: for, after the main stations are determined, and every thing adjoining to them, then the estate must be subdivided into two or three parts by new station lines; taking inner stations at proper places, where you can have the best view. Measure these station lines as you did the first, and all their interfections with hedges, and offsets to such objects as appear. Then proceed to survey the adjoining fields, by taking the angles that the sides make with the station line, at the interfections, and measuring the distances to each corner, from the interfections. For the station lines will be the bases to all the future operations; the situation of all parts being entirely dependent upon them; and therefore they should be taken of as great length as possible; and it is best for them to run along some of the hedges or boundaries of one or more fields, or to pass through some of their angles. All things being determined for these stations, you must take more inner stations, and continue to divide and subdivide till at last you come to single fields; repeating the same work for the inner stations, as for the outer ones, till all is done; and close the work as often as you can, and in as few lines as possible.

5. An estate may be so situated that the whole cannot be surveyed together; because one part of the estate cannot be seen from another. In this case, you may divide it into three or four parts, and survey the parts separately, as if they were lands belonging to different persons; and at last join them together.

6. As it is necessary to protract or lay down the work as you proceed in it, you must have a scale of a due length to do it by. To get such a scale, measure the whole length of the estate in chains; then consider how many inches long the map is to be; and from these will be known how many chains you must have in an inch; then make the scale accordingly, or choose one already made.

*The New Method of Surveying.*

IN the former method of measuring a large estate, the accuracy of it depends on the correctness of the instruments used in taking the angles. To avoid the errors incident to such a multitude of angles, other methods have of late years been used by some few skilful surveyors: the most practical, expeditious, and correct, seems to be the following.

Choose two or more eminences, as grand stations, and measure a principal base line from one station to the other, noting every hedge, brook, or other remarkable object as you pass by it; measuring also such short perpendicular lines to the bends of hedges as may be near at hand. From the extremities of this base line, or from any convenient parts of the same, go off with other lines to some remarkable object situated towards the sides of the estate, without regarding the angles they make with the base line or with one another; still remembering to note every hedge, brook or other object that you pass by. These lines, when laid down by intersections, will with the base line form a grand triangle on the estate; several of which, if need be, being thus laid down, you may proceed to form other smaller triangles and trapezoids on the sides of the former: and so on, until you finish with the enclosures individually.

This grand triangle being completed, and laid down on the rough plan paper, the parts, exterior as well as interior, are to be completed by smaller triangles and trapezoids.

In countries where the lands are enclosed with high hedges, and where many lanes pass through an estate, a theodolite may be used to advantage, in measuring the angles of such lands;

lands; by which means, a kind of skeleton of the estate may be obtained, and the lane-lines serve as the bases of such triangles and trapezoids as are necessary to fill up the interior parts.

The field-book is ruled into three columns. In the middle one are set down the distances on the chain line at which any mark, offset, or other observation is made; and in the right and left hand columns are entered the offsets and observations made on the right and left hand respectively of the chain line.

It is of great advantage, both for brevity and perspicuity, to begin at the bottom of the leaf, and write upwards; denoting the crossing of fences, by lines drawn across the middle column, or only a part of such a line on the right and left opposite the figures, to avoid confusion; and the corners of fields, and other remarkable turns in the fences where offsets are taken to, by lines joining in the manner the fences do, as will be best seen by comparing the book with the plan annexed to the field-book following, p. 74.

The letter in the left-hand corner at the beginning of every line, is the mark or place measured *from*; and, that at the right hand corner at the end, is the mark measured *to*: But when it is not convenient to go exactly from a mark, the place measured from, is described *such a distance from one mark towards another*; and where a mark is not measured to, the exact place is ascertained by saying, turn to the right or left hand, *such a distance to such a mark*, it being always understood that those distances are taken in the chain line.

The characters used, are  $\lrcorner$  for *turn to the right hand*,  $\llcorner$  for *turn to the left hand*, and  $\wedge$  placed over an offset, to shew that it is not taken at right angles with the chain line, but in the line with some straight fence; being chiefly used when crossing their directions, and it is a better way of obtaining their true places than by offsets at right angles.

When a line is measured whose position is determined, either by former work (as in the case of producing a given line, or measuring from one known place or mark to another) or by itself (as in the third side of a triangle) it is called a *fast line*, and a double line across the book is drawn at the conclusion of it; but if its position is not determined (as in the second side of a triangle) it is called a *loose line*, and a single line is drawn across the book. When a line becomes determined in position, and is afterwards continued, a double line half through the book is drawn.

When

When a loose line is measured, it becomes absolutely necessary to measure some line that will determine its position. Thus, the first line  $ab$ , being the base of a triangle, is always determined; but the position of the second side  $bj$ , does not become determined, till the third side  $jb$  is measured; then the triangle may be constructed, and the position of both is determined.

At the beginning of a line, to fix a loose line to the mark or place measured from, the sign of turning to the right or left hand must be added (as at  $j$  in the third line); otherwise a stranger, when laying down the work, may as easily construct the triangle  $hjb$  on the wrong side of the line  $ab$ , as on the right one: but this error cannot be fallen into, if the sign above named be carefully observed.

In choosing a line to fix a loose one, care must be taken that it does not make a very acute or obtuse angle; as in the triangle  $pBr$ , by the angle at  $B$  being very obtuse, a small deviation from truth, even the breadth of a point at  $p$  or  $r$ , would make the error at  $B$ , when constructed, very considerable; but by constructing the triangle  $pBq$ , such a deviation is of no consequence.

Where the words *leave off* are written in the field-book, it is to signify that the taking of offsets is from thence discontinued; and of course something is wanting between that and the next offset.

The field-book for this method, and the plan drawn from it, are contained in the four following pages, engraven on copper-plates.

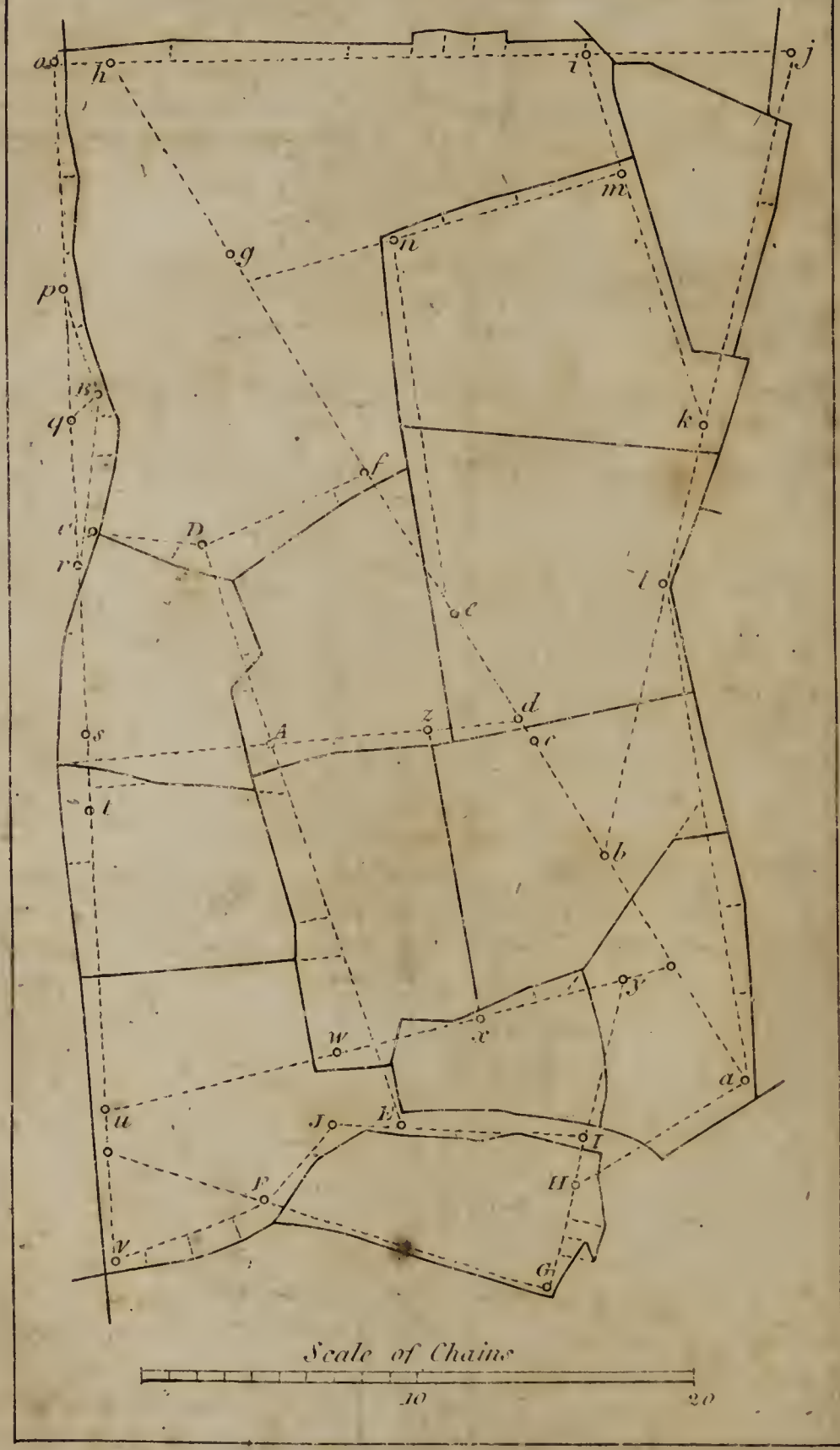
<p><i>n</i></p>	<p>1370 836 684</p>	<p>---156 to e 56 50</p>
<p><i>m</i></p>	<p>1180 960 930 700 400</p>	<p>90 to g 24 n 48 30</p>
<p><i>k</i></p>	<p>1430 1290 1004 980 610 280</p>	<p>to i 40 36 m 24 32</p>
<p><i>a</i></p>	<p>1794 1464 1050 920 650 350 0</p>	<p>to l 22 32 60 48 14</p>
<p><i>j</i></p>	<p>3074 2494 2100 2072 1730 1530 1420 1170 620 280</p>	<p>to b l k 40</p>
<p><i>h</i></p>	<p>2574 2494 2000 1880 1840 1794 1464 1328 1240 1130 860 190</p>	<p><i>j</i> 44 50 <i>i</i></p>
<p><i>a</i></p>	<p>4450 3570 2620 2590 2210 2080 1574 1550 1510 990 806</p>	<p><i>h</i> <i>g</i> <i>f</i> <i>e</i> <i>d</i> <i>c</i> <i>b</i></p>

		768	to A
		526	70
		496	
	70	460	
	40	124	
D		100	
		455	D
		400	76
C		48	10
		600	to r
		432	C
	56	160	
B	44	36	
		152	to q
B		480	B
P	24	160	
		1750	
		1600	44 to s
		1028	
		930	A
	44	666	
	70	310	z
d	60	236	
		2148	480 to b
		1950	y
		1836	
	128	1724	
	60	1600	
	30	1480	x
	0	1320	
	50	1110	
		1080	
		840	w
u		750	50
		4440	36
		4420	v
		3884	u
		3380	60
		2992	90
		2692	t
	120	2624	
		2592	
		2500	s
		2070	56
		1900	leave off
		1840	r
	60	1770	
		1320	q
		808	p
	leave off 40	650	
	80	360	
o	20	170	
		220	0
h produce by i		190	46



		580	to v
	40	500	
	76	300	
F	76	100	
		390	to F
J	20	150	
	15	954	J
		850	
	30	730	to E
		490	
		340	60
	0	280	
I	20	170	50
		744	to H
		672	0
	70	450	0
a	50	15	
		1160	to v
		1000	
		890	
		780	32
		590	40
		570	I
		530	40
		376	H
		256	150
		190	64
		144	130
G			leave off
		1676	G
		1676	30
		896	24
		632	
		620	50
180 from u towards v		588	F
		644	to f
D		488	32
		2280	
		2270	E
	20	2230	
	56	2050	
		2030	
		1970	130 to W
		1552	180
		1380	96
		930	110
		860	54

*Plan from the foregoing Field Book.*



## PROBLEM XII.

*To survey a County, or large Tract of Land.*

1. CHOOSE two, three, or four eminent places, for stations; such as the tops of high hills or mountains, towers, or church steeples, which may be seen from one another; from which most of the towns and other places of note may also be seen; and so as to be as far distant from one another as possible. Upon these places raise beacons, or long poles, with flags of different colours flying at them, so as to be visible from all the other stations.

2. At all the places, which you would set down in the map, plant long poles with flags at them, of several colours, to distinguish the places from one another; fixing them on the tops of church steeples, or the tops of houses, or in the centres of lesser towns.

These marks then being set up at a convenient number of places, and such as may be seen from both stations; go to one of these stations, and, with an instrument to take angles, standing at that station, take all the angles between the other station and each of these marks. Then go to the other station, and take all the angles between the first station and each of the former marks, setting them down with the others, each against his fellow with the same colour. You may, if you can, also take the angles at some third station, which may serve to prove the work, if the three lines intersect in that point where any mark stands. The marks must stand till the observations are finished at both stations; and then they must be taken down, and set up at fresh places. The same operations must be performed, at both stations, for these fresh places; and the like for others. The instrument for taking angles must be an exceeding good one, made on purpose with telescopic sights; and of a good length of radius. A circumferentor is reckoned a good instrument for this purpose.

3. And though it be not absolutely necessary to measure any distance, because a stationary line being laid down from any scale, all the other lines will be proportional to it; yet it is better to measure some of the lines, to ascertain the distances of places in miles, and to know how many geometrical miles there are in any length; as also from thence to make a scale to measure any distance in miles. In measuring any distance, it will not be exact enough to go along the high roads; by reason of their turnings and windings,  
hardly

hardly ever lying in a right line between the stations, which must cause infinite reductions, and create endless trouble to make it a right line; for which reason it can never be exact. But a better way is to measure in a right line with a chain, between station and station, over hills and dales or level fields, and all obstacles. Only in case of water, woods, towns, rocks, banks, &c, where one cannot pass, such parts of the line must be measured by the methods of inaccessible distances; and besides, allowing for ascents and descents, when they are met with. A good compass that shews the bearing of the two stations, will always direct you to go straight, when you do not see the two stations; and in the progress, if you can go straight, offsets may be taken to any remarkable places, likewise noting the intersection of the station line with all roads, rivers, &c.

4. From all the stations, and in the whole progress, be very particular in observing sea-coasts, river-mouths, towns, castles, houses, churches, mills, trees, rocks, sands, roads, bridges, fords, ferries, woods, hills, mountains, rills, brooks, parks, beacons, sluices, floodgates, locks, &c, and in general every thing that is remarkable.

5. After you have done with the first and main station lines, which command the whole county; you must then take inner stations, at some places already determined; which will divide the whole into several partitions: and from these stations you must determine the places of as many of the remaining towns as you can. And if any remain in that part, you must take more stations, at some places already determined; from which you may determine the rest. And thus go through all the parts of the county, taking station after station, till we have determined all we want. And in general the station distances must always pass through such remarkable points as have been determined before, by the former stations.

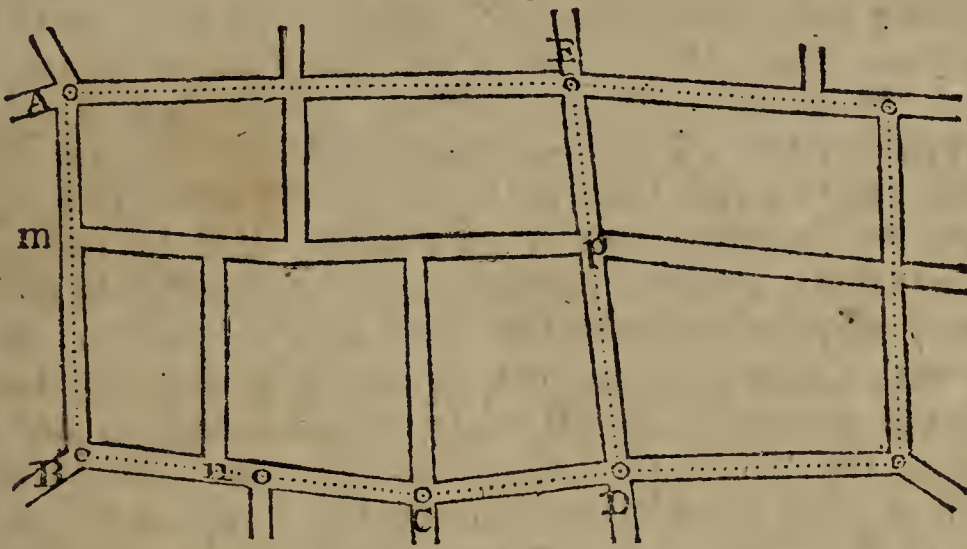
#### PROBLEM XIII.

##### *To survey a Town or City.*

THIS may be done with any of the instruments for taking angles, but best of all with the plain table, where every minute part is drawn while in sight. It is best also to have a chain of 50 feet long, divided into 50 links of one foot each, and an offset-staff of 10 feet long.

Begin

Begin at the meeting of two or more of the principal streets, through which you can have the longest prospects, to get the longest station lines: There having fixed the instrument, draw lines of direction along those streets, using two men as marks, or poles set in wooden pedestals, or perhaps some remarkable places in the houses at the further ends, as windows, doors, corners, &c. Measure these lines with the chain, taking offsets with the staff, at all corners of streets, bendings, or windings, and to all remarkable things, as churches, markets, halls, colleges, eminent houses, &c. Then remove the instrument to another station, along one of these lines; and there repeat the same process as before. And so on till the whole is finished.



Thus, fix the instrument at A, and draw lines in the direction of all the streets meeting there; then measure AB, noting the street on the left at m. At the second station B, draw the directions of the streets meeting there; and measure from B to C, noting the places of the streets at n and o as you pass by them. At the 3d station C, take the direction of all the streets meeting there, and measure CD. At D do the same, and measure DE, noting the place of the cross streets at p. And in this manner go through all the principal streets. This done, proceed to the smaller and intermediate streets; and lastly, to the lanes, alleys, courts, yards, and every part that it may be thought proper to represent in the plan.

## SECTION III.

## OF PLANNING, COMPUTING, AND DIVIDING.

## PROBLEM I.

*To lay down the Plan of any Survey.*

IF the survey was taken with a plain table, you have a rough plan of it already on the paper which covered the table. But if the survey was with any other instrument, a plan of it is to be drawn from the measures that were taken in the survey, and first of all a rough plan on paper.

To do this, you must have a set of proper instruments, for laying down both lines and angles, &c; as scales of various sizes, the more of them, and the more accurate, the better; scales of chords, protractors, perpendicular and parallel rulers, &c. Diagonal scales are best for the lines, because they extend to three figures, or chains and links, which are hundredth parts of chains. But in using the diagonal scale, a pair of compasses must be employed to take off the lengths of the principal lines very accurately. But a scale with a thin edge divided, is much readier for laying down the perpendicular offsets to crooked hedges, and for marking the places of those offsets upon the station line; which is done at only one application of the edge of the scale to that line, and then pricking off all at once the distances along it. Angles are to be laid down either with a good scale of chords, which is perhaps the most accurate way; or with a large protractor, which is much readier when many angles are to be laid down at one point, as they are pricked off all at once round the edge of the protractor.

In general, all lines and angles must be laid down on the plan in the same order in which they were measured in the field, and in which they are written in the field-book; laying down first the angles for the position of lines, next the lengths of the lines, with the places of the offsets, and then the lengths of the offsets themselves, all with dry or obscure lines; then a black line drawn through the extremities of all the offsets, will be the hedge or bounding line of the field, &c. After the principal bounds and lines are laid down, and made to fit or close properly, proceed next to the smaller objects, till you have entered every thing that ought to appear in the plan, as houses, brooks, trees, hills, gates, stiles, roads, lanes, mills, bridges, woodlands, &c. &c.

The

The north side of a map or plan is commonly placed uppermost, and a meridian somewhere drawn, with the compass or flower-de-luce pointing north. Also, in a vacant part, a scale of equal parts or chains is drawn, with the title of the map in conspicuous characters, and embellished with a compartment. Hills are shadowed, to distinguish them in the map. Colour the hedges with different colours; represent hilly grounds by broken hills and valleys; draw single dotted lines for foot-paths, and double ones for horse or carriage roads. Write the name of each field and remarkable place within it, and, if you choose, its content in acres, roods, and perches.

In a very large estate, or a county, draw vertical and horizontal lines through the map, denoting the spaces between them by letters placed at the top, and bottom, and sides, for readily finding any field or other object mentioned in a table.

In mapping counties, and estates that have uneven grounds of hills and valleys, reduce all oblique lines, measured uphill and down-hill, to horizontal straight lines, if that was not done during the survey, before they were entered in the field-book, by making a proper allowance to shorten them. For which purpose there is commonly a small table engraven on some of the instruments for surveying.

#### PROBLEM II.

##### *To compute the Contents of Fields.*

1. COMPUTE the contents of the figures, whether triangles, or trapeziums, &c, by the proper rules for the several figures laid down in measuring; multiplying the lengths by the breadths, both in links, and divide by 2; the quotient is acres, after you have cut off five figures on the right for decimals. Then bring these decimals to roods and perches, by multiplying first by 4, and then by 40. An example of which is given in the description of the chain, pag. 50.

2. In small and separate pieces, it is usual to compute their contents from the measures of the lines taken in surveying them, without making a correct plan of them.

3. In pieces bounded by very crooked and winding hedges, measured by offsets, all the parts between the offsets are most accurately measured separately as small trapezoids.

4. Sometimes such pieces as that last mentioned, are computed by finding a mean breadth, by dividing the sum of the offsets

offsets by the number of them, accounting that for one of them where the boundary meets the station line; then multiply the length by that mean breadth.

But this method is commonly in some degree erroneous.

5. But in larger pieces, and whole estates, consisting of many fields, it is the common practice to make a rough plan of the whole, and from it compute the contents quite independent of the measures of the lines and angles that were taken in surveying. For then new lines are drawn in the fields in the plan, so as to divide them into trapeziums and triangles, the bases and perpendiculars of which are measured on the plan by means of the scale from which it was drawn, and so multiplied together for the contents. In this way, the work is very expeditiously done, and sufficiently correct; for such dimensions are taken as afford the most easy method of calculation; and, among a number of parts, thus taken and applied to a scale, it is likely that some of the parts will be taken a small matter too little, and others too great; so that they will, upon the whole, in all probability, very nearly balance one another. After all the fields, and particular parts, are thus computed separately, and added all together into one sum; calculate the whole estate independent of the fields, by dividing it into large and arbitrary triangles and trapeziums, and add these also together. Then if this sum be equal to the former, or nearly so, the work is right; but if the sums have any considerable difference, it is wrong, and they must be examined, and recomputed, till they nearly agree.

6. But the chief secret in computing, consists in finding the contents of pieces bounded by curved or very irregular lines, or in reducing such crooked sides of fields or boundaries to straight lines, that shall inclose the same or equal area with those crooked sides, and so obtain the area of the curved figure by means of the right-lined one, which will commonly be a trapezium. Now this reducing the crooked sides to straight ones, is very easily and accurately performed in this manner:—Apply the straight edge of a thin, clear piece of lath or horn to the crooked line which is to be reduced, in such a manner, that the small parts cut off from the crooked figure by it, may be equal to those which are taken in: which equality of the parts included and excluded you will presently be able to judge of very nicely by a little practice: then with a pencil, or point of a tracer, draw a line by the straight edge of the horn. Do the same by the other sides of the field or figure. So shall you have a straight sided

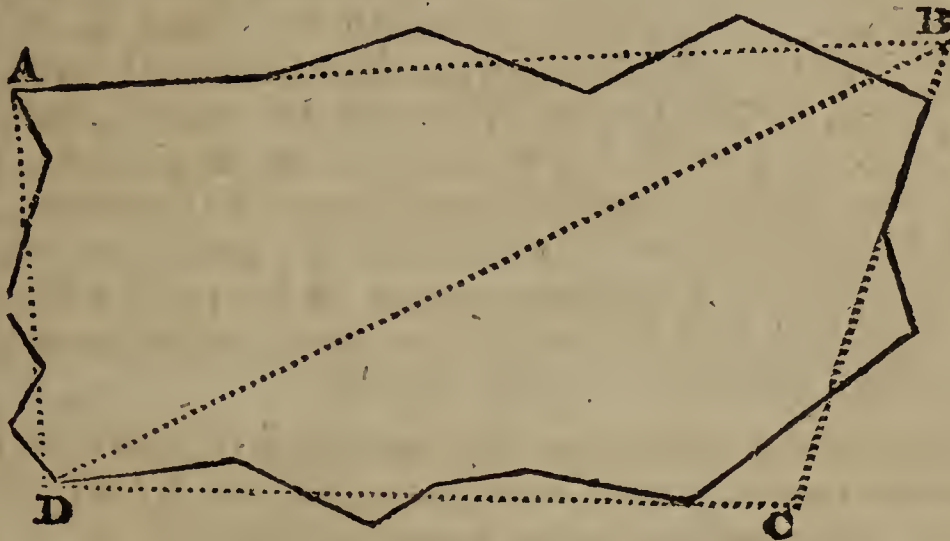


sided figure equal to the curved one; the content of which, being computed as before directed, will be the content of the curved figure proposed.

Or, instead of the straight edge of the horn, a horse-hair may be applied across the crooked sides in the same manner; and the easiest way of using the hair, is to string a small slender bow with it, either of wire, or cane, or whale-bone, or suchlike slender elastic matter; for, the bow keeping it always stretched, it can be easily and neatly applied with one hand, while the other is at liberty to make two marks by the side of it, to draw the straight line by.

## E X A M P L E.

Thus, let it be required to find the contents of the same figure as in prob. IX of the last section, pag. 65, to a scale of 4 chains to an inch.



Draw the four dotted straight lines AB, BC, CD, DA, cutting off equal quantities on both sides of them, which they do as near as the eye can judge: so is the crooked figure reduced to an equivalent right-lined one of four sides ABCD. Then draw the diagonal BD, which, by applying a proper scale to it, measures 1256. Also the perpendicular, or nearest distance, from A to this diagonal, measures 456; and the distance of C from it, is 428.

Then, half the sum of 456 and 428, multiplied by the diagonal 1256, gives 555152 square links, or 5 acres, 2 roods, 8 perches, the content of the trapezium, or of the irregular crooked piece.

## PROBLEM III.

*To transfer a Plan to another Paper, &c.*

AFTER the rough plan is completed, and a fair one is wanted; this may be done by any of the following methods.

*First Method.*—Lay the rough plan on the clean paper, keeping them always pressed flat and close together, by weights laid on them. Then, with the point of a fine pin or pricker, prick through all the corners of the plan to be copied. Take them afunder, and connect the pricked points, on the clean paper, with lines; and it is done. This method is only to be practised in plans of such figures as are small and tolerably regular, or bounded by right lines.

*Second Method.*—Rub the back of the rough plan over with black lead powder; and lay the said black part on the clean paper on which the plan is to be copied, and in the proper position. Then with the blunt point of some hard substance, as brass, or such like, trace over the lines of the whole plan; pressing the tracer so much as that the black lead under the lines may be transferred to the clean paper: after which, take off the rough plan, and trace over the leaden marks with common ink, or with Indian ink—Or, instead of blacking the rough plan, you may keep constantly a blacked paper to lay between the plans.

*Third Method.*—Another method of copying plans, is by means of squares. This is performed by dividing both ends and sides of the plan which is to be copied, into any convenient number of equal parts, and connecting the corresponding points of division with lines; which will divide the plan into a number of small squares. Then divide the paper, upon which the plan is to be copied, into the same number of squares, each equal to the former when the plan is to be copied of the same size, but greater or less than the others, in the proportion in which the plan is to be increased or diminished, when of a different size. Lastly, copy into the clean squares the parts contained in the corresponding squares of the old plan; and you will have the copy, either of the same size, or greater or less in any proportion.

*Fourth Method.*—A fourth method is by the instrument called a pentagraph, which also copies the plan in any size required.

*Fifth Method.*—But the neatest method of any, at least in copying a fair plan, is this. Procure a copying frame or glass, made in this manner; namely, a large square of the best window glass, set in a broad frame of wood, which can be raised up to any angle, when the lower side of it rests on a table. Set this frame up to any angle before you, facing a strong light; fix the old plan and clean paper together with several pins quite around, to keep them together, the clean paper being laid uppermost, and over the face of the plan to be copied. Lay them, with the back of the old plan, on the glass, namely, that part which you intend to begin at to copy first; and, by means of the light shining through the papers, you will very distinctly perceive every line of the plan through the clean paper. In this state then trace all the lines on the paper with a pencil. Having drawn that part which covers the glass, slide another part over the glass and copy it in the same manner. Then another part. And so on, till the whole is copied.

Then, take them asunder, and trace all the pencil-lines over with a fine pen and Indian ink, or with common ink.

And thus you may copy the finest plan, without injuring it in the least.

When the lines are copied on the clean paper, the next business is to write such names, remarks, or explanations as may be judged necessary; laying down the scale for taking the lengths of any parts, a flower-de-luce to point out the direction, and the proper title ornamented with a compartment; illustrating or colouring every part in the manner that shall seem most natural, such as shading rivers or brooks with crooked lines; drawing the representations of trees, bushes, hills, woods, hedges, houses, gates, roads, &c, in their proper places; running a single dotted line for a foot-path, and a double one for a carriage road; and either representing the bases or the elevations of buildings, &c.

## OF ARTIFICERS' WORKS,

AND

## TIMBER MEASURING.

## I. OF THE CARPENTER'S OR SLIDING RULE.

THE Carpenter's or Sliding Rule, is an instrument much used in measuring of timber and artificers' works, both for taking the dimensions, and computing the contents.

The instrument consists of two equal pieces, each a foot in length, which are connected together by a folding joint.

One side or face of the rule, is divided into inches, and eighths, or half quarters. On the same face also are several plane scales, divided into twelfth parts by diagonal lines; which are used in planning dimensions that are taken in feet and inches. The edge of the rule is commonly divided decimally, or into tenths; namely, each foot into 10 equal parts, and each of these into 10 parts again: so that by means of this last scale, dimensions are taken in feet, tenths and hundredths, and multiplied as common decimal numbers, which is the best way.

On the one part of the other face are four lines, marked A, B, C, D; the two middle ones B and C being on a slider, which runs in a groove made in the stock. The same numbers serve for both these two middle lines, the one being above the numbers, and the other below.

These four lines are logarithmic ones, and the three A, B, C, which are all equal to one another, are double lines, as they proceed twice over from 1 to 10. The other or lowest line D, is a single one, proceeding from 4 to 40. It is also called the girt line, from its use in computing the contents of trees and timber; and upon it are marked WG at 17.15, and AG at 18.95, the wine and ale gage points, to make this instrument serve the purpose of a gaging rule.

On the other part of this face, there is a table of the value of a load, or 50 cubic feet of timber, at all prices, from 6 pence to 2 shillings a foot.

When

When 1 at the beginning of any line is accounted 1, then the 1 in the middle will be 10, and the 10 at the end 100; but when 1 at the beginning is accounted 10, then the 1 in the middle is 100, and the 10 at the end 1000; and so on. And all the smaller divisions are altered proportionally.

## II. ARTIFICERS' WORK.

ARTIFICERS compute the contents of their works by several different measures. As,

Glazing and masonry by the foot;

Painting, plastering, paving, &c, by the yard, of 9 square feet;

Flooring, partitioning, roofing, tiling, &c, by the square, of 100 square feet.

And brickwork, either by the yard of 9 square feet, or by the perch, or square rod or pole, containing  $272\frac{1}{4}$  square feet, or  $30\frac{1}{4}$  square yards, being the square of the rod or pole of  $16\frac{1}{2}$  feet or  $5\frac{1}{2}$  yards long.

As this number  $272\frac{1}{4}$  is troublesome to divide by, the  $\frac{1}{4}$  is often omitted in practice, and the content in feet divided only by the 272. But when the exact divisor  $272\frac{1}{4}$  is to be used, it will be easier to multiply the feet by 4, and then divide successively by 9, 11, and 11. Also to divide square yards by  $30\frac{1}{4}$ , first multiply them by 4, and then divide twice by 11.

All works, whether superficial or solid, are computed by the rules proper to the figure of them, whether it be a triangle, or rectangle, a parallelopiped, or any other figure.

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## III. BRICKLAYERS' WORK.

BRICKWORK is estimated at the rate of a brick and a half thick. So that, if a wall be more or less than this standard thickness, it must be reduced to it, as follows:

Multiply the superficial content of the wall by the number of half bricks in the thickness, and divide the product by 3.

The dimensions of a building are usually taken by measuring half round on the outside and half round it on the inside; the sum of these two gives the compass of the wall, to be multiplied by the height, for the content of the materials.

Chimnies are by some measured as if they were solid, deducting only the vacuity from the hearth to the mantle, on account of the trouble of them.

And by others they are girt or measured round for their breadth, and the height of the story is their height, taking the depth of the jambs for their thickness. And in this case, no deduction is made for the vacuity from the floor to the mantle-tree, because of the gathering of the breast and wings, to make room for the hearth in the next story.

To measure the chimney shafts, which appear above the building; girt them about with a line for the breadth, to multiply by their height. And account their thickness half a brick more than it really is, in consideration of the plastering and scaffolding.

All windows, doors, &c, are to be deducted out of the contents of the walls in which they are placed. But this deduction is made only with regard to materials; for the whole measure is taken for workmanship, and that all outside measure too, namely, measuring quite round the outside of the building, being in consideration of the trouble of the returns or angles. There are also some other allowances, such as double measure for feathered gable ends, &c.

#### EXAMPLES.

EXAM. I. How many yards and rods of standard brickwork are in a wall whose length or compass is 57 feet 3 inches, and height 24 feet 6 inches; the walls being  $2\frac{1}{2}$  bricks or 5 half bricks thick?      Ans. 8 rods,  $17\frac{2}{3}$  yards.

EXAM. II. Required the content of a wall 62 feet 6 inches long, and 14 feet 8 inches high, and  $2\frac{1}{2}$  bricks thick?      Ans. 169.753 yards.

EXAM. III. A triangular gable is raised  $17\frac{1}{2}$  feet high, on an end wall whose length is 24 feet 9 inches, the thickness being 2 bricks; required the reduced content?      Ans.  $32.08\frac{1}{3}$  yards.

EXAM. IV. The end wall of a house is 28 feet 10 inches long, and 55 feet 8 inches high, to the eaves; 20 feet high is  $2\frac{1}{2}$  bricks thick, other 20 feet high is 2 bricks thick, and the remaining 15 feet 8 inches is  $1\frac{1}{2}$  brick thick; above which is a triangular gable, of 1 brick thick, which rises 42 courses of bricks, of which every 4 courses make a foot. What is the whole content in standard measure?      Ans. 253.626 yards.

#### IV. MASONS' WORK.

To masonry belong all sorts of stone-work; and the measure made use of is a foot, either superficial or solid.

Walls, columns, blocks of stone or marble, &c, are measured by the cubic foot; and pavements, slabs, chimney-pieces, &c, by the superficial or square foot.

Cubic or solid measure is used for the materials, and square measure for the workmanship.

In the solid measure, the true length, breadth, and thickness, are taken, and multiplied continually together. In the superficial, there must be taken the length and breadth of every part of the projection, which is seen without the general upright face of the building.

##### EXAMPLES.

EXAM. I. Required the solid content of a wall, 53 feet 6 inches long, 12 feet 3 inches high, and 2 feet thick?  
 Anf.  $1310\frac{3}{4}$  feet.

EXAM. II. What is the solid content of a wall, the length being 24 feet 3 inches, height 10 feet 9 inches, and 2 feet thick?  
 Anf. 521.375 feet.

EXAM. III. Required the value of a marble slab, at 8s. per foot; the length being 5 feet 7 inches, and breadth 1 foot 10 inches?  
 Anf. 4l. 1s.  $10\frac{1}{2}$ d.

EXAM. IV. In a chimney-piece, suppose the  
 length of the mantle and slab, each 4 feet 6 inches  
 breadth of both together - 3 2  
 length of each jamb - 4 4  
 breadth of both together - 1 9  
 Required the superficial content? Anf. 21 feet 10 inches.

#### V. CARPENTERS' AND JOINERS' WORK.

To this branch belongs all the wood-work of a house, such as flooring, partitioning, roofing, &c.

Large and plain articles are usually measured by the square foot or yard, &c; but enriched mouldings, and some other articles, are often estimated by running or lineal measure, and some things are rated by the piece.

In measuring of joists, it is to be observed, that only one of their dimensions is the same with that of the floor; for the other exceeds the length of the room by the thickness of the wall, and  $\frac{1}{3}$  of the same, because each end is let into the wall about  $\frac{2}{3}$  of its thickness.

No deductions are made for hearths, on account of the additional trouble and waste of materials.

*Partitions* are measured from wall to wall for one dimension, and from floor to floor, as far as they extend, for the other.

No deduction is made for door-ways, on account of the trouble of framing them.

In measuring of joiners' work, the string is made to ply close to every part of the work over which it passes.

*The measure of centering for cellars* is found, by making a string pass over the surface of the arch for the breadth, and taking the length of the cellar for the length; but in groin centering, it is usual to allow double measure, on account of their extraordinary trouble.

*In roofing*, the length of the house in the inside, together with  $\frac{2}{3}$  of the thickness of one gable, is to be considered as the length; and the breadth is equal to double the length of a string which is stretched from the ridge down the rafter, and along the eaves-board, till it meets with the top of the wall.

*For stair-cases*, take the breadth of all the steps, by making a line ply close over them, from the top to the bottom, and multiply the length of this line by the length of a step, for the whole area.—By the length of a step is meant the length of the front and the returns at the two ends; and by the breadth, is to be understood the girt of its two outer surfaces, or the tread and riser.

*For the balustrade*, take the whole length of the upper part of the hand-rail, and girt over its end till it meet the top of the newel-post, for the length; and twice the length of the baluster upon the landing, with the girt of the hand-rail, for the breadth.

*For wainscoting*, take the compass of the room for the length; and the height from the floor to the ceiling, making the string ply close into all the mouldings, for the breadth.—Out of this must be made deductions for windows, doors, and chimneys, &c; but workmanship is counted for the whole, on account of the extraordinary trouble.

*For doors*, it is usual to allow for their thickness, by adding it into both the dimensions of length and breadth, and then multiply



multiply them together for the area.—If the door be paneled on both sides, take double its measure for the workmanship; but if one side only be paneled, take the area and its half for the workmanship.—*For the surrounding architrave,* gird it about the outermost part for its length; and measure over it, as far as it can be seen when the door is open, for the breadth.

*Window shutters, bases, &c,* are measured in the same manner.

In the measuring of roofing for workmanship alone, holes for chimney shafts and sky-lights are generally deducted.

But in measuring for work and materials, they commonly measure in all sky-lights, luthern-lights, and holes for the chimney shafts, on account of their trouble and waste of materials.

#### EXAMPLES.

EXAM. I. Required the content of a floor 48 feet 6 inches long, and 24 feet 3 inches broad?      Anf. 11 sq.  $76\frac{1}{8}$  feet.

EXAM. II. A floor being 36 feet 3 inches long, and 16 feet 6 inches broad, how many squares are in it?

Anf. 5 sq.  $98\frac{1}{8}$  feet.

EXAM. III. How many squares are there in 173 feet 10 inches in length, and 10 feet 7 inches height, of partitioning?

Anf. 18.3973 squares.

EXAM. IV. What cost the roofing of a house at 10s. 6d. a-square; the length, within the walls, being 52 feet 8 inches, and the breadth 30 feet 6 inches; reckoning the roof  $\frac{3}{2}$  of the flat?

Anf. 12l. 12s.  $11\frac{3}{4}$ d.

EXAM. V. To how much at 6s. per square yard, amounts the wainscoting of a room; the height, taking in the cornice and mouldings, being 12 feet 6 inches, and the whole compass 83 feet 8 inches; also the three window shutters are each 7 feet 8 inches by 3 feet 6 inches, and the door 7 feet by 3 feet 6 inches; the door and shutters, being worked on both sides, are reckoned work and half work?

Anf. 36l. 12s.  $2\frac{1}{2}$ d.

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#### VI. SLATERS' AND TILERS' WORK.

IN these articles, the content of a roof is found by multiplying the length of the ridge by the girt over from eaves to eaves; making allowance in this girt for the double row of slates at the bottom, or for how much one row of slates or tiles is laid over another.

When

When the roof is of a true pitch, that is, forming a right angle at top; then the breadth of the building with its half added, is the girt over both sides.

In angles formed in a roof, running from the ridge to the eaves, when the angle bends inwards, it is called a valley; but when outwards, it is called a hip.

Deductions are made for chimney shafts or window holes.

#### EXAMPLES.

EXAM. I. Required the content of a slated roof, the length being 45 feet 9 inches, and the whole girt 34 feet 3 inches?

Ans.  $174\frac{1}{10}$  yards.

EXAM. II. To how much amounts the tiling of a house, at 25s. 6d. per square; the length being 43 feet 10 inches, and the breadth on the flat 27 feet 5 inches, also the eaves projecting 16 inches on each side, and the roof of a true pitch?

Ans. 24l. 9s.  $5\frac{3}{4}$ d.

### VII. PLASTERERS' WORK.

PLASTERERS' work is of two kinds, namely, ceiling, which is plastering upon laths; and rendering, which is plastering upon walls: which are measured separately.

The contents are estimated either by the foot or yard, or square of 100 feet. Inriched mouldings, &c, are rated by running or lineal measure.

Deductions are to be made for chimneys, doors, windows, &c. But the windows are seldom deducted, as the plastered returns at the top and sides are allowed to compensate for the window opening.

#### EXAMPLES.

EXAM. I. How many yards contains the ceiling, which is 43 feet 3 inches long, and 25 feet 6 inches broad?

Ans.  $122\frac{1}{2}$ .

EXAM. II. To how much amounts the ceiling of a room, at 10d. per yard; the length being 21 feet 8 inches, and the breadth 14 feet 10 inches?

Ans. 1l. 9s.  $8\frac{3}{4}$ d.

EXAM. III. The length of a room is 18 feet 6 inches, the breadth 12 feet 3 inches, and height 10 feet 6 inches; to how much amounts the ceiling and rendering, the former at 8d. and the latter at 3d. per yard; allowing for the door of 7 feet by 3 feet 8, and a fire-place of 5 feet square?

Ans. 1l. 13s.  $3\frac{1}{4}$ d.

EXAM.

EXAM. IV. Required the quantity of plastering in a room, the length being 14 feet 5 inches, breadth 13 feet 2 inches, and height 9 feet 3 inches to the under side of the cornice, which girts  $8\frac{1}{2}$  inches, and projects 5 inches from the wall on the upper part next the ceiling; deducting only for a door 7 feet by 4?

Ans. 53 yards 5 feet 3 inches of rendering  
 18            5            6            of ceiling  
                  39             $0\frac{1}{2}$             of cornice.

VIII. PAINTERS' WORK.

PAINTERS' work is computed in square yards. Every part is measured where the colour lies; and the measuring line is forced into all the mouldings, and corners.

Windows are done at so much a piece. And it is usual to allow double measure for carved mouldings, &c.

EXAMPLES.

EXAM. I. How many yards of painting contains the room which is 65 feet 6 inches in compass, and 12 feet 4 inches high? Ans.  $89\frac{3}{4}$  yards.

EXAM. II. The length of a room being 20 feet, its breadth 14 feet 6 inches, and height 10 feet 4 inches; how many yards of painting are in it, deducting a fire-place of 4 feet by 4 feet 4 inches, and two windows each 6 feet by 3 feet 2 inches? Ans.  $73\frac{2}{7}$  yards.

EXAM. III. What cost the painting of a room, at 6d. per yard; its length being 24 feet 6 inches, its breadth 16 feet 3 inches, and height 12 feet 9 inches; also the door is 7 feet by 3 feet 6, and the window shutters to two windows each 7 feet 9 by 3 feet 6, but the breaks of the windows themselves are 8 feet 6 inches high, and 1 foot 3 inches deep: deducting the fire-place of 5 feet by 5 feet 6? Ans. 3l. 3s.  $10\frac{3}{4}$ d.

IX. GLAZIERS' WORK.

GLAZIERS take their dimensions, either in feet, inches and parts, or feet, tenths, and hundredths. And they compute their work in square feet.

In

In taking the length and breadth of a window, the cross bars between the squares are included. Also windows of round or oval forms are measured as square, measuring them to their greatest length and breadth, on account of the waste in cutting the glafs.

## E X A M P L E S.

EXAM. I. How many square feet contains the window which is 4.25 feet long, and 2.75 feet broad?      Ans.  $11\frac{2}{3}$ .

EXAM. II. What will the glazing a triangular sky-light come to, at 10d. per foot; the base being 12 feet 6 inches, and the perpendicular height 6 feet 9 inches?

Ans. 1l. 15s.  $1d\frac{3}{4}$ .

EXAM. III. There is a house with three tier of windows, three windows in each tier, their common breadth 3 feet 11 inches;

now the height of the first tier is    7 feet 10 inches  
   of the second        6        8  
   of the third         5        4

Required the expence of glazing at 14d. per foot?

Ans. 13l. 11s.  $10\frac{1}{2}d$ .

EXAM. IV. Required the expence of glazing the windows of a house at 13d. a foot; there being three stories, and three windows in each story;

the height of the lower tier is 7 feet 9 inches  
   of the middle        6        6  
   of the upper         5         $3\frac{1}{4}$

and of an oval window over the door    1         $10\frac{1}{2}$   
 the common breadth of all the windows being 3 feet 9 inches?

Ans. 12l. 5s. 6d.

## X. PAVERS' WORK.

PAVERS' work is done by the square yard. And the content is found by multiplying the length by the breadth.

## E X A M P L E S.

EXAM. I. What cost the paving a foot-path at 3s. 4d. a-yard; the length being 35 feet 4 inches, and breadth 8 feet 3 inches?      Ans. 5l. 7s.  $11\frac{1}{2}d$ .

EXAM. II. What cost the paving a court, at 3s. 2d. per yard; the length being 27 feet 10 inches, and the breadth 14 feet 9 inches?      Ans. 7l. 4s.  $5\frac{1}{4}d$ .

EXAM. III.

EXAM. III. What will be the expence of paving a rectangular court-yard, whose length is 63 feet, and breadth 45 feet; in which there is laid a foot-path of 5 feet 3 inches broad, running the whole length, with broad stones, at 3s. a-yard; the rest being paved with pebbles at 2s. 6d. a-yard?

Anf. 40l. 5s. 10½d.

## XI. PLUMBERS' WORK.

PLUMBERS' work is rated at so much a pound, or else by the hundred weight, of 112 pounds.

Sheet lead used in roofing, guttering, &c, is from 7 to 10lb. to the square foot. And a pipe of an inch bore, is commonly 13 or 14lb. to the yard in length.

### EXAMPLES.

EXAM. I. How much weighs the lead which is 39 feet 6 inches long, and 3 feet 3 inches broad, at 8½lb. to the square foot? Anf. 1091<sup>3</sup>/<sub>16</sub>lb.

EXAM. II. What cost the covering and guttering a roof with lead, at 18s. the cwt.; the length of the roof being 43 feet, and breadth or girt over it 32 feet; the guttering 57 feet long, and 2 feet wide; the former 9.831lb, and the latter 7.373lb. to the square foot? Anf. 115l. 9s. 1½d.

## XII. TIMBER MEASURING.

### PROBLEM I.

*To find the Area, or Superficial Content, of a Board or Plank.*

MULTIPLY the length by the mean breadth.

*Note.* When the board is tapering, add the breadths at the two ends together, and take half the sum for the mean breadth.

*By the Sliding Rule.*

Set 12 on B to the breadth in inches on A; then against the length in feet on B, is the content on A, in feet and fractional parts.

EXAM-

## EXAMPLES.

EXAM. I. What is the value of a plank, at  $1\frac{1}{2}$ d. per foot, whose length is 12 feet 6 inches, and mean breadth 11 inches?

Anf. 1s. 5d.

EXAM. II. Required the content of a board, whose length is 11 feet 2 inches, and breadth 1 foot 10 inches?

Anf. 20 feet 5 inches 8".

EXAM. III. What is the value of a plank, which is 12 feet 9 inches long, and 1 foot 3 inches broad, at  $2\frac{1}{2}$ d. a foot?

Anf. 3s.  $3\frac{3}{4}$ d.

EXAM. IV. Required the value of 5 oaken planks at 3d. per foot, each of them being  $17\frac{1}{2}$  feet long; and their several breadths are as follows, namely, two of  $13\frac{1}{2}$  inches in the middle, one of  $14\frac{1}{2}$  inches in the middle, and the two remaining ones, each 18 inches at the broader end, and  $11\frac{1}{4}$  at the narrower.

Anf. 1l. 5s.  $9\frac{1}{2}$ d.

## PROBLEM II.

*To find the Solid Content of Squared or Four-sided Timber.*

MULTIPLY the mean breadth by the mean thickness, and the product again by the length, and the last product will give the content.

*By the Sliding Rule.*

C                      D                      D                      C

As length : 12 or 10 :: quarter girt : solidity.

That is, as the length in feet on C, is to 12 on D when the quarter girt is in inches, or to 10 on D when it is in tenths of feet; so is the quarter girt on D, to the content on C.

*Note, 1.* If the tree taper regularly from the one end to the other; either take the mean breadth and thickness in the middle, or take the dimensions at the two ends, and half their sum will be the mean dimensions.

2. If the piece do not taper regularly, but is unequally thick in some parts and small in others, take several different dimensions, add them all together, and divide their sum by the number of them, for the mean dimensions.

3. The quarter girt is a geometrical mean proportional between the mean breadth and thickness, that is the square root of their product. Sometimes unskilful measurers use the arithmetical mean instead of it, that is half their sum;

but

but this is always attended with error, and the more so as the breadth and depth differ the more from each other.

## EXAMPLES.

EXAM. I. The length of a piece of timber is 18 feet 6 inches, the breadths at the greater and less end 1 foot 6 inches and 1 foot 3 inches, and the thickness at the greater and less end 1 foot 3 inches and 1 foot: required the solid content? Ans. 28 feet 7 inches.

EXAM. II. What is the content of the piece of timber, whose length is  $24\frac{1}{2}$  feet, and the mean breadth and thickness each 1.04 feet? Ans.  $26\frac{1}{2}$  feet.

EXAM. III. Required the content of a piece of timber, whose length is 20.38 feet, and its ends unequal squares, the side of the greater being  $19\frac{1}{8}$ , and the side of the less  $9\frac{7}{8}$ ? Ans. 29.75625 feet.

EXAM. IV. Required the content of the piece of timber, whose length is 27.36 feet; at the greater end the breadth is 1.78, and thickness 1.23; and at the less end the breadth is 1.04, and thickness 0.91? Ans. 41.278 feet.

## PROBLEM III.

*To find the Solidity of Round or Unsquared Timber.*

MULTIPLY the square of the quarter girt, or of  $\frac{1}{4}$  of the mean circumference, by the length, for the content.

*By the Sliding Rule.*

As the length upon C : 12 or 10 upon D ::  
quarter girt, in 12ths or 10ths, on D : content on C.

*Note, 1.* When the tree is tapering, take the mean dimensions as in the former problems, either by girding it in the middle, for the mean girt, or at the two ends, and take half the sum of the two. But when the tree is very irregular, divide it into several lengths, and find the content of each part separately.

2. This rule, which is commonly used, gives the answer about  $\frac{1}{4}$  less than the true quantity in the tree, or nearly what the quantity would be after the tree is hewed square in the usual way: so that it seems intended to make an allowance for the squaring of the tree.

## EXAMPLES.

EXAM. I. A piece of round timber being 9 feet 6 inches long, and its mean quarter girt 42 inches; what is the content? Ans.  $116\frac{1}{3}$  feet.

EXAM. II. The length of a tree is 24 feet, its girt at the thicker end 14 feet, and at the smaller end 2 feet; required the content? Ans. 96 feet.

EXAM. III. What is the content of a tree, whose mean girt is 3.15 feet, and length 14 feet 6 inches? Ans. 8.9922 feet.

EXAM. IV. Required the content of a tree, whose length is  $17\frac{1}{4}$  feet, which girts in five different places as follows, namely, in the first place 9.43 feet, in the second 7.92, in the third 6.15, in the fourth 4.74, and in the fifth 3.16? Ans. 42.519525.

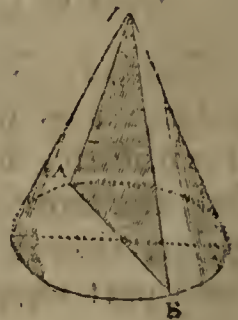
## CONIC SECTIONS.

## DEFINITIONS.

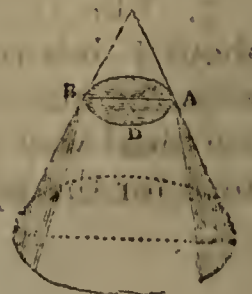
1. CONIC SECTIONS are the figures made by the mutual interfection of a cone and a plane.

2. According to the different positions of the cutting plane, there arise five different figures or sections, namely, a triangle, a circle, an ellipse, an hyperbola, and a parabola: the three last of which only are peculiarly called Conic Sections.

3. If the cutting plane pass through the vertex of the cone, and any part of the base, the section will evidently be a triangle; as VAB.



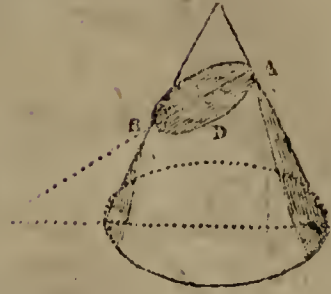
4. If the plane cut the cone parallel to the base, or make no angle with it, the section will be a circle; as ABD.



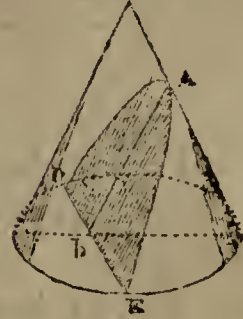
The



5. The section  $DAB$  is an ellipse, when the cone is cut obliquely through both sides, or when the plane is inclined to the base in a less angle than the side of the cone is.

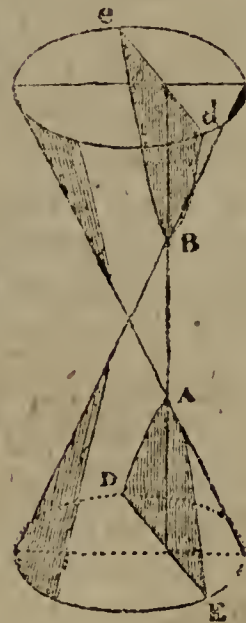


6. The section is a parabola, when the cone is cut by a plane parallel to the side, or when the cutting plane and the side of the cone make equal angles with the base.



7. The section is an hyperbola, when the cutting plane makes a greater angle with the base than the side of the cone makes.

8. And if all the sides of the cone be continued through the vertex, forming an opposite equal cone, and the plane be also continued to cut the opposite cone, this latter section will be the opposite hyperbola to the former; as  $dBe$ .



9. The vertices of any section, are the points where the cutting plane meets the opposite sides of the cone, or the sides of the vertical triangular section; as  $A$  and  $B$ .

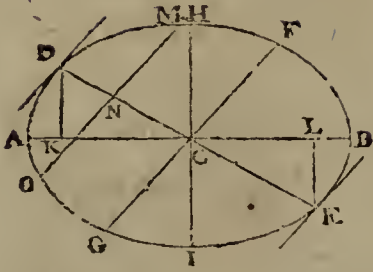
Hence the ellipse and the opposite hyperbolas, have each two vertices; but the parabola only one; unless we consider the other as at an infinite distance.

10. The Axis, or Transverse Diameter, of a conic section, is the line or distance  $AB$  between the vertices.

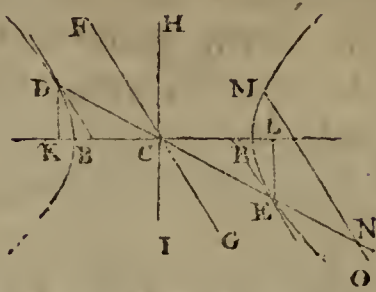
Hence the axis of a parabola is infinite in length,  $Ab$  being only a part of it.

Ellipse.

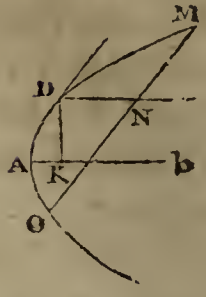
Ellipse.



Oppof. Hyperb.



Parabola,



11. The Centre  $c$  is the middle of the axis.

Hence the centre of a parabola is infinitely distant from the vertex. And of an ellipse, the axis and centre lie within the curve; but of an hyperbola, without.

12. A Diameter is any right line, as  $AB$  or  $DE$ , drawn through the centre, and terminated on each side by the curve; and the extremities of the diameter, or its interfections with the curve, are its vertices.

Hence all the diameters of a parabola are parallel to the axis, and infinite in length. And hence also every diameter of the ellipse and hyperbola have two vertices; but of the parabola, only one; unless we consider the other as at an infinite distance.

13. The Conjugate to any diameter, is the line drawn through the centre, and parallel to the tangent of the curve at the vertex of the diameter. So,  $FG$ , parallel to the tangent at  $E$ , is the conjugate to  $DE$ ; and  $HI$ , parallel to the tangent at  $A$ , is the conjugate to  $AB$ .

Hence the conjugate  $HI$ , of the axis  $AB$ , is perpendicular to it.

14. An Ordinate to any diameter, is a line parallel to its conjugate, or to the tangent at its vertex, and terminated by the diameter and curve. So  $DK$ ,  $EL$  are ordinates to the axis  $AB$ ; and  $MN$ ,  $NO$  ordinates to the diameter  $DE$ .

Hence the ordinates of the axis are perpendicular to it.

15. An Absciss is a part of any diameter contained between its vertex and an ordinate to it; as  $AK$  or  $BK$ , or  $DN$  or  $EN$ .

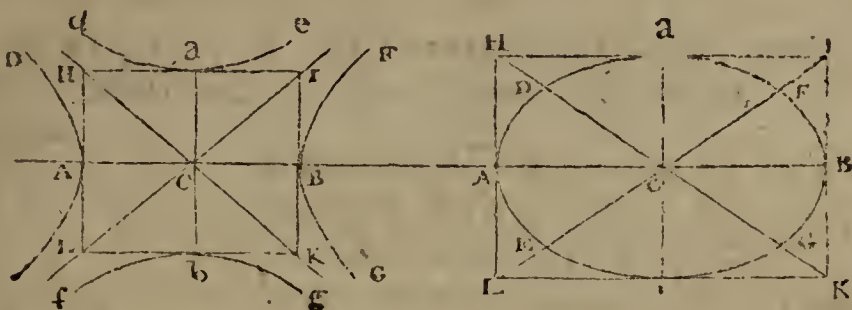
Hence, in the ellipse and hyperbola, every ordinate has two abscisses; but in the parabola, only one; the other vertex of the diameter being infinitely distant.

16. The Parameter of any diameter, is a third proportional to that diameter and its conjugate.

17. The

17. The Focus is the point in the axis where the ordinate is equal to half the parameter. As  $K$  and  $L$ , where  $DK$  or  $EL$  is equal to the semi-parameter.

Hence, the ellipse and hyperbola have each two foci; but the parabola only one.



18. If  $DAE$ ,  $FBG$  be two opposite hyperbolas, having  $AB$  for their first or transverse axis, and  $ab$  for their second or conjugate axis. And if  $dae$ ,  $fbg$  be two other opposite hyperbolas having the same axes, but in the contrary order, namely,  $ab$  their first axis, and  $AB$  their second; then these two latter curves  $dae$ ,  $fbg$ , are called the conjugate hyperbolas to the two former  $DAE$ ,  $FBG$ ; and each pair of opposite curves mutually conjugate to the other.

19. And if tangents be drawn to the four vertices of the curves, or extremities of the axes, forming the inscribed rectangle  $HIKL$ ; the diagonals  $HCK$ ,  $ICL$  of this rectangle, are called the asymptotes of the curves. And if these asymptotes intersect at right angles, or the inscribed rectangle be a square, or the two axes  $AB$  and  $ab$  be equal, then the hyperbolas are said to be right-angled, or equilateral.

SCHOLIUM.

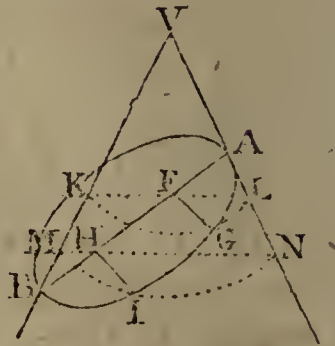
The rectangle inscribed between the four conjugate hyperbolas, is similar to a rectangle circumscribed about an ellipse, by drawing tangents, in like manner, to the four extremities of the two axes; and the asymptotes or diagonals in the hyperbola, are analagous to those in the ellipse, cutting this curve in similar points, and making the pair of equal conjugate diameters. Moreover, the whole figure, formed by the four hyperbolas, is, as it were, an ellipse turned inside out, cut open at the extremities  $D$ ,  $E$ ,  $F$ ,  $G$ , of the said equal conjugate diameters, and those four points drawn out to an infinite distance, the curvature being turned the contrary way, but the axes, and the rectangle passing through their extremities, continuing fixed.

OF THE ELLIPSE.

THEOREM I.

The Squares of the Ordinates of the Axis are to each other as the Rectangles of their Abscisses.

LET AVB be a plane passing through the axis of the cone; AGIH another section of the cone perpendicular to the plane of the former; AB the axis of this elliptic section; and FG, HI ordinates perpendicular to it. Then it will be, as  $FG : HI :: AF \cdot FB : AH \cdot HB$ .



For, through the ordinates FG, HI draw the circular sections KGL, MIN parallel to the base of the cone, having KL, MN for their diameters, to which FG, HI are ordinates, as well as to the axis of the ellipse.

Now, by the similar triangles AFL, AHN, and BFK, BHM, it is  $AF : AH :: FL : HN$ ,  
and  $FB : HB :: KF : MH$ ;

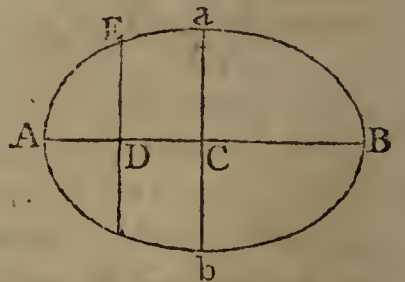
hence, taking the rectangles of the corresponding terms, it is, the rect.  $AF \cdot FB : AH \cdot HB :: KF \cdot FL : MH \cdot HN$ .

But, by the circle,  $KF \cdot FL = FG^2$ , and  $MH \cdot HN = HI^2$  ;  
Therefore the rect.  $AF \cdot FB : AH \cdot HB :: FG^2 : HI^2$ . Q. E. D.

THEOREM II.

As the Square of the Transverse Axis :  
Is to the Square of the Conjugate ::  
So is the Rectangle of the Abscisses :  
To the Square of their Ordinate.

That is,  $AB^2 : ab^2$  or  
 $AC^2 : ac^2 :: AD \cdot DB : DE^2$ .



For, by theor. I,  $AC \cdot CB : AD \cdot DB :: ca^2 : DE^2$  ;  
But, if c be the centre, then  $AC \cdot CB = AC^2$ , and ca is the semi-conj.

Therefore

Therefore  $AC^2 : AD \cdot DB :: ac^2 : DE^2$ ;  
 or, by permutation,  $AC^2 : ac^2 :: AD \cdot DB : DE^2$ ;  
 or, by doubling,  $AB^2 : ab^2 :: AD \cdot DB : DE^2$ . Q. E. D.

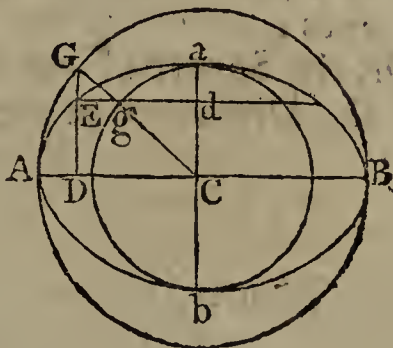
*Corol.* Or, by div.  $AB : \frac{ab^2}{AB} :: CA^2 - CD^2 : DE^2$ ,  
 that is,  $AB : p :: AD \cdot DB$  or  $CA^2 - CD^2 : DE^2$ ;  
 where  $p$  is the parameter  $\frac{ab^2}{AB}$  by the definition of it.

That is, As the transverse,  
 Is to its parameter,  
 So is the rectangle of the abscisses,  
 To the square of their ordinate.

THEOREM III.

As the Square of the Conjugate Axis :  
 Is to the Square of the Transverse Axis ::  
 So is the Rectangle of the Abscisses of the Conjugate, or  
 the Difference of the Squares of the Semi-conjugate and  
 Distance of the Centre from any Ordinate of that Axis :  
 To the Square of their Ordinate.

That is,  
 $ca^2 : CB^2 :: ad \cdot db$  or  $ca^2 - cd^2 : de^2$ .



For, draw the ordinate ED to the transverse AB.  
 Then, by theor. I,  $ca^2 : CA^2 :: DE^2 : AD \cdot DB$  or  $CA^2 - CD^2$ ,  
 or - - -  $ca^2 : CA^2 :: cd^2 : CA^2 - de^2$ .  
 But - - -  $ca^2 : CA^2 :: ca^2 : CA^2$ ,  
 theref. by division,  $ca^2 : CA^2 :: ca^2 - cd^2$  or  $ad \cdot db : de^2$ .  
 Q. E. D.

*Corol.* I. If two circles be described on the two axes as diameters, the one inscribed within the ellipse, and the other circumscribed about it; then an ordinate in the circle will be to the corresponding ordinate in the ellipse, as the axis of this ordinate, is to the other axis.

That is,  $CA : ca :: DG : DE$ ,  
 and  $ca : CA :: dg : de$ .

For, by the nature of the circle,  $AD \cdot DB = DG^2$ ; theref.  
by the nature of the ellipse,  $CA^2 : ca^2 :: AD \cdot DB$  or  $DG^2 : DE^2$ ,  
or  $CA : ca :: DG : DE$ .

In like manner  $ca : CA :: dg : dE$ .

Moreover, by equality,  $DG : DE$  or  $cd :: dE$  or  $DC : dg$ .

Therefore  $cgg$  is a continued straight line.

*Corol. 2.* Hence also, as the ellipse and circle are made up of the same number of corresponding ordinates, which are all in the same proportion of the two axes, it follows that the areas of the whole circle and ellipse, as also of any like parts of them, are in the same proportion of the two axes, or as the square of the diameter to the rectangle of the two axes; that is, the areas of the two circles, and of the ellipse, are as the square of each axis and the rectangle of the two, and therefore the ellipse is a mean proportional between the two circles.

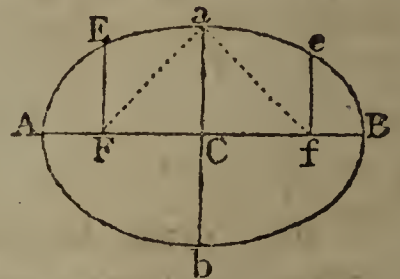
#### THEOREM IV.

The Square of the Distance of the Focus from the Centre, is equal to the Difference of the Squares of the Semi-axes;

Or, the Square of the Distance between the Foci, is equal to the Difference of the Squares of the two Axes.

$$\text{That is, } CF^2 = CA^2 - ca^2,$$

$$\text{or } Ff^2 = AB^2 - ab^2.$$



For, to the focus  $F$  draw the ordinate  $FE$ ; which, by the definition, will be the semi-parameter. Then, by the nature of the curve  $CA^2 : ca^2 :: CA^2 - CF^2 : FE^2$ ;  
and by the def. of the para.  $CA^2 : ca^2 :: ca^2 : FE^2$ ;  
therefore  $ca^2 = CA^2 - CF^2$ ;  
and by addit. and subtr.  $CF^2 = CA^2 - ca^2$ ;  
or, by doubling,  $Ff^2 = AB^2 - ab^2$ . Q. E. D.

*Corol. 1.* The two semi-axes, and the focal distance from the centre, are the sides of a right-angled triangle  $Cfa$ ; and the distance  $fa$  from the focus to the extremity of the conjugate axis, is  $= AC$  the semi-transverse.

*Corol. 2.*

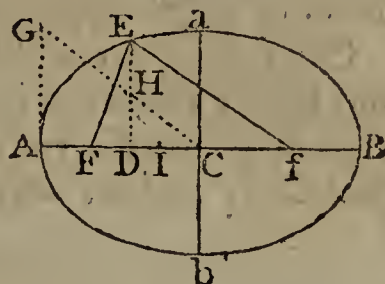
*Corol. 2.* The conjugate semi-axis  $ca$  is a mean proportional between  $AF$ ,  $FB$ , or between  $Af$ ,  $fB$ , the distances of either focus from the two vertices.

$$\text{For } ca^2 = CA^2 - CF^2 = CA + CF \cdot CA - CF = AF \cdot FB.$$

THEOREM V.

The Sum of two Lines drawn from the two Foci, to meet at any Point in the Curve, is equal to the Transverse Axis.

That is,  
 $FE + fe = AB.$



For, draw  $AG$  parallel and equal to  $ca$  the semi-conjugate; and join  $CG$  meeting the ordinate  $DE$  in  $H$ ; also take  $CI$  a 4th proportional to  $CA$ ,  $CF$ ,  $CD$ .

Then, by theor. 2,  $CA^2 : AG^2 :: CA^2 - CD^2 : DE^2$ ;  
 and, by sim. tri.  $CA^2 : AG^2 :: CA^2 - CD^2 : AG^2 - DH^2$ ;  
 consequently  $DE^2 = AG^2 - DH^2 = ca^2 - DH^2$ .

Also  $FD = CF \oslash CD$ , and  $FD^2 = CF^2 - 2CF \cdot CD + CD^2$ ;  
 And, by right-angled triangles,  $FE^2 = FD^2 + DE^2$ ;  
 therefore  $FE^2 = CF^2 + ca^2 - 2CF \cdot CD + CD^2 - DH^2$ .

But by theor. 4,  $CF^2 + ca^2 = CA^2$ ,  
 and, by supposition,  $2CF \cdot CD = 2CA \cdot CI$ ;  
 theref.  $FE^2 = CA^2 - 2CA \cdot CI + CD^2 - DH^2$ .

Again, by supp.  $CA^2 : CD^2 :: CF^2$  or  $CA^2 - AG^2 : CI^2$ ;  
 and, by sim. tri.  $CA^2 : CD^2 :: CA^2 - AG^2 : CD^2 - DH^2$ ;  
 therefore  $CI^2 = CD^2 - DH^2$ ;  
 consequently  $FE^2 = CA^2 - 2CA \cdot CI + CI^2$ .

And the root or side of this square is  $FE = CA - CI = AI$ .

In the same manner it is found that  $fe = CA + CI = BI$ .

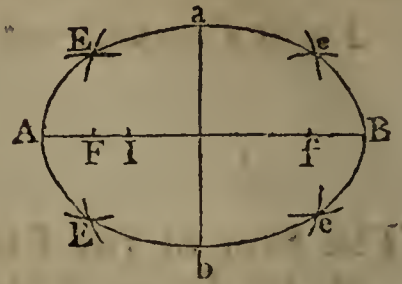
Conseq. by addit.  $FE + fe = AI + BI = AB$ . Q. E. D.

*Corol. 1.* Hence  $CI$  or  $CA - FE$  is a 4th proportional to  $CA$ ,  $CF$ ,  $CD$ .

*Corol. 2.* And  $fe - FE = 2CI$ ; that is, the difference between two lines drawn from the foci, to any point in the curve, is double the 4th proportional to  $CA$ ,  $CF$ ,  $CD$ .

*Corol. 3.* Hence is derived the common method of describing this curve mechanically by points, or with a thread, thus :

In the transverse take the foci  $F, f$ , and any point  $I$ . Then with the radii  $AI, BI$ , and centres  $F, f$ , describe arcs intersecting in  $E$ , which will be a point in the curve. In like manner, assuming other points  $i$ , as many other points will be found in the curve.



Then with a steady hand, the curve line may be drawn through all the points of intersection  $E$ .

Or, take a thread of the length of  $AB$  the transverse axis, and fix its two ends in the foci  $F, f$ , by two pins. Then carry a pen or pencil round by the thread, keeping it always stretched, and its point will trace out the curve line.

THEOREM VI.

If from any Point  $I$  in the Axis produced, a Line  $IL$  be drawn touching the curve in one Point  $L$ ; and the Ordinate  $LM$  be drawn; and if  $C$  be the Centre or Middle of  $AB$ : Then shall  $CM$  be to  $CI$  as the Square of  $AM$  to the Square of  $AI$ .

That is,  
 $CM : CI :: AM^2 : AI^2$ .



For, from the point  $I$  draw any other line  $IEH$  to cut the curve in two points  $E$  and  $H$ ; from which let fall the perpendiculars  $ED$  and  $HG$ ; and bisect  $DG$  in  $K$ .

Then, by theor. I,  $AD \cdot DB : AG \cdot GB :: DE^2 : GH^2$ ,  
 and by sim. tri.  $ID^2 : IG^2 :: DE^2 : GH^2$ ;  
 theref. by equality,  $AD \cdot DB : AG \cdot GB :: ID^2 : IG^2$ .

But  $DB = CB + CD = AC + CD = AG + DC - CG = 2CK + AG$ ,  
 and  $GB = CB - CG = AC - CG = AD + DC - CG = 2CK + AD$ ;  
 theref.  $AD \cdot 2CK + AD \cdot AG : AG \cdot 2CK + AD \cdot AG :: ID^2 : IG^2$ ,  
 and, by div.  $DG \cdot 2CK : IG^2 - ID^2$  or  $DG \cdot 2IK :: AD \cdot 2CK + AD : AG : ID^2$ .

or -  $2CK : 2IK :: AD \cdot 2CK + AD \cdot AG : ID^2$ ,

or -  $AD \cdot 2CK : AD \cdot 2IK :: AD \cdot 2CK + AD \cdot AG : ID^2$ ;

theref. by div.  $CK : IK :: AD \cdot AG : ID^2 - AD \cdot 2IK$ ,

and, by comp.  $CK : CI :: AD \cdot AG : ID^2 - AD \cdot ID + IA$ ,

or -  $CK : CI :: AD \cdot AG : AI^2$ .

But,

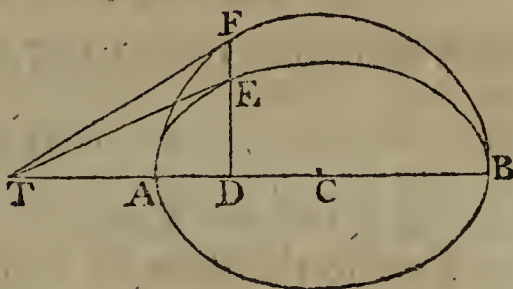


But, when the line IH, by revolving about the point I, comes into the position of the tangent IL, then the points E and H meet in the point L, and the points D, K, G, coincide with the point M; and then the last proportion becomes  $CM : CI :: AM^2 : AI^2$ .

THEOREM VII.

If a Tangent and Ordinate be drawn from any Point in the Curve, meeting the Transverse Axis; the Semi-transverse will be a Mean Proportional between the Distances of the said Two Intersections from the Centre.

That is,  
CA is a mean proportional between CD and CT; or CD, CA, CT are continued proportionals.



For, by theor. 6,  $CD : CT :: AD^2 : AT^2$ ,  
that is,  $CD : CT :: (CA - CD)^2 : (CT - CA)^2$ ,  
or -  $CD : CT :: CD^2 + CA^2 : CA^2 + CT^2$ ,  
and -  $CD : DT :: CD^2 + CA^2 : CT^2 - CD^2$ ,  
or -  $CD : DT :: GD^2 + CA^2 : (CT + CD) DT$ ,  
or -  $CD^2 : CD \cdot CT :: CD^2 + CA^2 : CD \cdot DT + CT \cdot DT$ ,  
hence -  $CD^2 : CA^2 :: CD \cdot DT : CT \cdot DT$ ,  
and -  $CD^2 : CA^2 :: CD : CT$ .  
therefore (th. 78, Geom.)  $CD : CA :: CA : CT$ . Q. E. D.

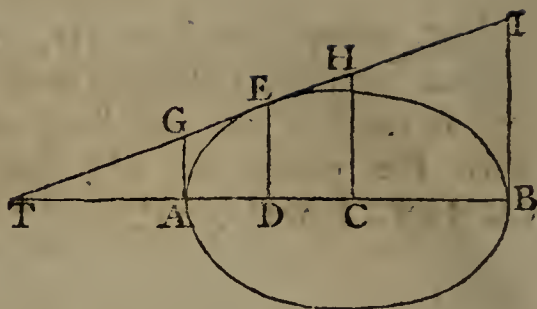
*Corol.* Since CT is always a third proportional to CD, CA; if the points D, A, remain constant, then will the point T be constant also; and therefore all the tangents will meet in this point T, which are drawn from the point E, of every ellipse described on the same axis AB, where they are cut by the common ordinate DEE drawn from the point D.

THEOREM VIII.

If there be any Tangent meeting Four Perpendiculars to the Axis drawn from these four points, namely the Centre, the two Extremities of the Axis, and the Point of Contact; those Four Perpendiculars will be Proportionals.

That

That is,  
 $AG : DE :: CH : BI.$



For, by theor. 7,  $TC : AC :: AC : DC$ ,  
 theref. by div.  $TA : AD :: TC : AC$  or  $CB$ ,  
 and by comp.  $TA : TD :: TC : TB$ ,  
 and by sim. tri.  $AG : DE :: CH : BI.$  Q. E. D.

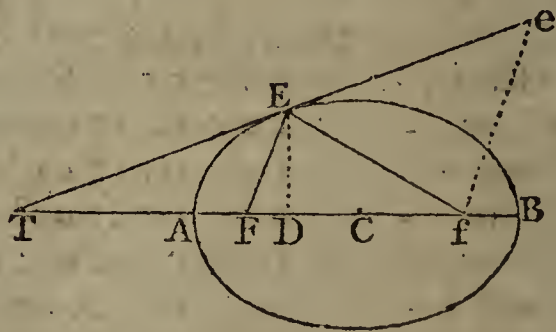
*Corol.* Hence  $TA, TD, TC, TB$   
 and  $TG, TE, TH, TI$  } are also proportionals.

For these are as  $AG, DE, CH, BI$ , by similar triangles.

#### THEOREM IX.

If there be any Tangent, and two Lines drawn from the Foci to the Point of Contact; these two Lines will make equal Angles with the Tangent.

That is,  
 the  $\angle FET = \angle fee.$



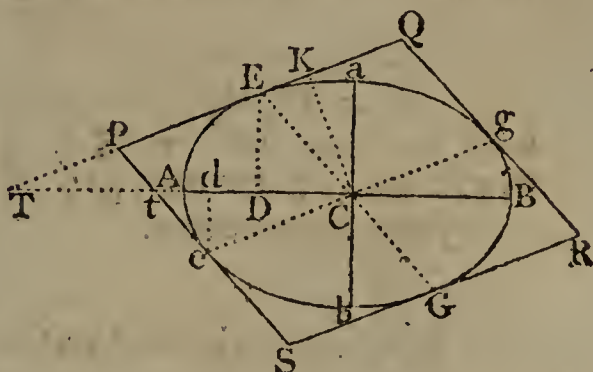
For, draw the ordinate  $DE$ , and  $fe$  parallel to  $FE$ .  
 By cor. 1, theor. 5,  $CA : CD :: CF : CA - FE$ ,  
 and by theor. 7,  $CA : CD :: CT : CA$ ;  
 therefore  $CT : CF :: CA : CA - FE$ ;  
 and by add. and sub.  $TF : Tf :: FE : 2CA - FE$  or  $FE$  by th. 5,  
 But by sim. tri.  $TF : Tf :: FE : fe$ ;  
 therefore  $fe = fe$ , and conseq.  $\angle e = \angle fee$ .  
 But, because  $FE$  is parallel to  $fe$ , the  $\angle e = \angle FET$ ;  
 therefore the  $\angle FET = \angle fee.$  Q. E. D.

*Corol.* As opticians find that the angle of incidence is equal to the angle of reflection, it appears from this theorem, that rays of light issuing from the one focus, and meeting the curve in every point, will be reflected into lines drawn from those points to the other focus. So the ray  $FE$  is reflected into  $FE$ . And this is the reason why the points  $F, f$  are called *foci*, or burning points.

THEOREM X.

All the Parallelograms circumscribed about an Ellipse are equal to one another, and each equal to the Rectangle of the two Axes.

That is,  
the parallelogram PQRS =  
the rectangle AB . ab.



Let EG, eg be two conjugate diameters parallel to the sides of the parallelogram, and dividing it into four lesser and equal parallelograms. Also, draw the ordinates DE, dc, and CK perpendicular to PQ; and let the axis CA produced meet the sides of the parallelogram, produced if necessary, in T and t.

Then, by theor. 7,  $CT : CA :: CA : CD$ ,  
and  $ct : CA :: CA : cd$ ;  
theref. by equality,  $CT : ct :: cd : CD$ ;  
but, by sim. triangles,  $CT : ct :: TD : cd$ ,  
theref. by equality,  $TD : cd :: cd : CD$ ,  
and the rectangle  $TD . DC$  is = the square  $cd^2$ .

Again, by theor. 7,  $CD : CA :: CA : CT$ ,  
or, by division,  $CD : CA :: DA : AT$ ,  
and by composition,  $CD : DB :: AD : DT$ ;  
conseq. the rectangle  $CD . DT = cd^2 = AD . DB^*$ .

But, by theor. 1,  $CA^2 : ca^2 :: (AD . DB \text{ or } ) cd^2 : DE^2$ ,  
therefore  $CA : ca :: cd : DE$ ;

In like manner,  $CA : ca :: CD : dc$ ,  
or  $ca : dc :: CA : CD$ .

But, by theor. 7,  $CT : CA :: CA : CD$ ;  
theref. by equality,  $CT : CA :: ca : dc$ .

But, by sim. tri.  $CT : CK :: ce : dc$ ;  
theref. by equality,  $CK : CA :: ca : ce$ ,  
and the rectangle  $CK . ce = CA . ca$ .

But the rect.  $CK . ce =$  the parallelogram CEPE;  
theref. the rect.  $CA . ca =$  the parallelogram CEPE,  
and conseq. the rect.  $AB . ab =$  the paral. PQRS. Q. E. D.

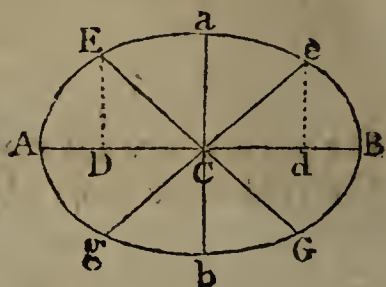
\* Corol. Because  $cd^2 = AD . DB = CA^2 - CD^2$ ,  
therefore  $CA^2 = CD^2 + cd^2$ .

In like manner,  $ca^2 = DE^2 + dc^2$ .

## THEOREM XI.

The Sum of the Squares of every Pair of Conjugate Diameters, is equal to the same constant Quantity, namely, the Sum of the Squares of the two Axes.

That is,  
 $AB^2 + ab^2 = EG^2 + eg^2$ ;  
 where  $EG, eg$  are any pair of conjugate diameters.



For, draw the ordinates  $ED, ed$ .

Then, by cor. to theor. 10,  $CA^2 = CD^2 + cd^2$ ,  
 and  $ca^2 = DE^2 + de^2$ ;  
 therefore the sum  $CA^2 + ca^2 = CD^2 + DE^2 + cd^2 + de^2$ .  
 But, by right-angled  $\Delta$ s,  $CE^2 = CD^2 + DE^2$ ,  
 and  $ce^2 = cd^2 + de^2$ ;  
 therefore the sum  $CE^2 + ce^2 = CD^2 + DE^2 + cd^2 + de^2$ .  
 consequently  $CA^2 + ca^2 = CE^2 + ce^2$ ;  
 or, by doubling,  $AB^2 + ab^2 = EG^2 + eg^2$ . Q. E. D.

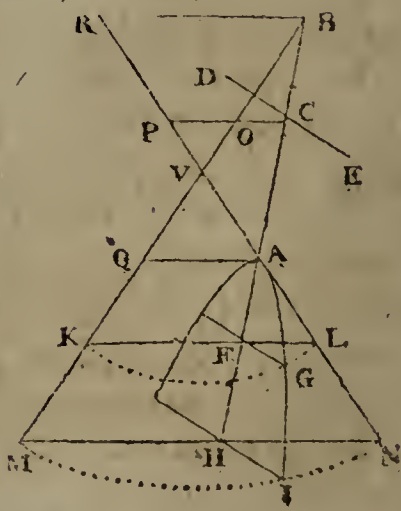
## OF THE HYPERBOLA.

## THEOREM I.

The Squares of the Ordinates of the Axis are to each other as the Rectangles of their Abscisses.

LET  $\Delta VNB$  be a plane passing through the vertex and axis of the opposite cones;  $AGIH$  another section of them perpendicular to the plane of the former;  $AB$  the axis of the hyperbolic sections; and  $FG, HI$  ordinates perpendicular to it. Then it will be, as  
 $FG^2 : HI^2 :: AF \cdot FB : AH \cdot HB$ .

For, through the ordinates  $FG, HI$  draw the circular sections  $KGL, MIN$



*MN* parallel to the base of the cone, having *KL*, *MN* for their diameters, to which *FG*, *HI* are ordinates, as well as to the axis of the hyperbola.

Now, by the similar triangles *AFL*, *AHN*, and *BFK*, *BHM*,  
it is  $AF : AH :: FL : HN$ ,  
and  $FB : HB :: KF : MH$ ;

hence, taking the rectangles of the corresponding terms,  
it is, the rect.  $AF \cdot FB : AH \cdot HB :: KF \cdot FL : MH \cdot HN$ .

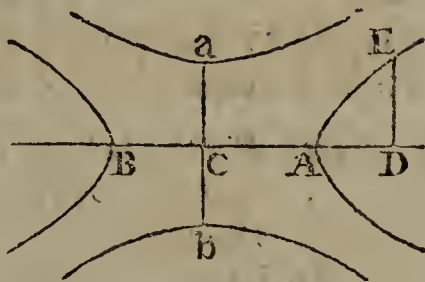
But, by the circle,  $KF \cdot FL = FG^2$ , and  $MH \cdot HN = HI^2$ ;  
Therefore the rect.  $AF \cdot FB : AH \cdot HB :: FG^2 : HI^2$ .

Q. E. D.

THEOREM II.

As the Square of the Transverse Axis :  
Is to the Square of the Conjugate ::  
So is the Rectangle of the Absciffes :  
To the Square of their Ordinate.

That is,  $AB^2 : ab^2$  or  
 $AC^2 : ac^2 :: AD \cdot DB : DE^2$ .



