UNIVERSITY OF CALIFORNIA

Los Angeles

Probabilistic perspectives on dispersive partial differential equations

A dissertation submitted in partial satisfaction

of the requirements for the degree

Doctor of Philosophy in Mathematics

by

Bjoern Bringmann

2021

ABSTRACT OF THE DISSERTATION

Probabilistic perspectives on dispersive partial differential equations

by

Bjoern Bringmann
Doctor of Philosophy in Mathematics
University of California, Los Angeles, 2021
Professor Terence Chi-Shen Tao, Chair

This thesis treats nonlinear dispersive equations with random initial data. First, we study the defocusing energy-critical nonlinear wave equation on Euclidean space. We prove that the scattering mechanism, which is well-understood for smooth initial data, is stable under rough and random perturbations. The main ingredients are Bourgain's bush argument, flux estimates, and a wave packet decomposition of the random linear evolution. Second, we study the three-dimensional wave equation with a Hartree nonlinearity. The main theorem proves the existence and invariance of the Gibbs measure. The novelty lies in the singularity of the Gibbs measure with respect to the Gaussian free field. The argument combines techniques from several areas of mathematics, such as dispersive equations, harmonic analysis, and random matrix theory.

The dissertation of Bjoern Bringmann is approved.

Georg Menz

Rowan Brett Killip

Monica Visan

Terence Chi-Shen Tao, Committee Chair

University of California, Los Angeles

2021

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ACKNOWLEDGMENTS

I would like to express my deepest gratitude to my advisor Terence Tao for his constant encouragement and support. Under his guidance, I not only learned mathematics but how to be a mathematician. I am also very grateful to Rowan Killip and Monica Visan, who taught me much during their lectures, our discussions, and our joint work. I would like to thank Angela Stevens for sending me on the road towards partial differential equations.

I would like to thank the dispersive PDE community for several interesting discussions during conferences and seminars. A special thank you goes to Yu Deng, Ben Dodson, Justin Forlano, Zaher Hani, Herbert Koch, Jonas Lührmann, Dana Mendelson, Andrea Nahmod, Tadahiro Oh, Gigliola Staffilani, Daniel Tataru, Leonardo Tolomeo, Nikolay Tzvetkov, and Haitian Yue. Furthermore, I want to thank Nikolay Barashkov, Massimiliano Gubinelli, Martin Hairer, Felix Otto, and Nikolas Perkowski for helpful discussions on singular SPDEs.

I would like to thank my friends and colleagues at UCLA, including Adam Azzam, Laura Cladek, Mitia Duerinckx, John Garnett, Michael Hitrik, Asgar Jamneshan, Siting Liu, Georg Menz, Maria Ntekoume, Yuejiao Sun, and Blaine Talbut, for creating a stimulating environment. My special thanks go to my friends Gyu Eun Lee, Zane Li, and Redmond McNamara from the "harmonic analysis reading group".

I wish to thank my parents and sister for letting me move to the other end of the world to pursue my dream. Finally, I want to thank Fei Feng for all the happiness she brings into me life. Without her support, I would not be where I am today.

VITA

2014	B.Sc. (Mathematics), University of Muenster
2016	M.Sc. (Mathematics), Technical University of Munich
2018	Ernst Adolf Marum Fellowship, University of California, Los Angeles
2020	Dissertation Year Fellowship, University of California, Los Angeles

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Bjoern Bringmann, Dana Mendelson, *An eigensystem approach to Anderson localization for multi-particle systems*, To appear in *Ann. Henri Poincaré (A Journal of Theoretical and Mathematical Physics)*

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CHAPTER 1

Introduction[1]

The research in this thesis lies at the intersection of *nonlinear dispersive equations* and *probability theory*. Dispersive equations model physical systems in which waves of different frequencies propagate through a medium at different velocities. For instance, dispersive equations are used as theoretical models in nonlinear optics [Abl11], quantum many-body systems [ESY07, ESY09, ESY10], and water waves [TW12, Tot15]. The most illustrative consequence of dispersive effects is a rainbow, which occurs when light passes through rain droplets and is split up into different colors. In addition to dispersive effects, many physical models contain disorder and randomness. In statistical mechanics, many central questions concern statistical equilibria, such as Gibbs measures. More generally, randomness can be a consequence of microscopic fluctuations in densities or temperatures. Physical models involving dispersion and randomness appear in plasma physics [LRS88] and the study of water waves [MMQ19].

The mathematical interest in the interface of dispersive equations and probability theory is due to its connections to several active areas of research. Far from just combining two areas of mathematics, recent advances also build on ideas from analytic number theory [BGH19], differential geometry [KLS20], harmonic analysis [GKO18b, Bri20a], random matrix theory [Bou96, DNY20], and quantum field theory [BG20b, BG20a]. In broad terms, the main question in the study of

[1]The first chapter of the thesis is partially based on the author's research statement for postdoctoral research positions, which is available on request.

random dispersive equations can be phrased as follows:

Main question. *How does randomness affect the flow of nonlinear dispersive equations?*

In case the randomness of the physical system enters only through the initial data, the main question concerns the push-forward of the initial distribution under the (nonlinear) data-to-solution map. As stated here, however, the main question is far too general for a direct answer. It may depend on

(1) the form of the dispersive equation,

(2) the distribution of the randomness,

(3) and which properties of the flow are under consideration.

Regarding (1), the answer or method of proof may depend on the spatial dimension, the linear dispersive symbol, or the strength and structure of the nonlinearity. Regarding (2), the answer differs for statistical equilibria, such as Gibbs measures, and out-of-equilibrium dynamics, as in wave turbulence. Regarding (3), the answer may depend on whether randomness is measured at a single time, multiple times, or in the large-time limit.

In the following two subsections, we briefly describe the two manifestations of our main questions which are addressed in this thesis. A more detailed description of the relevant literature and our arguments, however, is postponed until the introduction of the individual chapters.

1.1 Random perturbations

In this section, we discuss the defocusing energy-critical nonlinear wave equation in dimension $d \geqslant 3$, which is given by

$$\begin{cases} -\partial_{tt} u + \Delta u = |u|^{\frac{4}{d-2}} u \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^d \ , \\ u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1 \ . \end{cases} \tag{1.1.1}$$

In this thesis, we restrict ourselves to spatial dimensions $d = 3$ or $d = 4$, in which (1.1.1) has the polynomial nonlinearities u^5 and u^3, respectively. In higher dimensions, the low-regularity of the nonlinearity $F(u) = |u|^{\frac{4}{d-2}} u$ can lead to technical obstructions and we refer the reader to [BCL13, Vis07] for a more detailed discussion. The nonlinear wave equation (1.1.1) has the conserved energy

$$E[u](t) := \int_{\mathbb{R}^d} \frac{|\nabla u(t,x)|^2}{2} + \frac{(\partial_t u(t,x))^2}{2} + \frac{d-2}{2d} |u(t,x)|^{\frac{2d}{d-2}} \mathrm{d}x. \tag{1.1.2}$$

By using the Sobolev embedding $\dot{H}^1(\mathbb{R}^d) \hookrightarrow L^{\frac{2d}{d-2}}(\mathbb{R}^d)$ of the homogeneous Sobolev space, we obtain that the energy is finite if and only if $(u(t), \partial_t u(t)) \in \dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. The nonlinear wave equation (1.1.1) is invariant under the scaling symmetry

$$u(t,x) \mapsto u_\lambda(t,x) := \lambda^{\frac{d-2}{2}} u(\lambda t, \lambda x). \tag{1.1.3}$$

In addition to preserving the evolution equation, the scaling symmetry (1.1.3) also preserves the energy of the solution. Thus, the energy is scaling-critical, which is the reason for calling (1.1.1) energy-critical.

Our focus lies on the asymptotic behavior of solutions to (1.1.1). For initial data in the energy-space $\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$, this problem is now well-understood and we summarize the known results in the next theorem.

Theorem 1.1.1 (Global well-posedness and scattering [BG99, Gri90, Gri92, Rau81, SS93, SS94, Str68, Str88, Tao06b])**.** Let $3 \leqslant d \leqslant 4$ and let $(u_0, u_1) \in \dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. Then, there exists a unique global solution of (1.1.1) satisfying

$$u \in \Big(C_t^0 \dot{H}_x^1 \bigcap L_t^{\frac{d+2}{d-2}} L_x^{\frac{2(d+2)}{d-2}} \Big)(\mathbb{R} \times \mathbb{R}^d) \qquad \text{and} \qquad \partial_t u \in C_t^0 L_x^2(\mathbb{R} \times \mathbb{R}^d).$$

Furthermore, the solution u scatters. To be precise, there exist scattering states $(u_0^{\pm}, u_1^{\pm}) \in \dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ s.t.

$$\lim_{t \to \pm\infty} \|(u(t) - W(t)(u_0^{\pm}, u_1^{\pm}), \partial_t u(t) - \partial_t W(t)(u_0^{\pm}, u_1^{\pm}))\|_{\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)} = 0 \ .$$

Here, $W(t)(u_0^{\pm}, u_1^{\pm})$ denotes the solution to the linear wave equation with initial data $(u_0^{\pm}, u_1^{\pm})$.

In fact, Theorem 1.1.1 has not only been proven for the energy-critical nonlinear wave equation (1.1.1), but also serves as a blueprint for other defocusing dispersive equations at critical regularity. In the case of the defocusing energy-critical nonlinear Schrödinger equation, which is given by

$$\begin{cases} i\partial_t + \Delta u = |u|^{\frac{4}{d-2}} u \qquad (t, x) \in \mathbb{R} \times \mathbb{R}^d \ , \\ u|_{t=0} = u_0, \end{cases} \tag{1.1.4}$$

similar results have been obtained in the seminal works [Bou99, CKS08, Vis07, RV07].

In the spirit of our main question, we now ask whether the scattering mechanism (as proven in Theorem 1.1.1) is stable under rough and random perturbations. To define the random perturbations, we first let $\mathcal{J}$ be a countable index set. Then, we let $(P_j)_{j \in \mathcal{J}}$ be a sequence of operators (on certain Sobolev spaces) such that $\sum_{j \in \mathcal{J}} P_j$ strongly converges to the identity. In the literature, $(P_j)_{j \in \mathcal{J}}$ is chosen as a unit-scale decomposition in frequency space [BOP15b, LM14, Bri20b], physical space [Mur19], or, as in this work, phase space [Bri18]. We also let $(X_j)_{j \in \mathcal{J}}$ be a sequence of independent (sub-)Gaussian random variables. For more details regarding the operators or random

	Wave	Schrödinger
radial	$d = 4$: [DLM20, DLM19]. d=3: [Bri20b].	$d = 4$: [KMV19, DLM19].
non-radial	$d = 4$: [Bri18].	Open.

Figure 1.1: Almost sure scattering for defocusing energy-critical dispersive equations.

variables, we refer the reader to the introduction of Chapter 2. Finally, we let $0 \leqslant s < 1$ and let $(f_0, f_1) \in H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)$ be rough initial data. Then, the randomized initial data (f_0^ω, f_1^ω) is defined as

$$f_i^\omega = \sum_{j \in \mathcal{J}} X_j P_j f_i \qquad \text{for } i = 0, 1. \tag{1.1.5}$$

In order to study the stability of the scattering mechanism under random perturbations, we consider the random data Cauchy problem

$$\begin{cases} -\partial_{tt} u + \Delta u = |u|^{\frac{4}{d-2}} u & (t,x) \in \mathbb{R} \times \mathbb{R}^d \ , \\ u|_{t=0} = u_0 + f_0^\omega, \quad \partial_t u|_{t=0} = u_1 + f_1^\omega. \end{cases} \tag{1.1.6}$$

In recent years, there has been tremendous interest in (2.1.5) and related problems for the nonlinear Schrödinger equations (1.1.4). While we provide a brief overview of the literature in Figure 1.1, we postpone a more detailed discussion to Chapter 2. At this point, we only emphasize that there is a significant difference between the radial and non-radial case.

The first main result of this thesis proves the analogue of Theorem 1.1.1, which holds for smooth and deterministic initial data, for the random data Cauchy problem (1.1.6) in dimension $d = 4$.

Theorem 1.1.2 ([Bri18])**.** Let $(f_0, f_1) \in H^s(\mathbb{R}^4) \times H^{s-1}(\mathbb{R}^4)$, where $s > \frac{11}{12}$, let (f_0^ω, f_1^ω) be the microlocal randomization as in Definition 2.1.2, and let $(u_0, u_1) \in \dot{H}^1(\mathbb{R}^4) \times L^2(\mathbb{R}^4)$. Then, there exists a global solution $u\colon \mathbb{R}\times\mathbb{R}^4 \to \mathbb{R}$ of (1.1.6) such that

$$u \in W(t)(f_0^\omega, f_1^\omega) + \big(C_t^0 \dot{H}_x^1 \bigcap L_t^3 L_x^6\big)(\mathbb{R}\times\mathbb{R}^4) \quad \text{and} \quad \partial_t u \in \partial_t W(t)(f_0^\omega, f_1^\omega) + C_t^0 L_x^2(\mathbb{R}\times\mathbb{R}^4) \ .$$

Furthermore, the solution u scatters. To be precise, there exist scattering states $(u_0^\pm, u_1^\pm) \in \dot{H}^1(\mathbb{R}^d)\times L^2(\mathbb{R}^d)$ s.t.

$$\lim_{t\to\pm\infty} \| (u(t) - W(t)(u_0^\pm + f_0^\omega, u_1^\pm + f_1^\omega), \partial_t u(t) - \partial_t W(t)(u_0^\pm + f_0^\omega, u_1^\pm + f_1^\omega)) \|_{\dot{H}^1\times L^2} = 0 \ .$$

In contrast to the earlier literature, Theorem 1.1.2 does not require the radial symmetry of (f_0, f_1). The main novelty in our argument lies in a wave packet decomposition of the linear evolution of the random perturbation. Since a more detailed discussion of the main ideas requires additional notation, we postpone it until Chapter 2.

1.2 Invariant Gibbs measures

In this subsection, we are interested in invariant measures for nonlinear dispersive equations. In this invariant setting, the distribution of the initial data is preserved by the flow. From a physical perspective, the most natural candidates for invariant measures are Gibbs measures.

The proof of the existence and invariance of Gibbs measures is one of the most classical problems for dispersive equations. Once existence and invariance have been shown, the general theory of dynamical systems yields interesting information about the flow of the dispersive equation under consideration. For example, the Poincaré recurrence theorem proves that the evolution will return infinitely often to states arbitrarily close to its initial state.

In the following, we restrict ourselves to results for defocusing semilinear wave and Schrödinger equations with periodic boundary conditions, which are also displayed in Figure 1.2. In one dimension, the existence and invariance of the Gibbs measure was proven by Bourgain [Bou94],

Dimension & Nonlinearity	Wave	Schrödinger
$d = 1$, $\lvert u\rvert^{p-1}u$	[Fri85, Zhi94]	[Bou94]
$d = 2$, $\lvert u\rvert^2 u$	[OT20a]	[Bou96]
$d = 2$, $\lvert u\rvert^{p-1}u$		[DNY19]
$d = 3$, $(V_\beta * \lvert u\rvert^2)u$	$\beta > 1$: [OOT20] $\beta > 0$: [Bri20c, Bri20d]	$\beta > 2$: [Bou97] $\beta > 1-\epsilon$: [DNY21] $\beta > 0$: Open
$d = 3$, $\lvert u\rvert^2 u$	Open	Open

Figure 1.2: Invariant Gibbs measures[2].

Friedlander [Fri85], and Zhidkov [Zhi94]. Bourgain [Bou96] also solved this problem for the two-dimensional cubic nonlinear Schrödinger equation. For general power-type nonlinearities in two dimensions, however, this problem was only solved recently by Oh and Thomann [OT20a] and Deng, Nahmod, and Yue [DNY19] for wave and Schrödinger equations, respectively. Unfortunately, the invariance of Gibbs measures is still open for many important dispersive equations in three spatial dimensions. A more detailed discussion of both the earlier literature and remaining open problems is contained in Chapter 4.

In this thesis, we focus on the three-dimensional wave equation with a Hartree nonlinearity given by

$$-\partial_t^2 u - u + \Delta u = \,:(V_\beta * u^2)u: \qquad (t,x) \in \mathbb{R} \times \mathbb{T}^3. \tag{1.2.1}$$

Here, $\mathbb{T}^3$ is the three-dimensional flat torus. Regarding the potential $V_\beta \colon \mathbb{T}^3 \to \mathbb{R}$, which depends on a regularity parameter $\beta \in (0,3)$, we make the following assumption.

[2]Figure 1.2 is a modification of Figure 1 in [Bri20d], which also appears in Chapter 4 of this thesis.

Assumptions. *We assume that the interaction potential V_β satisfies*

(i) $V_\beta(x) = c_\beta |x|^{-(3-\beta)}$ *for some* $c_\beta > 0$ *and all* $x \in \mathbb{T}^3$ *satisfying* $\|x\| \leqslant 1/10$,

(ii) $V_\beta(x) \gtrsim_\beta 1$ *for all* $x \in \mathbb{T}^3$,

(iii) $V_\beta(x) = V_\beta(-x)$ *for all* $x \in \mathbb{T}^3$,

(iv) V_β *is smooth away from the origin.*

Finally, the nonlinearity $:(V_\beta * u^2)u:$ in (1.2.1) is a renormalization of $(V_\beta * u^2)u$, which is defined in Definition 3.2.6.

Similar as in Section 1.1, the Hartree nonlinear wave equation (1.2.1) also obeys an energy conservation law. In order to emphasize the Hamiltonian structure of (1.2.1), however, the energy is commonly referred to as the Hamiltonian and denoted by H_β. It is given by

$$H_\beta[u, \partial_t u](t) = \frac{1}{2}\Big(\|u(t,x)\|_{L_x^2}^2 + \|\nabla_x u(t,x)\|_{L_x^2}^2 + \|\partial_t u(t,x)\|_{L_x^2}^2 \Big) + \frac{1}{4}\int_{\mathbb{T}^3} :(V_\beta * u^2)\, u^2 : (t,x) \mathrm{d}x. \quad (1.2.2)$$

The Gibbs measure $\mu_\beta^{\otimes}$, which is associated with the Hamiltonian H_β and expected to be invariant under (1.2.1), is formally given by

$$\mathrm{d}\mu_\beta^{\otimes}(u, u_t) = \mathcal{Z}^{-1} \exp\Big(- H_\beta[u, \partial_t u] \Big) \mathrm{d}u \mathrm{d}u_t,$$

where $\mathcal{Z}$ is a normalization constant. The superscript $\otimes$ in $\mu_\beta^{\otimes}$ emphasizes that the Gibbs measure yields both a random initial position u and an initial velocity u_t. In Chapter 3 and 4, we prove the second main theorem of this thesis.

Theorem 1.2.1 ([Bri20c, Bri20d], Informal version)**.** The Gibbs measure μ_β exists and, for $\beta \in (0, 1/2)$, is singular with respect to the so-called Gaussian free field $\mathscr{g}$. Furthermore, μ_β is invariant under the evolution of (1.2.1).

For a more precise version of this theorem, we refer to Theorem 3.1.1 and Theorem 4.1.3 below. The Gaussian free field, which appears in Theorem 1.2.1, is a central object in probability theory. In our context, it is best understood as a random Fourier series with Gaussian coefficients, see (3.1.7) and (4.1.15). In all previous results on invariant Gibbs measures for dispersive equations, the Gibbs measure is absolutely continuous with respect to the Gaussian free field. In other words, Theorem 1.2.1 is the only available result with a singular Gibbs measure. The proof of Theorem 1.2.1 naturally splits into a measure-theoretic and dynamical part, which form Chapter 3 and Chapter 4, respectively.

The measure-theoretic part (Chapter 3) is based on ideas from stochastic quantization [Nel66, PW81]. The rigorous mathematical treatment [AK20, HM18, MW17, GH19] relies on recent advances in singular SPDEs, such as Hairer's regularity structures [Hai14] and Gubinelli, Imkeller, and Perkowski's para-controlled calculus [GIP15]. Inspired by stochastic quantization, Barashkov and Gubinelli [BG20b, BG20a] recently developed a new variational approach to the construction of Gibbs measures, which is used in this thesis. The main differences between [BG20b, BG20a] and Chapter 3 stem from the nonlocality of the Hartree nonlinearity. A more detailed discussion of the literature and main ideas is contained in the introduction of Chapter 3.

The dynamical part (Chapter 4), which is more difficult than the measure-theoretic part, is further split into a local and global theory. The local theory combines tools such as dispersive estimates, lattice point estimates, para-product decompositions, random operator bounds, and Wiener chaos estimates, which stem from different areas of mathematics. The global theory is based on an adaption of Bourgain's globalization argument [Bou94] to singular Gibbs measures. A detailed description of the main ideas is contained in the introduction of Chapter 4.

CHAPTER 2

Almost sure scattering for the energy critical nonlinear wave equation[3]

2.1 Introduction

We consider the defocusing cubic nonlinear wave equation in four space dimensions, that is,

$$\begin{cases} -\partial_{tt}u + \Delta u = u^3 \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^4 \ , \\ u|_{t=0} = u_0 \in \dot{H}^s(\mathbb{R}^4), \quad \partial_t u|_{t=0} = u_1 \in \dot{H}^{s-1}(\mathbb{R}^4) \ . \end{cases} \tag{2.1.1}$$

If u is a regular solution of (2.1.1), then it conserves the energy

$$E[u](t) := \int_{\mathbb{R}^4} \frac{|\nabla u(t,x)|^2}{2} + \frac{(\partial_t u(t,x))^2}{2} + \frac{u(t,x)^4}{4} \mathrm{d}x \ . \tag{2.1.2}$$

From the Sobolev embedding $\dot{H}^1(\mathbb{R}^4) \hookrightarrow L^4(\mathbb{R}^4)$, it follows that the initial data has finite energy if and only if $(u_0, u_1) \in \dot{H}^1(\mathbb{R}^4) \times L^2(\mathbb{R}^4)$. Thus, we also refer to $\dot{H}^1(\mathbb{R}^4) \times L^2(\mathbb{R}^4)$ as the energy space. In addition to the energy conservation law, (2.1.1) obeys the scaling symmetry $u(t,x) \mapsto u_\lambda(t,x) = \lambda u(\lambda t, \lambda x)$. Since the scaling leaves the energy invariant, the equation is called energy critical. Due to the positive sign in front of the potential term u^4, we call (2.1.1) defocusing. There also exists analogues of (2.1.1) with a power-type nonlinearity in any dimension

$d \geqslant 3$.

The Cauchy problem for (deterministic) initial data in the energy space is well-understood. We summarize the relevant results in the following theorem.

Theorem 2.1.1 (Global well-posedness and scattering [BG99, Gri90, Gri92, Rau81, SS93, SS94, Str68, Str88, Tao06b])**.** Let $(u_0, u_1) \in \dot{H}^1(\mathbb{R}^4) \times L^2(\mathbb{R}^4)$. Then, there exists a maximal time interval of existence I and a unique solution $u\colon I \times \mathbb{R}^4 \to \mathbb{R}$ of (2.1.1) such that $u \in C_t^0 \dot{H}_x^1(I \times \mathbb{R}^4) \cap L^3_{t,\mathrm{loc}} L^6_x(I \times \mathbb{R}^4)$ and $\partial_t u \in C_t^0 L_x^2(I \times \mathbb{R}^4)$. Furthermore, we have that

(i) u is global, i.e., $I = \mathbb{R}$.

(ii) u obeys a global space-time bound of the form

$$\|u\|_{L_t^3 L_x^6(\mathbb{R}\times\mathbb{R}^4)} \leqslant C(E[u_0, u_1]) \ .$$

(iii) u scatters to a solution of the linear wave equation. Thus, there exist scattering states $(u_0^\pm, u_1^\pm) \in \dot{H}^1(\mathbb{R}^4) \times L^2(\mathbb{R}^4)$ s.t.

$$\lim_{t\to\pm\infty} \|(u(t) - W(t)(u_0^\pm, u_1^\pm), \partial_t u(t) - \partial_t W(t)(u_0^\pm, u_1^\pm))\|_{\dot{H}^1 \times L^2} = 0 \ .$$

Here, $W(t)(u_0^\pm, u_1^\pm) = \cos(t|\nabla|)u_0^\pm + (\sin(t|\nabla|)/|\nabla|)u_1^\pm$ denotes the solution to the linear wave equation with initial data $(u_0^\pm, u_1^\pm)$.

Global well-posedness and scattering results such as Theorem 2.1.1 are known for many defocusing dispersive partial differential equations, and hold for the energy critical nonlinear Schrödinger equation [Bou99, CKS08, RV07, Vis07], the mass-critical nonlinear Schrödinger equation [Dod12, Dod16a, Dod16b, KTV09, KVZ08], the mass-critical generalized KdV [Dod17], and the $\dot{H}^{\frac{1}{2}}$-critical radial nonlinear wave equation [Dod18].

Since Theorem 2.1.1 provides a complete description of the Cauchy problem with initial data in the energy space, we now seek a similar result for initial data in a rough Sobolev space $H^s_x \times H^{s-1}_x$, where $s \in [0,1)$. However, since this leads to a scaling super-critical problem, all of the above properties can fail. In fact, [CCT03] proved that (2.1.1) exhibits norm inflation, which means that arbitrarily small data in $H^s \times H^{s-1}$ can grow arbitrarily fast. More precisely, we have for all $\epsilon > 0$ that there exists Schwartz initial data (u_0, u_1) and a time $0 < t_\epsilon < \epsilon$ such that $\|(u_0, u_1)\|_{H^s \times H^{s-1}} < \epsilon$ and $\|(u(t_\epsilon), \partial_t u(t_\epsilon))\|_{H^s \times H^{s-1}} > \epsilon^{-1}$. Using finite speed of propagation, one may then also construct solutions whose $H^s \times H^{s-1}$-norm blows up instantaneously.

2.1.1 The random data Cauchy problem

Many researchers in dispersive partial differential equations have recently examined whether blow-up behaviour, such as the norm-inflation described above, occurs for generic or only exceptional sets of rough initial data. To quantify this, one is quickly lead to random initial data. Indeed, one natural form of rough initial data is $(u_0 + f_0^\omega, u_1 + f_1^\omega)$, where the functions $(u_0, u_1) \in \dot{H}^1 \times L^2$ are regular and deterministic, while the functions $(f_0^\omega, f_1^\omega) \in H^s \times H^{s-1}$ are rough and random. An analogue of Theorem 2.1.1 in this case would imply the stability of the scattering mechanism under a perturbation by noise.

The literature on random dispersive partial differential equations is vast. We refer the interested reader to the survey [BOP19b], and mention the related works [BOP15a, BOP19a, Bou94, Bou96, BB14b, Bri20a, BT08a, BT08b, CCM20, LM14, LM16, NOR12, NPS13, Poc17]. In the following discussion, we focus on the Wiener randomization [BOP15b, LM14] of a function $f \in H^s(\mathbb{R}^d)$. Let $\varphi \in C^\infty(\mathbb{R}^d)$ be a smooth and symmetric function satisfying $\varphi|_{[-3/8,3/8]^d} = 1$, $\varphi|_{\mathbb{R}^d \setminus [-5/8,5/8]^d} = 0$, and $\sum_{k \in \mathbb{Z}^d} \varphi(\xi - k) = 1$ for all $\xi \in \mathbb{R}^d$. We then define the associated operator P_k by

$$\widehat{P_k f}(\xi) := \varphi(\xi - k) \widehat{f}(\xi) \ .$$

Since the translates $\{\varphi(\cdot - k)\}_k$ form a partition of unity, we have that

$$f = \sum_{k \in \mathbb{Z}^d} P_k f \ , \tag{2.1.3}$$

which is called the Wiener decomposition of f. The Wiener randomization is obtained by randomizing the coefficients in (2.1.3). Let $I \subseteq \mathbb{Z}^d$ by an index set such that $\mathbb{Z}^d = I \dot\cup \{0\} \dot\cup (-I)$. Let $\{X_k\}_{k \in I \cup \{0\}}$ be a sequence of symmetric, independent, and uniformly sub-gaussian random variables (see Definition 2.2.1). We set $X_{-k} := \overline{X_k}$ for all $k \in I$, and assume that X_0 is real-valued. Then, the Wiener randomization f^W is defined as

$$f^W := \sum_{k \in \mathbb{Z}^d} X_k \cdot P_k f \ . \tag{2.1.4}$$

The reason for introducing the set I is to preserve the real-valuedness of f. The Wiener randomization f^W is a random linear combination of functions with unit-scale frequency uncertainty, and therefore resembles a random Fourier series. We then examine the random data Cauchy problem

$$\begin{cases} -\partial_{tt} u + \Delta u = u^3 & (t,x) \in \mathbb{R} \times \mathbb{R}^4 \\ u|_{t=0} = u_0 + f_0^W, \quad \partial_t u|_{t=0} = u_1 + f_1^W \end{cases} . \tag{2.1.5}$$

We now seek an almost sure version of Theorem 2.1.1 for (2.1.5). Before we summarize the recent results, let us sketch the overall strategy, which was developed by Pocovnicu in [Poc17]. We let $F := \cos(t|\nabla|) f_0^W + (\sin(t|\nabla|)/|\nabla|) f_1^W$ be the solution of the linear wave equation with the rough and random initial data. We then define the nonlinear component v by $v := u - F$, and obtain the forced nonlinear wave equation

$$\begin{cases} -\partial_{tt} v + \Delta v = (v+F)^3 & (t,x) \in \mathbb{R} \times \mathbb{R}^4 \ , \\ v|_{t=0} = u_0, \quad \partial_t v|_{t=0} = u_1. \end{cases} \tag{2.1.6}$$

At the cost of introducing a rough forcing term, we have therefore removed the rough part of the initial data. This transformation is related to the Da Prato-Debussche trick [DD02]. Due to the smoothing effect of the Duhamel integral, we hope to control the nonlinear component v in the energy space. The local well-posedness of (2.1.6) follows readily from probabilistic Strichartz estimates (cf. [BOP15b, LM14]) and a contraction mapping argument. Thus our main interest lies in the global well-posedness and the long-time behaviour of the solution. Using the deterministic well-posedness theorem and stability theory, it can be shown (cf. [DLM20, Poc17]) that the solution to (2.1.6) exists as long as the energy of v remains bounded. Of course, due to the forcing term in (2.1.6), the energy is no longer conserved. In addition, a global bound on the energy of v implies a global bound on the $L_t^3 L_x^6$-norm, and hence also implies scattering. A short calculation shows that

$$\frac{\mathrm{d}}{\mathrm{d}t} E[v](t) = \int_{\mathbb{R}^4} (v^3 - (v+F)^3)\partial_t v \ \mathrm{d}x \approx -3 \int_{\mathbb{R}^4} F v^2 \partial_t v \mathrm{d}x \ . \tag{2.1.7}$$

In the formula above, we have neglected terms that contain more than a single factor of F, since they are simpler to estimate. Therefore, the remaining obstacle lies in the control of the right-hand side of (2.1.7). With this overall strategy in mind, we summarize the recent literature.

In [Poc17], Pocovnicu proved the almost sure global existence of solutions for all $s > 0$. Using a Gronwall-type argument and a probabilistic Strichartz estimate, (2.1.7) leads (at top order) to the growth estimate

$$E[v](T) \lesssim E[v](0) \exp(C \|F\|_{L_t^1 L_x^\infty([0,T]\times\mathbb{R}^4)}) \lesssim E[v](0) \exp(C_\omega T^{\frac{1}{2}}) \ . \tag{2.1.8}$$

Since this prevents the finite time blow-up of the energy, this yields an analogue of Theorem 2.1.1.(i). Similar theorems are also known in dimension five [Poc17], dimension three [OP16], and for the high-dimensional energy critical nonlinear Schrödinger equation [OOP19].

The bound (2.1.8), however, is not sufficient to obtain global control on the energy of v, and hence does not prove almost sure scattering. Assuming the regularity condition $s > \frac{1}{2}$ and that the (deterministic) data (f_0, f_1) is spherically symmetric, Dodson, Lührmann, and Mendelson proved almost sure scattering in [DLM20]. In their argument, the energy increment is estimated by

$$\Big| \int_0^T \int_{\mathbb{R}^4} F v^2 \partial_t v \mathrm{d}x \mathrm{d}t \Big| \lesssim \| |x|^{\frac{1}{2}} F \|_{L_t^2 L_x^\infty([0,T]\times\mathbb{R}^4)} \, \| |x|^{-\frac{1}{4}} v \|^2_{L_t^4 L_x^4([0,T]\times\mathbb{R}^4)} \, \| \partial_t v \|_{L_t^\infty L_x^2(\mathbb{R}\times\mathbb{R}^4)} \,. \tag{2.1.9}$$

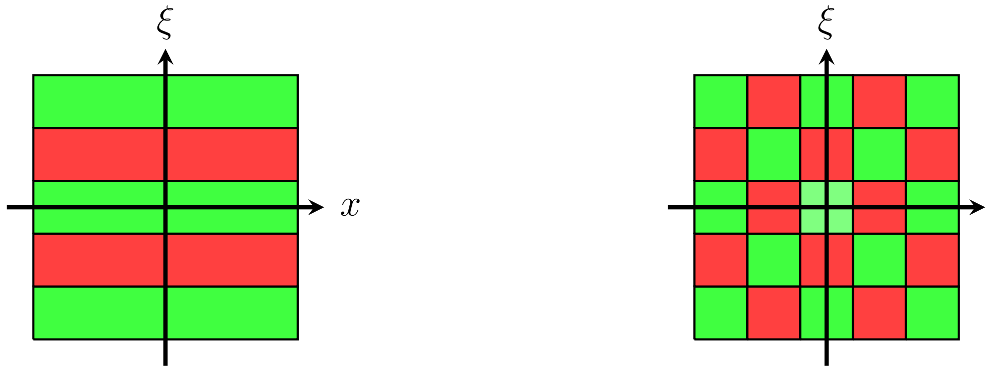
The first factor is controlled using Khintchine's inequality and a square-function estimate, and heavily relies on the spherical symmetry of f_0 and f_1. The main novelty lies in the control of the second factor, and involves a double bootstrap argument in the energy and a Morawetz term. Under the bootstrap hypothesis, one can then control the second factor in (2.1.9) by the square-root of the energy, and this eventually leads to a global bound.

The method of [DLM20] has since been used in several related works. In [DLM19], Dodson, Lührmann, and Mendelson used local energy decay to improve the regularity condition to $s > 0$. After replacing the cubes in the Wiener randomization by thin annuli, the author proved almost sure scattering for radial data in dimension three [Bri20b]. The main new ingredient is an interaction flux estimate between the linear and nonlinear components of the solution. Finally, the almost sure scattering for the radial energy critical nonlinear Schrödinger equation in four dimensions has been obtained in [DLM19, KMV19].

2.1.2 Main result and ideas

The remaining open question is concerned with almost sure scattering for non-radial data. In order to state the main result of this paper, we first need to introduce a microlocal randomization. While the Wiener randomization is based on a unit-scale decomposition in frequency space, the microlocal randomization is based on a unit-scale decomposition in phase space (see Figure 2.1).