For, by theor, I,  $AC \cdot CB : AD \cdot DB :: Ca^2 : DE^2$ ;  
But, if *c* be the centre, then  $AC \cdot CB = AC^2$ , and *ca* is the  
femi-conj.

Therefore  $AC^2 : AD \cdot DB :: ac^2 : DE^2$ ;  
or, by permutation,  $AC^2 : ac^2 :: AD \cdot DB : DE^2$ ;  
or, by doubling,  $AB^2 : ab^2 :: AD \cdot DB : DE^2$ . Q. E. D.

Corol. Or, by div.  $AB : \frac{ab^2}{AB} :: CD^2 - CA^2 : DE^2$ ,

that is,  $AB : p :: AD \cdot DB$  or  $CD^2 - CA^2 : DE^2$ ;

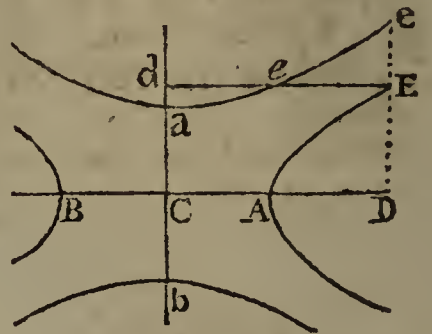
where *p* is the parameter  $\frac{ab^2}{AB}$  by the definition of it.

That is, As the transverse,  
Is to its parameter,  
So is the rectangle of the absciffes,  
To the square of their ordinate.

THEOREM III.

As the Square of the Conjugate Axis :  
 To the Square of the Transverse Axis ::  
 So is the Sum of the Squares of the Semi-conjugate, and  
 Distance of the Centre from any Ordinate of this Axis :  
 To the Square of their Ordinate.

That is,  
 $ca^2 : CA^2 :: ca^2 + cd^2 : de^2.$



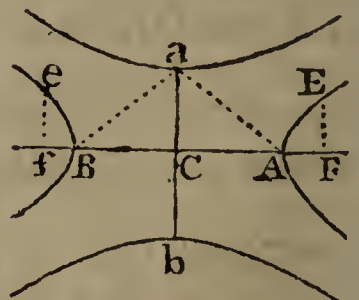
For, draw the ordinate ED to the transverse AB.  
 Then, by theor. I,  $ca^2 : CA^2 :: DE^2 : AD \cdot DB$  or  $CD^2 - CA^2$ ,  
 or - - -  $ca^2 : CA^2 :: cd^2 : de^2 - CA^2$ .  
 But - - -  $ca^2 : CA^2 :: ca^2 : CA^2$ ,  
 theref. by compos.  $ca^2 : CA^2 :: ca^2 + cd^2 : de^2$ .  
 In like manner,  $CA^2 : ca^2 :: CA^2 + CD^2 : DE^2$ . Q. E. D.

*Corol.* By the last theor.  $CA^2 : ca^2 :: CD^2 - CA^2 : DE^2$ ,  
 and by this theor.  $CA^2 : ca^2 :: CD^2 + CA^2 : de^2$ ,  
 therefore -  $DE^2 : de^2 :: CD^2 - CA^2 : CD^2 + CA^2$ .  
 In like manner,  $de^2 : de^2 :: cd^2 - ca^2 : cd^2 + ca^2$ .

THEOREM IV.

The Square of the Distance of the Focus from the Centre, is  
 equal to the Sum of the Squares of the Semi-Axes.  
 Or, the Square of the Distance between the Foci, is equal to  
 the Sum of the Squares of the two Axes.

That is,  
 $CF^2 = CA^2 + ca^2$ , or  
 $ff^2 = AB^2 + ab^2$ .



FOR, to the focus F draw the ordinate FE; which, by  
 the definition, will be the semi-parameter. Then, by the  
 nature

nature of the curve -  $CA^2 : ca^2 :: CF^2 - CA^2 : FE^2$ ;  
 and by the def. of the para.  $CA^2 : ca^2 :: ca^2 : FE^2$ ;  
 therefore -  $ca^2 = CF^2 - CA^2$ ;  
 and by addition, -  $CF^2 = CA^2 + ca^2$ ;  
 or, by doubling, -  $Ff^2 = AB^2 + ab^2$ . Q. E. D.

*Corol. 1.* The two femi-axes, and the focal distance from the centre, are the sides of a right-angled triangle  $CAa$ ; and the distance  $Aa$  is =  $CF$  the focal distance.

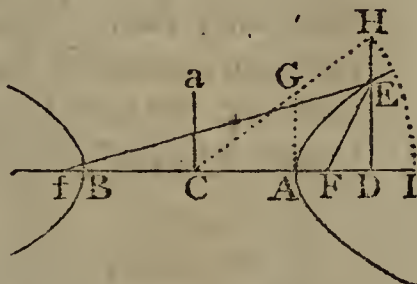
*Corol. 2.* The conjugate femi-axis  $ca$  is a mean proportional between  $AF$ ,  $FB$ , or between  $Af$ ,  $fb$ , the distances of either focus from the two vertices.

For  $ca^2 = CF^2 - CA^2 = CF + CA . CF - CA = AF . FB$ .

THEOREM V.

The Difference of two Lines drawn from the two Foci, to meet at any Point in the Curve, is equal to the Transverse Axis.

That is,  
 $fE - FE = AB$ .



FOR, draw  $AG$  parallel and equal to  $ca$  the femi-conjugate; and join  $CG$  meeting the ordinate  $DE$  produced in  $H$ ; also take  $CI$  a 4th proportional to  $CA$ ,  $CF$ ,  $CD$ .

Then, by th. 2,  $CA^2 : AG^2 :: CD^2 - CA^2 : DE^2$ ;  
 and, by sim.  $\Delta$ s,  $CA^2 : AG^2 :: CD^2 - CA^2 : DH^2 - AG^2$ ;  
 consequently  $DE^2 = DH^2 - AG^2 = DH^2 - ca^2$ .

Also,  $FD = CF \oslash CD$ , and  $FD^2 = CF^2 - 2CF . CD + CD^2$ ;

and, by right-angled triangles,  $FE^2 = FD^2 + DE^2$ ;

therefore  $FE^2 = CF^2 - ca^2 - 2CF . CD + CD^2 + DH^2$ .

But, by theor. 4,  $CF^2 - ca^2 = CA^2$ ,

and, by supposition,  $2CF . CD = 2CA . CI$ ;

theref.  $FE^2 = CA^2 - 2CA . CI + CD^2 + DH^2$ ;

Again, by suppos.  $CA^2 : CD^2 :: CF^2$  or  $CA^2 + AG^2 : CI^2$ ;

and, by sim. tri.  $CA^2 : CD^2 :: CA^2 + AG^2 : CD^2 + DH^2$ ;

therefore -  $CI^2 = CD^2 + DH^2 = CH^2$ ;

consequently  $FE^2 = CA^2 - 2CA . CI + CI^2$ .

And the root or side of this square is  $FE = CI - CA = AI$ .

In the same manner, it is found that  $fE = CI + CA = BI$ .

Conseq. by subtract.  $fE - FE = BI - AI = AB$ . Q. E. D.

*Corol. 1.*

*Corol. 1.* Hence  $CH = CI$  is a 4th proportional to  $CA$ ,  $CF$ ,  $CD$ .

*Corol. 2.* And  $fe + FE = 2CH$  or  $2CI$ ; or  $FE$ ,  $CH$ ,  $fe$  are in continued arithmetical progression, the common difference being  $CA$  the semi-transverse.

*Corol. 3.* Hence is derived the common method of describing this curve mechanically by points, thus:

In the transverse  $AB$ , produced, take the foci  $F$ ,  $f$ , and any point  $I$ . Then with the radii  $AI$ ,  $BI$ , and centres  $F$ ,  $f$ , describe arcs intersecting in  $E$ , which will be a point in the curve. In like manner, assuming other points  $I$ , as many other points will be found in the curve.

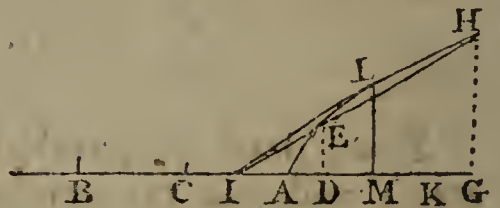
Then with a steady hand, the curve line may be drawn through all the points of intersection  $E$ .

In the same manner are constructed the other two hyperbolas, using the axis  $ab$  instead of  $AB$ .

#### THEOREM VI.

If from any Point  $I$  in the Axis, a Line  $IL$  be drawn touching the Curve in one Point  $L$ ; and the ordinate  $LM$  be drawn; and if  $C$  be the Centre or the Middle of  $AB$ : Then shall  $CM$  be to  $CI$  as the Square of  $AM$  to the Square of  $AI$ .

That is,  
 $CM : CI :: AM^2 : AI^2$ .



FOR, from the point  $I$  draw any line  $AEH$  to cut the curve in two points  $E$  and  $H$ ; from which let fall the perps.  $ED$ ,  $HG$ ; and bisect  $DG$  in  $K$ .

Then, by theor. I,  $AD \cdot DB : AG \cdot GB :: DE^2 : GH^2$ ,  
 and by sim. tri.  $ID^2 : IG^2 :: DE^2 : GH^2$ ;  
 theref. by equality,  $AD \cdot DB : AG \cdot GB :: ID^2 : IG^2$ ,  
 But  $DB = CB + CD = CA + CD = CG + CD - AG = 2CK - AG$ ,  
 and  $GB = CB + CG = CA + CG = CG + CD - AD = 2CK - AD$ ;  
 theref.  $AD \cdot 2CK - AD \cdot AG : AG \cdot 2CK - AD \cdot AG :: ID^2 : IG^2$ ,  
 and, by div.  $DG \cdot 2CK : IG^2 - ID^2$  or  $DG \cdot 2IK :: AD \cdot 2CK - AD \cdot AG : ID^2$ .

or  $2CK : 2IK :: AD \cdot 2CK - AD \cdot AG : ID^2$ ;  
 or  $AD \cdot 2CK : AD \cdot 2IK :: AD \cdot 2CK - AD \cdot AG : ID^2$ ;  
 theref. by div.  $CK : IK :: AD \cdot AG : AD \cdot 2IK - ID^2$ ,  
 and, by div.  $CK : CI :: AD \cdot AG : ID^2 - AD \cdot ID + IA$ ,  
 or  $CK : CI :: AD \cdot AG : AI^2$ .

But,



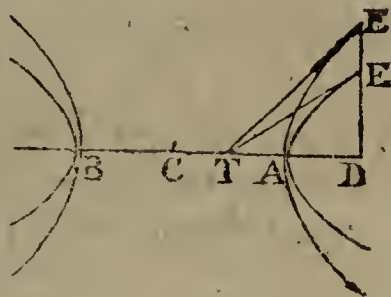
But, when the line IH, by revolving about the point I, comes into the position of the tangent IL, then the points E and H meet in the point L, and the points D, K, G, coincide with the point M; and then the last proportion becomes  $CM : CI :: AM^2 : AI^2$ . Q. E. D.

THEOREM VII.

If a Tangent and Ordinate be drawn from any Point in the Curve, meeting the Transverse Axis; the Semi-transverse will be a Mean Proportional between the Distances of the said Two Intersections from the Centre.

That is,

CA is a mean proportion between CD and CT; or CD, CA, CT are continued proportionals.



FOR, by th. 6,  $CD : CT :: AD^2 : AT^2$ ,  
 that is,  $CD : CT :: (CD - CA)^2 : (CA - CT)^2$ ,  
 or  $CD : CT :: CD^2 + CA^2 : CA^2 + CT^2$ ,  
 and  $CD : DT :: CD^2 + CA^2 : CD^2 - CT^2$ ,  
 or  $CD : DT :: CD^2 + CA^2 : (CD + CT) DT$ ,  
 or  $CD^2 : CD . DT :: CD^2 + CA^2 : CD . DT + CT . TD$ ;  
 hence  $CD^2 : CA^2 :: CD . DT : CT . TD$ ,  
 and  $CD^2 : CA^2 :: CD : CT$ ,  
 theref. (th. 78, Geom.)  $CD : CA :: CA : CD$ . Q. E. D.

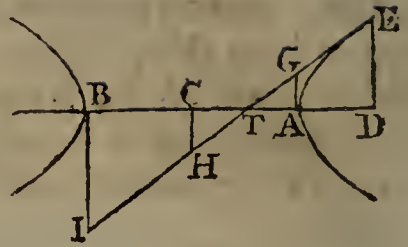
*Corol.* Since CT is always a third proportional to CD, CA; if the points D, A, remain constant, then will the point T be constant also; and therefore all the tangents will meet in this point T, which are drawn from the point E, of every hyperbola described on the same axis AB, where they are cut by the common ordinate DEE drawn from the point D.

THEOREM VIII.

If there be any Tangent meeting Four Perpendiculars to the Axis drawn from these four Points, namely, the Centre; the two Extremities of the Axis, and the Point of Contact; those Four Perpendiculars will be Proportionals.

That

That is,  
 $AG : DE :: CH : BI.$



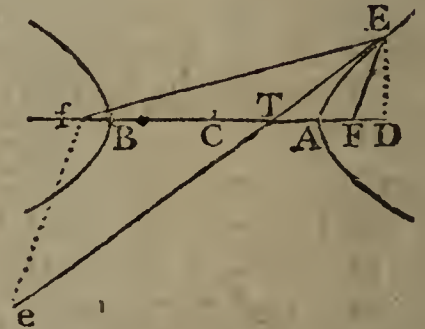
For, by theor. 7,  $TC : AC :: AC : DC,$   
 theref. by div.  $TA : AD :: TC : AC$  OR  $CB,$   
 and by comp.  $TA : TD :: TC : TB,$   
 and by sim. tri.  $AG : DE :: CH : BI.$  Q. E. D.

*Corol.* Hence  $TA, TD, TC, TB$   
 and  $TG, TE, TH, TI$  } are also proportionals.  
 For these are as  $AG, DE, CH, BI,$  by similar triangles.

THEOREM IX.

If there be any Tangent, and two Lines drawn from the Foci to the Point of Contact; these two Lines will make equal Angles with the Tangent.

That is,  
 the  $\angle FET = \angle fec.$



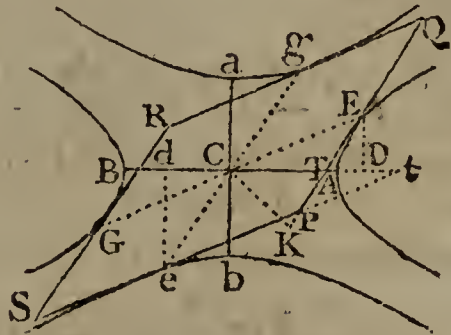
For, draw the ordinate  $DE,$  and  $fe$  parallel to  $FE.$   
 By cor. 1, theor. 5,  $CA : CD :: CF : CA + FE,$   
 and by th. 7,  $CA : CD :: CT : CA;$   
 therefore  $CT : CF :: CA : CA + FE;$   
 and by add. and sub.  $TF : Tf :: FE : 2CA + FE$  OR  $FE$  by th. 5.  
 But by sim. tri.  $TF : Tf :: FE : fe;$   
 therefore  $FE = fe,$  and conseq.  $\angle e = \angle fec.$   
 But, because  $FE$  is parallel to  $fe,$  the  $\angle e = \angle FET;$   
 therefore the  $\angle FET = \angle fec.$  Q. E. D.

*Corol.* As opticians find that the angle of incidence is equal to the angle of reflection, it appears from this proposition, that rays of light issuing from the one focus, and meeting the curve in every point, will be reflected into lines drawn from the other focus. So the ray  $FE$  is reflected into  $FE.$  And this is the reason why the points  $F, f$  are called *foci,* or burning points.

THEOREM X.

All the Parallelograms inscribed between the four Conjugate Hyperbolas are equal to one another, and each equal to the Rectangle of the two Axes.

That is,  
the parallelogram PQRS =  
the rectangle AB . ab.



Let EG, eg be two conjugate diameters parallel to the sides of the parallelogram, and dividing it into four lesser and equal parallelograms. Also, draw the ordinates DE, de, and ck perpendicular to PQ; and let the axis produced meet the sides of the parallelogram, produced, if necessary, in T and t.

Then, by theor. 7,  $CT : CA :: CA : CD$ ,  
and  $ct : cA :: CA : cd$ ;  
theref. by equality,  $CT : ct :: cd : CD$ ;  
but, by sim. triangles,  $CT : ct :: TD : cd$ ,  
theref. by equality,  $TD : cd :: cd : CD$ ,  
and the rectangle  $TD . DC$  is = the square  $cd^2$ .

Again, by theor. 7,  $CD : CA :: CA : CT$ ,  
or, by division  $CD : CA :: DA : AT$ ,  
and, by composition,  $CD : DB :: DA : DT$ ;  
conseq. the rectangle  $CD . DT = cd^2 = AD . DB$  \*.

But, by theor. 1,  $CA^2 : ca^2 :: (AD . DB \text{ or } ) cd^2 : DE^2$ ,  
therefore  $CA : ca :: cd : DE$ ;

In like manner,  $CA : ca :: CD : de$ ;  
or  $ca : de :: CA : CD$ .

But, by theor. 7,  $CT : CA :: CA : CD$ ;  
theref. by equality,  $CT : CA :: ca : de$ .

But, by sim. tri.  $CT : CK :: ce : de$ ;  
theref. by equality,  $CK : CA :: ca : ce$ ,  
and the rectangle  $CK . ce = CA . ca$ .

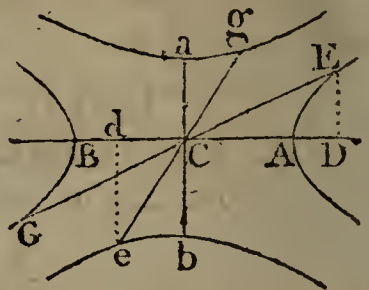
But the rect.  $CK . ce =$  the parallelogram CEPE,  
theref. the rect.  $CA . ce =$  the parallelogram CEPE,  
and conseq. the rect.  $AB . ab =$  the paral. PQRS. Q. E. D.

\* Corol. Because  $cd^2 = AD . DB = CD^2 - CA^2$ ,  
therefore  $CA^2 = CD^2 - cd^2$ .  
In like manner,  $ca^2 = de^2 - DE^2$ .

THEOREM XI.

The Difference of the Squares of every Pair of Conjugate Diameters, is equal to the same constant Quantity, namely the Difference of the Squares of the two Axes.

That is,  
 $AB^2 - ab^2 = EG^2 - eg^2$ ;  
 where EG, eg are any conjugate diameters.



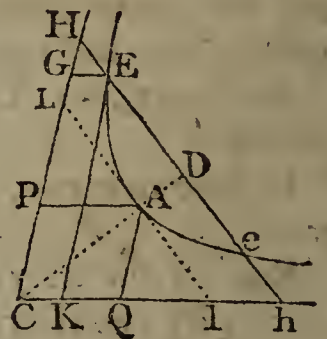
For, draw the ordinates ED, ed.

Then, by cor. to theor. 10,  $CA^2 = CD^2 - cd^2$ ,  
 and  $ca^2 = de^2 - DE^2$ ;  
 theref. the difference  $CA^2 - ca^2 = CD^2 + DE^2 - cd^2 - de^2$ .  
 But, by right-angled  $\Delta$ s,  $CE^2 = CD^2 + DE^2$ ,  
 and  $ce^2 = cd^2 + de^2$ ;  
 theref. the difference  $CE^2 - ce^2 = CD^2 + DE^2 - cd^2 - de^2$ .  
 consequently  $CA^2 - ca^2 = CE^2 - ce^2$ ;  
 or, by doubling,  $AB^2 - ab^2 = EG^2 - eg^2$ . Q. E. D.

THEOREM XII.

All the Parallelograms are equal which are formed between the Asymptotes and Curve, by Lines drawn Parallel to the Asymptotes.

That is, the lines GE, EK, AP, AQ being parallel to the asymptotes CH, cl; then the paral. CGEK = paral. CPAQ.



For, let A be the vertex of the curve, or extremity of the semi-transverse axis AC, perp. to which draw AL or Al, which will be equal to the semi-conjugate, by definition 19. Also, draw HEdeh parallel to Ll,

Then, by theor. 2,  $CA^2 : AL^2 :: CD^2 - CA^2 : DE^2$ ,  
 and, by parallels,  $CA^2 : AL^2 :: CD^2 : DH^2$ ;  
 theref. by subtract,  $CA^2 : AL^2 :: CA^2 : DH^2 - DE^2$  or  
 rect. HE . Eh;  
 confeq. the square  $AL^2 =$  the rect. HE . Eh.

But,

But, by sim. tri.  $PA : AL :: GE : EH$ ,  
 and, by the same,  $QA : Al :: EK : Eh$ ;  
 theref. by comp.  $PA . AQ : AL^2 :: GE . EK : HE . Eh$  ;  
 and, because  $AL^2 = HE . Eh$ , theref.  $PA . AQ = GE . EK$ .

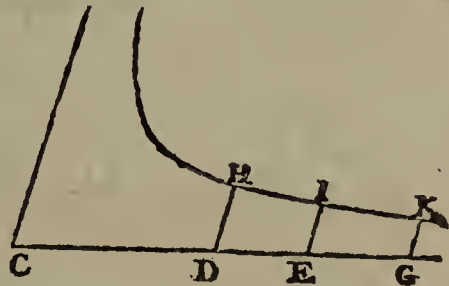
But the parallelograms  $CGEK$ ,  $CPAQ$ , being equiangular,  
 are as the rectangles  $GE . EK$  and  $PA . AQ$ .

Therefore the parallelogram  $GK =$  the paral.  $PQ$ .

That is, all the inscribed parallelograms are equal to one  
 another. Q. E. D.

*Corol. 1.* Because the rectangle  $GK$  or  $CGE$  is constant,  
 therefore  $GE$  is reciprocally as  $CG$ , or  $CG : CP :: PA : GE$ .  
 And hence the asymptote continually approaches towards the  
 curve, but never meets it : for  $GE$  decreases continually as  
 $CG$  increases ; and it is always of *some* magnitude, except  
 when  $CG$  is supposed to be infinitely great, for then  $GE$  is  
 infinitely small, or nothing. So that the asymptote  $CG$  may  
 be considered as a tangent to the curve at a point infinitely  
 distant from  $c$ .

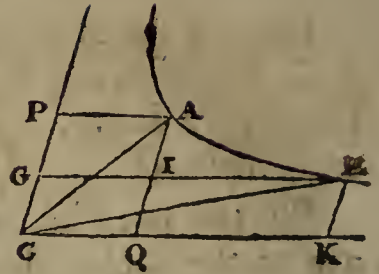
*Corol. 2.* If the abscisses  $CD$ ,  $CE$ ,  
 $CG$ , &c, taken on the one asymp-  
 tote, be in geometrical progression  
 increasing ; then shall the ordi-  
 nates  $DH$ ,  $EI$ ,  $GK$ , &c, parallel to  
 the other asymptote, be a decreas-  
 ing geometrical progression, hav-  
 ing the same ratio. For, all the  
 rectangles  $CDH$ ,  $CEI$ ,  $CGK$ , &c, being equal, the ordinates  
 $DH$ ,  $EI$ ,  $GK$ , &c, are reciprocally as the abscisses  $CD$ ,  $CE$ ,  $CG$ ,  
 &c, which are geometricals. And the reciprocals of geome-  
 tricals are also geometricals, and in the same ratio, but de-  
 creasing, or in converse order.



THEOREM XIII.

The three following Spaces, between the Asymptotes and  
 the Curve, are equal ; namely, the Sector or Trilinear  
 Space contained by an Arc of the Curve and two Radii,  
 or Lines drawn from its Extremities to the Centre ; and  
 each of the two Quadrilaterals, contained by the said Arc,  
 and two Lines drawn from its Extremities parallel to one  
 Asymptote, and the intercepted Part of the other Asymp-  
 tote.

That is,  
 The sector  $CAE = PAEG = QAEK$ ,  
 all standing on the same arc  $AE$ .



For, by theor. 12,  $CPAQ = CGEK$ ;  
 subtract the common space  $CGIQ$ ,  
 there remains the paral.  $PI =$  the par.  $IK$ ;  
 to each add the trilineal  $IAE$ ,  
 the sum is the quadr.  $PAEG = QAEK$ .

Again, from the quadrilateral  $CAEK$   
 take the equal triangles  $CAQ, CEK$ ,  
 and there remains the sector  $CAE = QAEK$ .  
 Therefore  $CAE = QAEK = PAEG$ .

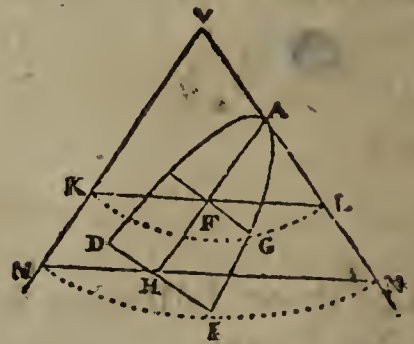
Q. E. D.

## OF THE PARABOLA.

### THEOREM I.

The Abscisses are Proportional to the Squares of their Ordinates,

LET  $AVM$  be a section through the axis of the cone, and  $AGIH$  a parabolic section by a plane perpendicular to the former, and parallel to the side  $VM$  of the cone; also let  $AFH$  be the common intersection of the two planes, or the axis of the parabola, and  $FG, HI$  ordinates perpendicular to it.



Then it will be, as  $AF : AH :: FG^2 : HI^2$ .

For, through the ordinates  $FG, HI$  draw the circular sections  $KGL, MIN$ , parallel to the base of the cone, having  $KL, MN$  for their diameters, to which  $FG, HI$  are ordinates, as well as to the axis of the parabola.

Then,

Then, by similar triangles,  $AF : AH :: FL : HN$  ;  
 but, because of the parallels,  $KF = MH$  ;  
 therefore  $AF : AH :: KF \cdot FL : MH \cdot HN$ .  
 But, by the circle,  $KF \cdot FL = FG^2$ , and  $MH \cdot HN = HI^2$  ;  
 Therefore  $AF : AH :: FG^2 : HI^2$ . Q. E. D.

*Corol.* Hence the third proportional  $\frac{FG^2}{AF}$  or  $\frac{HI^2}{AH}$  is a constant quantity, and is equal to the parameter of the axis by defin. 16.

Or  $AF : FG :: FG : P$  the parameter.  
 Or the rectangle  $P \cdot AF = FG^2$ .

THEOREM II.

As the Parameter of the Axis :  
 Is to the Sum of any Two Ordinates ::  
 So is the Difference of those Ordinates :  
 To the Difference of their Abscisses :

That is,

$$P : GH + DE :: GH - DE : DG,$$

Or,

$$P : KI :: IH : IE.$$



FOR, by cor. theor. I,  $P \cdot AG = GH^2$ ,  
 and  $P \cdot AD = DE^2$  ;  
 theref. by subtraction,  $P \cdot DG = GH^2 - DE^2$ .  
 Or,  $P \cdot DG = KI \cdot IH$ ,  
 therefore  $P : KI :: IH : DG$  or  $IE$ . Q. E. D.

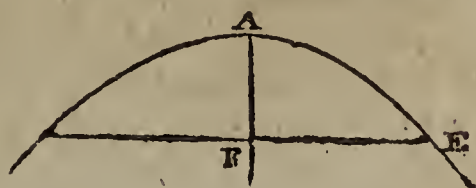
*Corol.* Hence, because  $P \cdot EI = KI \cdot IH$ ,  
 and, by cor. theor. I,  $P \cdot AG = GH^2$ ,  
 therefore  $AG : EI :: GH^2 : KI \cdot IH$ .

So that any diameter EI is as the rectangle of the segments KI, IH of the double ordinate KH.

THEOREM III.

The Distance from the Vertex to the Focus is equal to  $\frac{1}{4}$  of the Parameter, or to Half the Ordinate at the Focus.

That is,  
 $AF = \frac{1}{2}FE = \frac{1}{4}P$ ,  
 where F is the focus.



I 2 For,

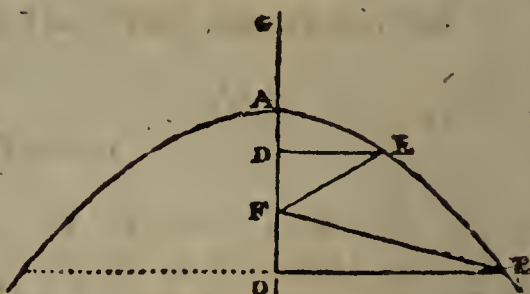
For, the general property is  $AF : FE :: FE : P$ .

But, by definition 17,  $FE = \frac{1}{2}P$ ;  
therefore also  $AF = \frac{1}{2}FE = \frac{1}{4}P$ . Q. E. D.

#### THEOREM IV.

A Line drawn from the Focus to any Point in the Curve, is equal to the Sum of the Focal Distance and the Absciss of the Ordinate to that Point.

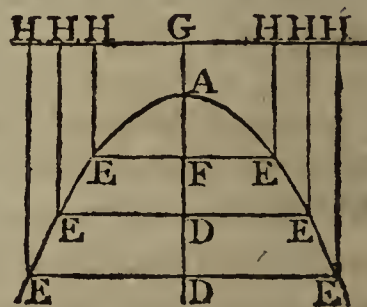
That is,  
 $FE = FA + AD = GD$ ,  
taking  $AG = AF$ .



FOR, since  $FD = AD \simeq AF$ ,  
theref. by squaring,  $FD^2 = AF^2 - 2AF \cdot AD + AD^2$ ,  
But, by cor. theor. I,  $DE^2 = P \cdot AD = 4AF \cdot AD$ ;  
theref. by addition,  $FD^2 + DE^2 = AF^2 + 2AF \cdot AD + AD^2$ ,  
But, by right-ang. tri.  $FD^2 + DE^2 = FE^2$ ;  
therefore  $FE^2 = AF^2 + 2AF \cdot AD + AD^2$ ,  
and the root or side is  $FE = AF + AD$ ,  
or  $FE = GD$ , by taking  $AG = AF$ .

Q. E. D.

*Corol. 1.* If, through the point G, the line HG be drawn perpendicular to the axis, it is called the directrix of the parabola. The property of which, from this theorem, it appears, is this: That drawing any lines HE parallel to the axis, HE is always equal to FE the distance of the focus from the point E.



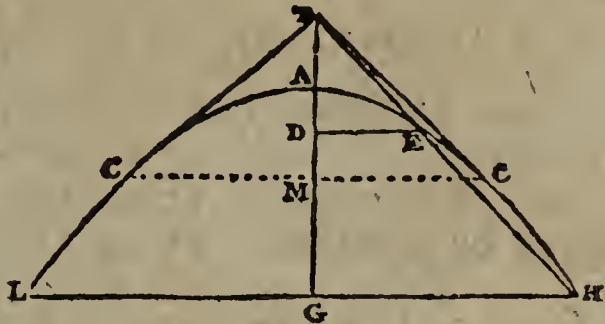
*Corol. 2.* Hence also the curve is easily described by points. Namely, in the axis produced take  $AG = AF$  the focal distance, and draw a number of lines EE perpendicular to the axis AD; then with the distances GD, GD, GD, &c, as radii, and the centre F, draw arcs crossing the parallel ordinates in E, E, E, &c. Then draw the curve through all the points E, E, E.



THEOREM V.

If a Tangent be drawn to any Point of the Parabola, meeting the Axis produced; and if an Ordinate to the Axis be drawn from the Point of Contact; then the Abscifs of that Ordinate will be equal to the External Part of the Axis.

That is,  
if TC touch the curve  
at the point C;  
then is  $AT = AM$ .



FOR, from the point T, draw any line cutting the curve in the two points E, H: to which draw the ordinates DE, GH; also draw the ordinate MC to the point of contact C.

Then, by th. I,  $AD : AG :: DE^2 : GH^2$ ;  
and, by sim. tri.  $TD^2 : TG^2 :: DE^2 : GH^2$ ;  
theref. by equality,  $AD : AG :: TD^2 : TG^2$ ;  
and, by division,  $AD : DG :: TD^2 : TG^2 - TD^2$  or  $DG \cdot TD + TG$ ,  
or  $AD : TD :: TD : TD + TG$ ;  
and, by division,  $AD : AT :: TD : TG$ ,  
and again by div.  $AD : AT :: AT : AG$ ;  
or  $AT$  is a mean propor. between  $AD, AG$ .

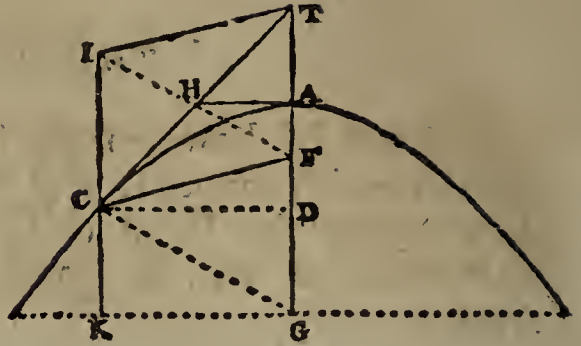
Now, if the line TH be supposed to revolve about the point T; then, as it recedes farther from the axis, the points E and H approach towards each other, the point E descending, and the point H ascending, till at last they meet in the point C, when the line becomes a tangent to the curve at C. And then the points D and G meet in the point M, and the ordinates DE, GH in the ordinate CM. Consequently AD, AG, becoming each equal to AM, their mean proportional AT will be equal to the abscifs AM. That is, the external part of the axis, cut off by a tangent, is equal to the abscifs of the ordinate to the point of contact. Q. E. D.

THEOREM VI.

If a tangent to the Curve meet the Axis produced; then the Line drawn from the Focus to the Point of Contact, will be equal to the Distance of the Focus from the Intersection of the Tangent and Axis.

That

That is,  
 $FC = FT.$



FOR, draw the ordinate DC to the point of contact c.  
 Then, by theor. 5,  $AT = AD$ ;  
 therefore  $FT = AF + AD.$   
 But, by theor. 4,  $FC = AF + AD$ ;  
 theref. by equality,  $FC = FT.$  Q. E. D.

*Corol. 1.* If CG be drawn perpendicular to the curve, or to the tangent, at c; then shall  $FG = FC = FT.$

For, draw FH perpendicular to TC, which will also bisect TC, because  $FT = FC$ ; and therefore, by the nature of the parallels, FH also bisects TG in F. And consequently  $FG = FT = FC.$

So that F is the centre of a circle passing through T, C, G,

*Corol. 2.* The tangent at the vertex AH, is a mean proportional between AF and AD.

For, because FHT is a right angle, therefore - AH is a mean between AF, AT, or between AF, AD, because  $AD = AT,$   
 Likewise, FH is a mean between FA, FT, or between FA, FC.

*Corol. 3.* The tangent TC makes equal angles with FC and the axis FT.

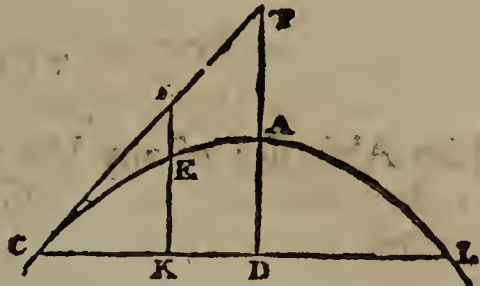
For, because  $FT = FC,$  therefore the  $\angle FCT = \angle FTC.$   
 Also, the angle  $GCF =$  the angle  $GCK,$  drawing  $ICK$  parallel to the axis AG.

*Corol. 4.* And because the angle of incidence  $GCK$  is = the angle of reflection  $GCF$ ; therefore a ray of light falling on the curve in the direction KC, will be reflected to the focus F. That is, all rays parallel to the axis, are reflected to the focus, or burning point.

THEOREM VII.

If there be any Tangent, and a Double Ordinate drawn from the Point of Contact, and also any Line parallel to the Axis, limited by the Tangent and Double Ordinate: then shall the Curve divide that Line in the same Ratio, as the Line divides the Double Ordinate.

That is,  
 $IE : EK :: CK : KL.$

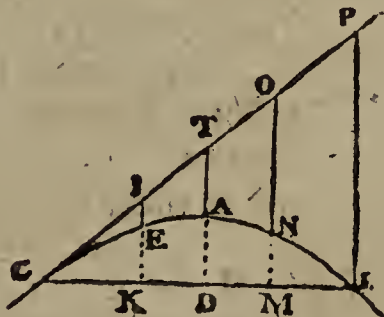


FOR, by sim. triangles,  $CK : KI :: CD : DT$  or  $2DA$  ;  
 but, by the def. the param.  $P : CL :: CD : 2DA$  ;  
 therefore, by equality,  $P : CK :: CL : KI.$   
 But, by theor. 2,  $P : CK :: KL : KE$  ;  
 therefore, by equality,  $CL : KL :: KI : KE$  ;  
 and, by division,  $CK : KL :: IE : EK.$  Q. E. D

THEOREM VIII.

The same being supposed as in theor. 7 ; then shall the External Part of the Line between the Curve and Tangent, be proportional to the Square of the intercepted Part of the Tangent, or to the Square of the intercepted Part of the Double Ordinate.

That is,  $IE$  is as  $CI^2$  or as  $CK^2$ .  
 and  $IE, TA, ON, PL, \&c,$   
 are as  $CI^2, CT^2, CO^2, CP^2, \&c,$   
 or as  $CK^2, CD^2, CM^2, CL^2, \&c.$



FOR, by theor. 7,  $IE : EK :: CK : KL,$   
 or, by equality,  $IE : EK :: CK^2 : CK . KL.$   
 But, by cor. th. 2,  $EK$  is as the rect.  $CK . KL,$   
 therefore  $IE$  is as  $CK^2,$  or as  $CI^2.$  Q. E. D.

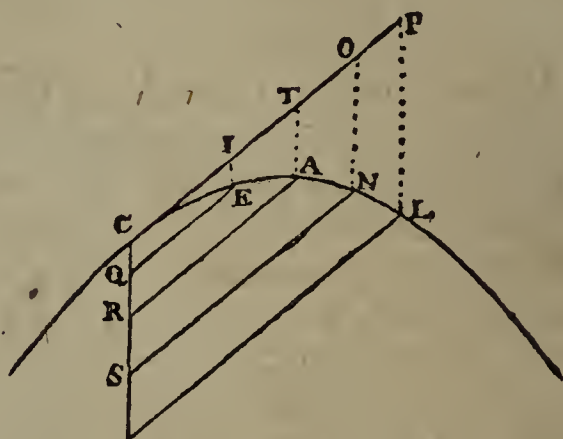
Corol

*Corol.* As this property is common to every position of the tangent, if the lines  $IE$ ,  $TA$ ,  $ON$ , &c, be appended to the points  $I$ ,  $T$ ,  $O$ , &c, and moveable about them, and of such lengths as that their extremities  $E$ ,  $A$ ,  $N$ , &c, be in the curve of a parabola in some one position of the tangent; then making the tangent revolve about the point  $C$ , it appears that the extremities  $E$ ,  $A$ ,  $N$ , &c, will always form the curve of some parabola, in every position of the tangent.

## THEOREM IX.

The Abscisses of any Diameter, are as the Squares of their Ordinates.

That is,  $CQ$ ,  $CR$ ,  $CS$ , &c,  
are as  $QE^2$ ,  $RA^2$ ,  $SN^2$ , &c.  
Or -  $CQ : CR :: QE^2 : RA^2$ ,  
&c.



FOR, draw the tangent  $CT$ , and the externals  $EI$ ,  $AT$ ,  $NO$ , &c, parallel to the axis, or to the diameter  $CS$ .

Then, because the ordinates  $QE$ ,  $RA$ ,  $SN$ , &c, are parallel to the tangent  $CT$ , by the definition of them, therefore all the figures  $IQ$ ,  $TR$ ,  $OS$ , &c, are parallelograms, whose opposite sides are equal,

namely,  $IE$ ,  $TA$ ,  $ON$ , &c,  
are equal to  $CQ$ ,  $CR$ ,  $CS$ , &c.

Therefore, by theor. 8,  $CQ$ ,  $CR$ ,  $CS$ , &c,  
are as -  $CI^2$ ,  $CT^2$ ,  $CO^2$ , &c,  
or as their equals  $QE^2$ ,  $RA^2$ ,  $SN^2$ , &c,

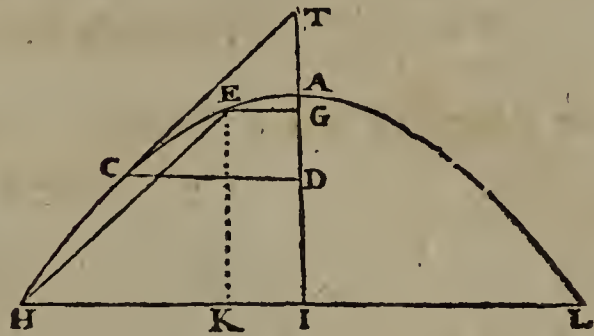
Q. E. D.

*Corol.* Here, like as in theor. 2, the difference of the abscisses is as the difference of the squares of their ordinates, or as the rectangles under the sum and difference of the ordinates, the rectangle under the sum and difference of the ordinates being equal to the rectangle under the difference of the abscisses and the parameter of that diameter, or a third proportional to any absciss and its ordinate,

THEOREM X.

If a Line be drawn parallel to any Tangent, and cut the Curve in two Points; then if two Ordinates be drawn to the Intersections, and a third to the Point of Contact, these three Ordinates will be in Arithmetical Progression, or the Sum of the Extremes will be equal to Double the Mean.

That is,  
 $EG + HI = 2CD.$

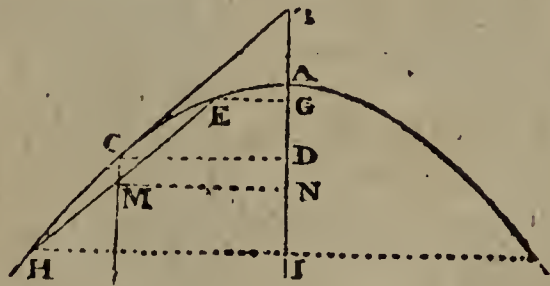


FOR, draw EK parallel to the axis, and produce HI to L.  
 Then, by sim. triangles,  $EK : HK :: TD \text{ or } 2AD : CD;$   
 but, by theor. 2,  $EK : HK :: KL : P \text{ the param.}$   
 theref. by equality,  $2AD : KL :: CD : P.$   
 But, by the defin.  $2AD : 2CD :: CD : P;$   
 theref. the 2d terms are equal,  $KL = 2CD,$   
 that is,  $EG + HI = 2CD.$  Q. E. D.

THEOREM XI.

Any Diameter bisects all its Double Ordinates, or Lines parallel to the Tangent at its Vertex.

That is,  
 $ME = MH.$



FOR, to the axis AI draw the ordinates EG, CD, HI, and MN parallel to them, which is equal to CD.  
 Then, by theor. 10,  $2MN \text{ or } 2CD = EG + HI,$   
 therefore M is the middle of EH.  
 And, for the same reason, all its parallels are bisected.

Q. E. D.

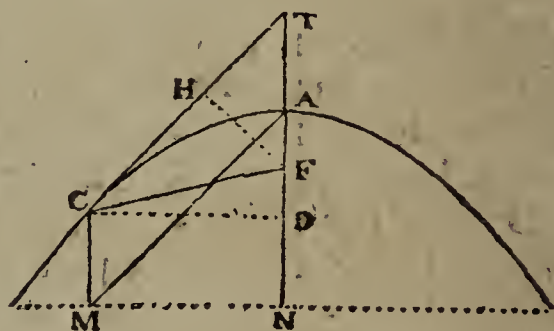
SCHOL.

SCHOL. Hence, as the absciffes of any diameter and their ordinates have the same relations as those of the axis, namely, that the ordinates are bisected by the diameter, and their squares proportional to the absciffes; so all the other properties of the axis and its ordinates and absciffes, before demonstrated, will likewise hold good for any diameter and its ordinates and absciffes. And also those of the parameters, understanding the parameter of any diameter, as a third proportional to any absciff and its ordinate. Some of the most material of which are demonstrated in the following theorems:

## THEOREM XII.

The Parameter of any Diameter is equal to four Times the Line drawn from the Focus to the Vertex of that Diameter.

That is,  $4FC = p$ ,  
the param. of the diam. CM.



FOR, draw the ordinate MA parallel to the tangent CT: as also CD, MN perpendicular to the axis AN, and FH perpendicular to the tangent CT.

Then the absciffes AD, CM or AT being equal, by theor. 5, the parameters will be as the squares of the ordinates CD, MA or CT, by the definition;

$$\text{that is, } p : p :: CD^2 : CT^2,$$

But, by sim. tri.  $FH : FT :: CD : CT$ ;  
therefore  $p : p :: FH^2 : FT^2$ .

But, by cor. 2, th. 6,  $FH^2 = FA \cdot FT$ ;  
therefore  $p : p :: FA \cdot FT : FT^2$ ,

or, by equality,  $p : p :: FA : FT$  or FC.

But, by theor. 3,  $p = 4FA$ ,  
and therefore  $p = 4FT$  or  $4FC$ .

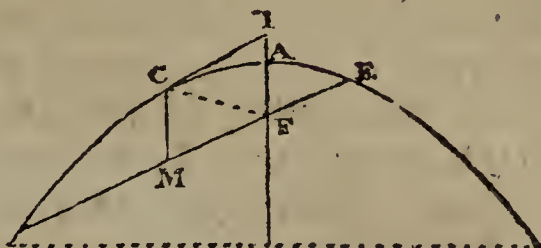
Q. E. D.

*Corol.* Hence the parameter  $p$  of the diameter CM is equal to  $4FA + 4AD$ , or to  $p + 4AD$ , that is, the parameter of the axis added to  $4AD$ ,

THEOREM XIII.

If an Ordinate to any Diameter, pass through the Focus, it will be equal to Half its Parameter; and its Absciss equal to One Fourth of the same Parameter.

That is,  $CM = \frac{1}{4}p$ ,  
and  $ME = \frac{1}{2}p$ .



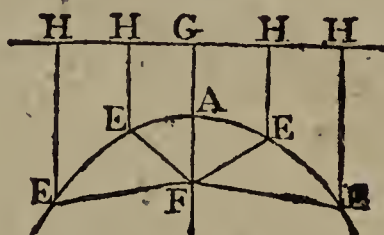
FOR, join FC, and draw the tangent CT.

By the parallels,  $CM = FT$ ;  
and, by theor. 6,  $FC = FT$ ;  
also, by theor. 12,  $FC = \frac{1}{4}p$ ;  
therefore  $CM = \frac{1}{4}p$ .

Again, by the defin.  $CM$  or  $\frac{1}{4}p : ME :: ME : p$ ,  
and consequently  $ME = \frac{1}{2}p = 2CM$ .

*Corol. 1.* Hence, of any diameter, the double ordinate which passeth through the focus, is equal to the parameter, or to quadruple its absciss.

*Corol. 2.* Hence, and from cor. 1 to theor. 4, and theor. 6 and 12, it appears, that if the directrix GH be drawn, and any lines HE, HE, parallel to the axis; then every parallel HE will be equal to EF, or  $\frac{1}{4}$  of the parameter of the diameter to the point E.

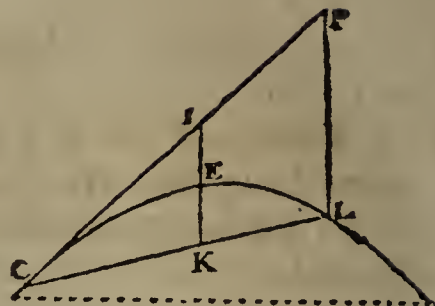


THEOREM XIV.

If there be a Tangent, and any Line drawn from the Point of Contact and meeting the Curve in some other Point, as also another Line parallel to the Axis, and limited by the First Line and the Tangent: then shall the Curve divide this Second Line in the same Ratio, as the Second Line divides the First Line.

That

That is,  
 $IE : EK :: CK : KL.$



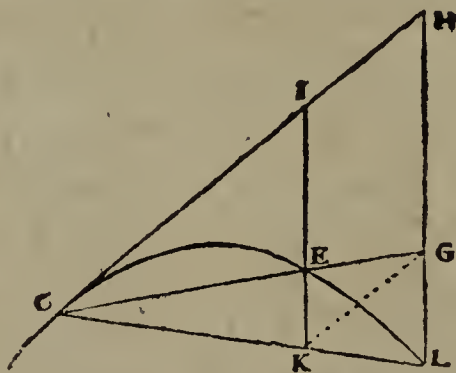
FOR, draw LP parallel to IK, or to the axis.

Then, by theor. 8,  $IE : PL :: CI^2 : CP^2,$   
 or, by sim. tri.  $IE : PL :: CK^2 : CL^2.$   
 Also, by sim. tri.  $IK : PL :: CK : CL,$   
 or  $IK : PL :: CK^2 : CK.CL;$   
 therefore by equality,  $IE : IK :: CK.CL : CL^2;$   
 or  $IE : IK :: CK : CL;$   
 and, by division,  $IE : EK :: CK : KL. \quad Q. E. D.$

Corol. When  $CK = KL,$  then  $IE = EK = \frac{1}{2}IK.$

THEOREM XV.

If from any Point of the Curve there be drawn a Tangent, and also Two Right Lines to cut the Curve; and Diameters be drawn through the Points of Interfection E and L, meeting those Two Right Lines in two other Points G and K: Then will the Line KG joining these last Two Points be parallel to the Tangent.



FOR, by theor. 14,  $CK : KL :: EI : EK;$   
 and by comp.  $CK : CL :: EI : KI;$   
 and by the parellels  $GH : LH;$   
 But, by sim. tri.  $CK : CL :: KI : LH;$   
 theref. by equal.  $KI : LH :: GH : LH:$   
 consequently  $KI = GH,$   
 and therefore  $KG$  is parallel and equal to  $IH. \quad Q. E. D.$



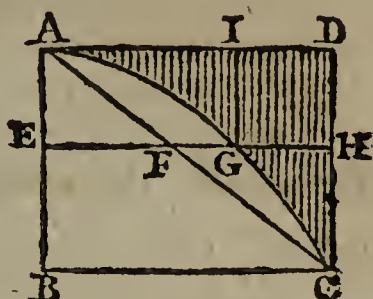
THEOREM XVI.

If a Rectangle be described about a Parabola, having the same Base and Altitude; and a diagonal Line be drawn from the Vertex to the Extremity of the Base of the Parabola, forming a right-angled Triangle, of the same Base and Altitude also; then any Line or Ordinate drawn across the three Figures, perpendicular to the Axis, will be cut in Continual Proportion by the Sides of those Figures.

That is,

$$EF : EG :: EG : EH,$$

Or, EF, EG, EH, are in continued proportion.



For, by theor. I,  $AB : AE :: BC^2 : EG^2$ ,  
 and, by sim. tri.  $AB : AE :: BC : EF$ ,  
 theref. of equality,  $EF : BC :: EG^2 : BC^2$ ,  
 that is,  $EF : EH :: EG^2 : EH^2$ ,  
 theref. by Geom. th. 78, EF, EG, EH are proportionals,  
 or  $EF : EG :: EG : EH$ . Q. E. D.

THEOREM XVII.

The Area or Space of a Parabola, is equal to Two-Thirds of its Circumscribing Parallelogram.

That is, the space ABCGA =  $\frac{2}{3}$ ABCD;  
 or, the space ADCGA =  $\frac{1}{3}$ ABCD.

FOR, conceive the space ADCGA to be composed of, or divided into, indefinitely small parts, by lines parallel to DC or AB, such as IG, which divide AD into like small and equal parts, the number or sum of which is expressed by the line AD. Then,

by the parabola,  $BC^2 : EG^2 :: AB : AE$ ,  
 that is,  $AD^2 : AI^2 :: DC : IG$ .

Hence it follows, that any one of these narrow parts, as IG, is =  $\frac{DC}{AD^2} \times AI^2$ , whence, AD and DC being given or constant quantities, it appears that the said parts IG, &c, are proportional to  $AI^2$ , &c, or proportional to a series of square numbers,

numbers, whose roots are in arithmetical progression, and the area  $ADCGA$  equal to  $\frac{DC}{AD^2}$  drawn into the sum of such a series of arithmeticals, the number of which is expressed by  $AD$ .

Now, by the remark at pag. 233, vol. i, the sum of the squares of such a series of arithmeticals, is expressed by  $\frac{1}{6}n \cdot n + 1 \cdot 2n + 1$ , where  $n$  denotes the number of them. In the present case,  $n$  represents an infinite number, and then the subfactors  $n + 1$ ,  $2n + 1$ , become only  $n$  and  $2n$ , omitting the 1 as inconsiderable in respect of the infinite number  $n$ : hence the expression above becomes barely  $\frac{1}{6}n \cdot n \cdot 2n = \frac{1}{3}n^3$ .

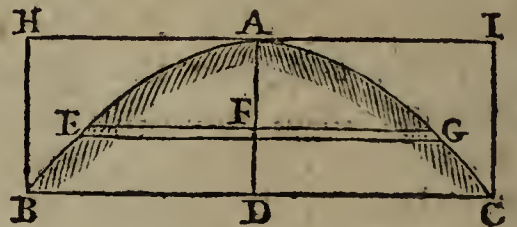
To apply this to the case above:  $n$  will denote  $AD$  or  $BC$ ; and the sum of all the  $AI^2$ 's becomes  $\frac{1}{3}AD^3$  or  $\frac{1}{3}BC^3$ ; consequently the sum of all the  $\frac{DC}{AD^2} \times AI^2$ 's, is  $\frac{DC}{AD^2} \times \frac{1}{3}AD^3 = \frac{1}{3}AD \cdot DC = \frac{1}{3}BD$ , which is the area of the exterior part  $ADCGA$ . That is, the said exterior part  $ADCGA$ , is  $\frac{1}{3}$  of the parallelogram  $ABCD$ ; and consequently the interior part  $ABCGA$  is  $\frac{2}{3}$  of the same parallelogram. Q. E. D.

*Corol.* The part  $AFCGA$ , inclosed between the curve and the right line  $AFC$ , is  $\frac{1}{6}$  of the same parallelogram.

THEOREM XVIII.

The Solid Content of a Paraboloid (or Solid generated by the Rotation of a Parabola about its Axis), is Half its Circumscribing Cylinder.

LET  $ABC$  be a paraboloid, generated by the rotation of the parabola  $AC$  about its axis  $AD$ . Suppose the axis  $AD$  be divided into an infinite number of equal parts, through which let circular planes pass, as  $EFG$ , all those circles making up the whole solid paraboloid.



Then, if  $c =$  the number 3.1416, then  $2c \times FG$  is the circumference of the circle  $EFG$  whose radius is  $FG$ ; therefore  $c \times FG^2$  is the area of that circle.

But, by cor. theor. I, Parabola,  $p \times AF = FG^2$ , where  $p$  denotes the parameter of the parabola; consequently  $pc \times AF$  will

will also express the same circular section EG, and therefore  $pc \times$  the sum of all the AF's will be the sum of all those circular sections, or the whole content of the solid paraboloid.

But all the AF's form an arithmetical progression, beginning at 0 or nothing, and having the greatest term and the sum of all the terms each expressed by the whole axis AD. And since the sum of all the terms of such a progression, is equal to  $\frac{1}{2}AD \times AD$  or  $\frac{1}{2}AD^2$ , half the product of the greatest term and the number of terms; therefore  $\frac{1}{2}AD^2$  is equal to the sum of all the AF's, and consequently,  $pc \times \frac{1}{2}AD^2$ , or  $\frac{1}{2}c \times p \times AD^2$ , is the sum of all the circular sections, or the content of the paraboloid.

But, by the parabola,  $p : DC :: DC : AD$  or  $p = \frac{DC^2}{AD}$ ; consequently  $\frac{1}{2}c \times p \times AD^2$  becomes  $\frac{1}{2}c \times AD \times DC^2$  for the solid content of the paraboloid. But  $c \times AD \times DC^2$  is equal to the cylinder BCIH; consequently the paraboloid is the half of its circumscribing cylinder. Q. E. D.

## THEOREM XIX.

The Solidity of the Frustrum BEGC of the Paraboloid, is equal to a Cylinder whose Height is DF, and its Base Half the Sum of the two Circular Bases EG, BC.

FOR, by last theor.  $\frac{1}{2}pc \times AD^2 =$  the solid ABC,  
and  $\frac{1}{2}pc \times AF^2 =$  the solid AEG,  
theref. the dif.  $\frac{1}{2}pc \times (AD^2 - AF^2) =$  the frust. BEGC.

But  $AD^2 - AF^2 = DF \times (AD + AF)$ ,  
theref.  $\frac{1}{2}pc \times DF \times (AD + AF) =$  the frust. BEGC.

But, by the parab.  $p \times AD = DC^2$ , and  $p \times AF = FG^2$ ;  
theref.  $\frac{1}{2}c \times DF \times (DC^2 + FG^2) =$  the frust. BEGC.

Q. E. D.

## OF MOTION, FORCES, &amp;c.

## DEFINITIONS.

*Art. 1.* BODY is the mass, or quantity of matter, in any material substance; and it is always proportional to its weight or gravity, whatever its figure may be.

2. Body is either Hard, Soft, or Elastic. A Hard Body is that whose parts do not yield to any stroke or percussion, but retains its figure unaltered. A Soft Body is that whose parts yield to any stroke or impression, without restoring themselves again; the figure of the body remaining altered. And an Elastic Body is that whose parts yield to any stroke, but which presently restore themselves again, and the body regains the same figure as before the stroke.

We know of no bodies that are absolutely, or perfectly, either hard, soft, or elastic; but all partaking these properties, more or less, in some intermediate degree.

3. Bodies are also either Solid or Fluid. A Solid Body, is that whose parts are not easily moved amongst one another, and which retains any figure you give it. But a Fluid Body is that whose parts yield to the slightest impression, being easily moved amongst one another; and its surface, when left to itself, is always observed to settle in a smooth plane at the top.

4. Density is the proportional weight or quantity of matter in any body.

So, in two spheres, or cubes, &c. of equal size or magnitude; if the one weigh only one pound, but the other two pounds; then the density of the latter is double the density of the former; if it weigh three pounds, its density is triple, and so on.

5. Motion is a continual and successive change of place.— If the body move equally, or pass over equal spaces in equal times, it is called Equable or Uniform Motion. But if it increase or decrease, it is Variable Motion; and it is called Accelerated Motion in the former case, and Retarded Motion in the latter.—Also, when the body moved is considered with  
respect

respect to some other body at rest, it is said to be Absolute Motion. But when compared with others in motion, it is called Relative Motion.

6. Velocity, or Celerity, is an affection of motion, by which a body passes over a certain space in a certain time. Thus, if a body in motion pass uniformly over 40 feet in 4 seconds of time, it is said to move with the velocity of 10 feet per second; and so on.

7. Momentum, or Quantity of Motion, is the power or force incident to moving bodies, by which they continually tend from their present places, or with which they strike any obstacle that opposes their motion.

8. Force is a power exerted on a body to move it, or to stop it. If the force act constantly, or incessantly, it is a Permanent Force: like pressure or the force of gravity. But if it act instantaneously, or but for an imperceptibly small time, it is called Impulse, or Percussion: like the smart blow of a hammer.

9. Forces are also distinguished into Motive, and Accelerative or Retarding. A Motive or Moving Force, is the power of an agent to produce motion; and it is equal or proportional to the momentum it will generate in any body, when acting, either by percussion, or for a certain time as a permanent force.

10. Accelerative, or Retardive Force, is commonly understood to be that which affects the velocity only; or it is that by which the velocity is accelerated or retarded; and it is equal or proportional to the motive force directly and to the mass or body moved inversely.—So, if a body of 2 pounds weight, be acted on by a motive force of 40; then the accelerating force is 20. But if the same force of 40 act on another body of 4 pounds weight; then the accelerating force in this latter case is only 10; and so is but half the former.

11. Gravity, or Weight, is that force by which a body endeavours to fall downwards. It is called Absolute Gravity, when the body is in empty space; and Relative Gravity, when immersed in a fluid.

12. Specific Gravity is the proportion of the weights of different bodies of equal magnitude; and so is proportional to the density of the body.

## A X I O M S.

13. EVERY body naturally endeavours to continue in its present state, whether it be at rest, or moving uniformly in a right line.

14. The Change or Alteration of Motion, by any external force, is always proportional to that force, and in the direction of the right line in which it acts.

15. Action and Re-action, between any two bodies, are equal and contrary. That is, by Action and Re-action, equal changes of motion are produced in bodies acting on each other; and these changes are directed towards opposite or contrary parts.

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 GENERAL LAWS OF MOTION, &c.

## PROPOSITION I.

16. *The Quantity of Matter, in all Bodies, is in the Compound Ratio of their Magnitudes and Densities.*

THAT is,  $b$  is as  $md$ ; where  $b$  denotes the body or quantity of matter,  $m$  its magnitude, and  $d$  its density.

For, by art. 4, in bodies of equal magnitude, the mass or quantity of matter is as the density. But, the densities remaining, the mass is as the magnitude; that is, a double magnitude contains a double quantity of matter, a triple magnitude a triple quantity, and so on. Therefore the mass is in the compound ratio of the magnitude and density.

17. *Corol. 1.* In similar bodies, the masses are as the densities and cubes of the diameters, or of any like linear dimensions.—For the magnitudes of bodies are as the cubes of the diameters, &c.

18. *Corol. 2.* The masses are as the magnitudes and specific gravities.—For, by art. 4 and 12, the densities of bodies are as the specific gravities.

19. *Scholium.* Hence, if  $b$  denote any body, or the quantity of matter in it,  $m$  its magnitude,  $d$  its density,  $g$  its specific gravity, and  $a$  its diameter or other dimension; then,  $\propto$  (pronounced or named *as*) being the mark for general proportion, from this proposition and its corollaries we have these general proportions:

$$\begin{aligned} b &\propto md \propto mg \propto a^3 d, \\ m &\propto \frac{b}{d} \propto \frac{b}{g} \propto a^3, \\ d &\propto \frac{b}{m} \propto g \propto \frac{mg}{a^3}, \\ a^3 &\propto \frac{b}{d} \propto m \propto \frac{mg}{d}. \end{aligned}$$

PROPOSITION II.

20. *The Momentum, or Quantity of Motion, generated by a Single Impulse, or any Momentary Force, is as the Generating Force.*

THAT is,  $m$  is as  $f$ ; where  $m$  denotes the momentum, and  $f$  the force.

For every effect is proportional to its adequate cause. So that a double force will impress a double quantity of motion; a triple force, a triple motion; and so on. That is, the motion impressed, is as the motive force which produces it.

PROPOSITION III.

21. *The Momenta, or Quantities of Motion, in Moving Bodies, are in the Compound Ratio of the Masses and Velocities.*

That is,  $m$  is as  $bv$ .

FOR, the motion of any body being made up of the motions of all its parts, if the velocities be equal, the momenta will be as the masses; for a double mass will strike with a double force; a triple mass, with a triple force; and so on. Again, when the mass is the same, it will require a double force to move it with a double velocity, a triple force with a triple velocity, and so on; that is, the motive force is as the velocity; but the momentum impressed, is as the force which produces it, by prop. 2; and therefore the momentum is as the velocity when the mass is the same. But the momentum was found to be as the mass when the velocity is the same.

Consequently, when neither are the same, the momentum is in the compound ratio of both the mass and velocity.

## PROPOSITION IV.

22. *In Uniform Motions, the Spaces described are in the Compound Ratio of the Velocities and the Times of their Description.*

That is,  $s$  is as  $tv$ .

FOR, by the nature of uniform motion, the greater the velocity, the greater is the space described in any one and the same time; that is, the space is as the velocity, when the times are equal. And when the velocity is the same, the space will be as the time; that is, in a double time a double space will be described; in a triple time, a triple space; and so on. Therefore universally, the space is in the compound ratio of the velocity, and the time of description.

23. *Corol. 1.* In uniform motions, the time is as the space directly, and velocity reciprocally; or as the space divided by the velocity. And when the velocity is the same, the time is as the space. But when the space is the same, the time is reciprocally as the velocity.

24. *Corol. 2.* The velocity is as the space directly and the time reciprocally; or as the space divided by the time. And when the time is the same, the velocity is as the space. But when the space is the same, the velocity is reciprocally as the time.

*Scholium.*

25. In uniform motions generated by momentary impulse, let  $b$  = any body or quantity of matter to be moved,  
 $f$  = force of impulse acting on the body  $b$ ,  
 $v$  = the uniform velocity generated in  $b$ ,  
 $m$  = the momentum generated in  $b$ ,  
 $s$  = the space described by the body  $b$ ,  
 $t$  = the time of describing the space  $s$  with the veloc.  $v$ .

Then from the last three propositions and corollaries, we have these three general proportions, namely,  $f \propto m$ ,  $m \propto bv$ , and  $s \propto tv$ ; from which is derived the following table of the general relations of those six quantities, in uniform motions, and impulsive or percussive forces:

$$f \propto m$$



$$f \propto m \propto bv \propto \frac{bs}{t}.$$

$$m \propto f \propto bv \propto \frac{bs}{t}.$$

$$b \propto \frac{f}{v} \propto \frac{m}{v} \propto \frac{ft}{s} \propto \frac{mt}{s}.$$

$$s \propto tv \propto \frac{tf}{b} \propto \frac{tm}{b}.$$

$$v \propto \frac{s}{t} \propto \frac{f}{b} \propto \frac{m}{b}.$$

$$t \propto \frac{s}{v} \propto \frac{bs}{f} \propto \frac{bs}{m}.$$

By means of which, may be resolved all questions relating to uniform motions, and the effects of momentary or impulsive forces.

## PROPOSITION V.

26. *The Momentum generated by a Constant and Uniform Force, acting for any Time, is in the Compound Ratio of the Force and Time of Acting.*

That is,  $m$  is as  $ft$ .

FOR, supposing the time divided into very small parts, by prop. 2, the momentum in each particle of time is the same, and therefore the whole momentum will be as the whole time, or sum of all the small parts. But, by the same prop. the momentum for each small time, is also as the motive force. Consequently the whole momentum generated, is in the compound ratio of the force and time of acting.

27. *Corol. 1.* The motion, or momentum, lost or destroyed in any time, is also in the compound ratio of the force and time. For whatever momentum any force generates in a given time; the same momentum will an equal force destroy in the same or equal time; acting in a contrary direction.

And the same is true of the increase or decrease of motion, by forces that conspire with, or oppose, the motion of bodies.

28. *Corol. 2.* The velocity generated, or destroyed, in any time, is directly as the force and time, and reciprocally as the body or mass of matter.—For, by this and the 3d prop. the compound ratio of the body and velocity, is as that of the force and time; and therefore the velocity is as the force and time divided by the body. And if the body and force be given, or constant, the velocity will be as the time.

## PROPOSITION VI.

29. *The Spaces passed over by Bodies, urged by any Constant and Uniform Forces, acting during any Times, are in the Compound Ratio of the Forces and Squares of the Times directly, and the Body or Mass reciprocally.*

*Or, the Spaces are as the Squares of the Times, when the Force and Body are given.*

THAT is,  $s$  is as  $\frac{ft^2}{b}$ , or as  $t^2$  when  $f$  and  $b$  are given.

For, let  $v$  denote the velocity acquired at the end of any time  $t$ , by any given body  $b$ , when it has passed over the space  $s$ . Then, because the velocity is as the time, by the last corol. therefore  $\frac{1}{2}v$  is the velocity at  $\frac{1}{2}t$ , or at the middle point of the time; and as the increase of velocity is uniform, the same space  $s$  will be described in the same time  $t$ , by the velocity  $\frac{1}{2}v$  uniformly continued from beginning to end. But, in uniform motions, the space is in the compound ratio of the time and velocity; therefore  $s$  is as  $\frac{1}{2}tv$ , or indeed  $s =$

$\frac{1}{2}tv$ . But, by the last corol. the velocity  $v$  is as  $\frac{ft}{b}$ , or as the force and time directly, and as the body reciprocally.

Therefore  $s$ , or  $\frac{1}{2}tv$ , is as  $\frac{ft^2}{b}$ , that is, the space is as the force and square of the time directly, and as the body reciprocally. Or  $s$  is as  $t^2$ , the square of the time only, when  $b$  and  $f$  are given.

30. *Corol. 1.* The space  $s$  is also as  $tv$ , or in the compound ratio of the time and velocity;  $b$  and  $f$  being given. For,  $s = \frac{1}{2}tv$  is the space actually described. But  $tv$  is the space which might be described in the same time  $t$ , with the last velocity  $v$ , if it were uniformly continued for the same or an equal time. Therefore the space  $s$ , or  $\frac{1}{2}tv$ , which is actually described, is just half the space, which would be described with the last or greatest velocity, uniformly continued for an equal time  $t$ .

31. *Corol. 2.* The space  $s$  is also as  $v^2$ , the square of the velocity; because the velocity  $v$  is as the time  $t$ .

*Scholium.*

32. Propositions 3, 4, 5, 6, give theorems for resolving all questions relating to motions uniformly accelerated. Thus, put  $b =$  any body or quantity of matter,  
 $f =$  the force constantly acting on it,

$t$  = the time of its acting,  
 $v$  = the velocity generating in the time  $t$ ,  
 $s$  = the space described in that time,  
 $m$  = the momentum at the end of the time.

Then, from these fundamental relations,  $m \propto bv$ ,  $m \propto ft$ ,  
 $s \propto tv$ , and  $v \propto \frac{ft}{b}$ , we obtain the following table of the  
 general relations of uniformly accelerated motions:

$$\begin{array}{l}
 m \propto bv \propto ft \propto \frac{bs}{t} \propto \frac{fs}{v} \propto \frac{ft^2v}{s} \propto \sqrt{bfs} \propto \sqrt{bftv}. \\
 b \propto \frac{m}{v} \propto \frac{ft}{v} \propto \frac{mt}{s} \propto \frac{ft^2}{s} \propto \frac{f^2t^3}{ms} \propto \frac{m^2}{fs} \propto \frac{m^2}{ftv} \propto \frac{fs}{v^2}. \\
 f \propto \frac{m}{t} \propto \frac{bv}{t} \propto \frac{mv}{s} \propto \frac{ms}{t^2v} \propto \frac{m^2}{bs} \propto \frac{m^2}{btv} \propto \frac{bv^2}{s} \propto \frac{bs}{t^2}. \\
 v \propto \frac{s}{t} \propto \frac{ft}{b} \propto \frac{m}{b} \propto \frac{ms}{ft^2} \propto \frac{fs}{m} \propto \frac{m^2}{bft} \propto \sqrt{\frac{fs}{b}} \propto \frac{f^2st}{m^2}. \\
 s \propto tv \propto \frac{ft^2}{b} \propto \frac{mt}{b} \propto \frac{ft^2v}{m} \propto \frac{mv}{f} \propto \frac{m^2}{bf} \propto \frac{bv^2}{f} \propto \frac{m^2v}{f^2t}. \\
 t \propto \frac{s}{v} \propto \frac{m}{f} \propto \frac{bv}{f} \propto \frac{bs}{m} \propto \sqrt{\frac{bs}{f}} \propto \sqrt{\frac{ms}{fv}} \propto \frac{m^2}{bfv} \text{ \&c.}
 \end{array}$$

33. And from these proportions those quantities are to be left out which are given, or which are proportional to each other. Thus, if the body or quantity of matter be always the same, then the space described is as the force and square of the time. And if the body be proportional to the force, as all bodies are in respect to their gravity; then the space described is as the square of the time, or square of the velocity; and in this case, if  $F$  be put =  $\frac{f}{b}$ , the accelerating force; then will

$$s \propto tv \propto Ft^2 \propto \frac{v^2}{F}.$$

$$v \propto \frac{s}{t} \propto Ft \propto \sqrt{Fs}.$$

$$t \propto \frac{s}{v} \propto \frac{v}{F} \propto \sqrt{\frac{s}{F}}.$$

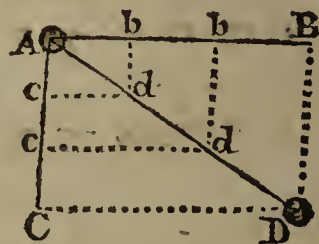
## THE COMPOSITION AND RESOLUTION OF FORCES.

34. COMPOSITION of FORCES, is the uniting of two or more forces into one, which shall have the same effect; or the finding of one force that shall be equal to several others taken together, in any different directions. And the Resolution of Forces, is the finding of two or more forces which, acting in any different directions, shall have the same effect as any given single force.

## PROPOSITION VII.

35. If a Body at A be urged in the Directions AB and AC, by any two Similar Forces, such that they would separately cause the Body to pass over the Spaces AB, AC in an equal time; then if both Forces act together, they will cause the Body to move in the same time, through AD the Diagonal of the Parallelogram ABCD.

DRAW  $cd$  parallel to  $AB$ , and  $bd$  parallel to  $AC$ . And while the body is carried over  $Ab$  or  $cd$  by the force in that direction, let it be carried over  $bd$  by the force in that direction; by which means it will be found at  $d$ . Now, if the forces be impulsive or momentary, the motions will be uniform, and the spaces described will be as the times of description;



theref.  $Ab$  or  $cd$  :  $AB$  or  $CD$  :: time in  $Ab$  : time in  $AB$ ,  
and  $cd$  or  $Ac$  :  $BD$  or  $AC$  :: time in  $Ac$  : time in  $AC$ ;  
but the time in  $Ab$  = time in  $Ac$ , and time in  $AB$  = time in  $AC$ , therefore  $Ab$  :  $bd$  ::  $AB$  :  $BD$  by equality.

And as this is always the case in every point  $d$ ,  $d$ , &c, therefore the path of the body is the straight line  $AdD$ , or the diagonal of the parallelogram.

But if the similar forces, by means of which the body is moved in the directions  $AB$ ,  $AC$ , be uniformly accelerating ones, then the spaces will be as the squares of the times; in which case, call the time in  $bd$  or  $cd$   $t$ , and the time in  $AB$  or  $AC$ ,  $T$ ; then

it will be  $Ab$  or  $cd$  :  $AB$  or  $CD$  ::  $t^2$  :  $T^2$ ,

and  $bd$  or  $AC$  :  $BD$  or  $AC$  ::  $t^2$  :  $T^2$ ,

theref. by equality,  $Ab$  :  $bd$  ::  $AB$  :  $BD$ ;

and so the body is always found in the diagonal, as before.

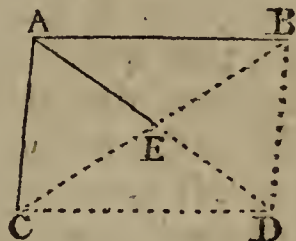
36. Corol.

36. *Corol. 1.* If the forces be not similar, by which the body is urged in the directions AB, AC, it will move in some curve line, depending on the nature of the forces.

37. *Corol. 2.* Hence it appears, that the body moves over the diagonal AD, by the compound motion, in the very same time that it would move over the side AB, by the single force impressed in that direction, or that it would move over the side AC by the force impressed in that direction.

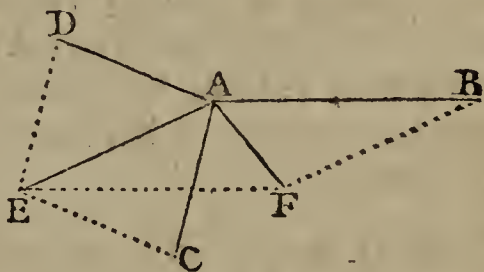
38. *Corol. 3.* The forces in the directions AB, AC, AD, are respectively proportional to the lines AB, AC, AD, and in these directions.

39. *Corol. 4.* The two oblique forces AB, AC, are equivalent to the single direct force AD, which may be compounded of these two, by drawing the diagonal of the parallelogram. Or they are equivalent to the double of AE drawn to the middle of the line BC.



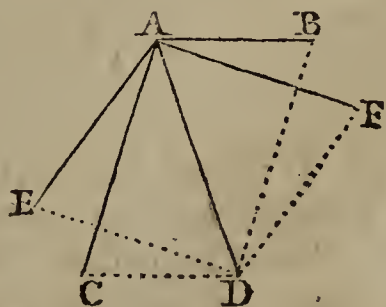
And thus any force may be compounded of two or more other forces; which is the meaning of the expression, *composition of forces.*

40. *Exam.* Suppose it were required to compound the three forces AB, AC, AD; or to find the direction and quantity of one single force, which shall be equivalent to, and have the same effect as if a body at A were



acted on by three forces in the directions AB, AC, AD, and proportional to these three lines. First reduce the two AC, AD to one AE, by completing the parallelogram ADEC. Then reduce the two AE, AB to one AF, by the parallelogram AEFB. So shall the single force AF be the direction, and as the quantity, which shall of itself produce the same effect, as if all the three AB, AC, AD acted together.

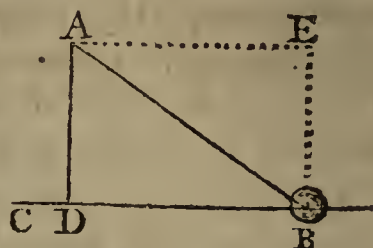
41. *Corol. 5.* Any single direct force AD, may be resolved into two oblique forces, whose quantities and directions are AB, AC, having the same effect, by describing any parallelogram whose diagonal may be AD: and this is called the resolution of forces. So the force AD



may be resolved into the two AB, AC, by the parallelogram ABDC;

ABDC ; or into the two AE, AF, by the parallelogram AEDF ; and so on, for any other two. And each of these may be resolved again into as many others as we please.

42. *Corol. 6.* Hence may be found the effect of any given force, in any other direction, besides that of the line in which it acts ; as of the force AB in any other given direction CB. For draw AD perpendicular to CB ; then shall DB be the effect of the force AB in the direction CB. For, the given force AB is equivalent to the two AD, DB, or AE ; of which the former AD or EB, being perpendicular, does not alter the velocity in the direction CB ; and therefore DB is the whole effect of AB in the direction CB. That is, a direct force expressed by the line DB acting in the direction DB, will produce the same effect or motion in a body B, in that direction, as the oblique force expressed by, and acting in, the direction AB, produces in the same direction CB. And hence a direct force DB, is to an oblique force AB, as AB to DB, or as radius to the cosine of the angle ABD of inclination of those forces. For the same reason, the force or effect in the direction AB, is to the force or effect in the direction AD or EB, as AB to AD ; or as radius to sine of the same angle ABD, or cosine of the angle DAB of those directions.



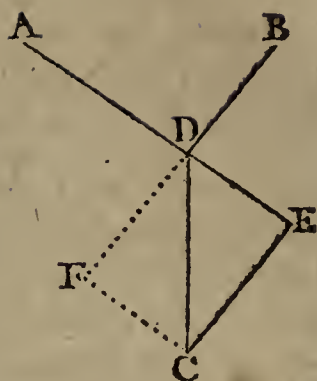
43. *Corol. 7.* Hence also, if the two given forces, to be compounded, act in the same line, either both the same way, or the one directly opposite to the other ; then their joint or compounded force will act in the same line also, and will be equal to the sum of the two when they act the same way, or to the difference of them when they act in opposite directions ; and the compound force, whether it be the sum or difference, will always act in the direction of the greater of the two.

#### PROPOSITION VIII.

44. *If Three Forces A, B, C, acting all together, in the same Plane, keep one another in Equilibrio ; they will be Proportional to the Three Sides DE, EC, CD, of a Triangle, which are drawn Parallel to the Directions of the Forces AD, DB, CD.*

PRODUCE AD, BD, and draw CF, CE parallel to them.

them. Then the force in  $CD$  is equivalent to the two  $AD$ ,  $BD$ , by the supposition; but the force  $CD$  is also equivalent to the two  $ED$  and  $CE$  or  $FD$ ; therefore, if  $CD$  represent the force  $C$ , then  $ED$  will represent its opposite force  $A$ , and  $CE$  or  $FD$  its opposite force  $B$ . Consequently the three forces  $A$ ,  $B$ ,  $C$ , are proportional to  $DE$ ,  $CE$ ,  $CD$ , the three lines parallel to the directions in which they act.

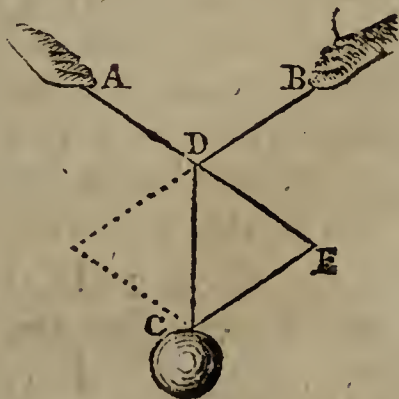


45. *Corol. 1.* Because the three sides  $CD$ ,  $CE$ ,  $DE$ , are proportional to the sines of their opposite angles  $E$ ,  $D$ ,  $C$ ; therefore the three forces, when in equilibrio, are proportional to the sines of the angles of the triangle made of their lines of direction; namely, each force proportional to the sine of the angle made by the directions of the other two.

46. *Corol. 2.* The three forces, acting against, and keeping one another in equilibrio, are also proportional to the sides of any other triangle made by drawing lines either perpendicular to the directions of the forces, or forming any given angle with those directions. For such a triangle is always similar to the former, which is made by drawing lines parallel to the directions; and therefore their sides are in the same proportion to one another.

— 47. *Corol. 3.* If any number of forces be kept in equilibrio by their actions against one another; they may be all reduced to two equal and opposite ones.—For, by cor. 4, prop. 7, any two of the forces may be reduced to one force acting in the same plane; then this last force and another may likewise be reduced to another force acting in their plane: and so on, till at last they be all reduced to the action of only two opposite forces; which will be equal, as well as opposite, because the whole are in equilibrio by the supposition.

48. *Corol. 4.* If one of the forces, as  $C$ , be a weight, which is sustained by two strings drawing in the directions  $DA$ ,  $DB$ : then the force or tension of the string  $AD$ , is to the weight  $C$ , or tension of the string  $DC$ , as  $DE$  to  $DC$ ; and the force or tension of the other string  $BD$ , is to the weight  $C$ , or tension of  $CD$ , as  $CE$  to  $CD$ .



49. *Corol.*

49. *Corol. 5.* If three forces be in equilibrio by their mutual actions; the line of direction of each force, as DC, passes through the opposite angle C of the parallelogram formed by the directions of the other two forces.

50. *Remark.* These properties, in this proposition and its corollaries, hold true of all similar forces whatever, whether they be instantaneous or continual, or whether they act by percussion, drawing, pushing, pressing, or weighing; and are of the utmost importance in mechanics and the doctrine of forces.

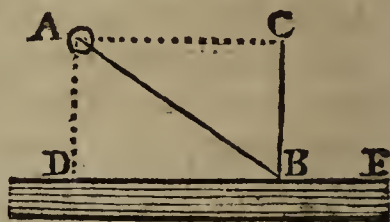
## ON THE COLLISION OF BODIES.

### PROPOSITION IX.

51. *If a Body impinge or act Obliquely on a Plane Surface; the Force or Energy of the Stroke, or Action, is as the Sine of the Angle of Incidence.*

*Or, the Force on the Surface is to the same if it had acted Perpendicularly, as the Sine of Incidence is to Radius.*

LET AB express the direction and the absolute quantity of the oblique force on the plane DE; or let a given body A, moving with a certain velocity, impinge on the plane at B; then its force will be to the action on the plane, as radius to the sine of the angle ABD, or as AB to BC, drawing BC perpendicular, and AC parallel to DE.



For, by prop. 7, the force AB is equivalent to the two forces AC, CB; of which the former AC does not act on the plane, because it is parallel to it. The plane is therefore only acted on by the direct force CB, which is to AB, as the sine of the angle BAC, or ABD, to radius.

52. *Corol. 1.* If a body act on another, in any direction, and by any kind of force, the action of that force on the second body, is made only in a direction perpendicular to the surface on which it acts.

For the force in AB acts on DE only by the force CB, and in that direction.

53. *Corol. 2.* If the plane DE be not absolutely fixed, it will move, after the stroke, in the direction perpendicular to its surface. For it is in that direction that the force is exerted.

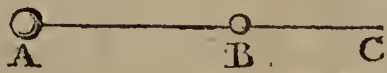


## PROPOSITION X.

54. *If one Body A, strike another Body B, which is either at Rest or moving towards the Body A, or moving from it, but with a less Velocity than that of A; then the Momenta, or Quantities of Motion, of the two Bodies, estimated in any one Direction, will be the very same after the Stroke that they were before it.*

FOR, because action and re-action are always equal, and in contrary directions, whatever momentum the one body gains one way by the stroke, the other must just lose as much in that same direction; and therefore the quantity of motion in that direction, resulting from the motions of both the bodies, remains still the same as it was before the stroke.

55. Thus, if A with a momentum of 10, strike B at rest, and communicate to it a momentum of 4, in the direction AB. Then A will have



only a momentum of 6 in that direction; which, together with the momentum of B, viz. 4, make up still the same momentum between them as before, namely 10.

56. If B were in motion before the stroke, with a momentum of 5, in the same direction, and receive from A an additional momentum of 2. Then the motion of A after the stroke will be 8, and that of B, 7; which between them make 15, the same as 10 and 5, the motions before the stroke.

57. Lastly, if the bodies move in opposite directions, and meet one another, namely, A with a motion of 10, and B, of 5; and A communicate to B a motion of 6 in the direction AB of its motion. Then, before the stroke, the whole motion from both, in the direction of AB, is  $10 - 5$  or 5. But, after the stroke, the motion of A is 4 in the direction AB, and the motion of B is  $6 - 5$  or 1 in the same direction AB; therefore the sum  $4 + 1$ , or 5, is still the same motion from both, as it was before.

## PROPOSITION XI.

58. *The Motion of Bodies included in a Given Space, is the same, with regard to each other, whether that Space be at Rest, or move uniformly in a Right Line.*

FOR, if any force be equally impressed both on the body and the line in which it moves, this will cause no change in  
the

the motion of the body along the right line. For the same reason, the motions of all the other bodies, in their several directions, will still remain the same. Consequently their motions among themselves will continue the same, whether the including space be at rest, or be moved uniformly forward. And therefore their mutual actions on one another, must also remain the same in both cases.

PROPOSITION XII.

59. *If a Hard and Fixed Plane be struck by either a Soft or a Hard Unelastic Body, the Body will adhere to it. But if the Plane be struck by a Perfectly Elastic Body, it will rebound from it again with the same Velocity with which it struck the Plane.*

FOR, since the parts which are struck, of the elastic body, suddenly yield and give way by the force of the blow, and as suddenly restore themselves again with a force equal to the force which impressed them, by the definition of elastic bodies; the intensity of the action of that restoring force on the plane, will be equal to the force or momentum with which the body struck the plane. And, as action and reaction are equal and contrary, the plane will act with the same force on the body, and so cause it to rebound or move back again with the same velocity as it had before the stroke.

But hard or soft bodies, being devoid of elasticity, by the definition, having no restoring force to throw them off again, they must necessarily adhere to the plane struck.

60. *Corol. 1.* The effect of the blow of the elastic body, on the plane, is double to that of the unelastic one, the velocity and mass being equal in each.

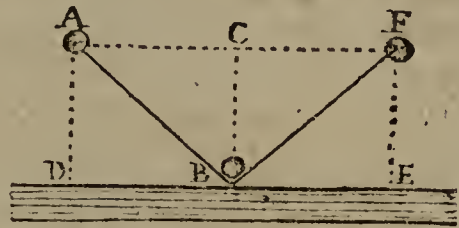
For the force of the blow from the unelastic body, is as its mass and velocity, which is only destroyed by the resistance of the plane. But in the elastic body, that force is not only destroyed and sustained by the plane; but another also equal to it is sustained by the plane, in consequence of the restoring force, and by virtue of which the body is thrown back again with an equal velocity. And therefore the intensity of the blow is doubled.

61. *Corol. 2.* Hence unelastic bodies lose, by their collision, only half the motion lost by elastic bodies; their mass and velocities being equal.—For the latter communicate double the motion of the former.

PROPOSITION XIII.

62. *If an Elastic Body A impinge on a Firm Plane DE at the Point B, it will rebound from it in an Angle equal to that in which it struck it; or the Angle of Incidence will be Equal to the Angle of Reflection; namely, the Angle ABD equal to the Angle FBE.*

LET AB express the force of the body A in the direction AB; which let be resolved into the two AC, CB, parallel and perpendicular to the plane.—Take BE and CF equal to AC, and draw BF.

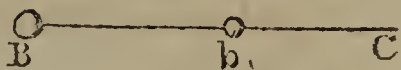


Now, action and re-action being equal, the plane will resist the direct force CB by another BC equal to it, and in a contrary direction; whereas the other AC, being parallel to the plane, is not acted on nor diminished by it, but still continues as before. The body is therefore reflected from the plane by two forces BC, BE, perpendicular and parallel to the plane, and therefore moves in the diagonal BF by composition. But, because AC is equal to BE or CF, and that BC is common, the two triangles BCA, BCF are mutually similar and equal; and consequently the angles at A and F are equal, as also their equal alternate angles ABD, FBE, which are the angles of incidence and reflection.

PROPOSITION XIV.

63. *To determine the Motion of Non-elastic Bodies, when they strike each other Directly, or in the Same Line of Direction.*

LET the non-elastic body B, moving with the velocity V in the direction Bb, and the body b with the velocity v, strike each other.



Then, because the momentum of any moving body is as the mass into the velocity,  $EV = M$  is the momentum of the body B, and  $bv = m$  the momentum of the body b, which let be the less powerful of the two motions. Then, by prop. 10, the bodies will both move together as one mass in the direction BC after the stroke, whether before the stroke the body b moved towards C or towards B. Now, according as that motion of b was from or towards B, that is, whether the motions were in the same or contrary ways, the momentum after the stroke, in direction BC, will be

be the sum or difference of the momentums before the stroke; namely, the momentum in direction BC will be

$BV + bv$ , if the bodies moved the same way, or  
 $BV - bv$ , if they moved contrary ways, and  
 $BV$  only, if the body  $b$  were at rest.

Then divide each momentum by the common mass of matter  $B + b$ , and the quotient will be the common velocity after the stroke in the direction BC; namely, the common velocity will be,

$\frac{BV + bv}{B + b}$  in the first case,  $\frac{BV - bv}{B + b}$  in the 2nd, and  $\frac{BV}{B + b}$  in the third.

64. For example, if the bodies, or weights,  $B$  and  $b$  be as 5 to 3, and their velocities  $V$  and  $v$ , as 6 to 4, or as 3 to 2, before the stroke; then 15 and 6 will be as their momentums, and 8 the sum of their weights; consequently after the stroke the common velocity will be as

$$\frac{15 + 6}{8} = \frac{21}{8} \text{ or } 2\frac{5}{8} \text{ in the first case,}$$

$$\frac{15 - 6}{8} = \frac{9}{8} \text{ or } 1\frac{1}{8} \text{ in the second, and}$$

$$\frac{15}{8} \text{ - - - or } 1\frac{7}{8} \text{ in the third.}$$

#### PROPOSITION XV.

65. *If two Perfectly Elastic Bodies impinge on one another; their Relative Velocity will be the same both Before and After the Impulse; that is, they will recede from each other with the Same Velocity with which they approached and met.*

FOR the compressing force is as the intensity of the stroke; which, in given bodies, is as the relative velocity with which they meet or strike. But perfectly elastic bodies restore themselves to their former figure by the same force by which they were compressed; that is, the restoring force is equal to the compressing force, or to the force with which the bodies approach each other before the impulse. But the bodies are impelled from each other by this restoring force; and therefore this force, acting on the same bodies, will produce a relative velocity equal to that which they had before; or it will make the bodies recede from each other with the same velocity

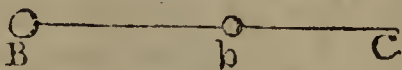
velocity with which they before approached, or so as to be equally distant from one another at equal times before and after the impact.

66. *Remark.* It is not meant by this proposition, that each body will have the same velocity after the impulse as it had before; for that will be varied according to the relation of the masses of the two bodies; but that the velocity of the one will be, after the stroke, as much increased as that of the other is decreased, in one and the same direction. So if the elastic body B move with a velocity  $V$ , and overtake the elastic body  $b$  moving the same way with the velocity  $v$ ; then their relative velocity, or that with which they strike, is  $V - v$ , and it is with this same velocity that they separate from each other after the stroke. But if they meet each other, or the body  $b$  move contrary to the body B; then they meet and strike with the velocity  $V + v$ , and it is with the same velocity that they separate and recede from each other after the stroke. But whether they move forward or backward after the impulse, and with what particular velocities, are circumstances that depend on the various masses and velocities of the bodies before the stroke, and which make the subject of the next proposition.

## PROPOSITION XVI.

67. *To determine the Motions of Elastic Bodies after Striking each other directly.*

LET the elastic body B move in the direction BC, with the velocity  $V$ ; and let the velocity of the other body  $b$  be  $v$  in the same direction; which latter velocity  $v$  will be positive if  $b$  move the same way as B, but negative if  $b$  move in the opposite direction to B. Then their relative velocity in the direction BC is  $V - v$ ; also the momenta before the stroke are  $BV$  and  $bv$ , the sum of which is  $BV + bv$  in the direction BC.



Again, put  $x$  for the velocity of B, and  $y$  for that of  $b$ , in the same direction BC, after the stroke; then their relative velocity is  $y - x$ , and the sum of their momenta  $Bx + by$  in the same direction.

But the momenta before and after the collision, estimated in the same direction, are equal, by prop. 10, as also the relative velocities, by the last prop. Whence arise these two equations,

$$BV + bv = Bx + by,$$

$$V - v = y - x;$$

the resolution of which equations gives

$$x = \frac{(B - b)V + 2bv}{B + b}, \text{ the velocity of } B,$$

$$y = \frac{-(B - b)v + 2BV}{B + b}, \text{ the velocity of } b.$$

both in the direction BC, when  $V$  and  $v$  are both positive, or the bodies both moved towards C before the collision. But if  $v$  be negative, or the body  $b$  moved in the contrary direction before collision, or towards B; then, changing the sign of  $v$ , the same theorems become

$$x = \frac{(B - b)V - 2bv}{B + b}, \text{ the velocity of } B,$$

$$y = \frac{(B - b)v + 2BV}{B + b}, \text{ the veloc. of } b, \text{ in the direction BC.}$$

And if  $b$  were at rest before the impact, making its velocity  $v = 0$ , the same theorems give

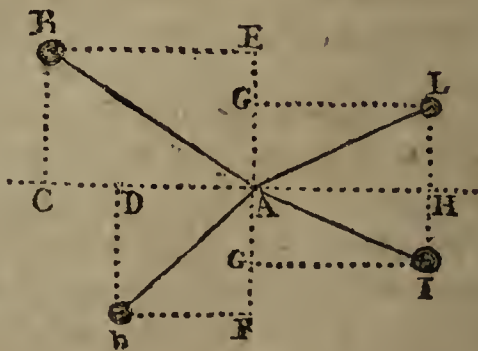
$$x = \frac{B - b}{B + b}V, \text{ and } y = \frac{2B}{B + b}V,$$

for the velocities in this case.

#### PROPOSITION XVII.

68. *If Bodies strike one another Obliquely, it is proposed to determine their Motions after the Stroke,*

LET the two bodies  $B, b$ , move in the oblique directions  $BA, bA$ , and strike each other at  $A$ , with velocities which are in proportion to the lines  $BA, bA$ ; to find their motions after the impact. Let  $CAH$  represent the plane in which the bodies touch in the point of concourse; to which draw the perpendiculars  $BC, bD$ , and complete the rectangles  $CE, DF$ . Then the motion in  $BA$  is resolved into the two  $BC, CA$ ; and the motion in  $bA$  is resolved into the two  $bD, DA$ ; of which the antecedents  $BC, bD$  are the velocities with which they directly meet, and the consequents  $CA, DA$  are parallel; therefore, by these the bodies do not impinge on each other, and consequently the motions, according to these directions,



rections, will not be changed by the impulse; so that the velocities with which the bodies meet, are as  $BC$  and  $bD$ , or their equals  $EA$  and  $FA$ . The motions, therefore, of the bodies  $B, b$ , directly striking each other with the velocities  $EA, FA$ , will be determined by prop. 16 or 14, according as the bodies are elastic or non-elastic; which being done, let  $AG$  be the velocity, so determined, of one of them, as  $A$ ; and since there remains also in the body a force of moving in the direction parallel to  $BE$ , with a velocity as  $BE$ , make  $AH$  equal to  $BE$ , and complete the rectangle  $GH$ : then the two motions in  $AH$  and  $AG$ , or  $HI$ , are compounded into the diagonal  $AI$ , which therefore will be the path and velocity of the body  $B$  after the stroke. And after the same manner is the motion of the other body  $b$  determined after the impact.

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THE LAWS OF GRAVITY; THE DESCENT OF HEAVY BODIES; AND THE MOTION OF PROJECTILES IN FREE SPACE.

PROPOSITION XVIII.

69. *All the Properties of Motion delivered in Proposition VI, its Corollaries and Scholium, for Constant Forces, are true in the Motions of Bodies freely descending by their own Gravity; namely, that the Velocities are as the Times, and the Spaces as the Squares of the Times, or as the Squares of the Velocities.*

FOR, since the force of gravity is uniform, and constantly the same, at all places near the earth's surface, or at nearly the same distance from the centre of the earth; and that this is the force by which bodies descend to the surface; they therefore descend by a force which acts constantly and equally; consequently all the motions freely produced by gravity, are as above specified, by that proposition, &c.

SCHOLIUM.

70. Now it has been found, by numberless experiments, that gravity is a force of such a nature, that all bodies, whether light or heavy, fall perpendicularly through equal spaces in the same time, abstracting from the resistance of the air; as lead or gold and a feather, which in an exhausted receiver fall from the top to the bottom in the same time. It is also

found, that the velocities acquired by descending, are in the exact proportion of the times of descent: and farther, that the spaces descended are proportional to the squares of the times, and therefore to the squares of the velocities. And hence it follows, that the weights, or gravities, of bodies near the surface of the earth, are proportional to the quantities of matter contained in them; and that the spaces, times, and velocities, generated by gravity, have the relations contained in the three general proportions before laid down. Moreover, as it is found, by accurate experiments, that a body in the latitude of London, falls nearly  $16\frac{1}{2}$  feet in the first second of time, and consequently that at the end of that time it has acquired a velocity double, or of  $32\frac{1}{2}$  feet, by corol. 1, prop. 6; therefore, if  $g$  denote  $16\frac{1}{2}$  feet, the space fallen through in one second of time, or  $2g$  the velocity generated in that time; then, because the velocities are directly proportional to the times, and the spaces to the squares of the times; therefore it will be,

$$\begin{aligned} \text{as } 1'' : t'' :: 2g : 2gt = v \text{ the velocity,} \\ \text{and } 1^2 : t^2 :: g : gt^2 = s \text{ the space.} \end{aligned}$$

So that for the descents of gravity, we have these general equations, namely,

$$s = gt^2 = \frac{v^2}{4g} = \frac{1}{2}tv.$$

$$v = 2gt = \frac{2s}{t} = 2\sqrt{gs}.$$

$$t = \frac{v}{2g} = \frac{2s}{v} = \sqrt{\frac{s}{g}}.$$

$$g = \frac{v}{2t} = \frac{s}{t^2} = \frac{v^2}{4s}.$$

Hence, because the times are as the velocities, and the spaces as the squares of either, therefore,

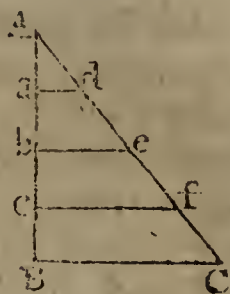
if the times be as the numbs. 1, 2, 3, 4, 5, &c,  
 the velocities will also be as 1, 2, 3, 4, 5, &c,  
 and the spaces as their squares 1, 4, 9, 16, 25, &c,  
 and the space for each time as 1, 3, 5, 7, 9, &c,

namely, as the series of the odd numbers, which are the differences of the squares denoting the whole spaces. So that, if the first series of natural numbers be seconds of time, namely, the times in seconds  $1''$ ,  $2''$ ,  $3''$ ,  $4''$ , &c,  
 the velocity in feet will be  $32\frac{1}{2}$ ,  $64\frac{1}{2}$ ,  $96\frac{1}{2}$ ,  $128\frac{1}{2}$ , &c,  
 the spaces in the whole times  $16\frac{1}{2}$ ,  $64\frac{1}{2}$ ,  $144\frac{3}{4}$ ,  $257\frac{1}{2}$ , &c,  
 and the space for each second  $16\frac{1}{2}$ ,  $48\frac{1}{4}$ ,  $80\frac{5}{8}$ ,  $112\frac{7}{8}$ , &c.

71. These

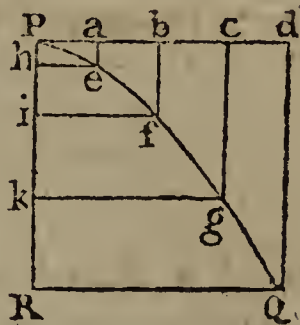


71. These relations of the times, velocities, and spaces, may be aptly represented by certain lines and geometrical figures. Thus, if the line AB denote the time of any body's descent, and BC at right angles to it, the velocity gained at the end of that time, by joining AC, and dividing the time AB into any number of parts at the points a, b, c; then



shall ad, be, cf, parallel to BC, be the velocities at the points of time a, b, c, or at the ends of the times Aa, Ab, Ac; because these latter lines, by similar triangles, are proportional to the former ad, be, cf, and the times are proportional to the velocities. Also, the area of the triangle ABC will represent the space descended by the force of gravity in the time AB, in which it generates the velocity BC; because that area is equal to  $\frac{1}{2}AB \times BC$ , and the space descended is  $s = \frac{1}{2}tv$ , or half the product of the time and the last velocity. And, for the same reason, the less triangles Aad, Abe, Acf, will represent the several spaces described in the corresponding times Aa, Ab, Ac, and velocities ad, be, cf; those triangles or spaces being also as the squares of their like sides Aa, Ab, Ac, which represent the times, or of ad, be, cf, which represent the velocities.

72. But as areas are rather unnatural representations of the spaces passed over by a body in motion, which are lines, the relations may better be represented by the abscisses and ordinates of a parabola. Thus, if PQ be a parabola, PR its axis, and RQ its ordinate; and Pa, Pb, Pc, &c, parallel to RQ, represent the times from the beginning, or the velocities,



then ae, bf, cg, &c, parallel to the axis PR, will represent the spaces described by a falling body in those times; for, in a parabola, the abscisses Ph, Pi, Pk, &c, or ae, bf, cg, &c, which are the spaces described, are as the squares of the ordinates he, if, kg, &c, or Pa, Pb, Pc, &c, which represent the times or velocities.

73. And because the laws for the destruction of motion, are the same as those for the generation of it, by equal forces, but acting in a contrary direction; therefore,

1<sup>st</sup>, A body thrown directly upwards, with any velocity, will lose equal velocities in equal times.

2<sup>d</sup>, If

2d, If a body be projected upwards, with the velocity it acquired in any time by descending freely; it will lose all its velocity in an equal time, and will ascend just to the same height from whence it fell, and will describe equal spaces in equal times, in rising and falling, but in an inverse order; and it will have equal velocities at any one and the same point of the line described, both in ascending and descending.

3d, If bodies be projected upwards; with any velocities, the height ascended to, will be as the squares of those velocities, or as the squares of the times of ascending, till they lose all their velocities.

74. To illustrate now the rules for the natural descent of bodies by a few examples, let it be required,

1st, To find the space descended by a body in 7 seconds of time, and the velocity acquired.

Anf.

2d, To find the time of generating a velocity of 100 feet per second, and the whole space descended.

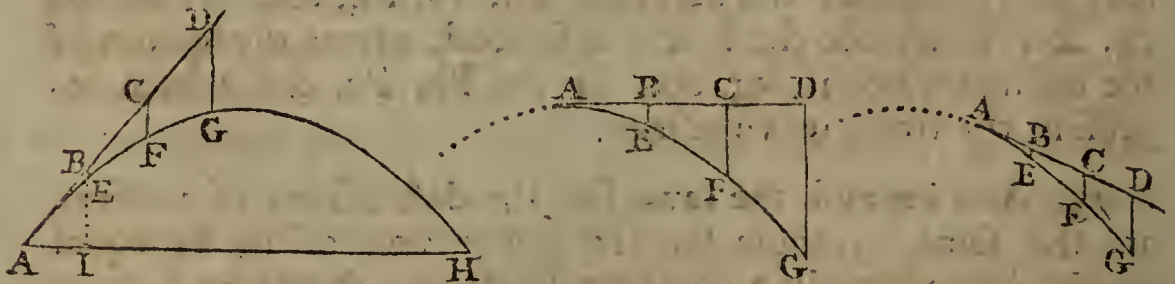
Anf.

3d, To find the time of descending 400 feet, and the velocity at the end of that time.

Anf.

#### PROPOSITION XIX.

75. If a Body be projected in Free Space, either Parallel to the Horizon, or in an Oblique Direction, by the Force of Gun Powder, or any other Impulse; it will, by this Motion, in Conjunction with the Action of Gravity, describe the Curve Line of a Parabola.



LET the body be projected from the point A, in the direction AD, with any uniform velocity; then, in any equal portions of time, it would, by prop. 4, describe the equal spaces

spaces AB, BC, CD, &c, in the line AD, if it were not drawn continually down below that line by the action of gravity. Draw BE, CF, DG, &c, in the direction of gravity, or perpendicular to the horizon, and equal to the spaces through which the body would descend by its gravity, in the same times in which it would uniformly pass over the corresponding spaces AB, AC, AD, &c, by the projectile motion. Then, since by these two motions the body is carried over the space AB, in the same time as over the space BE, and the space AC in the same time as the space CF, and the space AD in the same time as the space DG, &c; therefore by the composition of motions, at the end of those times, the body will be found respectively in the points E, F, G, &c; and consequently the real path of the projectile will be the curve line AEF, G, &c. But the spaces AB, AC, AD, &c, described by uniform motion, are as the times of description; and the spaces BE, CF, DG, &c, described in the same times by the accelerating force of gravity, are as the squares of the times; consequently the perpendicular descents are as the squares of the spaces in AD, that is BE, CF, DG, &c, are respectively proportional to  $AB^2$ ,  $AC^2$ ,  $AD^2$ , &c; which is the property of the parabola by theor. 8, Con. Sect. Therefore the path of the projectile is the parabolic line AEF, G, &c, to which AD is a tangent at the point A.

76. *Corol. 1.* The horizontal velocity of a projectile, is always the same constant quantity, in every point of the curve; because the horizontal motion is in a constant ratio to the motion in AD, which is the uniform projectile motion. And the constant horizontal velocity, is in proportion to the projectile velocity, as radius to the cosine of the angle DAH, or angle of elevation or depression of the piece above or below the horizontal line AH.

77. *Corol. 2.* The velocity of the projectile in the direction of the curve, or of its tangent at any point A, is as the secant of its angle BAI of direction above the horizon. For the motion in the horizontal direction AI is constant, and AI is to AB, as radius to the secant of the angle A; therefore the motion at A, in AB, is every where as the secant of the angle A.

78. *Corol. 3.* The velocity in the direction DG of gravity, or perpendicular to the horizon, at any point G of the curve, is to the first uniform projectile velocity at A, or point of contact of a tangent, as  $2GD$  is to AD. For, the times in AD and DG being equal, and the velocity acquired by freely descending

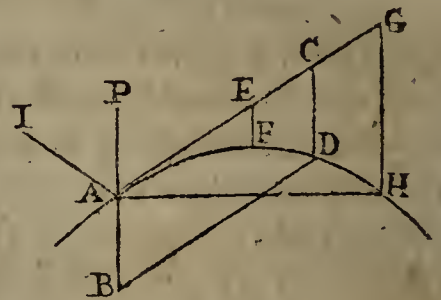
descending through  $DG$  being such as would carry the body uniformly over twice  $DG$  in an equal time, and the spaces described with uniform motions being as the velocities, therefore the space  $AD$  is to the space  $2DG$ , as the projectile velocity at  $A$ , to the perpendicular velocity at  $G$ .

## PROPOSITION XX.

79. *The Velocity in the Direction of the Curve, at any Point of it, as  $A$ , is equal to that which is generated by Gravity in freely descending through a Space which is equal to One-Fourth of the Parameter of the Diameter of the Parabola at that Point.*

LET  $PA$  or  $AB$  be the height due to the velocity of the projectile at any point  $A$ , in the direction of the curve or tangent  $AC$ , or the velocity acquired by falling through that height; and complete the parallelogram  $ACDB$ .

Then is  $CD = AB$  or  $AP$ , the height due to the velocity in the curve at  $A$ ; and  $CD$  is also the height due to the perpendicular velocity at  $D$ , which must be equal to the former: but, by the last corol. the velocity at  $A$  is to the perpendicular velocity at  $D$ , as  $AC$  to  $2CD$ ; and as these velocities are equal, therefore  $AC$  or  $BD$  is equal to  $2CD$ , or  $2AB$ ; and hence  $AB$  or  $AP$  is equal to  $\frac{1}{2}BD$ , or  $\frac{1}{4}$  of the parameter of the diameter  $AB$ , by corol. to theor. 13 of the Parabola.



80. *Corol. 1.* Hence, and from cor. 2, theor. 13 of the Parabola, it appears that, if from the directrix of the parabola which is the path of the projectile, several lines  $HE$  be drawn perpendicular to the directrix, or parallel to the axis; then the velocity of the projectile in the direction of the curve, at any point  $E$ , is always equal to the velocity acquired by a body falling freely through the perpendicular line  $HE$ .



81. *Corol. 2.* If a body, after falling through the height  $PA$  (last fig. but one), which is equal to  $AB$ , and when it arrives at  $A$ , have its course changed, by reflection from an elastic plane  $AI$ , or otherwise, into any direction  $AC$ , without altering the velocity; and if  $AC$  be taken  $= 2AP$  or  $2AB$ ,  
and

and the parallelogram be completed; then the body will describe the parabola passing through the point D.

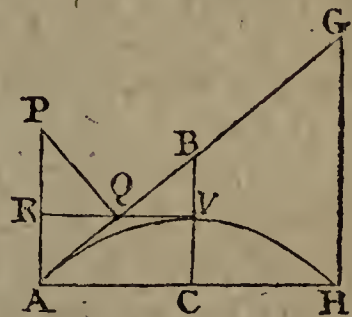
82. *Corol. 3.* Because  $AC = 2AB$  or  $2CD$  or  $2AP$ , therefore  $AC^2 = 2AP \times 2CD$  or  $AP \cdot 4CD$ ; and because all the perpendiculars  $EF, CD, GH$  are as  $AE^2, AC^2, AG^2$ ; therefore also  $AP \cdot 4EF = AE^2$ , and  $AP \cdot 4GH = AG^2$ , &c; and, because the rectangle of the extremes is equal to the rectangle of the means of four proportionals, therefore always

it is  $AP : AE :: AE : 4EF$ ,  
and  $AP : AC :: AC : 4CD$ ,  
and  $AP : AG :: AG : 4GH$ ,  
and so on.

## PROPOSITION XXI.

83. *Having given the Direction, and the Impetus, or Altitude due to the First Velocity of a Projectile; to determine the Greatest Height to which it will rise, and the Random or Horizontal Range.*

LET  $AP$  be the height due to the projectile velocity at  $A$ ,  $AG$  the direction, and  $AH$  the horizon. Upon  $AG$  let fall the perpendicular  $PQ$ , and on  $AP$  the perpendicular  $QR$ ; so shall  $AR$  be equal to the greatest altitude  $CV$ , and  $4QR$  equal to the horizontal range  $AH$ . Or, having drawn  $PQ$  perp. to  $AG$ , take  $AG = 4AQ$ , and draw  $GH$  perp. to  $AH$ ; then  $AH$  is the range.

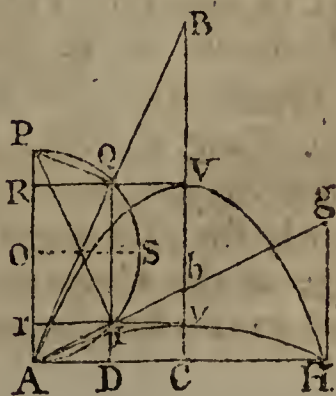


For, by the last corollary,  $AP : AG :: AG : 4GH$ ;  
and, by similar triangles,  $AP : AG :: AQ : GH$ ,  
or -  $AP : AG :: 4AQ : 4GH$ ;  
therefore  $AG = 4AQ$ ; and, by similar triangles,  $AH = 4QR$ .

Also, if  $V$  be the vertex of the parabola, then  $AB$  or  $\frac{1}{2}AG = 2AQ$ , or  $AQ = QB$ ; consequently  $AR = BV$ , which is  $= CV$  by the property of the parabola.

84. *Corol. 1.* Because the angle  $Q$  is a right angle, which is the angle in a semicircle, therefore if, upon  $AP$ , as a diameter; a semicircle be described, it will pass through the point  $Q$ .

85. *Corol. 2.* If the horizontal range and the projectile velocity be



be given, the direction of the piece so as to hit the object H, will be thus easily found: Take  $AD = \frac{1}{4}AH$ , draw  $DQ$  perpendicular to  $AH$ , meeting the semicircle, described on the diameter  $AP$ , in  $Q$  and  $q$ ; then  $AQ$  or  $Aq$  will be the direction of the piece. And hence it appears, that there are two directions  $AB, Ab$ , which, with the same projectile velocity, give the very same horizontal range  $AH$ . And these two directions make equal angles  $qAD, QAP$  with  $AH$  and  $AP$ , because the arc  $PQ =$  the arc  $Aq$ .

86. *Corol. 3.* Or, if the range  $AH$ , and direction  $AB$ , be given; to find the altitude and velocity or impetus. Take  $AD = \frac{1}{4}AH$ , and erect the perpendicular  $DQ$ , meeting  $AB$  in  $Q$ ; so shall  $DQ$  be equal to the greatest altitude  $CV$ . Also, erect  $AP$  perpendicular to  $AH$ , and  $QP$  to  $AQ$ ; so shall  $AP$  be the height due to the velocity.

87. *Corol. 4.* When the body is projected with the same velocity, but in different directions; the horizontal ranges  $AH$  will be as the sines of double the angles of elevation.— Or, which is the same, as the rectangle of the sine and cosine of elevation. For  $AD$  or  $RQ$ , which is  $\frac{1}{4}AH$ , is the sine of the arc  $AQ$ , which measures double the angle  $QAD$  of elevation.

And when the direction is the same, but the velocities different; the horizontal ranges are as the square of the velocities, or as the height  $AP$ , which is as the square of the velocity; for the sine  $AD$  or  $RQ$  or  $\frac{1}{4}AH$  is as the radius, or as the diameter  $AP$ .

Therefore, when both are different, the ranges are in the compound ratio of the squares of the velocities, and the sines of double the angles of elevation.

88. *Corol. 5.* The greatest range is when the angle of elevation is  $45^\circ$ , or half a right angle; for the double of  $45$  is  $90$ , which has the greatest sine. Or the radius  $OS$ , which is  $\frac{1}{4}$  of the range, is the greatest sine.

And hence the greatest range, or that at an elevation of  $45^\circ$ , is just double the altitude  $AP$  which is due to the velocity, or equal to  $4VC$ . And consequently, in that case,  $C$  is the focus of the parabola, and  $AH$  its parameter. Also, the ranges are equal, at angles equally above and below  $45^\circ$ .

89. *Corol. 6.* When the elevation is  $15^\circ$ , the double of which, or  $30^\circ$ , has its sine equal to half the radius; consequently then its range will be equal to  $AP$ , or half the greatest range at the elevation of  $45^\circ$ ; that is, the range at  $15^\circ$ , is equal to the impetus or height due to the projectile velocity.

90. *Corol. 7.*

90. *Corol. 7.* The greatest altitude  $CV$ , being equal to  $AK$ , is as the versed sine of double the angle of elevation, and also as  $AP$  or the square of the velocity. Or as the square of the sine of elevation, and the square of the velocity; for the square of the sine is as the versed sine of the double angle.

91. *Corol. 8.* The time of flight of the projectile, which is equal to the time of a body falling freely through  $GH$  or  $4CV$ , four times the altitude, is therefore as the square root of the altitude, or as the projectile velocity and sine of the elevation.

## SCHOLIUM.

92. From the last proposition, and its corollaries, may be deduced the following set of theorems, for finding all the circumstances of projectiles on horizontal planes, having any two of them given. Thus, let  $s, c, t$  denote the sine, cosine, and tangent of elevation;  $f, v$  the sine and versed sine of the double elevation;  $R$  the horizontal range,  $T$  the time of flight,  $V$  the projectile velocity,  $H$  the greatest height of the projectile,  $g = 16\frac{1}{2}$  feet, and  $a$  the impetus, or the altitude due to the velocity  $V$ . Then,

$$R = 2af = 4asc = \frac{fV^2}{2g} = \frac{scV^2}{g} = \frac{gcT^2}{s} = \frac{gT^2}{t} = \frac{4H}{t},$$

$$V = \sqrt{4ag} = \sqrt{\frac{2gR}{f}} = \sqrt{\frac{gR}{sc}} = \frac{gT}{s} = \frac{2}{s}\sqrt{gH},$$

$$T = \frac{sV}{g} = 2s\sqrt{\frac{a}{g}} = \sqrt{\frac{tR}{g}} = \sqrt{\frac{sR}{gc}} = 2\sqrt{\frac{H}{g}},$$

$$H = as^2 = \frac{1}{2}av = \frac{1}{4}tR = \frac{sR}{4c} = \frac{s^2V^2}{4g} = \frac{vV^2}{8g} = \frac{g}{4}T^2.$$

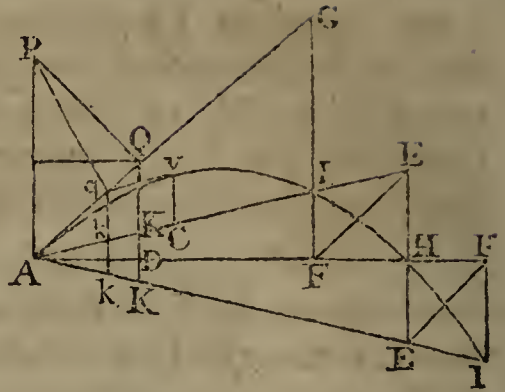
And from any of these, the angle of direction may be found. Also, in these theorems,  $g$  may, in many cases, be taken  $= 16$ , without the small fraction  $\frac{1}{2}$ , which will be near enough for common use.

## PROPOSITION XXII.

93. *To determine the Range on an Oblique Plane; having given the Impetus or Velocity, and the Angle of Direction.*

LET  $AE$  be the oblique plane, at a given angle, either above or below the horizontal plane  $AH$ ;  $AG$  the direction of

of the piece, and AP the altitude due to the projectile velocity at A.



By the last proposition, find the horizontal range AH to the given velocity and direction; draw HE perpendicular to AH, meeting the oblique plane in E; draw EF parallel to AG, and FI parallel to HE; so shall

the projectile pass through I, and the range on the oblique plane will be AI. As is evident by theor. 15 of the Parabola, where it is proved, that if AH, AI be any two lines terminated at the curve, and IF, HE parallel to the axis; then is EF parallel to the tangent AG.

94. *Otherwise*, without the Horizontal Range.

Draw PQ perp. to AG, and QD perp. to the horizontal plane AF, meeting the inclined plane in K; take  $AE = 4AK$ , draw EF parallel to AG, and FI parallel to AP or DQ; so shall AI be the range on the oblique plane. For,  $AH = 4AD$ , therefore EH is parallel to FI, and so on, as above.

*Otherwise*,

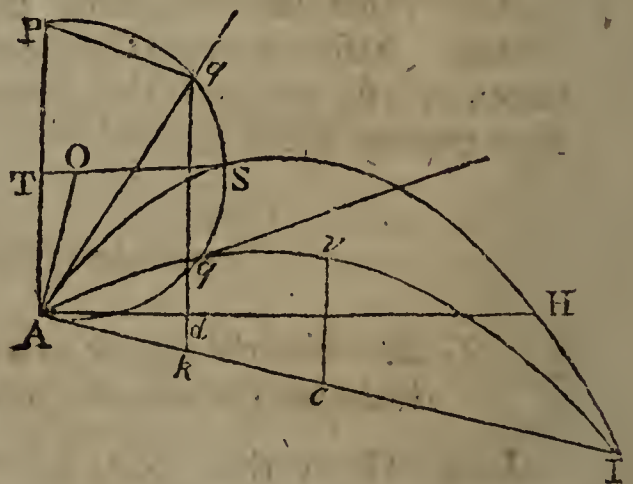
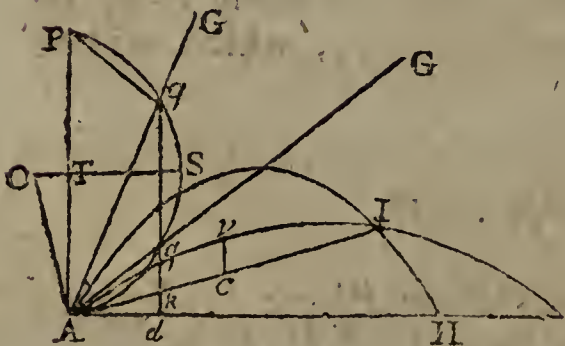
95. Draw Pq making the angle  $APq =$  the angle  $GAI$ ; then take  $AG = 4Aq$ , and draw GI perp. to AH. Or, draw qk perp. to AH, and take  $AI = 4Ak$ . Also, kq will be equal to cv the greatest height above the plane.

For, by cor. 2, prop. 20,  $AP : AG :: AG : 4GI$ ;

and, by sim. triangles,  $AP : AG :: Aq : GI$ ,  
or  $AP : AG :: 4Aq : 4GI$ ;

therefore  $AG = 4Aq$ ; and by sim. triangles,  $AI = 4Ak$ .

Also, qk, or  $\frac{1}{4}GI$ , is = to cv by theor. 13 of the Parabola.



96. *Corol. 1.* If AO be drawn perp. to the plane AI, and AP



AP be bisected by the perpendicular STO; then with the centre O describing a circle through A and P, the same will also pass through q, because the angle GAI, formed by the tangent AI and AG, is equal to the angle APq, which will therefore stand on the same arc Aq.

97. *Corol. 2.* If there be given the range and velocity, or the impetus, the direction will hence be easily found thus: Take  $Ak = \frac{1}{4}AI$ , draw  $kq$  perp. to AH, meeting the circle described with the radius AO in two points q and q; then Aq or Aq will be the direction of the piece. And hence it appears, that there are two directions, which, with the same impetus, give the very same range AI. And these two directions make equal angles with AI and AP, because the arc Pq is equal the arc Aq. They also make equal angles with a line drawn from A through S, because the arc Sq is equal the arc Sq.

98. *Corol. 3.* Or, if there be given the range AI, and the direction Aq; to find the velocity or impetus. Take  $Ak = \frac{1}{4}AI$ , and erect  $kq$  perp. to AH, meeting the line of direction in q; then draw  $qP$  making the  $\angle AqP = \angle Akq$ ; so shall AP be the impetus, or the altitude due to the projectile velocity.

99. *Corol. 4.* The range on an oblique plane, with a given elevation, is directly as the rectangle of the cosine of the direction of the piece above the horizon, and the sine of the direction above the oblique plane, and reciprocally as the square of the cosine of the angle of the plane above or below the horizon.

For, put  $s = \sin. \angle qAI$  or  $APq$ ,

$c = \cos. \angle qAH$  or  $\sin. PAq$ .

$C = \cos. \angle IAH$  or  $\sin. Akd$  or  $Akq$  or  $AqP$ .

Then, in the triangle APq,  $C : s :: AP : Aq$ ;

and in the triangle Akq,  $C : c :: Aq : Ak$ ;

theref. by composition,  $C^2 : cs :: AP : Ak = \frac{1}{4}AI$ .

So that the oblique range  $AI = \frac{cs}{C^2} \times 4AP$ .

100. The range is the greatest when Ak is the greatest; that is, when  $kq$  touches the circle in the middle point S; and then the line of direction passes through S, and bisects the angle formed by the oblique plane and the vertex. Also, the ranges are equal at equal angles above and below this direction for the maximum.

101. *Corol. 5.* The greatest height cv or  $kq$  of the projectile,

tile, above the plane, is equal to  $\frac{s^2}{c^2} \times AP$ . And therefore it is as the impetus and square of the sine of direction above the plane directly, and square of the cosine of the plane's inclination reciprocally.

For  $C$  (fin. AqP) :  $s$  (fin. APq) :: AP : Aq,  
and  $C$  (fin. Akq) :  $s$  (fin. kAq) :: Aq : kq,  
theref. by comp.  $C^2$  :  $s^2$  :: AP : kq.

102. *Corol. 6.* The time of flight in the curve AvI is =  $\frac{2s}{C} \sqrt{\frac{AP}{g}}$ , where  $g = 16\frac{1}{2}$  feet. And therefore it is as the velocity and sine of direction above the plane directly, and cosine of the plane's inclination reciprocally. For the time of describing the curve, is equal to the time of falling freely through GI or 4kq or  $\frac{4s}{C^2} \times AP$ . Therefore, the time being as the square root of the distance,

$$\sqrt{g} : \frac{2s}{C} \sqrt{AP} :: 1'' : \frac{2s}{C} \sqrt{\frac{AP}{g}}, \text{ the time of flight.}$$

## S C H O L I U M.

103. From the foregoing corollaries may be collected the following set of theorems, relating to projects made on any given inclined planes, either above or below the horizontal plane. In which the letters denote as before, namely,

$c$  = cof. of direction above the horizon,  
 $C$  = cof. of inclination of the plane,  
 $s$  = fin. of direction above the plane,  
 $R$  the range on the oblique plane,  
 $T$  the time of flight,  
 $V$  the projectile velocity,  
 $H$  the greatest height above the plane,  
 $a$  the impetus, or alt. due to the velocity  $V$ ,  
 $g = 16\frac{1}{2}$  feet. Then,

$$R = \frac{cs}{C^2} \times 4a = \frac{cs}{C^2 g} V^2 = \frac{gc}{s} T^2 = \frac{4c}{s} H.$$

$$H = \frac{s^2}{C^2} a = \frac{s^2 V^2}{4g C^2} = \frac{sR}{4c} = \frac{g}{4} T^2.$$

$$V = \sqrt{4ag} = C \sqrt{\frac{gR}{cs}} = \frac{gC}{s} T = \frac{2C}{s} \sqrt{gH}.$$

$$T = \frac{2s}{C} \sqrt{\frac{a}{g}} = \frac{sV}{gC} = \sqrt{\frac{sR}{gc}} = 2\sqrt{\frac{H}{g}}.$$

And from any of these, the angle of direction may be found.

## PRACTICAL GUNNERY.

104. THE two foregoing propositions contain the whole theory of projectiles, with theorems for all the cases, regularly arranged for use, both for oblique and horizontal planes. But, before they can be applied to use in resolving the several cases in the practice of gunnery, it is necessary that some more data be laid down, as derived from good experiments made with balls or shells discharged from cannon or mortars, by gunpowder, under different circumstances. For, without such experiments and data, those theorems can be of very little use in real practice, on account of the imperfections and irregularities in the firing of gunpowder, and the expulsion of balls from guns, but more especially on account of the enormous resistance of the air to all projectiles that are made with any velocities that are considerable. As to the cases in which projectiles are made with small velocities, or such as do not exceed 200 or 300, or 400 feet, per second of time, they may be resolved tolerably near the truth, especially for the larger shells, by the parabolic theory laid down above. But, in cases of great projectile velocities, that theory is quite inadequate, without the aid of several data drawn from many and good experiments. For so great is the effect of the resistance of the air to projectiles of considerable velocity, that some of those which in the air range only between 2 and 3 miles at the most, would in vacuo range about ten times as far, or between 20 and 30 miles.

The effects of this resistance are also various, according to the velocity, the diameter, and the weight of the projectile. So that the experiments made with one size of ball or shell, will not serve for another size, though the velocity should be the same; neither will the experiments made with one velocity serve for other velocities, though the ball be the same. And therefore it is plain that, to form proper rules for practical gunnery, we ought to have good experiments made with each size of mortar, and with every variety of charge, from the least to the greatest. And not only so, but these ought also to be repeated at many different angles of elevation, namely, for every single degree between  $30^{\circ}$  and  $60^{\circ}$  elevation, and at intervals of  $5^{\circ}$  above  $60^{\circ}$  and below  $30^{\circ}$ , from the vertical direction to point blank. By such a course of experiments it will be found, that the greatest range, instead of being constantly that for an elevation of  $45^{\circ}$ , as in the parabolic theory, will be at all intermediate degrees between 30 and 45, being

being more or less, both according to the velocity and the weight of the projectile; the smaller velocities and larger shells ranging farthest when projected almost at an elevation of  $45^\circ$ ; while the greatest velocities, especially with the smaller shells, range farthest with an elevation of about  $30^\circ$ .

105. There have, at different times, been made certain small parts of such a course of experiments as is hinted at above. Such as the experiments or practice carried on in the year 1773, on Woolwich Common; in which all the sizes of mortars were used, and a variety of small charges of powder. But they were all at the elevation of  $45^\circ$ ; and consequently these are defective in the higher charges, and in all the other angles of elevation.

Other experiments were also carried on in the same place in the year 1784 and 1786, with various angles of elevation indeed, but with only one size of mortar, and only one charge of powder, and that but a small one too: so that all those nearly agree with the parabolic theory. Other experiments have also been carried on with the ballistic pendulum, at different times; from which have been obtained some of the laws for the quantity of powder, the weight and velocity of the ball, the length of the gun, &c. Namely, that the velocity of the ball varies as the square root of the charge directly, and as the square root of the weight of ball reciprocally; and that, some rounds being fired with a medium length of one pounder gun, at  $15^\circ$  and  $45^\circ$  elevation, and with 2, 4, 8, and 12 ounces of powder, gave nearly the velocities, ranges, and times of flight, as they are here set down in the following Table.

Powder.	Elevation. of gun.	Velocity of ball.	Range.	Time of flight.
oz.		feet.	feet.	
2	$15^\circ$	860	4100	9''
4	15	1230	5100	12
8	15	1640	6000	$14\frac{1}{2}$
12	15	1680	6700	$15\frac{1}{2}$
2	45	860	5100	21

106. But as we are not yet provided with a sufficient number and variety of experiments, on which to establish true rules for practical gunnery, independent of the parabolic theory, we must content ourselves with the data of some one certain

certain experimented range and time of flight, at a given angle of elevation; and then by help of these, and the rules in the parabolic theory, determine the like circumstances for other elevations that are not greatly different from the former, as in the following examples:

107. *Example 1.* If a ball of 1lb. acquire a velocity of 1600 feet per second, when fired with 8 ounces of powder; it is required to find with what velocity each of the several kinds of shells will be discharged by the full charges of powder, viz.

Nature of the shells in inches	13	10	8	$5\frac{1}{2}$	$4\frac{2}{5}$
Their weight in lbs. - -	196	90	48	16	8
Charge of powder in lbs. -	9	4	2	1	$\frac{1}{2}$
Ans. the velocities are -	485	477	462	566	566

108. *Exam. 2.* If a shell be found to range 1000 yards, when discharged at an elevation of  $45^\circ$ ; how far will it range when the elevation is  $30^\circ 16'$ , the charge of powder being the same? *Ans.* 2612 feet or 871 yards.

109. *Exam. 3.* The range of a shell, at  $45^\circ$  elevation, being found to be 3750 feet: at what elevation must the piece be set, to strike an object at the distance of 2810 feet, with the same charge of powder? *Ans.* at  $24^\circ 16'$ , or at  $65^\circ 44'$ .

110. *Exam. 4.* With what impetus, velocity, and charge of powder, must a 13 inch shell be fired, at an elevation of  $32^\circ 12'$ , to strike an object at the distance of 3250 feet? *Ans.* impetus 1802, veloc. 340, charge 4lb.  $7\frac{1}{2}$  oz.

111. *Exam. 5.* A shell being found to range 3500 feet, when discharged at an elevation of  $25^\circ 12'$ ; how far then will it range at an elevation of  $36^\circ 15'$  with the same charge of powder? *Ans.* 4332 feet.

112. *Exam. 6.* If, with a charge of 9lb. of powder, a shell range 4000 feet; what charge will suffice to throw it 3000 feet, the elevation being  $45^\circ$  in both cases? *Ans.*  $6\frac{3}{4}$  lb. of powder.

113. *Exam. 7.* What will be the time of flight for any given range, at the elevation of  $45^\circ$ ? *Ans.* the time in secs. is  $\frac{1}{4}$  the 1q. root of the range in feet.

114. *Exam. 8.* In what time will a shell range 3250 feet, at an elevation of  $32^\circ$ ? *Ans.*  $11\frac{1}{4}$  sec. nearly.

VOL. II. M 115. *Exam. 9.*

115. *Exam. 9.* How far will a shot range on a plane which ascends  $8^{\circ} 15'$ , and on another which descends  $8^{\circ} 15'$ ; the impetus being 3000 feet, and the elevation of the piece  $32^{\circ} 30'$ ?

Ans. 4244 feet on the ascent,  
and 6745 feet on the descent.

116. *Exam. 10.* How much powder will throw a 13 inch shell 4244 feet on an inclined plane, which ascends  $8^{\circ} 15'$ , the elevation of the mortar being  $32^{\circ} 30'$ ?

Ans. 7 $\frac{3}{4}$ 765lb. or 7lb. 6 oz.

117. *Exam. 11.* At what elevation must a 13 inch mortar be pointed, to range 6745 feet, on a plane which descends  $8^{\circ} 15'$ ; the charge being  $7\frac{3}{8}$ lb. of powder?

Ans.  $32^{\circ} 30'$ .

118. *Exam. 12.* In what time will a 13 inch shell strike a plane which rises  $8^{\circ} 30'$ , when elevated  $45^{\circ}$ , and discharged with an impetus of 2304 feet?

Ans.  $14\frac{2}{5}$  seconds.

## THE DESCENT OF BODIES ON INCLINED PLANES AND CURVE SURFACES.—THE MOTION OF PENDULUMS.

### • PROPOSITION XXIII.

119. *If a Weight W be sustained on an Inclined Plane AB, by a Power P, acting in a Direction WP, parallel to the Plane. Then*

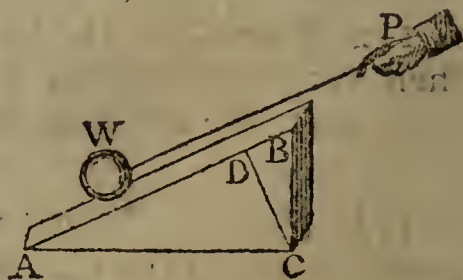
The Weight of the Body, W,  
The Sustaining Power, P, and  
The Pressure on the Plane, p,  
are respectively as

The Length AB,  
The Height BC, and  
The Base AC,  
of the Plane.

FOR, draw CD perpendicular to the plane. Now here are three forces, keeping one another in equilibrio; namely, the weight, or force of gravity, acting perpendicular to AC, or parallel to BC;

the power acting parallel to DB;

and the pressure perpendicular to AB, or parallel to DC: but when three forces keep one another in equilibrio, they are proportional to the sides of the triangle CBD, made by lines



lines in the direction of those forces, by prop. 8 ; therefore those forces are to one another as  $CB, BD, DC$ . But the two triangles  $ABC, CBD$  are equiangular, and have their like sides proportional ; therefore the three  $CB, BD, DC$ , are to one another respectively as the three  $AB, BC, AC$  ; which therefore are as the three forces  $W, P, p$ .

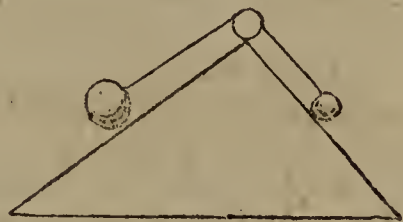
120. *Corol. 1.* Hence the weight  $W$ , power  $P$ , and pressure  $p$ , are respectively as radius, sine, and cosine, of the plane's elevation  $BAC$  above the horizon.

For, since the sides of triangles are as the sines of their opposite angles, therefore the three  $AB, BC, AC$ , are respectively as - - - sine  $C$ , sine  $A$ , sine  $B$ , or as - - - - - radius, sine, cosine, of the angle  $A$  of elevation.

Or, the three forces are as  $AC, CD, AD$  ; perpendicular to their directions.

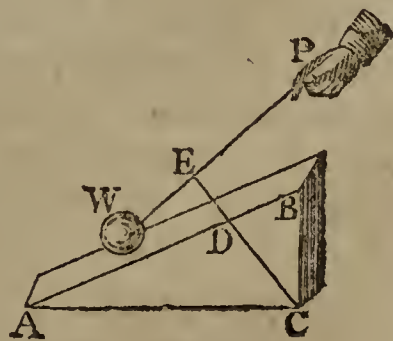
121. *Corol. 2.* The power or relative weight that urges a body  $W$  down the inclined plane, is  $= \frac{BC}{AB} \times W$  ; or the force with which it descends, or endeavours to descend, is as the sine of the angle  $A$  of inclination.

122. *Corol. 3.* Hence, if there be two planes of the same height, and two bodies be laid upon them which are proportional to the lengths of the planes ; they will have an equal tendency to descend down the planes.



And consequently they will mutually sustain each other if they be connected by a string acting parallel to the planes.

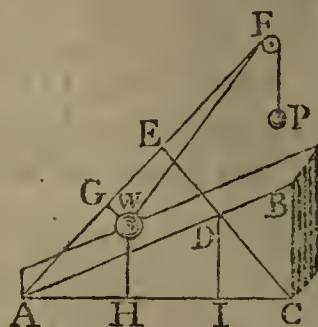
123. *Corol. 4.* In the same manner, when the power  $P$  acts in any other direction whatever,  $WP$  ; by drawing  $CDE$  perpendicular to the direction  $WP$ , the three forces in equilibrio, namely, the weight  $W$ , the power  $P$ , and the pressure on the plane, will still be respectively as  $AC, CD, AD$ , drawn perpendicular to the direction of those forces.



## PROPOSITION XXIV.

124. *If a Weight W on an Inclined Plane AB, be in Equilibrium with another Weight P hanging freely; then if they be set a moving, their Perpendicular Velocities, in that Place, will be Reciprocally as those Weights.*

LET the weight W descend a very small space, from W to A, along the plane, by which the string PFW, will come into the position PFA. Draw WH perpendicular to the horizon AC, and WG perpendicular to AF: then WH will be the space perpendicularly descended by the weight W; and AG, or the difference between FA and FW, will be the space perpendicularly ascended by the weight P; and their perpendicular velocities are as those spaces WH and AG passed over in those directions, in the same time. Draw CDE perpendicular to AF, and DI perpendicular to AC.



Then,

in the sim. figs, AGWH and AEDI,	$AG:WH::AE:DI;$
and in the sim. tri. AEC, DIC,	$AC:CD::AE:DI;$
but, by cor. 4, prop. 23,	$AC:CD::W:P;$
therefore, by equality,	$AG:WH::W:P.$

That is, their perpendicular spaces, or velocities, are reciprocally as their weights or masses.

125. *Corol. 1.* Hence it follows, that if any two bodies be in equilibrio on two inclined planes, and if they be set a moving, their perpendicular velocity will be reciprocally as their weights. Because the perpendicular weight which sustains the one, would also sustain the other.

126. *Corol. 2.* And hence also, if two bodies sustain each other in equilibrio, on any planes, and they be put in motion; then each body multiplied by its perpendicular velocity, will give equal products.

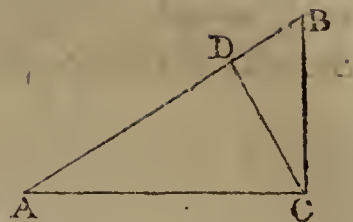
## PROPOSITION XXV.

127. *The Velocity acquired by a Body descending freely down an Inclined Plane AB, is to the Velocity acquired by a Body falling Perpendicularly, in the same Time; as the Height of the Plane BC, is to its Length AB.*

FOR the force of gravity, both perpendicularly and on the plane,



plane, is constant; and these two, by corol. 2, prop. 23, are to each other as  $AB$  to  $BC$ . But, by art. 28, the velocities generated by any constant forces, in the same time, are as those forces. Therefore the velocity down  $BA$  is to the velocity down  $BC$ , in the same time, as the force on  $BA$  to the force on  $BC$ : that is, as  $BC$  to  $BA$ .



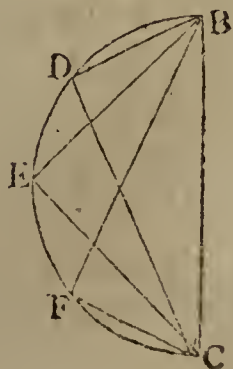
128. *Corol. 1.* Hence, as the motion down an inclined plane is produced by a constant force, it will be an uniformly accelerated motion; and therefore the laws before laid down for accelerated motions in general, hold good for motions on inclined planes; such, for instance, as the following: That the velocities are as the times of descending from rest; that the spaces descended are as the squares of the velocities, or squares of the times; and that, if a body be thrown up an inclined plane, with the velocity it acquired in descending, it will lose all its motion, and ascend to the same height, in the same time, and will repass any point of the plane with the same velocity as it passed it in descending.

129. *Corol. 2.* Hence also, the space descended down an inclined plane, is to the space descended perpendicularly, in the same time, as the height of the plane  $CB$ , to its length  $AB$ , or as the sine of inclination to radius. For the spaces described by any forces, in the same time, are as the forces, or as the velocities.

130. *Corol. 3.* Consequently the velocities and spaces descended, by bodies down different inclined planes, are as the sines of elevation of the planes.

131. *Corol. 4.* If  $CD$  be drawn perpendicular to  $AB$ ; then, while a body falls freely through the perpendicular space  $BC$ , another body will, in the same time, descend down the part of the plane  $BD$ . For, by similar triangles,  $BC : BD :: BA : BC$ , that is, as the spaces descended, by corol. 2.

Or, in any right-angled triangle  $BDC$ , having its hypotenuse  $BC$  perpendicular to the horizon, a body will descend down any of its three sides  $BD$ ,  $BC$ ,  $DC$ , in the same time. And therefore, if upon the diameter  $BC$  a circle be described, the times of descending down any chords  $BD$ ,  $BE$ ,  $BF$ ,  $DC$ ,  $EC$ ,  $FC$ , &c., will be all equal, and each

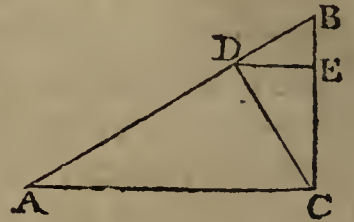


each equal to the time of falling freely through the perpendicular diameter BC.

## PROPOSITION XXVI.

132. *The Time of descending down the Inclined Plane BA, is to the Time of falling through the Height of the Plane BC, as the Length BA to the Height BC.*

DRAW CD perpendicular to AB. Then the times of describing BD and BC are equal, by the last corol. Call that time  $t$ , and the time of describing BA call T.



Now, because the spaces described by constant forces, are as the squares of the times; therefore  $t^2 : T^2 :: BD : BA$ .

But the three BD, BC, BA are in continual proportion; therefore  $BD : BA :: BC^2 : BA^2$ ; hence, by equality,  $t^2 : T^2 :: BC^2 : BA^2$ ,  
or  $t : T :: BC : BA$ .

133. *Corol.* Hence the times of descending down different planes, of the same height, are to one another as the lengths of the planes.

## PROPOSITION XXVII.

134. *A Body acquires the Same Velocity in descending down any Inclined Plane BA, as by falling perpendicularly through the Height of the Plane BC.*

FOR, the velocities generated by any constant forces, are in the compound ratio of the forces and times of acting. But if we put

F to denote the whole force of gravity in BC,  
 $f$  the force on the plane AB,  
 $t$  the time of describing BC, and  
 T the time of descending down AB;  
 then, by art. 119,  $F : f :: BA : BC$ ;  
 and by art. 132,  $t : T :: BC : BA$ ;  
 theref. by comp.  $Ft : fT :: 1 : 1$ .

That is, the compound ratio of the forces and times, or the ratio of the velocities, is a ratio of equality.

135. *Corol.* 1.

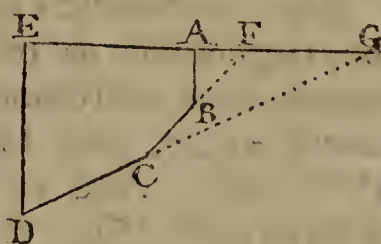
135. *Corol. 1.* Hence the velocities acquired, by bodies descending down any planes, from the same height, to the same horizontal line, are equal.

136. *Corol. 2.* If the velocities be equal, at any two equal altitudes, D, E; they will be equal, at all other equal altitudes A, C.

137. *Corol. 3.* Hence also, the velocities acquired by descending down any planes, are as the square roots of the heights.

PROPOSITION XXVIII.

138. *If a Body descend down any Number of Contiguous Planes AB, BC, CD; it will at last acquire the Same Velocity, as a Body falling perpendicularly through the Same Height ED, supposing the Velocity not altered by changing from one Plane to another.*



PRODUCE the planes DC, CB to meet the horizontal line EA produced in F and G. Then, by art. 135, the velocity at B is the same, whether the body descend through AB or FB. And therefore the velocity at C will be the same, whether the body descend through ABC or through FC, which is also again, by art. 135, the same as by descending through GC. Consequently it will have the same velocity at D, by descending through the planes AB, CD, as by descending through the plane GD; supposing no obstruction to the motion by the body impinging on the planes at B and C: and this again is the same velocity as by descending through the same perpendicular height ED.

139. *Corol. 1.* If the lines ABCD, &c, be supposed indefinitely small, they will form a curve line, which will be the path of the body; from which it appears, that a body acquires also the same velocity in descending along any curve, as in falling perpendicularly through the same height.

140. *Corol. 2.* Hence also, bodies acquire the same velocity, by descending from the same height, whether they descend perpendicularly, or down any planes, or down any curve or curves. And if their velocities be equal, at any one height, they will be equal at all other equal heights. Therefore the velocity acquired by descending down any lines or curves, are as the square roots of the perpendicular heights.

141. *Corol.*

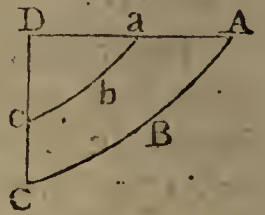
141. *Corol. 3.* And a body, after its descent through any curve, will acquire a velocity which will carry it to the same height through an equal curve, or through any other curve, either by running up the smooth concave side, or by being retained in the curve by a string, and vibrating like a pendulum: Also, the velocities will be equal, at all equal altitudes; and the ascent and descent will be performed in the same time, if the curves be the same.

## PROPOSITION XXIX.

142. *The Times in which Bodies descend through Similar Parts of Similar Curves, ABC, abc, placed alike, are as the Square Roots of their Lengths.*

THAT is, the time in AC is to the time in ac, as  $\sqrt{AC}$  to  $\sqrt{ac}$ .

For, as the curves are similar, they may be considered as made up of an equal number of corresponding parts, which are every where, each to each, proportional to the whole. And as they are placed alike, the corresponding small similar parts will also be parallel to each other. But the time of describing each of these pairs of corresponding parallel parts, by art. 128, are as the square roots of their lengths, which, by the supposition, are as  $\sqrt{AC}$  to  $\sqrt{ac}$ , the roots of the whole curves. Therefore, the whole times are in the same ratio of  $\sqrt{AC}$  to  $\sqrt{ac}$ .



143. *Corol. 1.* Because the axes DC, Dc, of similar curves, are as the lengths of the similar parts AC, ac; therefore the times of descent in the curves AC, ac, are as  $\sqrt{DC}$  to  $\sqrt{Dc}$ , or the square roots of their axes.

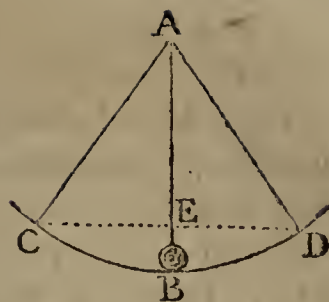
144. *Corol. 2.* As it is the same thing, whether the bodies run down the smooth concave side of the curves, or be made to describe those curves by vibrating like a pendulum, the lengths being DC, Dc; therefore the times of the vibration of pendulums, in similar arcs of any curves, are as the square roots of the lengths of the pendulums.

## SCHOLIUM.

145. Having, in the last corollary, mentioned the pendulum, it may not be improper here to add some remarks concerning it.

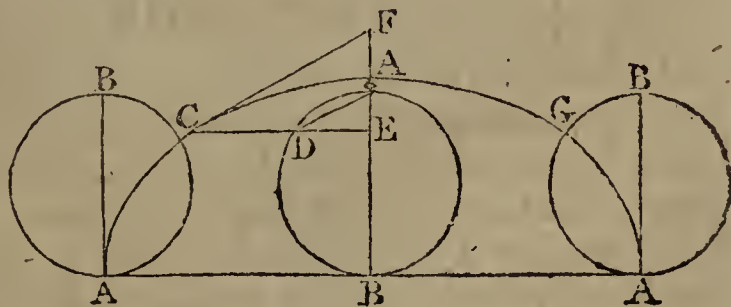
A pen-

A pendulum consists of a ball, or any other heavy body B, hung by a fine string or thread, moveable about a centre A, and describing the arc CBD; by which vibration the same motion happens to this heavy body, as would happen to any body descending by its gravity along the spherical superficies CBD, if that superficies was perfectly hard and smooth. If the pendulum be carried to the situation AC, and then let fall, the ball in descending will describe the arc CB; and in the point B it will have that velocity which is acquired by descending through CB, or by a body falling freely through EB. This velocity will be sufficient to cause the ball to ascend through an equal arc BD, to the same height D from whence it fell at C: having there lost all its motion, it will again begin to descend by its own gravity; and in the lowest point B it will acquire the same velocity as before; which will cause it to reascend to C: and thus, by ascending and descending, it will perform continual vibrations in the circumference CBD. And if the motions of pendulums met with no resistance from the air, and if there were no friction at the centre of motion A, the vibrations of pendulums would never cease. But from those obstructions, though small, it happens, that the velocity of the ball in the point B is a little diminished in every vibration; and consequently it does not return precisely to the same points C or D, but the arcs described continually become shorter and shorter, till at length they grow insensible; unless the motion be assisted by a mechanical contrivance, as in clocks, called a maintaining power.



DEFINITION.

146. If the circumference of a circle be rolled on a right line, beginning at any point A, and continued till the same point A arrive at the line



again, making just one revolution, and thereby measuring out a straight line ABA equal to the circumference of the circle, while the point A in the circumference traces out a curve line ACAGA; then this curve is called a cycloid; and some of its properties are contained in the following lemma.

LEM-

## L E M M A.

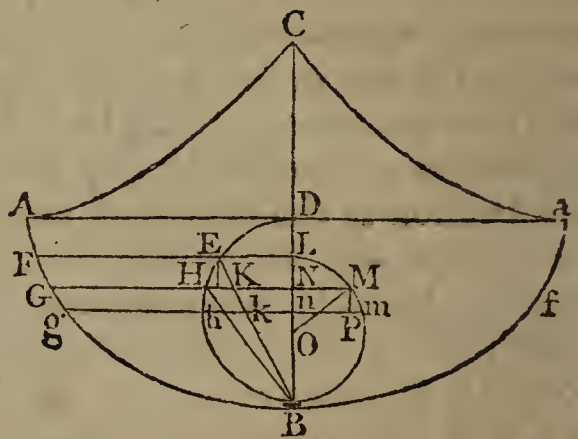
147. If the generating or revolving circle be placed in the middle of the cycloid, its diameter coinciding with the axis AB, and from any point there be drawn the tangent CF, the ordinate CDE perp. to the axis, and the chord of the circle AD: Then the chief properties are these:

- The right-line CD = the circular arc AD;
- The cycloidal arc AC = double the chord AD;
- The semi-cycloid ACA = double the diameter AB, and
- The tangent CF is parallel to the chord AD.

## PROPOSITION XXX.

148. When a Pendulum vibrates in a Cycloid; the Time of one Vibration, is to the Time in which a Body falls through Half the Length of the Pendulum, as the Circumference of a Circle is to its Diameter.

LET ABa be the cycloid; DB its axis, or the diameter of the generating semicircle DEB; CB = 2DB the length of the pendulum, or radius of curvature at B. Let the ball descend from F, and, in vibrating, describe the arc FBf. Divide FB into innumerable small parts, one of which is Gg; draw FEL, GM, gm perpendicular to DB. On LB describe the semi-circle LMB, whose centre is O; draw Mp parallel to DB; also draw the chords BE, BH, EH, and the radius OM.



Now the triangles BEH, BHK are equiangular; therefore  $BK : BH :: BH : BE$ , or  $BH^2 = BK \cdot BE$ , or  $BH = \sqrt{BK \cdot BE}$ . And the equiangular triangles Mmp, MON, give  $Mp : Mm :: MN : MO$ . Also, by the nature of the cycloid, Hh is equal and parallel to Gg.

If another body descend down the chord EB, it will have the same velocity as the ball in the cycloid has at the same height. So that Kk and Gg are passed over with the same velocity, and consequently the time in passing them will be as their lengths Gg, Kk, or as Hh to Kk, or BH to BK by similar triangles, or  $\sqrt{BK \cdot BE}$  to BK, or  $\sqrt{BE}$  to  $\sqrt{BK}$ , or as  $\sqrt{BL}$  to  $\sqrt{BN}$  by similar triangles.

That is, the time in Gg : time in Kk ::  $\sqrt{BL} : \sqrt{BN}$ .

Ag an,

Again, the time of describing any space with an uniform motion, is directly as the space, and reciprocally as the velocity; also, the velocity in K or Kk, is to the velocity at B, as  $\sqrt{EK}$  to  $\sqrt{EB}$ , or as  $\sqrt{LN}$  to  $\sqrt{LB}$ ; and the uniform velocity for EB is equal to half that at the point B, therefore the

time in Kk : time in EB ::  $\frac{Kk}{\sqrt{LN}}$  :  $\frac{EB}{\frac{1}{2}\sqrt{LB}}$  :: (by sim. tri.)

$\frac{Nn}{\sqrt{LN}}$  :  $\frac{LB}{\frac{1}{2}\sqrt{LB}}$  :: Nn or Mp :  $2\sqrt{BL \cdot LN}$ .

That is, the time in Kk : time in EB :: Mp :  $2\sqrt{BL \cdot LN}$ .

But it was, time in Gg : time in Kk ::  $\sqrt{BL}$  :  $\sqrt{BN}$ ; theref.

by comp. time in Gg : time in EB :: Mp :  $2\sqrt{BN \cdot NL}$  or  $2NM$ .

But, by sim. tri. Mm :  $2OM$  or  $BL$  :: Mp :  $2NM$ .

Theref. time in Gg : time in EB :: Mm :  $BL$ .

Consequently the sum of all the times in all the Gg's, is to the time in EB, or the time in DB, which is the same thing, as the sum of all the Mm's, is to  $LB$ ;

that is, the time in Fg : time in DB :: Lm :  $LB$ ,

and the time in FB : time in DB :: LMB :  $LB$ ,

or the time in FBf : time in DB ::  $2LMB$  :  $LB$ .

That is, the time of one whole vibration,  
is to the time of falling through half CB,  
as the circumference of any circle,  
is to its diameter.

149. *Corol. I.* Hence all the vibrations of a pendulum in a cycloid, whether great or small, are performed in the same time, which time is to the time of falling through the axis, or half the length of the pendulum, as  $3.1416$  to  $1$ , the ratio of the circumference to its diameter; and hence that time is easily found thus. Put  $p = 3.1416$ , and  $l$  the length of the pendulum, also  $g$  the space fallen by a heavy body in  $1''$  of time.

then  $\sqrt{g}$  :  $\sqrt{\frac{1}{2}l}$  ::  $1''$  :  $\sqrt{\frac{l}{2g}}$  the time of falling through  $\frac{1}{2}l$ ,

theref.  $1$  :  $p$  ::  $\sqrt{\frac{l}{2g}}$  :  $p\sqrt{\frac{l}{2g}}$ , which therefore is the time of one vibration of the pendulum.

150. And if the pendulum vibrate in a small arc of a circle; because that small arc nearly coincides with the small cycloidal arc at the vertex B; therefore the time of vibration in the small arc of a circle, is nearly equal to the time of vibration in  
the

the cycloidal arc ; and consequently the time of vibration in a small circular arc, is equal to  $p\sqrt{\frac{l}{2g}}$ , where  $l$  is the radius of the circle.

151. So that, if one of these,  $g$  or  $l$ , be found by experiment, this theorem will give the other. Thus, if  $g$ , or the space fallen through by a heavy body in 1'' of time, be found, then this theorem will give the length of the seconds pendulum. Or, if the length of the seconds pendulum be observed by experiment, which is the easier way ; this theorem will give  $g$  the descent of gravity in 1''. Now, in the latitude of London, the length of a pendulum which vibrates seconds, has been found to be  $39\frac{1}{8}$  inches ; and this being written for  $l$  in the theorem, it gives  $p\sqrt{\frac{39\frac{1}{8}}{2g}} = 1''$  : from hence is found  $g = \frac{1}{2}p^2l = \frac{1}{2}p^2 \times 39\frac{1}{8} = 193.07$  inches =  $16\frac{1}{2}$  feet, for the descent of gravity in 1'' ; which it has also been found to be, very exactly, by many accurate experiments.

## S C H O L I U M.

152. Hence is found the length of a pendulum that shall make any number of vibrations in a given time. Or, the number of vibrations that shall be made by a pendulum of a given length. Thus, suppose it were required to find the length of a half seconds pendulum, or a quarter-seconds pendulum ; that is, a pendulum to vibrate twice in a second, or 4 times in a second. Then, since the time of vibration is as the square root of the length,

$$\text{therefore } 1 : \frac{1}{2} :: \sqrt{39\frac{1}{8}} : \sqrt{l},$$

$$\text{or } 1 : \frac{1}{4} :: 39\frac{1}{8} : \frac{39\frac{1}{8}}{4} = 9\frac{3}{4} \text{ inches nearly, the}$$

length of the half-seconds pendulum.

And  $1 : \frac{1}{6} :: 39\frac{1}{8} : 2\frac{4}{9}$  inches, the length of the quarter-seconds pendulum.

Again, if it were required to find how many vibrations a pendulum of 80 inches long will make in a minute. Here

$$\sqrt{80} : \sqrt{39\frac{1}{8}} :: 60'' \text{ or } 1' : 60\sqrt{\frac{39\frac{1}{8}}{80}} = 7\frac{1}{2}\sqrt{31.3} = 41.95987, \text{ or almost } 42 \text{ vibrations in a minute.}$$

153. In these propositions, the thread is supposed to be very fine, or of no sensible weight, and the ball very small, or all the matter united in one point ; also, the length of the pendulum, is the distance from the point of suspension, or centre of motion, to this point, or centre of the small ball.



ball. But if the ball be large, or the string very thick, or the vibrating body be of any other figure; then the length of the pendulum is different, and is measured, from the centre of motion, not to the centre of magnitude of the body, but to such a point, as that if all the matter of the pendulum were collected into it, it would then vibrate in the same time as the compound pendulum; and this point is called the Centre of Oscillation; a point which will be treated of in what follows.

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THE MECHANICAL POWERS, &c.

154. **WEIGHT** and Power, when opposed to each other, signify the body to be moved, and the body that moves it; or the patient and agent. The power is the agent, which moves, or endeavours to move, the patient or weight.

155. **Equilibrium**, is an equality of action or force, between two or more powers or weights, acting against each other, by which they destroy each other's effects, and remain at rest.

156. **Machine, or Engine**, is any mechanical instrument contrived to move bodies. And it is composed of the mechanical powers.

157. **Mechanical Powers**, are certain simple machines, which are commonly employed for raising greater weights, or overcoming greater resistances, than could be effected by the natural strength without them.—These are usually accounted six in number, viz. the Lever and Balance, the Pulley, the Wheel and Axle, the Wedge, the Inclined Plane, and the Screw.

158. **Mechanics**, is the science of forces, and the effects they produce, when applied to machines, in the motion of bodies.

159. **Statics**, is the science of weights, especially when considered in a state of equilibrium.

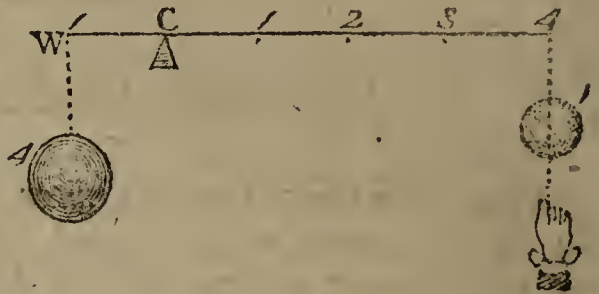
160. **Centre of Motion**, is the fixed point about which a body moves. And the **Axis of Motion**, is the fixed line about which it moves.

161. **Centre of Gravity**, is a certain point, upon which a body being freely suspended, it will rest in any position.

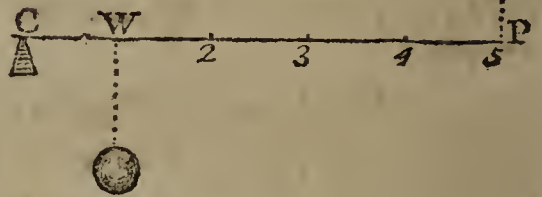
OF THE LEVER.

162. A LEVER is any inflexible rod, bar, or beam, which serves to raise weights, while it is supported at a point by a fulcrum or prop, which is the centre of motion. The lever is supposed to be void of gravity or weight, to render the demonstrations easier and simpler. There are four kinds of levers.

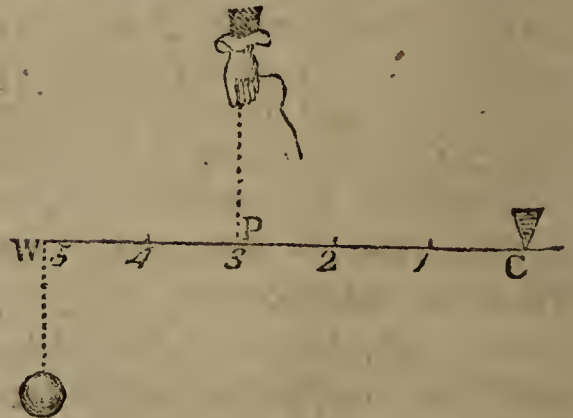
163. A Lever of the First Kind has the prop C between the weight W and the power P. And of this kind are balances, scales, crow's, hand-spikes, scissars, pinchers, &c.



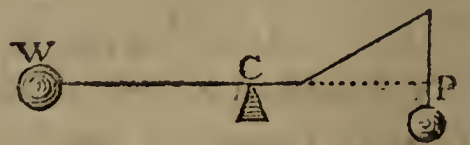
164. A Lever of the Second Kind has the weight between the power and the prop. Such as oars, rudders, cutting knives that are fixed at one end, &c.



165. A Lever of the Third Kind has the power between the weight and the prop. Such as tongs, the bones and muscles of animals, a man rearing a ladder, &c.



166. A Fourth Kind is sometimes added, called the Bended Lever. As a hammer drawing a nail.

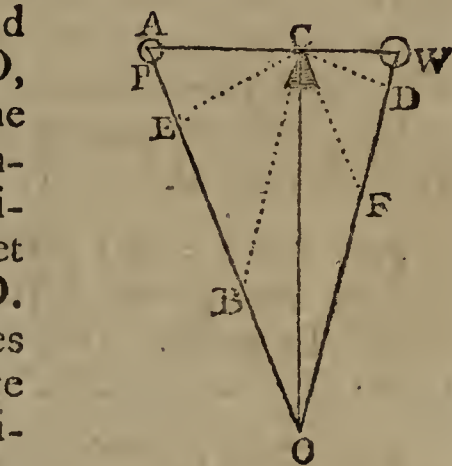


167. In all these machines, the power may be represented by a weight, which is its most natural measure, acting downward: but having its direction changed, when necessary, by means of a fixed pulley.

PROPOSITION XXXI.

168. *When the Weight and Power keep the Lever in Equilibrio, they are to each other Reciprocally as the Distances of their Lines of Direction from the Prop. That is,  $P : W :: CD : CE$ ; where  $CD$  and  $CE$  are perpendicular to  $WO$  and  $AO$ , which are the Directions of the two Weights, or the Weight and Power  $W$  and  $A$ .*

FOR, draw  $CF$  parallel to  $AO$ , and  $CB$  parallel to  $WO$ : Also, join  $CO$ , which will be the direction of the pressure on the prop  $C$ ; for there cannot be an equilibrium unless the directions of the three forces all meet in, or tend to, the same point, as  $O$ . Then, because these three forces keep each other in equilibrio, they are proportional to the sides of the triangle  $CBO$  or  $CFO$ , which are drawn in the direction of those forces; therefore



$$P : W :: CF : FO \text{ or } CB.$$

But, because of the parallels, the two triangles  $CDF$ ,  $CEB$  are equiangular, therefore

$$CD : CE :: CF : CB.$$

Hence, by equality,  $P : W :: CD : CE$ .

That is, each force is reciprocally proportional to the distance of its direction from the fulcrum.

And it will be found that this demonstration will serve for all the other kinds of levers, by drawing the lines as directed.

169. *Corol. 1.* When the two forces act perpendicularly on the lever, as two weights, &c; then, in case of an equilibrium,  $D$  coincides with  $W$ , and  $E$  with  $P$ ; consequently then the above proportion becomes  $P : W :: CW : CP$ , or the distances of the two forces from the fulcrum, taken on the lever, are reciprocally proportional to those forces.

170. *Corol. 2.* If any force  $P$  be applied to a lever at  $A$ ; its effect on the lever, to turn it about the centre of motion  $C$ , is as the length of the lever  $CA$ , and the sine of the angle of direction  $CAE$ . For the perp.  $CE$  is as  $CA \times s. \angle A$ .

171. *Corol. 3.* Because the product of the extremes is equal to the product of the means, therefore the product of the

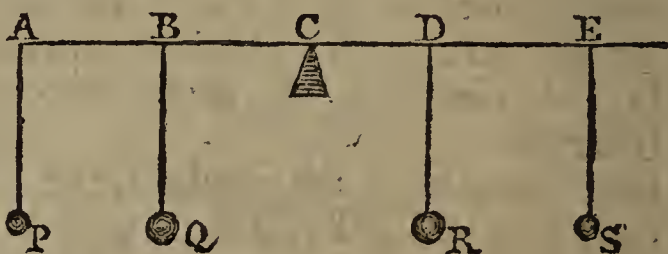
the power by the distance of its direction, is equal to the product of the weight by the distance of its direction.

That is,  $P \times CE = W \times CD$ .

172. *Corol. 4.* If the lever, with the weight and power fixed to it, be made to move about the centre  $C$ ; the momentum of the power will be equal to the momentum of the weight; and their velocities will be in reciprocal proportion to each other. For the weight and power will describe circles whose radii are the distances  $CD, CE$ ; and since the circumferences or spaces described, are as the radii, and also as the velocities, therefore the velocities are as the radii  $CD, CE$ ; and the momenta, which are as the masses and velocities, are as the momenta and radii; that is, as  $P \times CE$  and  $W \times CD$ , which are equal by cor. 3.

173. *Corol. 5.* In a straight lever, kept in equilibrio by a weight and power acting perpendicularly; then, of these three, the power, weight, and pressure on the prop, any one is as the distance of the other two.

174. *Corol. 6.* If several weights  $P, Q, R, S$ , act on a straight lever, and keep it in equilibrio; then the sum of the products on one side of the prop,

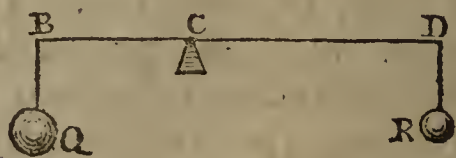


will be equal to the sum on the other side, made by multiplying each weight by its distance; namely,

$$P \times AC + Q \times BC = R \times DC + S \times EC.$$

For, the effect of each weight to turn the lever, is as the weight multiplied by its distance; and in the case of an equilibrium, the sums of the effects, or of the products on both sides, are equal.

175. *Corol. 7.* Because, when two weights  $Q$  and  $R$  are in equilibrio;  $Q : R :: CD : CB$ ;



therefore, by composition,  $Q + R : Q :: BD : CD$ ,  
and  $Q + R : R :: BD : CB$ .

That is, the sum of the weights is to either of them, as the sum of their distances is to the distance of the other.

SCHOLIUM.

176. On the foregoing principles depends the nature of scales and beams, for weighing all sorts of commodities. For, if the weights be equal, then will the distances be equal also, which gives the construction of the common scales, which ought to have these properties:



1<sup>st</sup>, That the points of suspension of the scales and the centre of motion of the beam, A, B, C, should be in a straight line: 2<sup>d</sup>, That the arms AB, BC be of an equal length: 3<sup>d</sup>, That the centre of gravity be in the centre of motion B, or a little below it: 4<sup>th</sup>, That they be in equilibrio when empty: 5<sup>th</sup>, That there be as little friction as possible at the centre B. A defect in any of these properties, makes the scales either imperfect or false. But it often happens that the one side of the beam is made shorter than the other, and the defect covered by making that scale the heavier, by which means the scales hang in equilibrio when empty; but when they are charged with any weights, so as to be still in equilibrio, those weights are not equal; but the deceit will be detected by changing the weights to the contrary sides, for then the equilibrio will be immediately destroyed.

177. To find the true weight of any body by such a false balance:—First weigh the body in one scale, and afterwards weigh it in the other; then the mean proportional between these two weights, will be the true weight required. For, if any body *b* weigh *W* pounds or ounces in the scale D, and only *w* pounds or ounces in the scale E: then we have these two equations, namely,  $AB \cdot b = BC \cdot W$ .

$$\text{and } BC \cdot b = AB \cdot w;$$

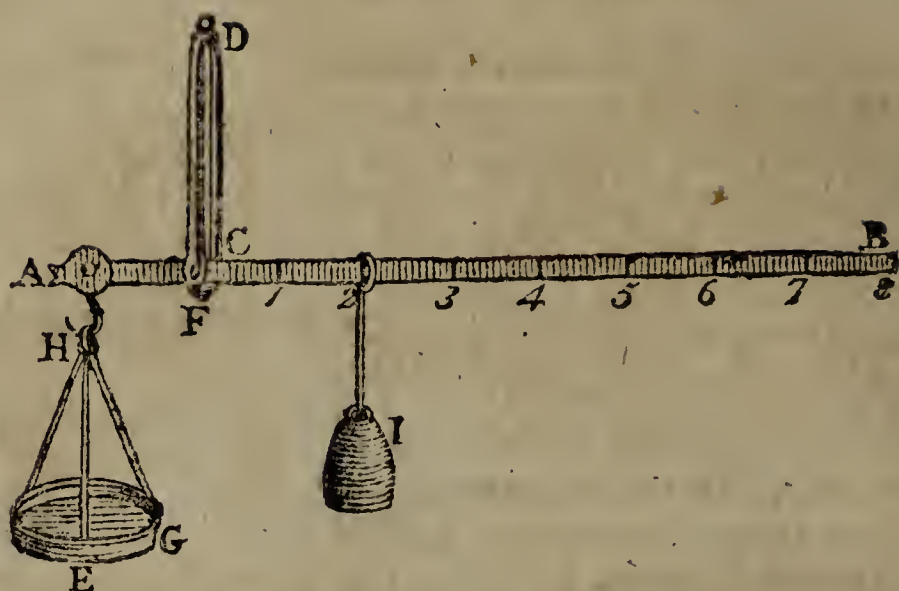
the product of the two is  $AB \cdot BC \cdot b^2 = AB \cdot BC \cdot Ww$ ;

hence then  $b^2 = Ww$ ,

and  $b = \sqrt{Ww}$ ,

the mean proportional, which is the true weight of the body *b*.

178. The Roman Statera, or Steelyard, is also a lever, but of unequal brachia or arms, so contrived, that one weight only may serve to weigh a great many, by sliding it backward and forward, to different distances on the longer arm of the lever; and it is thus constructed:



Let  $AB$  be the steelyard, and  $C$  its centre of motion, from whence the divisions must commence if the two arms just balance each other: if not, slide the constant moveable weight  $I$  along from  $B$  towards  $C$ , till it just balance the other end without a weight, and there make a notch in the beam, marking it with a cypher  $0$ . Then hang on at  $A$  a weight  $W$  equal to  $I$ , and slide  $I$  back towards  $B$  till they balance each other; there notch the beam, and mark it with  $1$ . Then make the weight  $W$  double of  $I$ , and sliding  $I$  back to balance it, there mark it with  $2$ . Do the same at  $3, 4, 5, \&c$ , by making  $W$  equal to  $3, 4, 5, \&c$ , times  $I$ ; and the beam is finished. Then, to find the weight of any body  $b$  by the steelyard; take off the weight  $W$ , and hang on the body  $b$  at  $A$ ; then slide the weight  $I$  backward and forward till it just balance the body  $b$ , which suppose to be at the number  $5$ ; then is  $b$  equal to  $5$  times the weight of  $I$ . So, if  $I$  be one pound, then  $b$  is  $5$  pounds; but if  $I$  be  $2$  pounds, then  $b$  is  $10$  pounds; and so on.

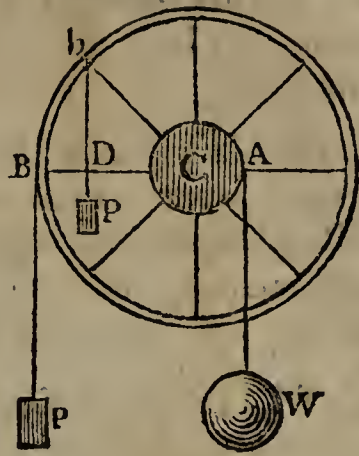
## OF THE WHEEL AND AXLE.