Definition 2.1.2 (Microlocal randomization)**.** Let $\{X_{k,l}\}_{k\in I\cup\{0\}, l\in\mathbb{Z}^d}$ be a sequence of symmetric, independent, and uniformly sub-gaussian random variables. We set $X_{-k,l} := \overline{X_{k,l}}$ for all $k \in I$, and



We display a partition of the phase space $\mathbb{R}^d \times \mathbb{R}^d$ into horizontal strips, which forms the basis of the Wiener randomization, and a partition into cubes, which forms the basis of the microlocal randomization. A similar figure has been used in the author's previous work [Bri20b, Figure 1].

Figure 2.1: Partions of phase space

assume that $X_{0,l}$ is real-valued. For any $f \in H^s(\mathbb{R}^d)$, we define its microlocal randomization f^ω by

$$f^\omega(x) := \sum_{k,l \in \mathbb{Z}^d} X_{k,l} P_k(\varphi(\cdot - l) f)(x) \ . \tag{2.1.10}$$

The microlocal randomization is inspired by [Mur19], which used a randomization in physical space.

Theorem 2.1.3 (Almost sure scattering for the microlocal randomization)**.** Let $(u_0, u_1) \in \dot{H}^1(\mathbb{R}^4) \times L^2(\mathbb{R}^4)$, and let $(f_0, f_1) \in H^s(\mathbb{R}^4) \times H^{s-1}(\mathbb{R}^4)$, where $s > \frac{11}{12}$. Then, there exists a random maximal time interval of existence I and a solution $u\colon I \times \mathbb{R}^4 \to \mathbb{R}$ of (2.1.5) such that

$$u \in W(t)(f_0^\omega, f_1^\omega) + \left(C_t^0 \dot{H}_x^1(I \times \mathbb{R}^4) \bigcap L_{t,\mathrm{loc}}^3 L_x^6(I \times \mathbb{R}^4)\right) \text{ and } \partial_t u \in \partial_t W(t)(f_0^\omega, f_1^\omega) + C_t^0 L_x^2(I \times \mathbb{R}^4) \ .$$

Furthermore, we have that

(i) u is almost surely global, i.e., $I = \mathbb{R}$.

(ii) u almost surely satisfies the global space-time bound $\|u\|_{L_t^3 L_x^6(\mathbb{R}\times\mathbb{R}^4)} < \infty$.

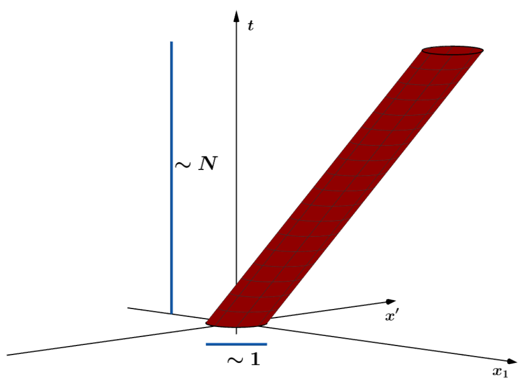
(iii) u almost surely scatters to a solution of the linear wave equation. Thus, there exist random scattering states $(u_0^\pm, u_1^\pm) \in \dot{H}^1(\mathbb{R}^4) \times L^2(\mathbb{R}^4)$ s.t.

$$\lim_{t\to\pm\infty} \|(u(t) - W(t)(u_0^\pm + f_0^\omega, u_1^\pm + f_1^\omega), \partial_t u(t) - \partial_t W(t)(u_0^\pm + f_0^\omega, u_1^\pm + f_1^\omega))\|_{\dot{H}^1\times L^2} = 0 .$$

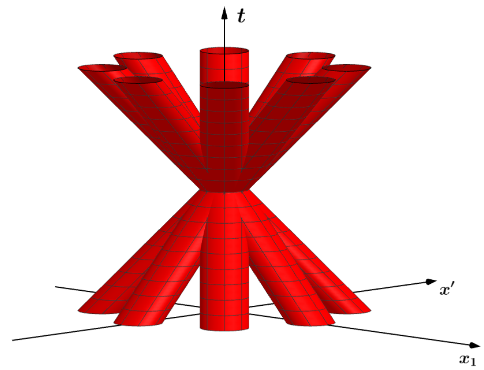
While Theorem 2.1.3 is only proven for the microlocal randomization, the majority of our argument directly applies to the Wiener randomization.

The main novelty in this paper lies in the application of a wave packet decomposition. To illustrate this idea, fix some $k \in \mathbb{Z}^d$ with $\|k\|_\infty \sim N$, and assume that $\widehat{f}_k(\xi) = N^{-s}\varphi(\xi - k)$. Then, f_k will essentially be unaffected by both the Wiener and microlocal randomizations, and hence forms an important example. From the method of non-stationary phase, it follows for all times $t \in [0, N]$ that the evolution $\exp(\pm it|\nabla|) f_k$ is concentrated in the ball $|x \pm tk/\|k\|_2| \lesssim 1$, and has amplitude $\sim N^{-s}$. In space-time, we can therefore view the evolution as a tube, see Figure 2.2a. For larger times, the dispersion of the evolution becomes significant, and the physical localization deteriorates. The wave packet perspective also explains the effect of the frequency randomization on the evolution. In Figure 2.2b, we display a bush (cf. [Bou91]), which is a collection of wave packets intersecting at a single point. If all wave packets in the bush have comparable amplitudes and the data is deterministic, one expects that the $L_t^\infty L_x^\infty$-norm is proportional to the number of wave packets. For random data, however, the phases of the wave packets are all independent, and the central limit theorem predicts that the $L_t^\infty L_x^\infty$-norm should instead be proportional to the square-root of the number of wave packets.

The examples in Figure 2.2 also illustrates an important heuristic: The natural timescale for the randomized evolution at frequency N is $T = N$. This differs from the natural timescale predicted by the (deterministic) bump-function heuristic, which is $T = N^{-1}$. We therefore decompose the



(a) Single wave packet



(b) Bush

In (a), we display the evolution $\exp(\pm it|\nabla|)f_k$ on the time-interval $[0, N]$. The space-time support can be viewed as a tube of length $\sim N$ and width ~ 1. The spatial center travels in a fixed direction at the speed of light, which has been normalized to 1. Furthermore, the amplitude of the evolution is given by $\sim N^{-s}$. In (b), we display a so-called bush, which is a collection of wave packets intersecting at a single point.

Figure 2.2: Wave packet heuristic

positive time-interval as

$$[0, \infty) = \Big(\bigcup_{n=0}^{\lfloor N^\theta \rfloor} [nN, (n+1)N) \Big) \cup [N^{1+\theta}, \infty) \ , \tag{2.1.11}$$

where $\theta > 0$ is a parameter. Our argument then splits into two separate parts.

On the long-time interval $[N^{1+\theta}, \infty)$, we use the additional decay obtained through the physical randomization. The basic idea is that after such a long time, the linear evolution could only be concentrated through constructive interference of a large portion of the initial data, which is highly unlikely due to the physical randomness (see Figure 2.4). To make this rigorous, we prove an $L_t^1 L_x^\infty([N^{1+\theta},\infty)\times\mathbb{R}^4)$-bound on $P_N F$, and this is sufficient to control the energy increment. This part of the proof requires the condition $s > 1-\theta/2$.

The majority of this paper focusses on time intervals such as $[0,N)$. This part of the argument does not rely on the physical randomness, and therefore also applies to the Wiener randomization. On this interval, we decompose the evolution into a family of wave packets, see Figure 2.3. As can be seen from a single wave packet, we cannot (always) control the evolution in $L_t^1 L_x^\infty$. Instead, we use the following dichotomy: Either F consists of only a few wave packets, in which case its support lies on a few light-cones, or it consists of many wave packets, in which case the $L_t^\infty L_x^\infty$-norm should be small.

We now present a heuristic and simplified version of the main argument. In order to illustrate the ideas, let us first assume that all wave packets belong to a single frequency $k \in \mathbb{Z}^d$. After a dyadic decomposition, we may further assume that all wave packets have amplitudes comparable to 2^m. Using the same notation as in Section 2.4, we denote the number of wave packets with this amplitude by $\#\mathscr{A}_m$. Due to the L^2-orthogonality of the wave packets, we have that $2^m(\#\mathscr{A}_m)^{\frac{1}{2}} \lesssim N^{-s}$.

In the case of only a few wave packets, we control the contribution on each tube separately. We have that

$$\Big|\int_0^N F_N v^2 \partial_t v \mathrm{d}x\mathrm{d}t\Big| \lesssim (\#\mathscr{A}_m) N^{\frac{1}{2}} 2^m \big(\sup_{\text{tubes } T} \|v\|^2_{L_t^4 L_x^4(T)}\big) \|\partial_t v\|_{L_t^\infty L_x^2([0,N)\times\mathbb{R}^4)} \lesssim N^{\frac{1}{2}} 2^m \#\mathscr{A}_m \sup_{t\in[0,N)} E[v](t) .$$

The supremum ranges over all tubes of length $\sim N$, width ~ 1, and unit-speed direction inside $[0,N)\times\mathbb{R}^4$. Using a flux estimate and a bootstrap argument, we controll this supremum by the square-root of the energy.

In the case of many wave packets (with the same direction), we use that their supports are disjoint, and obtain that

$$\Big| \int_0^N F_N v^2 \partial_t v \mathrm{d}x\mathrm{d}t \Big| \lesssim N \| F_N \|_{L_t^\infty L_x^\infty([0,N)\times \mathbb{R}^4)} \| v \|^2_{L_t^\infty L_x^4([0,N)\times\mathbb{R}^4)} \| \partial_t v \|_{L_t^\infty L_x^2([0,N)\times \mathbb{R}^4)} \lesssim N 2^m \sup_{t\in[0,N)} E[v](t) \ .$$

By combining both estimates, it follows that

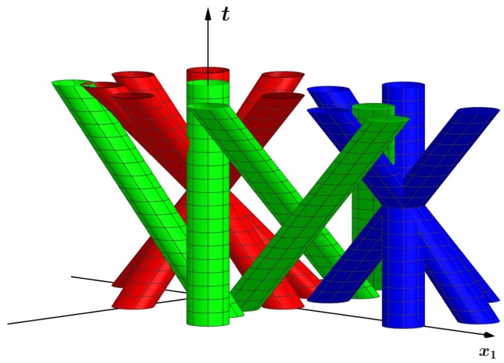
$$\Big| \int_0^N F_N v^2 \partial_t v \mathrm{d}x\mathrm{d}t \Big| \lesssim \min(N^{\frac{1}{2}} 2^m \# \mathcal{A}_m, N 2^m) \sup_{t\in[0,N)} E[v](t) \leqslant N^{\frac{3}{4}} 2^m (\#\mathcal{A}_m)^{\frac{1}{2}} \sup_{t\in[0,N)} E[v](t) \ .$$

We insert the bound $2^m(\#\mathcal{A}_m)^{\frac{1}{2}} \lesssim N^{-s}$, sum over N^θ intervals, and arrive at the condition $s > 3/4 + \theta$. In order to match the conditions from the intervals $[nN,(n+1)N)$ and the long-time interval $[N^{1+\theta},\infty)$, we choose $\theta = 1/6$, and obtain the regularity condition $s > 11/12$.

In order to remove the restriction to a single frequency, we need to consider both multiple directions and multiple scales. For this, we rely on techniques from the literature on the Kakeya and restriction conjectures. In order to control multiple directions, we use Bourgain's bush argument [Bou91]. The basic idea is to distinguish points which lie in multiple tubes from points which lie only in a few tubes. To this end, we group the wave packets into several bushes and a collection of (almost) non-overlapping wave packets (see Figure 2.3). We then almost argue as for a single frequency, but also use that each bush lies on the surface of a light-cone, which is crucial for the flux estimate. In order to control multiple scales, we rely on Wolff's induction on scales strategy [Wol01]. To fix ideas, let us try to bound the energy increment $E[v](N) - E[v](0)$. We have already described the estimates for wave packets of length greater than or equal to N, but the space-time region $[0,N] \times \mathbb{R}^4$ also contains many shorter wave packets. By induction on scales, we can already close the bootstrap argument at these shorter scales, which greatly reduces the complexity of the proof. We postpone a more detailed discussion to the Sections 2.4 and 2.6.

Acknowledgements:

I want to thank my advisor Terence Tao for his invaluable guidance and support. In particular,



We illustrate the wave packet decomposition of the linear evolution. We partition the wave packets into three groups: Two separate bushes (red and blue) and a collection of almost non-overlapping wave packets (green).

Figure 2.3: Wave packet decomposition

he proposed the greedy selection algorithm in Section 2.4. Furthermore, I want to thank Rowan Killip and Monica Visan for several interesting discussions. The figures in this paper have been created using TikZ and GeoGebra.

2.2 Notation and preliminaries

For the rest of this paper, the positive integer d denotes the dimension of physical space. In the analysis of the nonlinear evolution, we will eventually specialize to $d = 4$. Furthermore, we also fix positive absolute constants δ, θ, and η. The parameter δ will be used to deal with the spatial tails of the wave packets and certain summability issues. The parameter θ is used in the division of time (see (2.1.11)). We will eventually choose $\theta = 1/6$, but prefer to keep θ as a free parameter until the end of the argument. Finally, η describes the size of the frequency truncated data, see

Proposition 2.4.8.

If A, B are positive quantities, we write $A \lesssim B$ if and only if there exist a constant $C = C(\delta, \theta)$ such that $A \leqslant CB$. Furthermore, most capital letters, such as N, M, and R, will denote dyadic numbers greater than or equal to 1.

Finally, we define the Fourier transform $\widehat{f}$ of an integrable function $f \colon \mathbb{R}^d \to \mathbb{C}$ by

$$\widehat{f}(\xi) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \exp(-ix\xi) f(x) \mathrm{d}x \ .$$

We now summarize a few basic results from probability theory, harmonic analysis, and dispersive partial differential equations.

2.2.1 Probability theory

We recall a few basic estimates for sub-gaussian random variables. For an accessible introduction, we refer the reader to [Ver18].

Definition 2.2.1 (Sub-gaussian random variable)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $X \colon (\Omega, \mathcal{F}) \to \mathbb{R}$ be a random variable. We then define the sub-gaussian norm by

$$\|X\|_{\Psi_2} := \sup_{p \geqslant 1} \frac{(\mathbb{E}[|X|^p])^{\frac{1}{p}}}{\sqrt{p}} \tag{2.2.1}$$

We call a random variable X sub-gaussian if and only if $\|X\|_{\Psi_2} < \infty$. Furthermore, we call a family of random variables $\{X_i\}_{i \in I}$ uniformly sub-gaussian if and only if $\sup_{i \in I} \|X_i\|_{\Psi_2} < \infty$.

The relationship to the Gaussian distribution may not be obvious from (2.2.1). However, it follows from [Ver18, Proposition 2.52] that (2.2.1) implies

$$\mathbb{P}(|X| > x) \leqslant 2 \exp\left(-c \frac{x^2}{\|X\|_{\Psi_2}^2}\right) \qquad \forall x > 0 \ .$$

Many concentration inequalities for the sums of independent sub-gaussian random variables can be found in the literature. In the following, we mainly rely on Khintchine's inequality.

Lemma 2.2.2 (Khintchine's inequality, [Ver12, Corollary 5.12] or [Ver18, Proposition 2.6.1 and Exercise 2.6.5])**.** Let $(X_j)_{j=1,\dots,J}$ be a finite sequence of independent sub-gaussian random variables with zero mean. Then, it holds for all deterministic sequences $(a_j)_{j=1,\dots,J}$, and all $p \geqslant 1$, that

$$\left(\mathbb{E}\Big[|\sum_{j=1}^{J} a_j X_j|^p\Big]\right)^{\frac{1}{p}} \lesssim \sqrt{p}\left(\max_{j=1,\dots,J}\|X_j\|_{\Psi_2}\right)\left(\sum_{j=1}^{J}|a_j|^2\right)^{\frac{1}{2}} \tag{2.2.2}$$

In particular, the sum $\sum_{j=1}^{J} a_j X_j$ is sub-gaussian.

In this paper, Khintchine's inequality will often be combined with Minkowski's integral inequality, which we recall below.

Lemma 2.2.3 (Minkowski's integral inequality)**.** Let (X,μ) and (Y,ν) be two σ-finite measure spaces, and let $1 \leqslant p \leqslant q \leqslant \infty$. Then, we have for all measurable functions $f\colon X \times Y \to \mathbb{R}$ that

$$\|\|f(x,y)\|_{L^p(X)}\|_{L^q(Y)} \leqslant \|\|f(x,y)\|_{L^q(Y)}\|_{L^p(X)} \ .$$

The special case $p = 1$ is the standard Minkowski inequality, and it can be found in most real analysis books (see e.g. [LL01, Theorem 2.4]). Since Lemma 2.2.3 is central to many arguments in this paper, we prove the general statement from this special case.

Proof. Since $q/p \geqslant 1$, we have that

$$\|\|f(x,y)\|_{L^p(X)}\|_{L^q(Y)} = \|\|f(x,y)^p\|_{L^1(X)}\|^{\frac{1}{p}}_{L^{\frac{q}{p}}(Y)} \leqslant \|\|f(x,y)^p\|_{L^{\frac{q}{p}}(Y)}\|^{\frac{1}{p}}_{L^1(X)} = \|\|f(x,y)\|_{L^q(Y)}\|_{L^p(X)} \ .$$

□

We will also need a crude bound on the maximum of dependent sub-gaussian random variables

Lemma 2.2.4 (Suprema of dependent sub-gaussian random variables [Ver18, Exercise 2.5.10])**.** Assume that $(X_j)_{j=1,\ldots,J}$ are (possibly dependent) sub-gaussian random variables. Then,

$$\mathbb{E}\Big[\max_{j=1,\ldots,J}|X_j|\Big] \lesssim \sqrt{\log(2+J)}\max_{j=1,\ldots,J}\|X_j\|_{\Psi_2} \ . \tag{2.2.3}$$

Proof. Let $1 \leqslant p < \infty$. Using Hölder's inequality, we obtain that

$$\mathbb{E}\Big[\max_{j=1,\ldots,J}|X_j|\Big] \leqslant \Big(\mathbb{E}\Big[\max_{j=1,\ldots,J}|X_j|^p\Big]\Big)^{\frac{1}{p}} \leqslant \Big(\sum_{j=1}^{J}\mathbb{E}\big[|X_j|^p\big]\Big)^{\frac{1}{p}} \leqslant J^{\frac{1}{p}}\sqrt{p}\max_{j=1,\ldots,J}\|X_j\|_{\Psi_2} \ .$$

After choosing $p = \log(2+J)$, we arrive at (2.2.3). □

2.2.2 Harmonic analysis

Let $N \in 2^{\mathbb{N}_0}$ and $k \in \mathbb{Z}^d$. As in the introduction, we let $\varphi \in C_c^\infty(\mathbb{R}^d)$ be a smooth and symmetric function satisfying $\varphi|_{[-\frac{3}{8},\frac{3}{8}]^d} = 1$, $\varphi|_{\mathbb{R}^d\backslash[-\frac{5}{8},\frac{5}{8}]^d} = 0$, and $\sum_{k\in\mathbb{Z}^d}\varphi(\cdot - k) = 1$. We also define $\psi(\xi) = \varphi(\xi) - \varphi(2\xi)$. Then, the re-centered Littlewood-Paley operators are defined as

$$\widehat{P_{N;k}f}(\xi) = \begin{cases} \psi\big(\frac{\xi-k}{N}\big)\widehat{f}(\xi) & \text{if } N > 1 \\ \varphi(\xi-k)\widehat{f}(\xi) & \text{if } N = 1 \end{cases} .$$

With this choice of ψ, it holds that $P_{N;k}P_{M;k} = 0$ if $M \geqslant 4N$ or $N \geqslant 4M$. To simplify the notation, we also set $P_N := P_{N;0}$ and $P_k := P_{1;k}$. Furthermore, we define the fattened Littlewood-Paley operators

$$\widetilde{P}_{N;k} := \sum_{2^{-10}N\leqslant M\leqslant 2^{10}N} P_{M;k} \ . \tag{2.2.4}$$

Lemma 2.2.5 (Bernstein's inequalities)**.** Let $N \in 2^{\mathbb{N}_0}$, $k \in \mathbb{Z}^d$, and $1 \leqslant p \leqslant q \leqslant \infty$. Then, we for all $f \in L_x^p(\mathbb{R}^d)$ that

$$\|P_{N;k}f\|_{L_x^q(\mathbb{R}^d)} \lesssim N^{d(\frac{1}{p}-\frac{1}{q})}\|P_{N;k}f\|_{L_x^p(\mathbb{R}^d)} \ , \tag{2.2.5}$$

$$\||\nabla|P_{N;0}f\|_{L_x^p(\mathbb{R}^d)} \lesssim N\|P_{N;0}f\|_{L_x^p(\mathbb{R}^d)} \ . \tag{2.2.6}$$

We emphasize that the constant in (2.2.5) is independent of $k \in \mathbb{Z}^d$, since the phase $\exp(ikx)$ does not affect the L_x^p-norms.

We also record the following standard consequence of Bernstein's inequality and the uncertainty principle.

Lemma 2.2.6. Let $N \geqslant 1$, let $a, b \in \mathbb{R}$, and assume that $b \geqslant a + 1/N$. Then, we have for all $f \in L_x^2(\mathbb{R}^d)$, all $1 \leqslant q < \infty$, and all $1 \leqslant p \leqslant \infty$, that

$$\|\exp(\pm it|\nabla|)P_N f\|_{L_t^\infty L_x^p([a,b]\times\mathbb{R}^d)} \lesssim N^{\frac{1}{q}}\|\exp(\pm it|\nabla|)P_N f\|_{L_t^q L_x^p([a,b]\times\mathbb{R}^d)} \ .$$

The argument is essentially taken from [Bri20b].

Proof. Pick $t_0 \in [a,b]$, and let I be any interval such that $t_0 \in I \subseteq [a,b]$. For all $t \in I$, it holds that

$$\exp(\pm it_0|\nabla|)P_N f^\omega = \exp(\pm it|\nabla|)P_N f^\omega \pm i\int_t^{t_0} |\nabla|\exp(\pm it'|\nabla|)P_N f^\omega \mathrm{d}t'$$

From Bernstein's inequality, we obtain that

$$\begin{aligned}\|\exp(\pm it_0|\nabla|)P_N f^\omega\|_{L_x^p(\mathbb{R}^d)} &\leqslant \|\exp(\pm it|\nabla|)P_N f^\omega\|_{L_x^p(\mathbb{R}^d)} + |I|^{1-\frac{1}{q}}\||\nabla|\exp(\pm it|\nabla|)P_N f^\omega\|_{L_t^q L_x^p(I\times\mathbb{R}^d)} \\ &\lesssim \|\exp(\pm it|\nabla|)P_N f^\omega\|_{L_x^p(\mathbb{R}^d)} + N|I|^{1-\frac{1}{q}}\|\exp(\pm it|\nabla|)P_N f^\omega\|_{L_t^q L_x^p(I\times\mathbb{R}^d)}\end{aligned}$$

Taking the q-th power and averaging over all $t \in I$, we obtain that

$$\|\exp(\pm it_0|\nabla|)P_N f^\omega\|_{L_x^p(\mathbb{R}^d)} \lesssim \left(|I|^{-\frac{1}{q}} + N|I|^{1-\frac{1}{q}}\right)\|\exp(\pm it|\nabla|)P_N f^\omega\|_{L_t^q L_x^p(I\times\mathbb{R}^d)} \ .$$

By choosing $|I| = N^{-1}$, and taking the supremum over all $t_0 \in [a,b]$, it follows that

$$\| \exp(\pm it|\nabla|) P_N f^\omega \|_{L_t^\infty L_x^p([a,b]\times\mathbb{R}^d)} \lesssim N^{\frac{1}{q}} \| \exp(\pm it|\nabla|) P_N f^\omega \|_{L_t^q L_x^p([a,b]\times\mathbb{R}^d)}.$$

□

The following estimate, which appeared in the almost sure scattering problem for the nonlinear Schrödinger equation [KMV19], is useful in combination with Khintchine's inequality.

Lemma 2.2.7 (Square function estimate [KMV19, Lemma 2.8])**.** Let $f \in L_x^2(\mathbb{R}^d)$ and let φ be as above. Then, it holds that

$$\sum_{k\in\mathbb{Z}^d} |P_k f(x)|^2 \lesssim (|\check{\varphi}| * |f|^2)(x) \; . \tag{2.2.7}$$

In addition to the dyadic decomposition in frequency, we also need a dyadic decomposition in physical space. To avoid confusion, we denote the cut-off function in physical space by χ. More precisely, we set $\chi_1(x) := \varphi(x)$ and $\chi_L(x) := \psi(x/L)$, where $L \geqslant 2$.

Lemma 2.2.8 (Mismatch estimates)**.** Let $1 \leqslant p \leqslant \infty$ and $M, N, L \geqslant 1$. We further assume the mismatch conditions $\max(M/N, N/M) \geqslant 8$ and $L \geqslant 8$. Then, it holds for all absolute constants $D > 0$ that

$$\|\chi_1 P_M \chi_L\|_{L_x^p \to L_x^p} \lesssim_D (ML)^{-D} \quad , \tag{2.2.8}$$

$$\|P_N \chi_1 P_M\|_{L_x^p \to L_x^p} \lesssim_D (NM)^{-D} \; . \tag{2.2.9}$$

Proof. The inequality (2.2.8) can be found in [DLM19, Lemma 5.10]. An inequality similar to (2.2.9) can be found in [DLM19, Lemma 5.11], and we present a different argument.

Using duality and $(P_N\chi_1P_M)^* = P_M\chi_1P_N$, we can assume that $N \geqslant M$. From the mismatch condition, it then follows that $N \geqslant 8M$. Thus, we obtain for all $f \in L_x^p(\mathbb{R}^d)$ that

$$\|P_N(\chi_1 P_M f)\|_{L_x^p} = \|P_N((P_{\geqslant N/8}\chi_1) P_M f)\|_{L_x^p} \lesssim \|P_{\geqslant N/8}\chi_1\|_{L_x^\infty} \|P_M f\|_{L_x^p} \lesssim N^{-2D} \|f\|_{L_x^p} \; .$$

□

The following auxiliary lemma will be helpful in the proof of probabilistic Strichartz estimates.

Lemma 2.2.9 ($\ell^2_{k,l}$-estimate)**.** Let $s \in \mathbb{R}$ and let $f \in H^s_x(\mathbb{R}^d)$. For any $2 \leqslant p \leqslant \infty$, we have that

$$\|P_k(\varphi_l f)\|_{\ell^2_l \ell^2_k L^{p'}_x(\mathbb{Z}^d \times \{k \in \mathbb{Z}^d :\, \|k\|_\infty \in (N/2,N]\} \times \mathbb{R}^d)} \lesssim \|\widetilde{P}_N f\|_{L^2_x(\mathbb{R}^d)} + N^{-s-10d}\|f\|_{H^s_x(\mathbb{R}^d)} \tag{2.2.10}$$

Remark 2.2.10. The error term $N^{-s-10d}\|f\|_{H^s_x(\mathbb{R}^d)}$ is a result of the non-compact support of $\widehat{\varphi}_l$, but may essentially be ignored. On a heuristic level, each $P_k(\varphi_l f)$ is supported on a spatial region of volume ~ 1, and thus (2.2.10) should follow from Hölder's inequality. To make this argument rigorous, we use the square-function estimate and the mismatch estimates above.

Proof. Let $\widetilde{P}_N$ be the fattened Littlewood-Paley operator as in (2.2.4). We write $M \not\sim N$ if either $M < 2^{-10}N$ or $M > 2^{10}N$. In the following, we implicitly assume that $\|k\|_\infty \in (N/2, N]$. We then estimate

$$\|P_k(\varphi_l f)\|_{\ell^2_l \ell^2_k L^{p'}_x} \leqslant \|P_k(\varphi_l \widetilde{P}_N f)\|_{\ell^2_l \ell^2_k L^{p'}_x} + \sum_{M \not\sim N} \|P_k(\varphi_l P_M f)\|_{\ell^2_l \ell^2_k L^{p'}_x} . \tag{2.2.11}$$

We begin by controlling the first summand in (2.2.11). Using Minkowski's integral inequality and the square-function estimate (Lemma 2.2.7), we obtain that

$$\begin{aligned} &\|P_k(\varphi_l \widetilde{P}_N f)\|_{\ell^2_l \ell^2_k L^{p'}_x} = \|\varphi_{l'} P_k(\varphi_l \widetilde{P}_N f)\|_{\ell^2_l \ell^2_k \ell^1_{l'} L^{p'}_x} \lesssim \|\varphi_{l'} P_k(\varphi_l \widetilde{P}_N f)\|_{\ell^2_l \ell^2_k \ell^1_{l'} L^2_x} \lesssim \|\varphi_{l'} P_k(\varphi_l \widetilde{P}_N f)\|_{\ell^2_l \ell^1_{l'} L^2_x \ell^2_k} \\ &= \left\| \varphi_{l'} \left(|\check{\varphi}| * |\varphi_l \widetilde{P}_N f|^2 \right)^{\frac{1}{2}} \right\|_{\ell^2_l \ell^1_{l'} L^2_x} . \end{aligned} \tag{2.2.12}$$

Using simple support considerations, we have that

$$\begin{aligned} &\left\| \varphi_{l'} \left(|\check{\varphi}| * |\varphi_l \widetilde{P}_N f|^2 \right)^{\frac{1}{2}} \right\|_{L^2_x}^2 \leqslant \left\| \varphi_{l'} \left(|\check{\varphi}| * |\varphi_l \widetilde{P}_N f|^2 \right) \right\|_{L^1_x} \lesssim \langle l' - l \rangle^{-10d} \|(\varphi_l \widetilde{P}_N f)^2\|_{L^1_x} \\ &\lesssim \langle l' - l \rangle^{-10d} \|\varphi_l \widetilde{P}_N f\|_{L^2_x}^2 . \end{aligned}$$

Inserting this back into (2.2.12), we obtain that

$$\|P_k(\varphi_l \widetilde{P}_N f)\|_{\ell_l^2 \ell_k^2 L_x^{p'}} \lesssim \|\langle l'-l\rangle^{-5d}\varphi_l \widetilde{P}_N f\|_{\ell_l^2 \ell_{l'}^1 L_x^2} \lesssim \|\widetilde{P}_N f\|_{L_x^2} \ .$$

Thus, this yields the first term in (2.2.10). We now control the second summand in (2.2.11). First, note that $P_k = \sum_{2^{-5}N \leqslant N' \leqslant 2^5 N} P_{N'} P_k$. Since there exist only $\sim N^d$ frequencies of magnitude $\sim N$, we have that

$$\|P_k(\varphi_l P_M f)\|_{\ell_k^2 \ell_l^2 L_x^{p'}} \lesssim N^{\frac{d}{2}} \sum_{2^{-5}N \leqslant N' \leqslant 2^5 N} \|P_{N'}(\varphi_l P_M f)\|_{\ell_l^2 L_x^{p'}} \ .$$

It now suffices to prove for all $g \in S(\mathbb{R}^d)$, all $M \not\sim N$, and all absolute constants $D > 0$ that

$$\|P_{N'}(\varphi_l P_M g)\|_{L_x^{p'}} \lesssim_D (NM)^{-D} \|\langle x-l\rangle^{-D} g\|_{L_x^2} \ . \tag{2.2.13}$$

Using spatial translation invariance, we may set $l = 0$. Let $\{\chi_L\}_{L \geqslant 1}$ denote the dyadic decomposition in physical space. Using the mismatch estimates (Lemma 2.2.8), we obtain

$$\begin{aligned}\|P_{N'}(\varphi_0 P_M g)\|_{L_x^{p'}} &\leqslant \sum_{L \geqslant 1} \|P_{N'}(\varphi_0 P_M \chi_L g)\|_{L_x^{p'}} \leqslant \sum_{L \geqslant 1} \|P_{N'} \varphi_0 P_M \chi_L\|_{L_x^{p'} \to L_x^{p'}} \|\widetilde{\chi}_L g\|_{L_x^{p'}} \\ &\lesssim (NM)^{-D} \sum_{L \geqslant 1} L^{-2D} \|\widetilde{\chi}_L g\|_{L_x^{p'}} \lesssim (NM)^{-D} \|\langle x\rangle^{-D} g\|_{L_x^2} \ .\end{aligned}$$

□

As a direction consequence of (2.2.9), we also obtain the following estimate on the H^s-norm of the microlocal randomization.

Lemma 2.2.11 (H_x^s-norm of f^ω)**.** Let $f \in H_x^s(\mathbb{R}^d)$ and let f^ω be its microlocal randomization. We further set

$$f_1^\omega := \sum_{l \in \mathbb{Z}^d} X_{0,l} P_0(\varphi_l f) \qquad \text{and} \qquad f_N^\omega := \sum_{\substack{k,l \in \mathbb{Z}^d \\ \|k\|_\infty \in (N/2, N]}} X_{k,l} P_k(\varphi_l f) \ , \quad \text{where } N \geqslant 2 \ . \tag{2.2.14}$$

Then, we have for all $2 \leqslant r < \infty$ that

$$\|f^\omega\|_{L_\omega^r H_x^s} \simeq \|N^s f_N^\omega\|_{L_\omega^r \ell_N^2 L_x^2} \lesssim \sqrt{r} \|f\|_{H_x^s} \ . \tag{2.2.15}$$

Proof. The first equivalence in (2.2.15) is a direct consequence of the definition of the H^s_x-norm. Now, we prove the bound in (2.2.15). From Minkowski's integral inequality, Khintchine's inequality, and Lemma 2.2.9, we have for all $N \geqslant 2$ that

$$\begin{aligned}\|N^s f^\omega_N\|_{L^r_\omega L^2_x} &\leqslant \|N^s \sum_{\|k\|_\infty \in (N/2,N]} X_{k,l} P_k(\varphi_l f)\|_{L^2_x L^r_\omega} \\ &\leqslant \sqrt{r} N^s \|P_k(\varphi_l f)\|_{L^2_x \ell^2_{k,l}(\|k\|_\infty \in (N/2,N]} \\ &\lesssim \sqrt{r}\left(N^s \|\widetilde{P}_N f\|_{L^2_x(\mathbb{R}^d)} + N^{-10d}\|f\|_{H^s_x(\mathbb{R}^d)}\right)\end{aligned}$$

The same argument also applies to $N = 1$. After taking the ℓ^2_N-norm, this completes the proof. □

2.2.3 Strichartz estimates

The individual blocks in the microlocal randomization or the Wiener randomization have frequency support inside a unit-sized cube (at a large distance from the origin). Since this rules out the Knapp example, one expects a refined dispersive estimate. The following lemma is due to Klainerman and Tataru [KT99], and it has first been used in the probabilistic context by [DLM20].

Lemma 2.2.12 (Refined dispersive estimate by Klainerman-Tataru [KT99])**.** Let $f \in L^1(\mathbb{R}^d)$, let $k \in \mathbb{Z}^d$ satisfy $\|k\|_\infty \in (N/2, N]$, and let $M \leqslant N$. Then it holds for all $t \in \mathbb{R}$ and $2 \leqslant p \leqslant \infty$ that

$$\|\exp(\pm it|\nabla|)P_{M;k} f\|_{L^p_x(\mathbb{R}^d)} \lesssim \frac{M^{d(1-\frac{2}{p})}}{(1 + \frac{M^2}{N}|t|)^{(d-1)(\frac{1}{2}-\frac{1}{p})}} \|f\|_{L^{p'}_x(\mathbb{R}^d)} \ . \tag{2.2.16}$$

As stated, the inequality (2.2.16) essentially follows from [KT99]. For the sake of completeness, we present the modification below.

Proof. By interpolation against the energy estimate $\|\exp(\pm it|\nabla|)P_{M;k} f\|_{L^2_x(\mathbb{R}^d)} \leqslant \|f\|_{L^2_x(\mathbb{R}^d)}$, it suffices to prove (2.2.16) for $p = \infty$. The inequality [KT99, (A.66)], where $\mu = M/N$, and a scaling argument yield

$$\| \exp(\pm it|\nabla|) P_{M;k} f\|_{L_x^\infty(\mathbb{R}^d)} \lesssim \frac{MN^{d-1}}{(1+N|t|)^{\frac{d-1}{2}}} \|f\|_{L_x^1(\mathbb{R}^d)} \ . \tag{2.2.17}$$

We now distinguish two cases. If $|t| \lesssim N/M^2$, then Bernstein's inequality (Lemma 2.2.5) yields that

$$\| \exp(\pm it|\nabla|) P_{M;k} f\|_{L_x^\infty(\mathbb{R}^d)} \lesssim M^{\frac{d}{2}} \| \exp(\pm it|\nabla|) P_{M;k} f\|_{L_x^2(\mathbb{R}^d)} = M^{\frac{d}{2}} \|P_{M;k} f\|_{L_x^2(\mathbb{R}^d)} \lesssim M^d \|P_{M;k} f\|_{L_x^1(\mathbb{R}^d)} \ .$$

If $|t| \lesssim N/M^2$, then (2.2.17) yields that

$$\| \exp(\pm it|\nabla|) P_{M;k} f\|_{L_x^\infty(\mathbb{R}^d)} \lesssim \frac{MN^{d-1}}{(N|t|)^{\frac{d-1}{2}}} \|f\|_{L_x^1(\mathbb{R}^d)} = \frac{M^d}{(\frac{M^2}{N}t)^{\frac{d-1}{2}}} \|f\|_{L_x^1(\mathbb{R}^d)} \lesssim \frac{M^d}{(1+\frac{M^2}{N}t)^{\frac{d-1}{2}}} \|f\|_{L_x^1(\mathbb{R}^d)} \ .$$

□

In this paper, we are mainly concerned with the case $M = 1$. Then, (2.2.16) describes the linear evolution on short time intervals more accurately than (2.2.17). As a corollary of the refined dispersive estimate, we obtain the following refined Strichartz estimate.

Let $2 \leqslant q, p \leqslant \infty$. We call the pair (q,p) wave-admissible if

$$\frac{1}{q} + \frac{d-1}{2p} \leqslant \frac{d-1}{4} \qquad \text{and} \qquad (q,p,d) \neq (2,\infty,3) \ .$$

Corollary 2.2.13 (Refined Strichartz estimates [KT99])**.** Let $f \in L_x^2(\mathbb{R}^d)$, let $k \in \mathbb{Z}^d$ satisfy $\|k\|_\infty \in (N/2, N]$, and let $M \leqslant N$. Then, we have for all wave-admissible pairs (q,p) that

$$\| \exp(\pm it|\nabla|) P_{M;k} f\|_{L_t^q L_x^p(\mathbb{R}\times\mathbb{R}^d)} \lesssim M^{\frac{d}{2}-\frac{2}{q}-\frac{d}{p}} N^{\frac{1}{q}} \|P_{M;k} f\|_{L_x^2(\mathbb{R}^d)} \tag{2.2.18}$$

The derivation of the refined Strichartz estimate from Lemma 2.2.12 follows from a standard TT^*-argument, and we therefore omit the proof. For the endpoint $(2, 2(d-1)/(d-3))$, we also refer to [KT98]. Let us emphasize two special cases: If $M = N$, we obtain the usual scaling factor $N^{\frac{d}{2}-\frac{1}{q}-\frac{d}{p}}$, and if $M = 1$, we obtain the factor $N^{\frac{1}{q}}$, which does not depend on p.

2.3 Probabilistic Strichartz estimates

In this section, we derive probabilistic Strichartz estimates (cf. [BOP15b, DLM20, LM14]) and a probabilistic long-time decay estimate (cf. [Mur19]). To keep the exposition self-contained, we include the (short) proofs. Recall from (2.2.14) that

$$f_1^\omega := \sum_{l\in\mathbb{Z}^d} X_{0,l} P_0(\varphi_l f) \qquad \text{and} \qquad f_N^\omega := \sum_{\substack{k,l\in\mathbb{Z}^d \\ \|k\|_\infty\in(N/2,N]}} X_{k,l} P_k(\varphi_l f) \ , \quad \text{where } N \geqslant 2 \ .$$

Lemma 2.3.1 (Probabilistic Strichartz estimate)**.** Let $f \in H_x^s(\mathbb{R}^d)$ and let f^ω be its microlocal randomization. Then, it holds for all $N \geqslant 1$, all wave-admissible exponent pairs (q,p), and all $1 \leqslant r < \infty$ that

$$\| \exp(\pm it|\nabla|) f_N^\omega \|_{L_\omega^r L_t^q L_x^p(\Omega\times\mathbb{R}\times\mathbb{R}^d)} \lesssim \sqrt{r} N^{\frac{1}{q}-s+} \|f\|_{H_x^s(\mathbb{R}^d)} \ . \tag{2.3.1}$$

This estimate has previously appeared for the Wiener randomization in [DLM20].

Proof. In the following, we implicitly assume that $k \in \mathbb{Z}^d$ always satisfies $\|k\|_\infty \in (N/2, N]$. First, we assume that $2 \leqslant p, q < \infty$, and that $\max(p,q) \leqslant r < \infty$. Using Minkowski's integral inequality (Lemma 2.2.3), Khintchine's inequality (Lemma 2.2.2), and the refined dispersive estimate (Lemma 2.2.12), we have that

$$\begin{aligned} &\| \exp(\pm it|\nabla|) f_N^\omega \|_{L_\omega^r L_t^q L_x^p} \leqslant \| \exp(\pm it|\nabla|) f_N^\omega \|_{L_t^q L_x^p L_\omega^r} \lesssim \sqrt{r} \| \exp(\pm it|\nabla|) P_k(\varphi_l f) \|_{L_t^q L_x^p \ell_{k,l}^2} \\ &\leqslant \sqrt{r} \| \exp(\pm it|\nabla|) P_k(\varphi_l f) \|_{\ell_{k,l}^2 L_t^q L_x^p} \lesssim \sqrt{r} N^{\frac{1}{q}} \| P_k(\varphi_l f) \|_{\ell_{k,l}^2 L_x^2} \lesssim \sqrt{r} N^{\frac{1}{q}-s} \|f\|_{H_x^s} \ . \end{aligned}$$

In the last inequality, we have also used Lemma 2.2.9. The estimate for $1 \leqslant r \leqslant \max(p,q)$ then follows from Hölder's inequality. Thus, it remains to treat the cases $q = \infty$ and/or $p = \infty$. This is a know technical issue, see [Bri20b, Remark 3.8] for a discussion. Both cases can be reduced to the previous estimate by using Lemma 2.2.6 and Bernstein's inequality. □

Lemma 2.3.2 (Probabilistic long-time decay)**.** Let $f \in L^2_x(\mathbb{R}^d)$ and let f^ω be its microlocal randomization. Furthermore, let $1 \leqslant q < \infty$ and $2 \leqslant p \leqslant \infty$ be such that

$$\frac{1}{q} + \frac{d-1}{p} < \frac{d-1}{2} \ . \tag{2.3.2}$$

Then, we have for all $1 \leqslant r < \infty$ that

$$\| \exp(\pm it|\nabla|) f^\omega_N \|_{L^r_\omega L^q_t L^p_x(\Omega \times [T,\infty) \times \mathbb{R}^d)} \lesssim_{q,p} \sqrt{r} N^{\frac{1}{q}-s+} \left(1 + \frac{T}{N}\right)^{\frac{1}{q}+\frac{d-1}{p}-\frac{d-1}{2}} \|f\|_{H^s_x(\mathbb{R}^d)}. \tag{2.3.3}$$

Lemma 2.3.2 has previously been used for a physical space randomization in [Mur19, Proposition 3.1]. In contrast to the standard Strichartz estimates, which are time-translation invariant, (2.3.3) provides a quantitative decay rate. The motivation behind this estimate is illustrated in Figure 2.4. In this paper, we only require the following special case.

Corollary 2.3.3. Let $f \in L^2_x(\mathbb{R}^4)$ and let f^ω be its microlocal randomization. Then, we have for all $\theta > 0$ that

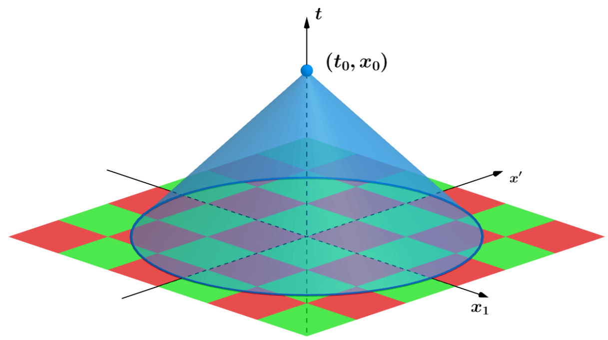
$$\| \exp(\pm it|\nabla|) f^\omega_N \|_{L^r_\omega L^1_t L^\infty_x(\Omega \times [N^{1+\theta},\infty) \times \mathbb{R}^4)} \lesssim \sqrt{r} N^{1-\frac{\theta}{2}+} \|f_N\|_{L^2_x(\mathbb{R}^4)} \ . \tag{2.3.4}$$

Remark 2.3.4. Due to (2.3.2), the $L^1_t L^\infty_x$-estimate fails logarithmically in three dimensions.

Proof of Lemma 2.3.2. We essentially follow the argument in [Mur19]. Let us first assume that $2 \leqslant q, p < \infty$.

We further assume that $r \geqslant \max(q,p)$, the corresponding estimate for $1 \leqslant r < \max(q,p)$ then follows from Hölder's inequality. Using Minkowski's integral inequality (Lemma 2.2.3), Khintchine's inequality (Lemma 2.2.2), and the refined dispersive estimate (Lemma 2.2.12), we have that

$$\begin{aligned}
&\| \exp(\pm it|\nabla|) f^\omega_N \|_{L^r_\omega L^q_t L^p_x(\Omega \times [T,\infty) \times \mathbb{R}^d)} \\
&\leqslant \| \exp(\pm it|\nabla|) f^\omega_N \|_{L^q_t L^p_x L^r_\omega([T,\infty) \times \mathbb{R}^d \times \Omega)}
\end{aligned}$$



This figure illustrates the effect of the physical randomization on the linear evolution. At the point (t_0, x_0), the linear evolution depends on the initial data in a large region of space. Due to the physical randomization, the initial data in different spatial regions cannot constructively interfere, and hence we expect an improved decay.

Figure 2.4: Effect of physical randomization

$$
\begin{aligned}
&\lesssim \sqrt{r} \| \exp(\pm it|\nabla|) P_k(\varphi_l f) \|_{L_t^q L_x^p \ell_{k,l}^2([T,\infty)\times\mathbb{R}^d\times\mathbb{Z}^{d+d})} \\
&\lesssim \sqrt{r} \| \exp(\pm it|\nabla|) P_k(\varphi_l f) \|_{\ell_{k,l}^2 L_t^q L_x^p(\mathbb{Z}^{d+d}\times[T,\infty)\times\mathbb{R}^d)} \\
&\lesssim \sqrt{r} \left\| \left(1+\frac{|t|}{N}\right)^{-(d-1)(\frac{1}{2}-\frac{1}{p})} \| P_k(\varphi_l f) \|_{L_x^{p'}} \right\|_{\ell_{k,l}^2 L_t^q(\mathbb{Z}^{d+d}\times[T,\infty))} \\
&\lesssim \sqrt{r} \left\| \left(1+\frac{|t|}{N}\right)^{-(d-1)(\frac{1}{2}-\frac{1}{p})} \right\|_{L_t^q([T,\infty))} \| P_k(\varphi_l f) \|_{\ell_{k,l}^2 L_x^{p'}(\mathbb{Z}^{d+d}\times\mathbb{R}^d)} .
\end{aligned}
$$

Using condition (2.3.2), we obtain

$$
\left\| \left(1+\frac{|t|}{N}\right)^{-(d-1)(\frac{1}{2}-\frac{1}{p})} \right\|_{L_t^q([T,\infty))} \lesssim N^{\frac{1}{q}} \left(1+\frac{T}{N}\right)^{\frac{1}{q}+\frac{d-1}{p}-\frac{d-1}{2}}
$$

Finally, from Lemma 2.2.9 we have that

$$
\| P_k(\varphi_l f) \|_{\ell_{k,l}^2 L_x^{p'}(\mathbb{Z}^{d+d}\times\mathbb{R}^d)} \lesssim N^{-s} \| f \|_{H_x^s} .
$$

This finishes the proof in the case $2 \leqslant q, p < \infty$. Using Bernstein's inequality, we can reduce the case $p = \infty$ to $p < \infty$. Thus, it remains to treat the range $1 \leqslant q < 2$. Using a dyadic decomposition in time, we have for all $T \geqslant N$ that

$$
\begin{aligned}
&\| \exp(\pm it|\nabla|) f_N^\omega \|_{L_\omega^r L_t^q L_x^p(\Omega \times [T,\infty) \times \mathbb{R}^d)} \\
&\leqslant \sum_{m=0}^\infty \| \exp(\pm it|\nabla|) f_N^\omega \|_{L_\omega^r L_t^q L_x^p(\Omega \times [2^m T, 2^{m+1}T) \times \mathbb{R}^d)} \\
&\leqslant \sum_{m=0}^\infty (2^m T)^{\frac{1}{q}-\frac{1}{2}} \| \exp(\pm it|\nabla|) f_N^\omega \|_{L_\omega^r L_t^2 L_x^p(\Omega \times [2^m T, 2^{m+1}T) \times \mathbb{R}^d)} \\
&\lesssim \sum_{m=0}^\infty (2^m T)^{\frac{1}{q}-\frac{1}{2}} N^{\frac{1}{2}-s+} \left(\frac{2^m T}{N} \right)^{\frac{1}{2}+\frac{d-1}{p}-\frac{d-1}{2}} \|f\|_{H_x^s} \\
&\lesssim N^{\frac{1}{q}-s+} \left(\frac{T}{N} \right)^{\frac{1}{q}+\frac{d-1}{p}-\frac{d-1}{2}} \|f\|_{H_x^s} .
\end{aligned}
$$

In the second last line, we used condition (2.3.2). For $T \leqslant N$, we also have that

$$
\| \exp(\pm it|\nabla|) f_N^\omega \|_{L_\omega^r L_t^q L_x^p(\Omega \times [T,N) \times \mathbb{R}^d)} \lesssim N^{\frac{1}{q}-\frac{1}{2}} \| \exp(\pm it|\nabla|) f_N^\omega \|_{L_\omega^r L_t^2 L_x^p(\Omega \times [0,N) \times \mathbb{R}^d)} \lesssim N^{\frac{1}{q}-s+} \|f\|_{H_x^s} .
$$

□

Definition 2.3.5 (Auxiliary norm)**.** Let $0 \leqslant s < 1$, let $(f_0, f_1) \in H^s(\mathbb{R}^4) \times H^{s-1}(\mathbb{R}^4)$, and let $N_0 \geqslant 1$. We then define

$$
\begin{aligned}
\|(f_0, f_1)\|_{Z(N_0)} := &\sum_{N \geqslant N_0} N^{s+\frac{\theta}{2}-1-\delta} \left\| \cos(t|\nabla|) f_{0,N} + \frac{\sin(t|\nabla|)}{|\nabla|} f_{1,N} \right\|_{L_t^1 L_x^\infty([N^{1+\theta},\infty) \times \mathbb{R}^4)} \\
&+ \sum_{N \geqslant N_0} N^{s-\delta} \left\| \cos(t|\nabla|) f_{0,N} + \frac{\sin(t|\nabla|)}{|\nabla|} f_{1,N} \right\|_{L_t^\infty L_x^\infty([0,\infty) \times \mathbb{R}^4)}
\end{aligned}
$$

From Proposition 2.3.1 and Corollary 2.3.3, it follows that

$$
\|(f_0^\omega, f_1^\omega)\|_{L_\omega^r Z(1)} \lesssim \sqrt{r} \|(f_0, f_1)\|_{H_x^s \times H_x^{s-1}} .
$$

2.4 Wave packet decomposition

In this section, we use a wave packet decomposition to better understand the (random) linear evolution.

This part of the argument does not rely on the additional randomization in physical space. We therefore phrase all results in a way that applies to both the microlocal and the Wiener randomization, and hope that this facilitates future applications. With this in mind, we now rewrite the microlocal randomization in a form that resembles the Wiener randomization.

Let the random variables $\{X_{k,l}\}_{k,l\in\mathbb{Z}^d}$ be as in Definition 2.1.2, let $\{\epsilon_k\}_{k\in I\cup\{0\}}$ be a family of independent random signs, and set $\epsilon_{-k} = \epsilon_k$ for all $k \in I$. We can then define $Y_{k,l} := \epsilon_k X_{k,l}$. For a sequence of multi-indices $k, l_1, \ldots, l_J \in \mathbb{Z}^d$ and any sequence of Borel-measurable sets $A_1, \ldots, A_J \subseteq \mathbb{R}$, we have that

$$\begin{aligned}\mathbb{P}(\epsilon_k = 1, Y_{k,l_1} \in A_1, \ldots, Y_{k,l_J} \in A_J) &= \mathbb{P}(\epsilon_k = 1, X_{k,l_1} \in A_1, \ldots, X_{k,l_J} \in A_J)\\ &= \mathbb{P}(\epsilon_k = 1)\prod_{j=1}^{J}\mathbb{P}(X_{k,l_j} \in A_j) = \mathbb{P}(\epsilon_k = 1)\prod_{j=1}^{J}\mathbb{P}(Y_{k,l_j} \in A_j)\ .\end{aligned}$$

In the last equality, we have used that the random variables $X_{k,l}$ are symmetric. Therefore, for a fixed $k \in \mathbb{Z}^d$, the family $\{\epsilon_k\} \cup \{Y_{k,l}\}_{l\in\mathbb{Z}^d}$ is independent. From this, it then easily follows that the whole family $\{\epsilon_k\}_{k\in I\cup\{0\}} \cup \{Y_{k,l}\}_{k\in I\cup\{0\},l\in\mathbb{Z}^d}$ is independent. We then rewrite the microlocal randomization as

$$f^\omega = \sum_{k,l\in\mathbb{Z}^d} X_{k,l}P_k(\varphi_l f) = \sum_{k\in\mathbb{Z}^d}\epsilon_k f_k, \qquad \text{where} \quad f_k := P_k\Big(\sum_{l\in\mathbb{Z}^d} Y_{k,l}\varphi_l f\Big)\ . \tag{2.4.1}$$

Due to the independence properties discussed above, we can regard the functions $\{f_k\}$ as deterministic by conditioning on the random variables $\{Y_{k,l}\}_{k,l}$, and only utilize the randomness through the random signs $\{\epsilon_k\}_k$. Note that (2.4.1) closely resembles the Wiener randomization.

To motivate the wave packet decomposition below, we now rewrite the linear evolution with initial data (f_0^ω, f_1^ω). Using the notation from (2.4.1), we first introduce the half-wave operators by writing

$$
\begin{aligned}
&\cos(t|\nabla|)f_0^\omega + \frac{\sin(t|\nabla|)}{|\nabla|} f_1^\omega \\
&= \sum_{k\in\mathbb{Z}^d} \epsilon_k \left(\cos(t|\nabla|) f_{0;k}^\omega + \frac{\sin(t|\nabla|)}{|\nabla|} f_{1;k}^\omega \right) \\
&= \sum_{k\in\mathbb{Z}^d} \epsilon_k \left[\exp(it|\nabla|) \left(\frac{f_{0;k} + i|\nabla|^{-1} f_{1;k}}{2} \right) + \exp(-it|\nabla|) \left(\frac{f_{0;k} - i|\nabla|^{-1} f_{1;k}}{2} \right) \right] \\
&=: \sum_{k\in\mathbb{Z}^d} \epsilon_k \left[\exp(it|\nabla|) f_k^+ + \exp(-it|\nabla|) f_k^- \right] .
\end{aligned}
\tag{2.4.2}
$$

As in (2.2.14), we also decompose dyadically in frequency space, and write

$$
F_N^\pm := \sum_{\|k\|_\infty \in (N/2,N]} \epsilon_k \exp(\pm it|\nabla|) f_k^\pm \qquad \text{and} \qquad F_N := F_N^+ + F_N^- . \tag{2.4.3}
$$

Let $k \in \mathbb{Z}^d$ with $\|k\|_\infty \in (N/2, N]$, and let $l \in \mathbb{Z}^d$. We define the tubes $T_{k,l}^\pm$ by

$$
T_{k,l}^\pm := \{(t,x) \in [0,N] \times \mathbb{R}^d \colon \|x - (l \mp t \cdot k/\|k\|_2)\|_2 \leqslant 1\} . \tag{2.4.4}
$$

Here, the superscripts $\pm$ are chosen so that the tubes correspond to the operators $\exp(\pm it|\nabla|)$. The dimensions and the shape of the tubes are illustrated in the introduction, see Figure 2.2. Motivated by the Doppler effect, the tubes $T_{k,l}^+$ are sometimes called red tubes, and the tubes $T_{k,l}^-$ are sometimes called blue tubes.

Proposition 2.4.1 (Spatial wave packet decomposition)**.** Let $k \in \mathbb{Z}^d$ with $\|k\|_\infty \in (N/2, N]$. Let $f_k \in L_x^2(\mathbb{R}^d)$ be a function such that $\operatorname{supp} \widehat{f_k} \subseteq k + [-1,1]^d$. Then, there exists a decomposition

$$
f_k = \sum_{l\in\mathbb{Z}^d} f_{k,l}
$$

such that

(i) $\operatorname{supp} \widehat{f_{k,l}} \subseteq k + [-4,4]^d$ for all $l \in \mathbb{Z}$,

(ii) the family $\{f_{k,l}\}_{l\in\mathbb{Z}^d}$ satisfies the almost-orthogonality condition

$$\sum_{l\in\mathbb{Z}^d} \|f_{k,l}\|^2_{L^2_x(\mathbb{R}^d)} \lesssim \|f\|^2_{L^2_x(\mathbb{R}^d)} \ , \tag{2.4.5}$$

(iii) and for any $D \geqslant 1$, any $l \in \mathbb{Z}^d$, and all $(t,x) \in [0,N] \times \mathbb{R}^d$, it holds that

$$|\exp(\pm it|\nabla|)f_{k,l}(x)| \lesssim_D (1 + \operatorname{dist}((t,x), T^{\pm}_{k,l}))^{-D} \|f_k\|_{L^2_x(\mathbb{R}^d)} \ . \tag{2.4.6}$$

Wave packet decomposition as in Proposition 2.4.1 have been used extensively in the literature, see e.g. [Bou91, Cor77, Fef73, Gut16, Wol01] and the survey [Tao04]. We present the details below, but encourage the expert reader to skip ahead to the end of the proof.

Proof. We define the fattened projection

$$\widetilde{P}_k := \sum_{\|k'-k\|_\infty \leqslant 2} P_{k'} \ .$$

Then, it holds that

$$f = \widetilde{P}_k f = \sum_{l\in\mathbb{Z}^d} \widetilde{P}_k(\varphi_l f) =: \sum_{l\in\mathbb{Z}^d} f_{k,l} \ .$$

The frequency support condition (i) directly follows from the definition of $\widetilde{P}_k$. Furthermore, the almost orthogonality (ii) follows from

$$\sum_{l\in\mathbb{Z}^d} \|f_{k,l}\|^2_{L^2_x(\mathbb{R}^d)} \lesssim \sum_{l\in\mathbb{Z}^d} \|\varphi_l f_k\|^2_{L^2_x(\mathbb{R}^d)} \lesssim \|f_k\|^2_{L^2_x(\mathbb{R}^d)} \ .$$

Thus, it remains to prove the decay estimate (iii). We only treat the operator $\exp(it|\nabla|)$, since the proof for $\exp(-it|\nabla|)$ is similar. If $N \lesssim 1$, the estimate is trivial. Thus, we may assume that $N \gg 1$. The argument is based on the method of non-stationary phase. For all $t \in [0,N]$ and

$x \in \mathbb{R}^d$, we have that

$$
\begin{aligned}
&\exp(it|\nabla|) f_{k,l}(x) \\
&= \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \exp(ix\xi + it|\xi|) \widetilde{\psi}(\xi - k)(\widehat{\varphi}_l * \widehat{f})(\xi) \mathrm{d}\xi \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \exp(ix\xi + it|\xi|) \widetilde{\psi}(\xi - k) \exp(-iy(\xi - \eta)) \varphi(y - l) \widehat{f}(\eta) \mathrm{d}y \mathrm{d}\eta \mathrm{d}\xi \\
&= \frac{1}{(2\pi)^d} \exp(ixk + it|k|) \int_{\mathbb{R}^d} K(\eta; t, x) \exp(-il(k - \eta)) \widehat{f}(\eta) \mathrm{d}\eta \ ,
\end{aligned}
$$

where the kernel $K(\eta; t, x)$ is given by

$$
K(\eta; t, x) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \exp(i(x - l)\xi + it(|\xi + k| - |k|)) \exp(-iy(\xi + k - \eta)) \widetilde{\psi}(\xi) \varphi(y) \mathrm{d}\xi \mathrm{d}y \ .
$$

Since $\operatorname{supp} \varphi \subseteq [-1, 1]^d$, the function $a(\xi; y, \eta) := \exp(-iy(\xi + k - \eta)) \widetilde{\psi}(\xi) \varphi(y)$ has uniformly bounded derivatives in ξ, i.e, we have for all $\alpha \in \mathbb{N}_0^d$ that $|\partial_\xi^\alpha a(\xi; y, \eta)| \lesssim_\alpha 1$. Using the support conditions in the variables y and η, it thus suffices to prove for all $a \in C_c^\infty([-2, 2]^d)$ that

$$
\left| \int_{\mathbb{R}^d} \exp(i(x - l)\xi + it(|\xi + k| - |k|)) a(\xi) \mathrm{d}\xi \right| \lesssim_M (1 + |x - l + t \cdot k/\|k\|_2)^{-M} \ . \tag{2.4.7}
$$

Due to the compact support of $a(\xi)$, we restrict to $|\xi| \leqslant 2$. The bound for $|x - l + t \cdot k/\|k\|_2| \lesssim 1$ is trivial. Thus, we may assume that $|x - l + t \cdot k/\|k\|_2| \gg 1$. We define the phase function

$$
\Phi_k(\xi) = \Phi_k(\xi; t, x, l) = i(x - l)\xi + it(|\xi + k| - |k|) \ .
$$

Then, we have that

$$
\begin{aligned}
\nabla_\xi \Phi_k(\xi) &= \nabla_\xi \Big((x - l)\xi + t(|\xi + k| - |k|) \Big) \\
&= x - l + t \frac{\xi + k}{|\xi + k|} \\
&= x - l + t \frac{k}{|k|} + t \Big(\frac{|k| - |\xi - k|}{|\xi + k||k|} \ k + \frac{\xi}{|\xi + k|} \Big) \ .
\end{aligned}
$$

From the assumption $|t| \leqslant N$, it follows that $\nabla_\xi \Phi_k = x - l + tk/|k| + \mathcal{O}(1)$. We also write

$$\nabla_\xi \Phi_k(\xi) = x - l + t\frac{k}{|k|} + t\Psi_k\big(\frac{\xi}{|k|}\big)\frac{k}{|k|} \ ,$$

where

$$\Psi_k(\nu) := \frac{1 - |\nu - k/|k||}{|\nu + k/|k||} + \frac{\nu}{|k/|k| + \nu|} \ .$$

From rotation invariance, it follows easily that $|\nabla^\alpha \Psi_k(\nu)| \lesssim_\alpha 1$ for all $|\nu| \leqslant 1/10$, uniformly in k. This leads to

$$|\nabla^\alpha \Phi_k(\xi)| \lesssim_\alpha \frac{|t|}{|k|^{|\alpha|-1}} \lesssim 1 \qquad \text{for all } |\alpha| \geqslant 2 \ .$$

We then rewrite the integral in (2.4.7) as

$$\begin{aligned}
&\int_{\mathbb{R}^d} \exp(i(x-l)\xi + it(|\xi + k| - |k|))a(\xi)\mathrm{d}\xi \\
&= \int_{\mathbb{R}^d} \left(\left(\frac{(-i)\nabla_\xi \Phi_k(\xi) \nabla_\xi}{|\nabla \Phi_k(\xi)|^2} \right)^M \exp(i\Phi_k(\xi)) \right) \cdot a(\xi)\mathrm{d}\xi \\
&= \int_{\mathbb{R}^d} \exp(i\Phi_k(\xi)) \left(\nabla_\xi \cdot \frac{i\nabla_\xi \Phi_k(\xi)}{|\nabla \Phi_k(\xi)|^2} \right)^M a(\xi)\mathrm{d}\xi \ .
\end{aligned}$$

The inequality (2.4.7) then follows from the bounds on the phase function above.

□

The wave packet decomposition in Proposition 2.4.1 is valid on the time interval $[0, N]$, and the physical localization deteriorates for larger times. When analyzing the linear evolution on an interval of the form $[t_0, t_0 + N)$, with $t_0 \in N\mathbb{N}_0$, we therefore use the wave packet decomposition of $\exp(\pm it_0|\nabla|)f_k$. To state the result, we set

$$T^{\pm}_{k,l;t_0} := \{(t,x) \in [t_0, t_0+N] \times \mathbb{R}^d \colon \|x - (l \mp (t - t_0)\cdot k/\|k\|_2)\|_2 \leqslant 1\} \ .$$

Corollary 2.4.2 (Time-translated spatial wave packet decomposition)**.** Let $k \in \mathbb{Z}^d$ with $\|k\|_\infty \in (N/2, N]$, and let $t_0 \in N\mathbb{N}_0$. Let $f_k \in L^2_x(\mathbb{R}^d)$ be a function satisfying $\operatorname{supp} \widehat{f}_k \subseteq k + [-1,1]^d$. Then, there exists a decomposition

$$\exp(\pm it_0|\nabla|)f_k = \sum_{l\in\mathbb{Z}^d} f^{\pm}_{k,l;t_0}$$

such that

(i) $\operatorname{supp} \widehat{f^{\pm}_{k,l;t_0}} \subseteq k + [-2,2]^d$ for all $l \in \mathbb{Z}$,

(ii) the family $\{f^{\pm}_{k,l;t_0}\}_{l\in\mathbb{Z}^d}$ satisfies the almost-orthogonality condition

$$\sum_{l\in\mathbb{Z}^d} \|f^{\pm}_{k,l;t_0}\|^2_{L^2_x(\mathbb{R}^d)} \lesssim \|f\|^2_{L^2_x(\mathbb{R}^d)} \ , \tag{2.4.8}$$

(iii) and for any $D \geqslant 1$, any $l \in \mathbb{Z}^d$, and all $(t,x) \in [t_0, t_0+N] \times \mathbb{R}^d$, it holds that

$$|\exp(\pm i(t-t_0)|\nabla|)f^{\pm}_{k,l;t_0}(x)| \lesssim_D (1 + \operatorname{dist}((t,x), T^{\pm}_{k,l;t_0}))^{-D} \|f_k\|_{L^2_x(\mathbb{R}^d)} \ . \tag{2.4.9}$$

Proof. We apply Proposition 2.4.1 to $\exp(\pm it_0|\nabla|)f_k$. □

As discussed in the introduction, we now group the wave packets into bushes and a (nearly) non-overlapping collection (see Figure 2.3). This argument is inspired by Bourgain's bush argument from [Bou91], and we also refer the reader to [Wol99, Proposition 2.2].

Before we state main proposition, we define the truncated and fattened ℓ^∞-cone

$$\widetilde{K}^N_{t_0,x_0} := \{(t,x) \in [t_0, t_0+N] \times \mathbb{R}^4 \colon \|x - x_0\|_\infty \leqslant 16N - |t - t_0|\} \ . \tag{2.4.10}$$

The significance of $\widetilde{K}^N_{t_0,x_0}$ will be explained in Section 2.5.2 and Section 2.6. For now, we encourage the reader to treat $\widetilde{K}^N_{t_0,x_0}$ as space-time cube of scale N.

Proposition 2.4.3 (Wave packet decomposition and bushes)**.** Let $\{f^{\pm}_k\}_k \subseteq L^2_x(\mathbb{R}^d)$ be a family of functions, where $\|k\|_\infty \in (N/2, N]$, and $\operatorname{supp} \widehat{f^{\pm}_k} \subseteq k + [-1,1]^d$. Let $t_0 \in N\mathbb{N}_0$, let $x_0 \in N\mathbb{Z}^d$, and let the wave packets $\{f^{\pm}_{k,l;t_0}\}$ be as in Corollary 2.4.2. Furthermore, let $\mathcal{Q}^N_{t_0,x_0}$ be a collection of disjoint space-time cubes with sidelength $\sim N^{\delta}$ covering $\widetilde{K}^N_{t_0,x_0}$. We group the wave packets according to their amplitude by setting

$$\mathscr{A}_m = \mathscr{A}^{N,\pm}_{m,t_0,x_0} := \{(k,l) \in \mathbb{Z}^d \times \mathbb{Z}^d \colon \|f^{\pm}_{k,l;t_0}\|_{L^2_x(\mathbb{R}^d)} \in [2^m, 2^{m+1}], \|l - x_0\|_\infty \leqslant 3N\} \ . \tag{2.4.11}$$

Then, there exists a family of bushes $\{\mathscr{B}_{j,m}\}_j = \{\mathscr{B}^{N,\pm}_{j,m,t_0,x_0}\}_j$, where $j = 1, \ldots, J^{N,\pm}_{m,t_0,x_0}$, and a nearly non-overlapping set $\mathscr{D}_m = \mathscr{D}^{N,\pm}_{m,t_0,x_0}$, depending only on the set $\mathscr{A}_m$, so that the following holds:

(i) The sets form a partition of $\mathscr{A}_m$, i.e.,

$$\mathscr{A}_m = \mathscr{D}_m \ \dot{\bigcup} \ \left(\dot{\bigcup}_{j=1,\ldots,J} \mathscr{B}_{j,m}\right) \tag{2.4.12}$$

(ii) We have the bound on the number of wave packets

$$\sum_{x_0 \in N\mathbb{Z}^d} \sum_{m \in \mathbb{Z}} 2^{2m} \# \mathscr{A}^{N,\pm}_{m,t_0,x_0} \lesssim \sum_k \|f_k\|^2_{L^2_x(\mathbb{R}^d)} \ . \tag{2.4.13}$$

(iii) Each bush $\mathscr{B}_{j,m}$ contains at least $\mu = \mu(N,m) := N^{-\frac{1}{2}} \# \mathscr{A}_m$ wave packets.