### PROPOSITION XXXII.

179. *In the Wheel-and-Axle; the Weight and Power will be in Equilibrio, when the Power  $P$  is to the Weight  $W$ , Reciprocally as the Radii of the Circles where they act; that is, as the Radius of the Axle  $CA$ , where the Weight hangs, to the Radius of the Wheel  $CB$ , where the Power acts. That is,  $P : W :: CA : CB$ .*

HERE the cord, by which the power  $P$  acts, goes about the  
the

the circumference of the wheel, while that of the weight  $W$  goes round its axle, or another smaller wheel, attached to the larger, and having the same axis or centre  $C$ . So that  $BA$  is a lever moveable about the point  $C$ , the power  $P$  acting always at the distance  $BC$ , and the weight  $W$  at the distance  $CA$ ; therefore  $P : W :: CA : CB$ .

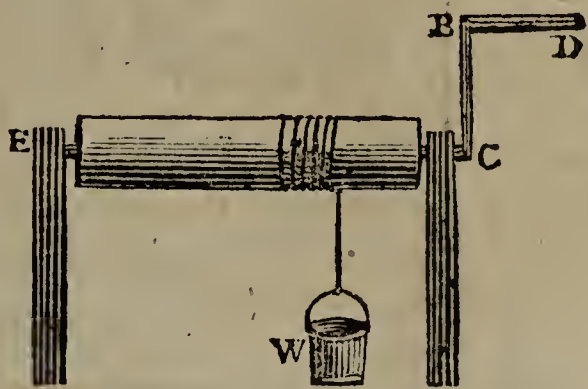


180. *Corol. 1.* If the wheel be put in motion; then, the spaces moved being as the circumferences, or as the radii, the velocity of  $W$  will be to the velocity of  $P$ , as  $CA$  to  $CB$ ; that is, the weight is moved as much slower, as it is heavier than the power; so that what is gained in power, is lost in time. And this is the universal property of all machines and engines.

181. *Corol. 2.* If the power do not act at right angles to the radius  $Cb$ , but obliquely; draw  $CD$  perpendicular to the direction of the power; then, by the nature of the lever,  $P : W :: CA : CD$ .

SCHOLIUM.

182. To this power belong all turning or wheel machines, of different radii. Thus, in the roller turning on the axis or spindle  $CE$ , by the handle  $CBD$ ; the power applied at  $B$  is to the weight  $W$  on the roller, as the radius of the roller is to the radius  $CB$  of the handle.



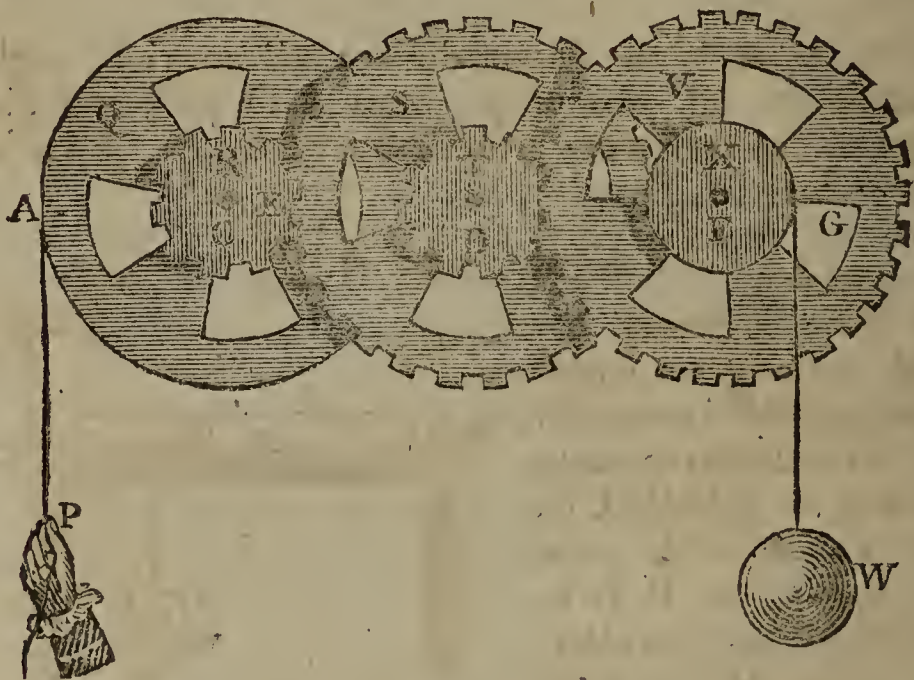
183. And the same for all cranes, capstans, windlasses, and such like; the power being to the weight, always as the radius or lever at which the weight acts, to that at which the power acts; so that they are always in the reciprocal ratio of their velocities. And to the same principle may be referred the gimblet and augur for boring holes.

184. But all this, however, is on supposition that the ropes or cords, sustaining the weights, are of no sensible thickness. For, if the thickness be considerable, or if there be several folds of them, over one another, on the roller or barrel; then we must measure to the middle of the outermost rope, for

the radius of the roller; or, to the radius of the roller we must add half the thickness of the cord, when there is but one fold.

185. The wheel-and-axle has a great advantage over the simple lever, in point of convenience. For a weight can be raised but a little way by the lever; whereas, by the continual turning of the wheel and roller, the weight may be raised to any height, or from any depth.

186. By increasing the number of wheels too, the power may be multiplied to any extent, making always the less wheels to turn greater ones, as far as we please: and this is commonly called Tooth and Pinion Work, the teeth of one circumference working in the rounds or pinions of another, to turn the wheel. And then, in case of an equilibrium, the power is to the weight, as the continual product of the radii of all the axles, to that of all the wheels. So, if the power P



turn the wheel Q, and this turn the small wheel or axle R, and this turn the wheel S, and this turn the axle T, and this turn the wheel V, and this turn the axle X, which raises the weight W; then  $P : W :: CB \cdot DE \cdot FG : AC \cdot BD \cdot EF$ . And in the same proportion is the velocity of W slower than that of P. Thus, if each wheel be to its axle, as 10 to 1; then  $P : W :: 1^3 : 10^3$  or as 1 to 1000. So that a power of one pound will balance a weight of 1000 pounds; but then, when put in motion, the power will move 1000 times faster than the weight.



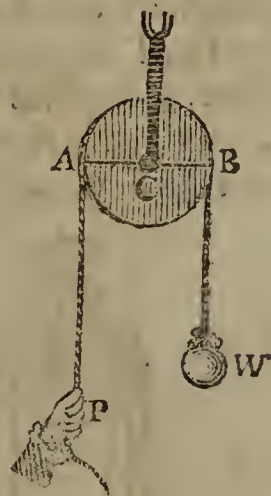
## OF THE PULLEY.

187. A PULLEY is a small wheel, commonly made of wood or brass, which turns about an iron axis passing through the centre, and fixed in a block, by means of a cord passed round its circumference, which serves to draw up any weight. The pulley is either single, or combined together, to increase the power. It is also either fixed or moveable, according as it is fixed to one place, or moves up and down with the weight and power.

## PROPOSITION XXXIII.

188. *If a Power sustain a Weight by means of a Fixed Pulley: the Power and Weight are Equal.*

FOR through the centre  $C$  of the pulley draw the horizontal diameter  $AB$ : then will  $AB$  represent a lever of the first kind, its prop being the fixed centre  $C$ ; from which the points  $A$  and  $B$ , where the power and weight act, being equally distant; the power  $P$  is consequently equal to the weight  $W$ .



189. *Corol.* Hence, if the pulley be put in motion, the power  $P$  will descend as fast as the weight  $W$  ascends. So that the power is not increased by the use of the fixed pulley, even though the rope go over several of them. It is, however, of great service in the raising of weights, both by changing the direction of the force, for the convenience of acting, and by enabling a person to raise a weight to any height without moving from his place, and also by permitting a great many persons at once to exert their force on the rope at  $P$ , which they could not do to the weight itself; as is evident in raising the hammer or weight of a pile-driver, as well as on many other occasions.

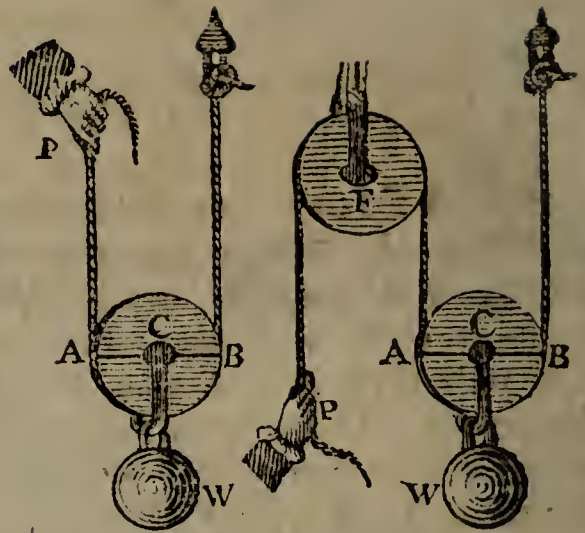
## PROPOSITION XXXIV.

190. *If a Power sustain a Weight by means of One Moveable Pulley; the Power is but Half the Weight.*

FOR, here  $AB$  may be considered as a lever of the second kind,

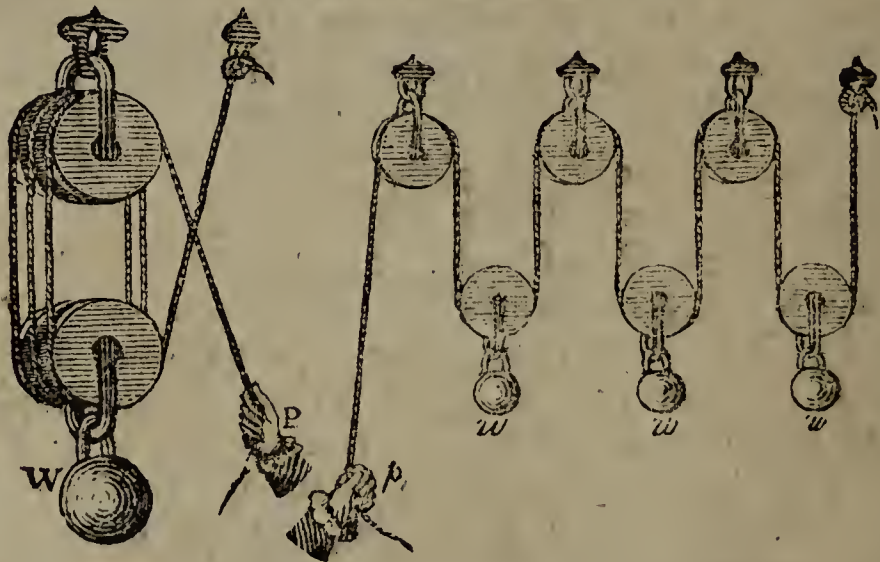
kind, the power acting at A, the weight at C, and the prop or fixed point at B; and because  $P : W :: CB : AB$ , and  $CB = \frac{1}{2}AB$ , therefore  $P = \frac{1}{2}W$ , or  $W = 2P$ .

191. *Corol. 1.* Hence it is evident, that, when the pulley is put in motion, the velocity of the power will be double the velocity of the weight, as the point P descends twice as fast as the point C and weight W rises.



It is also evident, that the fixed pulley F makes no difference in the power P, but is only used to change the direction of it, from upwards to downwards.

192. *Corol. 2.* Hence we may estimate the effect of a combination of any number of fixed and moveable pulleys; by which we shall find that every cord going over a moveable pulley, always doubles the power, since each end of the rope bears an equal share of the weight; while each rope that is fixed to a pulley, only increases the power by unity.



Here  $P = \frac{1}{6}W$ .

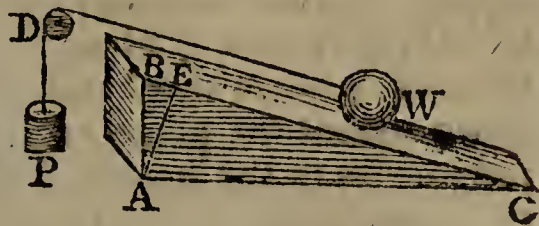
Here  $P = \frac{1}{2}w = \frac{w + w + w}{6}$ .

OF THE INCLINED PLANE.

193. THE INCLINED PLANE, is a plane inclined to the horizon, or making an angle with it. It is often reckoned one of the simple mechanic powers; and the double inclined plane makes the wedge. It is employed to advantage in raising heavy bodies in certain situations, diminishing their weights by laying them on the inclined planes.

PROPOSITION XXXV.

194. *The Power gained by the Inclined Plane, is in Proportion as the Length of the Plane is to its Height. That is, when a Weight W is sustained on an Inclined Plane BC, by a Power P, acting in the Direction DW, parallel to the Plane; then the Weight W, is in proportion to the Power P, as the Length of the Plane is to its height; that is,  $W : P :: BC : AB$ .*



FOR, draw AE perp. to the plane BC, or to DW. Then we are to consider that the body W is sustained by three forces, viz. 1st, its own weight or the force of gravity, acting perp. to AC, or parallel to BA; 2d, by the power P, acting in the direction WD, parallel to BC, or BE; and 3dly, by the re-action of the planes, perp. to its face, or parallel to the line EA. But when a body is kept in equilibrio by the action of three forces, it has been proved, that the intensities of these forces are proportional to the sides of the triangle, ABE, made by lines drawn in the directions of their actions; therefore those forces are to one another as the three lines - - - AB, BE, AE; that is, the weight of the body W is as the line AB, the power P is as the line BE, and the pressure on the plane as the line AE.

But the two triangles ABE, ABC are equiangular, and have therefore their like sides proportional; that is, the three lines - - - AB, BE, AE, are to each other respectively as the three BC, AB, AC, or also as the three - - - BC, AE, CE, which therefore are as the three forces W, P, p, where p denotes the pressure on the plane. That is,  $W : P :: BC : AB$ , or the weight is to the power, as the length of the plane is to its height.

See more on the Inclined Plane, at pa. 167, &c.

195. *Scholium.* The inclined plane comes into use in some situations in which the other mechanical powers cannot be conveniently applied, or in combination with them. As, in sliding heavy weights either up or down a plank or other plane laid sloping; or letting large casks down into a cellar, or drawing them out of it. Also, in removing earth from a lower situation to a higher, by means of wheel-barrows, or otherwise, as in making fortifications, &c; inclined planes, made of boards, laid aslope, serve for the barrows to run upon.

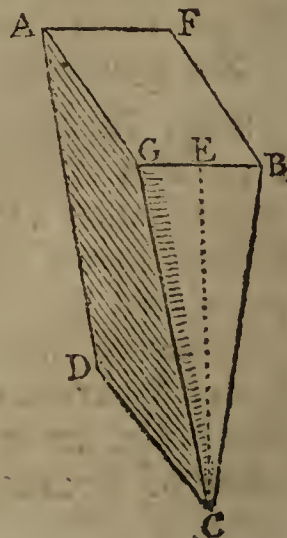
Of all the various directions of drawing bodies up an inclined plane, or sustaining them on it, the most favourable is where it is parallel to the plane  $BC$ , and passing through the centre of the weight: a direction which is easily given to it, by fixing a pulley at  $D$ , so that a cord passing over it, and fixed to the weight, may act or draw parallel to the plane. In every other position, it would require a greater power to support the body on the plane, or to draw it up. For if one end of the line be fixed at  $W$ , and the other end inclined down towards  $B$ , below the direction  $WD$ , the body would be drawn down against the plane, and the power must be increased in proportion to the greater difficulty of the traction. And, on the other hand, if the line were carried above the direction of the plane, the power must be also increased; but here only in proportion as it endeavours to lift the body off the plane.

If the length  $BC$  of the plane be equal to any number of times its perp. height  $AB$ , as suppose 3. times; then a power  $P$  of 1 pound, hanging freely, will balance a weight  $W$  of 3 pounds, laid on the plane; and a power  $P$  of 2 pounds, will balance a weight  $W$  of 6 pounds; and so on, always 3 times as much. But then if they be set a moving, the perp. descent of the power  $P$ , will be equal to 3 times as much as the perp. ascent of the weight  $W$ . For, although the weight  $W$  ascends up the direction of the oblique plane  $BC$ , just as fast as the power  $P$  descends perpendicularly, yet the weight rises only the perp. height  $AB$ , while it ascends up the whole length of the plane  $BC$ , which is 3 times as much; that is, for every foot of the perp. rise of the weight, it ascends 3 feet up in the direction of the plane, and the power  $P$  descends just as much, or 3 feet.

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### OF THE WEDGE.

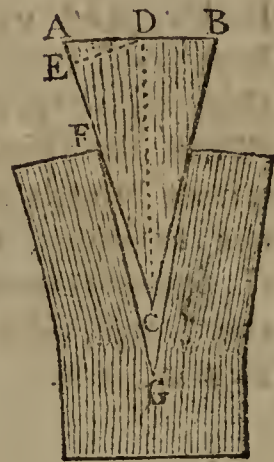
196. THE WEDGE is a piece of wood or metal, in form of half a rectangular prism.  $AF$  or  $BG$  is the breadth of its back;  $CE$  its height;  $GC$ ,  $BC$  its sides; and its end  $GBC$  is composed of two equal inclined planes  $GCE$ ,  $BCE$ .



PROPOSITION XXXVI.

197. *When a Wedge is in Equilibrio; the Power acting against the Back, is to the Force acting Perpendicularly against either Side, as the Breadth of the Back AB, is to the Length of the Side AC or BC.*

FOR, any three forces, which sustain one another in equilibrio, are as the corresponding sides of a triangle drawn perpendicular to the directions in which they act. But AB is perp. to the force acting on the back, to urge the wedge forward; and the sides AC, BC are perp. to the forces acting upon them; therefore the three forces are as AB, AC, BC.



198. *Corol.* The force on the back,  $\left\{ \begin{array}{l} AB, \\ \text{Its effect in direct. perp. to AC,} \\ \text{And its effect parallel to AB;} \end{array} \right. \left. \begin{array}{l} AC, \\ DC, \end{array} \right\}$  are as the three lines  $\left\{ \begin{array}{l} AC, \\ DC, \end{array} \right\}$  (which are perp. to them.

And therefore the thinner a wedge is, the greater is its effect, in splitting any body, or in overcoming any resistance against the sides of the wedge.

SCHOLIUM.

199. But it must be observed, that the resistance, or the forces above mentioned, respect one side of the wedge only. For if those against both sides be taken in, then, in the foregoing proportions, we must take only half the back AD, or else we must take double the line AC or DC.

In the wedge, the friction against the sides is very great, at least equal to the force to be overcome, because the wedge retains any position to which it is driven; and therefore the resistance is doubled by the friction. But then the wedge has a great advantage over all the other powers, arising from the force of percussion or blow with which the back is struck, which is a force incomparably greater than any dead weight or pressure, such as is employed in other machines. And accordingly we find it produces effects vastly superior to those of any other power; such as the splitting and raising the largest and hardest rocks, the raising and lifting the largest ship, by driving a wedge below it, which a man can do by the blow of a mallet: and thus it appears that the small blow of a hammer, on the back of a wedge, is incomparably greater than any mere pressure, and will overcome it.

## OF THE SCREW.

200. THE SCREW is one of the six mechanical powers, chiefly used in pressing or squeezing bodies close, though sometimes also in raising weights.

The screw is a spiral thread or groove cut round a cylinder, and every where making the same angle with the length of it. So that if the surface of the cylinder, with this spiral thread on it, were unfolded and stretched into a plane, the spiral thread would form a straight inclined plane, whose length would be to its height, as the circumference of the cylinder, is to the distance between two threads of the screw: as is evident by considering that, in making one round, the spiral rises along the cylinder the distance between the two threads.

## PROPOSITION XXXVII.

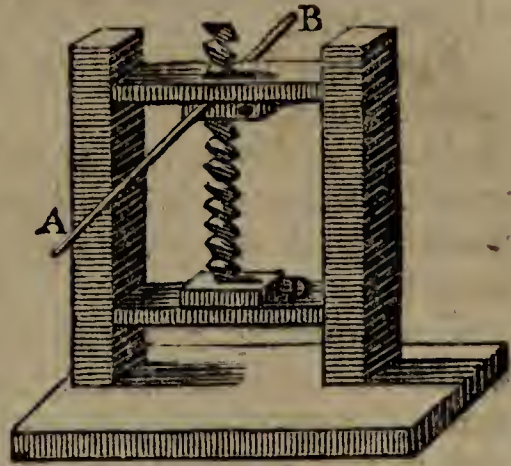
201. *The Force of a Power applied to turn a Screw round, is to the Force with which it presses upward or downward, setting aside the Friction, as the Distance between two Threads, is to the Circumference where the Power is applied.*

THE screw being an inclined plane, or half wedge, whose height is the distance between two threads, and its base the circumference of the screw; and the force in the horizontal direction, being to that in the vertical one, as the lines perpendicular to them, namely, as the height of the plane, or distance of the two threads, is to the base of the plane, or circumference of the screw; therefore the power is to the pressure, as the distance of two threads is to that circumference. But, by means of a handle or lever, the gain in power is increased in the proportion of the radius of the screw to the radius of the power, or length of the handle, or as their circumferences. Therefore, finally, the power is to the pressure, as the distance of the threads, is to the circumference described by the power.

202. *Corol.* When the screw is put in motion; then the power is to the weight which would keep it in equilibrio, as the velocity of the latter is to that of the former; and hence their two momenta are equal, which are produced by multiplying each weight or power by its own velocity. So that this is a general property in all the mechanical powers, namely, that the momentum of a power is equal to that of the weight which would balance it in equilibrio; or that each of them is reciprocally proportional to its velocity.

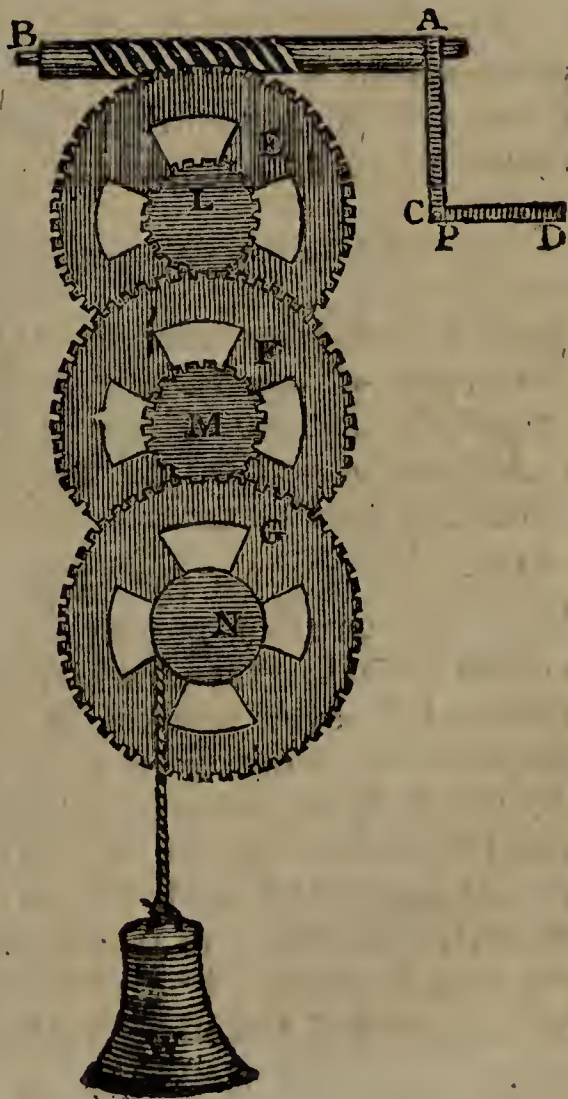
## SCHOLIUM.

203. Hence we can easily compute the force of any machine turned by a screw. Let the annexed figure represent a press driven by a screw, whose threads are each a quarter of an inch asunder; and let the screw be turned by a handle of 4 feet long, from A to B;



then, if the natural force of a man, by which he can lift, pull, or draw, be 150 pounds; and it be required to determine with what force the screw will press on the board at D, when the man turns the handle at A and B, with his whole force. Now the diameter AB of the power being 4 feet or 48 inches, its circumference is  $48 \times 3.1416$  or  $150\frac{4}{5}$  nearly; and the distance of the threads being  $\frac{1}{4}$  of an inch; therefore the power is to the pressure, as 1 to  $603\frac{1}{5}$ : but the power is equal to 150lb; theref. as  $1 : 603\frac{1}{5} :: 150 : 90480$ ; and consequently the pressure at D is equal to a weight of 90480 pounds; independent of friction.

204. Again, if the endless screw AB be turned by a handle AC of 20 inches, the threads of the screw being distant half an inch each; and the screw turn a toothed wheel E, whose pinion L turns another wheel F, and the pinion M of this another wheel G, to the pinion or barrel of which is hung a weight W; it is required to determine what weight the man will be able to raise, working at the handle C; supposing the diameters of the wheels to be 18 inches, and those of the pinions and barrel 2 inches; the teeth and pinions being all of a size.



Here

Here  $20 \times 3.1416 \times 2 = 125.664$ , is the circumference of the power.

And  $125.664$  to  $\frac{1}{2}$ , or  $251.328$  to  $1$ , is the force of the screw alone.

Also,  $18$  to  $2$ , or  $9$  to  $1$ , being the proportion of the wheels to the pinions; and as there are three of them, therefore  $9^3$  to  $1$ , or  $729$  to  $1$ , is the power gained by the wheels.

Consequently  $251.328 \times 729$  to  $1$ , or  $183218\frac{1}{9}$  to  $1$  nearly, is the ratio of the power to the weight, arising from the advantage both of the screw and the wheels.

But the power is  $150$  lb; therefore  $150 \times 183218\frac{1}{9}$ , or  $27482716$  pounds, is the weight the man can sustain, which is equal to  $12269$  tons weight.

But the power has to overcome, not only the weight, but also the friction of the screw, which is very great, in some cases equal to the weight itself, since it is sometimes sufficient to sustain the weight, when the power is taken off.

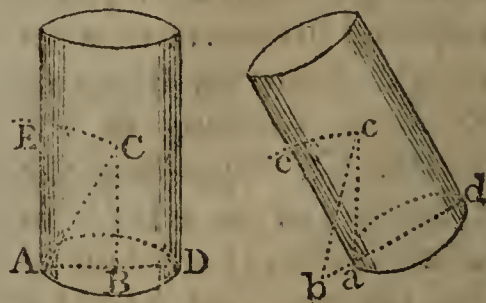
### ON THE CENTRE OF GRAVITY.

205. THE CENTRE of GRAVITY of a body, is a certain point within it, on which the body being freely suspended, it will rest in any position; and it will always descend to the lowest place to which it can get, in other positions.

#### PROPOSITION XXXVIII.

206. *If a Perpendicular to the Horizon, from the Centre of Gravity of any Body, fall Within the Base of the Body, it will rest in that Position; but if the Perpendicular fall Without the Base, the Body will not rest in that Position; but will tumble down.*

FOR, if  $CB$  be the perp. from the centre of gravity  $C$ , within the base: then the body cannot fall over towards  $A$ ; because, in turning on the point  $A$ , the centre of gravity  $C$  would describe an arc which would rise from  $C$  to  $E$ ; con-



trary to the nature of that centre, which only rests when in the lowest place. For the same reason, the body will not fall towards  $D$ . And therefore it will stand in that position.

But



But if the perpendicular fall without the base, as  $Cb$ ; then the body will tumble over on that side; because, in turning on the point  $a$ , the centre  $c$  descends by describing the descending arc  $ce$ .

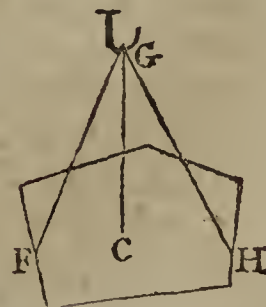
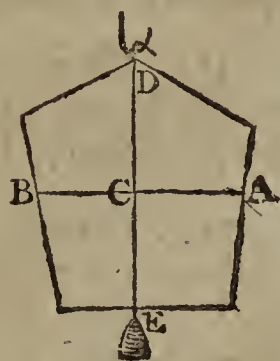
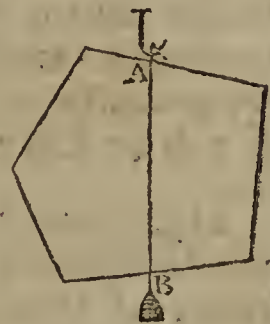
207. *Corol. 1.* If a perpendicular, drawn from the centre of gravity, fall just on the extremity of the base; the body may stand; but any the least force will cause it to fall that way. And the nearer the perpendicular is to any side, or the narrower the base is, the easier it will be made to fall, or be pushed over that way; because the centre of gravity has the less height to rise: which is the reason that a globe is made to roll on a smooth plane by any the least force. But the nearer the perpendicular is to the middle of the base, or the broader the base is, the firmer the body stands.

208. *Corol. 2.* Hence, if the centre of gravity of a body be supported, the whole body is supported. And the place of the centre of gravity must be accounted the place of the body; for into that point the whole matter of the body may be supposed to be collected, and therefore all the force also with which it endeavours to descend.

209. *Corol. 3.* From the property which the centre of gravity has, of always descending to the lowest point, is derived an easy mechanical method of finding that centre.

Thus, if the body be hung up by any point  $A$ , and a plumb line  $AB$  be hung by the same point, it will pass through the centre of gravity; because that centre is not in the lowest point till it fall in the plumb line. Mark the line  $AB$  upon it. Then hang the body up by any other point  $D$ , with a plumb line  $DE$ , which will also pass through the centre of gravity, for the same reason as before; and therefore that centre must be at  $C$  where the two plumb lines cross each other.

210. Or, if the body be suspended by two or more cords  $GF$ ,  $GH$ , &c, then a plumb line from the point  $G$  will cut the body in its centre of gravity  $C$ .



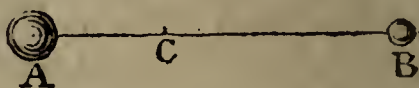
211. Like-

211. Likewise, because a body rests when its centre of gravity is supported, but not else; we hence derive another easy method of finding that centre mechanically. For, if the body be laid on the edge of a prism, and moved backward and forward till it rest, or balance itself; then is the centre of gravity just over the line of the edge. And if the body be then shifted into another position, and balanced on the edge again, this line will also pass by the centre of gravity; and consequently the intersection of the two will give the centre itself.

## PROPOSITION XXXIX.

212. *The Common Centre of Gravity C of any two Bodies A, B, divides the Line joining their Centres, into two Parts, which are Reciprocally as the Bodies.*

That is,  $AC : BC :: B : A$ .

FOR, if the centre of gravity C be supported, the two bodies A and B will be supported, and will rest in equilibrio. But,  by the nature of the lever, when two bodies are in equilibrio about a fixed point C, they are reciprocally as their distances from that point; therefore  $A : B :: CB : CA$ .

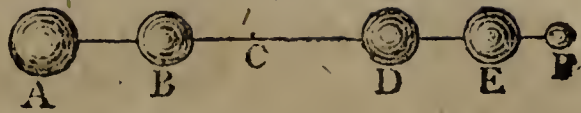
213. *Corol. 1.* Hence,  $AB : AC :: A + B : B$ ; or, the whole distance between the two bodies, is to the distance of either of them from the common centre, as the sum of the bodies is to the other body.

214. *Corol. 2.* Hence also,  $CA . A = CB . B$ ; or the two products are equal, which are made by multiplying each body by its distance from the centre of gravity.

215. *Corol. 3.* As the centre C is pressed with a force equal to both the weights A and B, while the points A and B are each pressed with the respective weights A and B. Therefore, if the two bodies be both united in their common centre C, and only the ends A and B of the line AB be supported, each will still bear, or be pressed by the same weights A and B as before. So that, if a weight of 100lb. be laid on a bar at C, supported by two men at A and B, distant from C, the one 4 feet, and the other 6 feet; then the nearer will bear the weight of 60lb, and the farther only 40lb. weight.

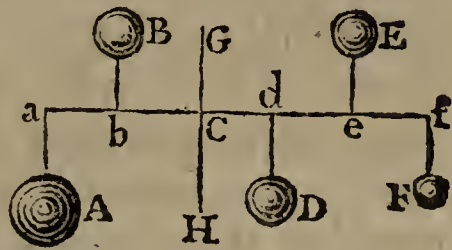
216. *Corol.*

216. *Corol. 4.* Since the effect of any body to turn a lever about the fixed point C, is as that body



and its distance from that point; therefore, if C be the common centre of gravity of all the bodies A, B, D, E, F, placed in the straight line AF; then is  $CA \cdot A + CB \cdot B = CD \cdot D + CE \cdot E + CF \cdot F$ ; or, the sum of the products on one side, equal to the sum of the products on the other, made by multiplying each body by its distance from that centre. And if several bodies be in equilibrium upon any straight lever, then the prop is in the centre of gravity.

217. *Corol. 5.* And although the bodies be not situated in a straight line, but scattered about in any promiscuous manner, the same property as in the last corollary still holds true, if perpendiculars to any line whatever af be drawn through the several



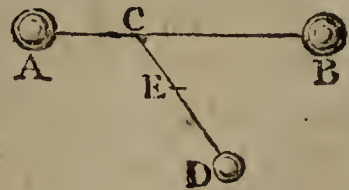
bodies and their common centre of gravity, namely, that  $Ca \cdot A + Cb \cdot B = Cd \cdot D + Ce \cdot E + Cf \cdot F$ . For the bodies have the same effect on the line af, to turn it about the point C, whether they are placed at the points a, b, d, e, f, or in any part of the perpendiculars Aa, Bb, Dd, Ee, Ff.

PROPOSITION XL.

218. *If there be three or more Bodies, and, if a Line be drawn from any one Body D to the Centre of Gravity of the rest C; then the Common Centre of Gravity E of all the Bodies, divides the line CD into two Parts in E, which are Reciprocally Proportional as the Body D to the Sum of all the other Bodies.*

That is,  $CE : ED :: D : A + B \text{ \&c.}$

FOR, suppose the bodies A and B to be collected into the common centre of gravity C, and let their sum be called S. Then, by the last prop.  $CE : ED :: D : S$  or  $A + B \text{ \&c.}$



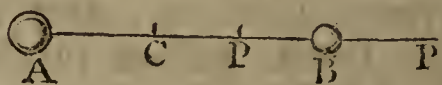
219. *Corol.* Hence we have a method of finding the common centre of gravity of any number of bodies; namely, by first finding the centre of any two of them, then the centre of that centre and a third, and so on for a fourth, or fifth, &c.

## PROPOSITION XLI.

220. If there be taken any Point P, in the Line passing through the Centres of two Bodies; then the Sum of the two Products, of each Body multiplied by its Distance from that Point, is equal to the Product of the Sum of the Bodies multiplied by the Distance of their Common Centre of Gravity C from the same Point P.

$$\text{That is, } PA \cdot A + PB \cdot B = PC \cdot \overline{A + B}.$$

For, by the 38th,  $CA \cdot A = CB \cdot B$ ,  
that is,  $\overline{PA - PC} \cdot A = \overline{PC - PB} \cdot B$ ;  
therefore, by adding,



$$PA \cdot A + PB \cdot B = PC \cdot \overline{A + B}.$$

221. *Corol. 1.* Hence, the two bodies A and B have the same force to turn the lever about the point P, as if they were both placed in C their common centre of gravity.

Or, if the line, with the bodies, move about the point P; the sum of the momenta of A and B, is equal to the momentum of the sum S or  $A + B$  placed at the centre C.

222. *Corol. 2.* The same is also true of any number of bodies whatever, as will appear by cor. 4, prop. 39, namely,  
 $PA \cdot A + PB \cdot B + PD \cdot D \&c. = PC \cdot \overline{A + B + D \&c.}$ ,  
where P is any point whatever in the line AC.

And, by cor. 5, prop. 39, the same thing is true when the bodies are not placed in that line, but any where in the perpendiculars passing through the points A, B, D, &c; namely,  
 $Pa \cdot A + Pb \cdot B + Pd \cdot D \&c. = PC \cdot \overline{A + B + D \&c.}$

223. *Corol. 3.* And if a plane pass through the point P perpendicular to the line CP; then the distance of the common centre of gravity from that plane, is

$$PC = \frac{Pa \cdot A + Pb \cdot B + Pd \cdot D \&c.}{A + B + D \&c.}, \text{ that is, equal to the}$$

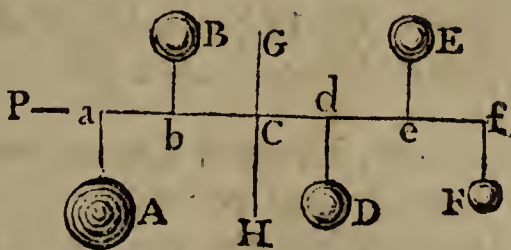
sum of all the forces divided by the sum of all the bodies. Or, if A, B, D, &c, be the several particles of one mass or compound body; then the distance of the centre of gravity of the body, below any given point P, is equal to the forces of all the particles divided by the whole mass or body, that is, equal to all the  $Pa \cdot A, Pb \cdot B, Pd \cdot D, \&c.$  divided by the body or sum of the particles A, B, D, &c.

PROPOSITION XLII.

224. To find the Centre of Gravity of any Body, or of any System of Bodies.

THROUGH any point P draw a plane, and let Pa, Pb, Pd, &c, be the distances of the bodies A, B, D, &c, from the plane; then, by the last cor. the distance of the common centre of gravity from the plane, will be

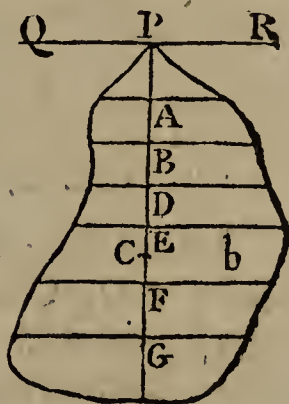
$$PC = \frac{Pa.A + Pb.B + Pd.D \&c}{A + B + D \&c}$$



225. Or, if *b* be any body, and QPR any plane; draw PAB &c, perpendicular to QR, and through A, B, &c, draw innumerable sections of the body *b* parallel to the plane QR. Let *s* denote any of these sections, and *d* = PA, or PB, &c, its distance from the plane QR. Then will the distance of the centre of gravity of the body from the plane be

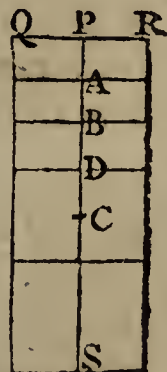
$$PC = \frac{\text{sum of all the } ds}{b}$$

And if the distance be thus found for two intersecting planes, they will give the point in which the centre is placed.



226. But the distance from one plane is sufficient for any regular body, because it is evident that, in such a figure, the centre of gravity is in the axis, or line passing through the centres of all the parallel sections.

Thus, if the figure be a parallelogram, or a cylinder, or any prism whatever; then the axis or line, or plane PS, which bisects all the sections parallel to QR, will pass through the centre of gravity of all those sections, and consequently through that of the whole figure C. Then, all the sections *s* being equal, and the body  $b = PS \cdot s$ , the distance of the centre



$$\text{will be } PC = \frac{PA \cdot s + PB \cdot s + \&c}{b} = \frac{PA + PB + \&c}{PS}$$

But  $PA + PB + \&c$ , is the sum of an arithmetical progression, beginning at 0, and increasing to the greatest term  $PS$ , the number of the terms being also equal to  $PS$ ; therefore the sum  $PA + BP + \&c = \frac{1}{2}PS \cdot PS$ ; and consequently  $PC = \frac{\frac{1}{2}PS \cdot PS}{PS} = \frac{1}{2}PS$ ; that is, the centre of gravity is in the middle of the axis of any figure whose parallel sections are equal.

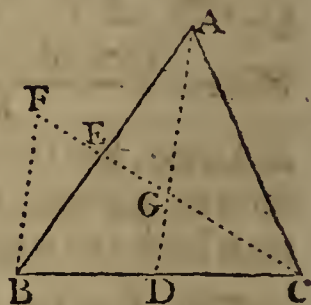
227. In other figures, whose parallel sections are not equal, but varying according to some general law, it will not be easy to find the sum of all the  $PA \cdot s$ ,  $PB \cdot s'$ ,  $PD \cdot s''$ , &c, except by the general method of Fluxions; which case, therefore, will be best reserved till we come to treat of that doctrine. It will be proper, however, to add here some examples of another method of finding the centre of gravity of a triangle, or any other right-lined plane figure.

PROPOSITION XLIII.

228. *To find the Centre of Gravity of a Triangle.*

FROM any two of the angles draw lines  $AD$ ,  $CE$ , to bisect the opposite sides; so will their intersection  $G$  be the centre of gravity of the triangle.

For, because  $AD$  bisects  $BC$ , it bisects also all its parallels, namely, all the parallel sections of the figure; therefore  $AD$  passes through the centres of gravity of all the parallel sections or component parts of the figure; and consequently the centre of gravity of the whole figure lies in the line  $AD$ . For the same reason, it also lies in the line  $CE$ . And consequently it is in their common point of intersection  $G$ .



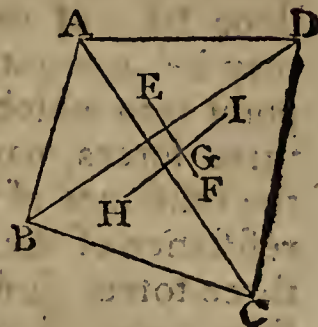
229. *Corol.* The distance of the point  $G$ , is  $AG = \frac{2}{3}AD$ , and  $CG = \frac{2}{3}CE$ ; or  $AG = 2GD$ , and  $CG = 2GE$ .

For, draw  $BF$  parallel to  $AD$ , and produce  $CE$  to meet it in  $F$ . Then the triangles  $AEG$ ,  $BEF$  are similar, and also equal, because  $AE = BE$ ; consequently  $AG = BF$ . But the triangles  $CDG$ ,  $CBF$  are also equiangular, and  $CB$  being  $= 2CD$ , therefore  $BF = 2GD$ . But  $BF$  is also  $= AG$ ; consequently  $AG = 2GD$  or  $\frac{2}{3}AD$ . In like manner,  $CG = 2GE$  or  $\frac{2}{3}CE$ .

PROPOSITION XLIV.

230. *To find the Centre of Gravity of a Trapezium.*

DIVIDE the trapezium ABCD into two triangles, by the diagonal BD, and find E, F the centres of gravity of these two triangles; then shall the centre of gravity of the trapezium lie in the line EF connecting them. And therefore if EF be divided, in G, in the alternate ratio of the two triangles, namely,  $EG : GF :: \text{triangle } BCD : \text{triangle } ABD$ , then G will be the centre of gravity of the trapezium.



231. Or, having found the two points E, F, if the trapezium be divided into two other triangles BAC, DAC, by the other diagonal AC, and the centres of gravity H and I of these two triangles be also found; then the centre of gravity of the trapezium will also lie in the line HI.

So that, lying in both the lines, EF, HI, it must necessarily lie in their intersection G.

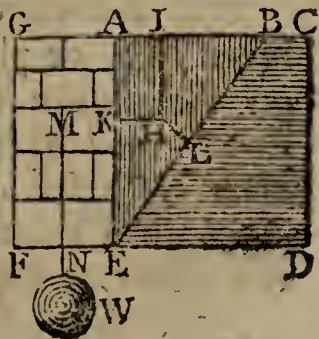
232. And thus we are to proceed for a figure of any greater number of sides, finding the centres of their component triangles and trapeziums, and then finding the common centre of every two of these, till they be all reduced into one only.

Of the use of the place of the centre of gravity, and the nature of forces, the following practical problem is added, to find the force of a bank of earth pressing against a wall, and the force of the wall to support it.

PROPOSITION XLV.

233. *To determine the Force with which a Bank of Earth, or such like, presses against a Wall, and the Dimensions of the Wall necessary to support it.*

LET ACDE be a vertical section of a bank of earth, and suppose that, if it were not supported, a triangular part of it, as ABE, would slide down, leaving it at what is called the natural slope BE; but that, by means of a wall ACFG, it is supported, and kept in its place.—It is required to find the force of ABE, and the dimensions of the wall ACFG.



Let  $H$  be the centre of gravity of the triangle  $ABE$ , from which draw  $HI$  perpendicular to the horizontal line  $AB$ ,  $HK$  to the upright face of the wall  $AE$ , and  $HL$  to the slope face of the earth  $BE$ . Now the centre of gravity  $H$  may be accounted the place of the triangle  $ABE$ , or the point into which it is all collected; and it is sustained by three forces, namely, its weight acting in the direction  $IH$ , the resistance of the plane  $BE$  in the direction  $LH$ , and the resistance of the plane  $AE$  in the direction  $KH$ ; and these three forces, sustaining the body in equilibrio, are as the three lines perpendicular to their directions, namely, as the three lines  $AB$ ,  $BE$ ,  $AE$ ; therefore the weight of the body  $ABE$ , is to its pressure against  $K$ , as  $AB$  is to  $AE$ . But  $\frac{1}{2}AE \cdot AB$  is the area of the triangle  $ABE$ ; and if  $m$  be the specific gravity of the earth, then  $\frac{1}{2}AE \cdot AB \cdot m$  is as its weight. Therefore, as  $AB : AE :: \frac{1}{2}AE \cdot AB \cdot m : \frac{1}{2}AE^2 \cdot m$ , the force or pressure against  $K$ : which therefore is proportional to the square of the altitude  $AE$ , whatever be the breadth  $AB$ , or the angle of the slope  $AEB$ . And the effect of this pressure to overturn the wall, is also as the length of the lever  $KE$  or  $\frac{2}{3}AE$ : consequently its effect is  $\frac{1}{2}AE^2 \cdot m \cdot \frac{2}{3}AE$ , or  $\frac{1}{3}AE^3 \cdot m$ . Which must be balanced by the counter resistance of the wall, in order that it may at least be supported.

Now, if  $M$  be the centre of gravity of the wall, into which its whole matter may be supposed to be collected, and acting in the direction  $MNW$ , its effect will be the same as if a weight  $W$  were suspended from the point  $N$  of the lever  $FN$ . Hence, if  $A$  be put for the area of the wall  $AEFG$ , and  $n$  its specific gravity; then  $A \cdot n$  will be equal to the weight  $W$ , and  $A \cdot n \cdot FN$  its effect on the lever to prevent it from turning about the point  $F$ . And as this effort must be equal to that of the triangle of earth, that it may just support it, which was before found equal to  $\frac{1}{3}AE^2 \cdot m$ ; therefore  $A \cdot n \cdot FN = \frac{1}{3}AE^3 \cdot m$ , in case of an equilibrium.

234. But now, both the breadth of the wall  $FE$ , and the lever  $FN$ , or place of the centre of gravity  $M$ , will depend on the figure of the wall. If the wall be rectangular, or as broad at top as bottom; then  $FN = \frac{1}{2}FE$ , and the area  $A = AE \cdot FE$ ; consequently the effort of the wall  $A \cdot n \cdot FN$  is  $= \frac{1}{2}FE^2 \cdot AE \cdot n$ ; which must be  $= \frac{1}{3}AE^3 \cdot m$ , the effort of the earth. And the resolution of this equation gives the breadth of the wall  $FE = AE \sqrt{\frac{2m}{3n}}$ . So that the

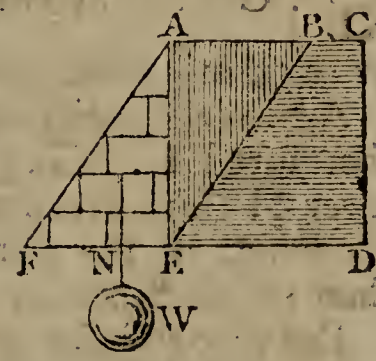
breadth



breadth of the wall is always proportional to its height, and is always the same at the same height, whatever the slope may be. But the breadth must be made a little more than the above value of it, that it may be more than a bare balance to the earth.

235. If the wall be of brick; its specific gravity is about 2000, and that of earth about 1984; namely,  $m$  to  $n$  as 1984 to 2000: then  $\sqrt{\frac{2m}{3n}} = .813 = \frac{13}{16}$  very nearly; and hence  $FE = \frac{13}{16} AE$ . That is, whenever a brick rectangular wall is made to support earth, its thickness must be at least  $\frac{13}{16}$  of its height. But if the wall be of stone, whose specific gravity is about 2520; then  $\sqrt{\frac{2m}{3n}} = .7246 = \frac{8}{11}$  nearly: that is, when the rectangular wall is of stone, the breadth must be at least  $\frac{8}{11}$  of its height.

236. But if the figure of the wall be a triangle, the outer side tapering to a point at top. Then the lever  $FN = \frac{2}{3}FE$ , and the area  $A = \frac{1}{2}FE \cdot AE$ ; consequently its effort  $A \cdot n \cdot FN$  is  $= \frac{1}{3}FE^2 \cdot AE \cdot n$ ; which being put  $= \frac{1}{3}AE^3 \cdot m$ , the equation gives  $FE = AE \sqrt{\frac{m}{n}}$  for the breadth



of the wall at the bottom, for an equilibrium in this case also. Where again  $FE$  is as  $AE$ , and indeed equal to it when the two specific gravities are equal; which is nearly the case when this wall is of brick. But when it is of stone; then  $\sqrt{\frac{m}{n}} = .8873 = \frac{8}{9}$  nearly: that is, the triangular stone wall must have its thickness at bottom equal to  $\frac{8}{9}$  of its height.

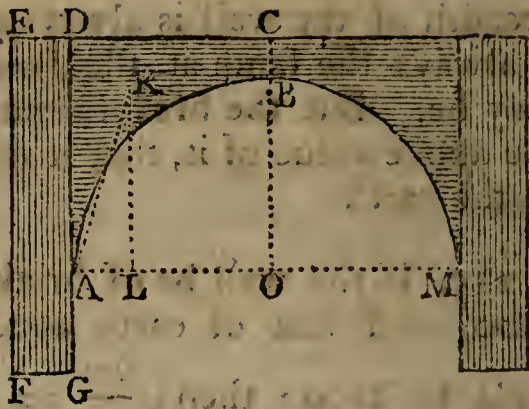
And in like manner, for other figures of the wall.

PROPOSITION XLVI.

237. To determine the Thickness of a Pier, necessary to support a Given Arch.

LET

LET  $ABCD$  be half the arch, and  $DEFG$  the pier. From the centre of gravity  $K$  of the arch draw  $KL$  perpendicular to the horizon. Then the weight of the arch in direction  $KL$  will be to the horizontal push at  $A$  in direction  $LA$ , as  $KL$  to  $LA$ .



For the weight of the arch in direction  $KL$ , the horizontal push or lateral pressure in direction  $LA$ , and the push in direction  $KA$ , will be as the three sides  $KL$ ,  $LA$ ,  $KA$ . So that if  $A$  denote the weight or area of the arch; then  $\frac{LA}{KL} \cdot A$  will be its force at  $A$  in the direction  $LA$ ; and  $\frac{LA}{KL} \cdot GA \cdot A$  its effect on the lever  $GA$  to overset the pier, or to turn it about the point  $F$ .

Again, the weight or area of the pier, is as  $EF \cdot FG$ ; and therefore  $EF \cdot FG \cdot \frac{1}{2}FG$ , or  $\frac{1}{2}EF \cdot FG^2$ , is its effect on the lever  $\frac{1}{2}FG$ , to prevent the pier from being overset; supposing the length of the pier, from point to point, to be no more than the thickness of the arch.

But that the pier and arch be in equilibrio, these two effects must be equal. Therefore we have  $\frac{1}{2}EF \cdot FG^2 = \frac{LA}{KL} \cdot GA \cdot A$ , and consequently  $FG = \sqrt{\frac{2GA \cdot AL}{EF \cdot KL}} \times A$  is the thickness of the pier as required.

*Example 1.* Suppose the arc  $ABM$  to be a semicircle; and that  $DC$  or  $AC = 45$ ,  $BC = 6$ , and  $GA = 18$  feet. Then  $KL$  will be found  $= 40$ ,  $AL = 15$  nearly, and  $EF = 69$ ; also, the area  $ABCD$  or  $A = 704\frac{1}{2}$ . Therefore  $FG = \sqrt{\frac{2GA \cdot AL}{EF \cdot KL}} \cdot A = \sqrt{\frac{36 \cdot 15}{69 \cdot 40}} \cdot 704\frac{1}{2} = 11\frac{2}{3}$  nearly, which is the thickness of the pier.

*Example 2.* Suppose, in the segment  $ABM$ ,  $AM = 100$ ,  $OB = 41\frac{1}{2}$ ,  $BC = 6\frac{1}{2}$ , and  $AG = 10$ . Then  $EF = 58$ ,  $KL = 35$ ,  $AL = 15$  nearly, and  $ABCD$  or  $A = 842$ . Therefore  $FG = \sqrt{\frac{2GA \cdot AL}{EF \cdot KL}} \cdot A = \sqrt{\frac{20 \cdot 15}{58 \cdot 35}} \cdot 842 = 11\frac{2}{3}$  nearly, is the thickness of the pier in this case.

ON THE CENTRES OF PERCUSSION, OSCILLATION,  
AND GYRATION.

238. THE CENTRE OF PERCUSSION, of a body, or a system of bodies, revolving about a point, or axis, is that point, which striking an immoveable object, the whole mass shall not incline to either side, but rest as it were in equilibrio, without acting on the centre of suspension.

239. The Centre of Oscillation is that point, in a body vibrating by its gravity, in which if any body be placed, or if the whole mass be collected, it will perform its vibrations in the same time, and with the same angular velocity, as the whole body, about the same point or axis of suspension.

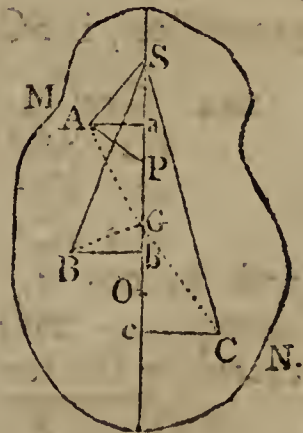
240. The Centre of Gyration, is that point, in which if the whole mass be collected, the same angular velocity will be generated in the same time, by a given force acting at any place, as in the body or system itself.

241. The angular motion of a body, or system of bodies, is the motion of a line connecting any point and the centre or axis of motion; and is the same in all parts of the same revolving body. And in different, unconnected bodies, each revolving about a centre, the angular velocity is as the absolute velocity directly, and distance from the centre inversely; so that, if their absolute velocities be as their radii or distances, the angular velocities will be equal.

## PROPOSITION XLVII.

242. *To find the Centre of Percussion of a Body, or System of Bodies.*

LET the body revolve about an axis passing through any point  $S$  in the line  $SGO$ , passing through the centres of gravity and percussion,  $G$  and  $O$ . Let  $MN$  be the section of the body, or the plane in which the axis  $SGO$  moves. And conceive all the particles of the body to be reduced to this plane, by perpendiculars let fall from them to the plane: a supposition which will not affect the centres  $G$ ,  $O$ , nor the angular motion of the body.



Let  $A$  be the place of one of the particles, so reduced;  
Join

join SA, and draw AP perpendicular to AS, and Aa perpendicular to SGO: then AP will be the direction of A's motion, as it revolves about S; and the whole mass being stopped at O, the body A will urge the point P forward, with a force proportional to its quantity of matter and velocity, or to its matter and distance from the point of suspension S; that is, as  $A \cdot SA$ ; and the efficacy of this force in a direction perpendicular to SO, at the point P, is as  $A \cdot Sa$ , by similar triangles; also, the effect of this force on the lever, to turn it about O, being as the length of the lever, is as  $A \cdot Sa \cdot PO = A \cdot Sa \cdot \overline{SO - SP} = A \cdot Sa \cdot SO - A \cdot Sa \cdot SP = A \cdot Sa \cdot SO - A \cdot SA^2$ . In like manner, the forces of B and C, to turn the system about O, are as

$$\begin{aligned} & B \cdot Sb \cdot SO - B \cdot SB^2, \text{ and} \\ & C \cdot Sc \cdot SO - C \cdot SC^2, \text{ \&c.} \end{aligned}$$

But, since the forces on the contrary sides of O destroy one another, by the definition of this force, the sum of the positive parts of these quantities, must be equal to the sum of the negative parts,

that is,  $A \cdot Sa \cdot SO + B \cdot Sb \cdot SO + C \cdot Sc \cdot SO \text{ \&c} = A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2 \text{ \&c}$ ; and

$$\text{hence } SO = \frac{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2 \text{ \&c}}{A \cdot Sa + B \cdot Sb + C \cdot Sc \text{ \&c}},$$

which is the distance of the centre of percussion below the axis of motion.

And here it must be observed that, if any of the points a, b, &c, fall on the contrary side of S, the corresponding product  $A \cdot Sa$ , or  $B \cdot Sb$ , &c, must be made negative.

243. *Corol. 1.* Since, by cor. 3; pr. 40,  $A + B + C \text{ \&c}$ , or the body  $b \times$  the distance of the centre of gravity, SG, is  $= A \cdot Sa + B \cdot Sb + C \cdot Sc \text{ \&c}$ , which is the denominator of the value of SO; therefore the distance of the centre of

percussion, is  $SO = \frac{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2 \text{ \&c}}{SG \times \text{body } b}$ .

244. *Corol. 2.* Since, by Geometry; theor. 36, 37,

$$\text{it is } SA^2 = SG^2 + GA^2 - 2SG \cdot Ga,$$

$$\text{and } SB^2 = SG^2 + GB^2 + 2SG \cdot Gb,$$

$$\text{and } SC^2 = SG^2 + GC^2 + 2SG \cdot Gc, \text{ \&c};$$

and, by cor. 5, pr. 40, the sum of the last terms is nothing, namely,  $- 2SG \cdot Ga + 2SG \cdot Gb + 2SG \cdot Gc \text{ \&c} = 0$ ; therefore the sum of the others, or  $A \cdot SA^2 + B \cdot SB^2 \text{ \&c}$

$$\begin{aligned} &= A + B \text{ \&c} \cdot SG^2 + A \cdot GA^2 + B \cdot GB^2 + C \cdot GC^2 \text{ \&c}, \\ \text{or } &= b \cdot SG^2 + A \cdot GA^2 + B \cdot GB^2 + C \cdot GC^2 \text{ \&c}; \end{aligned}$$

which being substituted in the numerator of the foregoing value of SO, gives

$$SO =$$

$$SO = \frac{b \cdot SG^2 + A \cdot GA^2 + B \cdot GB^2 + \&c}{b \cdot SG},$$

$$\text{or } SO = SG + \frac{A \cdot GA^2 + B \cdot GB^2 + C \cdot GC^2 \&c}{b \cdot SG}.$$

245. *Corol. 3.* Hence the distance of the centre of percussion, always exceeds the distance of the centre of gravity, and the excess is always  $GO = \frac{A \cdot GA^2 + B \cdot GB^2 \&c}{b \cdot SG}$ .

246. And hence also,  $SG \cdot GO = \frac{A \cdot GA^2 + B \cdot GB^2 \&c}{\text{the body } b}$ ;

that is,  $SG \cdot GO$  is always the same constant quantity, wherever the point of suspension  $S$  is placed; since the point  $G$  and the bodies  $A, B, \&c$ , are constant. Or  $GO$  is always reciprocally as  $SG$ , that is,  $GO$  is less, as  $SG$  is greater; and consequently the point rises upwards and approaches towards the point  $G$ , as the point  $S$  is removed to the greater distance; and they coincide when  $SG$  is infinite. But when  $S$  coincides with  $G$ , then  $GO$  is infinite, or  $O$  is at an infinite distance.

PROPOSITION XLVIII.

247. *If a Body A, at the Distance SA from an Axis passing through S, be made to revolve about that Axis by any Force of acting at P in the Line SP, Perpendicular to the Axis of Motion: It is required to determine the Quantity or Matter of another Body Q, which being placed at P, the Point where the Force acts, it shall be accelerated in the Same Manner, as when A revolved at the Distance SA; and consequently, that the Angular Velocity of A and Q about S, may be the Same in Both Cases.*

By the nature of the lever,  $SA : SP :: f :$

$\frac{SP}{SA} \cdot f$ , the effect of the force  $f$ , acting at  $P$ ,

on the body at  $A$ ; that is, the force  $f$  acting at  $P$ , will have the same effect on the body  $A$ , as the

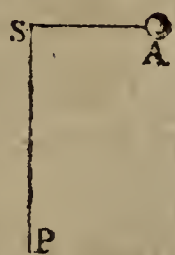
force  $\frac{SP}{SA} \cdot f$ , acting directly at the point  $A$ . But

as  $ASP$  revolves altogether about the axis at

$S$ , the absolute velocities of the points  $A$  and  $S$ , or of the bodies  $A$  and  $Q$ , will be as the radii  $SA, SP$ , of the circles described by them. Here then we have two bodies  $A$  and  $Q$ ,

which being urged directly by the forces  $f$  and  $\frac{SP}{SA} f$ , acquire

velocities



velocities which are as  $SP$  and  $SA$ . But the motive forces of bodies are as their mass and velocity: therefore

$$\frac{SP}{SA} f : f :: A \cdot SA : Q \cdot SP, \text{ and } SP^2 : SA^2 :: A : Q = \frac{SA^2}{SP^2} A,$$

which therefore expresses the mass of matter which, being placed at  $P$ , would receive the same angular motion from the action of any force at  $P$ , as the body  $A$  receives. So that the resistance of any body  $A$ , to a force acting at any point  $P$ , is directly as the square of its distance  $SA$  from the axis of motion, and reciprocally as the square of the distance  $SP$  of the point where the force acts.

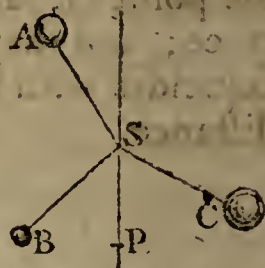
248. *Corol. 1.* Hence the force which accelerates the point  $P$ , is to the force of gravity, as  $\frac{f \cdot SP^2}{A \cdot SA^2}$  to 1, or as  $f \cdot SP^2$  to  $A \cdot SA^2$ .

249. *Corol. 2.* If any number of bodies  $A, B, C$ , be put in motion, about a fixed axis passing through  $S$ , by a force acting at  $P$ ; the point  $P$  will be accelerated in the same manner, and consequently the whole system will have the same angular velocity, if, instead of the bodies  $A, B, C$ , placed at the distances  $SA, SB, SC$ , there be substituted the bodies  $\frac{SA^2}{SP^2} A, \frac{SB^2}{SP^2} B, \frac{SC^2}{SP^2} C$ ; these being collected into the point  $P$ . And hence, the moving force being  $f$ , and the matter moved being  $A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2$ ; theref.  $\frac{f \cdot SP^2}{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}$

is the accelerating force; which therefore is to the accelerating force of gravity, as  $f \cdot SP^2$  to  $A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2$ .

250. *Corol. 3.* The angular velocity of the whole system of bodies, is as  $\frac{f \cdot SP}{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}$ . For the absolute velocity of the point  $P$ , is as the accelerating force, or directly as the motive force  $f$ , and inversely as the mass  $A \cdot SA^2$  &c  $\frac{f \cdot SP}{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}$ ; but the angular velocity is as the absolute

velocity directly, and the radius  $SP$  inversely; and therefore the angular velocity of  $P$ , or of the whole system, which is the same thing, is as  $\frac{f \cdot SP}{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}$ .

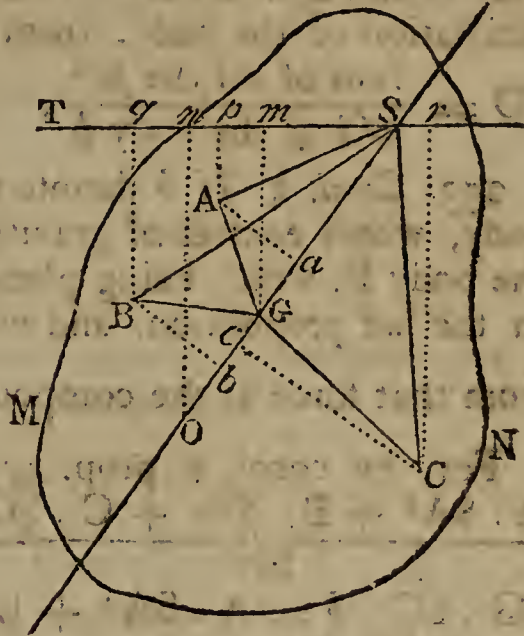


PROPOSITION XLIX.

251. To determine the Centre of Oscillation of any Compound Mass or Body MN, or of any System of Bodies A, B, C, &c.

LET MN be the plane of vibration, to which let all the matter be reduced, by letting fall perpendiculars from every particle, to this plane. Let

G be the centre of gravity, and O the centre of oscillation; through the axis S draw SGO, and the horizontal line Sq; then from every particle A, B, C, &c, let fall perpendiculars Aa, Ap, Bb, Bq, Cc, Cr, to these two lines; and join SA, SB, SC; also, draw Gm, On perpendicular to Sq. Now the forces of the weights A, B, C, to turn the body about the axis, are  $A \cdot Sp$ ,  $B \cdot Sq$ ,  $- C \cdot Sr$ ; therefore, by cor. 3, prop. 48, the angular



motion generated by all these forces is  $\frac{A \cdot Sp + B \cdot Sq - C \cdot Sr}{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}$ .

Also, the angular veloc. any particle  $p$ , placed in O, generates in the system, by its weight, is  $\frac{p \cdot Sn}{p \cdot SO^2}$  or  $\frac{Sn}{SO^2}$ , or  $\frac{Sm}{SG \cdot SO}$ , because of the similar triangles  $SGm$ ,  $SON$ . But, by the problem, the vibrations are performed alike in both cases, and therefore these two expressions must be equal to

each other, that is  $\frac{Sm}{SG \cdot SO} = \frac{A \cdot Sp + B \cdot Sq - C \cdot Sr}{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}$ ;

and hence  $SO = \frac{Sm}{SG} \times \frac{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}{A \cdot Sp + B \cdot Sq - C \cdot Sr}$ .

But, by cor. 2, pr. 41, the sum  $A \cdot Sp + B \cdot Sq - C \cdot Sr = (A + B + C) \cdot Sm$ ; therefore the distance  $SO = \frac{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}{SG \cdot (A + B + C)}$  =  $\frac{A \cdot Sa + B \cdot Sb + C \cdot Sc}{A \cdot Sa + B \cdot Sb + C \cdot Sc}$

by prop. 42, which is the distance of the centre of oscillation O, below the axis of suspension; where any of the products  $A \cdot Sa$ ,  $B \cdot Sb$ , must be negative, when  $a$ ,  $b$ , &c, lie on the other side of S. So that this is the same expression as that for the distance of the centre of percussion, found in prop. 47.

Hence

Hence it appears, that the centres of percussio and of oscillation, are in the very same point. And therefore the properties in all the corollaries there found for the former, are to be here understood of the latter.

252. *Corol. 1.* If  $p$  be any particle of a body  $b$ , and  $d$  its distance from the axis of motion  $S$ ; also  $G$ ,  $O$  the centres of gravity and oscillation. Then the distance of the centre of oscillation of the body, from the axis of motion, is

$$SO = \frac{\text{sum of all the } pd^2}{SG \times \text{the body } b}$$

253. *Corol. 2.* If  $b$  denote the matter in any compound body, whose centres of gravity and oscillation are  $G$  and  $O$ ; the body  $P$ , which being placed at  $P$ , where the force acts as in the last proposition, and which receives the same motion from that force as the compound body  $b$ , is  $P = \frac{SG \cdot SO}{SP^2} \cdot b$ .

For, by corol. 2, prop. 47, this body  $P$  is =  $\frac{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}{SP^2}$ . But, by corol. 1, prop. 46,

$$SG \cdot SO \cdot b = A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2; \text{ therefore}$$

$$P = \frac{SG \cdot SO}{SP^2} \cdot b.$$

#### SCHOLIUM.

254. By the method of Fluxions, the centre of oscillation, for a regular body, will be found from cor. 1. But for an irregular one; suspend it at the given point; and hang up also a simple pendulum of such a length, that, making them both vibrate, they may keep time together. Then the length of the simple pendulum, is equal to the distance of the centre of oscillation of the body, below the point of suspension.

255. Or it will be still better found thus: Suspend the body very freely by the given point, and make it vibrate in small arcs, counting the number of vibrations it makes in any time, as a minute, by a good stop watch; and let that number of vibrations made in a minute be called  $n$ : Then

shall the distance of the centre of oscillation, be  $SO = \frac{140850}{nn}$

inches. For, the length of the pendulum vibrating seconds, or 60 times in a minute, being  $39\frac{1}{8}$  inches; and the lengths of pendulums being reciprocally as the square of the number of vibrations made in the same time; therefore

$$n^2 : 60^2 :: 39\frac{1}{8} : \frac{60^2 \times 39\frac{1}{8}}{nn} = \frac{140850}{nn}, \text{ the length of the pendulum}$$



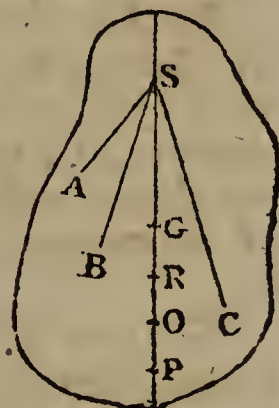
pendulum which vibrates  $n$  times in a minute, or the distance of the centre of oscillation below the axis of motion.

256. The foregoing determination of the point, into which all the matter of a body being collected, it shall oscillate in the same manner as before, only respects the case in which the body is put in motion by the gravity of its own particles, and the point is the centre of oscillation: but when the body is put in motion by some other extraneous force, instead of its gravity, then the point is different from the former, and is called the Centre of Gyration; which is determined in the following manner:

PROPOSITION L.

257. To determine the Centre of Gyration of a Compound Body or of a System of Bodies.

LET  $R$  be the centre of gyration, or the point into which all the particles  $A$ ,  $B$ ,  $C$ , &c, being collected, it shall receive the same angular motion from a force  $f$  acting at  $P$ , as the whole system receives.



Now, by cor. 3, pr. 47, the angular velocity generated in the system by the force  $f$ , is as  $\frac{f \cdot SP}{A \cdot SA^2 + B \cdot SB^2 \text{ \&c}}$ ; and

by the same, the angular velocity of the system placed in  $R$ , is  $\frac{f \cdot SP}{(A + B + C \text{ \&c}) \cdot SR^2}$ : then, by making these two expressions equal to each other, the equation gives

$SR = \sqrt{\frac{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}{A + B + C}}$ , for the distance of the centre of gyration below the axis of motion.

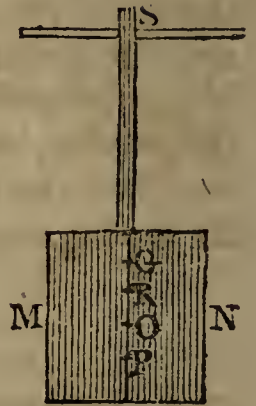
258. *Corol. 1.* Because  $A \cdot SA^2 + B \cdot SB^2 \text{ \&c} = SG \cdot SO \cdot b$ , where  $G$  is the centre of gravity,  $O$  the centre of oscillation, and  $b$  the body  $A + B + C \text{ \&c}$ ; therefore  $SR^2 = SG \cdot SO$ ; that is, the distance of the centre of gyration, is a mean proportional between those of gravity and oscillation.

259. *Corol. 2.* If  $p$  denote any particle of a body  $b$ , at  $d$  distance from the axis of motion; then  $SR^2 = \frac{\text{sum of all the } pd^2}{\text{body } b}$ .

## PROPOSITION LI.

260. To determine the Velocity with which a Ball moves, which, being shot against a Ballistic Pendulum, causes it to vibrate through a Given Angle.

THE Ballistic Pendulum, is a heavy block of wood MN, suspended vertically by a strong horizontally iron axis at S, to which it is connected by a firm iron stem. This problem is the application of the last proposition, or of prop. 47, and was invented by the very ingenious Mr. Robins, to determine the initial velocities of military projectiles; a circumstance very useful in that science; and it is the best method yet known for determining them with any degree of accuracy.



Let G, R, O be the centres of gravity, gyration, and oscillation, as determined by the foregoing propositions; and let P be the point where the ball strikes the face of the pendulum; the momentum of which, or the product of its weight and velocity, is expressed by the force  $f$ , acting at P, in the foregoing propositions. Now,

Put  $p$  = the whole weight of the pendulum,  
 $b$  = the weight of the ball,  
 $g$  = SG the dist. of the centre of gravity,  
 $o$  = SO the dist. of the centre of oscillation,  
 $r$  = SR =  $\sqrt{go}$  the dist. of the centre of gyration,  
 $i$  = SP the distance of the point of impact,  
 $v$  = the velocity of the ball,  
 $u$  = the velocity of the point of impact P,  
 $c$  = chord of the arc described by the point O.

By prop. 49, if the mass  $p$  be placed all at R, the pendulum will receive the same motion from the blow in the point P: and as  $SP^2 : SR^2 :: p : \frac{SR^2}{SP^2} \cdot p$  or  $\frac{r^2}{i^2} p$  or  $\frac{go}{ii} p$ , (prop. 47), the mass which being placed at P, the pendulum will still receive the same motion as before. Here then are two quantities of matter, namely,  $b$  and  $\frac{go}{ii} p$ , the former moving with the velocity  $v$ , and striking the latter at rest; to determine their common velocity  $u$ , with which they will jointly proceed forward together after the stroke. In which case,

case, by the law of the impact of non-elastic bodies, we have  $\frac{g_0}{ii} p + b : b :: v : u$ , and therefore  $v = \frac{bii + g_0 p}{bii} u$  the velocity of the ball in terms of  $u$ , the velocity of the point P, and the known dimensions and weights of the bodies.

But now to determine the value of  $u$ , we must have recourse to the angle through which the pendulum vibrates; for when the pendulum descends down again to the vertical position, it will have acquired the same velocity with which it began to ascend, and, by the laws of falling bodies, the velocity of the centre of oscillation is such, as a heavy body would acquire by freely falling through the versed sine of the arc described by the same centre O. But the chord of that arc is  $c$ , and its radius is  $o$ ; and, by the nature of the circle, the chord is a mean proportional between the versed sine and diameter, therefore  $2o : c :: c : \frac{c^2}{2o}$ , the versed sine of the arc described by O. Then, by the laws of falling bodies,  $\sqrt{16\frac{1}{2}} : \sqrt{\frac{c^2}{2o}} :: 32\frac{1}{6} : c\sqrt{\frac{2a}{o}}$ , the velocity acquired by the point O in descending through the arc whose chord is  $c$ , where  $a = 16\frac{1}{2}$  feet: and therefore  $o : i :: c\sqrt{\frac{2a}{o}} : \frac{ci}{o}\sqrt{\frac{2a}{o}}$ , which is the velocity  $u$ , of the point P.

Then, by substituting this value for  $u$ , the velocity of the ball, before found, becomes  $v = \frac{bii + g_0 p}{bio} \times c\sqrt{\frac{2a}{o}}$ . So that the velocity of the ball, is directly as the chord of the arc described by the pendulum in its vibration.

SCHOLIUM.

261. In the foregoing solution, the change in the centre of oscillation is omitted, which is caused by the ball lodging in the point P. But the allowance for that small change, and that of some other small quantities, may be seen in my Tracts, where all the circumstances of this method are treated at full length.

262. For an example in numbers of this method, suppose the weights and dimensions to be as follow: namely,

$p =$

$p = 570 \text{ lb,}$ $b = 180z \text{ } 1\frac{1}{2} \text{ dr}$ $= 1.131 \text{ lb,}$ $g = 78\frac{1}{2} \text{ inc.}$ $o = 84\frac{7}{9} \text{ inc.}$ $= 7.065 \text{ feet}$ $i = 94\frac{3}{10} \text{ inc.}$ $c = 18.73 \text{ inc.}$	<div style="text-align: right; margin-bottom: 10px;">Then</div> $\frac{bii + gop}{bio} \times c = \frac{1.131 \times 94.3^2 + 78\frac{1}{2} \times 84\frac{7}{9} \times 570}{1.131 \times 94\frac{3}{10} \times 84\frac{7}{9}}$ $\times \frac{18.73}{12} = 656.56.$ $\text{And } \sqrt{\frac{2a}{o}} = \sqrt{\frac{32\frac{1}{6}}{7.065}} = \sqrt{\frac{193}{42.39}} = 2.1337.$
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Therefore  $656.56 \times 2.1337$ , or 1401 feet, is the velocity, per second, with which the ball moved when it struck the pendulum.

### OF HYDROSTATICS.

263. **HYDROSTATICS** is the science which treats of the pressure, or weight, and equilibrium of water and other fluids, especially those that are non-elastic.

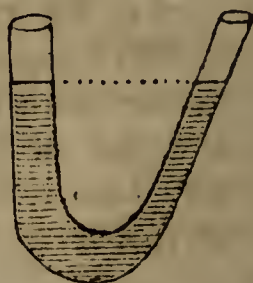
264. A fluid is elastic, when it can be reduced into a less bulk by compression, and which restores itself to its former bulk again when the pressure is removed; as air. And it is non-elastic, when it is non compressible or expansible; as water, &c.

#### PROPOSITION LII.

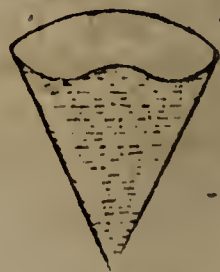
265. *If any Part of a Fluid be raised higher than the rest, by any Force, and then left to itself; the higher Parts will descend to the lower Places, and the Fluid will not rest, till its Surface be quite even and level.*

FOR, the parts of a fluid being easily moveable every way, the higher parts will descend by their superior gravity, and raise the lower parts, till the whole come to rest in a level or horizontal plane.

266. *Corol. 1.* Hence, water which communicates with other water, by means of a close canal or pipe, will stand at the same height in both places. Like as water in the two legs of a syphon.



267. *Corol. 2.* For the same reason, if a fluid gravitate towards a centre; it will dispose itself into a spherical figure, the centre of which is the centre of force. Like as the sea in respect of the earth.



## PROPOSITION LIII.

268. *When a Fluid is at rest in a Vessel, the Base of which is Parallel to the Horizon; Equal Parts of the Base are Equally Pressed by the Fluid.*

FOR, on every equal part of the base there is an equal column of the fluid supported by it. And as all the columns are of equal height, by the last proposition, they are of equal weight, and therefore they press the base equally; that is, equal parts of the base sustain an equal pressure.

269. *Corol. 1.* All parts of the fluid press equally at the same depth.

For, if a plane parallel to the horizon be conceived to be drawn at that depth; then, the pressure being the same in any part of that plane, by the proposition, therefore, the parts of the fluid, instead of the plane, sustain the same pressure at the same depth.

270. *Corol. 2.* The pressure of the fluid at any depth, is as the depth of the fluid.

For the pressure is as the weight, and the weight is as the height of the fluid.

271. *Corol. 3.* The pressure of the fluid on any horizontal surface or plane, is equal to the weight of a column of the fluid, whose base is equal to that plane, and altitude is its depth below the upper surface of the fluid.

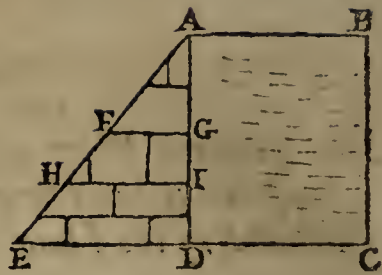
## PROPOSITION LIV.

272. *When a Fluid is pressed by its own Weight, or by any other Force; at any Point it presses Equally, in All Directions whatever.*

THIS arises from the nature of fluidity, by which it yields to any force in any direction. If it cannot recede from any force applied, it will press against other parts of the fluid in the direction of that force. And the pressure in all directions will be the same: for if it were less in any part, the fluid would move that way, till the pressure be equal every way.

273. *Corol. 1.* In a vessel containing a fluid; the pressure is the same against the bottom, as against the sides, or even upwards, at the same depth.

274. *Corol. 2.* Hence, and from the last proposition, if  $ABCD$  be a vessel of water, and there be taken, in the base produced,  $DE$ , to represent the pressure at the bottom; joining  $AE$ , and drawing any parallels to the base, as  $FG$ ,  $HI$ ; then shall  $FG$  represent the pressure at the depth  $AG$ , and  $HI$  the pressure at the depth  $AI$ , and so on; because the parallels  $FG$ ,  $HI$ ,  $ED$ , by sim. triangles, are as the depths  $AG$ ,  $AI$ ,  $AD$ ; which are as the pressures, by the proposition.



And hence the sum of all the  $FG$ ,  $HI$ , &c, or area of the triangle  $ADE$ , is as the pressure against all the points  $G$ ,  $I$ , &c, that is, against the line  $AD$ . But as every point in the line  $CD$  is pressed with a force as  $DE$ , and that thence the pressure on the whole line  $CD$  is as the rectangle  $ED \cdot DC$ , while that against the side is as the triangle  $ADE$  or  $\frac{1}{2}AD \cdot DE$ ; therefore the pressure on the horizontal line  $DC$ , is to the pressure against the vertical line  $DA$ , as  $DC$  to  $\frac{1}{2}DA$ . And hence, if the vessel be an upright rectangular one, the pressure on the bottom, or whole weight of the fluid, is to the pressure against one side, as the base is to half that side. And therefore the weight of the fluid is to the pressure against all the four upright sides, as the base is to half the upright surface. And the same holds true also in any upright vessel, whatever the sides be, or in a cylindrical vessel. Or, in the cylinder, the weight of the fluid, is to the pressure against the upright surface, as the radius of the base is to double the altitude.

Moreover, when the rectangular prism becomes a cube, it appears that the weight of the fluid on the base, is double the pressure against one of the upright sides, or half the pressure against the whole upright surface.

275. *Corol. 3.* The pressure of a fluid against any upright surface, as the gate of a sluice or canal, is equal to half the weight of a column of the fluid whose base is the surface pressed, and its altitude the same as the altitude of that surface.

For the pressure on a horizontal base equal to the upright surface, is equal to that column; and the pressure on the upright surface is but half that on the base, of the same area.

So that, if  $b$  denote the breadth, and  $d$  the depth of such a gate or upright surface; then the pressure against it, is equal to the weight of the fluid whose magnitude is  $\frac{1}{2}bd^2 = \frac{1}{2}AB \cdot AD^2$ .

If

If the fluid be water, a cubic foot of which weighs 1000 ounces, or  $62\frac{1}{2}$  pounds; and if the depth AD be 12 feet, the breadth AB 20 feet; then the content, or  $\frac{1}{2}AB \cdot AD^2$ , is 1440 feet; and the pressure is 1440000 ounces, or 90000 pounds, or  $40\frac{1}{2}$  tons weight nearly.

PROPOSITION LV.

276. *The Pressure of a Fluid on a Surface any how immersed in it, either Perpendicular, or Horizontal, or Oblique; is equal to the Weight of a Column of the Fluid, whose Base is equal to the Surface pressed, and its Altitude equal to the Depth of the Centre of Gravity of the Surface pressed below the Top or Surface of the Fluid.*

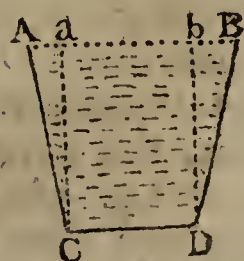
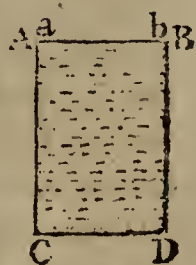
FOR, conceive the surface pressed to be divided into innumerable sections parallel to the horizon; and let  $s$  denote any one of those horizontal sections, also  $d$  its distance or depth below the top surface of the fluid. Then, by art. 272, the pressure of the fluid on the section is equal to the weight of  $ds$ ; consequently the total pressure on the whole surface is equal to all the weights  $ds$ . But, if  $b$  denote the whole surface pressed, and  $g$  the depth of its centre of gravity below the top of the fluid; then, by art. 222 or 225,  $bg$  is equal to the sum of all the  $ds$ . Consequently the whole pressure of the fluid on the body or surface  $b$ , is equal to the weight of the bulk  $bg$  of the fluid, that is, of the column whose base is the given surface  $b$ , and its height is  $g$  the depth of the centre of gravity in the fluid.

PROPOSITION LVI.

277. *The Pressure of a Fluid, on the Base of the Vessel in which it is contained, is as the Base and Perpendicular Altitude; whatever be the Figure of the Vessel that contains it.*

IF the sides of the base be upright, so that it be a prism of an uniform width throughout; then the case is evident; for then the base supports the whole fluid, and the pressure is just equal to the weight of the fluid.

But if the vessel be wider at top than bottom; then the bottom sustains, or is pressed by, only the part contained within the upright lines  $aC$ ,  $bD$ ; because the parts  $ACa$ ,  $BDb$  are supported by the sides  $AC$ ,  $BD$ ; and those parts have no other effect on the part  $abDC$  than keeping it in its position, by the lateral pressure against  $aC$  and  $bD$ , which does not alter its perpendicular pressure down-



wards.

wards. And thus the pressure on the bottom is less than the weight of the contained fluid.

And if the vessel be widest at bottom; then the bottom is still pressed with a weight which is equal to that of the whole upright column  $ABDC$ . For, as the parts of the fluid are in equilibrio, all the parts have an equal pressure at the same depth; so that the parts within  $Cc$  and  $dD$  press equally as those in  $cd$ , and therefore equally the same as if the sides of the vessel had gone upright to  $A$  and  $B$ , the defect of fluid in the parts  $ACa$  and  $BDb$  being exactly compensated by the downward pressure or resistance of the sides  $aC$  and  $bD$  against the contiguous fluid. And thus the pressure on the base may be made to exceed the weight of the contained fluid, in any proportion whatever.



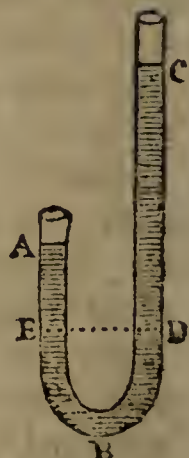
So that, in general, be the vessels of any figure whatever, regular or irregular, upright or sloping, or variously wide and narrow in different parts, if the bases and perpendicular altitudes be but equal, the bases always sustain the same pressure. And as that pressure, in the regular upright vessel, is the whole column of the fluid, which is as the base and altitude; therefore the pressure in all figures is in the same ratio.

278. *Corol. 1.* Hence, when the heights are equal, the pressures are as the bases. And when the bases are equal, the pressure is as the heights. But when both the heights and bases are equal, the pressures are equal in all, though their contents be ever so different.

279. *Corol. 2.* The pressure on the base of any vessel, is the same as on that of a cylinder, of an equal base and height.

280. *Corol. 3.* If there be an inverted syphon, or bent tube,  $ABC$ , containing two different fluids  $CD$ ,  $ABD$ , that balance each other, or rest in equilibrio; then their heights in the two legs,  $AE$ ,  $CD$ , above the point of meeting, will be reciprocally as their densities.

For, if they do not meet at the bottom, the part  $BD$  balances the part  $BE$ , and therefore the part  $CD$  balances the part  $AE$ ; that is, the weight of  $CD$  is equal to the weight of  $AE$ . And as the surface at  $D$  is the same, where they act against each other, therefore  $AE : CD :: \text{density of } CD : \text{density of } AE$ .



So,



So, if CD be water, and AE quicksilver, which is near 14 times heavier; then CD will be  $= 14AE$ ; that is, if AE be 1 inch, CD will be 14 inches; if AE be 2 inches, CD will be 28 inches; and so on.

## PROPOSITION LVII.

281. *If a Body be immersed in a Fluid of the Same Density or Specific Gravity; it will rest in any Place where it is put. But a Body of Greater Density will Sink; and one of a Less Density will Ascend to the Top, and Float.*

THE body, being of the same density, or of the same weight with the like bulk of the fluid, will press the fluid under it, just as much as if its space was filled with the fluid itself. The pressure then all around it will be the same as if the fluid were in its place; consequently there is no force, neither upward nor downward, to put the body out of its place. And therefore it will remain wherever it is put.

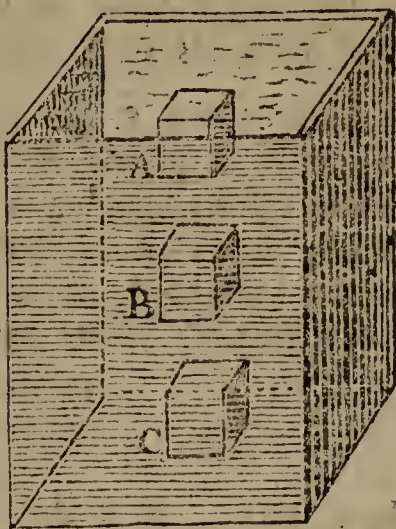
But if the body be lighter; its pressure downward will be less than before, and less than the water upward at the same depth; therefore the greater force will overcome the less, and push the body upward to A.

And if the body be heavier, the pressure downward will be greater than the fluid at the same depth; and therefore the greater force will prevail, and carry the body down to the bottom at C.

282. *Corol. 1.* A body immersed in a fluid, loses as much weight, as an equal bulk of the fluid weighs. And the fluid gains the same weight.

Thus, if the body be of equal density with the fluid, it loses all its weight, and so requires no force but the fluid to sustain it. If it be heavier, its weight in the water will be only the difference between its own weight and the weight of the same bulk of water; and it requires a force to sustain it just equal to that difference. But if it be lighter, it requires a force equal to the same difference of weights to keep it from rising up in the fluid.

283. *Corol. 2.* The weights lost, by immerging the same body in different fluids, are as the specific gravities of the fluids.



fluids. And bodies of equal weight, but different bulk, lose, in the same fluid, weights which are reciprocally as the specific gravities of the bodies, or directly as their bulks.

284. *Corol. 3.* The whole weight of a body which will float in a fluid, is equal to as much of the fluid, as the immersed part of the body takes up, when it floats.

For the pressure under the floating body, is just the same as so much of the fluid as is equal to the immersed part; and therefore the weights are the same.

285. *Corol. 4.* Hence the magnitude of the whole body, is to the magnitude of the part immersed, as the specific gravity of the fluid, is to that of the body.

For, in bodies of equal weight, the densities, or specific gravities, are reciprocally as their magnitudes.

286. *Corol. 5.* And because, when the weight of a body taken in a fluid, is subtracted from its weight out of the fluid, the difference is the weight of an equal bulk of the fluid; this therefore is to its weight in the air, as the specific gravity of the fluid, is to that of the body,

Therefore, if  $W$  be the weight of a body in air,  
 $w$  its weight in water, or any fluid,  
 $S$  the specific gravity of the body, and  
 $s$  the specific gravity of the fluid;  
 then  $W - w : W :: s : S$ , which proportion will give either of those specific gravities, the one from the other.

Thus  $S = \frac{W}{W - w} s$ , the specific gravity of the body;

and  $s = \frac{W - w}{W} S$ , the specific gravity of the fluid.

So that the specific gravities of bodies, are as their weights in the air directly, and their loss in the same fluid inversely.

287. *Corol. 6.* And hence, for two bodies connected together, or mixed together into one compound, of different specific gravities, we have the following equations, denoting their weights and specific gravities, as below, viz.

$H$	= weight of the heavier body in air,	}	$S$ its spec. gravity;
$h$	= weight of the same in water,		
$L$	= weight of the lighter body in air,	}	$s$ its spec. gravity;
$l$	= weight of the same in water,		
$C$	= weight of the compound in air,	}	$\rho$ its spec. gravity;
$c$	= weight of the same in water,		
$w$	= the specific gravity of water. Then		

1st,  $(H - b)S = Hw,$   
 2d,  $(L - l)s = Lw,$   
 3d,  $(C - c)f = Cw,$   
 4th,  $H + L = C,$   
 5th,  $b + l = c,$   
 6th,  $\frac{H}{S} + \frac{L}{s} = \frac{C}{f}.$

From which equations may be found any of the above quantities, in terms of the rest.

Thus, from one of the first three equations, is found the specific gravity of any body, as  $s = \frac{Lw}{L - l}$ , by dividing the absolute weight of the

body by its loss in water, and multiplying by the specific gravity of water.

But if the body  $L$  be lighter than water; then  $l$  will be negative, and we must divide by  $L + l$  instead of  $L - l$ , and to find  $l$  we must have recourse to the compound mass  $C$ ; and because, from the 4th and 5th equations,  $L - l = C - c -$

$\frac{H - b}{S}$ , therefore  $s = \frac{Lw}{(C - c) - (H - b)}$ ; that is, divide

the absolute weight of the light body, by the difference between the losses in water, of the compound and heavier body, and multiply by the specific gravity of water. Or

thus,  $s = \frac{S/L}{C/S - H/f}$ , as found from the last equation.

Also, if it were required to find the quantities of two ingredients mixed in a compound, the 4th and 6th equations would give their values as follows, viz.

$$H = \frac{(f - s)S}{(S - s)f} C, \text{ and } L = \frac{(S - f)s}{(S - s)f} C,$$

the quantities of the two ingredients  $H$  and  $L$ , in the compound  $C$ . And so for any other demand.

PROPOSITION LVIII.

*To find the Specific Gravity of a Body.*

288. CASE I.—When the body is heavier than water; weigh it both in water and out of water, and take the difference, which will be the weight lost in water. Then, by corol. 6, prop. 57,  $s = \frac{Bw}{B - b}$ , where  $B$  is the weight of the body out of water,  $b$  its weight in water,  $s$  its specific gravity, and  $w$  the specific gravity of water. That is,

As the weight lost in water,  
 Is to the whole or absolute weight,  
 So is the specific gravity of water,  
 To the specific gravity of the body.

EXAMPLE. If a piece of stone weigh 10lb, but in water only  $6\frac{1}{4}$ lb, required its specific gravity, that of water being 1000? Ans. 3077.

289. CASE II.—*When the body is lighter than water, so that it will not sink; annex to it a piece of another body, heavier than water, so that the mass compounded of the two may sink together. Weigh the denser body and the compound mass, separately, both in water, and out of it; then find how much each loses in water, by subtracting its weight in water from its weight in air; and subtract the less of these remainders from the greater. Then say, by proportion,*

As the last remainder,  
Is to the weight of the light body in air,  
So is the specific gravity of water,  
To the specific gravity of the body.

That is, the specific gravity is  $s = \frac{Lw}{(C - c) - (H - h)}$ ,  
by cor. 6, prop. 57.

EXAMPLE. Suppose a piece of elm weighs 15lb in air; and that a piece of copper, which weighs 18lb in air and 16lb in water, is affixed to it, and that the compound weighs 6lb in water; required the specific gravity of the elm? Ans. 600.

290. CASE III.—*For a fluid of any sort.*—Take a piece of a body of known specific gravity; weigh it both in and out of the fluid, finding the loss of weight by taking the difference of the two; then say,

As the whole or absolute weight,  
Is to the loss of weight,  
So is the specific gravity of the solid,  
To the specific gravity of the fluid.

That is, the spec. grav.  $w = \frac{B - b}{B}s$ , by cor. 6, pr. 57.

EXAMPLE. A piece of cast iron weighed  $34\frac{61}{100}$  ounces in a fluid, and 40 ounces out of it; of what specific gravity is that fluid? Ans. 1000.

#### PROPOSITION LIX.

291. *To find the Quantities of Two Ingredients in a Given Compound.*

TAKE the three differences of every pair of the three specific

cific gravities, namely, the specific gravities of the compound and each ingredient; and multiply each specific gravity by the difference of the other two. Then say, by proportion,

As the greatest product,  
Is to the whole weight of the compound,  
So is each of the other two products,  
To the weights of the two ingredients.

That is, the one  $H = \frac{(f - s) S}{(S - s) f} C$ ; and the other - -

$L = \frac{(S - f) s}{(S - s) f} C$ , by cor. 6, prop. 57.

EXAMPLE. A composition of 112lb being made of tin and copper, whose specific gravity is found to be 8784; required the quantity of each ingredient, the specific gravity of tin being 7320, and that of copper 9000?

Answer, there is 100lb of copper, } in the composition,  
and consequently 12lb of tin.

SCHOLIUM.

292. The specific gravities of several sorts of matter, as found from experiments, are expressed by the numbers annexed to their names in the following Table:

*A Table of the Specific Gravities of Bodies.*

Platina (pure)	-	22000	Clay	-	-	2160
Fine gold	-	19640	Brick	-	-	2000
Standard gold	-	18888	Common earth	-	-	1984
Quicksilver (pure)		14000	Nitre	-	-	1900
Quicksilver (common)		13600	Ivory	-	-	1825
Lead	-	11325	Brimstone	-	-	1810
Fine silver	-	11091	Solid gunpowder			1745
Standard silver	-	10535	Sand	-	-	1520
Copper	-	9000	Coal	-	-	1250
Copper halfpence		8915	Box-wood	-	-	1030
Gun metal	-	8784	Sea-water	-	-	1030
Cast brass	-	8000	Common water	-	-	1000
Steel	-	7850	Oak	-	-	925
Iron	-	7645	Gunpowder, close shaken			937
Cast iron	-	7425	Ditto, in a loose heap			836
Tin	-	7320	Ash	-	-	800
Clear crystal glass		3150	Maple	-	-	755
Granite	-	3000	Elm	-	-	600
Marble and hard stone		2700	Fir	-	-	550
Common green glass		2600	Charcoal	-	-	
Flint	-	2570	Cork	-	-	240
Common stone	-	2520	Air at a mean state			1 $\frac{2}{9}$

293. Note.

293. *Note.* The several sorts of wood are supposed to be dry. Also, as a cubic foot of water weighs just 1000 ounces averdupois, the numbers in this table express, not only the specific gravities of the several bodies, but also the weight of a cubic foot of each, in averdupois ounces; and therefore, by proportion, the weight of any other quantity, or the quantity of any other weight, may be known, as in the next two propositions.

## PROPOSITION LX.

294. *To find the Magnitude of any Body, from its Weight.*

As the tabular specific gravity of the body,  
Is to its weight in averdupois ounces,  
So is one cubic foot, or 1728 cubic inches,  
To its content in feet, or inches, respectively:

*Example 1.* Required the content of an irregular block of common stone, which weighs 1 cwt, or 112lb?

Ans.  $1228\frac{2}{5}\frac{1}{2}\frac{6}{8}$  cubic inches.

*Example 2.* How many cubic inches of gunpowder are there in 1lb weight?

Ans. 30 cubic inches nearly.

*Example 3.* How many cubic feet are there in a ton weight of dry oak?

Ans.  $38\frac{1}{8}\frac{3}{5}$  cubic feet.

## PROPOSITION LXI.

295. *To find the Weight of a Body, from its Magnitude.*

As one cubic foot, or 1728 cubic inches,  
Is to the content of the body,  
So is its tabular specific gravity,  
To the weight of the body.

*Example 1.* Required the weight of a block of marble, whose length is 63 feet, and breadth and thickness each 12 feet; being the dimensions of one of the stones in the walls of Balbeck?

Ans.  $683\frac{4}{8}$  ton, which is nearly equal to the burthen of an East-India ship.

*Example 2.* What is the weight of 1 pint, ale measure, of gunpowder?

Ans. 19 oz. nearly.

*Example 3.* What is the weight of a block of dry oak, which measures 10 feet in length, 3 feet broad, and  $2\frac{1}{2}$  feet deep or thick?

Ans.  $4335\frac{1}{8}\frac{5}{16}$  lb.

OF HYDRAULICS.

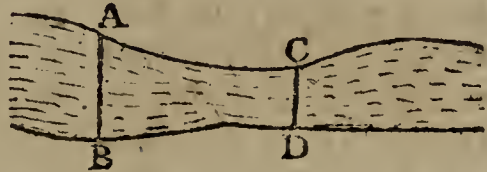
296. HYDRAULICS is the science which treats of the motion of fluids, and the forces with which they act upon bodies.

PROPOSITION LXII.

297. *If a Fluid run through a Canal or River, or Pipe of various Widths, always filling it; the Velocity of the Fluid in different Parts of it, AB, CD, will be Reciprocally as the Transverse Sections in those Parts.*

THAT is, veloc. at A : veloc.

at C ::  $\frac{1}{AB} : \frac{1}{CD}$ ; or :: CD : AB; where AB and CD denote, not the diameters at A and B, but the areas or sections, there.



For, as the channel is always equally full, the quantity of water running through AB is equal to the quantity running through CD, in the same time; that is, the column through AB is equal to the column through CD, in the same time; or  $AB \times \text{length of its column} = CD \times \text{length of its column}$ ; therefore  $AB : CD :: \text{length of column through CD} : \text{length of column through AB}$ . But the uniform velocity of the water, is as the space run over, or length of the columns; therefore  $AB : CD :: \text{velocity through CD} : \text{velocity through AB}$ .

298. *Corol.* Hence, by observing the velocity at any place AB, the quantity of water discharged in a second, or any other time, will be found, namely, by multiplying the section AB by the velocity there.

But if the channel be not a close pipe or tunnel, kept always full, but an open canal or river; then the velocity in all parts of the section will not be the same, because the velocity towards the bottom and sides will be diminished by the friction against the bed or channel; and therefore a medium among the three ought to be taken. So, if the velocity at the top be - 100 feet per minute,  
 that at the bottom - 60  
 and that at the sides - 50

$$\begin{array}{r} 3 \ ) \ 210 \ \text{sum;} \\ \hline \end{array}$$

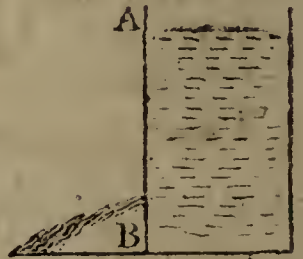
dividing their sum by 3 gives 70 for the mean velocity, which is to be multiplied by the section, to give the quantity discharged in a minute.

## PROPOSITION LXIII.

299. *The Velocity with which a Fluid runs out by a Hole in the Bottom or Side of a Vessel, kept always full, is equal to that which is generated by Gravity through the Height of the Water above the Hole; that is, the Velocity of a Heavy Body acquired by falling freely through the Height AB.*

DIVIDE the altitude AB into a great number of very small parts, each being  $1$ , their number  $a$ , or  $a =$  the altitude AB.

Now, by prop. 54, the pressure of the fluid against the hole B, by which the motion is generated, is equal to the weight of the column of fluid above it, that is the column whose height is AB or  $a$ , and base the area of the hole B. Therefore the pressure on the hole, or small part of the fluid  $1$ , is to its weight, or the natural force of gravity, as  $a$  to  $1$ . But, by art. 28, the velocities generated in the same body in any time, are as those forces; and because gravity generates the velocity  $2$  in descending through the small space  $1$ , therefore  $1 : a :: 2 : 2a$ , the velocity generated by the pressure of the column of fluid in the same time. But  $2a$  is also, by corol. 1, prop. 6, the velocity generated by gravity in descending through  $a$  or AB. That is, the velocity of the issuing water, is equal to that which is acquired by a body in falling through the height AB.



300. *Corol. 1.* The velocity, and quantity run out, at different depths, are as the square roots of the depths. For the velocity acquired in falling through AB, is as  $\sqrt{AB}$ .

301. *Corol. 2.* The water spouts out with the same velocity, whether it be downward, or upward, or sideways; because the pressure of fluids is the same in all directions, at the same depth. And therefore, if the adjutage be turned upward, the jet will ascend to the height of the surface of the water in the vessel. And this is confirmed by experience, by which it is found that jets really ascend nearly to the height of the reservoir, abating a small quantity only, for the friction against the sides, and some resistance from the air and from the oblique motion of the water in the hole.

302. *Corol. 3.* The quantity run out in any time, is equal to a column or prism, whose base is the area of the hole, and its length the space described in that time by the velocity acquired by falling through the altitude of the fluid. And the



the quantity is the same, whatever be the figure of the orifice, if it is of the same area.

Therefore, if  $b$  denote the height of the fluid,

$a$  the area of the orifice, and

$g = 16\frac{1}{2}$  feet, or 193 inches;

then  $2a\sqrt{gb}$  will be the quantity of water discharged in a second of time; or nearly  $8\frac{1}{8}a\sqrt{b}$  cubic feet, when  $a$  and  $b$  are taken in feet.

So, for example, if the height  $b$  be 25 inches, and the orifice  $a = 1$  square inch; then  $2a\sqrt{gb} = 2\sqrt{25} \times 193 = 139$  cubic inches, which is the quantity that would be discharged per second.

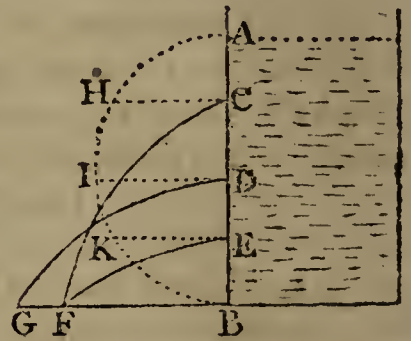
#### SCHOLIUM.

303. When the orifice is in the side of the vessel, then the velocity is different in the different parts of the hole, being less in the upper parts of it than in the lower. However, when the hole is but small, the difference is inconsiderable, and the altitude may be estimated from the centre of the hole, to obtain the mean velocity. But when the orifice is pretty large, then the mean velocity is to be more accurately computed by other principles, given in the next proposition.

304. It is not to be expected that experiments, as to the quantity of water run out, will exactly agree with this theory, both on account of the resistance of the air, the resistance of the water against the sides of the orifice, and the oblique motion of the particles of the water in entering it. For, it is not merely the particles situated immediately in the column over the hole, which enter it and issue forth, as if that column only were in motion; but also particles from all the surrounding parts of the fluid, which is in a commotion quite around; and the particles thus entering the hole in all directions, strike against each other, and impede one another's motion: from whence it happens, that the real velocity through the orifice, is somewhat less than that of a single body only, urged with the same pressure of the superincumbent column of the fluid. And experiments on the quantity of water discharged through apertures, shew that the quantity must be diminished, by those causes, rather more than the fourth part, when the orifice is small, or such as to make the mean velocity equal to that in a body falling through  $\frac{1}{2}$  the height of the fluid above the orifice. Or else, that the orifice is not quite full of particles that spout out with the whole velocity, assigned in the proposition.

305. Experiments have also been made on the extent to which

which the spout of water ranges on a horizontal plane, and compared with the theory, by calculating it as a projectile discharged with the velocity acquired by descending through the height of the fluid. For, when the aperture is in the side of the vessel, the fluid spouts out horizontally with a uniform velocity, which, combined with the perpendicular velocity from the action of gravity, causes the jet to form the curve of a parabola. Then the distances to which the jet will spout on the horizontal plane BG, will be as the roots of the rectangles of the segments AC, CB, AD, DB, AE, EB. For the spaces BF, BG, are as the times and horizontal velocities; but the velocity is as  $\sqrt{AC}$ , and the time of the fall, which is the same as the time of moving, is as  $\sqrt{CB}$ ; therefore the distance BF



is as  $\sqrt{AC \cdot CB}$ ; and the distance BG as  $\sqrt{AD \cdot DB}$ . And hence, if two holes are made equidistant from the top and bottom, they will project the water to the same distance; for if  $AC = EB$ , then the rectangle  $AC \cdot CB$  is equal the rectangle  $AE \cdot EB$ ; which makes  $EF$  the same for both. Or, if on the diameter  $AB$  a semicircle be described; then, because the squares of the ordinates  $CH, DI, EK$  are equal to the rectangles  $AC \cdot CB, \&c$ ; therefore the distances  $BF, BG$  are as the ordinates  $CH, DI$ . And hence also it follows, that the projection from the middle point  $D$  will be farthest, for  $DI$  is the greatest ordinate.

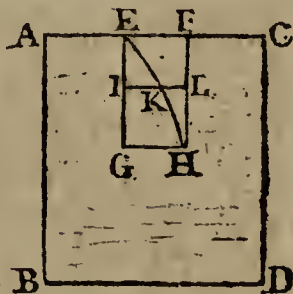
These are the *proportions* of the distances: but for the absolute distances, it will be thus. The velocity through any hole  $C$ , is such as will carry the water horizontally through a space equal to  $2AC$  in the time of falling through  $AC$ : but, after quitting the hole, it describes a parabola, and comes to  $F$  in the time a body will fall through  $CB$ ; and to find this distance, since the times are as the roots of the spaces, therefore  $\sqrt{AC} : \sqrt{CB} :: 2AC : 2\sqrt{AC \cdot CB} = 2CH = BF$ , the space ranged on the horizontal plane. And the greatest range  $BG = 2DI$ , or  $2AD$ , or equal to  $AB$ .

And as these ranges answer very exactly to the experiments, this confirms the theory, as to the velocity assigned.

## PROPOSITION LXIV.

306. If a Notch or Slit  $EH$  in form of a Parallelogram, be cut in the Side of a Vessel, full of Water,  $AD$ ; the Quantity of Water flowing through it, will be  $\frac{2}{3}$  of the Quantity flowing through an equal Orifice, placed at the whole Depth  $EG$ , or at the Base  $GH$ , in the Same Time; it being supposed that the Vessel is always kept full.

FOR the velocity at  $GH$  is to the velocity at  $IL$ , as  $\sqrt{EG}$  to  $\sqrt{EI}$ ; that is, as  $GH$  or  $IL$  to  $IK$ , the ordinate of a parabola  $EKH$ , whose axis is  $EG$ . Therefore the sum of the velocities at all the points  $I$ , is to as many times the velocity at  $G$ , as the sum of all the ordinates  $IK$ , to the sum of all the  $IL$ 's, namely, as the area of the parabola  $EGH$ , is to the area  $EGHF$ ; that is, the quantity running through the notch  $EH$ , is to the quantity running through an equal horizontal area placed at  $GH$ , as  $EGHKE$  to  $EGHF$ , or as 2 to 3; the area of a parabola being  $\frac{2}{3}$  of its circumscribing parallelogram.



307. *Corol. 1.* The mean velocity of the water in the notch, is equal to  $\frac{2}{3}$  of that at  $GH$ .

308. *Corol. 2.* The quantity flowing through the hole  $IGHL$ , is to that which would flow through an equal orifice placed as low as  $GH$ , as the parabolic frustum  $IGHK$ , is to the rectangle  $IGHL$ . As appears from the demonstration.

## OF PNEUMATICS.

309. PNEUMATICS is the science which treats of the properties of air, or elastic fluids.

## PROPOSITION LXV.

310. Air is a Heavy Fluid Body; and it Surrounds the Earth, and Gravitates on all Parts of its Surface.

THESE properties of air are proved by experience.—That it is a fluid, is evident from its easily yielding to any the least force impressed on it, without making a sensible resistance.

But when it is moved briskly, by any means, as by a fan or a pair of bellows; or when any body is moved very briskly through it; in these cases we become sensible of it as a body, by the resistance it makes in such motions, and likewise by its

its impelling or blowing away any light substances. So that, being capable of resisting, or moving other bodies by its impulse, it must itself be a body, and be heavy, like all other bodies, in proportion to the matter it contains; and therefore it will press on all bodies that are placed under it.

Also, as it is a fluid, it will spread itself all over on the earth; and, like other fluids, it will gravitate and press every where on the earth's surface.

311. The gravity and pressure of the air is also evident from many experiments. Thus, for instance, if water, or quicksilver, be poured into the tube ACE, and the air be suffered to press on it, in both ends of the tube, the fluid will rest at the same height in both legs of the tube: but if the air be drawn out of one end as E, by any means; then the air pressing on the other end A, will press down the fluid in this leg at B, and raise it up in the other to D, as much higher than at B, as the pressure of the air is equal to. By which it appears, not only that the air does really press, but also what the quantity of that pressure is equal to. And this is the principle of the barometer.

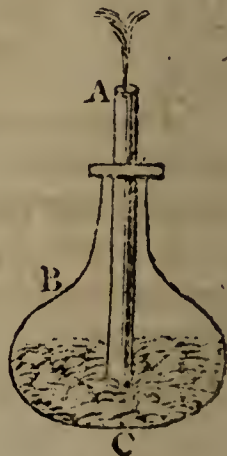


#### PROPOSITION LXVI.

312. *The Air is also an Elastic Fluid, being Condensible and Expansible. And the Law it observes is this, that its Density is proportional to the Force which compresses it.*

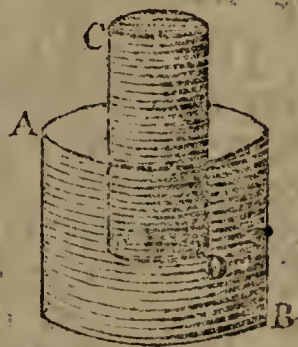
THIS property of the air is proved by many experiments. Thus, if the handle of a syringe be pushed inwards, it will condense the inclosed air into less space, thereby shewing its condensibility. But the included air, thus condensed, will be felt to act strongly against the hand, resisting the force compressing it more and more; and, on withdrawing the hand, the handle is pushed back again to where it was at first. Which shews that the air is elastic.

313. Again, fill a strong bottle half full of water, and then insert a pipe into it, putting its lower end down near to the bottom, and cementing it very close round the mouth of the bottle. Then, if air be strongly injected through the pipe, as by blowing with the mouth or otherwise, it will pass through the water from the lower end, ascending into the parts before occupied with air at B, and the whole

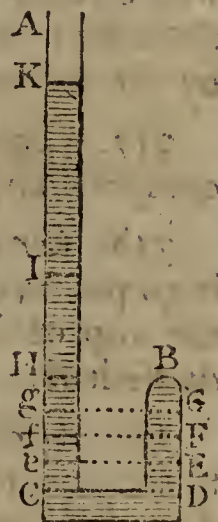


whole mass of air become there condensed, because the water is not compressible into a less space. But, on removing the force which injected the air at A, the water will begin to rise from thence in a jet, being pushed up the pipe by the increased elasticity of the air B, by which it presses on the surface of the water, and forces it through the pipe, till as much be expelled as there was air forced in; when the air at B will be reduced to the same density as at first, and, the balance being restored, the jet will cease.

314. Likewise, if into a jar of water AB, be inverted an empty glass tumbler CD, or such-like, the mouth downward; the water will enter it, and partly fill it, but not near so high as the water in the jar, compressing and condensing the air into a less space in the upper parts C, and causing the glass to make a sensible resistance to the hand in pushing it down. Then, on removing the hand, the elasticity of the internal condensed air throws the glass up again. All these shewing that the air is condensible and elastic.



315. Again, to shew the rate or proportion of the elasticity to the condensation; take a long crooked glass tube, equally wide throughout, or at least in the part BD, open at A, but close at the other end B. Pour in a little quicksilver at A, just to cover the bottom to the bend at CD, and to stop the communication between the external air and the air in BD. Then pour in more quicksilver, and mark the corresponding heights at which it stands in the two legs: so, when it rises to H in the open leg AC, let it rise to E in the close one, reducing its included air from the natural bulk BD to the contracted space BE,



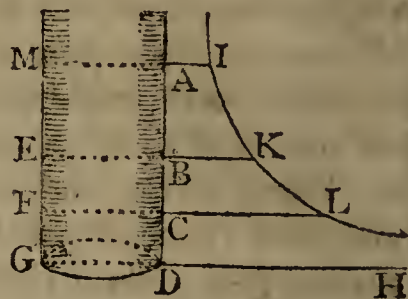
by the pressure of the column He; and when the quicksilver stands at I and K, in the open leg, let it rise to F and G in the other, reducing the air to the respective spaces BF, BG, by the weights of the columnus If, Kg. Then it is always found, that the condensations and elasticities are as the compressing weights, or columns, of the quicksilver, and the atmosphere together. So, if the natural bulk of the air BD be compressed into the spaces BE, BF, BG, which are  $\frac{3}{4}$ ,  $\frac{2}{4}$ ,  $\frac{1}{4}$  of BD, or as the numbers 3, 2, 1; then the atmosphere, together with the corresponding columns He, If, Kg, will also be

found to be in the same proportion reciprocally, viz. as  $\frac{2}{3}$ ,  $\frac{1}{2}$ ,  $\frac{1}{3}$ , or as the numbers 2, 3, 6. And then  $He = \frac{1}{3}A$ ,  $If = A$ , and  $Kg = 3A$ ; where  $A$  is the weight of the atmosphere, Which shews, that the condensations are directly as the compressing forces. And the elasticities are in the same ratio, since the columns in  $AC$  are sustained by the elasticities in  $BD$ .

From the foregoing principles may be deduced many useful remarks, as in the following corollaries, viz.

316. *Corol. 1.* The space which any quantity of air is confined in, is reciprocally as the force that compresses it. So, the forces which confine a quantity of air in the cylindrical spaces  $AG$ ,  $BG$ ,  $CG$ , are reciprocally as the same, or reciprocally as the heights  $AD$ ,  $BD$ ,  $CD$ .

And therefore, if to the two perpendicular lines  $DA$ ,  $DH$ , as asymptotes, the hyperbola  $IKL$  be described, and the ordinates  $AI$ ,  $BK$ ,  $CL$  be drawn; then the forces which confine the air in the spaces  $AG$ ,  $BG$ ,  $CG$ , will be directly as the corresponding ordinates  $AI$ ,  $BK$ ,  $CL$ , since these are reciprocally as the abscisses  $AD$ ,  $BD$ ,  $CD$ , by the nature of the hyperbola.



317. *Corol. 2.* All the air near the earth is in a state of compression, by the weight of the incumbent atmosphere.

318. *Corol. 3.* The air is denser near the earth, than in high places; or denser at the foot of a mountain than at the top of it. And the higher above the earth, the less dense it is.

319. *Corol. 4.* The spring or elasticity of the air, is equal to the weight of the atmosphere above it; and they will produce the same effects; since they always sustain and balance each other.

320. *Corol. 5.* If the density of the air be increased, preserving the same heat or temperature; its spring or elasticity will likewise be increased, and in the same proportion.

321. *Corol. 6.* By the gravity and pressure of the atmosphere, on the surface of fluids, the fluids are made to rise in any pipes or vessels, when the spring or pressure within is decreased or taken off.

## PROPOSITION LXVII.

322. *Heat Increases the Elasticity of the Air, and Cold Diminishes it. Or, Heat Expands, and Cold Condenses the Air.*

This property is also proved by experience.

323. **THUS**, tie a bladder very close with some air in it; and lay it before the fire: then as it warms, it will more and more distend the bladder, and at last burst it, if the heat be continued, and increased high enough. But if the bladder be removed from the fire, as it cools it will contract again, as before. And it was upon this principle, that the first air-balloons were made by Montgolfier: for, by heating the air within them, by a fire underneath, the hot air distends them to a size which occupies a space in the atmosphere, whose weight of common air exceeds that of the balloon.

324. Also, if a cup or glass, with a little air in it, be inverted into a vessel of water; and the whole be heated over the fire, or otherwise; the air in the top will expand till it fill the glass, and expel the water out of it; and part of the air itself will follow, by continuing or increasing the heat.

Many other experiments, to the same effect, might be adduced, all proving the properties mentioned in the proposition.

## SCHOLIUM.

325. So that, when the force of the elasticity of air is considered, regard must be had to its heat or temperature; the same quantity of air being more or less elastic, as its heat is more or less. And it has been found, by experiment, that the elasticity is increased by the 435th part, by each degree of heat, of which there are 180 between the freezing and boiling heat of water.

326. *N. B.* Water expands about the  $\frac{3}{20000}$  part, with each degree of heat. (Sir Geo. Shuckburgh, *Philos. Trans.* 1777, p. 560, &c.)

Also, the

Spec. grav. of air	1.201 or $1\frac{1}{5}$	} when the barom. is 29.5, and the thermom. is 55° which are their mean heights in this country.
water	1000	
mercury	13592	

Or thus,	air	1.222 or $1\frac{2}{9}$	} when the barom. is 30, and thermometer 55.
	water	1000	
	mercury	13600	

## PROPOSITION LXVIII.

327. *The Weight or Pressure of the Atmosphere, on any Base at the Earth's Surface, is equal to the Weight of a Column of Quicksilver, of the Same Base, and the Height of which is between 28 and 31 inches.*

THIS is proved by the barometer, an instrument which measures the pressure of the air, and which is described below. For, at some seasons, and in some places, the air sustains and balances a column of mercury, of about 28 inches: but at other times it balances a column of 29, or 30, or near 31 inches high; seldom in the extremes 28 or 31, but commonly about the means 29 or 30. A variation which depends partly on the different degrees of heat in the air near the surface of the earth, and partly on the commotions and changes in the atmosphere, from winds and other causes, by which it is accumulated in some places, and depressed in others, being thereby rendered denser and heavier, or rarer and lighter; which changes in its state are almost continually happening in any one place. But the medium state is commonly about  $29\frac{1}{2}$  or 30 inches.

328. *Corol. 1.* Hence the pressure of the atmosphere on every square inch at the earth's surface, at a medium, is very near 15 pounds averdupois, or rather  $14\frac{3}{4}$  pounds.

For, a cubic foot of mercury weighing 13600 ounces nearly, an inch of it will weigh 7.866 or almost 8 ounces, or near half a pound, which is the weight of the atmosphere for every inch of the barometer on a base of a square inch; and therefore 30 inches, or the medium height, weighs very near  $14\frac{3}{4}$  pounds.

329. *Corol. 2.* Hence also, the weight or pressure of the atmosphere, is equal to that of a column of water from 32 to 35 feet high, or on a medium 33 or 34 feet high.

For, water and quicksilver are in weight nearly as 1 to 13.6; so that the atmosphere will balance a column of water 13.6 times as high as one of quicksilver; consequently

$$13.6 \text{ times } 28 \text{ inches} = 381 \text{ inches, or } 31\frac{3}{4} \text{ feet,}$$

$$13.6 \text{ times } 29 \text{ inches} = 394 \text{ inches, or } 32\frac{5}{8} \text{ feet,}$$

$$13.6 \text{ times } 30 \text{ inches} = 408 \text{ inches, or } 34 \text{ feet,}$$

$$13.6 \text{ times } 31 \text{ inches} = 422 \text{ inches, or } 35\frac{1}{6} \text{ feet.}$$

And hence a common sucking pump will not raise water higher than about 33 or 34 feet. And a syphon will not run, if the perpendicular height of the top of it be more than about 33 or 34 feet.



330. *Corol. 3.* If the air were of the same uniform density at every height up to the top of the atmosphere, as at the surface of the earth; its height would be about  $5\frac{1}{4}$  miles at a medium.

For, the weights of the same bulk of air and water, are nearly as 1.222 to 1000; therefore as 1.222 : 1000 ::  $33\frac{3}{4}$  feet : 27600 feet, or  $5\frac{1}{4}$  miles nearly. And so high the atmosphere would be, if it were all of uniform density, like water. But, instead of that, from its expansive and elastic quality, it becomes continually more and more rare, the farther above the earth, in a certain proportion; which will be treated of below, as also the method of measuring heights by the barometer, which depends on it.

331. *Corol. 4.* From this proposition and the last it follows, that the height is always the same, of an uniform atmosphere above any place, which shall be all of the uniform density with the air there, and of equal weight or pressure with the real height of the atmosphere above that place, whether it be at the same place at different times, or at any different places or heights above the earth; and that height is always about  $5\frac{1}{4}$  miles, or 27600 feet, as above found. For, as the density varies in exact proportion to the weight of the column, therefore it requires a column of the same height in all cases, to make the respective weights or pressures. Thus, if  $W$  and  $w$  be the weights of atmosphere above any places,  $D$  and  $d$  their densities, and  $H$  and  $h$  the heights of the uniform columns, of the same densities and weights; Then  $H \times D = W$ ,

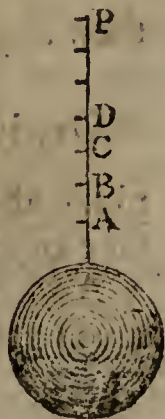
and  $b \times d = w$ ;

therefore  $\frac{W}{D}$  or  $H$  is equal to  $\frac{w}{d}$  or  $h$ . The temperature being the same.

PROPOSITION LXIX.

332. *The Density of the Atmosphere, at different Heights above the Earth, Decreases in such Sort, that when the Heights increase in Arithmetical Progression, the Densities decrease in Geometrical Progression.*

LET the indefinite perpendicular line AP, erected on the earth, be conceived to be divided into a great number of very small equal parts A, B, C, D, &c, forming so many thin strata of air in the atmosphere, all of different density, gradually decreasing from the greatest at A: then the density of the several strata A, B, C, D, &c, will be in geometrical progression decreasing.



For,

For, as the strata A, B, C, &c, are all of equal thickness, the quantity of matter in each of them, is as the density there; but the density in any one, being as the compressing force, is as the weight or quantity of all the matter from that place upward to the top of the atmosphere; therefore the quantity of matter in each stratum, is also as the whole quantity from that place upward. Now, if from the whole weight at any place as B, the weight or quantity in the stratum B be subtracted, the remainder is the weight at the next stratum C: that is, from each weight subtracting a part which is proportional to itself, leaves the next weight; or, which is the same thing, from each density subtracting a part which is always proportional to itself, leaves the next density. But when any quantities are continually diminished by parts which are proportional to themselves, the remainders form a series of continued proportionals; consequently these densities are in geometrical progression.

Thus, if the first density be D, and from each be taken its  $n$ th part; then there will remain its  $\frac{n-1}{n}$  part, or the  $\frac{m}{n}$  part, putting  $m$  for  $n-1$ ; and therefore the series of densities will be D,  $\frac{m}{n}D$ ,  $\frac{m^2}{n^2}D$ ,  $\frac{m^3}{n^3}D$ ,  $\frac{m^4}{n^4}D$ , &c, the common ratio of the series being that of  $n$  to  $m$ .

#### SCHOLIUM.

333. Because the terms of an arithmetical series, are proportional to the logarithms of the terms of a geometrical series; therefore different altitudes above the earth's surface, are as the logarithms of the densities, or of the weights of air, at those altitudes.

So that, if D denote the density at the altitude A, and  $d$  the density at the altitude  $a$ ; then A being as the log. of D, and  $a$  as the log. of  $d$ , the dif. of alt.  $A-a$  will be as the log.  $D-d$  or log.  $\frac{D}{d}$ .

And if  $A=0$ , or D the density at the surface of the earth; then any alt. above the surface  $a$ , is as the log. of  $\frac{D}{d}$ .

Or, in general, the log. of  $\frac{D}{d}$  is as the altitude of the one place above the other, whether the lower place be at the surface of the earth, or any where else.

And from this property is derived the method of determining the heights of mountains and other eminences, by the

the barometer, which is an instrument that measures the pressure or density of the air at any place. For, by taking, with this instrument, the pressure or density, at the foot of a hill for instance, and again at the top of it, the difference of the logarithms of these two pressures, or the logarithm of their quotient, will be as the difference of altitude, or as the height of the hill; supposing the temperatures of the air to be the same at both places, and the gravity of air not altered by the different distances from the earth's centre.

334. But as this formula expresses only the relations between different altitudes, with respect to their densities, recourse must be had to some experiment, to obtain the real altitude which corresponds to any given density, or the density which corresponds to a given altitude. And there are various experiments by which this may be done. The first, and most natural, is that which results from the known specific gravity of air, with respect to the whole pressure of the atmosphere on the surface of the earth. Now, as the alti-

tude  $a$  is always as  $\log. \frac{D}{d}$ ; assume  $b$  so that  $a = b \times \log. \frac{D}{d}$ ,

where  $b$  will be of one constant value for all altitudes; and to determine that value, let a case be taken in which we know the altitude  $a$  corresponding to a known density  $d$ ; as for instance, take  $a = 1$  foot, or 1 inch, or some such small altitude; then, because the density  $D$  may be measured by the pressure of the atmosphere, or the uniform column of 27600 feet, when the temperature is  $55^\circ$ ; therefore 27600 feet will denote the density  $D$  at the lower place, and 27599 the less density  $d$  at 1 foot above it; consequently  $1 = b \times \log. \frac{27600}{27599}$ ; which,

by the nature of logarithms, is nearly  $= b \times \frac{43429448}{27600}$

$= \frac{b}{63551}$  nearly; and hence  $b = 63551$  feet; which gives,

for any altitude in general, this theorem, viz.  $a = 63551 \times \log. \frac{D}{d}$ , or  $= 63551 \times \log. \frac{M}{m}$  feet, or  $10592 \times \log. \frac{M}{m}$

fathoms; where  $M$  is the column of mercury which is equal to the pressure or weight of the atmosphere at the bottom, and  $m$  that at the top of the altitude  $a$ ; and where  $M$  and  $m$  may be taken in any measure, either feet, or inches, &c.

335. Note, that this formula is adapted to the mean temperature of the air  $55^\circ$ : But, for every degree of temperature

ture different from this, in the medium between the temperatures at the top and bottom of the altitude  $a$ , that altitude will vary by its 435th part; which must be added when that medium exceeds  $55^\circ$ , otherwise subtracted.

336. Note also, that a column of 30 inches of mercury varies its length by about the  $\frac{1}{30}$  part of an inch for every degree of heat, or rather  $\frac{1}{9600}$  of the whole volume.

337. But the formula may be rendered much more convenient for use, by reducing the factor 10592 to 10000, by changing the temperature proportionally from  $55^\circ$ : thus, as the diff. 592 is the 18th part of the whole factor 10592; and as 18 is the 24th part of 435; therefore the corresponding change of temperature is  $24^\circ$ , which reduces the  $55^\circ$  to  $31^\circ$ . So that the formula is,  $a = 10000 \times \log. \frac{M}{m}$

fathoms, when the temperature is 31 degrees; and for every degree above that, the result is to be increased by so many times its 435th part.

338. *Exam. 1.* To find the height of a hill when the pressure of the atmosphere is equal to 29.68 inches of mercury at the bottom, and 25.28 at the top; the mean temperature being  $50^\circ$ ?  
Anf. 4378 feet, or 730 fathoms.

339. *Exam. 2.* To find the height of a hill when the atmosphere weighs 29.45 inches of mercury at the bottom, and 26.82 at the top, the mean temperature being  $33^\circ$ ?  
Anf. 2385 feet, or  $397\frac{1}{2}$  fathoms.

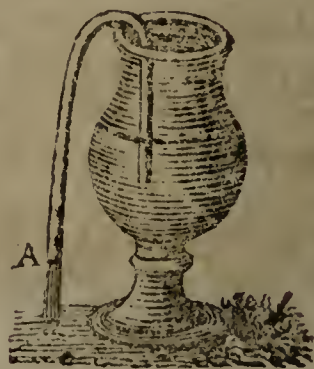
340. *Exam. 3.* At what altitude is the density of the atmosphere only the 4th part of what it is at the earth's surface?  
Anf. 6020 fathoms.

By the weight and pressure of the atmosphere, the effect and operations of pneumatic engines may be accounted for, and explained; such as syphons, pumps, barometers, &c; of which it may not be improper here to give a brief description.

#### OF THE SIPHON.

341. THE Siphon, or Syphon, is any bent tube, having its two legs either of equal or of unequal length.

If it be filled with water, and then inverted, with the two open ends downward, and held level in that position; the water will remain suspended



ed in it, if the two legs be equal. For the atmosphere will press equally on the surface of the water in each end, and support them, if they are not more than 34 feet high; and the legs being equal, the water in them is an exact counterpoise by their equal weights; so that the one has no power to move more than the other; and they are both supported by the atmosphere.

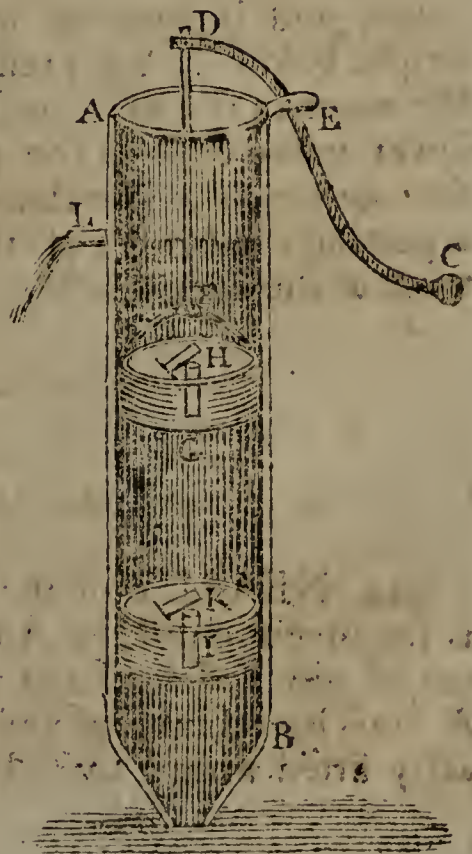
But if now the syphon be a little inclined to one side, so that the orifice of one end be lower than that of the other; or if the legs be of unequal length, which is the same thing; then the equilibrium is destroyed, and the water will all descend out by the lower end, and rise up in the higher. For, the air pressing equally, but the two ends weighing unequally, a motion must commence where the power is greatest, and so continue till all the water has run out by the lower end. And if the shorter leg be immersed into a vessel of water, and the syphon be set a running as above, it will continue to run till all the water be exhausted out of the vessel, or at least as low as that end of the syphon. Or, it may be set a running without filling the syphon as above, by only inverting it, with its shorter leg into the vessel of water; then, with the mouth applied to the lower orifice A, suck the air out; and the water will presently follow, being forced up into the syphon by the pressure of the air on the water in the vessel.

#### OF THE PUMP.

342. THERE are three sorts of PUMPS; the Sucking, the Lifting, and the Forcing Pump. By the former, water can be raised only to about 34 feet, viz. by the pressure of the atmosphere; but by the others, to any height; but then they require more apparatus and power.

The annexed figure represents a common sucking pump. AB is the barrel of the pump, being a hollow cylinder, made of metal, and smooth within, or of wood for very common purposes. CD is the handle, moveable about the pin D, by moving the end C up and down. DF

an



an iron rod turning about a pin D, which connects it to the end of the handle. This rod is fixed to the piston, bucket, or sucker, FG, by which this is moved up and down within the barrel, which it must fit very tight and close, that no air or water may pass between the piston and the sides of the barrel; and for this purpose it is commonly armed with leather. The piston is made hollow, or it has a perforation through it, the orifice of which is covered by a valve H opening upwards. I is a plug firmly fixed in the lower part of the barrel; also perforated, and covered by a valve K opening upwards.

343. When the pump is first to be worked, and the water is below the plug I; raise the end C of the handle, then the piston descending, compresses the air in HI, which by its spring shuts fast the valve K, and pushes up the valve H, and so enters into the barrel above the piston. Then putting the end C of the handle down again, raises the piston or sucker, which lifts up with it the column of air above it, the external atmosphere by its pressure keeping the valve H shut: the air in the barrel being thus exhausted, or rarefied, is no longer a counterpoise to that which presses on the surface of the water in the well, this is forced up the pipe, and through the valve K, into the barrel of the pump. Then pushing the piston down again into this water, now in the barrel, its weight shuts the lower valve K, and its resistance forces up the valve of the piston, and enters the upper part of the barrel, above the piston. Then, the bucket being raised, lifts up with it the water which had passed above its valve, and it runs out by the cock L; and taking off the weight below it, the pressure of the external atmosphere on the water in the well again forces it up through the pipe and lower valve close to the piston, all the way as it ascends, thus keeping the barrel always full of water. And thus, by repeating the strokes of the piston, a continued discharge is made at the cock L.

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#### OF THE AIR-PUMP.

344. **NEARLY** on the same principles as the water-pump, is the invention of the Air-pump, by which the air is drawn out of any vessel, like as water is drawn out by the former. A brass barrel is bored and polished truly cylindrical, and exactly fitted with a turned piston, so that no air can pass by  
the

the sides of it, and furnished with a proper valve opening upward. Then, by lifting up the piston, the air in the close vessel below it follows the piston, and fills the barrel; and being thus diffused through a larger space than before, when it occupied the vessel or receiver only, but not the barrel, it is made rarer than it was before, in proportion as the capacity of the barrel and receiver together, exceeds the receiver alone. Another stroke of the piston exhausts another barrel of this now rarer air, which again rarefies it in the same proportion as before. And so on, for any number of strokes of the piston, still exhausting in the same geometrical progression, of which the ratio is that which the capacity of the receiver and barrel together exceeds the receiver, till this is exhausted to any proposed degree, or as far as the nature of the machine is capable of performing; which happens when the elasticity of the included air is so far diminished, by rarefying, that it is too feeble to push up the valve of the piston, and escape.

345. From the nature of this exhausting, in geometrical progression; we may easily find how much the air in the receiver is rarefied by any number of strokes of the piston; or what number of such strokes is necessary, to exhaust the receiver to any given degree. Thus, if the capacity of the receiver and barrel together, be to that of the receiver alone, as  $c$  to  $r$ , and  $1$  denote the natural density of the air at first; then

$$c : r :: 1 : \frac{r}{c}, \text{ the density after 1 stroke of the piston,}$$

$$c : r :: \frac{r}{c} : \frac{r^2}{c^2}, \text{ the density after 2 strokes,}$$

$$c : r :: \frac{r^2}{c^2} : \frac{r^3}{c^3}, \text{ the density after 3 strokes,}$$

$$\&c, \text{ and } \frac{r^n}{c^n}, \text{ the density after } n \text{ strokes.}$$

So, if the barrel be equal to  $\frac{1}{4}$  of the receiver; then  $c : r :: 5 : 4$ ; and  $\frac{4^n}{5^n} = 0.8^n$  is  $= d$  the density after  $n$  turns.

And if  $n$  be 20, then  $0.8^{20} = .0115$  is the density of the included air after 20 strokes of the piston; which being the  $\frac{86\frac{7}{8}}{100}$  part of 1, or the first density, it follows that the air is  $\frac{86\frac{7}{8}}{100}$  times rarefied by the 20 strokes.

346. Or, if it were required to find the number of strokes necessary to rarefy the air any number of times; because

cause  $\frac{r^n}{c^n}$  is = the proposed density  $d$ ; therefore, taking the logarithms,  $n \times \log. \frac{r}{c} = \log. d$ , and  $n = \frac{\log. d}{1. r - 1. c}$ , the number of strokes required. So, if  $r$  be  $\frac{4}{5}$  of  $c$ , and it be required to rarefy the air 100 times: then  $d = \frac{1}{100}$  or .01; and hence  $n = \frac{\log. 100}{1. 5 - 1. 4} = 20\frac{3}{5}$  nearly. So that in  $20\frac{3}{5}$  strokes the air will be rarefied 100 times.

#### OF THE DIVING BELL AND CONDENSING MACHINE.

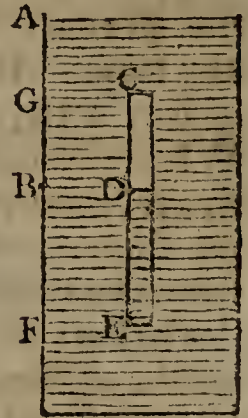
347. ON the same principles too, depend the operations and effect of the Condensing Engine, by which air may be condensed to any degree, instead of rarefied as in the air-pump. And, like as the air-pump rarefies the air, by extracting always one barrel of air after another; so, by this other machine, the air is condensed by throwing in or adding always one barrel of air after another; which it is evident may be done by only turning the valves of the piston and barrel, that is, making them to open the contrary way, and working the piston in the same manner; so that, as they both open upward, or outward, in the air-pump, or rarefier, they will both open downward, or inward, in the condenser.

348. And on the same principles, namely, of the compression and elasticity of the air, depends the use of the Diving Bell, which is a large vessel, in which a person descends to the bottom of the sea, the open end of the vessel being downward; only, in this case the air is not condensed by forcing more of it into the same space, as in the condensing engine; but by compressing the same quantity of air into a less space in the bell; by increasing always the force which compresses it.

349. If a vessel of any sort be inverted into water, and pushed or let down to any depth in it; then by the pressure of the water some of it will ascend into the vessel, but not so high as the water without, and will compress the air into less space, according to the difference between the heights of the internal and external water; and the density and elastic force of the air will be increased in the same proportion, as its space in the vessel is diminished.



So, if the tube CE be inverted, and pushed down into water, till the external water exceed the internal, by the height AB, and the air of the tube be reduced to the space CD; then that air is pressed both by a column of water of the height AB, and by the whole atmosphere which presses on the upper surface of the water; consequently the space CD is to the whole space CE, as the weight of the atmosphere, is to the weights both of the atmosphere and the column of water AB. So that, if AB be about 34 feet, which is equal to the force of the atmosphere, then CD will be equal to  $\frac{1}{2}$ CE; but if AB be double of that, or 68 feet, then CD will be  $\frac{1}{3}$ CE; and so on. And hence, by knowing the depth AF, to which the vessel is sunk, we can easily find the point D, to which the water will rise within it at any time. For let the weight of the atmosphere at that time be equal to that of 34 feet of water; also, let the depth AF be 20 feet, and the length of the tube CE 4 feet: then, putting the height of the internal water DE = x,



$$\begin{aligned} \text{it is} & - 34 + AB : 34 :: CE : CD, \\ \text{that is,} & 34 + AF - DE : 34 :: CE : CE - DE, \\ \text{or} & - 54 - x : 34 :: 4 : 4 - x; \end{aligned}$$

hence, multiplying extremes and means,  $216 - 58x + x^2 = 136$ , and the root is  $x = 1.414$  of a foot, or 17 inches nearly; being the height DE to which the water will rise within the tube.

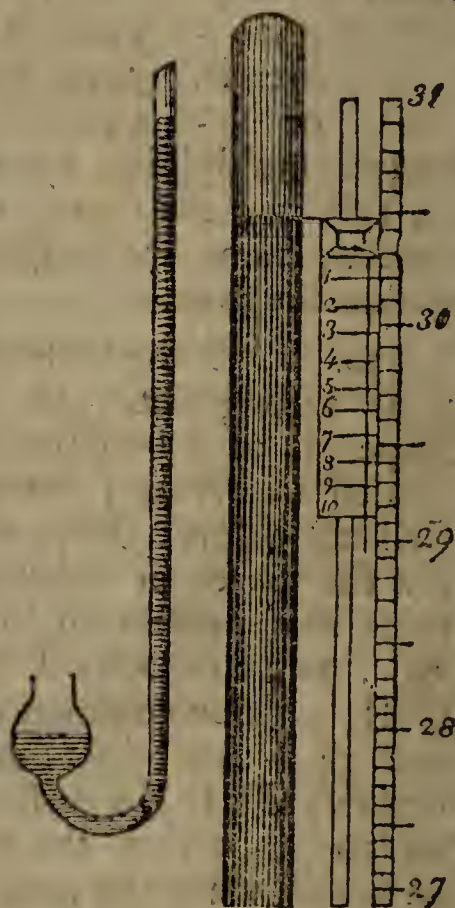
350. But if the vessel be not equally wide throughout, but of any other shape, as of a bell-like form, such as is used in diving; then the altitudes will not observe the proportion above, but the spaces or bulks only, will respect that proportion, namely,  $34 + AB : 34 :: \text{capacity CKL} : \text{capacity CHI}$ , if it be common or fresh water; and  $33 + AB : 33 :: \text{capacity CKL} : \text{capacity CHI}$ , if it be sea-water. From which proportion, the height DE may be found, when the nature or shape of the vessel or bell CKL is known.



## OF THE BAROMETER.

351. THE BAROMETER is an instrument for measuring the pressure of the atmosphere, and elasticity of the air, at any time. It is commonly made of a glass tube, of near 3 feet long, close at one end, and filled with mercury. When the tube is full, by stopping the open end with the finger, then inverting the tube, and immersing that end with the finger into a basin of quicksilver, on removing the finger from the orifice, the quicksilver in the tube will descend into the basin, till what remains in the tube be of the same weight with a column of the atmosphere, which is commonly between 28 and 31 inches of quicksilver; and leaving an entire vacuum in the upper end of the tube above the mercury. For, as the upper end of the tube is quite void of air, there is no pressure downwards but from the column of quicksilver, and therefore that will be an exact balance to the counter pressure of the whole column of atmosphere, acting on the orifice of the tube by the quicksilver in the basin. The upper three inches of the tube, namely, from 28 to 31 inches, have a scale attached to them, divided into inches, tenths, and hundredths, for measuring the length of the column at all times, by observing which division of the scale the top of the quicksilver is opposite to; as it ascends and descends within these limits, according to the state of the atmosphere.

So that the weight of the quicksilver in the tube, above that in the basin, is at all times equal to the weight or pressure of the column of atmosphere above it, and of the same base with the tube; and hence the weight of it may at all times be computed; being nearly at the rate of half a pound averdupois for every inch of quicksilver in the tube, on every square inch of base; or more exactly, it is  $\frac{59}{128}$  of a pound on the square inch, for every inch in the altitude of the quicksilver: for the cubic inch of quicksilver weighs just  $\frac{59}{128}$  lb, or nearly  $\frac{1}{2}$  a pound, in the mean temperature of 55° of



of heat. And consequently, when the barometer stands at 30 inches, or  $2\frac{1}{2}$  feet high, which is the medium or standard height, the whole pressure of the atmosphere is equal to  $14\frac{1}{2}$  pounds, on every square inch of the base. And so in proportion for other heights.

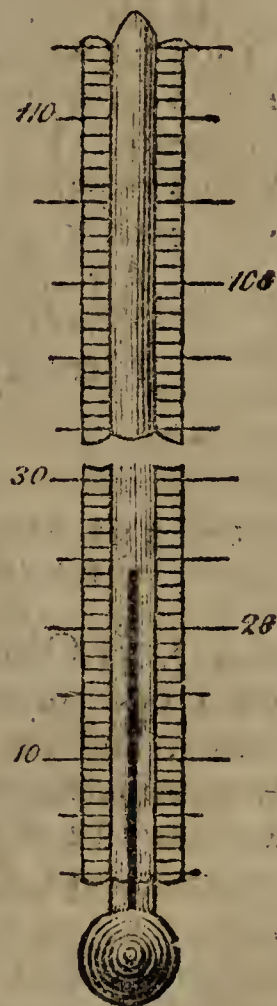
### OF THE THERMOMETER.

352. THE THERMOMETER is an instrument for measuring the temperature of the air, as to heat and cold.

It is found by experience, that all bodies expand by heat, and contract by cold; and hence the degrees of expansion become the measure of the degrees of heat. Fluids are more convenient for this purpose, than solids: and quicksilver is now most commonly used for it.

A very fine glass tube, having a pretty large hollow ball at the bottom, is filled about half way up with quicksilver: the whole being then heated very hot till the quicksilver rise quite to the top, the top is then hermetically sealed, so as perfectly to exclude all communication with the outward air. Then, in cooling, the quicksilver contracts, and consequently its surface descends in the tube, till it come to a certain point, correspondent to the temperature or heat of the air. And when the weather becomes warmer, the quicksilver expands, and its surface rises in the tube; and again contracts and descends when the weather becomes cooler. So that, by placing a scale of any divisions against the side of the tube, it will show the degrees of heat, by the expansion and contraction of the quicksilver in the tube; observing at what division of the scale the top of the quicksilver stands. And the method of preparing the scale, as used in

England, is thus:—Bring the thermometer into a temperature of just freezing, by immersing the ball in water just freezing, or in ice just thawing, and mark the scale where the mercury then stands, for the point of freezing. Next, immerse it in boiling water; and the quicksilver will rise to a certain height in the tube; which mark also on the scale,



for

for the boiling point, or the heat of boiling water. Then the distance between these two points is divided into 180 equal divisions, or degrees; and the like equal degrees are also continued to any extent below the freezing point, and above the boiling point. The divisions are then numbered as follows, namely, at the freezing point is set the number 32, and consequently 212 at the boiling point; and all the other numbers in their order.

This division of the scale, is commonly called Fahrenheit's. According to this division, 55 is at the mean temperature of the air in this country; and it is in this temperature, and in an atmosphere which sustains a column of 30 inches of quicksilver in the barometer, that all measures and specific gravities are taken, unless when otherwise mentioned; and in this temperature and pressure, the relative weights, or specific gravities, of air, water, and quicksilver, are as

$1\frac{2}{3}$ for air,	{	and these also are the weights of a cubic
1000 for water,		foot of each, in averdupois ounces, in
13600 for mercury;		that state of the barometer and thermometer. For other states of the thermometer, each of these

bodies expands or contracts according to the following rate, with each degree of heat; viz.

Air about	-	$\frac{1}{435}$	part of its bulk,
Water about		$\frac{1}{6666}$	part of its bulk,
Mercury about		$\frac{1}{9600}$	part of its bulk.

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#### ON THE MEASUREMENT OF ALTITUDES BY THE BAROMETER AND THERMOMETER.

353. FROM the principles laid down in the Scholium to prop. 69, concerning the measuring of altitudes by the barometer, and the foregoing descriptions of the barometer and thermometer, we may now collect together the precepts for the practice of such measurements, which are as follow:

*First*, Observe the height of the barometer at the bottom of any height, or depth, intended to be measured; with the temperature of the quicksilver by means of a thermometer attached to the barometer, and also the temperature of the air in the shade by a detached thermometer.

*Secondly*. Let the same thing be done also at the top of the said height or depth, and at the same time, or as near the same time as may be. And let those altitudes of barometer be reduced to the same temperature, if it be thought necessary, by correcting either the one or the other, that is,

augment

augment the height of the mercury in the colder temperature; or diminish that in the warmer, by its  $\frac{1}{9600}$  part for every degree of difference of the two.

*Third*, Take the difference of the common logarithms of the two heights of the barometer, corrected as above if necessary, cutting off three figures next the right hand for decimals, the rest being fathoms in whole numbers.

*Fourth*, Correct the number last found for the difference of temperature of the air, as follows: Take half the sum of the two temperatures, for the mean one; and for every degree which this differs from the temperature  $31^{\circ}$ , take so many times the  $\frac{1}{435}$  part of the fathoms above found, and add them if the mean temperature be above  $31^{\circ}$ , but subtract them if the mean temperature be below  $31^{\circ}$ ; and the sum or difference will be the true altitude in fathoms; or, being multiplied by 6, it will be the altitude in feet.

354. *Example 1.* Let the state of the barometers and thermometers be as follows; to find the altitude, viz.

Barom.	Thermom.		Ans. the alt. is
	attach.	detach.	
Lower 29.68	57	57	727 $\frac{3}{10}$ fath.
Upper 25.28	43	42	

355. *Exam. 2.* To find the altitude, when the state of the barometers and thermometers is as follows, viz.

Barom.	Thermom.		Ans. the alt. is
	attach.	detach.	
Lower 29.45	38	31	408 $\frac{1}{8}$ fath.
Upper 26.82	41	35	

ON THE RESISTANCE OF FLUIDS, WITH THEIR FORCES AND ACTION ON BODIES.

PROPOSITION LXX.

356. *If any Body move through a Fluid at Rest, or the Fluid move against the Body at Rest; the Force or Resistance of the Fluid against the Body, will be as the Square of the Velocity and the Density of the Fluid. That is,  $R \propto dv^2$ .*

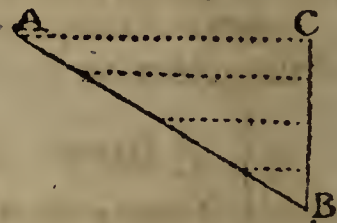
FOR, the force or resistance is as the quantity of matter or particles struck, and the velocity with which they are struck. But the quantity or number of particles struck, in any time,

are as the velocity and the density of the fluid. Therefore the resistance, or force of the fluid, is as the density and square of the velocity.

357. *Corol. 1.* The resistance to any plane, is also more or less, as the plane is greater or less; and therefore the resistance on any plane, is as the area of the plane  $a$ , the density of the medium, and the square of the velocity. That is,  $R \propto adv^2$ .

358. *Corol. 2.* If the motion be not perpendicular, but oblique to the plane, or to the face of the body; then the resistance, in the direction of motion, will be diminished in the triplicate ratio of radius to the sine of the angle of inclination of the plane to the direction of motion, or as the cube of radius to the cube of the sine of that angle. So that  $R \propto adv^2 s^3$ , putting  $r =$  radius, and  $s =$  sine of the angle of inclination  $CAB$ .

For, if  $AB$  be the plane,  $AC$  the direction of motion, and  $BC$  perpendicular to  $AC$ ; then no more particles meet the plane than what meet the perpendicular  $BC$ , and therefore their number is diminished as  $AB$  to  $BC$ , or as  $r$  to  $s$ . But the force of each particle, striking the plane obliquely in the direction  $CA$ , is also diminished as  $AB$  to  $BC$ , or as  $r$  to  $s$ ; therefore the resistance, which is perpendicular to the face of the plane by art. 52, is as  $r^2$  to  $s^2$ . But again, this resistance in the direction perpendicular to the face of the plane, is to that in the direction  $AC$ , by art. 51, as  $AB$  to  $BC$ , or as  $r$  to  $s$ . Consequently, on all these accounts, the resistance to the plane when moving perpendicular to its face, is to that when moving obliquely, as  $r^3$  to  $s^3$ , or  $r$  to  $s^3$ . That is, the resistance in the direction of the motion, is diminished as  $r$  to  $s^3$ , or in the triplicate ratio of radius to the sine of inclination.



PROPOSITION LXXI.

359. *The Real Resistance to a Plane, by a Fluid acting in a Direction Perpendicular to its Face, is equal to the Weight of a Column of the Fluid, whose Base is the Plane, and Altitude equal to that which is due to the Velocity of the Motion, or through which a Heavy Body must fall to acquire that Velocity.*

THE resistance to the plane moving through a fluid, is the same

same as the force of the fluid in motion with the same velocity, on the plane at rest. But the force of the fluid in motion, is equal to the weight or pressure which generates that motion; and this is equal to the weight or pressure of a column of the fluid, whose base is the area of the plane, and its altitude that which is due to the velocity.

360. *Corol. 1.* If  $a$  denote the area of the plane,  $v$  the velocity,  $n$  the density or specific gravity of the fluid, and  $g = 16\frac{1}{2}$  feet, or 193 inches. Then, the altitude due to the velocity  $v$  being  $\frac{v^2}{4g}$ , therefore  $a \times n \times \frac{v^2}{4g} = \frac{anv^2}{4g}$  will be the whole resistance, or motive force  $R$ .

361. *Corol. 2.* If the direction of motion be not perpendicular to the face of the plane, but oblique to it, in any angle, whose sine is  $s$ . Then the resistance to the plane will be  $\frac{anv^2s^3}{4g}$ .

362. *Corol. 3.* Also, if  $w$  denote the weight of the body, whose plane face  $a$  is resisted by the absolute force  $R$ ; then the retarding force  $f$ , or  $\frac{R}{w}$ , will be  $\frac{anv^2s^3}{4gw}$ .

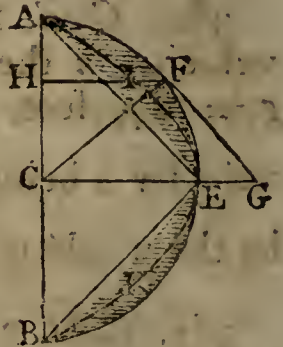
363. *Corol. 4.* And if the body be a cylinder, whose face or end is  $a$ , and radius  $r$ , moving in the direction of its axis: because then  $s = 1$ , and  $a = pr^2$ , where  $p = 3.1416$ ; then  $\frac{pnv^2r^2}{4g}$  will be the resisting force  $R$ , and  $\frac{pnv^2r^2}{4gw}$  the retarding force  $f$ .

364. *Corol. 5.* This is the value of the resistance when the end of the cylinder is a plane perpendicular to its axis, or to the direction of motion. But were its face an elliptic section, or a conical surface, or any other figure every where equally inclined to the axis, or direction of motion, the sine of inclination being  $s$ : then, the number of particles of the fluid striking the face being still the same, but the force of each, opposed to the direction of motion, diminished in the duplicate ratio of radius to the sine of inclination, the resisting force  $R$  would be  $\frac{pnr^2v^2s^2}{4g}$ .

## PROPOSITION LXXII.

365. *The Resistance to a Sphere moving through a Fluid, is but Half the Resistance to its Great Circle, or to the End of a Cylinder of the Same Diameter, moving with an equal Velocity.*

LET AFEB be half the sphere, moving in the direction CEG. Describe the paraboloid AIEKB on the same base. Let any particle of the medium meet the semicircle in F, to which draw the tangent FG, the radius FC, and the ordinate FIH. Then the force of any particle on the surface at F, is to its force on the base at H, as the square of the sine of the angle G, or its equal the angle FCH, to the square of radius, that is, as  $HF^2$  to  $CF^2$ . Therefore the force of all the particles, or the whole fluid, on the whole surface, is to its force on the circle of the base, as all the  $HF^2$  to as many times  $CF^2$ . But  $CF^2$  is  $= CA^2 = AC \cdot CB$ , and  $HF^2 = AH \cdot HB$  by the nature of the circle; also,  $AH \cdot HB : AC \cdot CB :: HI : CE$  by the nature of the parabola; consequently the force on the spherical surface, is to the force on its circular base, as all the HI's to as many CE's, that is, as the content of the paraboloid to the content of its circumscribed cylinder, namely, as 1 to 2.



366. *Corol.* Hence, the resistance to the sphere is  $R = \frac{pnu^2 r^2}{8g}$ , being the half of that of a cylinder of the same diameter. For example, a 9lb iron ball, whose diameter is 4 inches, when moving through the air with a velocity of 1600 feet per second, would meet a resistance which is equal to a weight of  $132\frac{2}{3}$ lb, over and above the pressure of the atmosphere, for want of the counterpoize behind the ball.

## PRACTICAL EXERCISES IN MENSURATION.

QUEST. 1. WHAT difference is there between a floor 28 feet long by 20 broad, and two others, each of half the dimensions; and what do all three come to at 45s. per square, or 100 square feet?

Ans. dif. 280 sq. feet. Amount 18 guineas.

QUEST. 2.



QUEST. 2. An elm plank is 14 feet 3 inches long, and I would have just a square yard slit off it; at what distance from the edge must the line be struck? Ans.  $7\frac{1}{7}$  inches.

QUEST. 3. A ceiling contains 114 yards 6 feet of plastering, and the room 28 feet broad; what is the length of it? Ans.  $36\frac{6}{7}$  feet.

QUEST. 4. A common joist is 7 inches deep, and  $2\frac{1}{2}$  thick; but I want a scantling just as big again, that shall be 3 inches thick; what will the other dimension be? Ans.  $11\frac{2}{3}$  inches.

QUEST. 5. A wooden cistern cost me 3s. 2d. painting within; at 6d. per yard; the length of it was 102 inches, and the depth 21 inches; what was the width? Ans.  $27\frac{1}{4}$  inches.

QUEST. 6. If my court-yard be 47 feet 9 inches square, and I have laid a foot-path with Purbeck stone, of 4 feet wide, along one side of it; what will paving the rest with flints come to; at 6d. per square yard? Ans. 5l. 16s.  $0\frac{1}{2}$ d.

QUEST. 7. A ladder, 40 feet long, may be so planted, that it shall reach a window 33 feet from the ground on one side of the street; and, by only turning it over, without moving the foot out of its place, it will do the same by a window 21 feet high on the other side: what is the breadth of the street? Ans. 56 feet  $7\frac{3}{4}$  inches.

QUEST. 8. The paving of a triangular court, at 18d. per foot, came to 100l; the longest of the three sides was 88 feet; required the sum of the other two equal sides? Ans. 106 85 feet.

QUEST. 9. There are two columns in the ruins of Persepolis left standing upright; the one is 64 feet above the plain, and the other 50: in a straight line between these stands an ancient small statue, the head of which is 97 feet from the summit of the higher, and 86 feet from the top of the lower column, the base of which measures just 76 feet to the centre of the figure's base. Required the distance between the tops of the two columns? Ans. 157 feet nearly.

QUEST. 10. The perambulator, or surveying wheel, is so contrived, as to turn just twice in the length of 1 pole, or  $16\frac{1}{2}$  feet; required the diameter? Ans. 2 626 feet.

QUEST. 11. In turning a one-horse chaise within a ring of a certain diameter, it was observed, that the outer wheel made two turns, while the inner made but one: the wheels

were both 4 feet high; and, supposing them fixed at the statutable distance of 5 feet asunder on the axletree, what was the circumference of the track described by the outer wheel? Ans. 62·832 feet.

QUEST. 12. What is the side of that equilateral triangle, whose area cost as much paving at 8d. a foot, as the palli-fading the three sides did at a guinea a yard? Ans. 72·746 feet.

QUEST. 13. In the trapezium ABCD, are given,  $AB = 13$ ,  $BC = 31\frac{1}{3}$ ,  $CD = 24$ , and  $DA = 18$ , also B a right angle; required the area? Ans. 410·122.

QUEST. 14. A roof, which is 24 feet 8 inches by 14 feet 6 inches, is to be covered with lead at 8lb. per square foot: what will it come to at 18s. per cwt. Ans. 22l. 19s. 10 $\frac{1}{4}$ d.

QUEST. 15. Having a rectangular marble slab, 58 inches by 27, I would have a square foot cut off parallel to the shorter edge; I would then have the like quantity divided from the remainder parallel to the longer side; and this alternately repeated, till there shall not be the quantity of a foot left: what will be the dimensions of the remaining piece? Ans. 20·7 inches by 6·086.

QUEST. 16. Given two sides of an obtuse-angled triangle, which are 20 and 40 poles; required the third side, that the triangle may contain just an acre of land? Ans. 58·876 or 23·099.

QUEST. 17. The end wall of a house is 24 feet 6 inches in breadth, and 40 feet to the eaves;  $\frac{1}{3}$  of which is 2 bricks thick,  $\frac{1}{3}$  more is  $1\frac{1}{2}$  brick thick, and the rest 1 brick thick. Now the triangular gable rises 38 courses of bricks, 4 of which usually make a foot in depth, and this is but  $4\frac{1}{2}$  inches, or half a brick thick: what will this piece of work come to at 5l. 10s. per statute rod? Ans. 20l. 11s. 7 $\frac{1}{2}$ d.

QUEST. 18. How many bricks will it take to build a wall, 10 feet high, and 500 feet long, of a brick and half thick; reckoning the brick 10 inches long, and 4 courses to the foot in height? Ans. 72000.

QUEST. 19. How many bricks will build a square pyramid of 100 feet on each side at the base, and also 100 feet perpendicular height: the dimensions of a brick being supposed 10 inches long, 5 inches broad, and 3 inches thick? Ans. 3840000.

QUEST. 20. If, from a right-angled triangle, whose base is 12, and perpendicular 16 feet, a line be drawn parallel to the

the perpendicular, cutting off a triangle whose area is 24 square feet; required the sides of this triangle?

Ans. 6, 8, and 10.

QUEST. 21. The ellipse in Grosvenor-square measures 840 links across the longest way, and 612 the shortest, within the rails: now the walls being 14 inches thick, what ground do they inclose, and what do they stand upon?

Ans.  $\left\{ \begin{array}{l} \text{inclose 4 ac. 0 r. 6 p.} \\ \text{stand on } 1760\frac{1}{2} \text{ sq. feet.} \end{array} \right.$

QUEST. 22. If a round pillar, 7 inches over, have 4 feet of stone in it; of what diameter is the column, of equal length, that contains 10 times as much?

Ans. 22.136 inches.

QUEST. 23. A circular fish-pond is to be made in a garden, that shall take up just half an acre; what must be the length of the chord that strikes the circle?

Ans.  $27\frac{3}{4}$  yards.

QUEST. 24. When a roof is of a true pitch, or making a right angle at the ridge, the rafters are nearly  $\frac{3}{4}$  of the breadth of the building: now supposing the eaves-boards to project 10 inches on a side, what will the new ripping a house cost, that measures 32 feet 9 inches long, by 22 feet 9 inches broad on the flat, at 15s. per square?

Ans. 8l. 15s.  $9\frac{1}{2}$ d.

QUEST. 25. A cable, which is 3 feet long, and 9 inches in compass, weighs 22lb; what will a fathom of that cable weigh, which measures a foot about?

Ans.  $78\frac{3}{4}$ lb.

QUEST. 26. My plumber has put 28lb. per square foot into a cistern, 74 inches and twice the thickness of the lead long, 26 inches broad, and 40 deep; he has also put three stays across it within, of the same strength, and 16 inches deep, and reckons 22s. per cwt, for work and materials. I, being a mason, have paved him a workshop, 22 feet 10 inches broad, with Purbeck stone, at 7d. per foot; and on the balance I find there is 3s. 6d. due to him; what was the length of the workshop, supposing sheet lead of  $\frac{1}{8}$  of an inch thick to weigh 5.899lb. the square foot.

Ans. 32 feet,  $0\frac{3}{4}$  inch.

QUEST. 27. The distance of the centres of two circles, whose diameters are each 50, being given, equal to 30; what is the area of the space inclosed by their circumferences?

Ans. 559.119.

QUEST. 28. If 20 feet of iron railing weigh half a ton, when the bars are an inch and quarter square; what will 50 feet come to at  $3\frac{1}{2}$ d. per lb, the bars being but  $\frac{7}{8}$  of an inch square?

Ans. 20l. os. 2d.

QUEST. 29:

QUEST. 29. The area of an equilateral triangle, whose base falls on the diameter, and its vertex in the middle of the arc of a semicircle, is equal to 100 : what is the diameter of the semicircle? Anf. 26.32148.

QUEST. 30. It is required to find the thickness of the lead in a pipe, of an inch and quarter bore, which weighs 14lb per yard in length; the cubic foot of lead weighing 11325 ounces? Anf. .20737 inches.

QUEST. 31. Supposing the expence of paving a semicircular plot, at 2s. 4d. per foot, come to 10l; what is the diameter of it? Anf. 14.7737 feet.

QUEST. 32. What is the length of a chord which cuts off  $\frac{1}{3}$  of the area from a circle whose diameter is 289? Anf. 278.6716.

QUEST. 33. My plumber has set me up a cistern, and, his shop-book being burnt, he has no means of bringing in the charge, and I do not choose to take it down to have it weighed; but by measure he finds it contains  $64\frac{3}{8}$  square feet, and that it is precisely  $\frac{1}{8}$  of an inch in thickness. Lead was then wrought at 21l. per fother of  $19\frac{1}{2}$  cwt. It is required from these items to make out the bill, allowing  $6\frac{2}{3}$  oz. for the weight of a cubic inch of lead? Anf. 4l. 11s. 2d.

QUEST. 34. What will the diameter of a globe be, when the solidity and superficial content are expressed by the same number? Anf. 6.

QUEST. 35. A sack, that would hold 3 bushels of corn, is  $22\frac{1}{2}$  inches broad when empty; what will another sack contain, which, being of the same length, has twice its breadth or circumference? Anf. 12 bushels.

QUEST. 36. A carpenter is to put an oaken curb to a round well, at 8d. per foot square; the breadth of the curb is to be  $7\frac{1}{4}$  inches, and the diameter within  $3\frac{1}{2}$  feet: what will be the expence? Anf. 5s.  $2\frac{1}{4}$ d.

QUEST. 37. A gentleman has a garden 100 feet long, and 80 feet broad; and a gravel walk is to be made of an equal width half round it: what must the breadth of the walk be, to take up just half the ground? Anf. 25.968 feet.

QUEST. 38. The top of a may-pole, being broken off by a blast of wind, struck the ground at 15 feet distance from the foot of the pole; what was the height of the whole may pole, supposing the length of the broken piece to be 39 feet? Anf. 75 feet.

QUEST. 39.

QUEST. 39. Seven men bought a grinding stone, of 60 inches diameter, each paying  $\frac{1}{7}$  part of the expence; what part of the diameter must each grind down for his share?

Ans. the 1st 4.4508, 2d 4.8400, 3d 5.3535, 4th 6.0765, 5th 7.2079, 6th 9.3935, 7th 22.6778 inches.

QUEST. 40. A maltster has a kiln, that is 16 feet 6 inches square: but he wants to pull it down, and build a new one, that may dry three times as much at once as the old one; what must be the length of its side? Ans. 28 feet, 7 inches.

QUEST. 41. How many 3 inch cubes may be cut out of a 12 inch cube? Ans. 64.

QUEST. 42. How long must the tether of a horse be, that will allow him to graze, quite around, just an acre of ground? Ans.  $39\frac{1}{4}$  yards.

QUEST. 43. What will the painting of a conical spire come to, at 8d. per yard; supposing the height to be 118 feet, and the circumference of the base 64 feet? Ans. 14l. os.  $8\frac{3}{4}$ d.

QUEST. 44. The diameter of a standard corn bushel is  $18\frac{1}{2}$  inches, and its depth 8 inches; then what must the diameter of that bushel be, whose depth is  $7\frac{1}{2}$  inches?

Ans. 19.1067 inches.

QUEST. 45. Suppose the ball on the top of St. Paul's church is 6 feet in diameter; what did the gilding of it cost at  $3\frac{1}{2}$ d per square inch? Ans. 237l. 10s. 1d.

QUEST. 46. What will a frustum of a marble cone come to, at 12s. per solid foot; the diameter of the greater end being 4 feet, that of the less end  $1\frac{1}{2}$ , and the length of the slant side 8 feet? Ans. 30l. 1s.  $10\frac{1}{4}$ d.

QUEST. 47. To divide a cone into three equal parts by sections parallel to the base, and to find the altitudes of the three parts, the height of the whole cone being 20 inches?

Ans. the upper part 13.867  
the middle part 3.605  
the lower part 2.528

QUEST. 48. A gentleman has a bowling-green, 300 feet long, and 200 feet broad, which he would raise 1 foot higher, by means of the earth to be dug out of a ditch that goes round it: to what depth must the ditch be dug, supposing its breadth to be every where 8 feet? Ans.  $7\frac{3}{8}$  feet.

QUEST. 49. How high above the earth must a person be raised, that he may see  $\frac{1}{3}$  of its surface?

Ans. to the height of the earth's diameter.

QUEST. 50.

QUEST. 50. A cubic foot of brass is to be drawn into wire, of  $\frac{1}{8}$  of an inch in diameter; what will the length of the wire be, allowing no loss in the metal?

Ans. 97784.797 yards, or 55 miles 984.797 yards.

QUEST. 51. Of what diameter must the bore of a cannon be, which is cast for a ball of 24lb weight, so that the diameter of the bore may be  $\frac{1}{8}$  of an inch more than that of the ball?

Ans. 5.647 inches.

QUEST. 52. Supposing the diameter of an iron 9lb ball to be 4 inches, as it is very nearly; it is required to find the diameters of the several balls weighing 1, 2, 3, 4, 6, 12, 18, 24, 32, 36, and 42lb, and the caliber of their guns, allowing  $\frac{1}{8}$  of the caliber, or  $\frac{1}{9}$  of the ball's diameter, for windage.

Answer.

Wt ball.	Diameter ball.	Caliber gun.
1	1.9230	1.9622
2	2.4228	2.4723
3	2.7734	2.8301
4	3.0526	3.1149
6	3.4943	3.5656
9	4.0000	4.0816
12	4.4026	4.4924
18	5.0397	5.1425
24	5.5469	5.6601
32	6.1051	6.2297
36	6.3496	6.4792
42	6.6844	6.8208

QUEST. 53. Supposing the windage of all mortars to be  $\frac{1}{8}$  of the caliber, and the diameter of the hollow part of the shell to be  $\frac{7}{8}$  of the caliber of the mortar: it is required to determine the diameter and weight of the shell, and the quantity or weight of powder requisite to fill it, for each of the several sorts of mortars, namely, the 13, 10, 8, 5.8, and 4.6 inch mortar.

Answer.

Calib. mort.	Diameter shell.	Wt shell empty.	Wt of powder.	Wt shell filled.
4.6	4.523	8.320	0.583	8.903
5.8	5.703	16.677	1.168	17.845
8	7.867	43.764	3.065	46.829
10	9.833	85.476	5.986	91.462
13	12.783	187.791	13.151	200.942

QUEST. 54. If a heavy sphere, whose diameter is 4 inches, be let fall into a conical glass, full of water, whose diameter is 5, and altitude 6 inches; it is required to determine how much water will run over?

Ans. 26.272 cubic inches, or nearly  $\frac{3}{4}$  of a pint.

QUEST. 55. The dimensions of the sphere and cone being the same as in the last question, and the cone only  $\frac{1}{5}$  full of water; required what part of the axis of the sphere is immersed in the water?

Ans. .546 parts of an inch.

QUEST. 56. The cone being still the same, and  $\frac{1}{5}$  full of water; required the diameter of a sphere which shall be just all covered by the water?

Ans. 2.445996 inches.

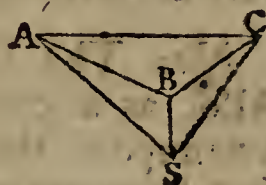
QUEST. 57. If a person, with an air balloon, ascend vertically from London, to such height that he can just see Oxford appear in the horizon; it is required to determine his height above the earth, supposing its circumference to be 25000 miles, and the distance between London and Oxford 49.5933 miles?

Ans.  $\frac{311}{1000}$  of a mile, or 547 yards 1 foot.

QUEST. 58. In a garrison there are three remarkable objects, A, B, C, the distances of which from one to another are known to be, AB 213, AC 424, and BC 262 yards; I am desirous of knowing my position and distance at a place or station S, from whence I observed the angle ASB  $13^{\circ} 30'$ , and the angle CSB  $29^{\circ} 50'$ , both by geometry and trigonometry.

Answer.

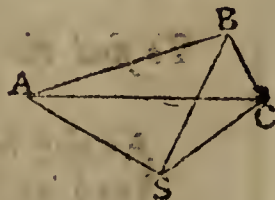
AS	605.7122;
BS	429.6814;
CS	524.2365.



QUEST. 59. Required the same as in the last question, when the point B is on the other side of AC, supposing AB 9, AC 12, and BC 6 furlongs; also the angle ASB  $33^{\circ} 45'$ , and the angle BSC  $22^{\circ} 30'$ .

Answer.

AS 10.64, BS 15.64, CS 14.01.