(iv) For each bush $\mathscr{B}_{j,m}$, all corresponding wave packets intersect in the same region of space-time. More precisely, there exists a cube $Q \in \mathcal{Q}^N_{t_0,x_0}$ s.t.

$$T^{\pm}_{k,l;t_0} \bigcap 2Q \neq \varnothing \qquad \forall (k,l) \in \mathscr{B}_{j,m} \ . \tag{2.4.14}$$

(v) At $\mu = N^{-\frac{1}{2}} \# \mathscr{A}_m$ wave packets in $\mathscr{D}_m$ overlap, i.e., we have for all cubes $Q \in \mathcal{Q}^N_{t_0,x_0}$ that

$$\#\{(k,l) \in \mathscr{D}_m \colon T^{\pm}_{k,l;t_0} \bigcap 2Q \neq \varnothing\} < P \ . \tag{2.4.15}$$

The choice of the number of packets/multiplicity $\mu = N^{-\frac{1}{2}} \# \mathscr{A}_m$ will be justified in the proof of Proposition 2.6.1, see (2.6.1). The parameter μ corresponds to the multiplicity parameter in Bourgain's bush argument, see [Wol99, Proposition 2.2].

Remark 2.4.4. We will later apply this proposition to a set of random functions $\{\epsilon_k f_k^{\pm}\}_k$. From (2.4.11), it follows that the sets $\mathscr{A}_m$, and hence also $\mathscr{B}_{j,m}$ and $\mathscr{D}_m$, do not depend on the random signs $\{\epsilon_k\}$.

Proof. Let us first prove the inequality in (ii). From Corollary 2.4.2, it follows that

$$\sum_{x_0 \in N\mathbb{Z}^d} \sum_{m \in \mathbb{Z}} 2^{2m} \# \mathscr{A}_{m,t_0,x_0}^{N,\pm} \lesssim \sum_{\substack{k,l \in \mathbb{Z}^d \\ \|k\|_\infty \in (N/2,N]}} \|f_{k,l;t_0}^{\pm}\|_{L_x^2}^2 \lesssim \sum_{\substack{k \in \mathbb{Z}^d \\ \|k\|_\infty \in (N/2,N]}} \|f_k^{\pm}\|_{L_x^2}^2 \ .$$

We now construct the sets $\mathscr{B}_{j,m,t_0,x_0}^{N,\pm}$ and $\mathscr{D}_{m,t_0,x_0}^{N,\pm}$. To simplify the expressions, we drop the super- and subscripts $\pm, N, t_0$, and x_0 from our notation. The basic idea is to form the bushes through a greedy selection algorithm. For any $Q \in \mathcal{Q}_{t_0,x_0}^N$, we define

$$\mathscr{T}_m(Q) := \{(k,l) \in \mathscr{A}_m \colon T_{k,l} \bigcap Q \neq \varnothing\} \ . \tag{2.4.16}$$

We further set $\mathscr{T}_m^{(1)}(Q) := \mathscr{T}_m(Q)$. We then choose a cube $Q_1 \in \mathcal{Q}_{t_0,x_0}^N$ such that

$$\#\mathscr{T}_m^{(1)}(Q_1) = \max_{Q \in \mathcal{Q}_{t_0,x_0}^N} \#\mathscr{T}_m^{(1)}(Q) \ ,$$

and define the first bush as $\mathscr{B}_{1,m} := \mathscr{T}_m^{(1)}(Q_1)$. By setting

$$\mathscr{T}_m^{(2)}(Q) = \mathscr{T}_m^{(1)}(Q) \backslash \mathscr{B}_{1,m} \ ,$$

we remove all of the wave packets in the first bush from the collection. We then iteratively define $\mathscr{B}_{j,m} := \mathscr{T}_m^{(j)}(Q_j)$, where

$$\#\mathscr{T}_m^{(j)}(Q_j) = \max_{Q \in \mathcal{Q}_{t_0,x_0}^N} \#\mathscr{T}_m^{(j)}(Q) \ ,$$

and the collections $\mathscr{T}_m^{(j)}(Q)$ are defined as $\mathscr{T}_m^{(j-1)}(Q) \backslash \mathscr{B}_{j-1,m}$. Once $\mathscr{T}_m^{(j+1)}(Q_{j+1}) < \mu$, we no longer create a new bush, and instead stop the algorithm. Since $\mathscr{A}_m$ contains at most $\sim N^8$ wave packets, the greedy selection algorithm terminates after finitely many steps. From the construction, we see that the sets $\mathscr{B}_{j,m} \subseteq \mathbb{Z}^d \times \mathbb{Z}^d$ are disjoint (even though the corresponding tubes may still overlap). Finally, we define the collection $\mathscr{D}_m$ by

$$\mathscr{D}_m := \mathscr{A}_m \backslash \bigcup_{j=1}^{J} \mathscr{B}_{j,m} \ .$$

The properties (i), (iii), (iv), and (v) then follow directly from the construction. □

We now prove a probabilistic estimate for the wave packets with random coefficients.

Proposition 2.4.5 (Square-root cancellation for wave packets)**.** Let $\{f_k^{\pm}\}_k \subseteq L_x^2(\mathbb{R}^d)$ be a deterministic family of functions, where $k \in \mathbb{Z}^d$ satisfies $\|k\|_\infty \in (N/2, N]$, and assume that $\sum_k \|f_k^{\pm}\|_{L_x^2(\mathbb{R}^d)}^2 \lesssim 1$. Let $\theta > 0$ be a parameter, and let $C_d > 0$ be any large absolute constant. Then, we have for all $m \in \mathbb{Z}$ satisfying $-C_d \log(N) \leqslant m \leqslant C_d \log(N)$ that

$$\mathbb{E}\Bigg[\sup_{\substack{t_0=0,\dots\lfloor N^\theta \rfloor N \\ x_0 \in N\mathbb{Z}^d}} \ \sup_{j=1,\dots,J^{N,\pm}_{m,t_n,x_0}} \frac{\Big\| \sum_{(k,l)\in \mathscr{B}^{N,\pm}_{j,m,t_0,x_0}} \epsilon_k \exp(\pm i(t-t_0)|\nabla|) f^{\pm}_{k,l;t_0} \Big\|_{L^\infty_{t,x}(\mathbb{R}\times\mathbb{R}^d)}}{2^m (\#\mathscr{B}^{N,\pm}_{j,m,t_0,x_0})^{\frac{1}{2}}} \Bigg] \lesssim N^{\delta d} \quad (2.4.17)$$

and

$$\mathbb{E}\Bigg[\sup_{\substack{t_0=0,\dots\lfloor N^\theta \rfloor N \\ x_0 \in N\mathbb{Z}^d}} \frac{\Big\| \sum_{(k,l)\in \mathscr{D}^{N,\pm}_{m,t_0,x_0}} \epsilon_k \exp(\pm i(t-t_0)|\nabla|) f^{\pm}_{k,l;t_0} \Big\|_{L^\infty_t L^\infty_x([t_0,t_0+N]\times\mathbb{R}^d)}}{2^m \mu^{\frac{1}{2}}} \Bigg] \lesssim N^{\delta d} \ . \quad (2.4.18)$$

Here, $\mu = N^{-\frac{1}{2}} \# \mathscr{A}^{N,\pm}_{m,t_0,x_0}$ is as in Proposition 2.4.3. To be perfectly precise, we use the convention $0/0 := 0$ in (2.4.18).

The expressions in (2.4.17) and (2.4.18) may seem complicated. To make sense of them, recall that the square function heuristic predicts that $\sum_k \epsilon_k a_k$ is roughly of size $\sim (\sum_k a_k^2)^{\frac{1}{2}}$. Then, Proposition 2.4.5 simply states that the square function heuristic can be justified for all relevant amplitudes, for all relevant times, all positions, all families of bushes, and all non-overlapping collections.

For instance, let us heuristically motivate (2.4.18). By the definition of $\mathscr{D}^{N,\pm}_{m,t_0,x_0}$, any fixed point in the space-time region $[t_0, t_0 + N] \times \mathbb{R}^d$ is contained in the (moral) support of at most μ wave packets. Since each of the wave packets has amplitude $\sim 2^m$, and they all correspond to different frequencies $k \in \mathbb{Z}^d$, the square-function heuristic predicts a contribution of size $\sim 2^m \mu^{\frac{1}{2}}$.

Proof. In this proof, we make extensive use of Lemma 2.2.4. First, we prove that the suprema in (2.4.17) and (2.4.18) are over at most $N^{\mathcal{O}(C_d)}$-many terms. From (2.4.13), it follows for all $m \geqslant -C_d \log(N)$ that

$$\sum_{x_0 \in N\mathbb{Z}^d} \#\mathscr{A}^{N,\pm}_{m,t_0,x_0} \lesssim 2^{-2m} \sum_{k,l} \| f^{\pm}_{k,l} \|^2_{L^2_x(\mathbb{R}^d)} \lesssim 2^{-2m} \sum_{k} \| f^{\pm}_{k} \|^2_{L^2_x(\mathbb{R}^d)} \lesssim N^{2C_d} \ .$$

Thus, this bounds the number of all wave packets with amplitude $\sim 2^m$. Since each bush $\mathscr{B}^{N,\pm}_{j,m,t_0,x_0}$ contains at least one wave packet, the supremum in (2.4.17) is over at most $\mathcal{O}(N^{2C_d})$ non-zero terms. The same applies to the non-overlapping families $\mathscr{D}^{N,\pm}_{m,t_0,x_0}$ in (2.4.18). From Lemma 2.2.4, it then suffices to obtain uniform sub-gaussian bounds on each individual term in (2.4.17) and (2.4.18).

We start with the contribution of the bushes. To simplify the notation, we write $\mathscr{B}_{j,m} = \mathscr{B}^{N,\pm}_{j,m,t_0,x_0}$. From Bernstein's inequality and Lemma 2.2.6, we have for all $2 \leqslant p < \infty$ that

$$\begin{aligned} &\Big\| \sum_{(k,l) \in \mathscr{B}_{j,m}} \epsilon_k \exp(\pm i (t - t_0) |\nabla|) f^{+}_{k,l;t_0} \Big\|_{L^\infty_t L^\infty_x(\mathbb{R} \times \mathbb{R}^d)} \\ &\leqslant N^{\frac{d+1}{p}} \Big\| \sum_{(k,l) \in \mathscr{B}_{j,m}} \epsilon_k \exp(\pm i (t - t_0) |\nabla|) f^{+}_{k,l;t_0} \Big\|_{L^p_t L^p_x(\mathbb{R} \times \mathbb{R}^d)} \ . \end{aligned}$$

Before we utilize the randomness, we observe that for each $k \in \mathbb{Z}^d$ at most $\mathcal{O}(N^{\delta d})$ tubes $T^{\pm}_{k,l;t_0}$ can intersect a space-time cube of sidelength $\sim N^{\delta}$. As a result, it follows from (2.4.14) that

$$\#\{ l \in \mathbb{Z}^d \colon (k,l) \in \mathscr{B}_{j,m} \} \lesssim N^{\delta d} \ .$$

For all $p \leqslant r < \infty$, we then obtain from Minkowski's integral inequality, Khintchine's inequality, and the refined Strichartz estimate (Corollary 2.2.13) that

$$\begin{aligned}
&\Big\| \sum_{(k,l)\in\mathscr{B}_{j,m}} \epsilon_k \exp(\pm i(t-t_0)|\nabla|) f^{+}_{k,l;t_0} \Big\|_{L^r_\omega L^\infty_t L^\infty_x(\Omega\times\mathbb{R}\times\mathbb{R}^d)} \\
&\lesssim N^{\frac{d+1}{p}} \Big\| \sum_{(k,l)\in\mathscr{B}_{j,m}} \epsilon_k \exp(\pm i(t-t_0)|\nabla|) f^{+}_{k,l;t_0} \Big\|_{L^p_t L^p_x L^r_\omega(\mathbb{R}\times\mathbb{R}^d\times\Omega)} \\
&\lesssim \sqrt{r} N^{\frac{d+1}{p}} \Big\| \Big(\sum_{k\in\mathbb{Z}^d} \Big(\sum_{l:\,(k,l)\in\mathscr{B}_{j,m}} \exp(\pm i(t-t_0)|\nabla|) f^{+}_{k,l;t_0} \Big)^2 \Big)^{\frac{1}{2}} \Big\|_{L^p_t L^p_x(\mathbb{R}\times\mathbb{R}^d)} \\
&\lesssim \sqrt{r} N^{\frac{d+1}{p}+\frac{\delta d}{2}} \Big\| \exp(\pm i(t-t_0)|\nabla|) f^{+}_{k,l;t_0} \Big\|_{L^p_t L^p_x \ell^2_{k,l}(\mathbb{R}\times\mathbb{R}^d\times\mathscr{B}_{j,m})} \\
&\lesssim \sqrt{r} N^{\frac{d+1}{p}+\frac{\delta d}{2}} \Big\| \exp(\pm i(t-t_0)|\nabla|) f^{+}_{k,l;t_0} \Big\|_{\ell^2_{k,l} L^p_t L^p_x(\mathscr{B}_{j,m}\times\mathbb{R}\times\mathbb{R}^d)} \\
&\lesssim \sqrt{r} N^{\frac{d+2}{p}+\frac{\delta d}{2}} \Big\| f^{+}_{k,l;t_0} \Big\|_{\ell^2_{k,l} L^2_x(\mathscr{B}_{j,m}\times\mathbb{R}^d)} \\
&\lesssim \sqrt{r} N^{\frac{d+2}{p}+\frac{\delta d}{2}} 2^m (\#\mathscr{B}_{j,m})^{\frac{1}{2}} .
\end{aligned}$$

By taking $p \geqslant 2$ to be sufficiently large, we then obtain the desired sub-gaussian bound. This completes the proof of (2.4.17).

We now control the contribution of a single non-overlapping family $\mathscr{D}_m = \mathscr{D}^{N,\pm}_{m,t_0,x_0}$. For the technical aspects of this part, recall that the collection $\mathbb{Q}^u_N$ from Proposition 2.4.3 covers $\widetilde{K}^N_{t_0,x_0}$, but due the definition of $\mathscr{A}_m$ in (2.4.11), all the tubes $T^{\pm}_{k,l;t_0}$ with indices in $\mathscr{D}_m$ are contained in the region $\|x-x_0\|_\infty \leqslant 6N$. This gives us sufficient room for the following argument.

We let $2 \leqslant p < \infty$. As before, it follows from Bernstein's inequality and Lemma 2.2.6 that

$$\begin{aligned}
&\Big\| \sum_{(k,l)\in\mathscr{D}^{N,\pm}_{m,t_0,x_0}} \epsilon_k \exp(\pm i(t-t_0)|\nabla|) f^{\pm}_{k,l;t_0} \Big\|_{L^\infty_t L^\infty_x([t_0,t_0+N]\times\mathbb{R}^d)} \\
&\lesssim N^{\frac{d+1}{p}} \Big\| \sum_{(k,l)\in\mathscr{D}^{N,\pm}_{m,t_0,x_0}} \epsilon_k \exp(\pm i(t-t_0)|\nabla|) f^{\pm}_{k,l;t_0} \Big\|_{L^p_t L^p_x([t_0,t_0+N]\times\mathbb{R}^d)}
\end{aligned}$$

For all $p \leqslant r < \infty$, we then obtain from Minkowski's integral inequality and Khintchine's inequality that

$$\begin{aligned}
&\Big\| \sum_{(k,l)\in\mathscr{D}_m} \epsilon_k \exp(\pm i(t-t_0)|\nabla|) f^{\pm}_{k,l;t_0} \Big\|_{L^r_\omega L^\infty_t L^\infty_x(\Omega\times[t_0,t_0+N]\times\mathbb{R}^d)} \\
&\lesssim N^{\frac{d+1}{p}} \Big\| \sum_{(k,l)\in\mathscr{D}_m} \epsilon_k \exp(\pm i(t-t_0)|\nabla|) f^{\pm}_{k,l;t_0} \Big\|_{L^p_t L^p_x L^r_\omega([t_0,t_0+N]\times\mathbb{R}^d\times\Omega)} \\
&\lesssim \sqrt{r} N^{\frac{d+1}{p}} \Big\| \Big(\sum_{k\in\mathbb{Z}^d} \big(\sum_{l\colon (k,l)\in\mathscr{D}_m} \exp(\pm i(t-t_0)|\nabla|) f^{\pm}_{k,l;t_0} \big)^2 \Big)^{\frac{1}{2}} \Big\|_{L^p_t L^p_x([t_0,t_0+N]\times\mathbb{R}^d)} \\
&\lesssim \sqrt{r} N^{\frac{d+1}{p}} \Big\| \Big(\sum_{k\in\mathbb{Z}^d} \big(\sum_{l\colon (k,l)\in\mathscr{D}_m} \exp(\pm i(t-t_0)|\nabla|) f^{\pm}_{k,l;t_0} \big)^2 \Big)^{\frac{1}{2}} \Big\|_{L^p_t L^p_x(\widetilde{K}^N_{t_0,x_0})} \\
&\quad + \sqrt{r} N^{\frac{d+1}{p}} \Big\| \Big(\sum_{k\in\mathbb{Z}^d} \big(\sum_{l\colon (k,l)\in\mathscr{D}_m} \exp(\pm i(t-t_0)|\nabla|) f^{\pm}_{k,l;t_0} \big)^2 \Big)^{\frac{1}{2}} \Big\|_{L^p_t L^p_x(([t_0,t_0+N]\times\mathbb{R}^d)\backslash\widetilde{K}^N_{t_0,x_0})}
\end{aligned}$$

Since $\mu = N^{-\frac{1}{2}} \#\mathscr{A}_m \geqslant N^{-\frac{1}{2}}$, the bound on $([t_0, t_0+N]\times\mathbb{R}^d)\backslash\widetilde{K}^N_{t_0,x_0}$ easily follows from the decay estimate (2.4.9). Thus, we now control the contribution on $\widetilde{K}^N_{t_0,x_0}$. From Hölder's inequality, we have that

$$\begin{aligned}
&\Big\| \Big(\sum_{k\in\mathbb{Z}^d} \big(\sum_{l\colon (k,l)\in\mathscr{D}_m} \exp(\pm i(t-t_0)|\nabla|) f^{\pm}_{k,l;t_0} \big)^2 \Big)^{\frac{1}{2}} \Big\|_{L^p_t L^p_x(\widetilde{K}^N_{t_0,x_0})} \\
&\lesssim N^{\frac{d+1}{p}} \Big\| \Big(\sum_{k\in\mathbb{Z}^d} \big(\sum_{l\colon (k,l)\in\mathscr{D}_m} \exp(\pm i(t-t_0)|\nabla|) f^{\pm}_{k,l;t_0} \big)^2 \Big)^{\frac{1}{2}} \Big\|_{L^\infty_t L^\infty_x(\widetilde{K}^N_{t_0,x_0})}
\end{aligned}$$

Now pick any cube $Q \in \mathbb{Q}^u_N$. In analogy to (2.4.16), we define the collection of "remaining" tubes by

$$\mathscr{T}^r_m(Q) := \{(k,l) \in \mathscr{D}_m \colon T^{\pm}_{k,l;t_0} \bigcap 2Q \neq \varnothing\} \ .$$

From Proposition 2.4.3, it follows that $\#\mathscr{T}^r_m(Q) \leqslant \mu$. As above, we have for each frequency $k \in \mathbb{Z}^d$ the bound $\#\{l \in \mathbb{Z}^d \colon (k,l) \in \mathscr{T}^r_m(Q)\} \lesssim N^{\delta d}$. Using the decay estimate (2.4.16) to treat distant wave packets, we obtain

$$
\begin{aligned}
&\Big\| \Big(\sum_{k\in\mathbb{Z}^d} \Big(\sum_{l\colon (k,l)\in\mathcal{D}_m} \exp(\pm i(t-t_0)|\nabla|) f^{\pm}_{k,l;t_0} \Big)^2 \Big)^{\frac{1}{2}} \Big\|_{L^\infty_t L^\infty_x(Q)} \\
&\lesssim \Big\| \Big(\sum_{k\in\mathbb{Z}^d} \Big(\sum_{l\colon (k,l)\in\mathcal{T}^r_m(Q)} \exp(\pm i(t-t_0)|\nabla|) f^{\pm}_{k,l;t_0} \Big)^2 \Big)^{\frac{1}{2}} \Big\|_{L^\infty_t L^\infty_x(Q)} \\
&\quad + \Big\| \Big(\sum_{k\in\mathbb{Z}^d} \Big(\sum_{l\colon (k,l)\in\mathcal{D}_m\backslash\mathcal{T}^r_m(Q)} \exp(\pm i(t-t_0)|\nabla|) f^{\pm}_{k,l;t_0} \Big)^2 \Big)^{\frac{1}{2}} \Big\|_{L^\infty_t L^\infty_x(Q)} \\
&\lesssim N^{\frac{\delta d}{2}} \Big\| \exp(\pm i(t-t_0)|\nabla|) f^{\pm}_{k,l;t_0} \Big\|_{L^\infty_t L^\infty_x \ell^2_{k,l}(Q\times\mathcal{T}^r_m(Q))} + N^{-10C_d} \\
&\lesssim N^{\frac{\delta d}{2}} 2^m (\#\mathcal{T}^r_m(Q))^{\frac{1}{2}} + N^{-10C_d} \\
&\lesssim N^{\frac{\delta d}{2}} 2^m \mu^{\frac{1}{2}} \ .
\end{aligned}
$$

After taking the supremum over all cubes $Q \in \mathbb{Q}^u_N$, we finally arrive at

$$
\Big\| \sum_{(k,l)\in\mathcal{D}_m} \epsilon_k \exp(\pm i(t-t_0)|\nabla|) f^{\pm}_{k,l;t_0} \Big\|_{L^r_\omega L^\infty_t L^\infty_x(\Omega\times[t_0,t_0+N]\times\mathbb{R}^d)} \lesssim \sqrt{r} N^{2\frac{d+2}{p}+\frac{\delta d}{2}} 2^m \mu^{\frac{1}{2}} \ .
$$

By choosing $p \geqslant 2$ sufficiently large, we arrive at the desired sub-gaussian bound. □

Definition 2.4.6 (Wave packet "norm")**.** Let $(f_0, f_1) \in H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)$ and let $f^{\pm}_k$ be defined as in (2.4.1) and (2.4.2). For any $N_0 \geqslant 1$, we then define the wave packet "norm" of the random data (f^ω_0, f^ω_1) by

$$
\begin{aligned}
&\|(f^\omega_0, f^\omega_1)\|_{\mathrm{WP}(N_0)} := \\
&\sum_{\substack{N\geqslant N_0 \\ |m|\leqslant C_d\log(N)}} N^{-2\delta d} \sup_{\substack{t_0=0,\dots\lfloor N^\theta\rfloor N \\ x_0\in N\mathbb{Z}^d}} \sup_{j=1,\dots,J^{N,\pm}_{m,t_0,x_0}} \frac{\Big\| \sum_{(k,l)\in\mathcal{B}^{N,\pm}_{j,m,t_0,x_0}} \epsilon_k \exp(\pm i(t-t_0)|\nabla|) f^{\pm}_{k,l;t_0} \Big\|_{L^\infty_t L^\infty_x(\mathbb{R}\times\mathbb{R}^d)}}{2^m (\#\mathcal{B}^{N,\pm}_{j,m,t_0,x_0})^{\frac{1}{2}}} \\
&\quad + \sum_{\substack{N\geqslant N_0 \\ |m|\leqslant C_d\log(N)}} N^{-2\delta d} \sup_{\substack{t_0=0,\dots\lfloor N^\theta\rfloor N \\ x_0\in N\mathbb{Z}^d}} \frac{\Big\| \sum_{(k,l)\in\mathcal{D}^{N,\pm}_{m,t_0,x_0}} \epsilon_k \exp(\pm i(t-t_0)|\nabla|) f^{\pm}_{k,l;t_0} \Big\|_{L^\infty_t L^\infty_x([t_0,t_0+N]\times\mathbb{R}^d)}}{2^m \mu^{\frac{1}{2}}}
\end{aligned}
$$

While the quantity $\|(f_0^\omega, f_1^\omega)\|_{\mathrm{WP}(N_0)}$ measures the size of the wave packets (over their expected size), it is certainly far from being an actual norm.

Corollary 2.4.7. Let $(f_0, f_1) \in H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)$ and let $f_k^\pm$ be defined as in (2.4.1) and (2.4.2). Furthermore, we assume that $N_0 = N_0(\{Y_{k,l}\})$ satisfies

$$\sum_{\|k\|_\infty \geqslant N_0/2} \left(\|f_{0;k}\|^2_{L^2_x(\mathbb{R}^d)} + \|f_{1;k}\|^2_{\dot{H}^{-1}_x(\mathbb{R}^d)} \right) \lesssim 1 \ .$$

Then, it holds that

$$\mathbb{E}_\epsilon \|(f_0^\omega, f_1^\omega)\|_{\mathrm{WP}(N_0)} < \infty \ ,$$

where $\mathbb{E}_\epsilon$ denotes the expectation over the random signs $\{\epsilon_k\}$.

Proof. This follows directly from Proposition 2.4.5 and Definition 2.4.6. □

For the bootstrap argument in Section 2.6, it will be convenient to create a small forcing term by truncating to high frequencies. If $N_{\mathrm{hi}} = N_{\mathrm{hi}}(\omega)$ is a (possibly random) frequency parameter, we set

$$F_{\mathrm{hi}} := \sum_{N \geqslant N_{\mathrm{hi}}} \left(\cos(t|\nabla|) f_{0,N}^\omega + \frac{\sin(t|\nabla|)}{|\nabla|} f_{1,N}^\omega \right) \ . \tag{2.4.19}$$

Proposition 2.4.8 (Truncation to high frequencies)**.** Let $0 < \eta \leqslant 1$ be an absolute constant and let $s > \frac{1}{3}$. Let $(f_0, f_1) \in H^s_x(\mathbb{R}^4) \times H^{s-1}_x(\mathbb{R}^4)$, let (f_0^ω, f_1^ω) be their microlocal randomizations, and let $\{f_k^\pm\}$ be as in (2.4.1) and (2.4.2). Then, there exists a random frequency parameter $N_{\mathrm{hi}} \geqslant \eta^{-1}$ such that

$$\begin{aligned} \|(P_{\geqslant N_{\mathrm{hi}}/4} f_0^\omega, P_{\geqslant N_{\mathrm{hi}}/4} f_1^\omega)\|_{H^s \times H^{s-1}} &\leqslant \eta \ , \\ \|(f_0^\omega, f_1^\omega)\|_{Z(N_{\mathrm{hi}})} &\leqslant \eta \ , \\ \|(f_0^\omega, f_1^\omega)\|_{\mathrm{WP}(N_{\mathrm{hi}})} &\leqslant \eta \ , \\ \|F_{\mathrm{hi}}\|_{L^3_t L^6_x(\mathbb{R} \times \mathbb{R}^4)} &\leqslant \eta \ . \end{aligned}$$

Proof. We only need to combine the previous estimates. From Lemma 2.2.11, it follows that

$$\sum_{N \geqslant 2} \sum_{\|k\|_\infty \in (N/2,N]} \Big(N^{2s} \|f_{0;k}\|_{L^2_x}^2 + N^{2(s-1)} \|f_{1;k}\|_{L^2_x}^2 \Big) < \infty \qquad \text{a.s.}$$

From dominated convergence, it then follows that there exists some random frequency $N_{\mathrm{hi},1}$, depending only on the random variables $\{Y_{k,l}\}$, satisfying

$$\sum_{N \geqslant N_{\mathrm{hi},1}} \sum_{\|k\|_\infty \in (N/2,N]} \Big(N^{2s} \|f_{0;k}\|_{L^2_x}^2 + N^{2(s-1)} \|f_{1;k}\|_{L^2_x}^2 \Big) \leqslant \eta \qquad \text{a.s.}$$

From Corollary 2.4.7, it then follows that

$$\mathbb{E}_\epsilon \|(f_0^\omega, f_1^\omega)\|_{\mathrm{WP}(N_{\mathrm{hi},1})} < \infty \ .$$

From dominated convergence, it follows that there exists a random frequency $N_{\mathrm{hi},2}$, depending on the random variables $\{\epsilon_k\}$ and $\{Y_{k,l}\}$, which satisfies

$$\|(f_0^\omega, f_1^\omega)\|_{\mathrm{WP}(N_{\mathrm{hi},2})} \leqslant \eta \qquad \text{a.s.}$$

By similar arguments, it also follows from Proposition 2.3.1 and Corollary 2.3.3 that there exists random frequencies $N_{\mathrm{hi},3}$ and $N_{\mathrm{hi},4}$ such that

$$\|(f_0^\omega, f_1^\omega)\|_{Z(N_{\mathrm{hi},3})} \leqslant \eta \quad \text{and} \quad \sum_{N \geqslant N_{\mathrm{hi},4}} \Big\| \cos(t|\nabla|) f_{0,N}^\omega + \frac{\sin(t|\nabla|)}{|\nabla|} f_{1,N}^\omega \Big\|_{L_t^3 L_x^6(\mathbb{R}\times\mathbb{R}^4)} \leqslant \eta$$

For the second inequality, we have used the condition $s > \frac{1}{3}$.
By choosing $N_{\mathrm{hi}} := \max(4N_{\mathrm{hi},1}, N_{\mathrm{hi},2}, N_{\mathrm{hi},3}, N_{\mathrm{hi},4}, \eta^{-1})$, we arrive at the desired conclusion. □

2.5 Nonlinear evolution: Local well-posedness, stability theory, and flux estimates

In this section, we first apply to Da Prato-Debussche trick [DD02] to the nonlinear wave equation with random initial data. Then, we recall certain properties of the (forced) energy critical nonlinear wave equation. In our exposition of the local well-posedness and stability theory, we mainly rely on [DLM20]. The flux estimate already played a major role in the author's work on almost sure scattering for the radial energy critical NLW [Bri20b], and we loosely follow parts of [Bri20b, Section 6].

Let N_{hi} be as in Proposition 2.4.8, and let $F = F_{\mathrm{hi}}$ be as in (2.4.19). We then decompose the solution u of (2.1.5) by setting $v := u - F$. Then, the nonlinear component v solves the forced nonlinear wave equation

$$\begin{cases} -\partial_{tt}v + \Delta v = (v+F)^3 \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^4 \ , \\ v|_{t=0} = v_0 \in \dot{H}^1, \quad \partial_t v|_{t=0} = v_1 \in L^2 \ , \end{cases} \tag{2.5.1}$$

where $v_0 := u_0 + f^{\omega}_{0,<N_{\mathrm{hi}}}$ and $v_1 := u_1 + f^{\omega}_{1,<N_{\mathrm{hi}}}$. The randomness in the initial data (v_0, v_1) is not important, and we treat it as arbitrary data in the energy space. For the rest of this section, we treat F as an arbitrary forcing term in $L^3_t L^6_x(\mathbb{R} \times \mathbb{R}^4)$, since the finer properties of F will only be relevant in Section 2.6.

2.5.1 Local well-posedness and stability theory

In this section, we recall the local well-posedness of (2.5.1). Using stability theory, we recall the reduction of Theorem 2.1.3 to an a-priori energy bound. These results are well-known in the literature, see e.g. [DLM20, Poc17].

Lemma 2.5.1 (Local well-posedness [DLM20, Lemma 3.1])**.** Let $(v_0, v_1) \in \dot{H}^1(\mathbb{R}^4) \times L^2(\mathbb{R}^4)$ and $F \in L^3_t L^6_x([0,\infty) \times \mathbb{R}^4)$. Then, there exists a time $T > 0$ and a unique solution $v \colon [0,T) \times \mathbb{R}^4 \to \mathbb{R}$ satisfying

$$v \in C^0_t \dot{H}^1_x([0,T) \times \mathbb{R}^4) \bigcap L^3_t L^6_x([0,T) \times \mathbb{R}^4) \qquad \text{and} \qquad \partial_t v \in C^0_t L^2_x([0,T) \times \mathbb{R}^4) \ .$$

Using stability theory, [DLM20] proved the following proposition.

Proposition 2.5.2 (Reduction to an a-priori energy bound [DLM20, Theorem 1.3])**.** Let $(v_0, v_1) \in \dot{H}^1(\mathbb{R}^4) \times L^2(\mathbb{R}^4)$ and $F \in L_t^3 L_x^6(\mathbb{R} \times \mathbb{R}^4)$. Let v be a solution of (2.5.1), and let $T_+ > 0$ be its maximal time of existence. Furthermore, assume the a-priori energy bound

$$\sup_{t \in [0,T_+)} E[v](t) < \infty \ .$$

Then, v is a global solution and satisfies the global space-time bound $\|v\|_{L_t^3 L_x^6([0,\infty)\times\mathbb{R}^4)} < \infty$. As a result, there exist a scattering states $(v_0^+, v_1^+) \in \dot{H}^1(\mathbb{R}^4) \times L^2(\mathbb{R}^4)$ such that

$$\lim_{t \to +\infty} \|(v(t) - W(t)(v_0^+, v_1^+), \partial_t v(t) - \partial_t W(t)(v_0^+, v_1^+))\|_{\dot{H}^1 \times L^2} = 0 \ .$$

Using Lemma 2.5.1 and Proposition 2.5.2, we have reduced the proof of Theorem 2.1.3 to an a-priori bound on the energy of v.

2.5.2 Flux estimates

As before, we let $v \colon I \times \mathbb{R}^4 \to \mathbb{R}$ be a solution to the forced equation (2.5.1). Recall that the (symmetric) energy-momentum tensor for the energy critical nonlinear wave equation is given by

$$\begin{aligned}
T^{00} &= T^{00}[v] := \frac{1}{2}\big((\partial_t v)^2 + |\nabla v|^2\big) + \frac{1}{4}v^4 \ , \\
T^{j0} &= T^{j0}[v] := -\partial_t v \cdot \partial_{x_j} v \ , \\
T^{jk} &= T^{jk}[v] := \partial_{x_j} v \partial_{x_k} v - \frac{\delta_{jk}}{4}(-\partial_{tt} + \Delta)(v^2) + \frac{\delta_{jk}}{2} v^4 \ .
\end{aligned}$$

The component T^{00} is the energy density, the component T^{j0} is the j-th momentum/energy flux, and the components T^{jk} are called the momentum flux. If v solves the energy critical nonlinear wave equation (2.1.1), then the energy-momentum tensor is divergence free. This fails for solutions to the forced equation (2.5.1); however, one can still expect that the error terms have lower order. Setting $\mathcal{N} = (v+F)^3 - v^3$, a short computation shows that

$$\partial_t T^{00} + \partial_{x_k} T^{0k} = -\mathcal{N}\partial_t v \ , \tag{2.5.2}$$

$$\partial_t T^{j0} + \partial_{x_k} T^{jk} = \mathcal{N}\partial_{x_j} v - \frac{1}{2}\partial_{x_j}(\mathcal{N}v) \ . \tag{2.5.3}$$

As in earlier work on almost sure scattering for radial data [Bri20b, DLM20, DLM19, KMV19], the main goal of this paper is to bound the energy of v. In terms of the energy-momentum tensor, the (total) energy can be written as

$$E[v](t) := \int_{\mathbb{R}^4} T^{00}(t,x)\mathrm{d}x \ .$$

For future use, we record the following consequence of (2.5.2).

Lemma 2.5.3 (Total energy increment)**.** Let $v\colon I \times \mathbb{R}^4 \to \mathbb{R}$ be a solution of (2.5.1), and let $a, b \in I$ with $a \leqslant b$. Then, we have that

$$E(b) - E(a) = -\int_a^b \int_{\mathbb{R}^4} \mathcal{N}\partial_t v \mathrm{d}x\mathrm{d}t \leqslant 6\int_a^b\int_{\mathbb{R}^4} |F||v|^2|\partial_t v|\mathrm{d}x\mathrm{d}t + 3\int_a^b\int_{\mathbb{R}^4} |F|^3|\partial_t v|\mathrm{d}x\mathrm{d}t \ . \tag{2.5.4}$$

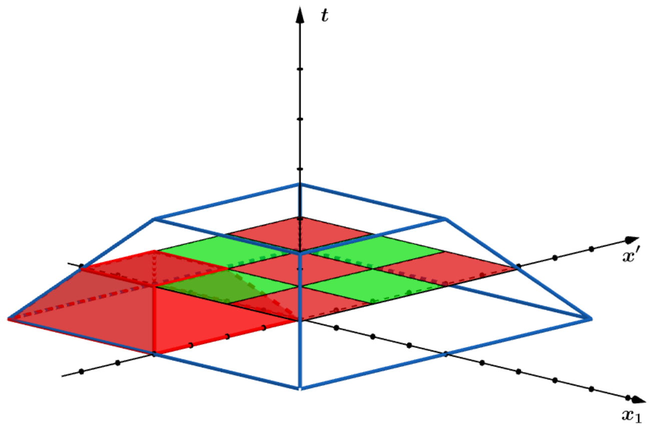
We will later see that the second summand on the right-hand side of (2.5.4) can be bounded directly using Hölder's inequality and probabilistic Strichartz estimates. In contrast, no such estimate is available for the first summand, and we need the wave packet decomposition to control this term.

Once we employ the wave packet decomposition, it will be natural to study the energy on a time and length scale $\sim N \geqslant 1$. We fix $t_0 \in N\mathbb{N}_0$ and $x_0 \in N\mathbb{Z}^4$, and define the local energy

$$E^N_{t_0,x_0}[v](t) := \int_{\|x-x_0\|_\infty \leqslant 2N - |t-t_0|} T^{00}(t,x)\mathrm{d}x, \qquad \text{where } t \in [t_0, t_0+N] \ . \tag{2.5.5}$$

Thus, this definition is adapted to the truncated ℓ^∞-cone $K^N_{t_0,x_0}$, which is given by

$$K^N_{t_0,x_0} := \{(t,x) \in [t_0,t_0+N] \times \mathbb{R}^4 \colon \|x-x_0\|_\infty \leqslant 2N - |t-t_0|\} \ . \tag{2.5.6}$$



In this figure, we illustrate the truncated ℓ^∞-cone, and its decomposition into smaller cones. The blue lines are the edges of a large ℓ^∞-cone. The red and green squares are the tops of smaller truncated cones. In the lower left corner, we have drawn a single one of these smaller cones. As can be seen from this figure, no smaller cone has to exit the large cone.

Figure 2.5: Decomposition of the ℓ^∞-cone

It might be more appropriate to call $K^N_{t_0,x_0}$ a pyramid (see Figure 2.5); unfortunately, the letter P is already heavily used in our notation, so that we decided to use the letter K. Our reason for using the ℓ^∞-norm, instead of the more common ℓ^2-norm, lies in the induction on scales argument (Proposition 2.5.4). Then, it will be an advantage to write $K^N_{t_0,x_0}$ as the union of finitely overlapping smaller cones $K^M_{\tau_0,y_0}$, which are contained in $K^N_{t_0,x_0}$. Using finite speed of propagation and the inequality $\|\cdot\|_{\ell^\infty(\mathbb{R}^4)} \leqslant \|\cdot\|_{\ell^2(\mathbb{R}^4)}$, one can still meaningfully restrict the nonlinear wave equation to $K^N_{t_0,x_0}$.

Lemma 2.5.4 (Local energy increment)**.** Let v be a solution to the forced equation (2.5.1), let

$N \geqslant 1$, let $t_0 \in N\mathbb{N}_0$, and let $x_0 \in N\mathbb{Z}^4$. Then, we have that

$$\sup_{t\in[t_0,t_0+N]} E^N_{t_0,x_0}[v](t) \leqslant E^N_{t_0,x_0}[v](t_0) + 6\int_{K^N_{t_0,x_0}} |F||v|^2|\partial_t v|\mathrm{d}x\mathrm{d}t + 3\int_{K^N_{t_0,x_0}} |F|^3|\partial_t v|\mathrm{d}x\mathrm{d}t \ . \quad (2.5.7)$$

Proof. Using (2.5.2), we have that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E^N_{t_0,x_0}[v](t) &= -\int\limits_{\substack{\|x-x_0\|_\infty\\=2N-|t-t_0|}} T^{00}\mathrm{d}\sigma(x) + \int\limits_{\substack{\|x-x_0\|_\infty\\\leqslant 2N-|t-t_0|}} \partial_t T^{00}\mathrm{d}x \\ &= \int\limits_{\substack{\|x-x_0\|_\infty\\=2N-|t-t_0|}} (-T^{00}+T^{0j}\nu_j)\mathrm{d}\sigma(x) - \int\limits_{\substack{\|x-x_0\|_\infty\\\leqslant 2N-|t-t_0|}} \mathcal{N}\partial_t v\mathrm{d}x\end{aligned}$$

Here, ν is the outward unit normal to the cube. From Cauchy-Schwarz, it follows that $|T^{0j}\nu_j| \leqslant |\partial_t v|\|\nabla v\|_2 \leqslant T^{00}$. After integrating in time, this completes the proof. □

To simplify the notation, we now write

$$\mathcal{E}^N_{t_0,x_0}[v] := \sup_{t\in[t_0,t_0+N]} E^N_{t_0,x_0}[v](t) \ . \quad (2.5.8)$$

In the following, we want to deduce a flux estimate for the solution v of the forced NLW (2.5.1). Here, we encounter a minor technical problem. Let $(t',x') \in K^N_{t_0,x_0}$ be a point in the truncated ℓ^∞-cone. We then want to control the potential $|v|^4$ on the truncated light-cone

$$C^N_{t',x'} := \{(t,x) \in [t_0,t_0+N]\times\mathbb{R}^4 \colon |t-t'| = \|x-x'\|_2\} \ .$$

Unfortunately, $C^N_{t',x'}$ may not be contained in $K^N_{t_0,x_0}$, and hence we cannot expect to bound this solely by $\mathcal{E}^N_{t_0,x_0}[v]$. Since the flux estimate is derived through a monotonicity formula for the local energy, this issue persists even if we are only interested in the portion of $C^N_{t',x'}$ intersecting $K^N_{t_0,x_0}$. To solve this problem, while still keeping the same energy increment as in (2.5.7), we introduce the notion of a locally forced solution.

Definition 2.5.5 (Locally forced solution)**.** Let $t_0 \in N\mathbb{N}_0$ and $x_0 \in N\mathbb{Z}^4$. We call $w\colon \mathbb{R}\times\mathbb{R}^4 \to \mathbb{R}$ a $K^N_{t_0,x_0}$-locally forced solution if it solves

$$\begin{cases} -\partial_{tt} w + \Delta w = (1_{K^N_{t_0,x_0}} F + w)^3 & (t,x) \in \mathbb{R}\times\mathbb{R}^4 \\ w|_{t=t_0} = w_0 \in \dot{H}^1(\mathbb{R}^4), \quad \partial_t w|_{t=t_0} = w_1 \in L^2(\mathbb{R}^4) \end{cases} . \tag{2.5.9}$$

We also require that the functions (w_0, w_1) agree with $(v(t_0), \partial_t v(t_0))$ on the cube $\|x - x_0\|_\infty \leqslant 2N$.

Remark 2.5.6. From finite speed of propagation, it follows that $w|_{K^N_{t_0,x_0}} = v|_{K^N_{t_0,x_0}}$.

For the same reasons as described in the last paragraph, we also use the energy on a slightly larger region. To this end, we define

$$\widetilde{E}^N_{t_0,x_0}[w](t) := \int_{\|x-x_0\|_\infty \leqslant 16N - |t-t_0|} T^{00}[w](t,x)\mathrm{d}x, \qquad \text{where } t \in [t_0, t_0+N] \ .$$

Thus, this definition is adapted to the fattened cone $\widetilde{K}^N_{t_0,x_0}$, which is defined in (2.4.10). We also set

$$\widetilde{\mathcal{E}}^N_{t_0,x_0}[w] := \sup_{t\in[t_0,t_0+N]} \widetilde{E}^N_{t_0,x_0}[w](t) \ . \tag{2.5.10}$$

Lemma 2.5.7 (Local flux estimate)**.** Let $t_0 \in N\mathbb{N}_0$, let $x_0 \in N\mathbb{Z}^4$, and let w be a $K^N_{t_0,x_0}$-locally forced solution. Then, we have that

$$\sup_{\substack{t'\in[t_0,t_0+N] \\ \|x'-x_0\|_{\ell^\infty} \leqslant 8N}} \int_{\substack{\|x-x'\|_2 = |t-t'| \\ t\in[t_0,t_0+N]}} \frac{w^4}{4} \,\mathrm{d}\sigma(t,x) \leqslant 4\widetilde{\mathcal{E}}^N_{t_0,x_0} + 12 \int_{K^N_{t_0,x_0}} |F|(|w|+|F|)^2 |\partial_t w| \mathrm{d}x\mathrm{d}t \ . \tag{2.5.11}$$

We emphasize that, even though the energy $\widetilde{\mathcal{E}}^N_{t_0,x_0}$ is measured on a truncated ℓ^∞-cone, the flux is still controlled on a light cone. The estimate (2.5.11), however, only controls w on a lower-dimensional surface in space-time, and thus cannot directly be used to bound the energy increment. In our main argument, we rely on the following averaged version.

Lemma 2.5.8 (Averaged local flux estimate)**.** Let $N \geqslant N_{\mathrm{hi}}$, let $t_0 \in N\mathbb{N}_0$, let $x_0 \in N\mathbb{Z}^4$, and let w be a $K^N_{t_0,x_0}$-locally forced solution. Then, we have that

$$\widetilde{\mathcal{F}}^N_{t_0,x_0} := \sup_{\substack{t'\in[t_0,t_0+N]\\ \|x'-x_0\|_\infty\leqslant 4N}} \int\limits_{\substack{|\|x-x'\|_2-|t-t'||\leqslant N^{10\delta}\\ t\in[t_0,t_0+N]}} \frac{w^4}{4}\,\mathrm{d}x\mathrm{d}t \tag{2.5.12}$$

$$\lesssim N^{40\delta}\left(\widetilde{\mathcal{E}}^N_{t_0,x_0} + \int\limits_{K^N_{t_0,x_0}} |F|(|w|+|F|)^2|\partial_t w|\mathrm{d}x\mathrm{d}t\right).$$

The appeareance of $N^{10\delta}$ is for technical reasons only, and the reader is encouraged to mentally replaced it by 1. This will later help us to deal with the spatial tails of the wave packets.

Proof of Lemma 2.5.7. For the duration of this proof, we define

$$e(t) := \int_{\|x-x'\|_2\leqslant|t-t'|} T^{00}(t,x)\mathrm{d}x$$

From finite speed of propagation, we expect $e(t)$ to be (nearly) non-increasing on $[t_0,t']$ and non-decreasing on $[t',t_0+N]$. From the assumptions above, it follows that $\|x-x'\|_2\leqslant|t-t'|$ implies

$$\|x-x_0\|_\infty \leqslant \|x-x'\|_\infty + \|x'-x_0\|_\infty \leqslant \|x-x'\|_2 + 8N \leqslant |t-t'|+8N \leqslant 10N-|t-t_0|\ .$$

Thus, we obtain that $e(t)\leqslant \widetilde{\mathcal{E}}^N_{t_0,x_0}$ for all $t\in[t_0,t_0+N]$. Using (2.5.2), we obtain for all $t\in[t',t_0+N]$ that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}e(t) &= \int_{\|x-x'\|_2\leqslant|t-t'|}\partial_t T^{00}(t,x)\mathrm{d}x + \int_{\|x-x'\|_2=|t-t'|} T^{00}(t,x)\mathrm{d}\sigma(x)\\
&= -\int_{\|x-x'\|_2\leqslant|t-t'|}\mathcal{N}\partial_t v\mathrm{d}x - \int_{\|x-x'\|_2\leqslant|t-t'|}\partial_{x_k}T^{0k}\mathrm{d}x + \int_{\|x-x'\|_2=|t-t'|} T^{00}(t,x)\mathrm{d}\sigma(x)\\
&= -\int_{\|x-x'\|_2\leqslant|t-t'|}\mathcal{N}\partial_t v\mathrm{d}x + \int_{\|x-x'\|_2=|t-t'|}(T^{00}(t,x)+T^{0k}\nu_k)\mathrm{d}\sigma(x)\\
&\geqslant -6\int_{\|x-x'\|_2\leqslant|t-t'|}|F|(|F|+|v|)^2|\partial_t v|\mathrm{d}x + \int_{\|x-x'\|_2=|t-t'|}\frac{v^4}{4}\mathrm{d}\sigma(x)\ .
\end{aligned}$$

Integrating this inequality in time, we obtain the result on $[t', t_0 + N]$. The bound on $[t_0, t']$ is similar. $\square$

Proof of Lemma 2.5.8. Since $N \geqslant N_{\mathrm{hi}} \geqslant \eta^{-1}$, we have that $N^{10\delta} \ll N$. We then simply integrate (2.5.11) over a spatial ball of radius $\sim N^{10\delta}$ around x'. $\square$

2.6 The energy increment and induction on scales

We are now ready to finally bound the energy increment of the nonlinear component v. The argument roughly splits into two parts: A single scale analysis and induction on scales.

For technical reasons, we define a flux-term involving a thinner neighborhood of the cone. More precisely, we let

$$\mathcal{F}^N_{t_0,x_0}[w] := \sup_{\substack{(t',x') \in [t_0, t_0+N] \times \mathbb{R}^4 \\ \|x'-x_0\|_\infty \leqslant 4N}} \int_{\substack{|\|x-x'\|_2 - |t-t'|| \leqslant N^{5\delta} \\ t \in [t_0, t_0+N]}} \frac{w^4}{4} \, \mathrm{d}x \mathrm{d}t$$

Recall that the light-cone in $\widetilde{\mathcal{F}}^N_{t_0,x_0}[w]$, as defined in (2.5.12), has width $N^{10\delta}$.

Proposition 2.6.1 (Single-scale energy increment)**.** Let $N \geqslant 1$. Let $t_0 \in N\mathbb{N}_0$, where $0 \leqslant t_0/N \leqslant \lfloor N^\theta \rfloor$, and let $x_0 \in N\mathbb{Z}^4$. Furthermore, let $w_1, w_2 \in L^\infty_t L^4_x(\mathbb{R} \times \mathbb{R}^4)$ and $w_3 \in L^\infty_t L^2_x(\mathbb{R} \times \mathbb{R}^4)$. Then,

$$\begin{aligned}
&\int_{K^N_{t_0,x_0}} |F_M||w_1||w_2||w_3| \mathrm{d}x\mathrm{d}t \\
&\lesssim \eta N^{\frac{3}{4}-s+8\delta} (\|w_1\|^4_{L^\infty_t L^4_x(\mathbb{R}\times\mathbb{R}^4)} + \mathcal{F}^N_{t_0,x_0}[w_1])^{\frac{1}{4}} (\|w_2\|^4_{L^\infty_t L^4_x(\mathbb{R}\times\mathbb{R}^4)} + \mathcal{F}^N_{t_0,x_0}[w_2])^{\frac{1}{4}} \|w_3\|_{L^\infty_t L^2_x(\mathbb{R}\times\mathbb{R}^4)} \,.
\end{aligned}$$

We have two separate reasons for introducing the auxiliary functions w_1, w_2, and w_3. First, it emphasizes that the proof does not depend on the evolution equation of the nonlinear component. Second, it allows us to pass to smaller spatial scales than N with minimal notational effort, see Corollary 2.6.2.

Proof. If $1 \leqslant N < N_{\mathrm{hi}}$, there is nothing to show. Thus, we may assume that $N \geqslant N_{\mathrm{hi}}$.

Step 1: Wave packet decomposition. Recall from (2.4.2) and (2.4.3) that

$$F_N^\omega = \sum_k \epsilon_k \exp(it|\nabla|) f_k^+ + \sum_k \epsilon_k \exp(-it|\nabla|) f_k^- \ .$$

We only control the contribution of $\sum_k \epsilon_k \exp(it|\nabla|) f_k^+$, the other estimate is nearly identical. Then, we may also drop the superscript + from our notation. We now apply Proposition 2.4.1 and Proposition 2.4.3 to the family $\{\epsilon_k f_k\}_k$, and let the sets $\mathscr{A}_m, \mathscr{B}_{j,m}$, and $\mathscr{D}_m$ be as in Proposition 2.4.3. As before, we implicitly restrict to $\|k\|_\infty \in (N/2, N]$. We also write $f_{k,l} = f_{k,l;t_0}$.

Step 2: Distant wave packets and extreme amplitudes. On a heuristic level, the wave packets whose tubes $T_{k,l}$ do not intersect $K^N_{t_0,x_0}$ should not contribute to the integral. We now make this precise using the decay estimate (2.4.9). Indeed,

$$\begin{aligned}
&\int_{K^N_{t_0,x_0}} \Big| \sum_{\substack{(k,l)\in\mathbb{Z}^4\times\mathbb{Z}^4: \\ \|l-x_0\|_\infty > 3N}} \epsilon_k \exp(i(t-t_0)|\nabla|) f_{k,l} \Big| |w_1||w_2||w_3| \mathrm{d}x\mathrm{d}t \\
&\lesssim N \Big\| \sum_{\substack{(k,l)\in\mathbb{Z}^4\times\mathbb{Z}^4: \\ \|l-x_0\|_\infty > 3N}} \epsilon_k \exp(i(t-t_0)|\nabla|) f_{k,l} \Big\|_{L_t^\infty L_x^\infty(K^N_{t_0,x_0})} \|w_1\|_{L_t^\infty L_x^4(\mathbb{R}\times\mathbb{R}^4)} \|w_2\|_{L_t^\infty L_x^4(\mathbb{R}\times\mathbb{R}^4)} \|w_3\|_{L_t^\infty L_x^2(\mathbb{R}\times\mathbb{R}^4)} \\
&\lesssim N \Big(\sum_{\substack{(k,l)\in\mathbb{Z}^4\times\mathbb{Z}^4: \\ \|l-x_0\|_\infty > 3N}} N^{-100} (1 + \|x_0 - l\|_2)^{-100} \|f_k\|_{L_x^2(\mathbb{R}^4)} \Big) \|w_1\|_{L_t^\infty L_x^4(\mathbb{R}\times\mathbb{R}^4)} \|w_2\|_{L_t^\infty L_x^4(\mathbb{R}\times\mathbb{R}^4)} \|w_3\|_{L_t^\infty L_x^2(\mathbb{R}\times\mathbb{R}^4)} \\
&\lesssim N^{-99+2} \Big(\sum_{k\in\mathbb{Z}^4} \|f_k\|^2_{L_x^2(\mathbb{R}^4)} \Big)^{\frac{1}{2}} \|w_1\|_{L_t^\infty L_x^4(\mathbb{R}\times\mathbb{R}^4)} \|w_2\|_{L_t^\infty L_x^4(\mathbb{R}\times\mathbb{R}^4)} \|w_3\|_{L_t^\infty L_x^2(\mathbb{R}\times\mathbb{R}^4)} \\
&\lesssim \eta N^{\frac{3}{4}-s} \|w_1\|_{L_t^\infty L_x^4(\mathbb{R}\times\mathbb{R}^4)} \|w_2\|_{L_t^\infty L_x^4(\mathbb{R}\times\mathbb{R}^4)} \|w_3\|_{L_t^\infty L_x^2(\mathbb{R}\times\mathbb{R}^4)} \ .
\end{aligned}$$

Thus, this contribution is acceptable. It remains to control the wave packets with indices in $\cup_{m\in\mathbb{Z}} \mathscr{A}_m$. We now use crude estimates to reduce to $\sim \log(N)$ amplitude scales. Let $m \leqslant -20\log(N)$. Since $\#\mathscr{A}_m \lesssim N^8$, we have that

$$\int_{K^N_{t_0,x_0}} \Big| \sum_{(k,l)\in\mathscr{A}_m} \epsilon_k \exp(i(t-t_0)|\nabla|) f_{k,l} \Big| |w_1||w_2||w_3| \mathrm{d}x\mathrm{d}t$$

$$\lesssim N\Big(\sum_{(k,l)\in\mathscr{A}_m}\|\exp(i(t-t_0)|\nabla|)f_{k,l}\|_{L_t^\infty L_x^\infty(\mathbb{R}\times\mathbb{R}^4)}\Big)\|w_1\|_{L_t^\infty L_x^4(\mathbb{R}\times\mathbb{R}^4)}\|w_2\|_{L_t^\infty L_x^4(\mathbb{R}\times\mathbb{R}^4)}\|w_3\|_{L_t^\infty L_x^2(\mathbb{R}\times\mathbb{R}^4)}$$
$$\lesssim N2^m\#\mathscr{A}_m\|w_1\|_{L_t^\infty L_x^4(\mathbb{R}\times\mathbb{R}^4)}\|w_2\|_{L_t^\infty L_x^4(\mathbb{R}\times\mathbb{R}^4)}\|w_3\|_{L_t^\infty L_x^2(\mathbb{R}\times\mathbb{R}^4)}$$
$$\lesssim 2^mN^9\|w_1\|_{L_t^\infty L_x^4(\mathbb{R}\times\mathbb{R}^4)}\|w_2\|_{L_t^\infty L_x^4(\mathbb{R}\times\mathbb{R}^4)}\|w_3\|_{L_t^\infty L_x^2(\mathbb{R}\times\mathbb{R}^4)}\ .$$

Summing this inequality over all $m\leqslant -20\log(N)$, we obtain that

$$\int_{K_{t_0,x_0}^N}\Big|\sum_{m\leqslant -20\log(N)}\sum_{(k,l)\in\mathscr{A}_m}\epsilon_k\exp(i(t-t_0)|\nabla|)f_{k,l}\Big||w_1||w_2||w_3|\mathrm{d}x\mathrm{d}t$$
$$\lesssim N^{-11}\|w_1\|_{L_t^\infty L_x^4(\mathbb{R}\times\mathbb{R}^4)}\|w_2\|_{L_t^\infty L_x^4(\mathbb{R}\times\mathbb{R}^4)}\|w_3\|_{L_t^\infty L_x^2(\mathbb{R}\times\mathbb{R}^4)}$$
$$\lesssim \eta N^{\frac{3}{4}-s}\|w_1\|_{L_t^\infty L_x^4(\mathbb{R}\times\mathbb{R}^4)}\|w_2\|_{L_t^\infty L_x^4(\mathbb{R}\times\mathbb{R}^4)}\|w_3\|_{L_t^\infty L_x^2(\mathbb{R}\times\mathbb{R}^4)}\ .$$

Finally, if $\mathscr{A}_m\neq\varnothing$, then $\#\mathscr{A}_m\geqslant 1$. This implies that

$$2^m\leqslant 2^m(\#\mathscr{A}_m)^{\frac{1}{2}}\lesssim\left(\sum_{k,l}\|f_{k,l}\|_{L^2(\mathbb{R}^4)}^2\right)^{\frac{1}{2}}\lesssim\eta N^{-s}\ .$$

For a sufficiently small absolute constant $\eta>0$, this implies that $m\leqslant 0$. This completes the crude part of the argument. *Step 3: Bushes.* First, we define the fattened tubes by

$$\widetilde{T}_{k,l}:=\{(t,x)\in[t_0,t_0+N]\times\mathbb{R}^d\colon |x-(l-t\cdot k/\|k\|_2)|\leqslant N^{2\delta}\}\ .$$

Furthermore, we define the collection of fattened tubes corresponding to a bush by

$$\widetilde{T}(\mathscr{B}_{j,m}):=\bigcup\nolimits_{(k,l)\in\mathscr{B}_{j,m}}\widetilde{T}_{k,l}\ .$$

With these definitions in hand, we now write

$$\int_{K_{t_0,x_0}^N}\Big|\sum_{j=1}^J\sum_{(k,l)\in\mathscr{B}_{j,m}}\epsilon_k\exp(i(t-t_0)|\nabla|)f_{k,l}\Big||w_1||w_2||w_3|\mathrm{d}x\mathrm{d}t$$
$$\leqslant\sum_{j=1}^J\int_{\widetilde{T}(\mathscr{B}_{j,m})}\Big|\sum_{(k,l)\in\mathscr{B}_{j,m}}\epsilon_k\exp(i(t-t_0)|\nabla|)f_{k,l}\Big||w_1||w_2||w_3|\mathrm{d}x\mathrm{d}t$$
$$+\sum_{j=1}^J\int_{K_{t_0,x_0}^N\backslash\widetilde{T}(\mathscr{B}_{j,m})}\Big|\sum_{(k,l)\in\mathscr{B}_{j,m}}\epsilon_k\exp(i(t-t_0)|\nabla|)f_{k,l}\Big||w_1||w_2||w_3|\mathrm{d}x\mathrm{d}t$$

Using that all tubes in $\mathscr{B}_{j,m}$ pass through the same space-time cube of size $\sim N^{\delta}$, we obtain from Proposition 2.4.8 that

$$\begin{aligned}
&\sum_{j=1}^{J}\int_{\widetilde{T}(\mathscr{B}_{j,m})}\Big|\sum_{(k,l)\in\mathscr{B}_{j,m}}\epsilon_k\exp(i(t-t_0)|\nabla|)f_{k,l}\Big||w_1||w_2||w_3|\mathrm{d}x\mathrm{d}t\\
&\lesssim\sum_{j=1}^{J}N^{\frac{1}{2}}\Big\|\sum_{(k,l)\in\mathscr{B}_{j,m}}\epsilon_k\exp(i(t-t_0)|\nabla|)f_{k,l}\Big\|_{L^{\infty}_{t,x}(\mathbb{R}\times\mathbb{R}^4)}\|w_1\|_{L^4_{t,x}(\widetilde{T}(\mathscr{B}_{j,m}))}\|w_2\|_{L^4_{t,x}(\widetilde{T}(\mathscr{B}_{j,m}))}\|w_3\|_{L^{\infty}_tL^2_x(\mathbb{R}\times\mathbb{R}^4)}\\
&\lesssim\eta N^{\frac{1}{2}+8\delta}2^m\Big(\sum_{j=1}^{J}(\#\mathscr{B}_{j,m})^{\frac{1}{2}}\Big)\mathcal{F}^N_{t_0,x_0}[w_1]^{\frac{1}{4}}\mathcal{F}^N_{t_0,x_0}[w_2]^{\frac{1}{4}}\|w_3\|_{L^{\infty}_tL^2_x(\mathbb{R}\times\mathbb{R}^4)}\\
&\lesssim\eta N^{\frac{1}{2}+8\delta}2^m\Big(\sum_{j=1}^{J}\mathscr{B}_{j,m}\Big)\mu^{-\frac{1}{2}}\mathcal{F}^N_{t_0,x_0}[w_1]^{\frac{1}{4}}\mathcal{F}^N_{t_0,x_0}[w_2]^{\frac{1}{4}}\|w_3\|_{L^{\infty}_tL^2_x(\mathbb{R}\times\mathbb{R}^4)}\\
&\lesssim\eta N^{\frac{1}{2}+8\delta}2^m(\#\mathscr{A}_m)\mu^{-\frac{1}{2}}\mathcal{F}^N_{t_0,x_0}[w_1]^{\frac{1}{4}}\mathcal{F}^N_{t_0,x_0}[w_2]^{\frac{1}{4}}\|w_3\|_{L^{\infty}_tL^2_x(\mathbb{R}\times\mathbb{R}^4)}\ .
\end{aligned}$$

Here, μ denotes the minimum number of packets inside a single bush, see Proposition 2.4.3. Using the decay estimate (2.4.9), we control the contributions outside the $\widetilde{T}(\mathscr{B}_{j,m})$ by

$$\begin{aligned}
&\sum_{j=1}^{J}\int_{K^N_{t_0,x_0}\backslash\widetilde{T}(\mathscr{B}_{j,m})}\Big|\sum_{(k,l)\in\mathscr{B}_{j,m}}\epsilon_k\exp(i(t-t_0)|\nabla|)f_{k,l}\Big||w_1||w_2||w_3|\mathrm{d}x\mathrm{d}t\\
&\lesssim N\sum_{j=1}^{J}\Big\|\sum_{(k,l)\in\mathscr{B}_{j,m}}\epsilon_k\exp(i(t-t_0)|\nabla|)f_{k,l}\Big\|_{L^{\infty}_{t,x}(K^N_{t_0,x_0}\backslash\widetilde{T}(\mathscr{B}_{j,m}))}\\
&\times\|w_1\|_{L^{\infty}_tL^4_x(\mathbb{R}\times\mathbb{R}^4)}\|w_2\|_{L^{\infty}_tL^4_x(\mathbb{R}\times\mathbb{R}^4)}\|w_3\|_{L^{\infty}_tL^2_x(\mathbb{R}\times\mathbb{R}^4)}\\
&\lesssim N^1N^{-100}2^m\Big(\sum_{j=1}^{J}\#\mathscr{B}_{j,m}\Big)\|w_1\|_{L^{\infty}_tL^4_x(\mathbb{R}\times\mathbb{R}^4)}\|w_2\|_{L^{\infty}_tL^4_x(\mathbb{R}\times\mathbb{R}^4)}\|w_3\|_{L^{\infty}_tL^2_x(\mathbb{R}\times\mathbb{R}^4)}\\
&\lesssim\eta N^{\frac{3}{4}-s}\|w_1\|_{L^{\infty}_tL^4_x(\mathbb{R}\times\mathbb{R}^4)}\|w_2\|_{L^{\infty}_tL^4_x(\mathbb{R}\times\mathbb{R}^4)}\|w_3\|_{L^{\infty}_tL^2_x(\mathbb{R}\times\mathbb{R}^4)}\ .
\end{aligned}$$

In the last line, we have used that $2^m(\#\mathscr{A}_m)^{\frac{1}{2}}\lesssim\eta$ and $\#\mathscr{A}_m\lesssim N^8$.

Step 4: Disjoint wave packets. We now control the contribution of the almost disjoint family $\mathscr{D}_m$. If $\mu<1$, then $\mathscr{D}_m$ is empty, and there is nothing to prove. If $\mu\geqslant1$, it follows from Proposition

2.4.8 that

$$\begin{aligned}
&\int_{K^N_{t_0,x_0}} \Big| \sum_{(k,l)\in\mathscr{D}_m} \epsilon_k \exp(i(t-t_0)|\nabla|) f_{k,l} \Big| |w_1||w_2||w_3| \mathrm{d}x\mathrm{d}t \\
&\lesssim N \Big\| \sum_{(k,l)\in\mathscr{D}_m} \epsilon_k \exp(i(t-t_0)|\nabla|) f_{k,l} \Big\|_{L^\infty_{t,x}(K^N_{x_0})} \|w_1\|_{L^\infty_t L^4_x(\mathbb{R}\times\mathbb{R}^4)} \|w_2\|_{L^\infty_t L^4_x(\mathbb{R}\times\mathbb{R}^4)} \|w_3\|_{L^\infty_t L^2_x(\mathbb{R}\times\mathbb{R}^4)} \\
&\lesssim \eta N^{1+8\delta} 2^m \mu^{\frac{1}{2}} \|w_1\|_{L^\infty_t L^4_x(\mathbb{R}\times\mathbb{R}^4)} \|w_2\|_{L^\infty_t L^4_x(\mathbb{R}\times\mathbb{R}^4)} \|w_3\|_{L^\infty_t L^2_x(\mathbb{R}\times\mathbb{R}^4)}.
\end{aligned}$$

Step 5: Finishing the proof.

In total, we have shown that

$$\begin{aligned}
&\int_{K^N_{t_0,x_0}} |F_M||w_1||w_2||w_3| \mathrm{d}x\mathrm{d}t \\
&\lesssim \eta N^{8\delta} \left(N^{\frac{3}{4}-s} + N 2^m \mu^{\frac{1}{2}} + N^{\frac{1}{2}} 2^m (\#\mathscr{A}_m) \mu^{-\frac{1}{2}} \right) \left(\|w_1\|^4_{L^\infty_t L^4_x(\mathbb{R}\times\mathbb{R}^4)} + \mathcal{F}^N_{t_0,x_0}[w_1] \right)^{\frac{1}{4}} \qquad (2.6.1) \\
&\quad \cdot \left(\|w_2\|^4_{L^\infty_t L^4_x(\mathbb{R}\times\mathbb{R}^4)} + \mathcal{F}^N_{t_0,x_0}[w_2] \right)^{\frac{1}{4}} \|w_3\|_{L^\infty_t L^2_x(\mathbb{R}\times\mathbb{R}^4)} \quad .
\end{aligned}$$

Due to our choice $\mu := N^{-\frac{1}{2}} \# \mathscr{A}_m$ in Proposition 2.4.3, this completes the proof. □

Corollary 2.6.2 (Coarse-scale energy increment)**.** Let $N \geqslant 1$. Let $t_0 \in N\mathbb{N}_0$, with $0 \leqslant t_0/N \leqslant \lfloor N^\theta \rfloor$, and let $x_0 \in N\mathbb{Z}^4$. Let w be a $K^N_{t_0,x_0}$-locally forced solution. Then, we have for all $M \geqslant N$ that

$$\int_{K^N_{t_0,x_0}} |F_M||w|^2 |\partial_t w| \mathrm{d}x\mathrm{d}t \lesssim \eta M^{\frac{3}{4}-s+8\delta} \widetilde{\mathcal{E}}^N_{t_0,x_0}[w]^{\frac{1}{2}} \left(\widetilde{\mathcal{E}}^N_{t_0,x_0}[w] + \widetilde{\mathcal{F}}^N_{t_0,x_0}[w] \right)^{\frac{1}{2}} \quad . \qquad (2.6.2)$$

We refer to Corollary 2.6.2 as a coarse scale estimate since the wave packets in F_M are atleast as long as the length of $K^N_{t_0,x_0}$ (in time).