QUEST. 60. It is required to determine the magnitude of a cube of gold, of the standard fineness, which shall be equal

equal to a sum of 480 million of pounds sterling; supposing a guinea to weigh 5 dwts  $9\frac{1}{2}$  grains.      Ans. 18.691 feet.

QUEST. 61. The ditch of a fortification is 1000 feet long, 9 feet deep, 20 feet broad at bottom, and 22 at top; how much water will fill the ditch?

Ans. 1158127 gallons nearly.

QUEST. 62. If the diameter of the earth be 7930 miles, and that of the moon 2160 miles; required the ratio of their surfaces, and also of their solidities; supposing them both to be globular, as they are very nearly?

Ans. the surfaces are as  $13\frac{1}{2}$  to 1 nearly;  
and the solidities as  $49\frac{1}{2}$  to 1 nearly.

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## PRACTICAL EXERCISES CONCERNING SPECIFIC GRAVITY.

THE Specific Gravities of Bodies, are their relative weights contained under the same given magnitude; as a cubic foot, or a cubic inch, &c.

The specific gravities of several sorts of matter, are expressed by the numbers annexed to their names in the Table of Specific Gravities, at page 221; from whence the numbers are to be taken, when wanted.

*Note.* The several sorts of wood are supposed to be dry. Also, as a cubic foot of water weighs just 1000 ounces averdupois, the numbers in the table express, not only the specific gravities of the several bodies, but also the weight of a cubic foot of each in averdupois ounces; and hence, by proportion, the weight of any other quantity, or the quantity of any other weight, may be known, as in the following problems:

### PROBLEM I.

*To find the Magnitude of any Body, from its Weight.*

As the tabular specific gravity of the body,  
Is to its weight in averdupois ounces,  
So is one cubic foot, or 1728 cubic inches,  
To its content in feet, or inches, respectively.



## EXAMPLES.

EXAM. 1. Required the content of an irregular block of common stone, which weighs 1cwt, or 112lb.

Anf.  $1228\frac{4}{5}$  cubic inches.

EXAM. 2. How many cubic inches of gunpowder are there in 1lb weight?

Anf. 30 cubic inches nearly.

EXAM. 3. How many cubic feet are there in a ton weight of dry oak?

Anf.  $38\frac{1\frac{3}{8}}{5}$  cubic feet.

## PROBLEM II.

*To find the Weight of a Body, from its Magnitude.*

As one cubic foot, or 1728 cubic inches,

Is to the content of the body,

So is its tabular specific gravity,

To the weight of the body.

## EXAMPLES.

EXAM. 1. Required the weight of a block of marble, whose length is 63 feet, and breadth and thickness each 12 feet; being the dimensions of one of the stones in the walls of Balbeck?

Anf.  $683\frac{7}{16}$  ton, which is nearly equal to the burthen of an East-India ship.

EXAM. 2. What is the weight of 1 pint, ale measure, of gunpowder?

Anf. 19 oz. nearly.

EXAM. 3. What is the weight of a block of dry oak, which measures 10 feet in length, 3 feet broad, and  $2\frac{1}{2}$  feet deep?

Anf.  $4335\frac{1}{16}$  lb.

## PROBLEM III.

*To find the Specific Gravity of a Body.*

CASE I. When the body is heavier than water, weigh it both in water and out of water, and take the difference, which will be the weight lost in water. Then say,

As the weight lost in water,

Is to the whole weight,

So is the specific gravity of water,

To the specific gravity of the body.

## EXAMPLE.

A piece of stone weighed 10lb, but in water only  $6\frac{1}{6}$ lb, required its specific gravity?

Anf. 2609.

CASE

CASE 2. When the body is lighter than water, so that it will not quite sink; affix to it a piece of another body, heavier than water, so that the mass compounded of the two may sink together. Weigh the denser body and the compound mass, separately, both in water, and out of it; then find how much each loses in water, by subtracting its weight in water from its weight in air; and subtract the less of these remainders from the greater. Then say,

As the last remainder,  
Is to the weight of the light body in air,  
So is the specific gravity of water,  
To the specific gravity of the body.

## EXAMPLE.

Suppose a piece of elm weighs 15lb in air; and that a piece of copper, which weighs 18lb in air, and 16lb in water, is affixed to it, and that the compound weighs 6lb in water; required the specific gravity of the elm?      Ans. 600.

## PROBLEM IV.

*To find the Quantities of Two Ingredients in a Given Compound.*

TAKE the three differences of every pair of the three specific gravities, namely, the specific gravities of the compound and each ingredient; and multiply the difference of every two specific gravities by the third. Then, as the greatest product, is to the whole weight of the compound, so is each of the other products, to the two weights of the ingredients.

## EXAMPLE.

A composition of 112lb being made of tin and copper, whose specific gravity is found to be 8784; required the quantity of each ingredient, the specific gravity of tin being 7320, and of copper 9000?

Answer, there is 100lb of copper } in the composition.  
and consequently 12lb of tin }

## OF THE WEIGHT AND DIMENSIONS OF BALLS AND SHELLS.

THE weight and dimensions of Balls and Shells might be found from the problems last given, concerning Specific gravity. But they may be found still easier by means of the experimented weight of a ball of a given size, from the known proportion of similar figures, namely, as the cubes of their diameters.

### PROBLEM I.

*To find the Weight of an Iron Ball, from its Diameter.*

An iron ball of 4 inches diameter weighs 9lb, and the weights being as the cubes of the diameters, it will be, as 64 (which is the cube of 4) is to 9, so is the cube of the diameter of any other ball, to its weight. Or, take  $\frac{9}{64}$  of the cube of the diameter, for the weight. Or, take  $\frac{1}{8}$  of the cube of the diameter, and  $\frac{1}{8}$  of that again, and add the two together, for the weight.

### EXAMPLES.

EXAM. 1. The diameter of an iron shot being 6.7 inches, required its weight? Ans. 42.294lb.

EXAM. 2. What is the weight of an iron ball, whose diameter is 5.54 inches? Ans. 24lb.

### PROBLEM II.

*To find the Weight of a Leaden Ball.*

A leaden ball of  $4\frac{1}{4}$  inches diameter weighs 17lb; therefore, as the cube of  $4\frac{1}{4}$  is to 17, or nearly as 9 is to 2, so is the cube of the diameter of a leaden ball, to its weight.

Or, take  $\frac{2}{9}$  of the cube of the diameter, for the weight, nearly.

### EXAMPLES.

EXAM. 1. Required the weight of a leaden ball of 6.6 inches diameter? Ans. 63.888lb.

EXAM. 2. What is the weight of a leaden ball of 5.24 inches diameter? Ans. 32lb nearly.

## PROBLEM III.

*To find the Diameter of an Iron Ball.*

MULTIPLY the weight by  $7\frac{1}{9}$ , and the cube root of the product will be the diameter.

## EXAMPLES.

EXAM. 1. Required the diameter of a 42lb iron ball?  
Ans. 6.685 inches.

EXAM. 2. What is the diameter of a 24lb iron ball?  
Ans. 5.54 inches.

## PROBLEM IV.

*To find the Diameter of a Leaden Ball.*

MULTIPLY the weight by 9, and divide the product by 2; then the cube root of the quotient will be the diameter.

## EXAMPLES.

EXAM. 1. Required the diameter of a 64lb leaden ball?  
Ans. 6.604 inches.

EXAM. 2. What is the diameter of an 8lb leaden ball?  
Ans. 3.302 inches.

## PROBLEM V.

*To find the Weight of an Iron Shell.*

TAKE  $\frac{9}{64}$  of the difference of the cubes of the external and internal diameter, for the weight of the shell.

That is, from the cube of the external diameter, take the cube of the internal diameter, multiply the remainder by 9, and divide the product by 64.

## EXAMPLES.

EXAM. 1. The outside diameter of an iron shell being 12.8, and the inside diameter 9.1 inches; required its weight?  
Ans. 188.941lb.

EXAM. 2. What is the weight of an iron shell, whose external and internal diameters are 9.8 and 7 inches?  
Ans.  $84\frac{1}{8}$ lb.

## PROBLEM VI.

*To find how much Powder will fill a Shell.*

Divide the cube of the internal diameter, in inches, by 57'3, for the lbs of powder.

## EXAMPLES.

EXAM. 1. How much powder will fill the shell whose internal diameter is 9'1 inches?      Anf.  $13\frac{2}{13}$  lb nearly.

EXAM. 2. How much powder will fill the shell whose internal diameter is 7 inches?      Anf. 6lb.

## PROBLEM VII.

*To find how much Powder will fill a Rectangular Box.*

FIND the content of the box in inches, by multiplying the length, breadth, and depth all together. Then divide by 30 for the pounds of powder.

## EXAMPLES.

EXAM. 1. Required the quantity of powder that will fill a box, the length being 15 inches, the breadth 12, and the depth 10 inches?      Anf. 60lb.

EXAM. 2. How much powder will fill a cubical box whose side is 12 inches?      Anf.  $57\frac{3}{5}$  lb.

## PROBLEM VIII.

*To find how much Powder will fill a Cylinder.*

MULTIPLY the square of the diameter by the length, then divide by 38'2 for the pounds of powder.

## EXAMPLES,

EXAM. 1. How much powder will the cylinder hold, whose diameter is 10 inches, and length 20 inches?      Anf.  $52\frac{1}{3}$  lb nearly.

EXAM. 2. How much powder can be contained in the cylinder, whose diameter is 4 inches, and length 12 inches?      Anf.  $5\frac{5}{91}$  lb.

## PROBLEM IX.

*To find the Size of a Shell to contain a given Weight of Powder.*

MULTIPLY the pounds of powder by 57.3, and the cube root of the product will be the diameter in inches.

## EXAMPLES.

EXAM. 1. What is the diameter of a shell that will hold  $13\frac{1}{2}$  lb of powder?      Ans. 9.1 inches.

EXAM. 2. What is the diameter of a shell to contain 6lb of powder?      Ans. 7 inches.

## PROBLEM X.

*To find the Size of a Cubical Box, to contain a given Weight of Powder.*

MULTIPLY the weight in pounds by 30, and the cube root of the product will be the side of the box in inches.

## EXAMPLES.

EXAM. 1. Required the side of a cubical box, to hold 50lb of gunpowder?      Ans. 11.44 inches.

EXAM. 2. Required the side of a cubical box, to hold 400lb of gunpowder?      Ans. 22.89 inches.

## PROBLEM XI.

*To find what Length of a Cylinder will be filled by a given Weight of Gunpowder.*

MULTIPLY the weight in pounds by 38.2, and divide the product by the square of the diameter in inches, for the length.

## EXAMPLES.

EXAM. 1. What length of a 36 pounder gun, of  $6\frac{2}{3}$  inches diameter, will be filled with 12lb of gunpowder?      Ans. 10.314 inches.

EXAM. 2. What length of a cylinder, of 8 inches diameter, may be filled with 20lb of powder?      Ans.  $11\frac{1}{6}$  inches.

OF THE PILING OF BALLS AND SHELLS.

IRON Balls and Shells are commonly piled, by horizontal courses, either in a pyramidical or wedge-like form; the base being either an equilateral triangle, a square, or a rectangle. In the triangle and square, the pile will finish in a single ball; but in the rectangle, it will finish in a single row of balls, like an edge.

In triangular and square piles, the number of horizontal rows, or courses, is always equal to the number of balls in one side of the bottom row. And in rectangular piles, the number of rows is equal to the number of balls in the breadth of the bottom row. Also, the number in the top row, or edge, is one more than the difference between the length and breadth of the bottom row.

PROBLEM I.

*To find the Number of Balls in a Triangular Pile.*

MULTIPLY continually together, the number in one side of the bottom row, that number increased by 1, and the same number increased by 2; and  $\frac{1}{6}$  of the last product will be the answer.

That is,  $\frac{n \cdot n + 1 \cdot n + 2}{6}$  is the number or sum; where  $n$  is the number in the bottom row.

EXAMPLES.

EXAM. 1. Required the number of balls in a triangular pile, each side of the base containing 30 balls? Ans. 4960.

EXAM. 2. How many balls are in the triangular pile, each side of the base containing 20? Ans. 1540.

PROBLEM II.

*To find the Number of Balls in a Square Pile.*

MULTIPLY continually together, the number in one side of the bottom course, that number increased by 1, and double the same number increased by 1; then  $\frac{1}{6}$  of the last product will be the answer.

That is,  $\frac{n \cdot n + 1 \cdot 2n + 1}{6}$  is the number.

EXAMPLES.

EXAM. 1. How many balls are in a square pile of 30 rows? Ans. 9455.

EXAM. 2. How many balls are in a square pile of 20 rows?  
 Ans. 2870.

## PROBLEM III.

*To find the Number of Balls in a Rectangular Pile.*

FROM 3 times the number in the length of the base row, subtract one less than the breadth of the same; multiply the remainder by the said breadth, and the product by one more than the same; and divide by 6 for the answer.

That is,  $\frac{b \cdot b + 1 \cdot 3l - b + 1}{6}$  is the number; where  $l$  is the length, and  $b$  the breadth, of the lowest course.

*Note.* In all the piles, the breadth of the bottom is equal to the number of courses. And, in the oblong or rectangular pile, the top row is one more than the difference between the length and breadth of the bottom.

## EXAMPLES.

EXAM. 1. Required the number of balls in a rectangular pile, the length and breadth of the base row being 46 and 15?  
 Ans. 4960.

EXAM. 2. How many shot are in a rectangular complete pile, the length of the bottom course being 59, and its breadth 20?  
 Ans. 11060.

## PROBLEM IV.

*To find the Number of Balls in an Incomplete Pile.*

FROM the number in the whole pile, considered as complete, subtract the number in the upper pile which is wanting at the top, both computed by the rule for their proper form; and the remainder will be the number in the frustum, or incomplete pile.

## EXAMPLES.

EXAM. 1. To find the number of shot in the incomplete triangular pile, one side of the bottom course being 40, and the top course 20?  
 Ans. 10150.

EXAM. 2. How many shot are in the incomplete triangular pile, the side of the base being 24, and of the top 8?  
 Ans. 2516.

EXAM. 3. How many balls are in the incomplete square pile, the side of the base being 24, and of the top 8?  
 Ans. 4760.

EXAM. 4. How many shot are in the incomplete rectangular pile, of 12 courses, the length and breadth of the base being 40 and 20?  
 Ans. 6146.



## OF DISTANCES BY THE VELOCITY OF SOUND.

By various experiments it has been found, that sound flies, through the air, uniformly at the rate of about 1142 feet in 1 second of time, or a mile in  $4\frac{2}{3}$  seconds. And therefore, by proportion, any distance may be found corresponding to any given time; namely, multiplying the given time, in seconds, by 1142, for the corresponding distance in feet; or taking  $\frac{3}{14}$  of the given time for the distance in miles.

*Note.* The time for the passage of sound in the interval between seeing the flash of a gun, or lightning, and hearing the report, may be observed by a watch, or a small pendulum. Or, it may be observed by the beats of the pulse in the wrist, counting, on an average, about 70 to a minute for persons in moderate health, or  $5\frac{1}{2}$  pulsations to a mile; and more or less according to circumstances.

## EXAMPLES.

EXAM. 1. After observing a flash of lightning, it was 12 seconds before the thunder was heard; required the distance of the cloud from whence it came?      Ans.  $2\frac{4}{7}$  miles.

EXAM. 2. How long, after firing the Tower guns, may the report be heard at Shooter's-Hill, supposing the distance to be 8 miles in a straight line?      Ans.  $37\frac{1}{3}$  seconds.

EXAM. 3. After observing the firing of a large cannon at a distance, it was 7 seconds before the report was heard; what was its distance?      Ans.  $1\frac{1}{2}$  mile.

EXAM. 4. Perceiving a man at a distance hewing down a tree with an axe, I remarked that 6 of my pulsations passed between seeing him strike and hearing the report of the blow; what was the distance between us, allowing 70 pulses to a minute?      Ans. 1 mile and 198 yards.

EXAM. 5. How far off was the cloud from which thunder issued, whose report was 5 pulsations after the flash of lightning; counting 75 to a minute?      Ans. 1523 yards.

EXAM. 6. If I see the flash of a cannon, fired by a ship in distress at sea, and hear the report 33 seconds after, how far is she off?      Ans.  $7\frac{1}{4}$  miles.

PRACTICAL EXERCISES IN MECHANICS, STATICS, HYDROSTATICS, SOUND, MOTION, GRAVITY, PROJECTILES, and other Branches of Natural Philosophy.

QUESTION 1. REQUIRED the weight of a cast iron ball of 3 inches diameter, supposing the weight of a cubic inch of the metal to be 258lb averdupois? Anf. 3.64739lb.

QUEST. 2. To determine the weight of a hollow spherical iron shell, 5 inches in diameter, the thickness of the metal being one inch? Anf. 13.2387 lb.

QUEST. 3. Being one day ordered to observe how far a battery of cannon was from me, I counted, by my watch, 17 seconds between the time of seeing the flash and hearing the report; what then was the distance? Anf.  $3\frac{2}{3}$  miles.

QUEST. 4. It is proposed to determine the proportional quantities of matter in the earth and moon; the density of the former being to that of the latter, as 10 to 7, and their diameters as 7930 to 2160. Anf. as 71 to 1 nearly.

QUEST. 5. What difference is there, in point of weight, between a block of marble containing 1 cubic foot and a half, and another of brass of the same dimensions? Anf. 496lb 14oz.

QUEST. 6. In the walls of Balbeck in Turkey, the ancient Heliopolis, there are three stones laid end to end, now in sight, that measured in length 61 yards; one of which in particular is 21 yards or 63 feet long; 12 feet thick, and 12 feet broad: now if this block be marble, what power would balance it, so as to prepare it for moving?

Anf.  $683\frac{7}{8}$  tons, the burthen of an East-India ship.

QUEST. 7. The battering-ram of Vespasian weighed, suppose 100000 pounds; and was moved, let us admit, with such a velocity, by strength of hand, as to pass through 20 feet in one second of time; and this was found sufficient to demolish the walls of Jerusalem. The question is, with what velocity a 32lb ball must move, to do the same execution? Anf. 62500 feet.

QUEST. 8. There are two bodies, of which the one contains 25 times the matter of the other, or is 25 times heavier; but the less moves with 1000 times the velocity of the

the greater: in what proportion then are the momenta, or forces, with which they are moved?

Ans. the less moves with a force 40 times greater.

QUEST. 9. A body, weighing 20lb, is impelled by such a force, as to send it through 100 feet in a second; with what velocity then would a body of 8lb weight move, if it were impelled by the same force?      Ans. 250 feet per second.

QUEST. 10. There are two bodies, the one of which weighs 100lb, the other 60; but the less body is impelled by a force 8 times greater than the other; the proportion of the velocities, with which these bodies move, is required?

Ans. the velocity of the greater to that of the less, as 3 to 40.

QUEST. 11. There are two bodies, the greater contains 8 times the quantity of matter in the less, and is moved with a force 48 times greater; the ratio of the velocities of these two bodies is required?

Ans. the greater is to the less, as 6 to 1.

QUEST. 12. There are two bodies, one of which moves 40 times swifter than the other; but the swifter body has moved only one minute, whereas the other has been in motion 2 hours: the ratio of the spaces described by these two bodies is required?

Ans. the swifter is to the slower, as 1 to 3.

QUEST. 13. Supposing one body to move 30 times swifter than another, as also the swifter to move 12 minutes, the other only 1: what difference will there be between the spaces described by them, supposing the last has moved 60 inches?      Ans. 1795 feet.

QUEST. 14. There are two bodies, the one of which has passed over 50 miles, the other only 5; and the first had moved with 5 times the celerity of the second: what is the ratio of the times they have been in describing those spaces?

Ans. as 2 to 1.

QUEST. 15. If a lever, 40 effective inches long, will, by a certain power thrown successively on it, in 13 hours, raise a weight 104 feet; in what time will two other levers, each 18 effective inches long, raise an equal weight 73 feet?

Ans. 10 hours  $8\frac{1}{3}$  minutes.

QUEST. 16. What weight will a man be able to raise, who presses with the force of a hundred and a half, on the end of an equipoised handspike, 100 inches long, meeting with  
a con-

a convenient prop exactly  $7\frac{1}{2}$  inches from the lower end of the machine?      Ans. 2072lb.

QUEST. 17. A weight of  $1\frac{1}{2}$ lb, laid on the shoulder of a man, is no greater burthen to him than its absolute weight, or 24 ounces: what difference will he feel, between the said weight applied near his elbow, at 12 inches from the shoulder, and in the palm of his hand, 28 inches from the same; and how much more must his muscles then draw, to support it at right angles, that is, having his arm stretched right out?      Ans. 24lb averdupois.

QUEST. 18. What weight hung on at 70 inches from the centre of motion of a steel-yard, will balance a small gun of  $9\frac{1}{2}$  cwt, freely suspended at 2 inches distance from the said centre on the contrary side?      Ans.  $30\frac{2}{3}$ lb.

QUEST. 19. It is proposed to divide the beam of a steel-yard, or to find the points of division where the weights of 1, 2, 3, 4, &c, lb, on the one side, will just balance a constant weight of 95lb at the distance of 2 inches on the other side of the fulcrum; the weight of the beam being 10lb, and its whole length 36 inches?

Ans. 30, 15, 10,  $7\frac{1}{2}$ , 6, 5,  $4\frac{2}{7}$ ,  $3\frac{3}{4}$ ,  $3\frac{1}{3}$ , 3,  $2^8$ ,  $2\frac{1}{2}$ , &c.

QUEST. 20. Two men carrying a burthen of 200lb weight between them, hung on a pole, the ends of which rest on their shoulders; how much of this load is borne by each man, the weight hanging 6 inches from the middle, and the whole length of the pole being 4 feet?      Ans. 125lb and 75lb.

QUEST. 21. If, in a pair of scales, a body weigh 90lb in one scale, and only 40lb in the other; required its true weight, and the proportion of the lengths of the two arms of the balance beam, on each side of the point of suspension?      Ans. the weight 60lb, and the proportion 3 to 2.

QUEST. 22. To find the weight of a beam of timber, or other body, by means of a man's own weight, or any other weight. For, instance, a piece of tapering timber, 24 feet long, being laid over a prop, or the edge of another beam, is found to balance itself when the prop is 13 feet from the less end; but removing the prop a foot nearer to the said end, it takes a man's weight of 210lb, standing on the less end, to hold it in equilibrium. Required the weight of the tree?      Ans. 2520lb.

QUEST. 23. If AB be a cane or walking-stick, 40 inches long, suspended by a string SD fastened to the middle point D: now

D: now a body being hung on at E, 6 inches distance from D, is balanced by a weight of 2lb, hung on at the larger end A; but removing the body to F, one inch nearer to D, the 2lb weight on the other side is moved to G, within 8 inches of D, before the cane will rest in equilibrio. Required the weight of the body?      Ans. 24lb.

QUEST. 24. If AB, BC be two inclined planes, of the lengths of 30 and 40 inches, and moveable about the joint at B: what will be the ratio of two weights P, Q, in equilibrio on the planes, in all positions of them: and what will be the altitude BD of the angle B above the horizontal plane AC, when this is 50 inches long?'

Ans.  $BD = 24$ ; and P to Q as AB to BC, or as 3 to 4.

QUEST. 25. A lever, of 6 feet long, is fixed at right angles in a screw, whose threads are one inch asunder, so that the lever turns just once round in raising or depressing the screw one inch. If then this lever be urged by a weight or force of 50lb, with what force will the screw press?

Ans.  $22619\frac{1}{2}$ lb.

QUEST. 26. If a man can draw a weight of 150lb up the side of a perpendicular wall, of 20 feet high; what weight will he be able to raise along a smooth plank of 30 feet long, laid aslope from the top of the wall?      Ans. 225lb.

QUEST. 27. If a force of 150lb be applied on the head of a rectangular wedge, its thickness being 2 inches, and the length of its side 12 inches; what weight will it raise or balance perpendicular to its side?      Ans. 900lb.

QUEST. 28. If a round pillar, of 30 feet diameter, be raised on a plane, inclined to the horizon in an angle of  $75^\circ$ , or the shaft inclining 15 degrees out of the perpendicular; what length will it bear before it overset?

Ans.  $30(2 + \sqrt{3})$  or 111.9615 feet.

QUEST. 29. If the greatest angle at which a bank of natural earth will stand, be  $45^\circ$ ; it is proposed to determine what thickness an upright wall of stone must be made throughout, just to support a bank of 12 feet high; the specific gravity of the stone being to that of earth, as 5 to 4.

Ans.  $12\sqrt{\frac{8}{15}}$ , or 8.76356 feet.

QUEST. 30. If the stone wall be made like a wedge or having its upright section a triangle, tapering to a point at top, but its side next the bank of earth perpendicular to the horizon; what is its thickness at the bottom, so as to support the same bank?      Ans.  $12\sqrt{\frac{4}{3}}$ , or 10.733126 feet.

QUEST. 31.

QUEST. 31. But if the earth will only stand at an angle of 30 degrees to the horizontal line; it is required to determine the thickness of wall in both the preceding cases?

Ans. the breadths are the same as before, because the area of the triangular bank of earth is increased in the same proportion as its horizontal push is decreased.

QUEST. 32. To find the thickness of an upright rectangular wall, necessary to support a body of water; the water being 10 feet deep, and the wall 12 feet high; also the specific gravity of the wall to that of the water, as 11 to 7.

Ans. 4.204374 feet.

QUEST. 33. To determine the thickness of the wall at the bottom, when the section of it is triangular, and the altitudes as before.

Ans. 5.1492866 feet.

QUEST. 34. Supposing the distance of the earth from the sun to be 95 millions of miles; I would know at what distance from him another body must be placed, so as to receive light and heat quadruple to that of the earth.

Ans. at half the distance, or  $47\frac{1}{2}$  millions.

QUEST. 35. If the mean distance of the sun from us be 106 of his diameters; how much hotter is it at the surface of the sun, than under our equator?

Ans. 11236 times hotter.

QUEST. 36. The distance between the earth and sun being accounted 95 millions of miles, and between Jupiter and the sun 495 millions; the degree of light and heat received by Jupiter, compared with that of the earth, is required?

Ans.  $\frac{361}{9801}$ , or nearly  $\frac{1}{27}$  of the earth's light and heat.

QUEST. 37. A certain body on the surface of the earth weighs a cwt, or 112lb; the question is, whither this body must be carried, that it may weigh only 10lb?

Ans. either at 3.3466 semi-diameters, or  $\frac{5}{6}$  of a semi-diameter, from the centre.

QUEST. 38. If a body weigh 1 pound, or 16 ounces, on the surface of the earth; what will its weight be at 50 miles above it, taking the earth's diameter at 7930 miles?

Ans. 15 oz.  $9\frac{5}{8}$  dr. nearly.

QUEST. 39. Whereabouts, in the line between the earth and moon, is their common centre of gravity; supposing the earth's diameter to be 7930 miles, and the moon's 2160; also the density of the former to that of the latter, as 99 to

68, or

68, or as 10 to 7 nearly, and their mean distance 30 of the earth's diameters?

Ans. at  $\frac{1}{2} \frac{0.5}{51}$  parts of a diameter from the earth's centre, or  $\frac{1}{5} \frac{1}{82}$  parts of a diameter, or 648 miles below the surface.

QUEST. 40. Whereabouts, between the earth and moon, are their attractions equal to each other? Or, where must another body be placed, so as to remain suspended in equilibrio, not being more attracted to the one than to the other, or having no tendency to fall either way? Their dimensions being as in the last question.

Ans. From the earth's centre  $26 \frac{9}{11}$  } of the earth's di-  
From the moon's centre  $3 \frac{2}{11}$  } ameters.

QUEST. 41. Suppose a stone, dropt into an abyss, should be stopped at the end of the 11th second after its delivery; what space would it have gone through? Ans.  $1946 \frac{1}{2}$  feet.

QUEST. 42. What is the difference between the depths of two wells, into each of which should a stone be dropped at the same instant, the one will strike the bottom at 6 seconds, the other at 10? Ans.  $1029 \frac{1}{3}$  feet.

QUEST. 43. If a stone be  $19 \frac{1}{2}$  seconds in descending from the top of a precipice to the bottom, what is its height?

Ans.  $6115 \frac{1}{6}$  feet.

QUEST. 44. In what time will a musket ball, dropped from the top of Salisbury steeple, said to be 400 feet high, reach the bottom? Ans. 5 seconds nearly.

QUEST. 45. If a heavy body be observed to fall through 100 feet in the last second of time, from what height did it fall, and how long was it in motion?

Ans. time  $3 \frac{2}{3} \frac{3}{8} \frac{5}{6}$  sec. and height  $209 \frac{4}{9} \frac{2}{2} \frac{7}{6} \frac{3}{4}$  feet.

QUEST. 46. A stone being let fall into a well, it was observed that, after being dropped, it was 10 seconds before the sound of the fall at the bottom reached the ear. What is the depth of the well? Ans. 1270 feet nearly.

QUEST. 47. It is proposed to determine the length of a pendulum vibrating seconds, in the latitude of London, where a heavy body falls through  $16 \frac{1}{2}$  feet in the first second of time? Ans. 39.11 inches.

By experiment this length is found to be  $39 \frac{1}{8}$  inches.

QUEST. 48. What is the length of a pendulum vibrating in 2 seconds; also in half a second, and in a quarter second?

Ans. the 2 second pendulum  $156 \frac{1}{2}$   
the  $\frac{1}{2}$  second pendulum  $9 \frac{2}{3} \frac{5}{2}$   
the  $\frac{1}{4}$  second pendulum  $2 \frac{5}{2} \frac{7}{8}$  inches.

QUEST. 49.

QUEST. 49. What difference will there be in the number of vibrations, made by a pendulum of 6 inches long, and another of 12 inches long, in an hour's time? Anf.  $2692\frac{1}{2}$ .

QUEST. 50. Observed that while a stone was descending, to measure the depth of a well, a string and plummet, that from the point of suspension, or the place where it was held, to the centre of oscillation, measured just 18 inches, had made 8 vibrations, when the sound from the bottom returned. What was the depth of the well? Anf. 412.61 feet.

QUEST. 51. If a ball vibrate in the arch of a circle, 10 degrees on each side of the perpendicular; or a ball roll down the lowest 10 degrees of the arch; required the velocity at the lowest point? the radius of the circle, or length of the pendulum, being 20 feet. Anf. 4.4213 feet per second.

QUEST. 52. If a ball descend down a smooth inclined plane, whose length is 100 feet, and altitude 10 feet; how long will it be in descending, and what will be the last velocity?

Anf. the veloc. 25.364 feet per sec. and time 7.8852 sec.

QUEST. 53. If a cannon ball, of 11b weight, be fired against a pendulous block of wood, and, striking the centre of oscillation, cause it to vibrate an arc whose chord is 30 inches; the radius of that arc, or distance from the axis to the lowest point of the pendulum, being 118 inches, and the pendulum vibrating in small arcs 40 oscillations per minute. Required the velocity of the ball, and the velocity of the centre of oscillation of the pendulum, at the lowest point of the arc; the whole weight of the pendulum being 500lb?

Anf. veloc. ball 1956.6054 feet per sec.  
and veloc. cent. oscil. 3.9054 feet per sec.

QUEST. 54. How deep will a cube of oak sink in common water; each side of the cube being 1 foot?

Anf.  $11\frac{1}{10}$  inches.

QUEST. 55. How deep will a globe of oak sink in water; the diameter being 1 foot?

Anf. 9.9867 inches.

QUEST. 56. If a cube of wood, floating in common water, have 3 inches of its height dry above the water, and  $4\frac{8}{103}$  inches dry when in sea-water; it is proposed to determine the magnitude of the cube, and what sort of wood it is made of?

Anf. the wood is oak, and each side 40 inches.

QUEST. 57. An irregular piece of lead ore weighs, in air 12 ounces, but in water only 7; and another fragment weighs



weighs in air  $14\frac{1}{2}$  ounces, but in water only 9; required their comparative densities, or specific gravities?

Anf. as 145 to 132.

QUEST. 58. An irregular fragment of glass, in the scale, weighs 171 grains, and another of magnet 102 grains; but in water the first fetches up no more than 120 grains, and the other 79: what then will their specific gravities turn out to be?

Anf. glass to magnet as 3933 to 5202  
or nearly as 10 to 13.

QUEST. 59. Hiero, king of Sicily, ordered his jeweller to make him a crown, containing 63 ounces of gold. The workmen thought that, substituting part silver was only a proper perquisite; which taking air, Archimedes was appointed to examine it; who, on putting it into a vessel of water, found it raised the fluid 8.2245 cubic inches: and having discovered that the inch of gold more critically weighed 10.36 ounces, and that of silver but 5.85 ounces, he found by calculation what part of the king's gold had been changed. And you are desired to repeat the process.

Anf. 28.8 ounces.

QUEST. 60. Supposing the cubic inch of common glass weigh 1.4921 ounces troy, the same of sea-water .59542, and of brandy .5368; then a seaman having a gallon of this liquor in a glass bottle, which weighs 3.84lb out of water, and, to conceal it from the officers of the customs, throws it overboard. It is proposed to determine, if it will sink, how much force will just buoy it up?

Anf. 14.1496 ounces.

QUEST. 61. Another person has half an anker of brandy, of the same specific gravity as in the last question; the wood of the cask suppose measures  $\frac{1}{8}$  of a cubic foot; it is proposed to assign what quantity of lead is just requisite to keep the cask and liquor under water?

Anf. 89.743 ounces.

QUEST. 62. Suppose, by measurement, it be found that a man of war, with its ordnance, rigging, and appointments, sinks so deep as to displace 50000 cubic feet of fresh water; what is the whole weight of the vessel?

Anf.  $1395\frac{1}{10}$  tons.

QUEST. 63. It is required to determine what would be the height of the atmosphere, if it were every where of the same density as at the surface of the earth, when the quicksilver in the barometer stands at 30 inches; and also, what would be the height of a water barometer at the same time?

Anf. height of the air  $29166\frac{2}{3}$  feet or 5.5240 miles,  
height of water 35 feet.

QUEST. 64.

QUEST. 64. With what velocity would each of those three fluids, viz. quicksilver, water, and air, issue through a small orifice in the bottom of vessels, of the respective heights of 30 inches, 35 feet, and 5.5240 miles; estimating the pressure by the half altitudes, and the air rushing into a vacuum?

Anf. the veloc. of quicksilver 8.967 feet.  
 the veloc. of water - 33.55  
 the veloc. of air - 968.6

But, estimating by the whole altitude, the veloc.

of air is - - - - - 1369.8

And the mean between these two is - 1169.2

which is nearly the velocity of sound, and also nearly equal to the velocity of a ball through the air when it suffers a resistance equal to the pressure of the atmosphere.

QUEST. 65. A very large vessel of 10 feet high (no matter what shape) being kept constantly full of water, by a large supplying cock at the top; if 9 small circular holes, each  $\frac{1}{5}$  of an inch diameter, be opened in its perpendicular side at every foot of the depth: it is required to determine the several distances to which they will spout on the horizontal plane of the base, and the quantity of water discharged by all of them in 10 minutes?

Anf. the distances are

$\sqrt{36}$  or 6.00000

$\sqrt{64}$  - 8.00000

$\sqrt{84}$  - 9.16515

$\sqrt{96}$  - 9.79796

$\sqrt{100}$  - 10.00000

$\sqrt{96}$  - 9.79796

$\sqrt{84}$  - 9.16515

$\sqrt{64}$  - 8.00000

$\sqrt{36}$  - 6.00000

and the quantity discharged in 10 min. 123.8849 gallons.

*Note.* In this solution, the velocity of the water is supposed to be equal to that which is acquired by a heavy body in falling through the whole height of the water above the orifice, and that it is the same in every part of the holes.

QUEST. 66. If the inner axis of a hollow globe of copper, exhausted of air, be 100 feet; what thickness must it be of, that it may just float in air?

Anf. .02688 of an inch thick.

QUEST. 67.

QUEST. 67. If a spherical balloon of copper, of  $\frac{1}{80}$  of an inch thick, have its cavity of 100 feet diameter, and be filled with inflammable air, of  $\frac{1}{8}$  of the gravity of common air, what weight will just balance it, and prevent it from rising up into the atmosphere?

Anf. 27981 lb.

QUEST. 68. If a glass tube, 36 inches long, close at top, be sunk perpendicularly into water, till its lower or open end be 30 inches below the surface of the water; how high will the water rise within the tube, the quicksilver in the common barometer at the same time standing at  $29\frac{1}{2}$  inches?

Anf. 2.26545 inches.

QUEST. 69. If a diving bell, of the form of a parabolic conoid, be let down into the sea to the several depths of 5, 10, 15, and 20 fathoms; it is required to assign the respective heights to which the water will rise within it: its axis and the diameter of its base being each 8 feet, and the quicksilver in the barometer standing at 30.9 inches?

Anf. at	5	fathoms	deep	the	water	rises	2.03546	feet,
	at	10	-	-	-	-	3.06393	
	at	15	-	-	-	-	3.70267	
	at	20	-	-	-	-	4.14658.	

## THE DOCTRINE OF FLUXIONS.

### DEFINITIONS AND PRINCIPLES.

*Art. I.* IN the Doctrine of Fluxions, magnitudes or quantities of all kinds, are considered, not as made up of a number of small parts, but as generated by continued motion, by means of which they increase or decrease. As, a line by the motion of a point; a surface by the motion of a line; and a solid by the motion of a surface. So likewise, time may be considered as represented by a line, increasing uniformly by the motion of a point. And quantities of all kinds whatever, which are capable of increase and decrease, may, in like manner, be represented by lines, surfaces, or solids, conceived to be generated by motion.

2. Any quantity, thus generated, and variable, is called a *Fluent*, or a *Flowing Quantity*. And the rate or proportion according to which any flowing quantity increases, at any position or instant, is the *Fluxion* of the said quantity, at that position or instant: and it is proportional to the magnitude by which the flowing quantity would be uniformly increased, in a given time with the generating celerity uniformly continued during that time.

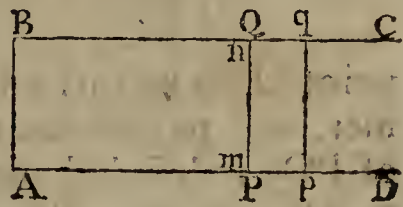
3. The small quantities that are actually generated, produced, or described, in any small given time, and by any continued motion, either uniform or variable, are called *Increments*.

4. Hence, if the motion of increase be uniform, by which increments are generated, the increments will in that case be proportional, or equal, to the measures of the fluxions: but if the motion of increase be accelerated, the increment so generated, in a given finite time, will exceed the fluxion: and if it be a decreasing motion, the increment, so generated, will be less than the fluxion. But if the time be indefinitely small, so that the motion be considered as uniform for that instant; then these nascent increments will always be proportional, or equal, to the fluxions, and may be substituted instead of them, in any calculation.

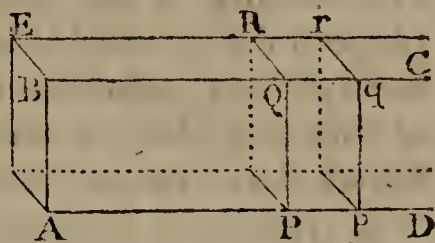
5. To

5. To illustrate these definitions: Suppose a point  $m$  be conceived to move from the position  $A$ , and to generate a line  $AP$ , by a motion any how regulated; and suppose the celerity of the point  $m$ , at any position  $P$ , to be such, as would, if from thence it should become or continue uniform, be sufficient to cause the point to describe, or pass uniformly over, the distance  $Pp$ , in the given time allowed for the fluxion: then will the said line  $Pp$  represent the fluxion of the fluent, or flowing line,  $AP$ , at that position.

6. Again, suppose the right line  $mn$  to move, from the position  $AB$ , continually parallel to itself; with any continued motion, so as to generate the fluent or flowing rectangle  $ABQP$ , whilst the point  $m$  describes the line  $AP$ : also, let the distance  $Pp$  be taken, as before, to express the fluxion of the line or base  $AP$ ; and complete the rectangle  $PQqp$ . Then, like as  $Pp$  is the fluxion of the line  $AP$ , so is  $Pq$  the fluxion of the flowing parallelogram  $AQ$ ; both these fluxions, or increments, being uniformly described in the same time.

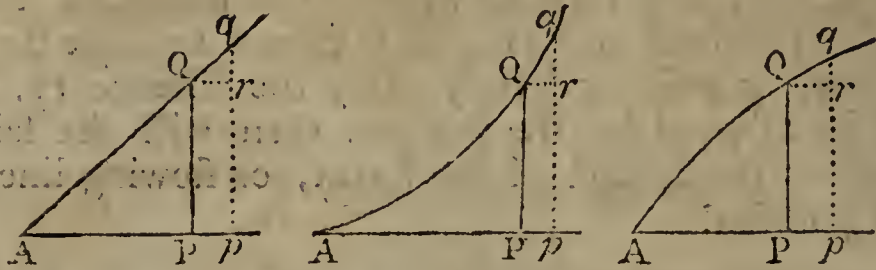


7. In like manner, if the solid  $AERP$  be conceived to be generated by the plane  $PQR$  moving, from the position  $ABE$ , always parallel to itself, along the line  $AD$ ; and if  $Pp$  denote the fluxion of the line  $AP$ : Then, like as the rectangle  $PQqp$ , or  $PQ \times Pp$ , expresses the fluxion of the flowing rectangle  $ABQP$ , so also shall the fluxion of the variable solid, or prism  $ABERQP$ , be expressed by the prism  $PQRrqp$ , or the plane  $PR \times Pp$ . And, in both these last two cases, it appears, that the fluxion of the generated rectangle, or prism, is equal to the product of the generating line, or plane, drawn into the fluxion of the line along which it moves.



8. Hitherto the generating line, or plane, has been considered as of a constant and invariable magnitude; in which case the fluent, or quantity generated, is a rectangle, or a prism, the former being described by the motion of a line, and the latter by the motion of a plane. So, in like manner are other figures, whether plane or solid, conceived to be

described, by the motion of a Variable Magnitude, whether it be a line or a plane. Thus let a variable line  $PQ$  be carried by a parallel motion along  $AP$ , or whilst a point  $P$  is carried along, and describes the line  $AP$ , suppose another



point  $Q$  to be carried by a motion perpendicular to the former, and to describe the line  $PQ$ : let  $pq$  be another position of  $PQ$ , indefinitely near to the former; and draw  $Qr$  parallel to  $AP$ . Now, in this case, there are several fluents, or flowing quantities, with their respective fluxions: namely, the line or fluent  $AP$ , the fluxion of which is  $Pp$  or  $Qr$ ; the line or fluent  $PQ$ , the fluxion of which is  $rq$ ; the curve or oblique line  $AQ$ , described by the oblique motion of the point  $Q$ , the fluxion of which is  $Qq$ ; and lastly, the surface  $APQ$ , described by the variable line  $PQ$ , the fluxion of which is the rectangle  $PQr$ , or  $PQ \times Pp$ . In the same manner may any solid be conceived to be described, by the motion of a variable plane parallel to itself, substituting the variable plane for the variable line; in which case, the fluxion of the solid, at any position, is represented by the variable plane, at that position, drawn into the fluxion of the line along which it is carried.

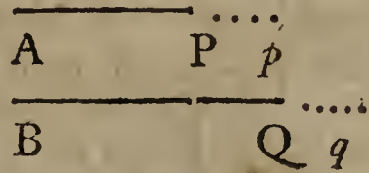
9. Hence then it follows in general, that the fluxion of any figure, whether plane or solid, at any position, is equal to the section of it, at that position, drawn into the fluxion of the axis, or line along which the variable section is supposed to be perpendicularly carried; that is, the fluxion of the figure  $AQP$ , is equal to the plane  $PQ \times Pp$ , when that figure is a solid, or to the ordinate  $PQ \times Pp$ , when the figure is a surface.

10. It also follows, from the same premises, that, in any curve, or oblique line,  $AQ$ , whose absciss is  $AP$ , and ordinate is  $PQ$ , the fluxions of these three form a small right-angled plane triangle  $Qqr$ ; for  $Qr = Pp$  is the fluxion of the absciss  $AP$ ,  $qr$  the fluxion of the ordinate  $PQ$ , and  $Qq$  the fluxion of the curve or right line  $AQ$ . And consequently that, in any curve, the square of the fluxion of the curve, is equal

equal to the sum of the squares of the fluxions of the absciss and ordinate, when these two are at right angles to each other.

11. From the premises it also appears, that contemporaneous fluents, or quantities that flow or increase together; which are always in a constant ratio to each other, have their fluxions also in the same constant ratio, at every position.

For, let AP and BQ be two contemporaneous fluents, described in the same time by the motion of the points P and Q, the contemporaneous positions being P, Q, and  $p, q$ ; and let AP be to BQ, or  $Ap$  to  $Bq$ , constantly in the ratio of 1 to  $n$ . Then is  $n \times AP = BQ$ ,



and  $n \times Ap = Bq$ ;

therefore, by subtraction,  $n \times Pp = Qq$ ;

that is, the fluxion  $Pp$  : fluxion  $Qq$  :: 1 :  $n$ ,

the same as the fluent AP : fluent BQ :: 1 :  $n$ ;

or, the fluxions and fluents are in the same constant ratio.

But, if the ratio of the fluents be variable, so will that of the fluxions be also, though not in the same variable ratio with the former, at every position.

NOTATION, &c.

12. To apply the foregoing principles to the determination of the fluxions of algebraical quantities, by means of which those of all other kinds are determined, it will be necessary first to premise the notation commonly used in this science, with some observations. As, first, that the final letters of the alphabet  $z, y, x, w, \&c$ , are used to denote variable or flowing quantities; and the initial letters,  $a, b, c, d, \&c$ , to denote constant or invariable ones: Thus, the variable base AP of the flowing rectangular figure ABQP, in art. 6, may be represented by  $x$ ; and the invariable altitude PQ, by  $a$ : also, the variable base or absciss AP, of the figures in art. 8, may be represented by  $x$ ; the variable ordinate PQ, by  $y$ ; and the variable curve or line AQ, by  $z$ .

Secondly, that the fluxion of a quantity denoted by a single letter, is represented by the same letter with a point over it: Thus, the fluxion of  $x$  is expressed by  $\dot{x}$ , the fluxion of  $y$  by  $\dot{y}$ , and the fluxion of  $z$  by  $\dot{z}$ . As to the fluxions of constant or invariable quantities, as of  $a, b, c, \&c$ , they are equal to nothing, because they do not flow or change their magnitude.

T 2

Thirdly,

Thirdly, that the increments of variable or flowing quantities, are also denoted by the same letters with a small  $\dot{\phantom{x}}$  over them: Thus, the increments of  $x, y, z$ , are  $\dot{x}, \dot{y}, \dot{z}$ .

13. From these notations, and the foregoing principles, the quantities, and their fluxions, there considered, will be denoted as below. Thus, in all the foregoing figures put

the variable or flowing line - - AP =  $x$ ,

in art. 6, the constant line - - PQ =  $a$ ,

in art. 8, the variable ordinate - PQ =  $y$ ,

also, the variable line or curve - AQ =  $z$ :

Then shall the several fluxions be thus represented, namely,

$\dot{x}$  = Pp the fluxion of the line AP,

$a\dot{x}$  = PQqp the fluxion of ABQP in art. 6,

$y\dot{x}$  = PQRp the fluxion of APQ in art. 8,

$\dot{z}$  = Qq =  $\sqrt{\dot{x}^2 + \dot{y}^2}$  the fluxion of AQ; and

$a\dot{x}$  = Pr the fluxion of the solid in art. 7, if  $a$  denote the constant generating plane PQR. Also,

$nx$  = BQ in the figure to art. 11, and

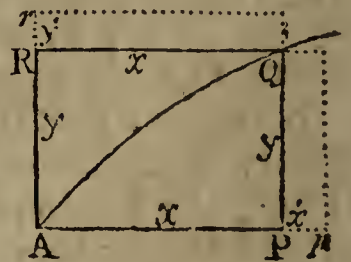
$n\dot{x}$  = Qq the fluxion of the same.

14. The principles and notation being now laid down, we may proceed to the practice and rules of this doctrine, which consists of two principal parts, called the Direct and Inverse Method of Fluxions; namely, the direct method, which consists in finding the fluxion of any proposed fluent or flowing quantity; and the inverse method, which consists in finding the fluent of any proposed fluxion. As to the former of these two problems, it can always be determined, and that in finite algebraical terms; but the latter, or finding of fluents, can only be effected in some certain cases, except by means of infinite series.—First then, of

### THE DIRECT METHOD OF FLUXIONS.

*To find the Fluxion of the Product or Rectangle of two Variable Quantities.*

15. LET ARQP, =  $xy$ , be the flowing or variable rectangle, generated by two lines PQ and RQ, moving always perpendicular to each other, from the positions AR and AP; denoting the one by  $x$ , and the other by  $y$ ; supposing  $x$  and  $y$  to be so related, that



the curve line AQ may always pass through the intersection Q of those lines, or the opposite angle of the rectangle.

Now,



Now, the rectangle consists of the two trilinear spaces  $APQ$ ,  $ARQ$ , of which the

fluxion of the former is  $PQ \times Pp$ , or  $y\dot{x}$ , and that of the latter is  $RQ \times Rr$ , or  $x\dot{y}$ , by art. 8; therefore the sum of the two,  $\dot{x}y + x\dot{y}$ , is the fluxion of the whole rectangle  $xy$  or  $ARQP$ .

*The Same Otherwise.*

16. Let the sides of the rectangle,  $x$  and  $y$ , by flowing, become  $x + \dot{x}$  and  $y + \dot{y}$ : then the product of these two, or  $xy + x\dot{y} + y\dot{x} + \dot{x}\dot{y}$  will be the new or contemporaneous value of the flowing rectangle  $PR$  or  $xy$ ; subtract the one value from the other, and the remainder,  $x\dot{y} + y\dot{x} + \dot{x}\dot{y}$ , will be the increment generated in the same time as  $\dot{x}$  or  $\dot{y}$ ; of which the last term  $\dot{x}\dot{y}$  is nothing, or indefinitely small, in respect of the other two terms, because  $\dot{x}$  and  $\dot{y}$  are indefinitely small in respect of  $x$  and  $y$ ; which term being therefore omitted, there remains  $x\dot{y} + y\dot{x}$  for the value of the increment: and hence, by substituting  $\dot{x}$  and  $\dot{y}$  for  $\dot{x}$  and  $\dot{y}$ , to which they are proportional, there arises  $x\dot{y} + y\dot{x}$  for the true value of the fluxion of  $xy$ ; the same as before.

17. Hence may be easily derived the fluxion of the powers and products of any number of flowing or variable quantities whatever; as of  $xyz$ , or  $wxyz$ , or  $vwxxyz$ , &c. And, first, for the fluxion of  $xyz$ : put  $p = xy$ , and the whole given fluent  $xyz = q$ , or  $q = xyz = pz$ . Then, taking the fluxions of  $q = pz$ , by the last article, they are  $\dot{q} = p\dot{z} + \dot{p}z$ ; but  $p = xy$ , and so  $\dot{p} = \dot{x}y + x\dot{y}$  by the same article; substituting therefore these values of  $p$  and  $\dot{p}$  instead of them, in the value of  $\dot{q}$ , this becomes  $\dot{q} = \dot{x}yz + x\dot{y}z + xy\dot{z}$ , the fluxion of  $xyz$  required; which is therefore equal to the sum of the products, arising from the fluxion of each letter, or quantity, multiplied by the product of the other two.

Again, to determine the fluxion of  $wxyz$ , the continual product of four variable quantities; put this product, namely  $wxyz$ , or  $qw$ , =  $r$ , where  $q = xyz$  as above. Then, taking the fluxions by the last article,  $\dot{r} = \dot{q}w + q\dot{w}$ ; which, by substituting for  $q$  and  $\dot{q}$  their values as above, becomes  $\dot{r} = wxyz + w\dot{x}yz + wx\dot{y}z + wxy\dot{z}$ , the fluxion of  $wxyz$  as required: consisting of the fluxion of each quantity, drawn into the products of the other three.

In the very same manner it is found, that the fluxion of  $vwxyz$  is  $\dot{v}wxyz + v\dot{w}xyz + vw\dot{x}yz + vwxy\dot{z} + vwxyz\dot{z}$ ; and so on, for any number of quantities whatever; in which it is always found, that there are as many terms as there are variable quantities in the proposed fluent; and that these terms consist of the fluxion of each variable quantity, multiplied by the product of all the rest of the quantities.

18. From hence is easily derived the fluxion of any power of a variable quantity, as of  $x^2$ , or  $x^3$ , or  $x^4$ , &c. For, in the product or rectangle  $xy$ , if  $x = y$ , then is  $xy = xx$  or  $x^2$ , and also its fluxion  $\dot{x}y + x\dot{y} = \dot{x}x + x\dot{x}$  or  $2x\dot{x}$ , the fluxion of  $x^2$ .

Again, if all the three  $x, y, z$  be equal; then is the product of the three  $xyz = x^3$ ; and consequently its fluxion  $\dot{x}yz + x\dot{y}z + xy\dot{z} = \dot{x}xx + x\dot{x}x + xx\dot{x}$  or  $3x^2\dot{x}$ , the fluxion of  $x^3$ .

In the same manner, it will appear that

the fluxion of  $x^4$  is  $= 4x^3\dot{x}$ ,

the fluxion of  $x^5$  is  $= 5x^4\dot{x}$ , and, in general,

the fluxion of  $x^n$  is  $= nx^{n-1}\dot{x}$ ;

where  $n$  is any positive whole number whatever.

That is, the fluxion of any positive integral power, is equal to the fluxion of the root ( $\dot{x}$ ), multiplied by the exponent of the power ( $n$ ), and by the power of the same root whose index is less by 1, ( $x^{n-1}$ ).

And thus, the fluxion of  $a + cx$  being  $c\dot{x}$ ,

that of  $(a + cx)^2$  is  $2c\dot{x} \times (a + cx)$  or  $2ac\dot{x} + 2c^2x\dot{x}$ ,

that of  $(a + cx^2)^2$  is  $4cx\dot{x} \times (a + cx^2)$  or  $4acx\dot{x} + 4c^2x^3\dot{x}$ ,

that of  $(x^2 + y^2)^2$  is  $(4x\dot{x} + 4y\dot{y}) \times (x^2 + y^2)$ ,

that of  $(x + cy^2)^3$  is  $(3\dot{x} + 6cy\dot{y}) \times (x + cy^2)^2$ .

19. From the conclusions in the same article, we may also derive the fluxion of any fraction, or the quotient of one variable quantity divided by another, as of

$\frac{x}{y}$ . For, put the quotient or fraction  $\frac{x}{y} = q$ ; then, multiplying by the denominator,  $x = qy$ ; and, taking the fluxions,

$$\dot{x} = \dot{q}y + q\dot{y}, \text{ or } \dot{q}y = \dot{x} - q\dot{y}; \text{ and, by division,}$$

$$\dot{q} = \frac{\dot{x}}{y} - \frac{q\dot{y}}{y} = (\text{by substituting the value of } q, \text{ or } \frac{x}{y}),$$

$$\frac{\dot{x}}{y} - \frac{x\dot{y}}{y^2} = \frac{\dot{x}y - x\dot{y}}{y^2}, \text{ the fluxion of } \frac{x}{y}, \text{ as required.}$$

That

That is, the fluxion of any fraction, is equal to the fluxion of the numerator drawn into the denominator, minus the fluxion of the denominator drawn into the numerator, and the remainder divided by the square of the denominator. So that the fluxion of  $\frac{ax}{y}$  is  $a \times \frac{\dot{x}y - x\dot{y}}{y^2}$  or  $\frac{a\dot{x}y - ax\dot{y}}{y^2}$ .

20. Hence, too, is easily derived the fluxion of any negative integer power of a variable quantity, as of  $x^{-n}$ , or  $\frac{1}{x^n}$ , which is the same thing. For here the numerator of the fraction is 1, whose fluxion is nothing; and therefore, by the last article, the fluxion of such a fraction, or negative power, is barely equal to minus the fluxion of the denominator, divided by the square of the said denominator. That is, the fluxion of  $x^{-n}$ , or  $\frac{1}{x^n}$ , is  $-\frac{nx^{n-1}\dot{x}}{x^{2n}}$  or  $-\frac{n\dot{x}}{x^{n+1}}$  or  $-nx^{n-1}\dot{x}$ ; or the fluxion of any negative integer power of a variable quantity, as  $x^{-n}$ , is equal to the fluxion of the root, multiplied by the exponent of the power, and by the next power less by 1.

The same thing is otherwise obtained thus: Put the proposed fraction, or quotient  $\frac{1}{x^n} = q$ ; then is  $qx^n = 1$ ; and, taking the fluxions, we have  $\dot{q}x^n + qnx^{n-1}\dot{x} = 0$ ; hence  $\dot{q}x^n = -qnx^{n-1}\dot{x}$ ; divide by  $x^n$ , then  $\dot{q} = -\frac{qn\dot{x}}{x} =$  (by substituting  $\frac{1}{x^n}$  for  $q$ ),  $-\frac{n\dot{x}}{x^{n+1}}$  or  $-nx^{n-1}\dot{x}$ ; the same as before.

Hence the fluxion of  $x^{-1}$  or  $\frac{1}{x}$  is  $-x^{-2}\dot{x}$ , or  $-\frac{\dot{x}}{x^2}$ ,  
 that of  $x^{-2}$  or  $\frac{1}{x^2}$  is  $-2x^{-3}\dot{x}$  or  $-\frac{2\dot{x}}{x^3}$ ,  
 that of  $x^{-3}$  or  $\frac{1}{x^3}$  is  $-3x^{-4}\dot{x}$  or  $-\frac{3\dot{x}}{x^4}$ ,  
 that of  $ax^{-4}$  or  $\frac{a}{x^4}$  is  $-4ax^{-5}\dot{x}$  or  $-\frac{4a\dot{x}}{x^5}$ ,  
 that of  $(a+x)^{-1}$  or  $\frac{1}{a+x}$  is  $-(a+x)^{-2}\dot{x}$  or  $-\frac{\dot{x}}{(a+x)^2}$ ,  
 and that of  $c(a+3x^2)^{-2}$  or  $\frac{c}{(a+3x^2)^2}$  is  $-12cx\dot{x} \times (a+3x^2)^{-3}$ ,  
 or  $-\frac{12cx\dot{x}}{(a+3x^2)^3}$ .

21. Much

21. Much in the same manner is obtained the fluxion of any fractional power of a fluent quantity, as of  $x^{\frac{m}{n}}$ , or  $\sqrt[n]{x^m}$ .

For, put the proposed quantity  $x^{\frac{m}{n}} = q$ ; then, raising each to the  $n$  power, gives  $x^m = q^n$ ;

take the fluxions, so shall  $mx^{m-1}\dot{x} = nq^{n-1}\dot{q}$ ; then

dividing by  $nq^{n-1}$ , gives  $\dot{q} = \frac{mx^{m-1}\dot{x}}{nq^{n-1}} = \frac{mx^{m-1}\dot{x}}{nx^{\frac{m-1}{n}}} = \frac{m}{n} x^{\frac{m-1}{n}} \dot{x}$ .

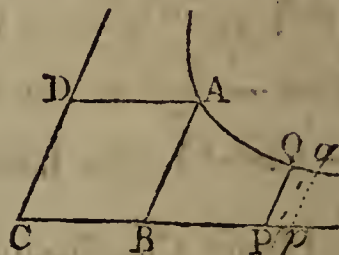
Which is still the same rule, as before, for finding the fluxion of any power of a fluent quantity, and which therefore is general, whether the exponent be positive or negative, integral or fractional. And hence the fluxion of  $ax^{\frac{3}{2}}$  is  $\frac{3}{2}ax^{\frac{1}{2}}\dot{x}$ ,

that of  $ax^{\frac{1}{2}}$  is  $\frac{1}{2}ax^{\frac{1}{2}-1}\dot{x} = \frac{1}{2}ax^{-\frac{1}{2}}\dot{x} = \frac{a\dot{x}}{2x^{\frac{1}{2}}} = \frac{a\dot{x}}{2\sqrt{x}}$ , and that of

$\sqrt{a^2 - x^2}$  or  $(a^2 - x^2)^{\frac{1}{2}}$  is  $\frac{1}{2}(a^2 - x^2)^{-\frac{1}{2}} \times -2x\dot{x} = \frac{-x\dot{x}}{\sqrt{a^2 - x^2}}$ .

22. Having now found out the fluxions of all the ordinary forms of algebraical quantities; it remains to determine those of logarithmic expressions, and also of exponential ones, that is, such powers as have their exponents variable or flowing quantities. And first, for the fluxion of Napier's, or the hyperbolic logarithm.

23. Now, to determine this from the nature of the hyperbolic spaces. Let A be the principal vertex of an hyperbola, having its asymptotes CD, CP, with the ordinates DA, BA, PQ, &c, parallel to them. Then, from the nature of the hyperbola and of logarithms, it is known that any space ABPQ is the log. of the



ratio of CB to CP, to the modulus ABCD. Now, put  $1 = CB$  or  $BA$  the side of the square or rhombus DB;  $m =$  the modulus, or  $CB \times BA$ ; or area of DB, or sine of the angle C to the radius 1; also the absciss  $CP = x$ , and the ordinate  $PQ = y$ . Then, by the nature of the hyperbola,  $CP \times PQ$  is always equal to DB, that is,  $xy = m$ ; hence  $y = \frac{m}{x}$ , and the fluxion of the space, or  $\dot{x}y$  is  $\frac{m\dot{x}}{x} = PQ \dot{q}p$

the fluxion of the log. of  $x$ , to the modulus  $m$ . And, in the hyperbolic logarithms, the modulus  $m$  being 1, therefore

$\frac{\dot{x}}{x}$  is

$\frac{\dot{x}}{x}$  is the fluxion of the hyp. log. of  $x$ ; which is therefore equal to the fluxion of the quantity, divided by the quantity itself.

Hence the fluxion of the hyp. log.

$$\text{of } 1 + x \text{ is } \frac{\dot{x}}{1 + x},$$

$$\text{of } 1 - x \text{ is } \frac{-\dot{x}}{1 - x},$$

$$\text{of } x + z \text{ is } \frac{\dot{x} + \dot{z}}{x + z},$$

$$\text{of } \frac{a + x}{a - x} \text{ is } \frac{\dot{x}(a - x) + \dot{x}(a + x)}{(a - x)^2} \times \frac{a - x}{a + x} = \frac{2a\dot{x}}{a^2 - x^2},$$

$$\text{of } ax^n \text{ is } \frac{nax^{n-1}\dot{x}}{ax^n} = \frac{n\dot{x}}{x}.$$

24. By means of the fluxions of logarithms, are usually determined those of exponential quantities, that is, quantities which have their exponent a flowing or variable letter. These exponentials are of two kinds, namely, when the root is a constant quantity, as  $e^x$ , and when the root is variable as well as the exponent, as  $y^x$ .

25. In the first case, put the exponential, whose fluxion is to be found, equal to a single variable quantity  $z$ , namely,  $z = e^x$ ; then take the logarithm of each, so shall  $\log. z = x \times \log. e$ ; take the fluxions of these, so shall  $\frac{\dot{z}}{z} = \dot{x} \times \log. e$  by the last article; hence  $\dot{z} = z\dot{x} \times \log. e = e^x\dot{x} \times \log. e$ , which is the fluxion of the proposed quantity  $e^x$  or  $z$ , and which therefore is equal to the said given quantity drawn into the fluxion of the exponent, and into the log. of the root.

Hence also, the fluxion of  $(a + c)^{ax}$  is  $(a + c)^{ax} \times n\dot{x} \times \log. (a + c)$ .

26. In like manner, in the second case, put the given quantity  $y^x = z$ ; then the logarithms give  $\log. z = x \times \log. y$ , and the fluxions give  $\frac{\dot{z}}{z} = \dot{x} \times \log. y + x \times \frac{\dot{y}}{y}$ ; hence  $\dot{z} = z\dot{x} \times \log. y + \frac{zy\dot{y}}{y} =$  (by substituting  $y^x$  for  $z$ )  $y^x\dot{x} \times \log. y + xy^{x-1}\dot{y}$ , which is the fluxion of the proposed quantity  $y^x$ ; and which therefore consists of two terms, of which the

the one is the fluxion of the given quantity considering the exponent as constant, and the other the fluxion of the same quantity considering the root as constant.

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### OF SECOND, THIRD, &c, FLUXIONS.

HAVING explained the manner of considering and determining the first fluxions of flowing or variable quantities; it remains now to consider those of the higher orders, as second, third, fourth, &c, fluxions.

27. If the rate or celerity with which any flowing quantity changes its magnitude, be constant, or the same at every position; then is the fluxion of it also constantly the same. But if the variation of magnitude be continually changing, either increasing or decreasing; then will there be a certain degree of fluxion peculiar to every point or position; and the rate of variation or change in the fluxion, is called the Fluxion of the Fluxion, or the Second Fluxion of the given fluent quantity. In like manner, the variation or fluxion of this second fluxion, is called the Third Fluxion of the first proposed fluent quantity; and so on.

These orders of fluxions are denoted by the same fluent letter; with the corresponding number of points over it: namely, two points for the second fluxion, three points for the third fluxion, four points for the fourth fluxion, and so on. So, the different orders of the fluxion of  $x$ , are  $\dot{x}$ ,  $\ddot{x}$ ,  $\dddot{x}$ ,  $\ddddot{x}$ , &c; where each is the fluxion of the one next before it.

28. This description of the higher orders of fluxions may be illustrated by the figures exhibited in page 274; where, if  $x$  denote the absciss AP, and  $y$  the ordinate PQ; and if the ordinate PQ or  $y$  flow along the absciss AP or  $x$ , with an uniform motion; then the fluxion of  $x$ , namely,  $\dot{x} = Pp$  or  $Qr$ , is a constant quantity, or  $\ddot{x} = 0$ , in all the figures. Also, in fig. 1, in which AQ is a right line,  $\dot{y} = rq$ , or the fluxion of PQ, is a constant quantity, or  $\ddot{y} = 0$ ; for, the angle Q, = the angle A, being constant,  $Qr$  is to  $rq$ , or  $\dot{x}$  to  $\dot{y}$ , in a constant ratio. But in the 2d fig.  $rq$ , or the fluxion of PQ, continually increases more and more; and

in fig. 3 it continually decreases more and more, and therefore in both these cases  $y$  has a second fluxion, being positive in fig. 2, but negative in fig. 3. And so on, for the other orders of fluxions.

Thus, if, for instance, the nature of the curve be such that  $x^3$  is every where equal  $a^2y$ ; then, taking the fluxions, it is  $a^2\dot{y} = 3x^2\dot{x}$ ; and, considering  $\dot{x}$  always as a constant quantity. and taking always the fluxions, the equations of the several orders of fluxions will be as below, viz.

the 1st fluxions  $a^2\dot{y} = 3x^2\dot{x}$ ,

the 2d fluxions.  $a^2\ddot{y} = 6x\dot{x}^2$ ,

the 3d fluxions  $a^2\dddot{y} = 6\dot{x}^3$ ,

the 4th fluxions  $a^2\overset{\circ}{y} = 0$ ,

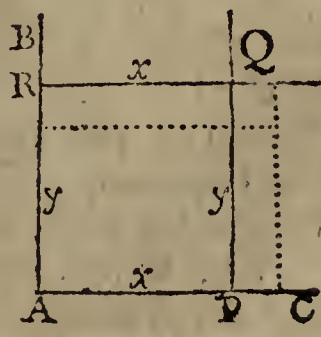
and all the higher fluxions also  $= 0$ , or nothing.

Also, the higher orders of fluxions are found in the same manner as the lower ones. Thus,

the first fluxion of $y^3$ is	- - - -	$3y^2\dot{y}$ ;
its 2d flux. or the flux. of $3y^2\dot{y}$ , con-	} -	$3y^2\ddot{y} + 6y\dot{y}^2$ ;
sidered as the rectangle of $3y^2$ and $\dot{y}$ , is		
and the flux. of this again, or the 3d	} -	$3\ddot{y}^2y + 18y\dot{y}\ddot{y} + 6\dot{y}^3$ .
flux. of $y^3$ , is		

29. In the foregoing articles, it has been supposed that the fluents increase, or that their fluxions are positive; but it often happens that some fluents decrease, and that therefore their fluxions are negative: and whenever this is the case, the sign of the fluxion must be changed, or made contrary to that of the fluent. So, of the rectangle  $xy$ , when both  $x$  and  $y$  increase together, the fluxion is  $\dot{x}y + x\dot{y}$ ; but if one of them, as  $y$ , decrease, while the other,  $x$ , increases; then, the fluxion of  $y$  being  $-\dot{y}$ , the fluxion of  $xy$  will in that case be  $\dot{x}y - x\dot{y}$ . This may be illustrated by the annexed rectangle  $APQR = xy$ , supposed to be generated by the motion of the line  $PQ$  from  $A$  towards  $C$ , and by the motion of the line  $RQ$  from  $B$  towards  $A$ : For, by the motion of  $PQ$ , from  $A$  towards  $C$ , the rectangle is increased, and its fluxion is  $+\dot{x}y$ ; but, by the motion of  $RQ$ , from  $B$  towards  $A$ , the rectangle is decreased, and the fluxion of the decrease is  $x\dot{y}$ ; therefore,

taking



taking the fluxion of the decrease from that of the increase, the fluxion of the rectangle  $xy$ , when  $x$  increases and  $y$  decreases, is  $\dot{x}y - x\dot{y}$ .

30. We may now collect all the rules together, which have been demonstrated in the foregoing articles, for finding the fluxions of all sorts of quantities. And,

1st, *For the fluxion of Any Power of a flowing quantity.*—Multiply all together the exponent of the power, the fluxion of the root, and the power next less by 1 of the same root.

2d, *For the fluxion of the Rectangle of two quantities.*—Multiply each quantity by the fluxion of the other, and connect the two products together by their proper signs.

3d, *For the fluxion of the Continual Product of any number of flowing quantities.*—Multiply the fluxion of each quantity by the product of all the other quantities, and connect all the products together by their proper signs.

4th, *For the fluxion of a Fraction.*—From the fluxion of the numerator drawn into the denominator, subtract the fluxion of the denominator drawn into the numerator, and divide the result by the square of the denominator.

5th, *Or, the 2d, 3d, and 4th cases may be all included under one, and performed thus.*—Take the fluxion of the given expression as often as there are variable quantities in it, supposing first only one of them variable, and the rest constant; then another variable, and the rest constant; and so on, till they have all in their turns been singly supposed variable: and connect all these fluxions together with their own signs.

6th, *For the fluxion of a Logarithm.*—Divide the fluxion of the quantity by the quantity itself, and multiply the result by the modulus of the system of logarithms.

*Note,* The modulus of the hyperbolic logarithms is 1, and the modulus of the common logs, is  $\approx 0.43429448$ .

7th, *For the fluxion of an Exponential quantity, having the root Constant.*—Multiply all together, the given quantity, the fluxion of its exponent, and the hyp. log. of the root.

8th, *For the fluxion of an Exponential quantity having the root Variable.*—To the fluxion of the given quantity, found by the 1st rule, as if the root only were variable, add the fluxion of the same quantity, found by the 7th rule, as if the exponent only were variable; and the sum will be the fluxion for both of them variable.

*Note,* When the given quantity consists of several terms, find the fluxion of each term separately, and connect them all together with their proper signs.



31. PRACTICAL EXAMPLES TO EXERCISE THE FOREGOING RULES.

1. The fluxion of  $axy$  is
2. The fluxion of  $bxyz$  is
3. The fluxion of  $cx \times (ax - cy)$  is
4. The fluxion of  $x^m y^n$  is
5. The fluxion of  $x^m y^n z^r$  is
6. The fluxion of  $(x + y) \times (x - y)$  is
7. The fluxion of  $2ax^2$  is
8. The fluxion of  $2x^3$  is
9. The fluxion of  $3x^4 y$  is
10. The fluxion of  $4x^{\frac{2}{3}} y^4$  is
11. The fluxion of  $ax^2 y - x^{\frac{1}{2}} y^3$  is
12. The fluxion of  $4x^4 - x^2 y + 3byz$  is
13. The fluxion of  $\sqrt[n]{x}$  or  $x^{\frac{1}{n}}$  is
14. The fluxion of  $\sqrt[n]{x^m}$  or  $x^{\frac{m}{n}}$  is
15. The fluxion of  $\frac{1}{\sqrt[n]{x^m}}$  or  $\frac{1}{x^{\frac{m}{n}}}$  or  $x^{-\frac{m}{n}}$  is
16. The fluxion of  $\sqrt{x}$  or  $x^{\frac{1}{2}}$  is
17. The fluxion of  $\sqrt[3]{x}$  or  $x^{\frac{1}{3}}$  is
18. The fluxion of  $\sqrt[3]{x^2}$  or  $x^{\frac{2}{3}}$  is
19. The fluxion of  $\sqrt{x^3}$  or  $x^{\frac{3}{2}}$  is
20. The fluxion of  $\sqrt[4]{x^3}$  or  $x^{\frac{3}{4}}$  is
21. The fluxion of  $\sqrt[3]{x^4}$  or  $x^{\frac{4}{3}}$  is
22. The fluxion of  $\sqrt{a^2 + x^2}$  or  $a^2 + x^2^{\frac{1}{2}}$  is
23. The fluxion of  $\sqrt{a^2 - x^2}$  or  $a^2 - x^2^{\frac{1}{2}}$  is
24. The fluxion of  $\sqrt{2rx - xx}$  or  $2rx - xx^{\frac{1}{2}}$  is
25. The fluxion of  $\frac{1}{\sqrt{a^2 - x^2}}$  or  $(a^2 - x^2)^{-\frac{1}{2}}$  is
26. The fluxion of  $\sqrt{ax - xx^{\frac{1}{3}}}$  is
27. The

27. The fluxion of  $2x\sqrt{a^2 + x^2}$  is
28. The fluxion of  $\frac{a^2 - x^2}{x^{\frac{3}{2}}}$  is
29. The fluxion of  $\sqrt{xz}$  or  $xz^{\frac{1}{2}}$  is
30. The fluxion of  $\sqrt{xz - zz}$  or  $xz - zz^{\frac{1}{2}}$  is
31. The fluxion of  $-\frac{1}{a\sqrt{x}}$  or  $-\frac{1}{a}x^{-\frac{1}{2}}$  is
32. The fluxion of  $\frac{ax^3}{a+x}$  is
33. The fluxion of  $\frac{x^m}{y^n}$  is
34. The fluxion of  $\frac{xy}{z}$  is
35. The fluxion of  $\frac{c}{xx}$  is
36. The fluxion of  $\frac{3x}{a-x}$  is
37. The fluxion of  $\frac{z}{x+z}$  is
38. The fluxion of  $\frac{x^2}{z^2}$  is
39. The fluxion of  $\frac{x^{\frac{2}{3}}}{y^{\frac{3}{2}}}$  is
40. The fluxion of  $\frac{axy^2}{z}$  is
41. The fluxion of  $\frac{3}{\sqrt{x^2 - y^2}}$  is
42. The fluxion of the hyp. log. of  $ax$  is
43. The fluxion of the hyp. log. of  $1+x$  is
44. The fluxion of the hyp. log. of  $1-x$  is
45. The fluxion of the hyp. log. of  $x^2$  is
46. The fluxion of the hyp. log. of  $\sqrt{z}$  is
47. The fluxion of the hyp. log. of  $x^m$  is
48. The

48. The fluxion of the hyp. log. of  $\frac{2}{x^2}$  is
49. The fluxion of the hyp. log. of  $\frac{1+x}{1-x}$  is
50. The fluxion of the hyp. log. of  $\frac{1-x}{1+x}$  is
51. The fluxion of  $c^x$  is
52. The fluxion of  $10^x$  is
53. The fluxion of  $\overline{a+c^x}$  is
54. The fluxion of  $100^{xy}$  is
55. The fluxion of  $x^z$  is
56. The fluxion of  $y^{10^x}$  is
57. The fluxion of  $x^x$  is
58. The fluxion of  $\overline{xy^{xz}}$  is
59. The fluxion of  $x\dot{y}$  is
60. The fluxion of  $\dot{x}\dot{y}^2$  is
61. The second fluxion of  $xy$  is
62. The second fluxion of  $xy$ , when  $\dot{x}$  is constant, is
63. The second fluxion of  $x^n$  is
64. The third fluxion of  $x^n$ , when  $\dot{x}$  is constant, is
65. The third fluxion of  $xy$  is

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THE INVERSE METHOD, OR THE FINDING OF FLUENTS.

32. It has been observed, that a Fluent, or Flowing Quantity, is the variable quantity which is considered as increasing or decreasing. Or, the fluent of a given fluxion, is such a quantity, that its fluxion, found according to the foregoing rules, shall be the same as the fluxion given or proposed.

33. It may farther be observed, that Contemporary Fluents, or Contemporary Fluxions, are such as flow together, or for the same time.—When contemporary fluents are always equal, or in any constant ratio; then also are their fluxions respectively either equal, or in that same constant ratio. That is, if  $x = y$ , then is  $\dot{x} = \dot{y}$ ; or if  $x : y :: n : 1$ , then is  $\dot{x} : \dot{y} :: n : 1$ ; or if  $x = ny$ , then is  $\dot{x} = n\dot{y}$ .

34. It

34. It is easy to find the fluxions to all the given forms of fluents; but, on the contrary, it is difficult to find the fluents of many given fluxions; and indeed there are numberless cases in which this cannot at all be done, excepting by the quadrature and rectification of curve lines, or by logarithms, or by infinite series. For, it is only in certain particular forms and cases that the fluents of given fluxions can be found; there being no method of performing this universally, *a priori*, by a direct investigation, like finding the fluxion of a given fluent quantity. We can only therefore lay down a few rules for such forms of fluxions as we know, from the direct method, belong to such and such kinds of flowing quantities: and these rules, it is evident, must chiefly consist in performing such operations as are the reverse of those by which the fluxions are found of given fluent quantities. The principal cases of which are as follow.

35. *To find the Fluent of a Simple Fluxion; or of that in which there is no variable quantity, and only one fluxional quantity.*

This is done by barely substituting the variable or flowing quantity instead of its fluxion; being the result or reverse of the notation only.—Thus,

The fluent of  $ax$  is  $ax$ .

The fluent of  $ay + 2y$  is  $ay + 2y$ .

The fluent of  $\sqrt{a^2 + x^2}$  is  $\sqrt{a^2 + x^2}$ .

36. *When any Power of a flowing quantity is Multiplied by the Fluxion of the Root.*

Then, having substituted, as before, the flowing quantity for its fluxion, divide the result by the new index of the power. Or, which is the same thing, take out, or divide by, the fluxion of the root; add 1 to the index of the power; and divide by the index so increased. Which is the reverse of the 1st rule for finding fluxions.

So, if the fluxion proposed be  $3x^5 \dot{x}$ .  
 Leave out, or divide by,  $\dot{x}$ , then it is  $3x^5$ ;  
 add 1 to the index, and it is  $3x^6$ ;  
 divide by the index 6, and it is  $\frac{3}{6}x^6$  or  $\frac{1}{2}x^6$ ,  
 which is the fluent of the proposed fluxion  $3x^5 \dot{x}$ .

In like manner,

The fluent of  $2ax\dot{x}$  is

The fluent of  $3x^2\dot{x}$  is

The

The fluent of  $4x^{\frac{1}{2}}\dot{x}$  is

The fluent of  $2y^{\frac{3}{4}}\dot{y}$  is

The fluent of  $az^{\frac{5}{6}}\dot{z}$  is

The fluent of  $x^{\frac{1}{2}}\dot{x} + 3y^{\frac{2}{3}}\dot{y}$  is

The fluent of  $x^{n-1}\dot{x}$  is

The fluent of  $ny^{n-1}\dot{y}$  is

The fluent of  $\frac{\dot{z}}{z^2}$ , or  $z^{-2}\dot{z}$  is

The fluent of  $\frac{a\dot{y}}{y^n}$  is

The fluent of  $(a+x)^4\dot{x}$  is

The fluent of  $(a^4+y^4)y^3\dot{y}$  is

The fluent of  $(a^3+z^3)^4z^2\dot{z}$  is

The fluent of  $(a^n+x^n)^m x^{n-1}\dot{x}$  is

The fluent of  $(a^2+y^2)^3 y\dot{y}$  is

The fluent of  $\frac{z\dot{z}}{\sqrt{a^2+z^2}}$  is

The fluent of  $\frac{\dot{x}}{\sqrt{a-x}}$  is

37. *When the Root under a Vinculum is a Compound Quantity; and the Index of the part or factor Without the Vinculum, increased by 1, is some Multiple of that Under the Vinculum:*

Put a single variable letter for the compound root; and substitute its powers and fluxion instead of those, of the same value, in the given quantity; so will it be reduced to a simpler form, to which the preceding rule can then be applied.

Thus, if the given fluxion be  $\dot{F} = (a^2 + x^2)^{\frac{2}{3}}x^3\dot{x}$ , where 3, the index of the quantity without the vinculum, increased by 1, making 4, which is just the double of 2, the exponent of  $x^2$  within the vinculum: therefore, putting  $z = a^2 + x^2$ , thence  $x^2 = z - a^2$ , the fluxion of which is  $2x\dot{x} = \dot{z}$ ; hence then  $x^3\dot{x} = \frac{1}{2}x^2\dot{z} = \frac{1}{2}\dot{z}(z - a^2)$ , and the given fluxion  $\dot{F}$ , or  $(a^2 + x^2)^{\frac{2}{3}}x^3\dot{x}$ , is  $= \frac{1}{2}z^{\frac{2}{3}}\dot{z}(z - a^2)$  or  $= \frac{1}{2}z^{\frac{5}{3}}\dot{z} - \frac{1}{2}a^2z^{\frac{2}{3}}\dot{z}$ ; and hence the fluent  $F$  is  $= \frac{3}{16}z^{\frac{8}{3}} - \frac{3}{10}a^2z^{\frac{5}{3}} = 3z^{\frac{5}{3}}(\frac{1}{16}z - \frac{3}{10}a^2)$ . Or, by substituting the value of  $z$  instead of it, the same fluent is  $3(a^2 + x^2)^{\frac{5}{3}} \times (\frac{1}{16}x^2 - \frac{3}{10}a^2)$ , or  $\frac{3}{16} \cdot \sqrt[3]{a^2 + x^2} \cdot x^2 - \frac{3}{10}a^2$ .

In like manner for the following examples.

To find the fluent of  $\sqrt{a+cx} \times x^3 \dot{x}$ .

To find the fluent of  $(a+cx)^{\frac{3}{4}} x^2 \dot{x}$ .

To find the fluent of  $(a+cx^2)^{\frac{1}{3}} \times dx^3 \dot{x}$ .

To find the fluent of  $\frac{cz \dot{z}}{\sqrt{a+z}}$  or  $(a+z)^{-\frac{1}{2}} cz \dot{z}$ .

To find the fluent of  $\frac{cz^{3n-1} \dot{z}}{\sqrt{a+z^n}}$  or  $(a+z^n)^{-\frac{1}{2}} cz^{3n-1} \dot{z}$ .

To find the fluent of  $\frac{\dot{z} \sqrt{a^2+z^2}}{z^6}$  or  $(a^2+z^2)^{\frac{1}{2}} z^{-6} \dot{z}$ .

To find the fluent of  $\frac{\dot{x} \sqrt{a-x^n}}{x^{\frac{7}{2}n-1}}$  or  $(a-x^n)^{\frac{1}{2}} x^{-\frac{7}{2}n-1} \dot{x}$ .

38. *When there are several Terms, involving Two or more Variable Quantities, having the Fluxion of each Multiplied by the other Quantity or Quantities :*

Take the fluent of each term, as if there was only one variable quantity in it, namely, that whose fluxion is contained in it, supposing all the others to be constant in that term; then, if the fluents of all the terms, so found, be the very same quantity in all of them, that quantity will be the fluent of the whole. Which is the reverse of the 5th rule for finding fluxions: Thus, if the given fluxion be  $\dot{x}y + x\dot{y}$ , then the fluent of  $\dot{x}y$  is  $xy$ , supposing  $y$  constant; and the fluent of  $x\dot{y}$  is also  $xy$ , supposing  $x$  constant; therefore  $xy$  is the required fluent of the given fluxion  $\dot{x}y + x\dot{y}$ :

In like manner,

The fluent of  $\dot{x}yz + x\dot{y}z + xy\dot{z}$  is

The fluent of  $2xy\dot{x} + x^2\dot{y}$  is

The fluent of  $\frac{1}{2}x^{-\frac{1}{2}}\dot{x}y^2 + 2x^{\frac{1}{2}}y\dot{y}$  is

The fluent of  $\frac{\dot{x}y - x\dot{y}}{y^2}$  or  $\frac{\dot{x}}{y} - \frac{x\dot{y}}{y^2}$  is

The fluent of  $\frac{2axx\dot{y}^{\frac{1}{2}} - \frac{1}{2}ax^2y^{-\frac{1}{2}}\dot{y}}{y}$  or  $\frac{2ax\dot{x}}{\sqrt{y}} - \frac{ax^2\dot{y}}{2y\sqrt{y}}$  is

39. *When*

39. When the given Fluxional Expression is in this Form  $\frac{\dot{x}y - x\dot{y}}{y^2}$ , namely, a Fraction, including Two Quantities, being the Fluxion of the former of them drawn into the latter, minus the Fluxion of the latter drawn into the former, and divided by the Square of the latter :

Then, the fluent is the fraction  $\frac{x}{y}$ , or the former quantity divided by the latter. That is,

The fluent of  $\frac{\dot{x}y - x\dot{y}}{y^2}$  is  $\frac{x}{y}$ . And, in like manner,

The fluent of  $\frac{2x\dot{x}y^2 - 2x^2y\dot{y}}{y^4}$  is  $\frac{x^2}{y^2}$ .

Though, indeed, the examples of this case may be performed by the foregoing one. Thus, the given fluxion

$\frac{\dot{x}y - x\dot{y}}{y^2}$  reduces to  $\frac{\dot{x}}{y} - \frac{x\dot{y}}{y^2}$ , or  $\frac{\dot{x}}{y} - x\dot{y}y^{-2}$ ; of which,

the fluent of  $\frac{\dot{x}}{y}$  is  $\frac{x}{y}$ , supposing  $y$  constant; and

the fluent of  $-x\dot{y}y^{-2}$  is also  $xy^{-1}$  or  $\frac{x}{y}$ , when  $x$  is constant;

therefore, by that case,  $\frac{x}{y}$  is the fluent of the whole  $\frac{\dot{x}y - x\dot{y}}{y^2}$ .

40. When the Fluxion of a Quantity is Divided by the Quantity itself:

Then the fluent is equal to the hyperbolic logarithm of that quantity; or, which is the same thing, the fluent is equal to 2.30258509 multiplied by the common logarithm of the same quantity.

So, the fluent of  $\frac{\dot{x}}{x}$  or  $x^{-1}\dot{x}$  is the hyp. log. of  $x$ .