Proof. As before, we may take $M \geqslant N_{\mathrm{hi}}$. We distinguish two different cases. If $M \geqslant N^{\frac{4}{3}}$, we obtain from Proposition 2.4.8 that

$$\int_{K^N_{t_0,x_0}} |F_M||w|^2 |\partial_t w| \mathrm{d}x\mathrm{d}t \leqslant N \|F_M\|_{L^\infty_t L^\infty_x(\mathbb{R}\times\mathbb{R}^4)} \|w\|^2_{L^\infty_t L^4_x(K^N_{t_0,x_0})} \|\partial_t w\|_{L^\infty_t L^2_x(K^N_{t_0,x_0})} \lesssim \eta M^{\frac{3}{4}-s+\delta} \widetilde{\mathcal{E}}^N_{t_0,x_0}[w].$$

Next, we let $M \leqslant N^{\frac{4}{3}}$. Then, there exist $\tau_0 \in M\mathbb{N}_0$ and $y_0 \in M\mathbb{Z}^4$ s.t. $K^N_{t_0,x_0} \subseteq K^M_{\tau_0,y_0}$. Furthermore, since $t_0/N \leqslant \lfloor N^\theta \rfloor$, it holds that $\tau_0/M \leqslant \lfloor M^\theta \rfloor$. Set $w_1 = w_2 := 1_{K^N_{t_0,x_0}} w$ and $w_3 := 1_{K^N_{t_0,x_0}} w$. Since $M \leqslant N^{\frac{4}{3}}$, we have that

$$\mathcal{F}^M_{\tau_0,y_0}[w_1] = \mathcal{F}^M_{\tau_0,y_0}[w_2] \leqslant \widetilde{\mathcal{F}}^N_{t_0,x_0}[w]$$

Using the single-scale energy increment (Proposition 2.6.1), we obtain that

$$\begin{aligned}
&\int_{K^N_{t_0,x_0}} |F_M||w|^2|\partial_t w| \mathrm{d}x\mathrm{d}t \\
&= \int_{K^M_{\tau_0,y_0}} |F_M||w_1||w_2||w_3| \mathrm{d}x\mathrm{d}t \\
&\lesssim \eta M^{\frac{3}{4}-s+\delta}(\|w_1\|^4_{L^\infty_t L^4_x(\mathbb{R}\times\mathbb{R}^4)} + \mathcal{F}^M_{\tau_0,y_0}[w_1])^{\frac{1}{4}}(\|w_2\|^4_{L^\infty_t L^4_x(\mathbb{R}\times\mathbb{R}^4)} + \mathcal{F}^M_{\tau_0,y_0}[w_2])^{\frac{1}{4}}\|w_3\|_{L^\infty_t L^2_x(\mathbb{R}\times\mathbb{R}^4)} \\
&\lesssim \eta M^{\frac{3}{4}-s+\delta} \widetilde{\mathcal{E}}^N_{t_0,x_0}[w]^{\frac{1}{2}} \left(\widetilde{\mathcal{E}}^N_{t_0,x_0}[w] + \widetilde{\mathcal{F}}^N_{t_0,x_0}[w]\right)^{\frac{1}{2}} .
\end{aligned}$$

□

Due to Proposition 2.6.1 and Corollary 2.6.2, we understand the energy increment at a single scale. Unfortunately, the cone $K^N_{t_0,x_0}$ may contain many wave packets on smaller scales. Similar problems are often encountered in restriction theory, and can sometimes be solved using Wolff's induction on scales strategy [Wol01]. The following argument can be seen as a (simple) implementation of this idea.

Proposition 2.6.3 (Induction on scales)**.** Let $s > \max(1-\theta/2, 3/4+\theta)$. Let $R \geqslant 1$ be a dyadic integer, and let F be as in Proposition 2.4.8. Let $t_0 \in R\mathbb{N}_0$, with $t_0/R \leqslant \lfloor R^\theta \rfloor$, $x_0 \in R\mathbb{Z}^4$, let w be a $K^R_{t_0,x_0}$-locally forced solution. For a large absolute constant $C_1 \geqslant 1$, we have that

$$\widetilde{\mathcal{E}}^R_{t_0,x_0}[w] \leqslant 2E_{|x-x_0|\leqslant 16R}[w](t_0) + C_1\|F\|^6_{L^3_t L^6_x(K^R_{t_0,x_0})} \tag{2.6.3}$$

and

$$\widetilde{\mathcal{F}}^R_{t_0,x_0}[w] \leqslant C_1 R^{50\delta}\left(E_{|x-x_0|\leqslant 16R}[w](t_0) + \|F\|^6_{L^3_t L^6_x(K^R_{t_0,x_0})}\right) . \tag{2.6.4}$$

Proof. We use induction on the dyadic integers $R \geqslant 1$.

Step 1: Base case $R = 1$. We have that

$$\begin{aligned}
\widetilde{\mathcal{E}}^1_{t_0,x_0}[w] &\leqslant E_{|x-x_0|\leqslant 16}[w](t_0) + C\int_{K^1_{t_0,x_0}} |F||w|^2|\partial_t w|\mathrm{d}x\mathrm{d}t + C\int_{K^1_{t_0,x_0}} |F|^3|\partial_t w|\mathrm{d}x\mathrm{d}t \\
&\leqslant E_{|x-x_0|\leqslant 16}[w](t_0) + C\|F\|_{L^\infty_t L^\infty_x(\mathbb{R}\times\mathbb{R}^4)}\|w\|^2_{L^\infty_t L^4_x(K^1_{t_0,x_0})}\|\partial_t w\|_{L^\infty_t L^2_x(K^1_{t_0,x_0})} \\
&\quad + C\|F\|^3_{L^3_t L^6_x(K^1_{t_0,x_0})}\|\partial_t w\|_{L^\infty_t L^2_x(K^1_{t_0,x_0})} \\
&\leqslant E_{|x-x_0|\leqslant 16}[w](t_0) + \frac{1}{4}\widetilde{\mathcal{E}}^1_{t_0,x_0}[w] + C\|F\|^3_{L^3_t L^6_x(K^1_{t_0,x_0})}\widetilde{\mathcal{E}}^1_{t_0,x_0}[w]^{\frac{1}{2}} \\
&\leqslant E_{|x-x_0|\leqslant 16}[w](t_0) + \frac{1}{2}\widetilde{\mathcal{E}}^1_{t_0,x_0}[w] + C\|F\|^6_{L^3_t L^6_x(K^1_{t_0,x_0})} \ .
\end{aligned}$$

Insert this bound into Lemma 2.5.8, we obtain that

$$\begin{aligned}
\widetilde{\mathcal{F}}^1_{t_0,x_0}[w] &\lesssim \widetilde{\mathcal{E}}^1_{t_0,x_0}[w] + \int_{K^1_{t_0,x_0}} |F||w|^2|\partial_t w|\mathrm{d}x\mathrm{d}t + C\int_{K^1_{t_0,x_0}} |F|^3|\partial_t w|\mathrm{d}x\mathrm{d}t \\
&\lesssim \widetilde{\mathcal{E}}^1_{t_0,x_0}[w] \\
&\lesssim E_{|x-x_0|\leqslant 16}[w](t_0) + \|F\|^6_{L^3_t L^6_x(K^1_{t_0,x_0})} \ .
\end{aligned}$$

By choosing C_1 sufficiently large, we obtain (2.6.3) and (2.6.4). This already determines our choice of C_1, which we now regard as a fixed constant. Let $R \geqslant 2$ be an arbitrary dyadic integer. Using the induction hypothesis, we can rely on the inequalities (2.6.3) and (2.6.4) for all scales $N \leqslant R/2$.

Step 2: Splitting the energy increment. From Lemma 2.5.4, we have that

$$\widetilde{\mathcal{E}}^R_{t_0,x_0}[w] \leqslant E_{|x-x_0|\leqslant 16R}[w](t_0) + C\int_{K^R_{t_0,x_0}} |F||w|^2|\partial_t w|\mathrm{d}x\mathrm{d}t + C\int_{K^R_{t_0,x_0}} |F|^3|\partial_t w|\mathrm{d}x\mathrm{d}t \ . \qquad (2.6.5)$$

The main term is the second summand in (2.6.5). We use a Littlewood-Paley type decomposition of the linear evolution and write

$$\begin{aligned}&\int_{K^R_{t_0,x_0}} |F||w|^2|\partial_t w|\mathrm{d}x\mathrm{d}t\\&\leqslant \sum_{N\geqslant R}\int_{K^R_{t_0,x_0}} |F_N||w|^2|\partial_t w|\mathrm{d}x\mathrm{d}t + \sum_{N\leqslant R/2}\int_{K^R_{t_0,x_0}} |F_N||w|^2|\partial_t w|\mathrm{d}x\mathrm{d}t\ .\end{aligned}$$

Step 3: High frequencies. The high frequencies can be controlled using the single-scale estimate from Proposition 2.6.1. Indeed, we have that

$$\begin{aligned}&\sum_{N\geqslant R}\int_{K^R_{t_0,x_0}} |F_N||w|^2|\partial_t w|\mathrm{d}x\mathrm{d}t\\&\lesssim \eta \sum_{N\geqslant R} N^{\frac{3}{4}-s+8\delta}\widetilde{\mathcal{E}}^R_{t_0,x_0}[w]^{\frac{1}{2}}\left(\widetilde{\mathcal{E}}^R_{t_0,x_0}[w]+\widetilde{\mathcal{F}}^R_{t_0,x_0}[w]\right)^{\frac{1}{2}}\\&\lesssim \eta R^{\frac{3}{4}-s+8\delta}\widetilde{\mathcal{E}}^R_{t_0,x_0}[w]^{\frac{1}{2}}\left(\widetilde{\mathcal{E}}^R_{t_0,x_0}[w]+\widetilde{\mathcal{F}}^R_{t_0,x_0}[w]\right)^{\frac{1}{2}}\ .\end{aligned} \tag{2.6.6}$$

Step 4: Low frequencies. For $\tau_n = Nn \in N\mathbb{N}_0$ and $y_j = Nj \in N\mathbb{Z}^4$, we write

$$\begin{aligned}&\int_{K^R_{t_0,x_0}} |F_N||w|^2|\partial_t w|\mathrm{d}x\mathrm{d}t\\&=\sum_{n=0}^{\lfloor N^\theta\rfloor}\left(\int_{([\tau_n,\tau_n+N]\times\mathbb{R}^4)\cap K^R_{t_0,x_0}} |F_N||w|^2|\partial_t w|\mathrm{d}x\mathrm{d}t\right) + \int_{([N^{1+\theta},\infty)\times\mathbb{R}^4)\cap K^R_{t_0,x_0}} |F_N||w|^2|\partial_t w|\mathrm{d}x\mathrm{d}t\\&=\sum_{n=0}^{\lfloor N^\theta\rfloor}\sum_{\substack{j\in\mathbb{Z}^4\\ K^N_{\tau_n,y_j}\subseteq K^R_{t_0,x_0}}}\int_{K^N_{\tau_n,y_j}} |F_N||w|^2|\partial_t w|\mathrm{d}x\mathrm{d}t + \int_{([N^{1+\theta},\infty)\times\mathbb{R}^4)\cap K^R_{t_0,x_0}} |F_N||w|^2|\partial_t w|\mathrm{d}x\mathrm{d}t\ .\end{aligned}$$

In the last line, we have used that

$$K^R_{t_0,x_0} = \bigcup_{\substack{(n,j)\in\mathbb{N}_0\times\mathbb{Z}^4\\ K^N_{\tau_n,y_j}\subseteq K^R_{t_0,x_0}}} K^N_{\tau_n,y_j}\ .$$

We first control the contributions on the time intervals $[\tau_n,\tau_n+N]$. To this end, we define $w^{(N,n,j)}$ as the $K^N_{\tau_n,y_j}$-locally forced solution with data

$$w^{(N,n,j)}(\tau_n) = w(\tau_n) \qquad \text{and} \qquad \partial_t w^{(N,n,j)}(\tau_n) = \partial_t w(\tau_n)\ .$$

Using finite speed of propagation, w and $w^{(N,n,j)}$ coincide on $K^N_{\tau_n,y_j}$. Applying Proposition 2.6.1 and the induction hypothesis to $w^{(N,n,j)}$, it follows that

$$\begin{aligned}
&\int_{K^N_{\tau_n,y_j}} |F_N||w|^2|\partial_t w|\mathrm{d}x\mathrm{d}t \\
&= \int_{K^N_{\tau_n,y_j}} |F_N||w^{(N,n,j)}|^2|\partial_t w^{(N,n,j)}|\mathrm{d}x\mathrm{d}t \\
&\lesssim \eta N^{\frac{3}{4}-s+8\delta}\widetilde{\mathcal{E}}^N_{\tau_n,y_j}[w^{(N,n,j)}]^{\frac{1}{2}}\left(\widetilde{\mathcal{E}}^N_{\tau_n,y_j}[w^{(N,n,j)}]^{\frac{1}{2}} + \widetilde{\mathcal{F}}^N_{\tau_n,y_j}[w^{(N,n,j)}]\right)^{\frac{1}{2}} \\
&\lesssim \eta N^{\frac{3}{4}-s+8\delta}\left(2E_{|x-y_j|\leqslant 16N}[w^{(N,n,j)}](\tau_n) + C_1\|F\|^6_{L^3_tL^6_x(K^N_{\tau_n,y_j})}\right)^{\frac{1}{2}} \\
&\quad \cdot N^{50\delta}\left(C_1E_{|x-y_j|\leqslant 16N}[w^{(N,n,j)}](\tau_n) + C_1\|F\|^6_{L^3_tL^6_x(K^N_{\tau_n,y_j})}\right)^{\frac{1}{2}} \\
&\lesssim \eta N^{\frac{3}{4}-s+60\delta}C_1\left(E_{|x-y_j|\leqslant 16N}[w^{(N,n,j)}](\tau_n) + \|F\|^6_{L^3_tL^6_x(K^N_{\tau_n,y_j})}\right) \\
&\lesssim \eta N^{\frac{3}{4}-s+60\delta}C_1\left(E_{|x-y_j|\leqslant 16N}[w](\tau_n) + \|F\|^6_{L^3_tL^6_x(K^N_{\tau_n,y_j})}\right) .
\end{aligned}$$

As a consequence, we obtain that

$$\begin{aligned}
&\sum_{n=0}^{\lfloor N^\theta\rfloor}\sum_{\substack{j\in\mathbb{Z}^4\\ K^N_{\tau_n,y_j}\subseteq K^R_{t_0,x_0}}}\int_{K^N_{\tau_n,y_j}} |F_N||w|^2|\partial_t w|\mathrm{d}x\mathrm{d}t \\
&\lesssim \eta N^{\frac{3}{4}-s+60\delta}C_1\sum_{n=0}^{\lfloor N^\theta\rfloor}\sum_{\substack{j\in\mathbb{Z}^4\\ K^N_{\tau_n,y_j}\subseteq K^R_{t_0,x_0}}}\left(E_{|x-y_j|\leqslant 16N}[w](\tau_n) + \|F\|^6_{L^3_tL^6_x(K^N_{\tau_n,y_j})}\right) \\
&\lesssim \eta N^{\frac{3}{4}-s+60\delta}C_1\sum_{n=\max(0,t_0/N)}^{\lfloor N^\theta\rfloor}\left(E_{|x-x_0|\leqslant 16R-|t-t_0|}[w](\tau_n) + \|F\|^6_{L^3_tL^6_x(([\tau_n,\tau_n+N]\times\mathbb{R}^4)\,\cap\, K^R_{t_0,x_0})}\right) \\
&\lesssim \eta N^{\frac{3}{4}+\theta-s+60\delta}C_1\widetilde{\mathcal{E}}^R_{t_0,x_0} + \eta N^{\frac{3}{4}-s+60\delta}C_1\|F\|^6_{L^3_tL^6_x(K^R_{t_0,x_0})} . \qquad (2.6.7)
\end{aligned}$$

Using the long-time decay estimate, we can control the contribution on the interval $[N^{1+\theta},\infty)$ by

$$\begin{aligned}
&\int_{([N^{1+\theta},\infty)\times\mathbb{R}^4)\cap K^R_{t_0,x_0}} |F_N||w|^2|\partial_t w|\mathrm{d}x\mathrm{d}t \\
&\lesssim \|F_N\|_{L^1_tL^\infty_x([N^{1+\theta},\infty)\times\mathbb{R}^4)}\|w\|^2_{L^\infty_tL^4_x(K^R_{t_0,x_0})}\|\partial_t w\|_{L^\infty_tL^2_x(K^R_{t_0,x_0})} \\
&\lesssim \eta N^{1-\frac{\theta}{2}-s+\delta}\widetilde{\mathcal{E}}^R_{t_0,x_0}\ .
\end{aligned} \tag{2.6.8}$$

Combining (2.6.7) and (2.6.8), it follows that

$$\begin{aligned}
&\sum_{N\leqslant R/2}\int_{K^R_{t_0,x_0}} |F_N||w|^2|\partial_t w|\mathrm{d}x\mathrm{d}t \\
&\lesssim \eta C_1 \sum_{N\leqslant R/2}\Big(N^{\frac{3}{4}+\theta-s+5\delta}+N^{1-\frac{\theta}{2}-s}\Big)\widetilde{\mathcal{E}}^R_{t_0,x_0}+\eta\Big(\sum_{N\leqslant R/2}N^{\frac{3}{4}-s+5\delta}\Big)C_1\|F\|^6_{L^3_tL^6_x(K^R_{t_0,x_0})} \\
&\lesssim \eta C_1\widetilde{\mathcal{E}}^R_{t_0,x_0}+\eta C_1\|F\|^6_{L^3_tL^6_x(K^R_{t_0,x_0})}\ .
\end{aligned} \tag{2.6.9}$$

Here, we have used that $s>\max(1-\frac{\theta}{2},\frac{3}{4}+\theta)$, and that $\delta=\delta(s,\theta)>0$ is sufficiently small.

Step 5: Finishing the proof. At this point, we have proven all the necessary estimates on w. It only remains to put them together, and use a "kick back" argument. From the energy increment (2.6.5), the high frequency estimate (2.6.6), the low-frequency estimate (2.6.9), and Hölder's inequality, it follows that

$$\widetilde{\mathcal{E}}^R_{t_0,x_0}[w]\leqslant E_{|x-x_0|\leqslant 16R}[w](t_0)+\eta C_1\widetilde{\mathcal{E}}^R_{t_0,x_0}[w]+\eta R^{\frac{3}{4}-s+8\delta}\widetilde{\mathcal{F}}^R_{t_0,x_0}[w]+\eta C_1\|F\|^6_{L^3_tL^6_x(K^R_{t_0,x_0})} \tag{2.6.10}$$

Inserting the same bound for the energy increment into (2.5.8), we also have that

$$\widetilde{\mathcal{F}}^R_{t_0,x_0}[w]\lesssim R^{40\delta}\Big(\widetilde{\mathcal{E}}^R_{t_0,x_0}[w]+\eta R^{\frac{3}{4}-s+8\delta}\widetilde{\mathcal{F}}^R_{t_0,x_0}[w]+\eta C_1\|F\|^6_{L^3_tL^6_x(K^R_{t_0,x_0})}\Big) \tag{2.6.11}$$

If the absolute constant $\eta=\eta(C_1,\delta)>0$ is chosen sufficiently small, then (2.6.11) implies that

$$\widetilde{\mathcal{F}}^R_{t_0,x_0}[w]\lesssim R^{40\delta}\Big(\widetilde{\mathcal{E}}^R_{t_0,x_0}[w]+\eta C_1\|F\|^6_{L^3_tL^6_x(K^R_{t_0,x_0})}\Big)\ . \tag{2.6.12}$$

Inserting this into (2.6.10), we obtain (2.6.3). Finally, (2.6.3) and (2.6.12) imply (2.6.4). This completes the proof of the induction step.

□

Using Proposition 2.6.3, we now provide a short proof of the main result.

Proof of Theorem 2.1.3. Assume that the statements in Proposition 2.4.8 hold for $\omega \in \Omega$. From Lemma 2.5.1, it follows that there exists a local solution to (2.5.1). From Proposition 2.6.3, it follows for all $R \geqslant 1$ that

$$\sup_{t \in [0,R]} \int_{|x| \leqslant 2R-t} T^{00}[v](t,x) \mathrm{d}x \leqslant 2E[v_0, v_1] + C_1 \ .$$

By letting $R \to \infty$, we obtain the a-priori energy bound

$$\sup_{t \in [0,\infty)} E[v](t) \leqslant 2E[v_0, v_1] + C_1 \ .$$

From Proposition 2.5.2, this implies the global space-time bound $\|v\|_{L_t^3 L_x^6([0,\infty)\times\mathbb{R}^4)} < \infty$ and the existence of scattering states $(v_0^+, v_1^+) \in \dot{H}^1(\mathbb{R}^4) \times L^2(\mathbb{R}^4)$. Since $u = F + v$, we obtain the global space-time bound $\|u\|_{L_t^3 L_x^6([0,\infty)\times\mathbb{R}^4)} < \infty$ and the scattering states $(u_0^+, u_1^+) = (v_0^+ - f_{0,<N_{\mathrm{hi}}}^\omega, v_1^+ - f_{1,<N_{\mathrm{hi}}}^\omega)$. This completes the proof for positive times. By time-reflection symmetry, we obtain the same result for negative times. □

CHAPTER 3

Invariant Gibbs measures for the three-dimensional wave equation with a Hartree nonlinearity I: Measures[4]

3.1 Introduction

In this chapter, we rigorously construct and study the Gibbs measure $\mu^{\otimes}$. We recall from (1.2.2) that the Hamiltonian is given by

$$H_\beta[\phi_0,\phi_1] = \frac{1}{2}\Big(\|\phi_0\|_{L^2_x}^2 + \|\nabla_x \phi_0\|_{L^2_x}^2 + \|\phi_1\|_{L^2_x}^2 \Big) + \frac{1}{4}\int_{\mathbb{T}^3} :(V_\beta * \phi_0^2)\ \phi_0^2:\, \mathrm{d}x.$$

Since the Hamiltonian $H[\phi_0,\phi_1]$ splits into a sum of functions in ϕ_0 and ϕ_1, we can rewrite the Gibbs measure $\mu^{\otimes}$ as

$$\begin{aligned} &\mathrm{d}\mu^{\otimes}(\phi_0,\phi_1) \\ =&\mathcal{Z}_0^{-1} \exp\Big(-\frac{1}{4}\int_{\mathbb{T}^3} :(V*\phi_0^2)\phi_0^2:\, \mathrm{d}x - \frac{1}{2}\|\phi_0\|_{L^2}^2 - \frac{1}{2}\|\nabla\phi_0\|_{L^2}^2 \Big) \mathrm{d}\phi_0 \otimes \mathcal{Z}_1^{-1} \exp\Big(-\frac{1}{2}\|\phi_1\|_{L^2}^2\Big)\mathrm{d}\phi_1. \end{aligned}$$

The construction and properties of the second factor are elementary (as will be explained below), and we now focus on the first factor. As a result, we are interested in the rigorous construction of a measure μ which is formally given by

$$\mathrm{d}\mu(\phi) = \mathcal{Z}^{-1} \exp\Big(-\frac{1}{4}\int_{\mathbb{T}^3} :(V*\phi^2)\phi^2:\, \mathrm{d}x - \frac{1}{2}\|\phi\|_{L^2(\mathbb{T}^3)}^2 - \frac{1}{2}\|\nabla\phi\|_{L^2(\mathbb{T}^3)}^2 \Big) \mathrm{d}\phi. \qquad (3.1.1)$$

[4] The content of this chapter has been published online in *Stochastics and Partial Differential Equations: Analysis and Computations* [Bri20c].

Our Gibbs measure μ is closely related to the Φ^4_d-models, which replace the three-dimensional torus $\mathbb{T}^3$ by the more general d-dimensional torus $\mathbb{T}^d$ and replace the integrand $:(V*\phi^2)\phi^2:$ by the renormalized quartic power $:\phi^4:$. Thus, the Φ^4_d-model is formally given by

$$\mathrm{d}\Phi^4_d(\phi) = \mathcal{Z}^{-1} \exp\Big(-\frac{1}{4}\int_{\mathbb{T}^d} :\phi^4: \,\mathrm{d}x - \frac{1}{2}\|\phi\|^2_{L^2(\mathbb{T}^d)} - \frac{1}{2}\|\nabla\phi\|^2_{L^2(\mathbb{T}^d)}\Big)\,\mathrm{d}\phi. \tag{3.1.2}$$

Aside from their connection to Hamiltonian PDEs, such as nonlinear wave and Schrödinger equations, the Φ^4_d-models are of independent interest in quantum field theory (cf. [Fol08]). In most rigorous constructions of measures such μ or the Φ^4_d-models, the first step consists of a regularization. For instance, one may insert a frequency-truncation in the nonlinearity or replace the continuous spatial domain by a discrete lattice. In a second step, one then proves the convergence of the regularized measures as the regularization is removed, either by direct estimates or compactness arguments.

With a particular focus on Φ^4_d-models, the question of convergence of the regularized measures has been extensively studied over several decades. The first proof of convergence was a major success of the constructive field theory program, which thrived during the 1970s and 1980s. We refer the reader to the excellent introduction of [GH19] for a detailed overview and the original works [BCG78, BFS83, FO76, GJ87, MS76, Par77, Sim74, Wat89].

In the 1990s, Bourgain [Bou94, Bou96] revisited the Φ^4_d-model in dimension $d = 1, 2$ using tools from harmonic analysis and introduced these problems into the dispersive PDE community. Bourgain's works [Bou94, Bou96] also contain important dynamical insights, which will be utilized in the second part of this series.

Based on the method of stochastic quantization, which was introduced by Nelson [Nel66, Nel67] and Parisi-Wu [PW81], the construction and properties of the Φ^4_d-models have also been studied over the last twenty years in the stochastic PDE community. The main idea behind stochastic quantization is that the Φ^4_d-measure is formally invariant under the stochastic nonlinear heat equation

$$\partial_t u + u - \Delta u = - :u^3: + \sqrt{2}\xi \qquad (t,x) \in \mathbb{R} \times \mathbb{T}^d, \tag{3.1.3}$$

where ξ is space-time white noise. After prescribing simple initial data, such as $u(0) = 0$, one hopes to obtain the Φ^4_d-measure as the limit of the law of $u(t)$ as $t \to \infty$. In spatial dimensions $d = 1, 2$, this approach was carried out by Iwata [Iwa87] and Da Prato-Debussche [DD03], respectively. In spatial dimension $d = 3$, however, (3.1.3) is highly singular and the local well-posedness theory of (3.1.3) is beyond classical methods in stochastic partial differential equations. In groundbreaking work [Hai14], Hairer introduced regularity structures, which provide a detailed description of the local dynamics of (3.1.3). Alternatively, the local well-posedness of (3.1.3) was also obtained by Catellier and Chouk in [CC18], which is based on the para-controlled calculus of Gubinelli, Imkeller, and Perkowski [GIP15]. In order to construct the Φ^4_3-model using (3.1.3), however, local control over the solution is not sufficient, and one needs a global well-posedness theory. The global theory has been addressed very recently in [AK20, GH19, HM18, MW17], which combine regularity structures or para-controlled calculus with further PDE arguments, such as the energy method. Using similar tools, Barashkov and Gubinelli [BG20b, BG20a] recently developed a variational approach to the Φ^4_3-model, which does not directly rely on the stochastic heat equation (3.1.3). Their work forms the basis of this paper and will be discussed in more detail below.

After this broad overview of the relevant literature, we now begin a more detailed discussion of the previous methods. Throughout this discussion we encourage the reader to think of the nonlinear wave equation as a Hamiltonian system of ordinary differential equations in Fourier space. We begin with the construction of the Gaussian free field. Then, we discuss the construction of the Φ^4_1 and Φ^4_2-models using harmonic analysis, similar as in Bourgain's works [Bou94, Bou96], and the construction of the Φ^4_3-model using the variational approach of Barashkov and Gubinelli [BG20b].

Given a function $\phi\colon \mathbb{T}^d \to \mathbb{R}$, its Fourier expansion is given by

$$\phi(x) = \sum_{n\in\mathbb{Z}^d} \widehat{\phi}(n) e^{i\langle n,x\rangle}. \tag{3.1.4}$$

Due to the real-valuedness of ϕ, the sequence $(\widehat{\phi}(n))_{n\in\mathbb{Z}^d}$ satisfies the symmetry condition $\overline{\widehat{\phi}}(n) = \widehat{\phi}(-n)$. In order to respect this symmetry, we let $\Lambda \subseteq \mathbb{Z}^d$ be such that $\mathbb{Z}^d = \{0\} \biguplus \Lambda \biguplus (-\Lambda)$, where $\biguplus$ denotes the disjoint union. For $n \in \Lambda$, we denote by $\mathrm{d}\widehat{\phi}(n)$ the Lebesgue measure on $\mathbb{C}$, and for $n = 0$, we denote by $\mathrm{d}\widehat{\phi}(0)$ the Lebesgue measure on $\mathbb{R}$. We can then formally identify the d-dimensional Gaussian free field

$$\mathrm{d}\mathscr{g}_d(\phi) = \mathcal{Z}^{-1} \exp\Big(-\frac{1}{2}\|\phi\|^2_{L^2(\mathbb{T}^d)} - \frac{1}{2}\|\nabla\phi\|^2_{L^2(\mathbb{T}^d)}\Big)\, \mathrm{d}\phi \tag{3.1.5}$$

as the push-forward under the Fourier transform of

$$\begin{aligned} &\mathcal{Z}^{-1} \exp\Big(-\frac{1}{2}\sum_{n\in\mathbb{Z}^d} (1+|n|^2)|\widehat{\phi}(n)|^2\Big) \bigotimes_{n\in\{0\}\cup\Lambda} \mathrm{d}\widehat{\phi}(n) \\ &= \frac{1}{2\pi} \exp\Big(-\frac{1}{2}|\widehat{\phi}(0)|^2\Big)\mathrm{d}\widehat{\phi}(0) \otimes \Big(\bigotimes_{n\in\Lambda} \frac{1}{\pi\langle n\rangle^2} \exp\Big(-\langle n\rangle^2|\widehat{\phi}(n)|^2\Big)\mathrm{d}\widehat{\phi}(n)\Big), \end{aligned} \tag{3.1.6}$$

where $\langle n\rangle^2 = 1 + |n|^2$. While (3.1.5) is entirely formal, the right-hand side of (3.1.6) is a well-defined product measure. Under the measure in (3.1.6), $\widehat{\phi}(0)$ is a standard real-valued Gaussian and $(\widehat{\phi}(n))_{n\in\Lambda}$ is a sequence of independent complex Gaussians satisfying $\mathbb{E}|\widehat{\phi}(n)|^2 = \langle n\rangle^{-2}$. Turning this formal discussion around, we let $(\Omega, \mathcal{F}, \mathbb{P})$ be an ambient probability space containing a sequence of independent complex-valued standard Gaussians $(g_n)_{n\in\Lambda}$ and a standard real-valued Gaussian g_0. Then, we can rigorously define the Gaussian free field $\mathscr{g}_d$ by

$$\mathrm{d}\mathscr{g}_d(\phi) = \Big(\sum_{n\in\mathbb{Z}^d} \frac{g_n}{\langle n\rangle} e^{i\langle n,x\rangle}\Big)_{\#}\mathbb{P}, \tag{3.1.7}$$

where the subscript # denotes the pushforward. Using the representation (3.1.7), we see that a typical sample of $\mathscr{g}_d$ almost surely lies in $H^s_x(\mathbb{T}^d)$ for all $s < 1 - d/2$ but not in $H^{1-d/2}_x(\mathbb{T}^d)$.

We now turn to the construction of the Φ^4_1 and Φ^4_2-models. Based on our formal expression of the Φ^4_1-model in (3.1.2), we would like to define

$$\mathrm{d}\Phi^4_1(\phi) \overset{\text{def}}{=} \mathcal{Z}^{-1} \exp\Big(-\frac{1}{4} \int_{\mathbb{T}} \phi^4(x) \mathrm{d}x \Big) \mathrm{d}g_1(\phi). \tag{3.1.8}$$

Using either Sobolev embedding or Khintchine's inequality, we obtain g_1-almost surely that $0 < \|\phi\|_{L^4(\mathbb{T})} < \infty$. This implies that the density $\mathrm{d}\Phi^4_1/\mathrm{d}g_1$ is well-defined, almost surely positive, and lies in $L^q(g_1)$ for all $1 \leqslant q \leqslant \infty$. In particular, the Φ^4_1-model is absolutely continuous with respect to the Gaussian free field g_1. We emphasize that the potential energy in (3.1.8) does not require a renormalization. Furthermore, we can define truncated Φ^4_1-models by

$$\mathrm{d}\Phi^4_{1;N}(\phi) \overset{\text{def}}{=} \mathcal{Z}_N^{-1} \exp\Big(-\frac{1}{4} \int_{\mathbb{T}} (P_{\leqslant N}\phi)^4(x) \mathrm{d}x \Big) \mathrm{d}g_1(\phi),$$

where N is a dyadic integer and $P_{\leqslant N}$ a Littlewood-Paley projection. As was shown in [Bou94], direct estimates yield the convergence of $\mathrm{d}\Phi^4_{1;N}/\mathrm{d}g_1$ in $L^q(g_1)$ for all $1 \leqslant q < \infty$ and hence $\Phi^4_{1;N}$ converges to Φ^4_1 in total variation as N tends to infinity.

In two spatial dimensions, however, we encounter a new difficulty. Since g_1-almost surely $\|\phi\|_{L^2} = \infty$, the potential energy $\|\phi\|^4_{L^4}$ is almost surely infinite. As a result, the potential energy requires a renormalization. A direct calculation using the definition of $P_{\leqslant N}$ in (3.1.14) below yields

$$\sigma_N^2 = \int_0^\infty \mathrm{d}g_2(\phi) \|P_{\leqslant N}\phi\|^2_{L^2(\mathbb{T}^2)} \sim \log(N).$$

We then replace the monomial $(P_{\leqslant N}\phi)^4$ by the Hermite polynomial

$$:(P_{\leqslant N}\phi)^4: = (P_{\leqslant N}\phi)^4 - 6\sigma_N^2 (P_{\leqslant N}\phi)^2 + 3\sigma_N^4.$$

This leads to the truncated Φ^4_2-model given by

$$\mathrm{d}\Phi^4_{2;N}(\phi) \overset{\text{def}}{=} \mathcal{Z}_N^{-1} \exp\Big(-\frac{1}{4} \int_{\mathbb{T}^2} :(P_{\leqslant N}\phi)^4: (x) \mathrm{d}x \Big) \mathrm{d}g_2(\phi).$$

After this renormalization, one can show (cf. [OT18]) that the densities $\mathrm{d}\Phi^4_{2;N}/\mathrm{d}\mathscr{g}_2$ converge in $L^q(\mathscr{g}_2)$ for all $1 \leqslant q < \infty$ and we can define Φ^4_2 as the limit (in total-variation) of $\Phi^4_{2;N}$ as $N \to \infty$. As in one spatial dimension, the Φ^4_2-model is absolutely continuous with respect to the Gaussian free field $\mathscr{g}_2$. Using similar tools as for the Φ^4_2-model, Bourgain [Bou97] constructed the Gibbs measure μ for the Hamiltonian with a Hartree interaction for $\beta > 2$, which corresponds to a relatively smooth interaction potential V. The key point of this paragraph is that the Φ^4_1-model, the Φ^4_2-model, and the Gibbs measure μ for a smooth interaction potential can be constructed through "hard" analysis. As a result, one obtains strong modes of convergence and absolute continuity with respect to Gaussian free field.

The construction of the Φ^4_3-model, however, is much more complicated. As will be described below, several of the "hard" conclusions, such as convergence in total-variation or absolute continuity with respect to the Gaussian free field, are either unavailable or fail. As a result, we have to (partially) replace hard estimates by softer compactness arguments. We now give a short overview of the variational approach in [BG20b, BG20a], which forms the basis of this paper.

In order to use techniques from stochastic control theory, we introduce a family of Gaussian processes $(W_t(x))_{t\geqslant 0}$ on an ambient probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying $\mathrm{Law}_{\mathbb{P}}(W_\infty) = \mathscr{g}_3$, which will be defined in Section 3.2.1. We view t as a stochastic time-variable which serves as a regularization parameter. Using this terminology, we obtain a truncated Φ^4_3-model by setting

$$\mathrm{d}\Phi^4_{3;T}(\phi) = (W_\infty)_\# \big(\mathrm{d}\overline{\Phi}^4_{3;T}(\phi)\big)$$

and

$$\mathrm{d}\overline{\Phi}^4_{3;T}(\phi) = \mathcal{Z}_T^{-1} \exp\Big(-\frac{1}{4}\int_{\mathbb{T}^3} W_T^4(x) - a_T W_T^2(x) - b_T \mathrm{d}x\Big)\mathrm{d}\mathbb{P}.$$

We emphasize already that the $\Phi^4_{3;T}$-measure does not correspond to a truncated Hamiltonian, which will be discussed in full detail in Section 3.2.1. In order to construct the Φ^4_3-model, the main step is to prove the tightness of the $\Phi^4_{3;T}$-measures. Using Prokhorov's theorem, this implies the weak convergence of a subsequence of $\Phi^4_{3;T}$ and we can define the Φ^4_3-measure as the weak limit. To prove tightness, Barashkov and Gubinelli obtain uniform bounds in T on the Laplace transform

$$f \in C(\mathcal{C}_x^{-\frac{1}{2}-}(\mathbb{T}^3);\mathbb{R}) \to \int \mathrm{d}\Phi^4_{3;T}(\phi)\ e^{-f(\phi)}.$$

The main ingredients for the uniform bounds are the Boué-Dupuis formula (Theorem 3.2.1) and the para-controlled calculus of Gubinelli, Imkeller, and Perkowski [GIP15], which has also been used in the stochastic quantization approach to the Φ^4_3-model (cf. [GH19]).

While the variational approach yields the existence of the Φ^4_3-measure, it only yields limited information regarding its properties. In spatial dimensions $d = 1, 2$, the Φ^4_d-model is absolutely continuous with respect to the Gaussian free field g_d, and hence the samples of Φ^4_d for many purposes behave like a random Fourier series with independent coefficients. This is an essential ingredient in almost all invariance arguments for random dispersive equations (see e.g. [Bou97, Bou96, DNY19, NOR12]). Unfortunately, the Φ^4_3-measure is singular with respect to the Gaussian free field g_3. This fact seems to be part of the folklore in mathematical physics, but it is surprisingly difficult to find a detailed reference. In an unpublished note available to the author [Hai], Martin Hairer proved the singularity using the stochastic quantization approach and regularity structures. Using the Girsanov-transformation, Barashkov and Gubinelli [BG20a] constructed a reference measure ν^4_3 for the Φ^4_3-model, which serves a similar purpose as the Gaussian free field for Φ^4_1 and Φ^4_2. The samples of ν^4_3 are given by an explicit Gaussian chaos and Φ^4_3 is absolutely continuous with respect to ν^4_3. Furthermore, Barashkov and Gubinelli proved that the reference measure ν^4_3 and the Gaussian free field g_3 are mutually singular, which yields a self-contained proof of the singularity of Φ^4_3 with respect to the Gaussian free field g_3.

3.1.1 Main results and methods

In the following, we simply write $g = g_3$ for the three-dimensional Gaussian free field. Let $N \geqslant 1$ be a dyadic integer and define the renormalized potential energy by

$$:\mathcal{V}_N^\lambda(\phi): \overset{\text{def}}{=} \frac{\lambda}{4} \int_{\mathbb{T}^3} \Big((V * \phi^2)\phi^2 - 2a_N \phi^2 - 4(\mathcal{M}_N \phi)\phi + \widehat{V}(0) a_N^2 + 2b_N \Big) \mathrm{d}x + c_N^\lambda. \tag{3.1.9}$$

The coupling constant $\lambda > 0$ is introduced for illustrative purposes, but the reader may simply set $\lambda = 1$ as in all previous discussions. The renormalization constants a_N, b_N, and c_N^λ are as in Definition 3.2.8 and Proposition 3.3.2 and the renormalization multiplier $\mathcal{M}_N$ is as in Definition 3.2.8. We emphasize that the renormalization in (3.1.9) goes beyond the usual Wick-ordering, which is only based on the mass $\|P_{\leqslant N}\phi\|_{L^2}^2$. The additional renormalization is contained in the renormalization constant c_N^λ, which is related to the mutual singularity of $\mu^\otimes$ and g (for $0 < \beta < 1/2$). The truncated and renormalized Hamiltonian H_N is given by

$$H_N[\phi_0, \phi_1] \overset{\text{def}}{=} \frac{1}{2}\Big(\|\phi_0\|_{L^2}^2 + \|\nabla \phi_0\|_{L^2}^2 + \|\phi_1\|_{L^2}^2 \Big) + :\mathcal{V}_N^\lambda(P_{\leqslant N}\phi_0): \ , \tag{3.1.10}$$

where we omit the dependence on $\lambda > 0$ from our notation. We emphasize that only the quartic term contains a frequency-truncation and renormalization, whereas the quadratic terms remain unchanged. As described in the beginning of the introduction, we focus on the first factor of the truncated Gibbs measure $\mu_N^\otimes$, which is given by

$$\mathrm{d}\mu_N(\phi) = \frac{1}{\mathcal{Z}_N^\lambda} \exp\Big(- :\mathcal{V}_N^\lambda(P_{\leqslant N}\phi): \Big) \mathrm{d}g(\phi). \tag{3.1.11}$$

Before we state our main result, we recall the assumptions on the interaction potential $V \colon \mathbb{T}^3 \to \mathbb{R}$ from the introduction to the thesis. In these assumptions, $0 < \beta < 3$ is a parameter.

Assumptions A. *We assume that the interaction potential V satisfies*

(i) $V(x) = c_\beta |x|^{-(3-\beta)}$ *for some* $c_\beta > 0$ *and all* $x \in \mathbb{T}^3$ *satisfying* $\|x\| \leqslant 1/10$,

(ii) $V(x) \gtrsim_\beta 1$ *for all* $x \in \mathbb{T}^3$,

(iii) $V(x) = V(-x)$ *for all* $x \in \mathbb{T}^3$,

(iv) V *is smooth away from the origin.*

We now state the conclusions of this paper which will be needed in the Chapter 4 of this thesis. A more comprehensive version of our results will then be stated in Theorem 3.1.3, Theorem 3.1.4, and Theorem 3.1.5 below. The additional results may be useful in further applications, such as invariant measures for a Schrödinger equation with a Hartree nonlinearity.

Theorem 3.1.1 (The Gibbs measure)**.** Let $\kappa > 0$ be a fixed positive parameter, let $0 < \beta < 3$ be a parameter, and let the interaction potential V be as in the Assumptions A. Then, the sequence of truncated Gibbs measures $(\mu_N)_{N\geqslant 1}$ converges weakly to a probability measure μ_∞ on $\mathcal{C}_x^{-1/2-\kappa}(\mathbb{T}^3)$, which is called the Gibbs measure. If in addition $0 < \beta < 1/2$, the Gibbs measure μ_∞ and the Gaussian free field g are mutually singular. Furthermore, there exists a sequence of reference measures $(\nu_N)_{N\geqslant 1}$ on $\mathcal{C}_x^{-1/2-\kappa}(\mathbb{T}^3)$ and an ambient probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying the following properties:

(i) (Absolute continuity and L^q-bounds) The truncated Gibbs measures μ_N are absolutely continuous with respect to the reference measures ν_N. More quantitatively, there exists a parameter $q > 1$ and a constant $C \geqslant 1$, depending only on β, such that

$$\mu_N(A) \leqslant C \nu_N(A)^{1-\frac{1}{q}}$$

for all Borel sets $A \subseteq \mathcal{C}_x^{-1/2-\kappa}(\mathbb{T}^3)$.

(ii) (Representation of ν_N) Let $\gamma = \min(1/2 + \beta, 1)$. There exists a large integer $k = k(\beta)$ and two random functions $\mathcal{G}, \mathcal{R}_N \colon (\Omega, \mathcal{F}) \to \mathcal{C}_x^{-1/2-\kappa}(\mathbb{T}^3)$ satisfying for all $p \geqslant 2$ that

$$\nu_N = \mathrm{Law}_{\mathbb{P}}\big(\mathcal{G} + \mathcal{R}_N\big), \quad g = \mathrm{Law}_{\mathbb{P}}\big(\mathcal{G}\big), \quad \text{and} \quad \|\mathcal{R}_N\|_{L^p_\omega \mathcal{C}_x^{\gamma-\kappa}(\Omega\times\mathbb{T}^3)} \leqslant p^{\frac{k}{2}}.$$

Remark 3.1.2. After the completion of the series [Bri20c, Bri20d], the author learned of independent work by Oh, Okamoto, and Tolomeo [OOT20], which discusses the focusing and defocusing three-dimensional (stochastic) nonlinear wave equation with a Hartree nonlinearity. In the focusing case, the authors provide a complete picture of the construction and properties of the focusing Gibbs measures, which distinguishes the three regimes $\beta > 2$, $\beta = 2$, and $\beta < 2$ (cf. [OOT20]). In the defocusing case, the authors construct the Gibbs measures for $\beta > 0$ and prove the singularity for $0 < \beta \leqslant 1/2$, which includes the endpoint $\beta = 1/2$. The reference measures are briefly discussed in [OOT20, Appendix C], but only play a minor role in their analysis. The L^q-bound in Theorem 3.1.1, which will be essential in the second part of this series [Bri20e], is not proven in [OOT20].

In the first version of the paper [Bri20c], we proved the tightness of the truncated Gibbs measures $(\mu_N)_{N\geqslant 1}$, which only implies that a subsequence of $(\mu_N)_{N\geqslant 1}$. In [OOT20], the authors proved the uniqueness of weak subsequential limits, which lead to the convergence of the full sequence. A version of the uniqueness argument from [OOT20], which has been modified to match our notation, has now been included in Appendix 3.6.3.

While the measure-theoretic part of [OOT20] treats all $\beta > 0$, the dynamical results are restricted to $\beta > 1$. In particular, the singular regime $0 < \beta < 1/2$ is not covered, which is the main object of the series [Bri20c, Bri20d].

In addition to the singular regime $0 < \beta < 1/2$, the most interesting cases in Theorem 3.1.1 are the Newtonian potential $|x|^{-2}$ (corresponding to $\beta = 1$) and the Coulomb potential $|x|^{-1}$ (corresponding to $\beta = 2$). As mentioned earlier in the introduction, Bourgain [Bou97] proved a version of Theorem 3.1.1 in the limited range $\beta > 2$, which corresponds to a relatively smooth interaction potential.

We now split the main theorem (Theorem 3.1.1) into three parts:

- the tightness and weak convergence of the truncated Gibbs measures μ_N,
- the construction and properties of the reference measures ν_N,
- the mutual singularity of the Gibbs measure and the Gaussian free field.

Theorem 3.1.3 (Tightness and convergence)**.** The truncated Gibbs measures $(\mu_N)_{N\geqslant 1}$ are tight on $\mathcal{C}_x^{-1/2-\kappa}(\mathbb{T}^3)$. Furthermore, the sequence $(\mu_N)_{N\geqslant 1}$ weakly converges to a limiting measure μ_∞.

The overall strategy of the proof of Theorem 3.1.3 is the same as in the variational approach of Barashkov and Gubinelli [BG20b]. In comparison with [BG20b], the terms in this paper often have a more complicated algebraic structure but obey better analytical estimates. As any reader familiar with regularity structures or para-controlled calculus may certify, the algebraic structure of most stochastic objects is already quite complicated, so this trade-off is not always favorable. In addition, the non-locality of the nonlinearity requires different analytical estimates and we mention the two most important examples:

(i) The coercive term $\|f\|_{L^4}^4$ in the variational problem for the Φ_3^4-model is replaced by the potential energy

$$\int_{\mathbb{T}^3} (V * f^2) f^2 \mathrm{d}x.$$

We emphasize that the coercive term in the variational problem does not contain a renormalization, which is a result of the binomial formula in Lemma 3.2.11. In order to use the potential energy in our estimates, we rely on a fractional derivative estimate of Visan [Vis07, (5.17)].

(ii) In the variational problem, we encounter mixed terms of the form

$$\int_{\mathbb{T}^3} \Big[\big(V * (P_{\leqslant N} W_\infty \cdot P_{\leqslant N} f_1) \cdot P_{\leqslant N} W_\infty \cdot P_{\leqslant N} f_2 - \big(\mathcal{M}_N P_{\leqslant N} f_1\big) P_{\leqslant N} f_2 \Big] \mathrm{d}x,$$

where $(W_t)_{t\geqslant 0}$ is the Gaussian process from the introduction. Based on the literature on random dispersive equations [Bou97, Bou96, DNY19, DNY20, GKO18a], it is tempting to bound this mixed term through Fourier-analytic and random matrix techniques. We instead develop a simpler and elegant physical-space approach.

The next theorem gives a more detailed description of the reference measures in Theorem 3.1.1. To simplify the notation, we allow the truncation parameter N to take the value ∞.

Theorem 3.1.4 (Reference measures)**.** There exists a family of reference measures $(\nu_N)_{1\leqslant N\leqslant\infty}$ and an ambient probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying the following properties:

(i) Absolute continuity and L^q-bounds: The truncated Gibbs measures μ_N are absolutely continuous with respect to the reference measures ν_N. More quantitatively, there exists a parameter $q > 1$ and a constant $C \geqslant 1$, depending only on β, such that

$$\mu_N(A) \leqslant C\nu_N(A)^{1-\frac{1}{q}}$$

for all Borel sets $A \subseteq \mathcal{C}_x^{-1/2-\kappa}(\mathbb{T}^3)$.

(ii) Representation of ν_N: We have that

$$\nu_N = \mathrm{Law}_{\mathbb{P}}\left(\mathcal{G}^{(1)} + \mathcal{G}_N^{(3)} + \mathcal{G}_N^{(n)}\right).$$

Here, $n = n(\beta)$ is a large integer and the linear, cubic, and n-th order Gaussian chaoses are explicitly given by

$$\begin{aligned}
\mathcal{G}^{(1)} &= W_\infty, \\
\mathcal{G}_N^{(3)} &= -\lambda P_{\leqslant N} \int_0^\infty J_s^2\Big(:(V * (P_{\leqslant N} W_s)^2) P_{\leqslant N} W_s: \Big) \mathrm{d}s, \\
\mathcal{G}_N^{(n)} &= P_{\leqslant N} \int_0^\infty \langle\nabla\rangle^{-\frac{1}{2}} J_s^2\Big(:(\langle\nabla\rangle^{-\frac{1}{2}} P_{\leqslant N} W_s)^n: \Big) \mathrm{d}s,
\end{aligned}$$

where we refer the reader to Section 3.2.1 and Definition 3.2.6 for the definitions of J_s and the renormalizations.

We emphasize that the representation of ν_N in Theorem 3.1.4 is much more detailed than stated in Theorem 3.1.1. This additional information is not required in our proof of global well-posedness and invariance in the Chapter 4. However, we believe that the more detailed representation way be relevant for the Schrödinger equation with a Hartree nonlinearity. The reason lies in low×low×high-interactions, which are more difficult in Schrödinger equations than in wave equations. In the last two years, we have seen new and intricate methods dealing with these interactions [Bri20a, DNY19, DNY20], but all of these papers heavily rely on the independence of the Fourier coefficients. In fact, overcoming this obstruction is mentioned as an open problem in [DNY20, Section 9.1].

The proof of Theorem 3.1.4 is based on the Girsanov-approach of Barashkov and Gubinelli [BG20a]. As mentioned earlier, however, we cannot use the same approximate Gibbs measures as in [BG20a], since they do not correspond to a frequency-truncated Hamiltonian. In Chapter 4, the frequency-truncated Hamiltonians are an essential ingredient in the proof of global well-posedness and invariance. This difference will be discussed in detail in Section 3.2.1. For now, we simply mention that there is a trade-off between desirable properties from a PDE or probabilistic perspective.

Our last theorem describes the relationship between the Gibbs measure μ_∞ and the Gaussian free field $\mathfrak{g}$.

Theorem 3.1.5 (Singularity)**.** If $0 < \beta < 1/2$, then the Gibbs measure μ_∞ and the Gaussian free field $\mathfrak{g}$ are mutually singular. If $\beta > 1/2$, then the Gibbs measure is absolutely continuous with respect to the Gaussian free field $\mathfrak{g}$.

Theorem 3.1.5 determines the exact threshold between absolute continuity and singularity of μ_∞

with respect to g. As mentioned in Remark 3.1.2, the singularity at the endpoint $\beta = 1/2$ has been obtained in independent work by Oh, Okamoto, and Tolomeo [OOT20]. The absolute continuity for $\beta > 1/2$ already follows from the variational estimates in our construction of μ_∞. The main step is the mutual singularity of μ_∞ and g for $0 < \beta < 1/2$. We provide an explicit event witnessing this singularity, which is based on the behaviour of the frequency-truncated potential energy

$$\int_{\mathbb{T}^3} :(V * (P_{\leqslant N}\phi)^2)(P_{\leqslant N}\phi)^2 : \mathrm{d}x$$

under the different measures.

Acknowledgements: The author thanks his advisor Terence Tao for his patience and invaluable guidance. The author also thanks Nikolay Barashkov, Martin Hairer, Redmond McNamara, Dana Mendelson, Tadahiro Oh, and Felix Otto for helpful discussions.

3.1.2 Overview

To orient the reader, let us review the rest of this paper. In Section 3.2.1, we introduce the stochastic control perspective and recall the Boué-Dupuis formula. In Section 3.2.2, we estimate several stochastic objects, such as the renormalized nonlinearity $:(V * W_t^2)W_t:$. Our main tools will be Itô's formula and Gaussian hypercontractivity. In Section 3.3, we prove the tightness of the truncated Gibbs measures μ_N and construct the limiting measure μ_∞. Using the Laplace transform and the Boué-Dupuis formula, the proof of tightness reduces to estimates for a variational problem, which occupy most of this section. In Section 3.4, we first construct the reference measures ν_N and then examine their properties. The main ingredients are Girsanov's transformation and our earlier variational estimates. Finally, in Section 3.5, we prove the singularity of the Gibbs measure μ_∞ with respect to the Gaussian free field g for all $0 < \beta < 1/2$.

3.1.3 Notation

In the rest of the paper, we use $\overset{\text{def}}{=}$ instead of := for definitions. The reason is that the colon in := may be confused with our notation for renormalized powers in Definition 3.2.6 below. With a slight abuse of notation, we write $\mathrm{d}x$ for the normalized Lebesgue measure on $\mathbb{T}^3$. That is, we implicitly normalize

$$\int_{\mathbb{T}^3} 1 \mathrm{d}x = 1.$$

We define the Fourier transform of a function $f\colon \mathbb{T}^3 \to \mathbb{C}$ by

$$\widehat{f}(n) \overset{\text{def}}{=} \int_{\mathbb{T}^3} f(x) e^{-inx} \mathrm{d}x.$$

For any $k \in \mathbb{N}$ and $n_1, \ldots, n_k \in \mathbb{Z}^3$, we define

$$n_{12\ldots k} \overset{\text{def}}{=} \sum_{j=1}^{k} n_j. \tag{3.1.12}$$

For instance, $n_{12} = n_1 + n_2$ and $n_{123} = n_1 + n_2 + n_3$.

We now introduce our frequency-truncation operators. We let $\rho\colon \mathbb{R}_{>0} \to [0,1]$ be a smooth, non-increasing function satisfying $\rho(y) = 1$ for all $0 \leqslant y \leqslant 1/4$ and $\rho(y) = 0$ for all $y \geqslant 4$. We also assume that $\min(\rho(y), -\rho'(y)) \gtrsim 1$ for all $1/2 \leqslant y \leqslant 2$. For any $t \geqslant 0$ and $n \in \mathbb{Z}^3$, we also define

$$\rho_t(n) \overset{\text{def}}{=} \rho\Big(\frac{\|n\|_2}{\langle t \rangle}\Big).$$

In particular, it holds that $t \mapsto \rho_t(\xi)$ is non-decreasing. In order to break up the frequency truncation, we also set

$$\sigma_t(n) \overset{\text{def}}{=} \Big(\frac{\mathrm{d}}{\mathrm{d}t}\rho_t(n)\Big)^{\frac{1}{2}}. \tag{3.1.13}$$

This continuous approach instead of the usual discrete decomposition will be essential in the stochastic control approach (Section 3.2.1). Nevertheless, we will sometimes use the usual dyadic

Littlewood-Paley operators. For any dyadic $N \geqslant 1$, we define $P_{\leqslant N}$ by

$$\widehat{P_{\leqslant N} f}(n) = \rho_N(n)\widehat{f}(n). \tag{3.1.14}$$

We further set

$$P_1 f = P_{\leqslant 1} f \qquad \text{and} \qquad P_N f = P_{\leqslant N} f - P_{\leqslant N/2} f \quad \text{for all } N \geqslant 2.$$

The corresponding Fourier multipliers are denoted by

$$\chi_1(n) = \rho_1(n) \qquad \text{and} \qquad \chi_N(n) = \rho_N(n) - \rho_{N/2}(n) \quad \text{for all } N \geqslant 2. \tag{3.1.15}$$

For any $s \in \mathbb{R}$, the $\mathcal{C}_x^s(\mathbb{T}^3)$-norm is defined as

$$\|f\|_{\mathcal{C}_x^s(\mathbb{T}^3)} \overset{\text{def}}{=} \sup_{N \geqslant 1} N^s \|P_N f\|_{L_x^\infty(\mathbb{T}^3)}. \tag{3.1.16}$$

We then define the corresponding space $\mathcal{C}_x^s(\mathbb{T}^3)$ by

$$\mathcal{C}_x^s(\mathbb{T}^3) \overset{\text{def}}{=} \big\{ f \colon \mathbb{T}^3 \to \mathbb{R} \big| \ \|f\|_{\mathcal{C}_x^s} < \infty, \lim_{N \to \infty} N^s \|P_N f\|_{L_x^\infty(\mathbb{T}^3)} = 0 \big\}. \tag{3.1.17}$$

Due to the additional constraint as $N \to \infty$, the space $\mathcal{C}_x^s(\mathbb{T}^3)$ is separable. This allows us to later use Prokhorov's theorem for families of measures on $\mathcal{C}_x^s(\mathbb{T}^3)$. We also define

$$\begin{aligned} &\mathcal{C}_t^0 \mathcal{C}_x^s([0,\infty] \times \mathbb{T}^3) \\ &\overset{\text{def}}{=} \big\{ f \colon [0,\infty) \times \mathbb{T}^3 \to \mathbb{R} \big| \ \sup_{t \geqslant 0} \|f(t,\cdot)\|_{\mathcal{C}_x^s(\mathbb{T}^3)} < \infty, \lim_{t \to \infty} f(t,\cdot) \text{ exists in } \mathcal{C}_x^s(\mathbb{T}^3) \big\}. \end{aligned} \tag{3.1.18}$$

Similar as above, the additional restriction as $t \to \infty$ makes $\mathcal{C}_t^0 \mathcal{C}_x^s([0,\infty] \times \mathbb{T}^3)$ separable. As a measure of tightness in $\mathcal{C}_t^0 \mathcal{C}_x^s([0,\infty] \times \mathbb{T}^3)$, we define for any $0 < \alpha < 1$ and $\eta > 0$ the norm

$$\|f\|_{\mathcal{C}_t^{\alpha,\eta} \mathcal{C}_x^s([0,\infty] \times \mathbb{T}^3)} \overset{\text{def}}{=} \|f(0)\|_{\mathcal{C}_x^s(\mathbb{T}^3)} + \sup_{0 \leqslant t, t' \leqslant \infty} \bigg(\min(\langle t \rangle, \langle t' \rangle)^\eta \frac{\|f(t) - f(t')\|_{\mathcal{C}_x^s(\mathbb{T}^3)}}{1 \wedge |t - t'|^\alpha} \bigg). \tag{3.1.19}$$

For $1 \leqslant r \leqslant \infty$, we also define the Sobolev space $\mathbb{W}_x^{s,r}(\mathbb{T}^3)$ as the completion of $C_x^\infty(\mathbb{T}^3)$ with respect to

$$\|f\|_{\mathbb{W}_x^{s,r}} = \|N^s P_N f\|_{\ell_N^r L_x^r}.$$

We hope that the subscript x prevents any confusion with the stochastic objects in Section 3.2.2.

3.2 Stochastic objects

In this section, we introduce the stochastic control framework and describe several stochastic objects. While the reader with a background in singular SPDE and advanced stochastic calculus can think of this section as standard, much of this section may be new to a reader with a primary background in dispersive PDE. As a result, we include full details for most standard arguments but encourage the expert to skip the proofs.

3.2.1 Stochastic control perspective

We let $(B_t^n)_{n\in\mathbb{Z}^3\backslash\{0\}}$ be a sequence of standard complex Brownian motions such that $B_t^{-n} = \overline{B_t^n}$ and B_t^n, B_t^m are independent for $n \neq \pm m$. We let B_t^0 be a standard real-valued Brownian motion independent of $(B_t^n)_{n\in\mathbb{Z}^3\backslash\{0\}}$. Furthermore, we let $B_t(\cdot)$ be the Gaussian process with Fourier coefficients $(B_t^n)_{n\in\mathbb{Z}^3}$, i.e.,

$$B_t(x) \stackrel{\text{def}}{=} \sum_{n\in\mathbb{Z}^3} e^{i\langle n,x\rangle} B_t^n. \tag{3.2.1}$$

For every $t \geqslant 0$, the Gaussian process formally satisfies $\mathbb{E}[B_t(x)B_t(y)] = t\cdot\delta(x-y)$ and hence $B_t(\cdot)$ is a scalar multiple of spatial white noise. We also let $(\mathcal{F}_t)_{t\geqslant 0}$ be the filtration corresponding to the family of Gaussian processes $(B_t^n)_{t\geqslant 0}$. For future use, we denote the ambient probability space by $(\Omega, \mathcal{F}, \mathbb{P})$.

The Gaussian free field g, however, has covariance $(1-\Delta)^{-1}$. To this end, we now introduce the Gaussian process $W_t(x)$. For $\sigma_t(x)$ as in (3.1.13) and any $n \in \mathbb{Z}^3$, we define

$$W_t^n \stackrel{\text{def}}{=} \int_0^t \frac{\sigma_s(n)}{\langle n\rangle} \mathrm{d}B_s^n\ . \tag{3.2.2}$$

We note that W_t^n is a complex Gaussian random variable with variance $\rho_t^2(n)/\langle n\rangle^2$. We finally set

$$W_t(x) \stackrel{\text{def}}{=} \sum_{n\in\mathbb{Z}^3} e^{i\langle n,x\rangle} W_t^n. \tag{3.2.3}$$

It is easy to see for any $\kappa > 0$ that $W \in \mathcal{C}_t^0 \mathcal{C}_x^{-1/2-\kappa}([0,\infty] \times \mathbb{T}^3)$ almost surely. With a slight abuse of notation, we write $\mathrm{d}\mathbb{P}(W)$ for the integration with respect to the law of W under $\mathbb{P}$, i.e., we omit the pushforward by W, and we write W for the canonical process on $\mathcal{C}_t^0 \mathcal{C}_x^{-1/2-\kappa}([0,\infty] \times \mathbb{T}^3)$. Comparing W_t and B_t, we have changed the covariance from $t\,\mathrm{Id}$ to $\rho_t(\nabla)^2 (I-\Delta)^{-1}$. For any fixed $T \geqslant 0$, we have that

$$\mathrm{Law}_{\mathbb{P}}(W_T) = \mathrm{Law}_{\mathbb{P}}(\rho_T(\nabla) W_\infty). \tag{3.2.4}$$

We already emphasize, however, that the processes $t \mapsto W_t$ and $t \mapsto \rho_t(\nabla) W_\infty$ have different laws, since only the first process has independent increments. This difference will be important in the definition of $\widetilde{\mu}_T$ below. To simplify the notation, we also introduce the Fourier multiplier J_t, which is defined by

$$\widehat{J_t f}(n) \overset{\text{def}}{=} \frac{\sigma_t(n)}{\langle n \rangle} \widehat{f}(n), \tag{3.2.5}$$

Using this notation, we can represent the Gaussian process W_t through the stochastic integral

$$W_t = \int_0^t J_s \, \mathrm{d}B_s.$$

In a similar spirit, we define for any $u \colon [0,\infty) \times \mathbb{T}^3 \to \mathbb{R}$ the integral $I_t[u]$ by

$$I_t[u] \overset{\text{def}}{=} \int_0^t J_s u_s \mathrm{d}s. \tag{3.2.6}$$

We now recall the Boué-Dupuis formula [BD98], where our formulation closely follows [BG20b, BG20a]. We let $\mathbb{H}_a$ be the space of $\mathcal{F}_t$-progressively measurable processes $u \colon \Omega \times [0,\infty) \times \mathbb{T}^3 \to \mathbb{R}$ which $\mathbb{P}$-almost surely belong to $L^2_{t,x}([0,\infty) \times \mathbb{T}^3)$.

Theorem 3.2.1 (Boué-Dupuis formula)**.** Let $0 < T < \infty$, let $F : C_t([0,T], C_x^\infty(\mathbb{T}^3)) \to \mathbb{R}$ be a Borel measurable function, and let $1 < p, q < \infty$. Assume that

$$\frac{1}{p} + \frac{1}{q} = 1, \quad \mathbb{E}_{\mathbb{P}}\big[|F(W)|^p\big] < \infty, \quad \text{and} \quad \mathbb{E}_{\mathbb{P}}\big[e^{-qF(W)}\big] < \infty, \tag{3.2.7}$$

where we regard W as an element of $C_t([0,T], C_x^\infty(\mathbb{T}^3))$. Then,

$$-\log \mathbb{E}_{\mathbb{P}}\Big[e^{-F(W)}\Big] = \inf_{u \in \mathbb{H}_a} \mathbb{E}_{\mathbb{P}}\Big[F(W+I(u)) + \frac{1}{2}\int_0^T \|u_s\|^2_{L^2(\mathbb{T}^3)} \mathrm{d}s\Big]. \tag{3.2.8}$$

Remark 3.2.2. The optimization problem in (3.2.8) and, more generally, the change of perspective from W_∞ to the whole process $t \mapsto W_t$, is reminiscent of stochastic control theory.

Due to the frequency projection in the definition of J_t, we have that $W_t, I_t[u] \in C_t([0,T], C_x^\infty(\mathbb{T}^3))$. In our arguments below, the smoothness can be used to verify (3.2.7) through soft methods. Of course, a soft method cannot yield uniform bounds in T, which are one of the main goals of this section.

In the introduction, we discussed the Gibbs measure μ_N corresponding to the truncated dynamics induced by H_N, which has been defined in (3.1.10). In the spirit of the stochastic control approach, we now change our notation and use the parameter T to denote the truncation. Since the law of W_∞ under $\mathbb{P}$ is the same as the Gaussian free field g and $P_{\leqslant T} = \rho_T(\nabla)$, we obtain that

$$\mathrm{d}\mu_T(\phi) = \frac{1}{\mathcal{Z}^{T,\lambda}} \exp\Big(-:\mathcal{V}^{T,\lambda}(\rho_T(\nabla)\phi):\Big)\, \mathrm{d}\big((W_\infty)_\#\mathbb{P}\big)(\phi). \tag{3.2.9}$$

The renormalized potential energy $\mathcal{V}^{T,\lambda}$ is as in (3.3.2). We view μ_T as a measure on the space $\mathcal{C}_x^{-1/2-\kappa}(\mathbb{T}^3)$ for any fixed $\kappa > 0$. In order to utilize the Boué-Dupuis formula, we lift μ_T to a measure on $\mathcal{C}_t^0\mathcal{C}_x^{-1/2-\kappa}([0,\infty] \times \mathbb{T}^3)$.

Definition 3.2.3. We define the measure $\widetilde{\mu}_T$ on $\mathcal{C}_t^0\mathcal{C}_x^{-1/2-\kappa}([0,\infty] \times \mathbb{T}^3)$ by

$$\mathrm{d}\widetilde{\mu}_T(W) \overset{\text{def}}{=} \frac{1}{\mathcal{Z}^{T,\lambda}} \exp\big(-:\mathcal{V}^{T,\lambda}(\rho_T(\nabla)W_\infty):\big)\, \mathrm{d}\mathbb{P}(W). \tag{3.2.10}$$

The content of the next lemma explains the relationship between $\widetilde{\mu}_T$ and μ_T.

Lemma 3.2.4. The Gibbs measure μ_T is the pushforward of $\widetilde{\mu}_T$ under W_∞, i.e.,

$$\mu_T = (W_\infty)_\#\widetilde{\mu}_T. \tag{3.2.11}$$

Due to its central importance to the rest of the paper, we prove this basic identity.