The fluent of  $\frac{2\dot{x}}{x}$  is  $2 \times$  hyp. log. of  $x$ , or  $=$  hyp. log.  $x^2$ .

The fluent of  $\frac{a\dot{x}}{x}$  is  $2 \times$  hyp. log.  $x$ , or  $=$  hyp. log. of  $x^2$ .

The fluent of  $\frac{\dot{x}}{a+x}$  is

The fluent of  $\frac{3x^2\dot{x}}{a+x^3}$  is

41. *Many fluents may be found by the Direct Method thus :*

Take the fluxion again of the given fluxion, or the second fluxion of the fluent sought ; into which substitute  $\frac{\dot{x}^2}{x}$  for  $\ddot{x}$ ,  $\frac{\dot{y}^2}{y}$  for  $\ddot{y}$ , &c ; that is, make  $x, \dot{x}, \ddot{x}$ , as also,  $y, \dot{y}, \ddot{y}$ , &c, to be in continual proportion, or so that  $x : \dot{x} :: \dot{x} : \ddot{x}$ , and  $y : \dot{y} :: \dot{y} : \ddot{y}$ , &c ; then divide the square of the given fluxional expression by the second fluxion, just found, and the quotient will be the fluent required in many cases.

*Or the same rule may be otherwise delivered thus :*

In the given fluxion  $\dot{F}$ , write  $x$  for  $\dot{x}$ ,  $y$  for  $\dot{y}$ , &c, and call the result  $G$ , taking also the fluxion of this quantity,  $\dot{G}$  ; then make  $\dot{G} : \dot{F} :: G : F$  ; so shall the fourth proportional  $F$  be the fluent sought in many cases.

It may be proved if this be the true fluent, by taking the fluxion of it again, which, if it agree with the proposed fluxion, will shew that the fluent is right ; otherwise, it is wrong.

#### EXAMPLES.

EXAM. 1. Let it be required to find the fluent of  $n\dot{x}^{n-1}$ .

Here  $\dot{F} = n\dot{x}^{n-1}$ , write  $x$  for  $\dot{x}$ , then  $n\dot{x}^{n-1}x$  or  $n\dot{x}^n = G$  ; the fluxion of this is  $\dot{G} = n^2\dot{x}^{n-1}$  ; therefore  $\dot{G} : \dot{F} :: G : F$ , becomes  $n^2\dot{x}^{n-1} : n\dot{x}^{n-1} :: n\dot{x}^n : x^n = F$ , the fluent sought.

EXAM. 2. To find the fluent of  $\dot{x}y + x\dot{y}$ .

Here  $\dot{F} = \dot{x}y + x\dot{y}$  ; then, writing  $x$  for  $\dot{x}$ , and  $y$  for  $\dot{y}$ , it is  $xy + xy$  or  $2xy = G$  ; hence  $\dot{G} = 2\dot{x}y + 2x\dot{y}$  ; then  $\dot{G} : \dot{F} :: G : F$ , becomes  $2\dot{x}y + 2x\dot{y} : \dot{x}y + x\dot{y} :: 2xy : xy = F$ , the fluent sought.

42. *To find Fluents by means of a Table of Forms of Fluxions and Fluents.*

In the following Table are contained the most usual forms of fluxions that occur in the practical solution of problems with their corresponding fluents set opposite to them ; by means of which, namely, by comparing any proposed fluxion with the corresponding form in the table, the fluent of it will be found.



Forms.	Fluxions.	Fluents.
I	$x^{n-1}\dot{x}$	$\frac{x^n}{n}$ , or $\frac{1}{n}x^n$
II	$(a \pm x^n)^{m-1}x^{n-1}\dot{x}$	$\pm \frac{1}{mn} (a \pm x^n)^m$
III	$\frac{x^{mn-1}\dot{x}}{(a \pm x^n)^m \pm 1}$	$\frac{1}{mna} \times \frac{x^{mn}}{(a \pm x^n)^m}$
IV	$\frac{(a \pm x^n)^{m-1}\dot{x}}{x^{mn} \pm 1}$	$\frac{\pm 1}{mna} \times \frac{(a \pm x^n)^m}{x^{mn}}$
V	$(my\dot{x} + nx\dot{y}) \times x^{m-1}y^{n-1}$ , or $(\frac{m\dot{x}}{x} + \frac{n\dot{y}}{y}) x^m y^n$	$x^m y^n$
VI	$mx^{m-1}\dot{x}y^n z^r + nx^m y^{n-1}\dot{y}z^r + rx^m y^n z^{r-1}\dot{z}$ , or $(m\dot{x}yz + nx\dot{y}z + rxy\dot{z})x^{m-1}y^{n-1}z^{r-1}$ , or $(\frac{m\dot{x}}{x} + \frac{n\dot{y}}{y} + \frac{r\dot{z}}{z}) x^m y^n z^r$ ,	$x^m y^n z^r$
VII	$\frac{\dot{x}}{x}$ or $x^{-1}\dot{x}$	log. of $x$
VIII	$\frac{x^{n-1}\dot{x}}{a \pm x^n}$	$\pm \frac{1}{n} \log. \text{ of } a \pm x^n$
IX	$\frac{x^{-1}\dot{x}}{a \pm x^n}$	$\frac{1}{na} \log. \text{ of } \frac{x^n}{a \pm x^n}$
X	$\frac{x^{\frac{1}{2}n-1}\dot{x}}{a - x^n}$	$\frac{1}{n\sqrt{a}} \log. \text{ of } \frac{\sqrt{a} + \sqrt{x^n}}{\sqrt{a} - \sqrt{x^n}}$
XI	$\frac{x^{\frac{1}{2}n-1}\dot{x}}{a + x^n}$	$\frac{2}{n\sqrt{a}} \times \text{arc to tan. } \sqrt{\frac{x^n}{a}}$ , or $\frac{1}{n\sqrt{a}} \times \text{arc to cosine } \frac{a - x^n}{a + x^n}$
XII	$\frac{x^{\frac{1}{2}n-1}\dot{x}}{\sqrt{a + x^n}}$	$\frac{2}{n} \log. \text{ of } \sqrt{x^n} + \sqrt{a + x^n}$

Forms.	Fluxions.	Fluents.
XIII	$\frac{x^{\frac{1}{2}n-1} \dot{x}}{\sqrt{a-x^n}}$	$\frac{2}{n} \times \text{arc to fin. } \sqrt{\frac{x^n}{a}}, \text{ or}$ $\frac{1}{n} \times \text{arc to verf. } \frac{2x^n}{a}$
XIV	$\frac{x^{-1} \dot{x}}{\sqrt{a+x^n}}$	$\frac{1}{n\sqrt{a}} \log. \text{ of } \frac{+\sqrt{a+x^n} + \sqrt{a}}{\sqrt{a+x^n} + \sqrt{a}}$
XV	$\frac{x^{-1} \dot{x}}{\sqrt{-a+x^n}}$	$\frac{2}{n\sqrt{a}} \times \text{arc to secant } \sqrt{\frac{x^n}{a}}, \text{ or}$ $\frac{1}{n\sqrt{a}} \times \text{arc to cofin. } \frac{2a-x^n}{x^n}$
XVI	$\dot{x} \sqrt{dx-x^2}$	$\frac{1}{2} \text{ circ. seg. to diam. } d \text{ \& verf. } x$
XVII	$c^{nx} \dot{x}$	$\frac{c^{nx}}{n \log. c}$
XVIII	$\dot{x} y^x \log. y + x y^{x-1} \dot{y}$	$y^x$

*Note.* The logarithms, in the above forms, are the hyperbolic ones, which are found by multiplying the common logarithms by 2.302585092994. And the arcs, whose sine, or tangent, &c, are mentioned, have the radius 1, and are those in the common tables of sines, tangents, and secants. Also, the numbers  $m$ ,  $n$ , &c, are to be some real quantities, as the forms fail when  $m = 0$ , or  $n = 0$ , &c.

*The Use of the foregoing Table of Forms of Fluxions and Fluents.*

43. In using the foregoing table, it is to be observed, that the first column serves only to shew the number of the form; in the second column are the several forms of fluxions, which are of different kinds or classes; and in the third or last column, are the corresponding fluents.

The method of using the table, is this. Having any fluxion given, to find its fluent: First, Compare the given fluxion with the several forms of fluxions in the second column of the table, till one of the forms be found that agrees

with

with it; which is done by comparing the terms of the given fluxion with the like parts of the tabular fluxion, namely, the radical quantity of the one, with that of the other; and the exponents of the variable quantities of each, both within and without the vinculum; all which, being found to agree or correspond, will give the particular values of the general quantities in the tabular form: then substitute these particular values in the general or tabular form of the fluent, and the result will be the particular fluent of the given fluxion; after it is multiplied by any co-efficient the proposed fluxion may have.

EXAMPLES.

EXAM. 1. To find the fluent of the fluxion  $3x^{\frac{5}{3}}\dot{x}$ .

This is found to agree with the first form. And, by comparing the fluxions, it appears that  $x = x$ , and  $n - 1 = \frac{5}{3}$ , or  $n = \frac{8}{3}$ ; which being substituted in the tabular fluent, or  $\frac{1}{n}x^n$ , gives, after multiplying by 3 the co-efficients,  $3 \times \frac{3}{8}x^{\frac{8}{3}}$ , or  $\frac{9}{8}x^{\frac{8}{3}}$ , for the fluent sought.

EXAM. 2. To find the fluent of  $5x^2\dot{x}\sqrt{c^3 - x^3}$ , or  $5x^2\dot{x}(c^3 - x^3)^{\frac{1}{2}}$ .

This fluxion, it appears, belongs to the 2d tabular form: for  $a = c^3$ , and  $-x^n = -x^3$ , and  $n = 3$  under the vinculum, also  $m - 1 = \frac{1}{2}$ , or  $m = \frac{3}{2}$ , and the exponent  $n-1$  of  $x^{n-1}$  without the vinculum, by using 3 for  $n$ , is  $n - 1 = 2$ , which agrees with  $x^2$  in the given fluxion: so that all the parts of the form are found to correspond. Then, substituting these values into the general fluent,  $-\frac{1}{m \cdot n}(a - x^n)^m$ ,

it becomes  $-\frac{5}{3} \times \frac{2}{3}(c^3 - x^3)^{\frac{3}{2}} = -\frac{10}{9}(c^3 - x^3)^{\frac{3}{2}}$ .

EXAM. 3. To find the fluent of  $\frac{x^2\dot{x}}{1 + x^3}$ .

This is found to agree with the 8th form; where  $-\frac{1}{n}x^n = +x^3$  in the denominator, or  $n = 3$ ; and the numerator  $x^{n-1}$  then becomes  $x^2$ , which agrees with the numerator in the given fluxion; also  $a = 1$ . Hence then, by substituting in the general or tabular fluent,  $\frac{1}{n} \log. \text{ of } a + x^n$ , it becomes  $\frac{1}{3} \log. \text{ of } 1 + x^3$ .

EXAM. 4. To find the fluent of  $ax^4\dot{x}$ .

EXAM. 5. To find the fluent of  $2(10 + x^2)^{\frac{2}{3}}x\dot{x}$ .

EXAM. 6. To find the fluent of  $\frac{ax\dot{x}}{c^2 + x^2}^{\frac{3}{2}}$ .

EXAM. 7. To find the fluent of  $\frac{3x^2\dot{x}}{a - \lambda}^+$ .

EXAM. 8.

EXAM. 8. To find the fluent of  $\frac{c^2 - x^2}{x^5} \dot{x}$ .

EXAM. 9. To find the fluent of  $\frac{1 + 3x}{2x^4} \dot{x}$ .

EXAM. 10. To find the fluent of  $\left(\frac{3\dot{x}}{x} + \frac{2\dot{y}}{y}\right) x^3 y^2$ .

EXAM. 11. To find the fluent of  $\left(\frac{\dot{x}}{x} + \frac{\dot{y}}{3y}\right) xy^{\frac{1}{3}}$ .

EXAM. 12. To find the fluent of  $\frac{3\dot{x}}{ax}$  or  $\frac{3}{a} x^{-1} \dot{x}$ .

EXAM. 13. To find the fluent of  $\frac{a\dot{x}}{3 - 2x}$ .

EXAM. 14. To find the fluent of  $\frac{3\dot{x}}{2x - x^2}$  or  $\frac{3x^{-1}\dot{x}}{2 - x}$ .

EXAM. 15. To find the fluent of  $\frac{2\dot{x}}{x - 3x^3}$  or  $\frac{2x^{-1}\dot{x}}{1 - 3x^2}$ .

EXAM. 16. To find the fluent of  $\frac{3x\dot{x}}{1 - x^4}$ .

EXAM. 17. To find the fluent of  $\frac{ax^{\frac{3}{2}}\dot{x}}{2 - x^5}$ .

EXAM. 18. To find the fluent of  $\frac{2x\dot{x}}{1 + x^4}$ .

EXAM. 19. To find the fluent of  $\frac{ax^{\frac{3}{2}}\dot{x}}{2 + x^5}$ .

EXAM. 20. To find the fluent of  $\frac{3x\dot{x}}{\sqrt{1 + x^4}}$ .

EXAM. 21. To find the fluent of  $\frac{a\dot{x}}{\sqrt{x^2 - 4}}$ .

EXAM. 22. To find the fluent of  $\frac{3x\dot{x}}{\sqrt{1 - x^4}}$ .

EXAM. 23. To find the fluent of  $\frac{a\dot{x}}{\sqrt{4 - x^2}}$ .

EXAM. 24. To find the fluent of  $\frac{2x^{-1}\dot{x}}{\sqrt{1 - x^2}}$ .

EXAM. 25. To find the fluent of  $\frac{a\dot{x}}{\sqrt{ax^2 + x^{\frac{1}{2}}}}$ .

EXAM. 26. To find the fluent of  $\frac{2x^{-1}\dot{x}}{\sqrt{x^2 - 1}}$ .

EXAM. 27.

EXAM. 27. To find the fluent of  $\frac{a\dot{x}}{\sqrt{x^{\frac{1}{2}} - ax^2}}$ .

EXAM. 28. To find the fluent of  $2\dot{x}\sqrt{2x - x^2}$ .

EXAM. 29. To find the fluent of  $a^x\dot{x}$ .

EXAM. 30. To find the fluent of  $3a^{2x}\dot{x}$ .

EXAM. 31. To find the fluent of  $3z^x\dot{x} \log. z + 3xz^{x-1}\dot{x}$ .

EXAM. 32. To find the fluent of  $(1 + x^3) x\dot{x}$ .

EXAM. 33. To find the fluent of  $(2 + x^4) x^{\frac{3}{2}}\dot{x}$ .

EXAM. 34. To find the fluent of  $x^2\dot{x}\sqrt{a^2 + x^2}$ .

*To find Fluents by Infinite Series.*

44. When a given fluxion, whose fluent is required, is so complex, that it cannot be made to agree with any of the forms in the foregoing table of cases, nor made out from the general rules before given; recourse is then to be had to the method of infinite series; which is thus performed:

Expand the radical or fraction, in the given fluxion, into an infinite series of simple terms, by the methods given for that purpose in books of algebra; viz. either by division, or extraction of roots, or by the binomial theorem, &c; and multiply every term by the fluxional letter, and by such simple variable factor as the given fluxional expression may contain. Then take the fluent of each term separately, by the foregoing rules, connecting them all together by their proper signs; and the series will be the fluent sought, after multiplied by any constant factor or co-efficient which may be contained in the given fluxional expression.

45. It is to be noted, however, that the quantities must be so arranged, as that the series produced may be a converging one, rather than diverging; and this is effected by placing the greater terms foremost in the given fluxion. When these are known or constant quantities, the infinite series will be an ascending one; that is, the powers of the variable quantity will ascend, or increase; but if the variable quantity be set foremost, the infinite series produced will be a descending one, or the powers of that quantity will decrease always more and more in the succeeding terms, or increase in the denominators of them, which is the same thing.

For

For example, to find the fluent of  $\frac{1-x}{1+x-x^2}$ .

Here, by dividing the numerator by the denominator, the proposed fluxion becomes  $\dot{x} - 2x\dot{x} + 3x^2\dot{x} - 5x^3\dot{x} + 8x^4\dot{x} - \&c$ ; then the fluents of all the terms, being taken, give  $x - x^2 + x^3 - \frac{5}{4}x^4 + \frac{8}{5}x^5 - \&c$ , for the fluent sought.

Again, to find the fluent of  $\dot{x}\sqrt{1-x^2}$ .

Here, by extracting the root, or expanding the radical quantity  $\sqrt{1-x^2}$ , the given fluxion becomes  $\dot{x} - \frac{1}{2}x^2\dot{x} - \frac{1}{8}x^4\dot{x} - \frac{1}{16}x^6\dot{x} - \&c$ . Then, the fluents of all the terms, being taken, give  $x - \frac{1}{6}x^3 - \frac{1}{40}x^5 - \frac{1}{112}x^7 - \&c$ , for the fluent sought.

#### OTHER EXAMPLES.

EXAM. 1. To find the fluent of  $\frac{bx\dot{x}}{a-x}$  both in an ascending and descending series.

EXAM. 2. To find the fluent of  $\frac{b\dot{x}}{a+x}$  in both series.

EXAM. 3. To find the fluent of  $\frac{3\dot{x}}{(a+x)^2}$ .

EXAM. 4. To find the fluent of  $\frac{1-x^2+2x^4}{1+x-x^2}\dot{x}$ .

EXAM. 5. Given  $\dot{z} = \frac{b\dot{x}}{a^2+x^2}$ , to find  $z$ .

EXAM. 6. Given  $\dot{z} = \frac{a^2+x^2}{a+x}\dot{x}$ , to find  $z$ .

EXAM. 7. Given  $\dot{z} = 3\dot{x}\sqrt{a+x}$ , to find  $z$ .

EXAM. 8. Given  $\dot{z} = 2\dot{x}\sqrt{a^2+x^2}$ , to find  $z$ .

EXAM. 9. Given  $\dot{z} = 4\dot{x}\sqrt{a^2-x^2}$ , to find  $z$ .

EXAM. 10. Given  $\dot{z} = \frac{5a\dot{x}}{\sqrt{x^2-a^2}}$ , to find  $z$ .

EXAM. 11. Given  $\dot{z} = 2\dot{x}^3\sqrt{a^3-x^3}$ , to find  $z$ .

EXAM. 12. Given  $\dot{z} = \frac{3a\dot{x}}{\sqrt{ax-xx}}$ , to find  $z$ .

EXAM. 13. Given  $\dot{z} = 2\dot{x}\sqrt{x^3+x^4+x^5}$ , to find  $z$ .

EXAM. 14. Given  $\dot{z} = 5\dot{x}\sqrt{ax-xx}$ , to find  $z$ .

*To Correct the Fluent of any Given Fluxion.*

46. The fluxion found from a given fluent, is always perfect and complete: but the fluent found from a given fluxion, is not always so; as it often wants a correction, to make it contemporaneous with that required by the problem under consideration, &c: for, the fluent of any given fluxion, as  $\dot{x}$ , may be either  $x$ , which is found by the rule, or it may be  $x + c$ , or  $x - c$ , that is,  $x$  plus or minus some constant quantity  $c$ ; because both  $x$  and  $x \pm c$  have the same fluxion  $\dot{x}$ , and the finding of the constant quantity  $c$ , to be added or subtracted with the fluent as found by the foregoing rules, is called *correcting* the fluent.

Now this correction is to be determined from the nature of the problem in hand, by which we come to know the relation which the fluent quantities have to each other at some certain point or time. Reduce, therefore, the general fluential equation, supposed to be found by the foregoing rules, to that point or time; then if the equation be true, it is correct; but if not, it wants a correction; and the quantity of the correction, is the difference between the two general sides of the equation when reduced to that particular point. Hence the general rule for the correction is this:

Connect the constant, but indeterminate, quantity  $c$ , with one side of the fluential equation, as determined by the foregoing rules; then, in this equation, substitute for the variable quantities, such values as they are known to have at any particular state, place, or time; and then, from that particular state of the equation, find the value of  $c$ , the constant quantity of the correction.

EXAMPLES.

47. EXAM. I. To find the correct fluent of  $\dot{z} = ax^3\dot{x}$ .

The general fluent is  $z = ax^4$ , or  $z = ax^4 + c$ , taking in the correction  $c$ .

Now, if it be known that  $z$  and  $x$  begin together, or that  $z$  is  $= 0$ , when  $x = 0$ ; then writing 0 for both  $x$  and  $z$ , the general equation becomes  $0 = 0 + c$ , or  $= c$ ; so that, the value of  $c$  being 0, the correct fluents are  $z = ax^4$ .

But if  $z$  be  $= 0$ , when  $x$  is  $= b$ , any known quantity; then substituting 0 for  $z$ , and  $b$  for  $x$ , in the general equation, it becomes  $0 = ab^4 + c$ , and from hence we find  $c = -ab^4$ ; which being written for  $c$  in the general fluential equation, it becomes  $z = ax^4 - ab^4$ , for the correct fluents.

Or, if it be known that  $z$  is = some quantity  $d$ , when  $x$  is = some other quantity as  $b$ ; then substituting  $d$  for  $z$ , and  $b$  for  $x$ , in the general fluential equation  $z = ax^4 + c$ , it becomes  $d = ab^4 + c$ ; and hence is deduced the value of the correction, namely,  $c = d - ab^4$ ; consequently, writing this value for  $c$  in the general equation, it becomes  $z = ax^4 - ab^4 + d$ , for the correct equation of the fluents in this case.

48. And hence arises another easy and general way of correcting the fluents, which is this: In the general equation of the fluents, write the particular values of the quantities which they are known to have at any certain time or position; then subtract the sides of the resulting particular equation from the corresponding sides of the general one, and the remainders will give the correct equation of the fluents sought.

So, the general equation being  $z = ax^4$ ;  
write  $d$  for  $z$ , and  $b$  for  $x$ , then  $d = ab^4$ ;  
hence, by subtraction,  $z - d = ax^4 - ab^4$ ,  
or  $z = ax^4 - ab^4 + d$ , the correct fluents, as before.

EXAM. 2. To find the correct fluents of  $\dot{z} = 5x\dot{x}$ ;  $z$  being = 0 when  $x$  is =  $a$ .

EXAM. 3. To find the correct fluents of  $\dot{z} = 3x\sqrt{a+x}$ ;  
 $z$  and  $x$  being = 0 at the same time.

EXAM. 4. To find the correct fluents of  $\dot{z} = \frac{3a\dot{x}}{a+x}$ ; sup-  
posing  $z$  and  $x$  to begin to flow together, or to be each  
= 0 at the same time.

EXAM. 5. To find the correct fluents of  $\dot{z} = \frac{2\dot{x}}{a^2+x^2}$ ; sup-  
posing  $z$  and  $x$  to begin together.

### OF MAXIMA AND MINIMA; OR, THE GREATEST AND LEAST MAGNITUDE OF VARIABLE OR FLOWING QUANTITIES.

49. MAXIMUM, denotes the greatest state or quantity attainable in any given case, or the greatest value of a variable quantity: by which it stands opposed to Minimum, which is the least possible quantity in any case.

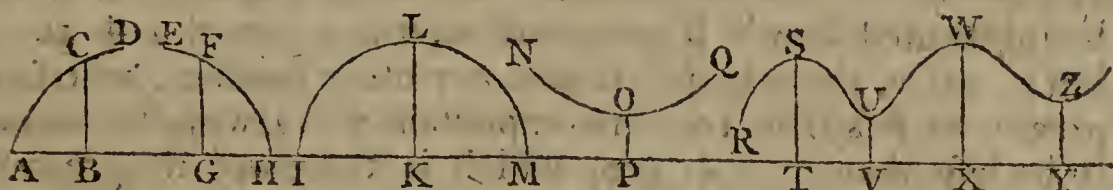
Thus,



Thus, the expression or sum  $a^2 + bx$ , evidently increases as  $x$ , or the term  $bx$ , increases; therefore the given expression will be the greatest, or a maximum, when  $x$  is the greatest, or infinite; and the same expression will be a minimum, or the least, when  $x$  is the least, or nothing.

Again, in the algebraic expression  $a^2 - bx$ , where  $a$  and  $b$  denote constant or invariable quantities, and  $x$  a flowing or variable one. Now, it is evident that the value of this remainder or difference,  $a^2 - bx$ , will increase, as the term  $bx$ , or as  $x$ , decreases; therefore the former will be the greatest, when the latter is the smallest; that is,  $a^2 - bx$  is a maximum, when  $x$  is the least, or nothing at all; and the difference is the least, when  $x$  is the greatest.

50. Some variable quantities increase continually; and so have no maximum, but what is infinite. Others again decrease continually; and so have no minimum, but what is of no magnitude, or nothing. But, on the other hand, some variable quantities increase only to a certain finite magnitude, called their Maximum, or greatest state, and after that they decrease again. While others decrease to a certain finite magnitude, called their Minimum, or least state, and afterwards increase again. And lastly, some quantities have several maxima and minima.



Thus, for example, the ordinate BC of the parabola; or such-like curve, flowing along the axis AB from the vertex A, continually increases, and has no limit or maximum. And the ordinate GF of the curve EFH, flowing from E towards H, continually decreases to nothing when it arrives at the point H. But in the circle ILM, the ordinate only increases to a certain magnitude, namely the radius, when it arrives at the middle as at KL, which is its maximum; and after that it decreases again to nothing, at the point M. And in the curve NOQ, the ordinate decreases only to the position OP, where it is least, or a minimum; and after that it continually increases towards Q. But in the curve RSU, &c. the ordinates have several maxima, as ST, WX, and several minima, as VU, YZ, &c.

51. Now,

51. Now, because the fluxion of a variable quantity, is the rate of its increase or decrease; and because the maximum or minimum of a quantity neither increases nor decreases, at those points or states; therefore such maximum or minimum has no fluxion, or the fluxion is then equal to nothing. From which we have the following rule.

*To find the Maximum or Minimum.*

52. From the nature of the question or problem, find an algebraical expression for the value, or general state, of the quantity whose maximum or minimum is required; then take the fluxion of that expression, and put it equal to nothing; from which equation, by dividing by, or leaving out, the fluxional letter and other common quantities, and performing other proper reductions, as in common algebra, the value of the unknown quantity will be obtained, determining the point of the maximum or minimum.

So, if it be required to find the maximum state of the compound expression  $100x - 5x^2 + c$ , or the value of  $x$  when  $100x - 5x^2 + c$  is a maximum. The fluxion of this expression is  $100\dot{x} - 10x\dot{x} = 0$ ; which being made  $= 0$ , and divided by  $10\dot{x}$ , the equation is  $10 - x = 0$ ; and hence  $x = 10$ . That is, the value of  $x$  is 10, when the expression  $100x - 5x^2 + c$  is the greatest. As is easily tried: for if 10 be substituted for  $x$  in that expression, it becomes  $+c + 500$ : but if, for  $x$ , there be substituted any other number, whether greater or less than 10; that expression will always be found to be less than  $+c + 500$ , which is therefore its greatest possible value, or its maximum.

53. It is evident, that if a maximum or minimum be any way compounded with, or operated upon, by a given constant quantity, the result will still be a maximum or minimum. That is, if a maximum or minimum be increased, or decreased, or multiplied, or divided, by a given quantity, or any given power or root of it be taken; the result will still be a maximum or minimum. Thus, if  $x$  be a maximum or minimum, then also is  $x + a$ , or  $x - a$ , or  $ax$ , or  $\frac{x}{a}$ , or  $x^2$ , or  $\sqrt{x}$ , still a maximum or minimum. Also, the logarithm of the same will be a maximum or a minimum. And therefore, if any proposed maximum or minimum can be made simpler by performing any of these operations, it is better to do so, before the expression is put into fluxions.

54. When

§4. When the expression for a maximum or minimum contains several variable letters or quantities; take the fluxion of it as often as there are variable letters; supposing first one of them only to flow, and the rest to be constant; then another only to flow, and the rest constant; and so on for all of them: then putting each of these fluxions = 0, there will be as many equations as unknown letters, from which these may be all determined. For the fluxion of the expression must be equal to nothing in each of these cases; otherwise the expression might become greater or less, without altering the values of the other letters, which are considered as constant.

So, if it be required to find the values of  $x$  and  $y$  when  $4x^2 - xy + 2y$  is a minimum. Then we have,

First  $- 8x\dot{x} - xy = 0$ , and  $8x - y = 0$ , or  $y = 8x$ .

Secondly,  $2\dot{y} - x\dot{y} = 0$ , and  $2 - x = 0$ , or  $x = 2$ .

And hence  $y$  or  $8x = 16$ .

55. *To find whether a proposed quantity admits of a Maximum or a Minimum.*

Every algebraic expression does not admit of a maximum or minimum, properly so called; for it may either increase continually to infinity, or decrease continually to nothing; and, in both these cases, there is neither a proper maximum nor minimum; for the true maximum is that finite value to which an expression increases, after which it decreases again: and the minimum is that finite value to which the expression decreases, and after that it increases again. Therefore, when the expression admits of a maximum, its fluxion is positive before that point, and negative after it; but when it admits of a minimum, its fluxion is negative before, and positive after it. Hence, taking the fluxion of the expression a little before the fluxion is equal to nothing, and again a little after the same; if the former fluxion be positive, and the latter negative, the middle state is a maximum; but if the former fluxion be negative, and the latter positive, the middle state is a minimum.

So, if we would find the quantity  $ax - x^2$  a maximum or minimum; make its fluxion equal to nothing, that is,  $- ax\dot{x} - 2x\dot{x} = 0$ , or  $(a - 2x)\dot{x} = 0$ ; dividing by  $\dot{x}$ , gives  $a - 2x = 0$ , or  $x = \frac{1}{2}a$  at that state. Now, if in the fluxion  $(a - 2x)\dot{x}$ , the value of  $x$  be taken rather less than its true value,  $\frac{1}{2}a$ , that fluxion will evidently be positive: but if  $x$  be taken somewhat greater than  $\frac{1}{2}a$ , the value of  $a - 2x$ , and consequently of the fluxion, is as evidently negative. Therefore, the fluxion of  $ax - x^2$  being positive before, and negative

gative after the state when its fluxion is  $= 0$ , it follows that at this state the expression is not a minimum, but a maximum.

Again, taking the expression  $x^3 - ax^2$ , its fluxion  $3x^2\dot{x} - 2ax\dot{x} = (3x - 2a)x\dot{x} = 0$ ; this divided by  $x\dot{x}$  gives  $3x - 2a = 0$ , and  $x = \frac{2}{3}a$ , its true value when the fluxion of  $x^3 - ax^2$  is equal to nothing. But now to know whether the given expression be a maximum or a minimum at that time, take  $x$  a little less than  $\frac{2}{3}a$  in the value of the fluxion  $(3x - 2a)x\dot{x}$ , and this will evidently be negative; and again, taking  $x$  a little more than  $\frac{2}{3}a$ , the value of  $3x - 2a$ , or of the fluxion, is as evidently positive. Therefore the fluxion of  $x^3 - ax^2$  being negative before that fluxion is  $= 0$ , and positive after it; it follows that in this state the quantity  $x^3 - ax^2$  admits of a minimum, but not of a maximum.

### 56. SOME EXAMPLES FOR PRACTICE.

EXAM. 1. To divide a line, or any other given quantity  $a$ , into two parts, so that their rectangle or product may be the greatest possible.

EXAM. 2. To divide the given quantity  $a$  into two parts such, that the product of the  $m$  power of one, by the  $n$  power of the other, may be a maximum.

EXAM. 3. To divide the given quantity  $a$  into three parts such, that the continual product of them all may be a maximum.

EXAM. 4. To divide the given quantity  $a$  into three parts such, that the continual product of the 1st, the square of the 2d, and the cube of the 3d, may be a maximum.

EXAM. 5. To determine a fraction such, that the difference between its  $m$  power and  $n$  power shall be the greatest possible.

EXAM. 6. To divide the number 80 into two such parts,  $x$  and  $y$ , that  $2x^2 + xy + 3y^2$  may be a minimum.

EXAM. 7. To find the greatest rectangle that can be inscribed in a given right-angled triangle.

EXAM. 8. To find the greatest rectangle that can be inscribed in the quadrant of a given circle.

EXAM. 9. To find the least right-angled triangle that can circumscribe the quadrant of a given circle.

EXAM. 10. To find the greatest rectangle inscribed in, and the least isosceles triangle circumscribed about, a given semi-ellipse.

EXAM. 11.

EXAM. 11. To determine the same for a given parabola.

EXAM. 12. To determine the same for a given hyperbola.

EXAM. 13. To inscribe the greatest cylinder in a given cone; or to cut the greatest cylinder out of a given cone.

EXAM. 14. To determine the dimensions of a rectangular cistern, capable of containing a given quantity  $a$  of water, so as to be lined with lead at the least possible expence.

EXAM. 15. Required the dimensions of a cylindrical tankard, to hold one quart of ale measure, that can be made of the least possible quantity of silver, of a given thickness.

EXAM. 16. To cut the greatest parabola from a given cone.

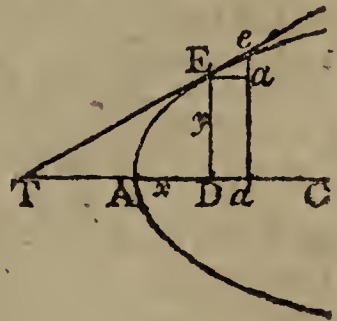
EXAM. 17. To cut the greatest ellipse from a given cone.

EXAM. 18. To find the value of  $x$  when  $x^x$  is a minimum.

### THE METHOD OF TANGENTS; OR, TO DRAW TANGENTS TO CURVES.

THE Method of Tangents, is a method of determining the quantity of the tangent and subtangent of any algebraic curve; the equation of the curve being given. Or, *vice versa*, the nature of the curve, from the tangent given.

57. If  $AE$  be any curve, and  $E$  be any point in it, to which it is required to draw a tangent  $TE$ . Draw the ordinate  $ED$ : then if we can determine the subtangent  $TD$ , limited between the ordinate and tangent, in the axis produced, by joining the points  $T$ ,  $E$ , the line  $TE$  will be the tangent sought.



58. Let  $dae$  be another ordinate, indefinitely near to  $DE$ , meeting the curve, or tangent produced, in  $e$ ; and let  $Ea$  be parallel to the axis  $AD$ . Then is the elementary triangle  $Eea$  similar to the triangle  $TDE$ ;

and therefore -  $ea : aE :: ED : DT.$

But -  $ea : aE :: \text{flux. ED} : \text{flux. AD}.$

Therefore -  $\text{flux. ED} : \text{flux. AD} :: DE : DT.$

That is, -  $\dot{y} : \dot{x} :: y : \frac{y\dot{x}}{\dot{y}} = DT,$

which is therefore the value of the subtangent sought ;  
where  $x$  is the absciss AD, and  $y$  the ordinate DE.

Hence we have this general rule.

#### GENERAL RULE.

59. By means of the given equation of the curve, when put into fluxions, find the value of either  $\dot{x}$  or  $\dot{y}$ , or of  $\frac{\dot{x}}{\dot{y}}$  ; which value substitute for it in the expression  $DT = \frac{y\dot{x}}{\dot{y}}$ , and, when reduced to its simplest terms, it will be the value of the subtangent sought.

#### EXAMPLES.

EXAM. 1. Let the proposed curve be that which is defined, or expressed, by the equation  $ax^2 + xy^2 - y^3 = 0.$

Here the fluxion of the equation of the curve is  $2ax\dot{x} + y^2\dot{x} + 2xy\dot{y} - 3y^2\dot{y} = 0$  ; then, by transposition,  $2ax\dot{x} + y^2\dot{x} = 3y^2\dot{y} - 2xy\dot{y}$  ; and hence, by division,  $\frac{\dot{x}}{\dot{y}} = \frac{3y^2 - 2xy}{2ax + y^2}$  ; consequently  $\frac{y\dot{x}}{\dot{y}} = \frac{3y^3 - 2xy^2}{2ax + y^2}$ , which is the value of the subtangent TD sought.

EXAM. 2. To draw a tangent to a circle ; the equation of which is  $ax - x^2 = y^2$  ; where  $x$  is the absciss,  $y$  the ordinate, and  $a$  the diameter.

EXAM. 3. To draw a tangent to a parabola ; its equation being  $ax = y^2$  ; where  $a$  denotes the parameter of the axis.

EXAM. 4. To draw a tangent to an ellipse ; its equation being  $c^2(ax - x^2) = a^2y^2$  ; where  $a$  and  $c$  are the two axes.

EXAM. 5. To draw a tangent to an hyperbola ; its equation being  $c^2(ax + x^2) = a^2y^2$  ; where  $a$  and  $c$  are the two axes.

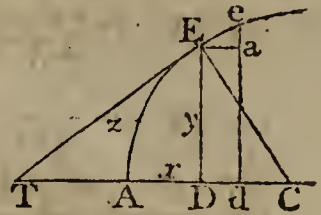
EXAM. 6. To draw a tangent to the hyperbola referred to the asymptote as an axis ; its equation being  $xy = a^2$  ; where  $a^2$  denotes the rectangle of the absciss and ordinate answering to the vertex of the curve.

## OF RECTIFICATIONS;

OR, TO FIND THE LENGTHS OF CURVE LINES.

RECTIFICATION, is the finding the length of a curve line, or finding a right line equal to a proposed curve.

60. By art. 10 it appears, that the elementary triangle  $Eae$ , formed by the increments of the absciss, ordinate, and curve, is a right-angled triangle, of which the increment of the curve is the hypotenuse; and therefore the square of the latter is equal to the sum of the squares of the two former; that is,  $Ee^2 = Ea^2 + ae^2$ . Or, substituting, for the increments, their proportional fluxions, it is  $\dot{z}\dot{z} = \dot{x}\dot{x} + \dot{y}\dot{y}$ , or  $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$ ; where  $z$  denotes any curve line  $AE$ ,  $x$  its absciss  $AD$ , and  $y$  its ordinate  $DE$ . Hence this rule.



## R U L E.

61. From the given equation of the curve put into fluxions, find the value of  $\dot{x}^2$  or  $\dot{y}^2$ , which value substitute instead of it in the equation  $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$ ; then the fluents, being taken, will give the value of  $z$ , or the length of the curve, in terms of the absciss or ordinate.

## E X A M P L E S.

EXAM. I. To find the length of the arc of a circle, in terms both of the sine, versed sine, tangent, and secant.

The equation of the circle may be expressed in terms of the radius, and either the sine, or the versed sine, or tangent, or secant, &c, of an arc. Let, therefore, the radius of the circle be  $CA$  or  $CE = r$ , the versed sine  $AD$  (of the arc  $AE$ )  $= x$ , the right sine  $DE = y$ , the tangent  $TE = t$ , and the secant  $CT = s$ ; then, by the nature of the circle, there arise these equations, viz.

$$y^2 = 2rx - x^2 = \frac{r^2 t^2}{r^2 + t^2} = \frac{s^2 - r^2}{s^2} r^2.$$

Then, by means of the fluxions of these equations, with the general fluxional equation  $\dot{z}^2 = \dot{x}^2 + \dot{y}^2$ , are obtained the following fluxional forms, for the fluxion of the curve; the fluent of any one of which will be the curve itself; viz.

$$\dot{z} = \frac{r\dot{x}}{\sqrt{2rx - xx}} = \frac{r\dot{y}}{\sqrt{r^2 - yy}} = \frac{r^2\dot{t}}{r^2 + t^2} = \frac{r^2\dot{s}}{\sqrt{s^2 - r^2}}.$$

X 2

Hence

Hence the value of the curve, from the fluent of each of these, gives the four following forms, in series, viz. putting  $d = 2r$  the diameter, the curve is  $z$

$$\begin{aligned}
 &= (1 + \frac{x}{2.3d} + \frac{3x^2}{2.4.5d^2} + \frac{3.5x^3}{2.4.6.7d^3} + \&c) \sqrt{dx}, \\
 &= (1 + \frac{y^2}{2.3r^2} + \frac{3y^4}{2.4.5r^4} + \frac{3.5y^6}{2.4.6.7r^6} + \&c) y, \\
 &= (1 - \frac{t^2}{3r^2} + \frac{t^4}{5r^4} - \frac{t^6}{7r^6} + \frac{t^8}{9r^8} - \&c) t, \\
 &= (\frac{s-r}{s} + \frac{s^3-r^3}{2.3s^3} + \frac{3(s^5-r^5)}{2.4.5s^5} + \&c) r.
 \end{aligned}$$

Now, it is evident that the simplest of these series, is the third in order, or that which is expressed in terms of the tangent. That form will therefore be the fittest to calculate an example by in numbers. And, for this purpose, it will be convenient to assume some arc whose tangent, or at least the square of it, is known to be some small simple number. Now, the arc of 45 degrees, it is known, has its tangent equal to the radius; and therefore, taking the radius  $r = 1$ , and consequently the tangent of 45°, or  $t = 1$  also, in this case the arc of 45° to the radius 1, or the arc of the quadrant to the diameter 1, will be equal to the infinite series  $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \&c$ .

But as this series converges very slowly, it will be proper to take some smaller arc, that the series may converge faster; such as the arc of 30 degrees, the tangent of which is  $= \sqrt{\frac{1}{3}}$ , or its square  $t^2 = \frac{1}{3}$ : which being substituted in the series, the length of the arc of 30° comes out - - - -

$(1 - \frac{1}{3.3} + \frac{1}{5.3^2} - \frac{1}{7.3^3} + \frac{1}{9.3^4} - \&c) \sqrt{\frac{1}{3}}$ . Hence, to compute these terms in decimal numbers, after the first, the succeeding terms will be found by dividing always by 3, and these quotients again by the absolute numbers 3, 5, 7, 9, &c; and lastly, adding every other term together, into two sums, the one the sum of the positive terms, and the other the sum of the negative ones; then lastly, the one sum taken from the other, leaves the length of the arc of 30 degrees; which being the 12th part of the whole circumference when the radius is 1, or the 6th part when the diameter is 1, consequently 6 times that arc will be the length of the whole circumference to the diameter 1. Therefore, multiplying the first term  $\sqrt{\frac{1}{3}}$  by 6, the product is  $\sqrt{12} = 3.4641016$ ; and hence the operation will be conveniently made as follows:

+ Terms.



		+ Terms.		- Terms.
1 )	3.4641016	{ 3.4641016		
3 )	1.1547005	{		0.3849002
5 )	3849002	{ 769800		
7 )	1283001	{		183286
9 )	427667	{ 47519		
11 )	142556	{		12960
13 )	47519	{ 3655		
15 )	15840	{		1056
17 )	5280	{ 311		
19 )	1760	{		93
21 )	587	{ 28		
23 )	196	{		8
25 )	65	{ 3		
27 )	22	{		1
		+ 3.5462332		- 0.4046406
		- 0.4046406		

So that at last 3.1415926 is the whole circumference to the diameter 1.

EXAM. 2. To find the length of a parabola.

EXAM. 3. To find the length of the semicubical parabola, whose equation is  $ax^2 = y^3$ .

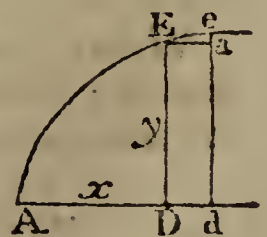
EXAM. 4. To find the length of an elliptical curve.

EXAM. 5. To find the length of an hyperbolic curve.

OF QUADRATURES;  
OR, FINDING THE AREAS OF CURVES.

62. THE Quadrature of Curves, is the measuring their areas, or finding a square, or other right-lined space, equal to a proposed curvilinear one.

By art. 9 it appears, that any flowing quantity being drawn into the fluxion of the line along which it flows, or in the direction of its motion, there is produced the fluxion of the quantity generated by the flowing. That is,  $Dd \times DE$  or  $y\dot{x}$  is the fluxion of the area ADE. Hence this rule.



R U L E.

## R U L E.

63. From the given equation of the curve, find the value either of  $\dot{x}$  or of  $y$ ; which value substitute instead of it in the expression  $y\dot{x}$ ; then the fluent of that expression, being taken, will be the area of the curve sought.

## E X A M P L E S.

EXAM. I. To find the area of the common parabola.

The equation of the parabola being  $ax = y^2$ ; where  $a$  is the parameter,  $x$  the absciss AD, or part of the axis, and  $y$  the ordinate DE.

From the equation of the curve is found  $y = \sqrt{ax}$ . This substituted in the general fluxion of the area  $y\dot{x}$ , gives  $\dot{x}\sqrt{ax}$  or  $a^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x}$  is the fluxion of the parabolic area; and the fluent of this, or  $\frac{2}{3}a^{\frac{1}{2}}x^{\frac{3}{2}} = \frac{2}{3}x\sqrt{ax} = \frac{2}{3}xy$ , is the area of the parabola ADE, and which is therefore equal to  $\frac{2}{3}$  of its circumscribing rectangle.

EXAM. 2. To square the circle, or find its area.

The equation of the circle being  $y^2 = ax - x^2$ , or  $y = \sqrt{ax - x^2}$ , where  $a$  is the diameter; by substitution, the general fluxion of the area  $y\dot{x}$ , becomes  $\dot{x}\sqrt{ax - x^2}$ , for the fluxion of the circular area. But as the fluent of this cannot be found in finite terms, the quantity  $\sqrt{ax - x^2}$  is thrown into a series, by extracting the root, and then the fluxion of the area becomes

$$\dot{x}\sqrt{ax} \times \left(1 - \frac{x}{2a} - \frac{x^2}{2.4a^2} - \frac{1.3x^3}{2.4.6a^3} - \frac{1.3.5x^4}{2.4.6.8a^4} - \&c\right);$$

then the fluent of every term being taken, it gives

$$x\sqrt{ax} \times \left(\frac{2}{3} - \frac{1.x}{5a} - \frac{1.x^2}{4.7a^2} - \frac{1.3x^3}{4.6.9a^3} - \&c\right)$$

for the general expression of the semifegment ADE.

When the point D arrives at the extremity of the diameter, then the space becomes a semicircle, and  $x = a$ ; and then the series above becomes barely

$$a^2\left(\frac{2}{3} - \frac{1}{5} - \frac{1}{4.7} - \frac{1.3}{4.6.9} - \&c\right)$$

for the area of the semicircle whose diameter is  $a$ .

EXAM. 3,

EXAM. 3. To find the area of any parabola, whose equation is  $a^m x^n = y^m + n$ .

EXAM. 4. To find the area of an ellipse.

EXAM. 5. To find the area of an hyperbola.

EXAM. 6. To find the area between the curve and asymptote of an hyperbola.

EXAM. 7. To find the like area in any other hyperbola whose general equation is  $x^m y^n = a^m + n$ .

### TO FIND THE SURFACES OF SOLIDS.

64. IN the solid formed by the rotation of any curve about its axis, the surface may be considered as generated by the circumference of an expanding circle, moving perpendicularly along the axis, but the expanding circumference moving along the arc or curve of the solid. Therefore, as the fluxion of any generated quantity, is produced, by drawing the generating quantity into the fluxion of the line or direction in which it moves, the fluxion of the surface will be found by drawing the circumference of the generating circle into the fluxion of the curve. That is, the fluxion of the surface BAE, is equal to AE drawn into the circumference BCEF, whose radius is the ordinate DE.



65. But, if  $c$  be  $= 3.1416$ , the circumference of a circle whose diameter is 1,  $x = AD$  the absciss,  $y = DE$  the ordinate, and  $z = AE$  the curve; then  $2y =$  the diameter BE, and  $2cy =$  the circumference BCEF; also,  $AE = z = \sqrt{x^2 + y^2}$ : therefore  $2cyz$  or  $2cy\sqrt{x^2 + y^2}$  is the fluxion of the surface. And consequently if, from the given equation of the curve, the value of  $\dot{x}$  or  $\dot{y}$  be found, and substituted in this expression  $2cy\sqrt{x^2 + y^2}$ , the fluent of the expression, being then taken, will be the surface of the solid required.

#### EXAMPLES.

EXAM. 1. To find the surface of a sphere, or of any segment.

In this case, AE is a circular arc, whose equation is  $y^2 = ax - x^2$ , or  $y = \sqrt{ax - x^2}$ .

The fluxion of this gives  $\dot{y} = \frac{a - 2x}{2\sqrt{ax - x^2}} \dot{x} = \frac{a - 2x}{2y} \dot{x}$ ;

hence  $\dot{y}^2 = \frac{a^2 - 4ax + 4x^2}{4y^2} \dot{x}^2 = \frac{a^2 - 4y^2}{4y^2} \dot{x}^2$ ;

conseq.  $\dot{x}^2 + \dot{y}^2 = \frac{a^2 \dot{x}^2}{4y^2}$ , and  $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2} = \frac{a\dot{x}}{2y}$ .

This value of  $\dot{z}$ , the fluxion of a circular arc, may be found more easily thus: In the fig. to art. 60, the two triangles EDC, Eae are equiangular, being each of them equiangular to the triangle ETC: conseq. ED : EC :: Ea : Ee, that is,

$y : \frac{1}{2}a :: \dot{x} : \dot{z} = \frac{a\dot{x}}{2y}$ , the same as before.

The value of  $\dot{z}$  being found, by substitution is obtained  $2cy\dot{z} = ac\dot{x}$  for the fluxion of the spherical surface, generated by the circular arc in revolving about the diameter AD. And the fluent of this gives  $acx$  for the said surface of the spherical segment BAE.

But  $ac$  is equal to the whole circumference of the generating circle; and therefore it follows, that the surface of any spherical segment, is equal to the said circumference of the generating circle, drawn into  $x$  or AD, the height of the segment.

Also, when  $x$  or AD becomes equal to the whole diameter  $a$ , the expression  $acx$  becomes  $aca$  or  $ca^2$ , or 4 times the area of the generating circle, for the surface of the whole sphere.

And these agree with the rules before found in Mensuration of Solids.

EXAM. 2. To find the surface of a spheroid.

EXAM. 3. To find the surface of a paraboloid.

EXAM. 4. To find the surface of an hyperboloid.

## TO FIND THE CONTENTS OF SOLIDS.

66. ANY solid which is formed by the revolution of a curve about its axis (see last fig.), may also be conceived to be generated by the motion of the plane of an expanding circle, moving perpendicularly along the axis. And therefore

fore

fore the area of that circle being drawn into the fluxion of the axis, will produce the fluxion of the solid. That is,  $AD \times$  area of the circle BCF, whose radius is DE, or diameter BE, is the fluxion of the solid, by art. 9.

67. Hence, if  $AD = x$ ,  $DE = y$ ,  $c = 3.1416$ ; because  $cy^2$  is equal to the area of the circle BCF; therefore  $cy^2 \dot{x}$  is the fluxion of the solid. And consequently if, from the given equation of the curve, the value of either  $y^2$  or  $\dot{x}$  be found, and that value substituted for it in the expression by  $cy^2 \dot{x}$ , the fluent of the resulting quantity, being taken, will be the solidity of the figure proposed.

## EXAMPLES.

EXAM. 1. To find the solidity of a sphere, or any segment.

The equation to the generating circle being  $y^2 = ax - x^2$ , where  $a$  denotes the diameter, by substitution, the general fluxion of the solid,  $cy^2 \dot{x}$ , becomes  $cax\dot{x} - cx^2\dot{x}$ , the fluent of which gives  $\frac{1}{2}cax^2 - \frac{1}{3}cx^3$ , or  $\frac{1}{6}cx^2(3a - 2x)$ , for the solid content of the spherical segment BAE, whose height AD is  $x$ .

When the segment becomes equal to the whole sphere, then  $x = a$ , and the above expression for the solidity, becomes  $\frac{1}{6}ca^3$  for the solid content of the whole sphere.

And these deductions agree with the rules before given and demonstrated in the Mensuration of Solids.

EXAM. 2. To find the solidity of a spheroid.

EXAM. 3. To find the solidity of a paraboloid.

EXAM. 4. To find the solidity of an hyperboloid.

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 TO FIND LOGARITHMS.

68. IT has been proved, art. 23, that the fluxion of the hyperbolic logarithm of a quantity, is equal to the fluxion of the quantity divided by the same quantity. Therefore, when any quantity is proposed, to find its logarithm; take the fluxion of that quantity, and divide it by the same quantity; then take the fluent of the quotient, either in a series or otherwise, and it will be the logarithm sought; when corrected as usual, if need be; that is, the hyperbolic logarithm.

69. But, for any other logarithm, multiply the hyperbolic logarithm, above found, by the modulus of the system, for the logarithm sought.

Note,

*Note,* The modulus of the hyperbolic logarithms, is 1; and the modulus of the common logarithms, is .43429448190, &c; and, in general, the modulus of any system, is equal to the logarithm of 10 in that system divided by the number 2.3025850929940, &c, which is the hyp. log. of 10.

EXAM. I. To find the log. of  $\frac{a+x}{a}$ .

Denoting any proposed number  $z$ , whose logarithm is required to be found, by the compound expression  $\frac{a+x}{a}$ , the fluxion of the number  $z$ , is  $\frac{\dot{x}}{a}$ , and the fluxion of the log.  $\frac{z}{z} = \frac{\dot{z}}{a+x} = \frac{\dot{x}}{a} - \frac{x\dot{x}}{a^2} + \frac{x^2\dot{x}}{a^3} - \frac{x^3\dot{x}}{a^4} + \&c.$

Then the fluents of these terms give the logarithm of  $z$  or logarithm of  $\frac{a+x}{a} = \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} \&c.$

writing  $-x$  for  $x$ , gives  $\log. \frac{a-x}{a} = -\frac{x}{a} + \frac{x^2}{2a^2} - \frac{x^3}{3a^3} + \frac{x^4}{4a^4} \&c.$

Mult. these numbs. and adding their logs. gives  $\log. \frac{a+x}{a-x} = \frac{2x}{a} + \frac{2x^3}{3a^3} + \frac{2x^5}{5a^5} \&c.$

Also, because  $\frac{a}{a+x} = 1 \div \frac{a+x}{a}$ , or  $\log. \frac{a}{a+x} = 0 - \log. \frac{a+x}{a}$ .

therefore  $\log. \frac{a}{a+x}$  is  $-\frac{x}{a} + \frac{x^2}{2a^2} - \frac{x^3}{3a^3} + \frac{x^4}{4a^4} \&c,$

and the  $\log. \frac{a}{a-x}$  is  $+\frac{x}{a} + \frac{x^2}{2a^2} + \frac{x^3}{3a^3} + \frac{x^4}{4a^4} \&c,$

the prod. gives  $\log. \frac{a^2}{a^2-x^2} = \frac{2x^2}{2a^2} + \frac{2x^4}{4a^4} + \frac{2x^6}{6a^6} + \&c.$

Now, for an example in numbers, suppose it were required to compute the common logarithm of the number 2. This will be best done by the series,

$$\log. \text{ of } \frac{a+x}{a-x} = 2m \times \left( \frac{x}{a} + \frac{x^3}{3a^3} + \frac{x^5}{5a^5} + \frac{x^7}{7a^7} \&c \right).$$

Making  $\frac{a+x}{a-x} = 2$ , gives  $a = 3x$ ; conseq.  $\frac{x}{a} = \frac{1}{3}$ , and  $\frac{x^2}{a^2} = \frac{1}{9}$ ; which is the constant factor for every succeeding term; also,  $2m = 2 \times .43429448190 = .868588964$ ; therefore the calculation will be conveniently made, by first dividing this number by 3, then the quotients successively by 9, and last-

ly

ly these quotients in order by the respective numbers 1, 3, 5, 7, 9, &c, and after that, adding all the terms together, as follows :

3 ) .868588964	1 ) .289529654	( .289529654
9 ) 289529654	3 ) 32169962	( 10723321
9 ) 32169962	5 ) 3574440	( 714888
9 ) 3574440	7 ) 397160	( 56737
9 ) 397160	9 ) 44129	( 4903
9 ) 44129	11 ) 4903	( 446
9 ) 4903	13 ) 545	( 42
9 ) 545	15 ) 61	( 4
9 ) 61		<hr style="width: 100%;"/>

Sum of the terms gives  $\log. 2 = \underline{\underline{.301029995}}$

EXAM. 2. To find the log. of  $\frac{a+x}{b}$ .

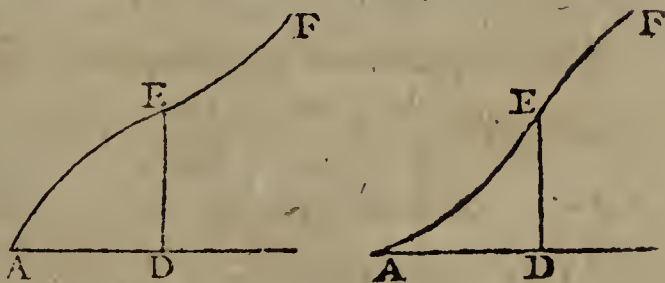
EXAM. 3. To find the log of  $a-x$ .

EXAM. 4. To find the log. of  $\frac{a+x}{a-x}$ .

EXAM. 5. To find the log. of  $\frac{a-x}{a+x}$ .

TO FIND THE POINTS OF INFLEXION,  
OR OF CONTRARY FLEXURE, IN CURVES.

70. THE Point of Inflexion in a curve, is that point of it which separates the concave from the convex part, lying between the two; or



where the curve changes from concave to convex, or from convex to concave, on the same side of the curve. Such as the point E in the annexed figures; where the former of the two is concave towards the axis AD, from A to E, but convex from E to F; and, on the contrary, the latter figure is convex from A to E, and concave from E to F.

71. From

71. From the nature of curvature, as has been remarked before at art. 28, it is evident that, when a curve is concave towards an axis, then the fluxion of the ordinate decreases, or is in a decreasing ratio, with regard to the fluxion of the absciss; but, on the contrary, that it increases, or is in an increasing ratio to the fluxion of the absciss, when the curve is convex towards the axis; and consequently those two fluxions are in a constant ratio at the point of inflexion, where the curve is neither convex nor concave; that is,  $\dot{x}$  is to  $\dot{y}$  in a constant ratio, or  $\frac{\dot{y}}{\dot{x}}$  or  $\frac{\ddot{x}}{\ddot{y}}$  is a constant quantity.

But constant quantities have no fluxion, or their fluxion is equal to nothing; so that, in this case, the fluxion of  $\frac{\dot{y}}{\dot{x}}$  or  $\frac{\ddot{x}}{\ddot{y}}$  is equal to nothing. And hence we have this general rule:

72. Put the given equation of the curve into fluxions; from which find either  $\frac{\dot{y}}{\dot{x}}$  or  $\frac{\ddot{x}}{\ddot{y}}$ . Then take the fluxion of this ratio, or fraction, and put it equal 0 or nothing; and from this last equation find also the value of the same  $\frac{\ddot{x}}{\ddot{y}}$  or  $\frac{\dot{y}}{\dot{x}}$ . Then put this latter value equal to the former, which will form an equation, from whence, and the first given equation, of the curve,  $x$  and  $y$  will be determined, being the absciss and ordinate answering to the point of inflexion in the curve, as required.

#### EXAMPLES.

EXAM. I. To find the point of inflexion in the curve whose equation is  $ax^2 = a^2y + x^2y$ .

This equation in fluxions is  $2ax\dot{x} = a^2\dot{y} + 2xy\dot{x} + x^2\dot{y}$ , which gives  $\frac{\dot{x}}{\dot{y}} = \frac{a^2 + x^2}{2ax - 2xy}$ . Then the fluxion of this quantity made = 0, gives  $2x\dot{x}(ax - xy) = (a^2 + x^2) \times (a\dot{x} - \dot{x}y - x\dot{y})$ ; and this again gives  $\frac{\dot{x}}{\dot{y}} = \frac{a^2 + x^2}{a^2 - x^2} \times \frac{x}{a - y}$ .

Lastly, this value of  $\frac{\dot{x}}{\dot{y}}$  being put equal the former, gives  $\frac{a^2 + x^2}{a^2 - x^2} \cdot \frac{x}{a - y} = \frac{a^2 + x^2}{2x} \cdot \frac{-1}{a - y}$ ; and hence  $2x^2 = a^2 - x^2$ , or  $3x^2 = a^2$ , and  $x = a\sqrt{\frac{1}{3}}$ , the absciss.

Hence



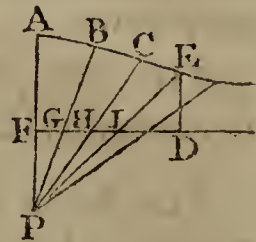
Hence also, from the original equation,

$$y = \frac{ax^2}{a^2 + x^2} = \frac{\frac{1}{3}a^3}{\frac{4}{3}a^2} = \frac{1}{4}a, \text{ the ordinate to the point of inflexion sought.}$$

EXAM. 2. To find the point of inflexion in a curve defined by the equation  $ay = a\sqrt{ax + xx}$ .

EXAM. 3. To find the point of inflexion in a curve defined by the equation  $ay^2 = a^2x + x^3$ .

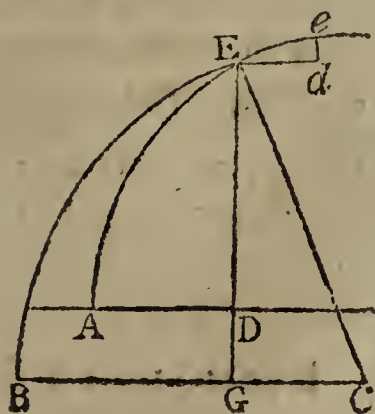
EXAM. 4. To find the point of inflexion in the Conchoid of Nicomedes, which is generated or constructed in this manner: From a fixed point P, which is called the pole of the conchoid, draw any number of right lines PA, PB, PC, PE, &c, cutting the given line FD in the points F, G, H, I, &c: then make the distances FA, GB, HC, IE, &c, equal to each other, and equal to a given line; and the curve line ABCE, &c, will be the conchoid; a curve so called by its inventor Nicomedes,



TO FIND THE RADIUS OF CURVATURE OF CURVES.

73. THE Curvature of a Circle is constant, or the same in every point of it, and its radius is the radius of curvature. But the case is different in other curves, every one of which has its curvature continually varying, either increasing or decreasing, and every point having a degree of curvature peculiar to itself; and the radius of a circle which has the same curvature with the curve at any given point, is the radius of curvature at that point; which radius it is the business of this chapter to find.

74. Let A*Ee* be any curve, concave toward its axis AD; draw an ordinate DE to the point E, where the curvature is required to be found; and suppose EC perpendicular to the curve, and equal to the radius of curvature sought, or equal to the radius of a circle having the same curvature there, and with that radius describe the said equally-curved circle BE*e*;



BEe; lastly, draw Ed parallel to AD, and de parallel and indefinitely near to DE; thereby making Ed the fluxion, or increment of the absciss AD, also de the fluxion of the ordinate DE, and Ee that of the curve AE. Then put  $x = AD$ ,  $y = DE$ ,  $z = AE$ , and  $r = CE$  the radius of curvature; then is  $Ed = \dot{x}$ ,  $de = \dot{y}$ , and  $Ee = \dot{z}$ .

Now, by sim. triangles, the three lines Ed, de, Ee, are respectively as the three  $\dot{x}, \dot{y}, \dot{z}$ , or  $GE, GC, CE$ ; therefore  $GC \cdot \dot{x} = GE \cdot \dot{y}$ ; and the flux. of this eq. is  $GC \cdot \ddot{x} + GC \cdot \dot{x} = GE \cdot \ddot{y} + GE \cdot \dot{y}$ , or, because  $GC = -BG$ , it is,  $GC \cdot \ddot{x} - BG \cdot \dot{x} = GE \cdot \ddot{y} + GE \cdot \dot{y}$ .

But, since the two curves AE and BE have the same curvature at the point E, their abscisses and ordinates have the same fluxions at that point, that is, Ed or  $\dot{x}$  is the fluxion both of AD and BG, and de or  $\dot{y}$  is the fluxion both of DE and GE. In the equation above, therefore, substitute  $\dot{x}$  for BG, and  $\dot{y}$  for GE, and it becomes

$$\text{or } GC\ddot{x} - \dot{x}\dot{x} = GE\ddot{y} + \dot{y}\dot{y},$$

$$GC\ddot{x} - GE\ddot{y} = \dot{x}^2 + \dot{y}^2 = \dot{z}^2.$$

Now, multiply the three terms of this equation respectively by these three quantities,  $\frac{\dot{y}}{GC}, \frac{\dot{x}}{GE}, \frac{\dot{z}}{CE}$ , which are equal, and it becomes  $\dot{y}\ddot{x} - \dot{x}\ddot{y} = \frac{\dot{z}^3}{CE}$  or  $\frac{\dot{z}^3}{r}$ ;

and hence is found  $r = \frac{\dot{z}^3}{\dot{y}\ddot{x} - \dot{x}\ddot{y}}$ , for the general value of the radius of curvature, for all curves whatever, in terms of the fluxions of the absciss and ordinate.

75. Farther, as in any case either  $x$  or  $y$  may be supposed to flow equably, that is, either  $\dot{x}$  or  $\dot{y}$  constant quantities, or  $\ddot{x}$  or  $\ddot{y}$  equal to nothing, it follows that, by this supposition, either of the terms in the denominator, of the value of  $r$ , may be made to vanish. Thus, when  $\dot{x}$  is supposed constant,  $\ddot{x}$  being then = 0, the value of  $r$  is barely  $\frac{\dot{z}^3}{-\dot{x}\ddot{y}}$ ; or  $r$  is =  $\frac{\dot{z}^3}{\dot{y}\ddot{x}}$  when  $\dot{y}$  is constant.

EXAMPLES.

EXAM.-I. To find the radius of curvature to any point of

of a parabola, whose equation is  $ax = y^2$ , its vertex being A, and axis AD.

Now, the equation to the curve being  $ax = y^2$ ; the fluxion of it is  $a\dot{x} = 2y\dot{y}$ ; and the fluxion of this again is  $a\ddot{x} = 2\dot{y}^2$ , supposing  $\dot{y}$  constant; hence then  $r$  or

$$\frac{\dot{z}^3}{y\ddot{x}} \text{ or } \frac{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}{y\ddot{x}} \text{ is } = \frac{(a^2 + 4y^2)^{\frac{3}{2}}}{2a^2} \text{ or } \frac{(a + 4x)^{\frac{3}{2}}}{2\sqrt{a}},$$

for the general value of the radius of curvature at any point E, the ordinate to which cuts off the absciss AD =  $x$ .

Hence, when the absciss  $x$  is nothing, the last expression becomes barely  $\frac{1}{2}a = r$ , for the radius of curvature at the vertex of the parabola; that is, the diameter of the circle of curvature at the vertex of a parabola, is equal to  $a$ , the parameter of the axis.

EXAM. 2. To find the radius of curvature of an ellipse, whose equation is  $a^2y^2 = c^2 \cdot ax - x^2$ .

$$\text{Ans. } r = \frac{(a^2c^2 + 4(a^2 - c^2) \times (ax - x^2))^{\frac{3}{2}}}{2a^4c}.$$

EXAM. 3. To find the radius of curvature of an hyperbola, whose equation is  $a^2y^2 = c^2 \cdot ax + x^2$ .

EXAM. 4. To find the radius of curvature of the cycloid.

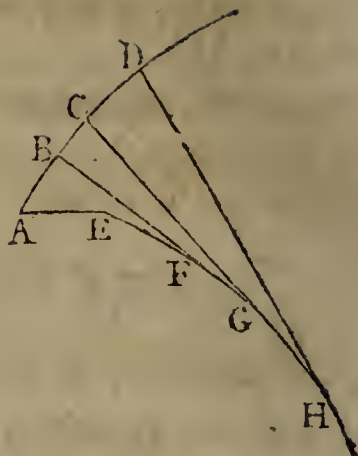
Ans.  $r = 2\sqrt{aa - ax}$ , where  $x$  is the absciss, and  $a$  the diameter of the generating circle.

## OF INVOLUTE AND EVOLUTE CURVES.

76. AN. Evolute is any curve supposed to be evolved or opened, by having a thread wrapped close about it, fastened at one end, and beginning to evolve or unwind the thread from the other end, keeping always tight stretched the part which is evolved or wound off: then this end of the thread will describe another curve, called the Involute. Or, the same involute is described in the contrary way, by wrapping the thread about the curve of the evolute, keeping it at the same time always stretched.

77. Thus,

77. Thus, if EFGH be any curve, and AE be either a part of the curve, or a right line: then if a thread be fixed to the curve at H, and be wound or plyed close to the curve, &c, from H to A, keeping the thread always stretched tight; the other end of the thread will describe a certain curve ABCD, called an Involute; the first curve EFGH being its evolute. Or, if the thread, fixed at H, be unwound from the curve, beginning at A, and keeping it always tight, it will describe the same involute ABCD.



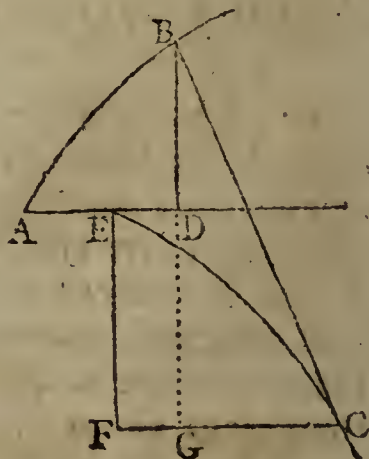
78. If AE, BF, CG, DH, &c, be any positions of the thread, in evolving or unwinding; it follows, that these parts of the thread are always the radii of curvature, at the corresponding points, A, B, C, D; and also equal to the corresponding lengths AE, AEF, AEEFG, AEEFGH, of the evolute; that is,

$$\begin{aligned} AE &= AE \text{ is the radius of curvature to the point A,} \\ BF &= AF \text{ is the radius of curvature to the point B,} \\ CG &= AG \text{ is the radius of curvature to the point C,} \\ DH &= AH \text{ is the radius of curvature to the point D.} \end{aligned}$$

79. It also follows, from the premises, that any radius of curvature, BF, is perpendicular to the involute at the point B, and is a tangent to the evolute curve at the point F. Also, that the evolute is the locus of the centre of curvature of the involute curve.

80. Hence, and from art. 74, it will be easy to find one of these curves, when the other is given. To this purpose, put

$$\begin{aligned} x &= AD, \text{ the absciss of the involute,} \\ y &= DB, \text{ an ordinate to the same,} \\ z &= AB, \text{ the involute curve,} \\ r &= BC, \text{ the radius of curvature,} \\ v &= EF, \text{ the absciss of the evolute EC,} \\ u &= FC, \text{ the ordinate of the same, and} \\ a &= AE, \text{ a certain given line.} \end{aligned}$$



Then,

Then, by the nature of the radius of curvature, it is

$$r = \frac{\dot{z}^3}{\dot{y}\ddot{x} - \dot{x}\ddot{y}} = BC = AE + EC; \text{ also, by sim triangles,}$$

$$\dot{z} : \dot{x} :: r : GB = \frac{r\ddot{x}}{\dot{z}} = \frac{\dot{x}\dot{z}^2}{\dot{y}\ddot{x} - \dot{x}\ddot{y}};$$

$$\dot{z} : \dot{y} :: r : GC = \frac{r\ddot{y}}{\dot{z}} = \frac{\dot{y}\dot{z}^2}{\dot{y}\ddot{x} - \dot{x}\ddot{y}}.$$

$$\text{Hence } EF = GB - DB = \frac{\dot{x}\dot{z}^2}{\dot{y}\ddot{x} - \dot{x}\ddot{y}} - y = v;$$

$$\text{and } FC = AD - AE + GC = x - a + \frac{\dot{y}\dot{z}^2}{\dot{y}\ddot{x} - \dot{x}\ddot{y}} = u;$$

which are the values of the absciss and ordinate of the evolute curve EC; from which, therefore, these may be found, when the involute is given.

On the contrary, if  $v$  and  $u$ , or the evolute, be given: then, putting the given curve  $EC = s$ ; since  $CB = AE + EC$ , or  $r = a + s$ , this gives  $r$  the radius of curvature. Also, by similar triangles, there arise these proportions, viz.

$$\dot{s} : \dot{v} :: r : \frac{r\dot{v}}{s} = \frac{a + s}{s} \dot{v} = GB,$$

$$\text{and } \dot{s} : \dot{u} :: r : \frac{r\dot{u}}{s} = \frac{a + s}{s} \dot{u} = GC;$$

$$\text{theref. } AD = AE + FC - GC = a + u - \frac{a + s}{s} \dot{u} = x,$$

$$\text{and } DB = GB - GD = GB - EF = \frac{a + s}{s} \dot{v} - v = y;$$

which are the absciss and ordinate of the involute curve, and which may therefore be found, when the evolute is given!

Where it may be noted, that  $\dot{s}^2 = \dot{v}^2 + \dot{u}^2$ , and  $\dot{z}^2 = \dot{x}^2 + \dot{y}^2$ . Also, either of the quantities  $x, y$ , may be supposed to flow equably, in which case the respective second fluxion,  $\ddot{x}$  or  $\ddot{y}$ , will be nothing, and the corresponding term in the denominator  $\dot{y}\ddot{x} - \dot{x}\ddot{y}$  will vanish, leaving only the other term in it; which will have the effect of rendering the whole operation simpler.

### 81. EXAMPLES.

EXAM. I. To determine the nature of the curve by whose evolution the common parabola AB is described.

VOL. II.

Y

Here

Here the equation of the given involute AB, is  $cx = y^2$ , where  $c$  is the parameter of the axis AD. Hence then  $y = \sqrt{cx}$ , and  $\dot{y} = \frac{1}{2}\dot{x}\sqrt{\frac{c}{x}}$ , also  $\ddot{y} = \frac{-\dot{x}^2}{4x}\sqrt{\frac{c}{x}}$  by making  $\dot{x}$  constant. Consequently the general values of  $v$  and  $u$ , or of the absciss and ordinate, EF and FC, above given, become, in that case,

$$EF = v = \frac{\dot{z}^2}{-\ddot{y}} - y = \frac{\dot{x}^2 + \dot{y}^2}{-\ddot{y}} - y = 4x\sqrt{\frac{x}{c}}; \text{ and}$$

$$FC = u = x - a + \frac{y\dot{z}^2}{-\dot{x}\ddot{y}} = 3x + \frac{1}{2}c - a.$$

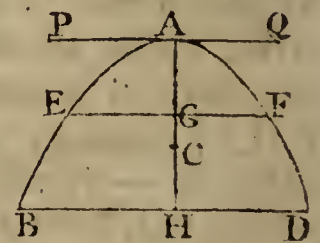
But the value of the quantity  $a$  or AE, by exam. 1 to art. 75, was found to be  $\frac{1}{2}c$ ; consequently the last quantity, FC or  $u$ , is barely  $= 3x$ .

Hence then, comparing the values of  $v$  and  $u$ , there is found  $3v\sqrt{c} = 4u\sqrt{x}$ , or  $27cv^2 = 16u^3$ ; which is the equation between the absciss and ordinate of the evolute curve EC, shewing it to be the semicubical parabola.

EXAM. 2. To determine the evolute of the common cycloid.      Anf. another cycloid, equal to the former.

### TO FIND THE CENTRE OF GRAVITY.

82. By referring to prop. 41, &c, in Mechanics, it is seen what are the principles and nature of the Centre of Gravity in any figure, and how it is generally expressed. It there appears, that, if PAQ be a line, or plane, drawn through any point, as suppose the vertex of any body, or figure, ABD, and if



$s$  denote any section EF of the figure,  $d = AG$ , its distance below PQ, and  $b =$  the whole body or figure ABD; then the distance AC, of the centre of

gravity below PQ, is universally denoted by  $\frac{\text{sum of all the } ds}{b}$ ;

whether ABD be a line, or a plane surface, or a curve superficies, or a solid.

But

But the sum of all the  $ds$ , is the same as the fluent of  $db$ , and  $b$  is the same as the fluent of  $\dot{b}$ ; therefore the general expression for the distance of the centre of gravity, is  $AC = \frac{\text{fluent of } x\dot{b}}{\text{fluent of } \dot{b}} = \frac{\text{fluent } x\dot{b}}{b}$ ; putting  $x = d$  the variable distance  $AG$ . Which will divide into the following four cases.

83. CASE 1. When  $AE$  is some line, as a curve suppose. In this case,  $\dot{b}$  is  $= \dot{z}$  or  $\sqrt{\dot{x}^2 + \dot{y}^2}$ , the fluxion of the curve; and  $b = z$ : theref.  $AC = \frac{\text{fluent of } x\dot{z}}{z} = \frac{\text{fluent of } x\sqrt{\dot{x}^2 + \dot{y}^2}}{z}$  is the distance of the centre of gravity in a curve.

84. CASE 2. When the figure  $ABD$  is a plane; then  $\dot{b} = y\dot{x}$ ; therefore the general expression becomes  $AC = \frac{\text{fluent of } yx\dot{x}}{\text{fluent of } y\dot{x}}$  for the distance of the centre of gravity in a plane.

85. CASE 3. When the figure is the superficies of a body generated by the rotation of a line  $AEB$ , about the axis  $AH$ .

Then, putting  $c = 3.14159$  &c,  $2cy$  will denote the circumference of the generating circle, and  $2cy\dot{z}$  the fluxion of the surface; therefore  $AC = \frac{\text{fluent of } 2cyx\dot{z}}{\text{fluent of } 2cy\dot{z}} = \frac{\text{fluent of } yx\dot{z}}{\text{fluent of } y\dot{z}}$  will be the distance of the centre of gravity for a surface generated by the rotation of a curve line  $z$ .

86. CASE 4. When the figure is a solid generated by the rotation of a plane  $ABH$ , about the axis  $AH$ .

Then, putting  $c = 3.14159$  &c, it is  $cy^2 =$  the area of the circle whose radius is  $y$ , and  $cy^2\dot{x} = \dot{b}$ , the fluxion of the solid; therefore

$$AC = \frac{\text{fluent of } x\dot{b}}{\text{fluent of } \dot{b}} = \frac{\text{fluent of } cy^2x\dot{x}}{\text{fluent of } cy^2\dot{x}} = \frac{\text{fluent of } y^2x\dot{x}}{\text{fluent of } y^2\dot{x}}$$

is the distance of the centre of gravity below the vertex in a solid.

87. EXAMPLES.

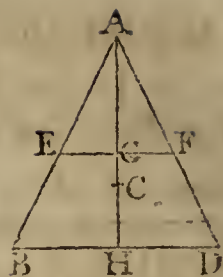
EXAM. 1. Let the figure proposed be the isosceles triangle  $ABD$ .

It is evident that the centre of gravity  $C$ , will be somewhere

where in the perpendicular AH. Now, if  $a$  denote AH,  $c = BD$ ,  $x = AG$ , and  $y = EF$  any line parallel to the base BD:

then as  $a : c :: x : y = \frac{cx}{a}$ ; therefore, by the

2d Case,  $AC = \frac{\text{fluent } yx\dot{x}}{\text{fluent } y\dot{x}} = \frac{\text{fluent } x^2\dot{x}}{\text{fluent } x\dot{x}} = \frac{\frac{1}{3}x^3}{\frac{1}{2}x^2}$   
 $= \frac{2}{3}x = \frac{2}{3}AH$ , when  $x$  becomes  $= AH$ : consequently  
 $CH = \frac{1}{3}AH$ .



In like manner, the centre of gravity of any other plane triangle, will be found to be at  $\frac{1}{3}$  of the altitude of the triangle; the same as it was found in prop. 42, Mechanics.

EXAM. 2. In a parabola; the distance from the vertex is  $\frac{3}{5}x$ , or  $\frac{3}{5}$  of the axis.

EXAM. 3. In a circular arc; the distance from the centre of the circle, is  $\frac{ac}{r}$ ; where  $a$  denotes the arc,  $c$  its chord, and  $r$  the radius.

EXAM. 4. In a circular sector; the distance from the centre of the circle, is  $\frac{cr}{a}$ : where  $a, c, r$ , are the same as in exam. 3.

EXAM. 5. In a circular segment; the distance from the centre of the circle is  $\frac{c^3}{12a}$ ; where  $c$  is the chord, and  $a$  the area, of the segment.

EXAM. 6. In a cone, or any other pyramid; the distance from the vertex is  $\frac{3}{4}x$ , or  $\frac{3}{4}$  of the altitude.

EXAM. 7. In the semisphere, or semispheroid; the distance from the centre is  $\frac{3}{8}r$ , or  $\frac{3}{8}$  of the radius.

EXAM. 8. In the parabolic conoid; the distance from the base is  $\frac{3}{5}x$ , or  $\frac{3}{5}$  of the axis.

EXAM. 9. In the segment of a sphere, or of a spheroid; the distance from the base is  $\frac{2a - x}{6a - 4x}x$ ; where  $x$  is the height of the segment, and  $a$  the whole axis, or diameter of the sphere.

EXAM. 10. In the hyperbolic conoid; the distance from the base is  $\frac{2a + x}{6a + 4x}x$ ; where  $x$  is the height of the conoid, and  $a$  the whole axis or diameter.

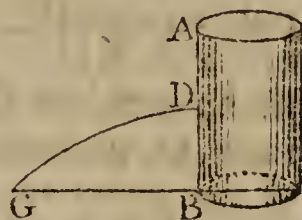


PRACTICAL QUESTIONS.

QUESTION I.

A LARGE vessel, of 10 feet, or any other given depth, and of any shape, being kept constantly full of water, by means of a supplying cock at the top; it is proposed to assign the place where a small hole must be made in the side of it, so that the water may spout through it to the greatest distance on the plane of the base.

Let AB denote the height or side of the vessel; D the required hole in the side, from whence the water spouts, in the parabolic curve DG, to the greatest distance BG, on the horizontal plane.



By the Scholium to prop. 61, Hydraulics, the distance BG is always equal to  $2\sqrt{AD \cdot DB}$ , which is equal to

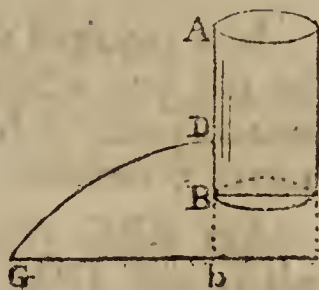
$2\sqrt{x(a-x)}$  or  $2\sqrt{ax-x^2}$ , if  $a$  be put to denote the whole height AB of the vessel, and  $x = AD$  the depth of the hole.

Hence  $2\sqrt{ax-x^2}$ , or  $ax-x^2$ , must be a maximum. In fluxions,  $a\dot{x} - 2x\dot{x} = 0$ , or  $a - 2x = 0$ , and  $2x = a$ , or  $x = \frac{1}{2}a$ . So that the hole D must be in the middle between the top and bottom; the same as before found at the end of the scholium above quoted.

QUESTION II.

If the same vessel, as in Quest. I, stand on high, with its bottom a given height above a horizontal plane below; it is proposed to determine where the small hole must be made, so as to spout farthest on the said plane.

Let the annexed figure represent the vessel as before, and bG the greatest distance spouted by the fluid DG, on the plane bG.



Here, as before,  $bG = 2\sqrt{AD \cdot Db}$   $= 2\sqrt{x(c-x)}$   $= 2\sqrt{cx-x^2}$ , by putting  $Ab = c$ , and  $AD = x$ . So that

$2\sqrt{cx-x^2}$  or  $cx-x^2$  must be a maximum. And hence, like as in the former question,

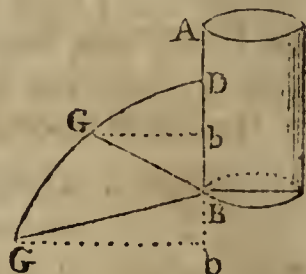
$x = \frac{1}{2}c = \frac{1}{2}AB$ . So that the hole D must be made in the middle

middle between the top of the vessel, and the given plane, that the water may spout farthest.

## QUESTION III.

But if the same vessel as before, stand on the top of an inclined plane, making a given angle, as suppose of 30 degrees, with the horizon; it is proposed to determine the place of the small hole, so as the water may spout the farthest on the said inclined plane.

Here again (D being the place of the hole, and BG the given inclined plane),  $bG = 2\sqrt{AD}$ .  $Db = 2\sqrt{x(a-x+z)}$ , putting  $z = Bb$ , and, as before,  $a = AB$ , and  $x = AD$ . Then  $bG$  must still be a maximum, as also  $Bb$ , being in a given ratio to the maximum  $BG$ , on account of the given angle B. Therefore  $ax - x^2 + xz$ , as well as  $z$ , is a maximum. Hence, by art. 54 of



the Fluxions,  $ax - 2x\dot{x} + z\dot{x} = 0$ , or  $a - 2x + z = 0$ ; conseq.  $z = 2x - a$ ; and hence  $bG = 2\sqrt{x(a-x+z)}$  becomes barely  $2x$ . But, as the given angle  $GBb$  is  $= 30^\circ$ , the sine of which is  $\frac{1}{2}$ ; therefore  $BG = 2Bb$  or  $2z$ , and  $bG^2 = BG^2 - Bb^2 = 3z^2 = 3(2x-a)^2$ , or  $bG = \pm(2x-a)\sqrt{3}$ .

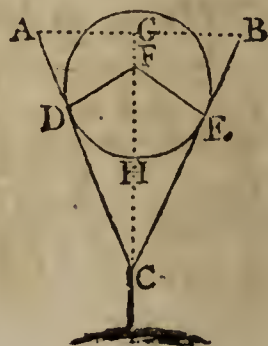
Putting now these two values of  $bG$  equal to each other, gives the equation  $2x = \pm(2x-a)\sqrt{3}$ , from which is found  $x = \frac{\frac{1}{2}a\sqrt{3}}{\sqrt{3}+1} = \frac{3+\sqrt{3}}{4}a$ , the value of AD required.

*Note.* In the Select Exercises, page 165, this answer is brought out  $\frac{6+\sqrt{6}}{10}a$ , by taking the velocity as the root of half the altitude.

## QUESTION IV.

It is required to determine the size of a ball, which, being let fall into a conical glass full of water, shall expel the most water possible from the glass; its depth being 6, and diameter 5 inches.

Let ABC represent the cone of the glass, and DHE the ball, touching the sides in the points D and E, the centre of the ball being at some point F in the axis GC of the cone.



Put

Put  $AG = GB = 2\frac{1}{2} = a,$   
 $GC = 6 = b,$

$AC = \sqrt{AG^2 + GC^2} = 6\frac{1}{2} = c,$

$FD = FE = FH = x$  the radius of the ball.

The two triangles  $ACG$  and  $DCF$  are equiangular; theref.

$AG : AC :: DF : FC$ , that is,  $a : c :: x : \frac{cx}{a} = FC$ ; hence

$GF = GC - FC = b - \frac{cx}{a}$ , and  $GH = GF + FH = b + x - \frac{cx}{a}$ ,

the height of the segment immersed in the water. Then (by rule 1 for the spherical segment, page 48), the content of the said immersed segment will be  $(6DF - 2GH) \times GH^2$

$\times .5236 = (2x - b + \frac{cx}{a}) \times (x + b - \frac{cx}{a})^2 \times 1.0472,$

which must be a maximum by the question; the fluxion of this made  $= 0$ , and divided by  $2x$  and the common factors,

gives  $\frac{2a + c}{a} \times (b - \frac{c - a}{a}x) - (\frac{2a + c}{a}x - b) \times \frac{c - a}{a} \times 2 = 0;$

this reduced gives  $x = \frac{abc}{(c - a) \times (c + 2a)} = 2\frac{1}{3}\frac{1}{2}$ , the radius of

the ball. Consequently its diameter is  $4\frac{1}{3}\frac{1}{2}$  inches, as required.

PRACTICAL EXERCISES, CONCERNING FORCES;  
 WITH THE RELATION BETWEEN THEM AND THE  
 TIME, VELOCITY, AND SPACE DESCRIBED.

BEFORE entering on the following problems, it will be convenient here, to lay down a synopsis of the theorems which express the several relations between any forces, and their corresponding times, velocities, and spaces described; which are all comprehended in the following 12 theorems, as collected from the principles in the foregoing parts of this work.

Let  $f, F$  be any two constant accelerative forces, acting on any body, during the respective times  $t, T$ , at the end of which are generated the velocities  $v, V$ , and described the spaces  $s, S$ . Then, because the spaces are as the times and velocities conjointly, and the velocities as the forces and times; we shall have,

I. *In Constant Forces.*

$$\begin{aligned}
 1. \quad \frac{s}{S} &= \frac{tv}{TV} = \frac{t^2 f}{T^2 F} = \frac{v^2 F}{V^2 f} \\
 2. \quad \frac{v}{V} &= \frac{ft}{FT} = \frac{sT}{St} = \sqrt{\frac{fs}{FS}} \\
 3. \quad \frac{t}{T} &= \frac{Fv}{fV} = \frac{sV}{Sv} = \sqrt{\frac{Es}{fS}} \\
 4. \quad \frac{f}{F} &= \frac{Tv}{tV} = \frac{T^2 s}{t^2 S} = \frac{v^2 S}{V^2 s}
 \end{aligned}$$

And if one of the forces, as  $F$ , be the force of gravity at the surface of the earth, and be called  $I$ , and its time  $T$  be  $= 1''$ ; then it is known by experiment that the corresponding space  $S$  is  $= 16\frac{1}{2}$  feet, and consequently its velocity  $V = 2S = 32\frac{1}{2}$ , which call  $2g$ , namely,  $g = 16\frac{1}{2}$  feet, or 193 inches. Then the above four theorems, in this case, become as here below:

$$\begin{aligned}
 5. \quad s &= \frac{1}{2}tv = \frac{1}{2}ft^2 = \frac{v^2}{4gf} \\
 6. \quad v &= \frac{2s}{t} = 2gft = \sqrt{4gfs} \\
 7. \quad t &= \frac{2s}{v} = \frac{v}{2gf} = \sqrt{\frac{s}{gf}} \\
 8. \quad f &= \frac{v}{2gt} = \frac{s}{gt^2} = \frac{v^2}{4gs}
 \end{aligned}$$

And from these are deduced the following four theorems, for variable forces, viz.

II. *In Variable Forces.*

$$\begin{aligned}
 9. \quad \dot{s} &= v\dot{t} = \frac{v\dot{v}}{2gf} \\
 10. \quad \dot{v} &= 2g\dot{f}t = \frac{2gfs\dot{s}}{v} \\
 11. \quad \dot{t} &= \frac{\dot{s}}{v} = \frac{\dot{v}}{2gf} \\
 12. \quad \dot{f} &= \frac{v\dot{v}}{2gs} = \frac{\dot{v}}{2gt}
 \end{aligned}$$

In

In these last four theorems, the force  $f$ , though variable, is supposed to be constant for the indefinitely small time  $t$ , and they are to be used in all cases of variable forces, as the former ones in constant forces; namely, from the circumstances of the problem under consideration, an expression is deduced for the value of the force  $f$ , which being substituted in one of these theorems, that may be proper to the case in hand; the equation thence resulting will determine the corresponding values of the other quantities, required in the problem.

When a motive force happens to be concerned in the question, it may be proper to observe, that the motive force  $m$ , of a body, is equal to  $f q$ , the product of the accelerative force, and the quantity of matter in it  $q$ ; and the relation between these three quantities being universally expressed by this equation  $m = qf$ , it follows that, by means of it, any one of the three may be expelled out of the calculation, or else brought into it.

Also, the momentum, or quantity of motion in a moving body, is  $qv$ , the product of the velocity and matter.

It is also to be observed, that the theorems equally hold good for the destruction of motion and velocity, by means of retarding forces, as for the generation of the same, by means of accelerating forces.

And to the following problems, which are all resolved by the application of these theorems, it has been thought proper to subjoin their solutions, for the better information and convenience of the student.

## PROBLEM I.

*To determine the time and velocity of a body descending, by the force of gravity, down an inclined plane; the length of the plane being 20 feet, and its height 1 foot.*

HERE, by Mechanics, the force of gravity being to the force down the plane, as the length of the plane is to its height, therefore as  $20 : 1 :: 1$  (the force of gravity)  $: \frac{1}{20} = f$ , the force on the plane.

Therefore, by theor. 6,  $v$  or  $\sqrt{4gfs}$  is  $\sqrt{4 \times 16\frac{1}{2} \times \frac{1}{20} \times 20} = \sqrt{4 \times 16\frac{1}{2}} = 2 \times 4\frac{1}{8} = 8\frac{1}{8}$  feet, the last velocity per second. And,

By theor. 7,  $t$  or  $\sqrt{\frac{s}{gf}}$ , is  $\sqrt{\frac{20}{16\frac{1}{2} \times \frac{1}{20}}} = \sqrt{\frac{400}{16\frac{1}{2}}} = \frac{20}{4\frac{1}{8}}$

$4\frac{1}{8} = 4\frac{7}{8}$  seconds, the time of descending.

## PROBLEM II.

If a cannon ball be fired with a velocity of 1000 feet per second, up a smooth inclined plane, which rises 1 foot in 20: it is proposed to assign the length which it will ascend up the plane, before it stops and begins to return down again, and the time of its ascent.

Here  $f = \frac{1}{20}$  as before.

Then, by theor. 5,  $s = \frac{v^2}{4gf} = \frac{1000^2}{4 \times 16\frac{1}{2} \times \frac{1}{20}} = \frac{60000000}{193}$   
 $= 310880\frac{1}{3}$  feet, or nearly 59 miles, the distance moved.

And, by theor. 7,  $t = \frac{v}{2gf} = \frac{1000}{2 \times 16\frac{1}{2} \times \frac{1}{20}} = \frac{120000}{193} =$   
 $621\frac{1}{3} = 10' 21\frac{1}{3}$ , the time of ascent.

## PROBLEM III.

If a ball be projected up a smooth inclined plane, which rises 1 foot in 10, and ascend 100 feet before it stop: required the time of ascent, and the velocity of projection.

FIRST, by theor. 6,  $v = \sqrt{4gfs} = \sqrt{4 \times 16\frac{1}{2} \times \frac{1}{10} \times 100} = 8\frac{1}{8} \sqrt{10} = 25.36408$  feet per second, the velocity.

And, by theor. 7,  $t = \sqrt{\frac{s}{gf}} = \sqrt{\frac{100}{16\frac{1}{2} \times \frac{1}{10}}} = \frac{10}{4\frac{1}{96}} \sqrt{10} =$   
 $\frac{192}{77} \sqrt{10} = 7.88516$  seconds, the time in motion.

## PROBLEM IV.

If a ball be observed to ascend up a smooth inclined plane, 100 feet in ten seconds, before it stop, to return back again: required the velocity of projection, and the angle of the plane's inclination.

FIRST, by theor. 6,  $v = \frac{2s}{t} = \frac{200}{10} = 20$  feet per second, the velocity.

And, by theor. 8,  $f = \frac{s}{gt^2} = \frac{100}{16\frac{1}{2} \times 100} = \frac{12}{193}$ . That is, the length of the plane is to its height, as 193 to 12.

Therefore,  $193 : 12 :: 100 : 6.2176$  the height of the plane, or the sine of elevation to radius 100, which answers to  $3^\circ 34'$ , the angle of elevation of the plane.

PROBLEM V.

By a mean of several experiments, I have found, that a cast iron ball, of 2 inches diameter, impinging perpendicularly on the face or end of a block of elm wood, or in the direction of the fibres, with a velocity of 1500 feet per second, penetrated 13 inches deep into its substance. It is proposed then to determine the time of the penetration, and the resisting force of the wood, as compared to the force of gravity, supposing that force to be a constant quantity.

FIRST, by theor. 7,  $t = \frac{2s}{v} = \frac{2 \times 13}{1500 \times 12} = \frac{1}{692}$  part of a second, the time in penetrating.

And, by theor. 8,  $f = \frac{v^2}{4gs} = \frac{1500^2}{4 \times 16\frac{1}{2} \times \frac{13}{12}} = \frac{81000000}{13 \times 193} = 32284$ . That is, the resisting force of the wood, is to the force of gravity, as 32284 to 1.

But this number will be different, according to the diameter of the ball, and its density or specific gravity. For, since  $f$  is as  $\frac{v^2}{s}$  by theor. 4, the density and size of the ball remaining the same; if the density, or specific gravity,  $n$ , vary, and all the rest be constant, it is evident that  $f$  will be as  $n$ ; and therefore  $f$  as  $\frac{nv^2}{s}$  when the size of the ball only is constant. But when only the diameter  $d$  varies, all the rest being constant, the force of the blow will vary as  $d^3$ , or as the magnitude of the ball; and the resisting surface, or force of resistance, varies as  $d^2$ ; therefore  $f$  is as  $\frac{d^3}{d^2}$ , or as  $d$  only when all the rest are constant. Consequently  $f$  is as  $\frac{dnv^2}{s}$  when they are all variable.

And so  $\frac{f}{F} = \frac{dnv^2S}{DNV^2s}$ , and  $\frac{s}{S} = \frac{dnv^2F}{DNV^2f}$ ; where  $f$  denotes the strength or firmness of the substance penetrated, and is here supposed to be the same, for all balls and velocities, in the same substance, which it is either accurately or nearly so. See page 264, &c, of my Tracts, vol. i.

Hence, taking the numbers in the problem, it is - - -  
 $f = \frac{dnv^2}{s} = \frac{\frac{2}{12} \times 7\frac{1}{3} \times 1500^2}{\frac{1}{12}} = \frac{44 \times 1500^2}{39} = 2538462$   
 the value of  $f$  for elm wood. Where the specific gravity of the

the ball is taken  $7\frac{1}{3}$ , which is a little less than that of solid cast iron, as it ought, on account of the air bubble which is found in all cast balls.

## PROBLEM VI.

*To find how far a 24lb ball of cast iron will penetrate into a block of sound elm, when fired with a velocity of 1600 feet per second.*

HERE, because the substance is the same as in the last problem, both of the balls and wood,  $N = n$ , and  $F = f$ ; therefore  $\frac{S}{s} = \frac{DV^2}{dv^2}$ , or  $S = \frac{DV^2s}{dv^2} = \frac{5.55 \times 1600^2 \times 13}{2 \times 1500^2} = 41\frac{2}{3}$  inches nearly, the penetration required.

## PROBLEM VII.

*It was found by Mr. Robins (vol. i. p. 273, of his works), that an 18 pounder ball, fired with a velocity of 1200 feet per second, penetrated 34 inches into sound dry oak. It is required to ascertain the comparative strength or firmness of oak and elm.*

THE diameter of an 18lb ball is 5.04 inches = D. Then, by the numbers given in this problem for oak, and in prob. 5 for elm, we have

$$\frac{f}{F} = \frac{dv^2 S}{DV^2 s} = \frac{2 \times 1500^2 \times 34}{5.04 \times 1200^2 \times 13} = \frac{100 \times 17}{5.04 \times 16 \times 13} = \frac{1700}{1048}$$

or =  $\frac{8}{5}$  nearly.

From which it would seem, that elm timber resists more than oak, in the ratio of about 8 to 5; which is not probable, as oak is a much firmer and harder wood. But it is to be suspected that Mr. Robins's great penetration was owing to the splitting of his timber in some degree.

## PROBLEM VIII.

*A 24 pounder ball being fired into a bank of firm earth, with a velocity of 1300 feet per second, penetrated 15 feet: It is required then to ascertain the comparative resistances of elm and earth.*

COMPARING the numbers here with those in prob. 5, it is

$$\frac{f}{F}$$



$$\frac{f}{F} = \frac{dv^2 S}{DV^2 s} = \frac{2 \times 1500^2 \times 15 \times 12}{5.55 \times 1300^2 \times 13} = \frac{15^2 \times 24}{13^3 \times 0.37} = \frac{1.8^{\circ} 7^{\circ} 1^{\circ}}{2^{\circ} 3^{\circ}} \text{ nearly} = 6\frac{2}{3} \text{ nearly.}$$

That is, elm timber resists about  $6\frac{2}{3}$  times more than earth.

PROBLEM IX.

To determine how far a leaden bullet, of  $\frac{3}{4}$  of an inch diameter, will penetrate dry elm; supposing it fired, with a velocity of 1700 feet per second, and that the lead does not change its figure by the stroke against the wood.

HERE  $D = \frac{3}{4}$ ,  $N = 11\frac{1}{3}$ ,  $n = 7\frac{1}{3}$ . Then, by the numbers and theorem in prob. 5, it is  $S =$  - - - - -

$$\frac{DNV^2 s}{dnv^2} = \frac{\frac{3}{4} \times 11\frac{1}{3} \times 1700^2 \times 13}{2 \times 7\frac{1}{3} \times 1500^2} = \frac{17^3 \times 13}{200 \times 33} = \frac{63869}{6600} = 9\frac{2}{3} \text{ inches nearly, the depth of penetration.}$$

But as Mr. Robins found this penetration, by experiment, to be only 5 inches; it follows, either that his timber must have resisted about twice as much; or else, which is much more probable, that the defect in his penetration arose from the change of figure in the leaden ball, from the blow against the wood.

PROBLEM X.

A one pound ball, projected with a velocity of 1500 feet per second, having been found to penetrate 13 inches deep into dry elm: It is required to ascertain the time of passing through every single inch of the 13, and the velocity lost at each of them; supposing the resistance of the wood constant or uniform.

THE velocity  $v$  being 1500 feet, or  $1500 \times 12 = 18000$  inches, and velocities and times being as the roots of the spaces, in constant retarding forces, as well as in accelerating ones, and  $t$  being  $= \frac{2s}{v} = \frac{26}{12 \times 1500} = \frac{13}{9000} = \frac{1}{692}$  part of a second, the whole time of passing through the 13 inches; therefore as

$$\sqrt{13} : \sqrt{13} - \sqrt{12} :: v :$$

	veloc. loft		Time in the
$\frac{\sqrt{13} - \sqrt{12}}{\sqrt{13}}$	$v = 58.9$	$:: t : \frac{\sqrt{13} - \sqrt{12}}{\sqrt{13}}$	$t = .00005$ 1st
$\frac{\sqrt{12} - \sqrt{11}}{\sqrt{13}}$	$v = 61.4$	$:: t : \frac{\sqrt{12} - \sqrt{11}}{\sqrt{13}}$	$t = .00006$ 2d
$\frac{\sqrt{11} - \sqrt{10}}{\sqrt{13}}$	$v = 64.2$	&c $\frac{\sqrt{11} - \sqrt{10}}{\sqrt{13}}$	$t = .00006$ 3d
$\frac{\sqrt{10} - \sqrt{9}}{\sqrt{13}}$	$v = 67.5$	$\frac{\sqrt{10} - \sqrt{9}}{\sqrt{13}}$	$t = .00007$ 4th
$\frac{\sqrt{9} - \sqrt{8}}{\sqrt{13}}$	$v = 71.4$	$\frac{\sqrt{9} - \sqrt{8}}{\sqrt{13}}$	$t = .00007$ 5th
$\frac{\sqrt{8} - \sqrt{7}}{\sqrt{13}}$	$v = 76.0$	$\frac{\sqrt{8} - \sqrt{7}}{\sqrt{13}}$	$t = .00007$ 6th
$\frac{\sqrt{7} - \sqrt{6}}{\sqrt{13}}$	$v = 81.7$	$\frac{\sqrt{7} - \sqrt{6}}{\sqrt{13}}$	$t = .00008$ 7th
$\frac{\sqrt{6} - \sqrt{5}}{\sqrt{13}}$	$v = 88.8$	$\frac{\sqrt{6} - \sqrt{5}}{\sqrt{13}}$	$t = .00008$ 8th
$\frac{\sqrt{5} - \sqrt{4}}{\sqrt{13}}$	$v = 98.2$	$\frac{\sqrt{5} - \sqrt{4}}{\sqrt{13}}$	$t = .00009$ 9th
$\frac{\sqrt{4} - \sqrt{3}}{\sqrt{13}}$	$v = 111.4$	$\frac{\sqrt{4} - \sqrt{3}}{\sqrt{13}}$	$t = .00011$ 10th
$\frac{\sqrt{3} - \sqrt{2}}{\sqrt{13}}$	$v = 132.2$	$\frac{\sqrt{3} - \sqrt{2}}{\sqrt{13}}$	$t = .00013$ 11th
$\frac{\sqrt{2} - \sqrt{1}}{\sqrt{13}}$	$v = 172.3$	$\frac{\sqrt{2} - \sqrt{1}}{\sqrt{13}}$	$t = .00017$ 12th
$\frac{\sqrt{1} - \sqrt{0}}{\sqrt{13}}$	$v = 416.0$	$\frac{\sqrt{1} - \sqrt{0}}{\sqrt{13}}$	$t = .00040$ 13th
	Sum <u>1500.0</u>		Sum $\frac{1}{6 \frac{1}{92}}$ or <u>.00144</u> inch.

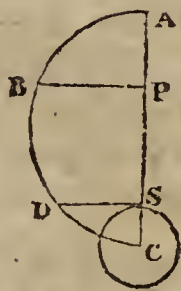
Hence, as the motion loft at the beginning is very small; and consequently the motion communicated to any body, as an inch plank, in passing through it, is very small also; we can conceive how such a plank may be shot through, without oversetting it.

## PROBLEM XI.

The force of attraction, above the earth, being inversely as the square of the distance from the centre; it is proposed to determine the time, velocity, and other circumstances, attending a heavy body falling from any given height; the descent at the earth's surface being  $16\frac{1}{2}$  feet, or 193 inches, in the first second of time.

Put

$r =$  CS the radius of the earth,  
 $a =$  CA the dist. fallen from,  
 $x =$  CP any variable distance,  
 $v =$  the velocity at P,  
 $t =$  time of falling there, and  
 $g = 16\frac{1}{2}$ , half the veloc. or force at S,  
 $f =$  the force at the point P.



Then we have the three following equations, viz.

$$x^2 : r^2 :: 1 : f = \frac{r^2}{x^2} \text{ the force at P, when the force of gravity is considered as } r;$$

$$tv = -\dot{x}, \text{ because } x \text{ decreases; and}$$

$$v\dot{v} = -2gf\dot{x} = -\frac{2gr^2\dot{x}}{x^2}.$$

The fluents of the last equation give  $v^2 = \frac{4gr^2}{x}$ .

But when  $x = a$ , the velocity  $v = 0$ ; therefore, by correction,  $v^2 = \frac{4gr^2}{x} - \frac{4gr^2}{a} = 4gr^2 \times \frac{a-x}{ax}$ ; or  $v =$

$\sqrt{\frac{4gr^2}{a} \times \frac{a-x}{x}}$ , a general expression for the velocity at any point P.

When  $x = r$ , this gives  $v = \sqrt{4gr \times \frac{a-r}{a}}$  for the greatest velocity, or the velocity when the body strikes the earth.

When  $a$  is very great in respect of  $r$ , the last velocity becomes  $(1 - \frac{r}{2a}) \times \sqrt{4gr}$  very nearly, or nearly  $\sqrt{4gr}$  only, which is accurately the greatest velocity by falling from an infinite height. And this, when  $r = 3965$  miles, is 6.9506 miles per second. Also, the velocity acquired in falling from the

the distance of the sun, or 12000 diameters of the earth, is 6.9505 miles per second. And the velocity acquired in falling from the distance of the moon, or 30 diameters, is 6.8927 miles per second.

Again, to find the time; since  $t v = -\dot{x}$ , therefore

$$t = \frac{-\dot{x}}{v} = \sqrt{\frac{a}{4gr^2}} \times \frac{-x\dot{x}}{\sqrt{ax - xx}}$$

which gives  $t = \sqrt{\frac{a}{4gr^2}} \times (\sqrt{ax - xx} + \text{arc to diameter } a \text{ and vers. } a - x)$ ; or the time of falling to any point P =  $\frac{1}{2r} \sqrt{\frac{a}{g}} \times (AB + BP)$ . And when  $x = r$ , this becomes  $t = \frac{1}{2} \sqrt{\frac{a}{g}} \times \frac{AD + DS}{SC}$  for the whole time of falling to the surface at S; which is evidently infinite when  $a$  or AC is infinite, although the velocity is then only the finite quantity  $\sqrt{4gr}$ .

When the height above the earth's surface is given =  $g$ ; because  $r$  is then nearly =  $a$ , and AD nearly = DS, the time  $t$  for the distance  $g$  will be nearly

$$\sqrt{\frac{1}{4gr^2}} \times 2DS = \sqrt{\frac{1}{4gr}} \times \sqrt{4gr} = 1'', \text{ as it ought to be.}$$

If a body at the distance of the moon at A fall to the earth's surface at S. Then  $r = 3965$  miles,  $a = 60r$ , and  $t = 416806'' = 4 \text{ da. } 19 \text{ h. } 46' 46''$ , which is the time of falling from the moon to the earth.

When the attracting body is considered as a point C; the whole time of descending to C will be - - - - -

$$\frac{1}{2r} \sqrt{\frac{a}{g}} \times ABDC = \frac{.7854a}{r} \sqrt{\frac{a}{g}}$$

#### PROBLEM XII.

*The force of attraction below the earth's surface being directly as the distance from the centre: it is proposed to determine the circumstances of velocity, time, and space fallen by a heavy body from the surface, through a perforation made straight to the centre of the earth: abstracting from the effect of the earth's rotation, and supposing it to be a homogeneous sphere of 3965 miles radius.*

Put

Put  $r = AC$  the radius of the earth,  
 $x = CP$  the dist. from the centre,  
 $v =$  the velocity at  $P$ ,  
 $t =$  the time there,  
 $g = 16\frac{1}{2}$ , half the force at  $A$ ,  
 $f =$  the force at  $P$ .



Then  $CA : CP :: 1 : f$ ; and the three equations are  $rf = x$ , and  $v\dot{v} = -2gf\dot{x}$ , and  $t\dot{v} = -\dot{x}$ .

Hence  $f = \frac{x}{r}$ , and  $v\dot{v} = \frac{-2gx\dot{x}}{r}$ ; the correct fluent of

which gives  $v = \sqrt{2g \times \frac{r^2 - x^2}{r}} = PD \sqrt{\frac{2g}{r}} = PD \sqrt{\frac{2g}{CE}}$ , the velocity at the point  $P$ ; where  $PD$  and  $CE$  are perpendicular to  $CA$ . So that the velocity at any point  $P$ , is as the perpendicular or sine  $PD$  at that point.

When the body arrives at  $C$ , then  $v = \sqrt{2gr} = \sqrt{2g} \cdot AC = 25950$  feet or  $4.9148$  miles per second, which is the greatest velocity, or that at the centre  $C$ .

Again, for the time;  $t = \frac{-\dot{x}}{v} = \sqrt{\frac{r}{2g}} \times \frac{-\dot{x}}{\sqrt{r^2 - x^2}}$ ; and

the fluents give  $t = \sqrt{\frac{r}{2g}} \times \text{arc to cosine } \frac{x}{r} = \sqrt{\frac{r}{2g}} \times \text{arc AD}$ . So that the time of descent to any point  $P$ , is as the corresponding arc  $AD$ .

When  $P$  arrives at  $C$ , the above becomes  $t =$

$\sqrt{\frac{r}{2g}} \times \text{quadrant AE} = \frac{AE}{AC} \sqrt{\frac{r}{2g}} = 1.5708 \sqrt{\frac{r}{2g}} = 1267\frac{1}{4}$  seconds  $= 21' 7''\frac{1}{4}$ , for the time of falling to the centre  $C$ .

The time of falling to the centre, is the same quantity  $1.5708 \sqrt{\frac{r}{2g}}$ , from whatever point in the radius  $AC$  the body begins to move. For, let  $n$  be any given distance from  $C$  at which the motion commences: then, by correction,

$v = \sqrt{\frac{2g}{r}(n^2 - x^2)}$ ; and hence  $t = \sqrt{\frac{r}{2g}} \times \frac{-\dot{x}}{\sqrt{n^2 - x^2}}$ , the

fluents of which give  $t = \sqrt{\frac{r}{2g}} \times \text{arc to cosine } \frac{x}{n}$ ; which,

when  $x = 0$ , gives  $t = \sqrt{\frac{r}{2g}} \times \text{quadrant} = 1.5708 \sqrt{\frac{r}{2g}}$ , for the time of descent to the centre  $C$ , the same as before.

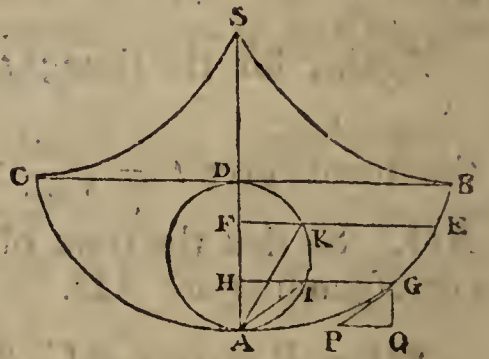
As an equal force, acting in contrary directions, generates or destroys an equal quantity of motion, in the same time; it follows that, after passing the centre, the body will just ascend to the opposite surface at B, in the same time in which it fell to the centre from A. Then from B it will return again in the same manner, through C to A; and so vibrate continually between A and B, the velocity being always equal at equal distances from C on both sides; and the whole time of a double oscillation, or of passing from A and arriving at A again, will be quadruple the time of passing over the radius AC, or  $= 2 \times 3.1416 \sqrt{\frac{r}{2g}} = 1\text{h. } 24' 29''$ .

### PROBLEM XIII.

*To find the Time of a Pendulum vibrating in the Arc of a Cycloid.*

Let

S be the point of suspension;  
SA, the length of pendulum;  
CAB, the whole cycloidal arc;  
AIKD, the generating circle,  
to which FKE, HIG are  
perpendiculars.  
SC, SB two other equal semi-  
cycloids, upon which  
the thread wrapping, the  
end A is made to describe  
the cycloid BAC.



Now, by the nature of the cycloid,  $AD = DS$ ; and  $SA = 2AD = SC = SB = SA = AB$ . Also, if at any point G be drawn the tangent GP; also GQ parallel and PQ perpendicular to AD. Then PG is parallel to the chord AI by the nature of the curve. And, by the nature of forces, the force of gravity : force in direction GP :: GP : GQ :: AI : AH :: AD : AI; in like manner, the force of gravity : force in the curve at E :: AD : AK; that is, the accelerative force in the curve, is every where as the corresponding chord AI or AK of the circle, or as the arc AG or AE of the cycloid, since AG is always  $= 2AI$ , by the nature of the curve. So that the process and conclusions, for the velocity and time of describing any arc in this case, will be the very same as in the last problem, the nature of the forces being the same, viz. as the distance to be passed over to the lowest point A.

From

From whence it follows, that the time of a femi-vibration, in all arcs, AG, AE, &c, is the same constant quantity

$$1.5708\sqrt{\frac{r}{2g}} = 1.5708\sqrt{\frac{AS}{2g}} = 1.5708\sqrt{\frac{l}{2g}}; \text{ and the time of}$$

a whole vibration from B to C, or from C to B, is  $3.1416\sqrt{\frac{l}{2g}}$ ;

where  $l = AS = AB$  is the length of the pendulum,  $g = 16\frac{1}{2}$  feet or 193 inches, and 3.1416 the circumference of a circle whose diameter is 1.

Since the time of a body's falling by gravity through  $\frac{1}{2}l$ , or half the length of the pendulum, by the nature of descents, is  $\sqrt{\frac{l}{2g}}$ , which being in proportion to  $3.1416\sqrt{\frac{l}{2g}}$ , as 1 is to 3.1416; therefore the diameter of a circle is to its circumference, as the time of falling through half the length of a pendulum, is to the time of one vibration.

If the time of the whole vibration be 1 second, this equation arises, viz.  $1'' = 3.1416\sqrt{\frac{l}{2g}}$ ; hence  $l = \frac{2g}{3.1416^2} = \frac{g}{4.9348}$ , and  $g = 3.1416^2 \times \frac{1}{2}l = 4.9348l$ . So that if one of these,  $g$  or  $l$ , be given by experiment, these equations will give the other. When  $g$ , for instance, is supposed to be given

$$= 16\frac{1}{2} \text{ feet, or } 193 \text{ inches; then is } l = \frac{g}{4.9348} = 39.11,$$

the length of a pendulum to vibrate seconds. Or if  $l = 39\frac{1}{8}$ , the length of the seconds pendulum for the latitude of London, by experiment; then is  $g = 4.9348l = 193.07$  inches  $= 16\frac{1.07}{1.200}$  feet, or nearly  $16\frac{1}{2}$  feet, for the space descended by gravity in the first second of time in the latitude of London; also agreeing with experiment.

Hence the times of vibration of pendulums, are as the square roots of their lengths; and the number of vibrations made in a given time, is reciprocally as the square roots of the lengths. And hence also, the length of a pendulum vibrating  $n$  times in a minute, or  $60''$ , is  $l = 39\frac{1}{8} \times \frac{60^2}{n^2} = \frac{140850}{n^2}$ .

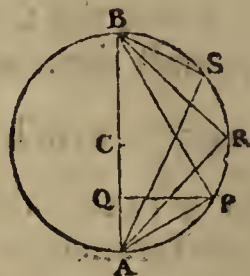
When a pendulum vibrates in a circular arc; as the length of the string is constantly the same, the time of vibration will be longer than in a cycloid; but the two times will approach nearer together as the circular arc is smaller; so that

when it is very small, the times of vibration will be nearly equal. And hence it happens that  $39\frac{1}{8}$  inches is the length of a pendulum vibrating seconds, in the very small arc of a circle.

PROBLEM XIV.

*To determine the Time of a Body descending down the Chord of a Circle.*

LET  $C$  be the centre ;  $AB$  the vertical diameter ;  $AP$  any chord, down which a body is to descend from  $P$  to  $A$  ; and  $PQ$  perpendicular to  $AB$ .



Now, as the natural force of gravity in the vertical direction  $BA$ , is to the force urging the body down the plane  $PA$ , as the length of the plane  $AP$ , is to its height  $AQ$  ; therefore the velocity in  $PA$  and  $QA$ , will be equal at all equal perpendicular distances below  $PQ$  ; and consequently the time in  $PA$  : time in  $QA$  ::  $PA$  :  $QA$  ::  $BA$  :  $PA$  ; but time in  $BA$  : time in  $QA$  ::  $\sqrt{BA}$  :  $\sqrt{QA}$  ::  $BA$  :  $PA$  ; hence, as three of the terms in each proportion are the same, the fourth terms must be equal, namely the time in  $BA$  = the time  $PA$ .

And, in like manner, the time in  $BP$  = the time in  $BA$ . So that, in general, the times of descending down all the chords  $BA$ ,  $BP$ ,  $BR$ ,  $BS$ , &c, or  $PA$ ,  $RA$ ,  $SA$ , &c, are all equal, and each equal to the time of falling freely through the diameter ; as before found at art. 131, Mechanics. Which time is  $\sqrt{\frac{2r}{g}}$ , where  $g = 16\frac{1}{2}$  feet, and  $r$  = the radius  $AC$  ;

for  $\sqrt{g} : \sqrt{2r} :: 1'' : \sqrt{\frac{2r}{g}}$ .

PROBLEM XV.

*To determine the Time of filling the Ditches of a Work with Water, at the Top, by a Sluice of 2 Feet square ; the Head of Water above the Sluice being 10 Feet, and the Dimensions of the Ditch being 20 Feet wide at Bottom, 22 at Top, 9 deep, and 1000 Feet long.*

THE capacity of the ditch is 189000 cubic feet.

But  $\sqrt{g} : \sqrt{10} :: 2g : 2\sqrt{10g}$  the velocity of the water through the sluice, the area of which is 4 square feet ;  
therefore



therefore  $8\sqrt{10g}$  is the quantity per second running through it; and consequently  $8\sqrt{10g} : 189000 :: 1'' : \frac{23625}{\sqrt{10g}} = 1863''$  or  $31' 3''$  nearly, which is the time of filling the ditch.

## PROBLEM XVI.

To determine the Time of emptying a Vessel of Water by a Sluice in the Bottom of it, or in the Side near the Bottom; the Height of the Aperture being very small in respect of the Altitude of the Fluid.

PUT  $a$  = the area of the aperture or sluice;

$2g$  =  $32\frac{1}{8}$  feet, the force of gravity;

$d$  = the whole depth of water;

$x$  = the variable altitude of the surface above the aperture;

$A$  = the area of the surface of the water.

Then  $\sqrt{g} : \sqrt{x} :: 2g : 2\sqrt{gx}$  the velocity with which the fluid will issue at the sluice; and hence  $A : a :: 2\sqrt{gx} : \frac{2a\sqrt{gx}}{A}$  the velocity with which the surface of the water will descend at the altitude  $x$ , or the space it would descend in 1 second with the velocity there. Now, in descending the space  $\dot{x}$ , the velocity may be considered as uniform; and uniform descents are as their times; therefore  $\frac{2a\sqrt{gx}}{A} : \dot{x} :: 1'' : \frac{A\dot{x}}{2a\sqrt{gx}}$  the time of descending  $\dot{x}$  space, or the fluxion of the time of exhausting. That is,  $t = \frac{-A\dot{x}}{2a\sqrt{gx}}$ ; which is made negative, because  $x$  is a decreasing quantity, or its fluxion negative.

Then, when the nature or figure of the vessel is given, the area  $A$  will be given in terms of  $x$ ; which value of  $A$  being substituted into this fluxion of the time, the fluent of the result will be the time of exhausting sought.

So if, for example, the vessel be any prism, or every where of the same breadth; then  $A$  is a constant quantity, and therefore the fluent is  $-\frac{A}{a}\sqrt{\frac{x}{g}}$ . But when  $x = d$ , this

becomes  $-\frac{A}{a}\sqrt{\frac{d}{g}}$ , and should be 0; therefore the correct

fluent is  $t = \frac{A}{a} \times \frac{\sqrt{d} - \sqrt{x}}{\sqrt{g}}$  for the time of the surface descending

ascending till the depth of the water be  $x$ . And when  $x=0$ , the whole time of exhausting is barely  $\frac{A}{a} \sqrt{\frac{d}{g}}$ .

Hence, if  $A$  be 10000 square feet,  $a=1$  square foot, and  $d=10$  feet; the time is  $7885\frac{1}{2}$  seconds, or 2h. 11' 25'' $\frac{1}{5}$ .

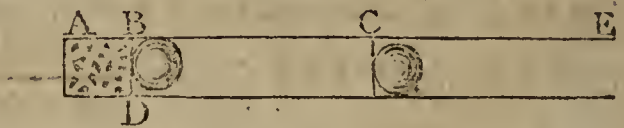
Again, if the vessel be a ditch, or canal, of 20 feet broad at the bottom, 22 at the top, 9 deep, and 1000 feet long; then is  $90 : 90 + x :: 20 : \frac{90 + x}{9} \times 2$  the breadth of the surface of the water when its depth in the canal is  $x$ ; and therefore  $A = \frac{90 + x}{9} \times 2000$  is the surface at that time.

Consequently  $i$  or  $\frac{-Ax}{2a\sqrt{gx}} = 1100 \times \frac{90 + x}{9} \times \frac{-x}{a\sqrt{gx}}$  is the fluxion of the time; the correct fluent of which, when  $x=0$ , is  $1000 \times \frac{180 + \frac{2}{3}d}{9a} \times \sqrt{\frac{d}{g}} = \frac{1000 \times 186 \times 3}{9 \times 4\frac{1}{6}} = 15459''\frac{2}{3}$  nearly, or 4h. 17' 39'' $\frac{2}{3}$ , being the whole time of exhausting by a sluice of 1 foot square.

#### PROBLEM XVII.

*To determine the Velocity with which a Ball is discharged from a Given Piece of Ordnance, with a Given Charge of Gunpowder.*

LET the annexed figure represent the bore of the gun; AD being the part filled with gunpowder. And put



- $a = AB$ , the part at first filled with powder;
- $b = AE$ , the whole length of the gun bore;
- $c = .7854$ , the area of a circle whose diameter is 1;
- $d = BD$ , the diameter of the bore;
- $e =$  the specific gravity of the ball, or weight of 1 cubic foot;
- $g = 16\frac{1}{2}$  feet, descended by a body in 1 second;
- $m = 230$  ounces, the pressure of the atmosphere on a sq. inch;
- $n$  to 1 the ratio of the first force of the fired powder, to the pressure of the atmosphere;
- $w =$  the weight of the ball. Also, let
- $x = AC$ , be any variable distance of the ball from A, in moving along the gun barrel.

First,

First,  $cd^2$  is  $\equiv$  the area of the circle BD of the bore ;  
 theref.  $mcd^2$  is the pressure of the atmosphere on BD ;  
 conseq.  $mnacd^2$  is the first force of the powder on BD.

But the force of the inflamed powder is proportional to its  
 density, and the density is inversely as the space it fills ; there-  
 fore the force of the powder on the ball at B, is to the force  
 on the same at C, as AC is to AB ; that is,

$$x : a :: mnacd^2 : \frac{mnacd^2}{x} = F, \text{ the motive force at C ;}$$

$$\text{conseq. } \frac{F}{w} = \frac{mnacd^2}{wx} = f, \text{ the accelerating force there.}$$

$$\text{Hence, theor. 10 of forces gives } v\dot{v} = 2gf\dot{x} = \frac{2gmnacd^2}{w} \times \frac{\dot{x}}{x} ;$$

$$\text{the fluent of which is } v^2 = \frac{4gmnacd^2}{w} \times \text{hyp. log. of } x.$$

But when  $v = 0$ ,  $x = a$  ; therefore, by correction,  
 $v^2 = \frac{4gmnacd^2}{w} \times \text{hyp. log. } \frac{x}{a}$  is the correct fluent ;

$$\text{conseq. } v = \sqrt{\left(\frac{4gmnacd^2}{w} \times \text{hyp. log. } \frac{x}{a}\right)} \text{ is the vel. of ball at C,}$$

and  $v = \sqrt{\left(\frac{4gmnbcd^2}{w} \times \text{hyp. log. } \frac{b}{a}\right)}$  the velocity with which  
 the ball issues from the muzzle at E ; where  $b$  denotes the  
 length of the cylinder filled with powder, and  $a$  the length  
 to the hinder part of the ball, which will be more than  $b$   
 when the powder does not touch the ball.

Or, by substituting the numbers for  $g$  and  $m$ , and chang-  
 ing the hyperbolic logarithms for the common ones, then  
 $v = \sqrt{\left(\frac{2230nbd^2}{w} \times \text{com. log. } \frac{b}{a}\right)}$  the veloc. at E, in feet.

But, the content of the ball being  $\frac{2}{3}cd^3$ , its weight is  
 $w = \frac{\frac{2}{3}cd^3e}{12^3} = \frac{ccd^3}{2592} = \frac{cd^3}{3300}$  ; which being substituted for  $w$ ,  
 in the value of  $v$ , it becomes

$$v = 2713 \sqrt{\left(\frac{nb}{de} \times \text{com. log. } \frac{b}{a}\right)}, \text{ the velocity at E.}$$

When the ball is of cast iron ; taking  $e = 7333$ , the rule becomes

$$v = 100 \sqrt{\left(\frac{nb}{10d} \times \text{log. } \frac{b}{a}\right)} \text{ for the veloc. of the cast-iron ball.}$$

Or, when the ball is of lead ; then

$$v = 80^{\frac{3}{5}} \sqrt{\left(\frac{nb}{10d} \times \text{log. } \frac{b}{a}\right)} \text{ for the veloc. of the leaden ball.}$$

*Corol.* From the general expression for the velocity  $v$ , above given, may be derived what must be the length of the charge of powder  $a$ , in the gun barrel, so as to produce the greatest possible velocity in the ball; namely, by making the value of  $v$  a maximum, or, by squaring and omitting the constant quantities, the expression  $a \times \text{hyp. log. of } \frac{b}{a}$  a maximum, or its fluxion equal to nothing; that is,  $\dot{a} \times \text{hyp. log. } \frac{b}{a} - \dot{a} = 0$ , or  $\text{hyp. log. of } \frac{b}{a} = 1$ ; hence  $\frac{b}{a} = 2.71828$ , the number whose hyp. log. is 1. So that  $a : b :: 1 : 2.71828$ , or as 7 : 19 very nearly; that is, the length of the charge, to produce the greatest velocity, is the  $\frac{7}{19}$ th part of the length of the bore.

By actual experiment it is found, that the charge for the greatest velocity, somewhat exceeds that which is here computed from theory; as may be seen by turning to page 269 of my volume of Tracts, where the corresponding parts are found to be, for four different lengths of gun, thus,  $\frac{3}{18}$ ,  $\frac{3}{12}$ ,  $\frac{3}{16}$ ,  $\frac{3}{20}$ ; the parts here varying, as the gun is longer, which allows time for the greater quantity of powder to be fired, before the ball is out of the bore.

SCHOLIUM.

In the calculation of the foregoing problem, the value of the constant quantity  $n$  remains to be determined. It denotes the first strength or force of the fired gunpowder, just before the ball is moved out of its place. This value is assumed, by Mr. Robins, equal to 1000, that is, 1000 times the pressure of the atmosphere, on any equal spaces.

But the value of the quantity  $n$ , may be derived much more accurately, from the experiments related in my Tracts, by comparing the velocities there found by experiment, with the rule for the value of  $v$ , or the velocity, as above computed by theory, viz. - - - - -

$$v = 100\sqrt{\left(\frac{na}{100d} \times \text{log. of } \frac{b}{a}\right)}, \text{ or } = 100\sqrt{\left(\frac{nb}{100d} \times \text{log. of } \frac{b}{a}\right)}.$$

Now, supposing that  $v$  is a given quantity, as well as all the other quantities, excepting only the number  $n$ , by reducing this equation, the value of the letter  $n$  is found to be as follows, viz. - - - - -

$$n = \frac{dvv}{1000a} \div \text{com. log. of } \frac{b}{a}, \text{ or } = \frac{dvv}{1000b} \div \text{log. of } \frac{b}{a},$$

when  $b$  is different from  $a$ .

Now,

Now, to apply this to the experiments. By page 257 of the Tracts, the velocity of the ball, of 1.96 inches diameter, with 4 ounces of powder, in the gun No. 1, was 1100 feet per second; and, by page 109, the length of the gun was 28.2; also, by page 237, the length of the charge was 3.12 inches: so that the values of the quantities in the rule, are thus:  $a = 3.12$ ;  $b = 28.2$ ;  $d = 1.96$ ; and  $v = 1100$ : then, by substituting these values instead of the letters, in the theorem  $n = \frac{dvv}{1000a} \div \text{com. log. of } \frac{b}{a}$ , it comes out  $n = 795$ , when  $b$  is considered as the same as  $a$ . And so on, for the other experiments there treated of.

It is here to be noted however, that there is a circumstance in the experiments delivered in the Tracts, just mentioned, which will alter the value of the letter  $a$  in this theorem, which is this, viz. that  $a$  denotes the distance of the shot from the bottom of the bore; and the length of the charge of powder alone ought to be the same thing; but in the experiments that length included, beside the length of real powder, the substance of the thin flannel bag in which it was always contained, of which the neck at least extended a considerable length, being the part where the open end was wrapped and tied close round with a thread. This circumstance causes the value of  $n$ , as found by the theorem above, to come out less than it ought to be, for it shews the strength of the inflamed powder when just fired, and the flame fills the whole space  $a$  before occupied both by the real powder and the bag, whereas it ought to shew the first strength of the flame when it is supposed to be contained in the space only occupied by the powder alone, without the bag. The formula will therefore bring out the value of  $n$  too little, in proportion as the real space filled by the powder is less than the space filled both by the powder and its bag. In the same proportion therefore must we increase the formula, that is, in the proportion of  $b$ , the length of real powder, to  $a$  the length of powder and bag together. When the theorem is so corrected, it becomes  $\frac{dvv}{1000b} \div \text{com. log. of } \frac{b}{a}$ .

Now, by pa. 237 of the Tracts, there are given both the lengths of all the charges, or values of  $a$ , including the bag, and also the length of the neck of the bag, which is 0.58 of an inch, which therefore must be subtracted from all the values of  $a$ , to give the corresponding values of  $b$ . This in the example above reduces 3.12 to 2.54.

Hence,

Hence, by increasing the above result 795, in proportion of 2.54 to 3.12, it becomes 977.

But it will be best to arrange the results in a table, with the several dimensions, from which they are computed, as here below.

*Table of Velocities of Balls, and First Force of Powder, &c.*

Gun.		Charge.			Velocity, or value of $v$ .	First force, or value of $n$ .
No.	Length, or value of $b$ .	Weight in ounces.	Length, or value.			
			of $a$ .	of $b$ .		
1	inches. 28.2	4	3.12	2.54	1100	977
		8	5.66	5.08	1430	1131
		16	10.74	10.16	1430	941
2	38.1	4	3.12	2.54	1180	989
		8	5.66	5.08	1580	1163
		16	10.74	10.16	1660	967
3	57.37	4	3.12	2.54	1300	1031
		8	5.66	5.08	1790	1229
		16	10.74	10.16	2000	1060
4	79.9	4	3.12	2.54	1370	1028
		8	5.66	5.08	1940	1263
		16	10.74	10.16	2200	1071

where the numbers in the column of velocities, 1420 and 2200, are a little increased, as, from a view of the table of experiments, they evidently required to be. Also the value of the letter is constantly 1.36 inch.

Hence it appears, that the value of the letter  $n$ , used in the theorem, though not greatly different from the number 1000, used by Mr. Robins, is rather various, both for the different lengths of the gun, and for the different charges with the same gun.

But this diversity in the value of the quantity  $n$ , or the first force of the inflamed gunpowder, is probably owing to the omission of a material datum in the calculation of the  
last

last problem, namely, the weight of the charge of powder, which has not at all been brought into the computation. For it is manifest, that the elastic fluid has not only the ball to move and impel before it, but its own weight of matter also. The computation may therefore be renewed, in the ensuing problem, to take that datum into the account.

## PROBLEM XVIII.

*To determine the same as in the last Problem; taking both the Weight of Powder and the Ball into the Calculation.*

BESIDE the notation used in the last problem, let  $2p$  denote the weight of the powder in the charge.

Now, because the inflamed powder occupies all the part of the gun bore which is behind the ball, its centre of gravity, or the middle part of the same, will move with only half the velocity that the ball moves with; and this will require the same force, as half the weight of the powder, moved with the same velocity as the ball. Therefore, in the conclusion derived in the last problem, we are now, instead of  $w$ , to substitute the quantity  $p + w$ ; and when that is done, the last velocity will come out,  $v = \sqrt{\left(\frac{2230nad^2}{p + w} \times \text{com. log. } \frac{b}{a}\right)}$ .

Now, to find the value of  $p + w$  in terms of  $ad$ , the dimensions of the ball and powder; it appears, in the first place, from the calculations in the last scholium, that about 5 inches in length of a gun, of near 2 inches diameter, contains just 8 ounces of powder; and as the contents of cylinders, or the weight of powder they will contain, are in the compound ratio of their length and square of the diameter; therefore as  $5 \times 2^2 : ad^2 :: 8 : \frac{2ad^2}{5}$ , which will be the weight of powder contained in the bore, or charge, whose length is  $a$ , and diameter  $d$ ; that is,  $2p = \frac{2ad^2}{5}$ , or  $p = \frac{ad^2}{5}$ , the value of the quantity  $p$ , in terms of the dimensions of the charge.

But the value of the quantity  $w$ , or weight of the ball, as found in the last problem, was  $w =$

$$\frac{ed^3}{3300}. \text{ Conseq. } p + w = \frac{ad^2}{5} + \frac{ed^3}{3300} = \frac{660ad^2 + ed^3}{3300}.$$

Then, substituting this value of  $p + w$ , in the expression above found for the velocity, gives

$$v =$$

$$v = 2713 \sqrt{\frac{na}{660a + de}} \times \log. \frac{b}{a}, \text{ for the last veloc. of ball.}$$

And, when the ball is of cast iron, using 7333 for  $n$ , the same theorem becomes

$$v = \sqrt{\frac{1000na}{d + .09a}} \times \log. \frac{b}{a}, \text{ or } = \sqrt{\frac{1000nb}{d + .09b}} \times \log. \frac{b}{a}, \text{ for}$$

the last velocity of the ball, when of cast iron.

*Corol.* To determine the maximum velocity, from the foregoing expression, by squaring, and expelling the constant quantities, there is obtained at length  $\frac{a}{d + .09a} \times \text{hyp. log. } \frac{b}{a} = a$  maximum. Then, by making the fluxion of this quantity = 0, there is produced  $\text{hyp. log. } \frac{b}{a} = 1 + .09a$ .

From which  $a$  may easily be found, in any particular case, by the method of trial-and-error, and a table of hyperbolic logarithms.

Thus, taking the first case in the corollary to the last problem, in which  $b = 28.2$ , and  $d = 2$  nearly; then, by double position, is soon found  $a = 7.4$ ; which is  $\frac{3}{11}$  of  $b$ , being very near  $\frac{3}{10}b$ , as found by experiment.

Again, taking the second case there mentioned, in which  $b = 38.1$ , and  $d = 2$ , as before. Then is easily found  $a = 9.3$ ; which is  $\frac{1}{4}$  or  $\frac{3}{12}$  of  $b$ , the very same as found by experiment.

Thirdly, taking the third case, in which  $b = 57.37$ , and  $d = 2$ . Then  $a$  comes out 12; which is between  $\frac{1}{4}$  and  $\frac{1}{5}$  of  $b$ ; which was found  $\frac{1}{6}b$  by experiment.

Lastly, taking the fourth case, in which  $b = 79.9$ , and  $d = 2$ . Then  $a$  comes out 15, which is  $\frac{3}{16}$  of  $b$ , but was found near  $\frac{3}{20}$  of  $b$  by experiment.

The near agreement of these calculations, from theory, with the experiments, is a confirmation of the truth, both of the one and the other. And this gives a reasonable ground to hope, that the value of the quantity  $n$ , or the first force of the fired gunpowder, when calculated from the last theorem for the velocity, may come nearly the same, when fired in the same quantity, by using different experiments. Which may now be tried in the following scholium.

#### SCHOLIUM.

Proceeding here as in the scholium to the last problem, to find the value of the letter  $n$ , in terms of  $v$  and the known quan-



quantities, from the theorem for the velocity last found, viz.

$$v = \sqrt{\frac{1000na}{d + .09a}} \times \log. \frac{b}{a}, \text{ or } = \sqrt{\frac{1000nb}{d + .09b}} \times \log. \frac{b}{a},$$

by reduction it comes out

$$n = \left( \frac{d + .09a}{1000a} \right) v^2 \div \log. \frac{b}{a}, \text{ or } \left( \frac{d + .09b}{1000b} \right) v^2 \div \log. \frac{b}{a}.$$

Then, calculating the value of  $n$ , by this theorem, from the same data as in the 12 experiments made use of in the scholium to the last problem, the values of  $n$  come out as they are here placed in the margin, which stand in the same order as those in the last column of the table of data and results in the scholium before mentioned. Also, the corresponding values of the same force  $n$ , from the former problem and scholium, are set after these, that their differences may be seen at one view.

Values of $n$ from this prob.	Values of $n$ from the former prob.
1091	977
1396	1131
1381	941
1104	989
1435	1163
1419	967
1152	1031
1516	1229
1556	1060
1149	1028
1558	1263
1572	1071

From a view of these results, it clearly appears, that the theorems, for the values of  $v$  and  $n$ , found and employed in this problem, and the corollary and scholium to the same, must be very near the truth, as the values of  $n$ , or first force of the powder, come out very uniform, for all the guns, when the same quantity of powder is fired.

To make this uniformity more obvious, they are here arranged, according to the like weight of powder in each column, and for each gun on the same line. From which it appears, that there is very little difference on account of the length of the gun; but a considerable one for the different quantities of powder, being, on an average, nearly, as follows, viz.

Length of gun.	Value of $n$ , with powder.		
	4 oz.	8 oz.	16 oz.
inc.			
28.2	1091	1396	1381
38.1	1104	1435	1449
57.37	1152	1516	1556
79.9	1149	1558	1572

That

That the first force of fired gunpowder, when 4 oz are fired, is about 1150 times } the strength of  
 when 8 oz are fired, is about 1500 times } the atmo-  
 when 16 oz are fired, is about 1550 times } sphere.

And this increase of strength is probably owing to the greater heat of the elastic fluid, arising from the greater quantity of powder that is fired in the gun.

## ON THE MOTION OF BODIES IN FLUIDS.

### PROBLEM XIX.

*To determine the Force of Fluids in Motion; and the Circumstances attending Bodies moving in Fluids.*

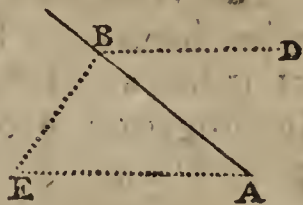
1. IT is evident that the resistance to a plane, moving perpendicularly through an infinite fluid, at rest, is equal to the pressure or force of the fluid on the plane at rest, and the fluid moving with the same velocity, and in the contrary direction, to that of the plane in the former case. But the force of the fluid in motion, must be equal to the weight or pressure which generates that motion; and which, it is known, is equal to the weight or pressure of a column of the fluid, whose base is equal to the plane, and its altitude equal to the height through which a body must fall, by the force of gravity, to acquire the velocity of the fluid: and that altitude is, for the sake of brevity, called the altitude due to the velocity. So that, if  $a$  denote the area of the plane,  $v$  the velocity, and  $n$  the specific gravity of the fluid; then, the altitude due to the velocity  $v$  being  $\frac{v^2}{4g}$ , the whole resistance, or motive force  $m$ , will be

$a \times n \times \frac{v^2}{4g} = \frac{anv^2}{4g}$ ;  $g$  being  $16\frac{1}{2}$  feet. And hence, *cæteris paribus*, the resistance is as the square of the velocity.

2. This ratio, of the square of the velocity, may be otherwise derived thus. The force of the fluid in motion, must be as the force of one particle multiplied by the number of them; but the force of a particle is as its velocity; and the number of them striking the plane in a given time, is also as the velocity; therefore the whole force is as  $v \times v$  or  $v^2$ , that is, as the square of the velocity.

3. If the direction of motion, instead of being perpendicular to the plane, as above supposed, be inclined to it in any

any angle, the sine of that angle being  $s$ , to the radius  $r$ : then the resistance to the plane, or the force of the fluid against the plane, in the direction of the motion, as assigned above, will be diminished in the triplicate ratio of radius to the sine of the angle of inclination, or in the ratio of  $r$  to  $s^3$ . For,  $AB$  being the direction of the plane, and  $BD$  that of the motion, making the angle  $ABD$ , whose sine is  $s$ ; the number of particles, or quantity of the fluid, striking the plane, will be diminished in the ratio of  $r$  to  $s$ , or of radius to the sine of the angle  $B$  of inclination; and the force of each particle will also be diminished in the same ratio of  $r$  to  $s$ : so that, on both these accounts, the whole resistance will be diminished in the ratio of  $r$  to  $s^2$ , or in the duplicate ratio of radius to the sine of the said angle. But again, it is to be considered that this whole resistance is exerted in the direction  $BE$  perpendicular to the plane; and any force in the direction  $BE$ , is to its effect in the direction  $AE$ , parallel to  $BD$ , as  $AE$  to  $BE$ , that is as  $r$  to  $s$ . So that finally, on all these accounts, the resistance in the direction of motion, is diminished in the ratio of  $r$  to  $s^3$ , or in the triplicate ratio of radius to the sine of inclination. Hence, comparing this with article 1, the whole resistance, or the motive force on



the plane, will be  $m = \frac{anv^2 s^3}{4g}$ .

4. Also, if  $w$  denote the weight of the body, whose plane face  $a$  is resisted by the absolute force  $m$ ; then the retarding force  $f$ , or  $\frac{m}{w}$ , will be  $\frac{anv^2 s^3}{4gw}$ .

5. And if the body be a cylinder, whose face or end is  $a$ , and diameter  $d$ , or radius  $r$ , moving in the direction of its axis; because then  $s = 1$ , and  $a = pr^2 = \frac{1}{4}pd^2$ , where  $p = 3.1416$ ; the resisting force  $m$  will be  $\frac{npd v^2}{16g} = \frac{npr^2 v^2}{4g}$ , and the retarding force  $f = \frac{npd^2 v^2}{16gw} = \frac{npr^2 v^2}{4gw}$ .

6. This is the value of the resistance when the end of the cylinder is a plane perpendicular to its axis, or to the direction of motion. But were its face a conical surface, or an elliptic section, or any other figure every where equally inclined to the axis, the sine of inclination being  $s$ : then, the number  
of

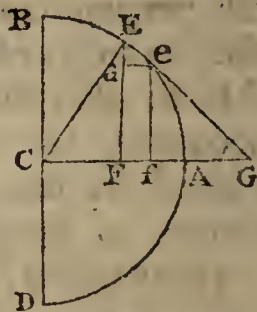
of particles of the fluid striking the face being still the same, but the force of each, opposed to the direction of motion, diminished in the duplicate ratio of radius to the sine of inclination, the resisting force  $m$  would be  $\frac{npd^2v^2s^2}{16g} = \frac{npr^2v^2s^2}{4g}$ .

But if the body were terminated by an end or face of any other form, as a spherical one, or such like, where every part of it has a different inclination to the axis; then a farther investigation becomes necessary, such as in the following proposition.

PROBLEM XX.

To determine the Resistance of a Fluid to any Body, moving in it, of a Curved End; as a Sphere, or a Cylinder with a Hemispherical End, &c.

I. LET BEAD be a section through the axis CA of the solid, moving in the direction of that axis. To any point of the curve draw the tangent EG, meeting the axis produced in G: also, draw the perpendicular ordinates EF, ef indefinitely near to each other; and draw ae parallel to CG.



Putting  $CF = x$ ,  $EF = y$ ,  $BE = z$ ,  $s = \text{sine } \angle G$  to radius 1, and  $p = 3.1416$ : then  $2py$  is the circumference whose radius is EF, or the circumference described by the point E, in revolving about the axis CA; and  $2py \times Ee$  or  $2pyz$  is the fluxion of the surface, or it is the surface described by Ee, in the said revolution about CA, and which is the quantity represented by  $u$  in art. 3 of the last problem: hence  $\frac{nv^2s^3}{4g} \times 2pyz$  or  $\frac{pnv^2s^3}{2g} \times yz$  is the resistance on that ring, or the fluxion of the resistance to the body, whatever the figure of it may be. And the fluent of which will be the resistance required.

2. In the case of a spherical form; putting the radius CA or CB =  $r$ , we have  $y = \sqrt{r^2 - x^2}$ ,  $s = \frac{EF}{EG} = \frac{CF}{CE} = \frac{x}{r}$ , and  $yz$  or  $EF \times Ee = CE \times ae = rx\dot{x}$ ; therefore the general fluxion  $\frac{pnv^2}{2g} \times s^3yz$  becomes  $\frac{pnv^2}{2g} \times \frac{x^3}{r^3} \times rx\dot{x} = \frac{pnv^2}{2gr^2} \times x^3\dot{x}$ ;

the

the fluent of which, or  $\frac{\rho v^2}{8gr^2} x^4$ , is the resistance to the spherical surface generated by BE. And when  $x$  or CF is  $= r$  or CA, it becomes  $\frac{\rho n v^2 r^2}{8g}$  for the resistance on the whole hemisphere; which is also equal to  $\frac{\rho n v^2 d^2}{32g}$ , where  $d = 2r$  the diameter.

3. But the perpendicular resistance to the circle of the same diameter  $d$  or BD, by art. 5 of the preceding problem, is  $\frac{\rho n v^2 d^2}{16g}$ ; which, being double the former, shews that the resistance to the sphere, is just equal to half the direct resistance to a great circle of it, or to a cylinder of the same diameter.

4. Since  $\frac{1}{6}\rho d^3$  is the magnitude of the globe; if N denote its density or specific gravity, its weight  $w$  will be  $= \frac{1}{6}\rho d^3 N$ , and therefore the retardive force  $f$  or  $\frac{m}{w} = \frac{\rho n v^2 d^2}{32g} \times \frac{6}{\rho N d^3} = \frac{3n v^2}{16g N d}$ ; which is also  $= \frac{v^2}{4gs}$  by art. 8 of the general

theorems in page 329; hence then  $\frac{3n}{4Nd} = \frac{1}{s}$ , and  $s = \frac{N}{n} \times \frac{4}{3}d$ ; which is the space that would be described by the globe, while its whole motion is generated or destroyed by a constant force which is equal to the force of resistance, if no other force acted on the globe to continue its motion. And if the density of the fluid were equal to that of the globe, the resisting force is such, as, acting constantly on the globe without any other force, would generate or destroy its motion in describing the space  $\frac{4}{3}d$ , or  $\frac{4}{3}$  of its diameter, by that accelerating or retarding force.

5. Hence the greatest velocity that a globe will acquire by descending in a fluid, by means of its relative weight in the fluid, will be found by making the resisting force equal to that weight. For, after the velocity is arrived at such a degree, that the resisting force is equal to the weight that urges it, it will increase no longer, and the globe will afterwards continue to descend with that velocity uniformly. Now, N and  $n$  being the separate specific gravities of the globe and fluid,  $N - n$  will be the relative gravity of the globe in the fluid, and therefore  $w = \frac{1}{6}\rho d^3 (N - n)$  is the

weight by which it is urged; also  $m = \frac{p n v^2 d^2}{32g}$  is the resistance; consequently  $\frac{p n v^2 d^2}{32g} = \frac{1}{6} p d^3 (N - n)$  when the velocity becomes uniform; from which equation is found  $v = \sqrt{4g \cdot \frac{4}{3}d \cdot \frac{N - n}{n}}$ , for the said uniform or greatest velocity.

And, by comparing this form with that in art. 6 of the general theorems in page 329, it will appear that its greatest velocity, is equal to the velocity generated by the accelerating force  $\frac{N - n}{n}$ , in describing the space  $\frac{4}{3}d$ , or equal to the velocity generated by gravity in freely describing the space  $\frac{N - n}{n} \times \frac{4}{3}d$ .—If  $N = 2n$ , or the specific gravity of the

globe be double that of the fluid, then  $\frac{N - n}{n} = 1 =$  the natural force of gravity; and then the globe will attain its greatest velocity in describing  $\frac{4}{3}d$ , or  $\frac{4}{3}$  of its diameter.—It is farther evident, that if the body be very small, it will very soon acquire its greatest velocity, whatever its density may be.

EXAM. If a leaden ball, of 1 inch diameter, descend in water, and in air of the same density as at the earth's surface, the three specific gravities being as  $11\frac{1}{3}$ , and 1, and  $\frac{2500}{2500}$ . Then  $v = \sqrt{4 \cdot 16\frac{1}{2} \cdot \frac{4}{36} \cdot 10\frac{1}{3}} = \frac{1}{9}\sqrt{31 \cdot 193} = 8.5944$  feet, is the greatest velocity per second the ball can acquire by descending in water. And  $v = \sqrt{4 \cdot \frac{193}{12} \cdot \frac{4}{36} \cdot \frac{34}{3} \cdot \frac{2500}{3}}$  nearly  $= \frac{50}{9}\sqrt{\frac{34 \cdot 193}{3}} = 259.82$  is the greatest velocity it can acquire in air.

But if the globe were only  $\frac{1}{100}$  of an inch diameter, the greatest velocities it could acquire, would be only  $\frac{1}{10}$  of these, namely  $\frac{86}{100}$  of a foot in water, and 26 feet nearly in air. And if the ball were still farther diminished, the greatest velocity would also be diminished, and that in the subduplicate ratio of the diameter of the ball.

PROBLEM XXI.

To determine the Relations of Velocity, Space, and Time, of a Ball moving in a Fluid, in which it is projected with a Given Velocity.

1. LET  $a$  = the first velocity of projection,  $x$  the space described in any time  $t$ , and  $v$  the velocity then. Now, by art. 4 of the last problem, the accelerative force  $f = \frac{3nv^2}{16gNd}$ , where  $N$  is the density of the ball,  $n$  that of the fluid, and  $d$  the diameter. Therefore the general equation  $v\dot{v} = 2gf's$  becomes  $v\dot{v} = -\frac{3nv^2}{8Nd}x$ ; and hence  $\frac{\dot{v}}{v} = -\frac{3n}{8Nd}x = -bx$ , putting  $b$  for  $\frac{3n}{8Nd}$ .

The correct fluent of this, is  $\log. a - \log. v$  or  $\log. \frac{a}{v} = bx$ . Or, putting  $c = 2.718281828$ , the number whose hyp. log. is 1, then is  $\frac{a}{v} = c^{bx}$ , and the velocity  $v = \frac{a}{c^{bx}} = ac^{-bx}$ .

2. The velocity  $v$  at any time being the  $c^{-bx}$  part of the first velocity, therefore the velocity lost in any time, will be the  $1 - c^{-bx}$  part, or the  $\frac{c^{bx} - 1}{c^{bx}}$  part of the first velocity.

EXAMPLES.

EXAM. 1. If a globe be projected, with any velocity, in a medium of the same density with itself, and it describe a space equal to  $3d$  or 3 of its diameters. Then  $x = 3d$ , and  $b = \frac{3n}{8Nd} = \frac{3}{8d}$ ; therefore  $bx = \frac{9}{8}$ , and  $\frac{c^{bx} - 1}{c^{bx}} = \frac{2.08}{3.08}$  is the velocity lost, or nearly  $\frac{2}{3}$  of the projectile velocity.

EXAM. 2. If an iron ball of 2 inches diameter were projected with a velocity of 1200 feet per second; to find the velocity lost after moving through any space, as suppose 500 feet of air: we should have  $d = \frac{2}{12} = \frac{1}{6}$ ,  $a = 1200$ ,  
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$x = 500$ ,  $N = 7\frac{1}{3}$ ,  $n = .0012$ ; and therefore  $bx = \frac{3nx}{8Nd} = \frac{3 \cdot 12 \cdot 500 \cdot 3 \cdot 6}{8 \cdot 22 \cdot 10000} = \frac{81}{440}$ , and  $v = \frac{1200}{c^{\frac{81}{440}}} = 998$  feet per second: having lost 202 feet, or nearly  $\frac{1}{6}$  of its first velocity.

EXAM. 3. If the earth revolved about the sun, in a medium as dense as the atmosphere near the earth's surface; and it were required to find the quantity of motion lost in a year. Then, since the earth's mean density is about  $4\frac{1}{2}$ , and its distance from the sun 12000 of its diameters, we have  $24000 \times 3.1416 = 75398$  diameters  $= x$ , and  $bx = \frac{3 \cdot 75398 \cdot 12 \cdot 2}{8 \cdot 10000 \cdot 9} = 7.5398$ ; hence  $\frac{c^{bx} - 1}{c^{bx}} = \frac{1.8880}{18881}$  parts are lost of the first motion in the space of a year, and only the  $\frac{1}{18881}$  part remains.

EXAM. 4. If it be required to determine the distance moved,  $x$ , when the globe has lost any part of its motion, as suppose  $\frac{1}{2}$ , and the density of the globe and fluid equal: The general equation gives  $x = \frac{1}{b} \times \log. \frac{a}{v} = \frac{8d}{3} \times \log. of 2 = 1.8483925d$ . So that the globe loses half its motion before it has described twice its diameter.

3. To find the time  $t$ ; we have  $\dot{t} = \frac{\dot{s}}{v} = \frac{\dot{x}}{v} = \frac{c^{bx} \dot{x}}{a}$ . Now, to find the fluent of this, put  $z = c^{bx}$ ; then is  $bx = \log. z$ , and  $b\dot{x} = \frac{\dot{z}}{z}$ , or  $\dot{x} = \frac{\dot{z}}{bz}$ ; conseq.  $\dot{t}$  or  $\frac{c^{bx} \dot{x}}{a} = \frac{z\dot{x}}{a} = \frac{\dot{z}}{ab}$ ; and hence  $t = \frac{z}{ab} = \frac{c^{bx}}{ab}$ . But as  $t$  and  $x$  vanish together, and when  $x = 0$ , the quantity  $\frac{c^{bx}}{ab}$  is  $= \frac{1}{ab}$ ; therefore, by correction,  $t = \frac{c^{bx} - 1}{ab} = \frac{1}{bv} - \frac{1}{ba} = \frac{1}{b} \left( \frac{1}{v} - \frac{1}{a} \right)$  the time sought; where  $b = \frac{3n}{8Nd}$ , and  $v = \frac{a}{c^{bx}}$  the velocity.

EXAM. If an iron ball of 2 inches diameter were projected in the air with a velocity of 1200 feet per second; and it were required to determine in what time it would pass over 500 yards



500 yards or 1500 feet, and what would be its velocity at the end of that time: We should have, as in exam. 2 above,  $b = \frac{3 \cdot 12 \cdot 3 \cdot 6}{8 \cdot 22 \cdot 10000} = \frac{1}{2716}$ , and  $bx = \frac{1500}{2716} = \frac{375}{679}$ ; hence  $\frac{1}{v} = \frac{2716}{1}$ , and  $\frac{1}{a} = \frac{1}{1200}$ , and  $\frac{1}{v} = \frac{c^{bx}}{a} = \frac{1 \cdot 7372}{1200} = \frac{1}{690}$  nearly. Consequently  $v = 690$  is the velocity; and  $t = \frac{1}{b} \left( \frac{1}{v} - \frac{1}{a} \right) = 2716 \times \left( \frac{1}{690} - \frac{1}{1200} \right) = 1 \frac{3}{4}$  seconds, is the time required, or 1'' and  $\frac{2}{3}$  nearly.

PROBLEM XXII.

*To determine the Relations of Space, Time, and Velocity, when a Globe descends, by its own Weight, in an infinite Fluid.*

THE foregoing notation remaining, viz.  $d =$  diameter,  $N$  and  $n$  the density of the ball and fluid, and  $v, s, t$ , the velocity, space, and time, in motion; we have  $\frac{1}{6}pd^3 =$  the magnitude of the ball, and  $\frac{1}{6}pd^3(N - n) =$  its weight in the fluid, also  $m = \frac{pnd^2v^2}{32g} =$  its resistance from the fluid; consequently  $\frac{1}{6}pd^3(N - n) - \frac{pnd^2v^2}{32g}$  is the motive force by which the ball is urged; which being divided by  $\frac{1}{6}Nd^3$ , the quantity of matter moved, gives  $f = 1 - \frac{n}{N} - \frac{3nv^2}{16gNd}$  for the accelerative force.

2. Hence  $v\dot{v} = 2gfs$ , and  $s = \frac{v\dot{v}}{2gf} = \frac{Nv\dot{v}}{2g(N - n) - \frac{3n}{8d}v^2}$   
 $= \frac{1}{b} \times \frac{v\dot{v}}{a - v^2}$ , putting  $b = \frac{3n}{8Nd}$ , and  $\frac{1}{a} = \frac{3n}{2g \cdot 8d(N - n)}$ ,  
 or  $ab = 2g$  nearly; the fluent of which is  $s = \frac{1}{2b} \times \log. \text{ of } \frac{a}{a - v^2}$ , an expression for the space  $s$ , in terms of the velocity  $v$ .

3. But now, to determine  $v$  in terms of  $s$ , put  $c = 2 \cdot 718281828$ ; then, since the log. of  $\frac{a}{a - v^2} = 2bs$ , therefore

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fore  $\frac{a}{a-v^2} = c^{2bs}$ , or  $\frac{a-v^2}{a} = c^{-2bs}$ ; hence  $v = \sqrt{a - ac^{-2bs}}$  is the velocity sought.

4. The greatest velocity is to be found, as in art. 5 of prob. 20, by making  $f$  or  $1 - \frac{n}{N} - \frac{3nv^2}{16gNd} = 0$ , which gives  $v = \sqrt{2g \cdot 8d \cdot \frac{N-n}{3n}} = \sqrt{a}$ . The same value is also obtained by making the fluxion of  $v^2$ , or of  $a - ac^{-2bs}$ ,  $= 0$ . And the same value of  $v$  is obtained by making  $s$  infinite, for then  $c^{-2bs} = 0$ . But this velocity  $\sqrt{a}$  cannot be attained in any finite time, and it only denotes the velocity to which the general value of  $v$  or  $\sqrt{a - ac^{-2bs}}$  continually approaches. It is evident, however, that it will approximate towards it the faster, the greater  $b$  is, or the less  $d$  is; and that, the diameters being very small, the bodies descend by nearly uniform velocities, which are directly in the subduplicate ratio of the diameters. See also art. 5, prob. 20, for other observations on this head.

5. To find the time  $t$ . Now  $i = \frac{\dot{s}}{v} = \sqrt{\frac{1}{a}} \times \frac{\dot{s}}{\sqrt{1-c^{-2bs}}}$ .

Then, to find the fluent of this fluxion, put  $z = \sqrt{1-c^{-2bs}} = \frac{v}{\sqrt{a}}$ , or  $z^2 = 1 - c^{-2bs}$ ; hence  $z\dot{z} = bsc^{-2bs}$ , and  $\dot{s} = \frac{z\dot{z}}{bc^{-2bs}} = \frac{1}{b} \cdot \frac{z\dot{z}}{1-z^2}$ ; consequently  $i = \frac{1}{b\sqrt{a}} \cdot \frac{\dot{z}}{1-z^2}$ ,

and therefore the fluent is  $t = \frac{1}{2b\sqrt{a}} \times \log. \frac{1+z}{1-z} = \frac{1}{2b\sqrt{a}} \times \log. \frac{1 + \sqrt{1-c^{-2bs}}}{1 - \sqrt{1-c^{-2bs}}} = \frac{1}{2b\sqrt{a}} \times \log. \frac{\sqrt{a} + v}{\sqrt{a} - v}$ , which is the general expression for the time.

EXAM. If it were required to determine the time and velocity, by descending in air 1000 feet, the ball being of lead, and 1 inch diameter.

Here  $N = 11\frac{1}{3}$ ,  $n = \frac{3}{2500}$ ,  $d = \frac{1}{12}$ , and  $s = 1000$ .

$$\text{Hence } a = \frac{2 \cdot 16\frac{1}{12} \cdot \frac{8}{16} \cdot 11\frac{1}{3}}{3 \cdot \frac{3}{2500}} = \frac{2 \cdot 193 \cdot 8 \cdot 34 \cdot 2500}{3 \cdot 3 \cdot 12 \cdot 12 \cdot 3} =$$

$$\frac{193 \cdot 34 \cdot 50^2}{9 \cdot 27}, \text{ and } b = \frac{3 \cdot \frac{3}{2500}}{8 \cdot 11\frac{1}{3} \cdot \frac{1}{2}} = \frac{3 \cdot 3 \cdot 3 \cdot 12}{8 \cdot 34 \cdot 2500} = \frac{9 \cdot 9}{68 \cdot 50^2};$$

$$\text{consequently } v = \sqrt{a} \times \sqrt{1 - c^{2bs}} = \sqrt{\frac{193 \cdot 34 \cdot 50^2}{9 \cdot 27}} \times$$

$$\sqrt{1 - c^{\frac{81}{85}}} = 203\frac{2}{3} \text{ the velocity. And } t = \frac{1}{2b\sqrt{a}} \times \log.$$

$$\frac{1 + \sqrt{1 - c^{2bs}}}{1 - \sqrt{1 - c^{2bs}}} = \sqrt{\frac{34 \cdot 2500}{27 \cdot 193}} \times \log. \frac{1 \cdot 78383}{0 \cdot 21617} = 8 \cdot 5236'',$$

the time.

*Note.* If the globe be so light as to ascend in the fluid; it is only necessary to change the signs of the first two terms in the value of  $f$ , or the accelerating force, by which it be-

comes  $f = \frac{n}{N} - 1 - \frac{3nv^2}{16gNd}$ ; and then proceeding in all respects as before.

#### SCHOLIUM.

To compare this theory, contained in the last four problems, with experiment, the few following numbers are here extracted from extensive tables of velocities and resistances, resulting from a course of many hundred very accurate experiments, made in the course of the year 1786.

In the first column are contained the mean uniform or greatest velocities acquired in air, by globes, hemispheres, cylinders, and cones, all of the same diameter, and the altitude of the cone nearly equal to the diameter also, when urged by the several weights, expressed in averdupois ounces, and standing on the same line with the velocities, each in their proper column. So, in the first line, the numbers shew, that, when the greatest or uniform velocity was accurately 3 feet per second, the bodies were urged by these weights, according as their different ends went foremost; namely, by .028 oz. when the vertex of the cone went foremost; by .064 oz. when the base of the cone went foremost; by .027 oz. for a whole sphere; by .050 oz. for a cylinder; by .051 oz. for the flat side of the hemisphere; and by .020 oz. for the round or convex side of the hemisphere. Also, at the bottom of all, are placed the mean proportions of the resistances of these figures in the nearest whole numbers. Note, the common diameter of all the figures, was  $6 \cdot 375$  or  $6\frac{3}{8}$  inches; so that the area of the circle of that diameter is just 32 square inches, or  $\frac{2}{9}$  of a square foot; and the altitude

tude of the cone was  $6\frac{5}{8}$  inches. Also, the diameter of the small hemisphere was  $4\frac{3}{4}$  inches, and consequently the area of its base is  $17\frac{3}{4}$  square inches, or  $\frac{1}{8}$  of a square foot nearly.

From the given dimensions of the cone, it appears, that the angle made by its side and axis, or direction of the path, is 26 degrees, very nearly.

The mean height of the barometer at the times of making the experiments, was nearly 30.1 inches, and of the thermometer  $62^{\circ}$ ; consequently the weight of a cubic foot of air was equal to  $1\frac{1}{5}$  oz. nearly, in those circumstances.

Veloc. per sec.	Cone.		Whole globe.	Cylin- der.	Hemisphere.		Small Hemisf. flat
	vertex	base.			flat	round	
feet.	oz.	oz.	oz.	oz.	oz.	oz.	oz.
3	.028	.064	.027	.050	.051	.020	.028
4	.048	.109	.047	.090	.096	.039	.048
5	.071	.162	.068	.143	.148	.063	.072
6	.098	.225	.094	.205	.211	.092	.103
7	.129	.298	.125	.278	.284	.123	.141
8	.168	.382	.162	.360	.368	.160	.184
9	.211	.478	.205	.456	.464	.199	.233
10	.260	.587	.255	.565	.573	.242	.287
11	.315	.712	.310	.688	.698	.297	.349
12	.376	.850	.370	.826	.836	.347	.418
13	.440	1.000	.435	.979	.988	.409	.492
14	.512	1.166	.505	1.145	1.154	.478	.573
15	.589	1.346	.581	1.327	1.336	.552	.661
16	.673	1.546	.663	1.526	1.538	.634	.754
17	.762	1.763	.752	1.745	1.757	.722	.853
18	.858	2.002	.848	1.986	1.998	.818	.959
19	.959	2.260	.949	2.246	2.258	.922	1.073
20	1.069	2.540	1.05	2.528	2.542	1.033	1.196
Propor. Numb.	126	291	124	285	288	119	140

From this table of resistances, several practical inferences may be drawn. As,

1. That the resistance is nearly as the surface; the resistance increasing but a very little above that proportion in the greater

greater surfaces. Thus, by comparing together the numbers in the 6th and last columns, for the bases of the two hemispheres, the areas of which are in the proportion of  $17\frac{3}{4}$  to 32, or as 5 to 9 very nearly; it appears that the numbers in those two columns, expressing the resistances, are nearly as 1 to 2, or as 5 to 10, as far as to the velocity of 12 feet; after which the resistances on the greater surface increase gradually more and more above that proportion. And the mean resistances are as 140 to 288, or as 5 to  $10\frac{2}{7}$ . This circumstance therefore agrees nearly with the theory.

2. The resistance to the same surface, is nearly as the square of the velocity; but gradually increases more and more above that proportion as the velocity increases. This is manifest from all the columns. And therefore this circumstance also differs but little from the theory, in small velocities.

3. When the hinder parts of bodies are of different forms, the resistances are different, though the fore parts be alike; owing to the different pressures of the air on the hinder parts. Thus, the resistance to the fore part of the cylinder, is less than that on the flat base of the hemisphere, or of the cone; because the hinder part of the cylinder is more pressed or pushed, by the following air, than those of the other two figures.

4. The resistance on the base of the hemisphere, is to that on the convex side, nearly as  $2\frac{2}{3}$  to 1, instead of 2 to 1, as the theory assigns the proportion. And the experimented resistance, in each of these, is nearly  $\frac{1}{4}$  part more than that which is assigned by the theory.

5. The resistance on the base of the cone is to that on the vertex, nearly as  $2\frac{3}{10}$  to 1. And in the same ratio is radius to the sine of the angle of the inclination of the side of the cone, to its path or axis. So that, in this instance, the resistance is directly as the sine of the angle of incidence, the transverse section being the same, instead of the square of the sine.

6. Hence we can find the altitude of a column of air, whose pressure shall be equal to the resistance of a body, moving through it with any velocity. Thus,

Let  $a$  = the area of the section of the body, similar to any of those in the table, perpendicular to the direction of motion;

$r$  = the resistance to the velocity, in the table; and

$x$  = the altitude sought, of a column of air, whose base is  $a$ , and its pressure  $r$ .

Then  $ax$  = the content of the column in feet, and  $\frac{1}{5}ax$  or  $\frac{6}{5}ax$  its weight in ounces;

therefore  $\frac{6}{5}ax = r$ , and  $x = \frac{5}{6} \times \frac{r}{a}$  is the altitude sought in feet, namely,  $\frac{5}{6}$  of the quotient of the resistance of any body divided by its transverse section; which is a constant quantity for all similar bodies, however different in magnitude, since the resistance  $r$  is as the section  $a$ , as was found in art. 1. When  $a = \frac{9}{16}$  of a foot, as in all the figures in the foregoing table, except the small hemisphere: then,  $x = \frac{5}{6} \times \frac{r}{a}$  becomes  $x = \frac{1}{4}r$ , where  $r$  is the resistance in the table, to the similar body.

If, for example, we take the convex side of the large hemisphere, whose resistance is .634 oz. to a velocity of 16 feet per second, then  $r = .634$ , and  $x = \frac{1}{4}r = 2.3775$  feet, is the altitude of the column of air whose pressure is equal to the resistance on a spherical surface, with a velocity of 16 feet. And to compare the above altitude with that which is due to the given velocity, it will be  $32^2 : 16^2 :: 16 : 4$ , the altitude due to the velocity 16; which is near double the altitude that is equal to the pressure. And as the altitude is proportional to the square of the velocity, therefore, in small velocities, the resistance to any spherical surface, is equal to the pressure of a column of air on its great circle, whose altitude is  $\frac{1}{3} \frac{9}{2}$  or .594 of the altitude due to its velocity.

But if the cylinder be taken, whose resistance  $r = 1.526$ : then  $x = \frac{1}{4}r = 5.72$ ; which exceeds the height, 4, due to the velocity in the ratio of 23 to 16 nearly. And the difference would be still greater, if the body were larger; and also if the velocity were more.

7. Also, if it be required to find with what velocity any flat surface must be moved, so as to suffer a resistance just equal to the whole pressure of the atmosphere.

The

The resistance on the whole circle whose area is  $\frac{2}{9}$  of a foot, is .051 oz. with the velocity of 3 feet per second, it is  $\frac{1}{9}$  of .051, or .0056 oz. only with a velocity of 1 foot. But  $2\frac{1}{2} \times 13600 \times \frac{2}{9} = 7555\frac{5}{9}$  oz. is the whole pressure of the atmosphere. Therefore, as  $\sqrt{.0056} : \sqrt{7556} :: 1 : 1162$  nearly, which is the velocity sought. Being almost equal to the velocity with which air rushes into a vacuum.

8. Hence may be inferred the great resistance suffered by military projectiles. For, in the table, it appears, that a globe of  $6\frac{3}{8}$  inches diameter, which is equal to the size of an iron ball weighing 36lb, moving with a velocity of only 16 feet per second, meets with a resistance equal to the pressure of  $\frac{2}{3}$  of an ounce weight; and therefore, computing only according to the square of the velocity, the least resistance that such a ball would meet with, when moving with a velocity of 1600 feet, would be equal to the pressure of 417lb, and that independent of the pressure of the atmosphere itself on the fore part of the ball, which would be 487lb more, as there would be no pressure from the atmosphere on the hinder part, in the case of so great a velocity as 1600 feet per second. So that the whole resistance would be more than 900lb to such a velocity.

9. Having said, in the last article, that the pressure of the atmosphere is taken entirely off the hinder part of the ball, moving with a velocity of 1600 feet per second; which must happen when the ball moves faster than the particles of air can follow by rushing into the place quitted and left void by the ball, or when the ball moves faster than the air rushes into a vacuum from the pressure of the incumbent air: let us therefore inquire what this velocity is. Now, the velocity with which any fluid issues, depends upon its altitude above the orifice, and is indeed equal to the velocity acquired by a heavy body in falling freely through that altitude. But, supposing the height of the barometer to be 30 inches, or  $2\frac{1}{2}$  feet, the height of an uniform atmosphere, all of the same density as at the earth's surface, would be  $2\frac{1}{2} \times 13.6 \times 833\frac{1}{3}$  or 28333 feet; therefore  $\sqrt{16} : \sqrt{28333} :: 32 : 8 \sqrt{28333} = 1346$  feet, which is the velocity sought. And therefore, with a velocity of 1600 feet per second, or any velocity above 1346 feet, the ball must continually leave a vacuum behind it, and so must sustain the whole pressure of the atmosphere on its fore part, as well as the resistance arising from the vis inertia of the particles of air struck by the ball.

10. Upon the whole, we find, that the resistance of the air, as determined by the experiments, differs very widely, both in respect to the quantity of it on all figures, and in respect to the proportions of it on oblique surfaces, from the same as determined by the preceding theory; which is the same as that of Sir Isaac Newton, and most modern philosophers. Neither should we succeed better if we have recourse to the theory given by Professor Gravesande, or others, as similar differences and inconsistencies still occur.

We conclude, therefore, that all the theories of the resistance of the air hitherto given, are very erroneous. And the preceding one is only laid down, till further experiments, on this important subject, shall enable us to deduce from them, another, that shall be more consonant to the true phenomena of nature.

F I N I S.

ERRATA.

Pa. 324, l. 14, for  $\frac{ac}{r}$ , read  $\frac{cr}{a}$

Pa. 346, l. 9 from the bottom, for 1'36, read 1'96.