Proof. For any measurable function $f\colon \mathcal{C}_x^{-\frac{1}{2}-\kappa}(\mathbb{T}^3) \to \mathbb{R}$, we have that

$$\begin{aligned}
\int f(\phi)\mathrm{d}\mu_T(\phi) &= \frac{1}{\mathcal{Z}^{T,\lambda}} \int f(\phi) \exp(- :\mathcal{V}^{T,\lambda}(\rho_T(\nabla)\phi):)\mathrm{d}\big((W_\infty)_\# \mathbb{P}\big)(\phi) \\
&= \frac{1}{\mathcal{Z}^{T,\lambda}} \int f(W_\infty) \exp(- :\mathcal{V}^{T,\lambda}(\rho_T(\nabla)W_\infty):)\mathrm{d}\mathbb{P}(W) \\
&= \int f(W_\infty)\mathrm{d}\widetilde{\mu}_T(W) \\
&= \int f(\phi)\mathrm{d}\big((W_\infty)_\# \widetilde{\mu}_T\big)(\phi).
\end{aligned}$$

This proves the desired identity (3.2.4). □

In [BG20b, BG20a], Barashkov and Gubinelli work with the lifted measure

$$\mathrm{d}\bar{\mu}_T(W) = \frac{1}{\mathcal{Z}^{T,\lambda}} \exp\big(- :\mathcal{V}^{T,\lambda}(W_T): \big)\, \mathrm{d}\mathbb{P}(W). \tag{3.2.12}$$

While W_T and $\rho_T(\nabla)W_\infty$ have the same distribution, the measures $\widetilde{\mu}_T$ and $\bar{\mu}_T$ do *not* coincide. Since this is an important difference between this paper and the earlier works [BG20b, BG20a], let us explain our motivation for working with $\widetilde{\mu}_T$ instead of $\bar{\mu}_T$. From a probabilistic stand-point, the measure $\bar{\mu}_T$ has better properties than $\widetilde{\mu}_T$. This is related to the independent increments of the process $t \mapsto W_t$ and we provide further comments in Remark 3.4.8 below. From a PDE perspective, however, $\bar{\mu}_T$ behaves much worse than $\widetilde{\mu}_T$. For the proof of global well-posedness and invariance in the second part of this series, it is essential that $\mu_T = (W_\infty)_\# \widetilde{\mu}_T$ is invariant under the Hamiltonian flow of (3.1.10). In contrast, the author is not aware of an explicit expression for the pushforward of $\bar{\mu}_T$ under W_∞. In particular, $(W_\infty)_\# \bar{\mu}_T$ is not directly related to μ_T and not necessarily invariant under the Hamiltonian flow of H_N. Alternatively, we could work with the pushforward of $\bar{\mu}_T$ under W_T. A similar calculation as in the proof of Lemma 3.2.4 shows that $(W_T)_\# \bar{\mu}_T = (\rho_T(\nabla))_\# \mu_T$. Unfortunately, $(\rho_T(\nabla))_\# \mu_T$ also does not seem to be invariant under a truncation of the nonlinear wave equation. To summarize, while the measure $\bar{\mu}_T$ has useful probabilistic properties, it lacks a direct relationship to the truncated dynamics and is ill-suited for our globalization and invariance arguments.

Since we rely on $\rho_T(\nabla)W_\infty$ in the definition of $\widetilde{\mu}_T$, the Gaussian process $t \mapsto \rho_T(\nabla)W_t$ will play an important role in the rest of this paper. As a result, we now deal with both values T and t simultaneously. In most arguments, T will remain fixed while we use Itô's formula and martingale properties in t. To simplify the notation, we now write

$$W_t^T \overset{\text{def}}{=} \rho_T(\nabla)W_t \qquad \text{and} \qquad W_t^{T,n} \overset{\text{def}}{=} \rho_T(n)W_t^n. \tag{3.2.13}$$

Since this will be convenient below, we also define

$$\rho_t^T(n) \overset{\text{def}}{=} \rho_T(n)\cdot\rho_t(n), \qquad \sigma_t^T(n) \overset{\text{def}}{=} \rho_T(n)\sigma_t(n), \qquad \text{and} \qquad J_t^T \overset{\text{def}}{=} \rho_T(\nabla)J_t. \tag{3.2.14}$$

Furthermore, we define the integral operator I_t^T by

$$I_t^T[u] = \rho_T(\nabla)I_t[u] = \int_0^t J_s^T u_s \mathrm{d}s. \tag{3.2.15}$$

3.2.2 Stochastic objects and renormalization

We now proceed with the construction and renormalization of several stochastic objects. Similar constructions are standard in the probability theory literature and a comprehensive and well-written introduction can be found in [GP18, MWX17, OT20b]. In order to make this section accessible to readers with a primary background in dispersive PDEs, however, we include full details. In a similar spirit, we follow a hands-on approach and mainly rely on Itô calculus. In Lemma 3.2.20, however, this approach becomes computationally infeasible and we also use multiple stochastic integrals (see [Nua06] or Section 3.6.1.2).

Lemma 3.2.5. Let S_N be the symmetric group on $\{1,\ldots,N\}$ and let $W_t^{T,n}$ be as in (3.2.13). Then, we have for all $n_1,n_2,n_3,n_4 \in \mathbb{Z}^3$ that

$$W_t^{T,n_1} = \int_0^t \mathrm{d}W_{t_1}^{T,n_1} \tag{3.2.16}$$

$$W_t^{T,n_1}W_t^{T,n_2} = \sum_{\pi\in S_2}\int_0^t\int_0^{t_1} \mathrm{d}W_{t_2}^{T,n_{\pi(2)}}\mathrm{d}W_{t_1}^{T,n_{\pi(1)}} + \delta_{n_1+n_2=0}\frac{\rho_t^T(n_1)^2}{\langle n_1\rangle^2}, \tag{3.2.17}$$

$$\begin{aligned} W_t^{T,n_1}W_t^{T,n_2}W_t^{T,n_3} &= \sum_{\pi\in S_3}\int_0^t\int_0^{t_1}\int_0^{t_2} \mathrm{d}W_{t_3}^{T,n_{\pi(3)}}\mathrm{d}W_{t_2}^{T,n_{\pi(2)}}\mathrm{d}W_{t_1}^{T,n_{\pi(1)}} \\ &+ \frac{1}{2}\sum_{\pi\in S_3}\delta_{n_{\pi(1)}+n_{\pi(2)}=0}\frac{\rho_t^T(n_{\pi(1)})^2}{\langle n_{\pi(1)}\rangle^2}W_t^{T,n_{\pi(3)}}, \end{aligned} \tag{3.2.18}$$

$$\begin{aligned} W_t^{T,n_1}W_t^{T,n_2}W_t^{T,n_3}W_t^{T,n_4} &= \sum_{\pi\in S_4}\int_0^t\int_0^{t_1}\int_0^{t_2}\int_0^{t_3} \mathrm{d}W_{t_4}^{T,n_{\pi(4)}}\mathrm{d}W_{t_3}^{T,n_{\pi(3)}}\mathrm{d}W_{t_2}^{T,n_{\pi(2)}}\mathrm{d}W_{t_1}^{T,n_{\pi(1)}} \\ &+ \frac{1}{4}\sum_{\pi\in S_4}\delta_{n_{\pi(1)}+n_{\pi(2)}=0}\frac{\rho_t^T(n_{\pi(1)})^2}{\langle n_{\pi(1)}\rangle^2}W_t^{T,n_{\pi(3)}}W_t^{T,n_{\pi(4)}} \\ &- \frac{1}{8}\sum_{\pi\in S_4}\delta_{n_{\pi(1)}+n_{\pi(2)}=n_{\pi(3)}+n_{\pi(4)}=0}\frac{\rho_t^T(n_{\pi(1)})^2}{\langle n_{\pi(1)}\rangle^2}\frac{\rho_t^T(n_{\pi(3)})^2}{\langle n_{\pi(3)}\rangle^2}. \end{aligned} \tag{3.2.19}$$

The integrals in (3.2.16)-(3.2.19) are iterated Itô integrals. This lemma is related to the product formula for multiple stochastic integrals, see e.g. [Nua06, Proposition 1.1.3].

Proof. The first equation (3.2.16) follows from the definition of the Itô derivative $\mathrm{d}W_t^n$.
The second equation (3.2.17) follows from Itô's product formula. Indeed, we have that

$$\begin{aligned} W_t^{T,n_1}W_t^{T,n_2} &= \int_0^t W_s^{T,n_2}\mathrm{d}W_s^{T,n_1} + \int_0^t W_s^{T,n_1}\mathrm{d}W_s^{T,n_2} + \int_0^t \mathrm{d}\langle W^{T,n_1},W^{T,n_2}\rangle_s \\ &= \int_0^t\Big(\int_0^s \mathrm{d}W_\tau^{T,n_2}\Big)\mathrm{d}W_s^{T,n_1} + \int_0^t\Big(\int_0^s \mathrm{d}W_\tau^{T,n_1}\Big)\mathrm{d}W_s^{T,n_2} + \delta_{n_1+n_2=0}\int_0^t \frac{\sigma_s^T(n_1)^2}{\langle n_1\rangle^2}\mathrm{d}s \\ &= \sum_{\pi\in S_2}\int_0^t\int_0^{t_1} \mathrm{d}W_{t_2}^{T,n_{\pi(2)}}\mathrm{d}W_{t_1}^{T,n_{\pi(1)}} + \delta_{n_1+n_2=0}\frac{\rho_t^T(n_1)^2}{\langle n_1\rangle^2}. \end{aligned}$$

The third equation (3.2.18) follows from Itô's formula and the second equation (3.2.17). Using

Itô's formula, we have that

$$\begin{aligned}
&W_t^{T,n_1}W_t^{T,n_2}W_t^{T,n_3}\\
&=\frac{1}{2}\sum_{\pi\in S_3}\int_0^t W_s^{T,n_{\pi(3)}}W_s^{T,n_{\pi(2)}}\mathrm{d}W_s^{T,n_{\pi(1)}}+\frac{1}{2}\sum_{\pi\in S_3}\int_0^t W_s^{T,n_{\pi(3)}}\mathrm{d}\langle W^{T,n_{\pi(2)}},W^{T,n_{\pi(1)}}\rangle_s.
\end{aligned}$$

The easiest way to keep track of the pre-factors throughout the proof is to compare the number of terms of each type and the cardinality of the symmetric group. In the formula above, we have three terms of each type and the cardinality $\#S_3=6$, so we need the pre-factor $1/2$. By inserting the second equation (3.2.17) and our expression for the cross-variation, we obtain

$$\begin{aligned}
&W_t^{T,n_1}W_t^{T,n_2}W_t^{T,n_3}\\
&=\sum_{\pi\in S_3}\int_0^t\int_0^{t_1}\int_0^{t_2}\mathrm{d}W_{t_3}^{T,n_{\pi(3)}}\mathrm{d}W_{t_2}^{T,n_{\pi(2)}}\mathrm{d}W_{t_1}^{T,n_{\pi(1)}}+\frac{1}{2}\sum_{\pi\in S_3}\delta_{n_{\pi(3)}+n_{\pi(2)}=0}\int_0^t\frac{\rho_s^T(n_{\pi(2)})^2}{\langle n_{\pi(2)}\rangle^2}\mathrm{d}W_s^{T,n_{\pi(1)}}\\
&+\frac{1}{2}\sum_{\pi\in S_3}\delta_{n_{\pi(1)}+n_{\pi(2)}=0}\int_0^t\frac{\sigma_s^T(n_{\pi(1)})^2}{\langle n_{\pi(1)}\rangle^2}W_s^{T,n_{\pi(3)}}\mathrm{d}s\\
&=\sum_{\pi\in S_3}\int_0^t\int_0^{t_1}\int_0^{t_2}\mathrm{d}W_{t_3}^{T,n_{\pi(3)}}\mathrm{d}W_{t_2}^{T,n_{\pi(2)}}\mathrm{d}W_{t_1}^{T,n_{\pi(1)}}\\
&+\frac{1}{2}\sum_{\pi\in S_3}\delta_{n_{\pi(1)}+n_{\pi(2)}=0}\int_0^t\left(\frac{\sigma_s^T(n_{\pi(1)})^2}{\langle n_{\pi(1)}\rangle^2}W_s^{T,n_{\pi(3)}}\mathrm{d}s+\frac{\rho_s^T(n_{\pi(1)})^2}{\langle n_{\pi(1)}\rangle^2}\mathrm{d}W_s^{T,n_{\pi(3)}}\right)\\
&=\sum_{\pi\in S_3}\int_0^t\int_0^{t_1}\int_0^{t_2}\mathrm{d}W_{t_3}^{T,n_{\pi(3)}}\mathrm{d}W_{t_2}^{T,n_{\pi(2)}}\mathrm{d}W_{t_1}^{T,n_{\pi(1)}}+\frac{1}{2}\sum_{\pi\in S_3}\delta_{n_{\pi(1)}+n_{\pi(2)}=0}\frac{\rho_t^T(n_{\pi(1)})^2}{\langle n_{\pi(1)}\rangle^2}W_t^{T,n_{\pi(3)}}.
\end{aligned}$$

For the second equality, we also used the permutation invariance of any sum over $\pi\in S_3$. This completes the proof of the third equation (3.2.18).

We now prove the fourth and final equation (3.2.19). The argument differs from the proof of the third equation only in its notational complexity. Using Itô's formula and the third equation

(3.2.18), we obtain that

$$
\begin{aligned}
&W_t^{T,n_1} W_t^{T,n_2} W_t^{T,n_3} W_t^{T,n_4} \\
&= \frac{1}{6} \sum_{\pi \in S_4} \int_0^t W_s^{T,n_{\pi(4)}} W_s^{T,n_{\pi(3)}} W_s^{T,n_{\pi(2)}} \mathrm{d}W_s^{T,n_{\pi(1)}} + \frac{1}{4} \sum_{\pi \in S_4} \int_0^t W_s^{T,n_{\pi(4)}} W_s^{T,n_{\pi(3)}} \mathrm{d}\langle W^{T,n_{\pi(2)}}, W^{T,n_{\pi(1)}} \rangle_s \\
&= \sum_{\pi \in S_4} \int_0^t \int_0^{t_1} \int_0^{t_2} \int_0^{t_3} \mathrm{d}W_{t_4}^{T,n_{\pi(4)}} \mathrm{d}W_{t_3}^{T,n_{\pi(3)}} \mathrm{d}W_{t_2}^{T,n_{\pi(2)}} \mathrm{d}W_{t_1}^{T,n_{\pi(1)}} \\
&+ \frac{1}{2} \sum_{\pi \in S_4} \frac{\delta_{n_{\pi(1)}+n_{\pi(2)}=0}}{\langle n_{\pi(1)} \rangle^2} \int_0^t \rho_s^T(n_{\pi(1)})^2 W_s^{T,n_{\pi(4)}} \mathrm{d}W_s^{T,n_{\pi(3)}} + \frac{1}{4} \sum_{\pi \in S_4} \int_0^t \sigma_s^T(n_{\pi(1)})^2 W_s^{T,n_{\pi(4)}} W_s^{T,n_{\pi(3)}} \mathrm{d}s \\
&= \sum_{\pi \in S_4} \int_0^t \int_0^{t_1} \int_0^{t_2} \int_0^{t_3} \mathrm{d}W_{t_4}^{T,n_{\pi(4)}} \mathrm{d}W_{t_3}^{T,n_{\pi(3)}} \mathrm{d}W_{t_2}^{T,n_{\pi(2)}} \mathrm{d}W_{t_1}^{T,n_{\pi(1)}} + \frac{1}{4} \sum_{\pi \in S_4} \Bigg[\frac{\delta_{n_{\pi(1)}+n_{\pi(2)}=0}}{\langle n_{\pi(1)} \rangle^2} \times \\
&\int_0^t \Big(\sigma_s^T(n_{\pi(1)})^2 W_s^{T,n_{\pi(4)}} W_s^{T,n_{\pi(3)}} \mathrm{d}s + \rho_s^T(n_{\pi(1)})^2 W_s^{T,n_{\pi(4)}} \mathrm{d}W_s^{T,n_{\pi(3)}} + \rho_s^T(n_{\pi(1)})^2 W_s^{T,n_{\pi(3)}} \mathrm{d}W_s^{T,n_{\pi(4)}} \Big) \Bigg].
\end{aligned}
$$

Using Itô's formula, we obtain that

$$
\begin{aligned}
&\int_0^t \Big(\sigma_s^T(n_{\pi(1)})^2 W_s^{T,n_{\pi(4)}} W_s^{T,n_{\pi(3)}} \mathrm{d}s + \rho_s^T(n_{\pi(1)})^2 W_s^{T,n_{\pi(4)}} \mathrm{d}W_s^{T,n_{\pi(3)}} + \rho_s^T(n_{\pi(1)})^2 W_s^{T,n_{\pi(3)}} \mathrm{d}W_s^{T,n_{\pi(4)}} \Big) \\
&= \rho_t^T(n_{\pi(1)})^2 W_t^{T,n_{\pi(3)}} W_t^{T,n_{\pi(4)}} - \delta_{n_{\pi(3)}+n_{\pi(4)}=0} \int_0^t \rho_s^T(n_{\pi(1)})^2 \frac{\sigma_s^T(n_{\pi(3)})^2}{\langle n_{\pi(3)} \rangle^2} \mathrm{d}s.
\end{aligned}
$$

The total contribution of the second summand is

$$
\begin{aligned}
&- \frac{1}{4} \sum_{\pi \in S_4} \frac{\delta_{n_{\pi(1)}+n_{\pi(2)}=n_{\pi(3)}+n_{\pi(4)}=0}}{\langle n_{\pi(1)} \rangle^2 \langle n_{\pi(3)} \rangle^2} \int_0^t \rho_s^T(n_{\pi(1)})^2 \sigma_s^T(n_{\pi(3)})^2 \mathrm{d}s \\
&= -\frac{1}{8} \sum_{\pi \in S_4} \frac{\delta_{n_{\pi(1)}+n_{\pi(2)}=n_{\pi(3)}+n_{\pi(4)}=0}}{\langle n_{\pi(1)} \rangle^2 \langle n_{\pi(3)} \rangle^2} \int_0^t \Big(\rho_s^T(n_{\pi(1)})^2 \sigma_s^T(n_{\pi(3)})^2 + \sigma_s^T(n_{\pi(1)})^2 \rho_s^T(n_{\pi(3)})^2 \Big) \mathrm{d}s \\
&= -\frac{1}{8} \sum_{\pi \in S_4} \delta_{n_{\pi(1)}+n_{\pi(2)}=n_{\pi(3)}+n_{\pi(4)}=0} \frac{\rho_t^T(n_{\pi(1)})^2}{\langle n_{\pi(1)} \rangle^2} \frac{\rho_t^T(n_{\pi(3)})^2}{\langle n_{\pi(3)} \rangle^2}.
\end{aligned}
$$

This completes the proof of the fourth equation (3.2.19). □

Definition 3.2.6 (Renormalization)**.** We define the renormalization constants $a_t^T, b_t^T \in \mathbb{R}$ and the multiplier $\mathcal{M}_t^T \colon L^2(\mathbb{T}^3) \to L^2(\mathbb{T}^3)$ by

$$
a_t^T \stackrel{\text{def}}{=} \sum_{n \in \mathbb{Z}^3} \frac{\rho_t^T(n)^2}{\langle n \rangle^2}, \qquad b_t^T \stackrel{\text{def}}{=} \sum_{n_1, n_2 \in \mathbb{Z}^3} \frac{\widehat{V}(n_1+n_2) \rho_t^T(n_1)^2 \rho_t^T(n_2)^2}{\langle n_1 \rangle^2 \langle n_2 \rangle^2}
$$

and

$$\widehat{\mathcal{M}_t^T f}(n) \overset{\text{def}}{=} \Big(\sum_{m\in\mathbb{Z}^3} \widehat{V}(n+m) \frac{\rho_t^T(m)^2}{\langle m\rangle^2} \Big) \widehat{f}(n).$$

Using this notation, we set

$$:f^2: \overset{def}{=} f^2 - a_t^T, \tag{3.2.20}$$

$$:(V*f^2)f: \overset{def}{=} (V*f^2)f - a_t^T \widehat{V}(0) f - 2\mathcal{M}_t^T f, \tag{3.2.21}$$

$$:(V*f^2)f^2: \overset{def}{=} (V*f^2)f^2 - a_t^T V*f^2 - a_t^T \widehat{V}(0) f^2 - 4(\mathcal{M}_t^T f) f + (a_t^T)^2 \widehat{V}(0) + 2b_t^T. \tag{3.2.22}$$

Remark 3.2.7. As is clear from the definition, the renormalized powers in (3.2.20), (3.2.21), and (3.2.22) depend on the regularization parameter t. This dependence will always be clear from context and we thus do not reflect it in our notation.

Definition 3.2.8 (Renormalization of the dynamics)**.** For any $N \geqslant 1$, we define

$$a_N \overset{\text{def}}{=} a_\infty^N = a_N^\infty, \quad b_N \overset{\text{def}}{=} b_\infty^N = b_N^\infty, \quad \text{and} \quad \mathcal{M}_N \overset{\text{def}}{=} \mathcal{M}_\infty^N = \mathcal{M}_N^\infty. \tag{3.2.23}$$

Throughout most of the paper, we will only work with the renormalization constants from Definition 3.2.6, which contain two finite parameters. The renormalization constants in Definition 3.2.8 will be more important in the second part of this series.

Proposition 3.2.9 (Stochastic integral representation of renormalized powers)**.** With n_{12}, n_{123}, and n_{1234} defined as in (3.1.12), we have that

$$:(W_t^T)^2: = 2 \sum_{n_1,n_2\in\mathbb{Z}^3} e^{i\langle n_{12},x\rangle} \int_0^t \int_0^{t_1} \mathrm{d}W_{t_2}^{T,n_2} \mathrm{d}W_{t_1}^{T,n_1} \tag{3.2.24}$$

$$:(V*(W_t^T)^2)W_t^T: = \sum_{\substack{n_1,n_2,n_3\in\mathbb{Z}^3\\ \pi\in S_3}} \widehat{V}(n_{\pi(1)}+n_{\pi(2)}) e^{i\langle n_{123},x\rangle} \int_0^t \int_0^{t_1} \int_0^{t_2} \mathrm{d}W_{t_3}^{T,n_3} \mathrm{d}W_{t_2}^{T,n_2} \mathrm{d}W_{t_1}^{T,n_1} \tag{3.2.25}$$

$$:(V * (W_t^T)^2)(W_t^T)^2: = \sum_{\substack{n_1,n_2,n_3,n_4 \in \mathbb{Z}^3 \\ \pi \in S_4}} \Big[\widehat{V}(n_{\pi(1)} + n_{\pi(2)}) e^{i\langle n_{1234}, x\rangle} \times \int_0^t \int_0^{t_1} \int_0^{t_2} \int_0^{t_3} \mathrm{d}W_{t_4}^{T,n_4} \mathrm{d}W_{t_3}^{T,n_3} \mathrm{d}W_{t_2}^{T,n_2} \mathrm{d}W_{t_1}^{T,n_1} \Big]. \tag{3.2.26}$$

Furthermore, it holds that

$$\int_{\mathbb{T}^3} :(V * (W_t^T)^2)(W_t^T)^2: \mathrm{d}x = 4 \int_0^t \int_{\mathbb{T}^3} :(V * (W_s^T)^2) W_s^T: \mathrm{d}W_s^T. \tag{3.2.27}$$

Remark 3.2.10. The "lower-order" terms in Definition 3.2.6 were chosen precisely to obtain the result in Proposition 3.2.9. The renormalized powers of W_t^T can be represented solely using iterated stochastic integrals, which have many desirable properties.

Proposition 3.2.9 essentially follows from Lemma 3.2.5, Definition 3.2.6, and a tedious calculation. For the sake of completeness, however, we provide full details.

Proof. We first prove (3.2.24). Using (3.2.17), we have that

$$\begin{aligned}
(W_t^T)^2 &= \sum_{n_1,n_2 \in \mathbb{Z}^3} e^{i\langle n_1+n_2, x\rangle} W_t^{T,n_1} W_t^{T,n_2} \\
&= \sum_{\pi \in S_2} \sum_{n_1,n_2 \in \mathbb{Z}^3} e^{i\langle n_1+n_2, x\rangle} \int_0^t \int_0^{t_1} \mathrm{d}W_{t_2}^{T,n_{\pi(2)}} \mathrm{d}W_{t_1}^{T,n_{\pi(1)}} + \sum_{n_1,n_2 \in \mathbb{Z}^3} \delta_{n_1+n_2=0} \frac{\rho_t^T(n_1)^2}{\langle n_1 \rangle^2} e^{i\langle n_1+n_2, x\rangle} \\
&= \sum_{\pi \in S_2} \sum_{n_1,n_2 \in \mathbb{Z}^3} e^{i\langle n_1+n_2, x\rangle} \int_0^t \int_0^{t_1} \mathrm{d}W_{t_2}^{T,n_{\pi(2)}} \mathrm{d}W_{t_1}^{T,n_{\pi(1)}} + a_t^T.
\end{aligned}$$

By subtracting a_t^T from both sides and symmetrizing, this leads to the desired identity.

We now turn to the proof of (3.2.25). From (3.2.18), we obtain that

$$
\begin{aligned}
(V*(W_t^T)^2)W_t^T &= \sum_{n_1,n_2,n_3\in\mathbb{Z}^3} \widehat{V}(n_1+n_2)e^{i\langle n_{123},x\rangle} W_t^{T,n_1}W_t^{T,n_2}W_t^{T,n_3} \\
&= \sum_{\pi\in S_3}\sum_{n_1,n_2,n_3\in\mathbb{Z}^3} \widehat{V}(n_1+n_2)e^{i\langle n_{123},x\rangle} \int_0^t\int_0^{t_1}\int_0^{t_2} \mathrm{d}W_{t_3}^{T,n_{\pi(3)}}\mathrm{d}W_{t_2}^{T,n_{\pi(2)}}\mathrm{d}W_{t_1}^{T,n_{\pi(1)}} \\
&+ \frac{1}{2}\sum_{\pi\in S_3}\sum_{n_1,n_2,n_3\in\mathbb{Z}^3} \widehat{V}(n_1+n_2)e^{i\langle n_{123},x\rangle}\delta_{n_{\pi(1)}+n_{\pi(2)}=0}\frac{\rho_t^T(n_{\pi(1)})^2}{\langle n_{\pi(1)}\rangle^2}W_t^{T,n_{\pi(3)}}, \\
&= \sum_{\pi\in S_3}\sum_{n_1,n_2,n_3\in\mathbb{Z}^3} \widehat{V}(n_1+n_2)e^{i\langle n_{123},x\rangle} \int_0^t\int_0^{t_1}\int_0^{t_2} \mathrm{d}W_{t_3}^{T,n_{\pi(3)}}\mathrm{d}W_{t_2}^{T,n_{\pi(2)}}\mathrm{d}W_{t_1}^{T,n_{\pi(1)}} \\
&+ \sum_{n_1,n_3\in\mathbb{Z}^3}\widehat{V}(0)e^{i\langle n_3,x\rangle}\frac{\rho_t^T(n_1)^2}{\langle n_1\rangle^2}W_t^{T,n_3} + 2\sum_{n_1,n_3\in\mathbb{Z}^3}\widehat{V}(n_1+n_3)e^{i\langle n_3,x\rangle}\frac{\rho_t^T(n_1)^2}{\langle n_1\rangle^2}W_t^{T,n_3} \\
&= \sum_{\pi\in S_3}\sum_{n_1,n_2,n_3\in\mathbb{Z}^3} \widehat{V}(n_1+n_2)e^{i\langle n_{123},x\rangle} \int_0^t\int_0^{t_1}\int_0^{t_2} \mathrm{d}W_{t_3}^{T,n_{\pi(3)}}\mathrm{d}W_{t_2}^{T,n_{\pi(2)}}\mathrm{d}W_{t_1}^{T,n_{\pi(1)}} \\
&+ a_t^T\widehat{V}(0)W_t^T + 2\mathcal{M}_t^T W_t^T.
\end{aligned}
$$

After symmetrizing and comparing with Definition 3.2.6, this leads to the desired identity. Next, we prove the identity (3.2.26). Using (3.2.19), we have that

$$
\begin{aligned}
&(V*(W_t^T)^2)(W_t^T)^2 \\
&= \sum_{n_1,n_2,n_3,n_4\in\mathbb{Z}^3} \widehat{V}(n_1+n_2)e^{i\langle n_{1234},x\rangle} W_t^{T,n_1}W_t^{T,n_2}W_t^{T,n_3}W_t^{T,n_4} \\
&= \sum_{\substack{n_1,n_2,n_3,n_4\in\mathbb{Z}^3\\ \pi\in S_4}} \widehat{V}(n_1+n_2)e^{i\langle n_{1234},x\rangle} \int_0^t\int_0^{t_1}\int_0^{t_2}\int_0^{t_3} \mathrm{d}W_{t_4}^{T,n_{\pi(4)}}\mathrm{d}W_{t_3}^{T,n_{\pi(3)}}\mathrm{d}W_{t_2}^{T,n_{\pi(2)}}\mathrm{d}W_{t_1}^{T,n_{\pi(1)}} \\
&+ \frac{1}{4}\sum_{\substack{n_1,n_2,n_3,n_4\in\mathbb{Z}^3\\ \pi\in S_4}} \widehat{V}(n_1+n_2)e^{i\langle n_{1234},x\rangle}\delta_{n_{\pi(1)}+n_{\pi(2)}=0}\frac{\rho_t^T(n_{\pi(1)})^2}{\langle n_{\pi(1)}\rangle^2}W_t^{T,n_{\pi(3)}}W_t^{T,n_{\pi(4)}} \qquad (3.2.28) \\
&- \frac{1}{8}\sum_{\substack{n_1,n_2,n_3,n_4\in\mathbb{Z}^3\\ \pi\in S_4}} \widehat{V}(n_1+n_2)e^{i\langle n_{1234},x\rangle}\delta_{n_{\pi(1)}+n_{\pi(2)}=n_{\pi(3)}+n_{\pi(4)}=0}\frac{\rho_t^T(n_{\pi(1)})^2}{\langle n_{\pi(1)}\rangle^2}\frac{\rho_t^T(n_{\pi(3)})^2}{\langle n_{\pi(3)}\rangle^2}. \qquad (3.2.29)
\end{aligned}
$$

It remains to simplify (3.2.28) and (3.2.29). Regarding (3.2.28), we have that

$$
\begin{aligned}
&\frac{1}{4} \sum_{\substack{n_1,n_2,n_3,n_4 \in \mathbb{Z}^3 \\ \pi \in S_4}} \widehat{V}(n_1+n_2) e^{i\langle n_{1234}, x\rangle} \delta_{n_{\pi(1)}+n_{\pi(2)}=0} \frac{\rho_t^T(n_{\pi(1)})^2}{\langle n_{\pi(1)}\rangle^2} W_t^{T,n_{\pi(3)}} W_t^{T,n_{\pi(4)}} \\
&= \sum_{n_1,n_2,n_3 \in \mathbb{Z}^3} \widehat{V}(n_1+n_2) \frac{\rho_t^T(n_3)^2}{\langle n_3\rangle^2} e^{i\langle n_1+n_2, x\rangle} W_t^{T,n_1} W_t^{T,n_2} \\
&+ 4 \sum_{n_1,n_2,n_3 \in \mathbb{Z}^3} \widehat{V}(n_1+n_2) \frac{\rho_t^T(n_2)^2}{\langle n_2\rangle^2} e^{i\langle n_1+n_3, x\rangle} W_t^{T,n_1} W_t^{T,n_3} \\
&+ \sum_{n_1,n_3,n_4 \in \mathbb{Z}^3} \widehat{V}(0) \frac{\rho_t^T(n_1)^2}{\langle n_1\rangle^2} e^{i\langle n_3+n_4, x\rangle} W_t^{T,n_3} W_t^{T,n_4} \\
&= a_t^T V * (W_t^T)^2 + 4(\mathcal{M}_t^T W_t^T) W_t^T + a_t^T \widehat{V}(0) (W_t^T)^2.
\end{aligned}
$$

Regarding (3.2.29), we note that

$$
\begin{aligned}
&-\frac{1}{8} \sum_{\pi \in S_4} \sum_{n_1,n_2,n_3,n_4 \in \mathbb{Z}^3} \widehat{V}(n_1+n_2) e^{i\langle n_{1234}, x\rangle} \delta_{n_{\pi(1)}+n_{\pi(2)}=n_{\pi(3)}+n_{\pi(4)}=0} \frac{\rho_t^T(n_{\pi(1)})^2}{\langle n_{\pi(1)}\rangle^2} \frac{\rho_t^T(n_{\pi(3)})^2}{\langle n_{\pi(3)}\rangle^2} \\
&= -\sum_{n_1,n_3 \in \mathbb{Z}^3} \widehat{V}(0) \frac{\rho_t^T(n_1)^2 \rho_t^T(n_3)^2}{\langle n_1\rangle^2 \langle n_3\rangle^2} - 2 \sum_{n_1,n_2 \in \mathbb{Z}^3} \frac{\widehat{V}(n_1+n_2) \rho_t^T(n_1)^2) \rho_t^T(n_2)^2}{\langle n_1\rangle^2 \langle n_2\rangle^2} \\
&= -\widehat{V}(0)(a_t^T)^2 - 2b_t^T.
\end{aligned}
$$

After symmetrizing, this completes the proof of (3.2.26).

Finally, it remains to prove (3.2.27). Since V is real-valued and even, we have that $\widehat{V}(n) = \overline{\widehat{V}(n)} = \widehat{V}(-n)$. As long as $n_{1234} = 0$, this implies

$$
\sum_{\pi \in S_4} \widehat{V}(n_{\pi(1)} + n_{\pi(2)}) = 4 \sum_{\pi \in S_3} \widehat{V}(n_{\pi(1)} + n_{\pi(2)}). \tag{3.2.30}
$$

Using (3.2.30), (3.2.27) follows after inserting (3.2.25) and (3.2.26) into the two sides of the identity.

□

Like the monomials and Hermite polynomials (further discussed below), the generalized and renormalized powers in Definition 3.2.6 satisfy a binomial formula.

Lemma 3.2.11 (Binomial formula)**.** For any $f \in H^1(\mathbb{T}^3)$, we have the binomial formulas

$$\begin{aligned}
&:(V * (W_t^T + f)^2)(W_t^T + f): \\
&=:(V * (W_t^T)^2)W_t^T: +(V* :(W_t^T)^2:)f + 2\big[(V * (W_t^T f))W_t^T - \mathcal{M}_t^T f\big] \\
&+ 2(V * (W_t^T f))f + (V * f^2)W_t^T + (V * f^2)f
\end{aligned} \tag{3.2.31}$$

and

$$\begin{aligned}
&\int_{\mathbb{T}^3} :(V * (W_t^T + f)^2)(W_t^T + f)^2: \mathrm{d}x \\
&= \int_{\mathbb{T}^3} :(V * (W_t^T)^2)(W_t^T)^2: \mathrm{d}x + 4\int_{\mathbb{T}^3} :(V * (W_t^T)^2)W_t^T: f\mathrm{d}x + 2\int_{\mathbb{T}^3} (V* :(W_t^T)^2:)f^2\mathrm{d}x \\
&+ 4\int_{\mathbb{T}^3} \Big[(V * (W_t^T f))W_t^T f - (\mathcal{M}_t^T f)f\Big]\mathrm{d}x + 4\int_{\mathbb{T}^3} (V * f^2)f\, W_t^T\mathrm{d}x + \int_{\mathbb{T}^3} (V * f^2)f^2\mathrm{d}x.
\end{aligned} \tag{3.2.32}$$

Remark 3.2.12. Overall, the terms in (3.2.32) obey better analytical estimates than their counterparts for the Φ^4_3-model in [BG20a]. However, their algebraic structure is more complicated. The most challenging term is

$$\int_{\mathbb{T}^3} \Big[(V * (W_t^T f))W_t^T f - (\mathcal{M}_t^T f)f\Big]\mathrm{d}x,$$

which requires a delicate random matrix estimate (Section 3.3.3).

Proof of Lemma 3.2.11: This follows from Definition 3.2.6 and the classical binomial formula. For the quartic binomial formula (3.2.32), we also used the self-adjointness of the convolution with V and the multiplier $\mathcal{M}_t^T$. □

While this is not reflected in our notation, it is clear from Definition 3.2.6 that the multiplier $\mathcal{M}_t^T$ depends linearly on the interaction potential V. In the proof of the random matrix estimate (Proposition 3.3.7), we will need to further decompose $\mathcal{M}_t^T$, both with respect to the interaction potential V and dyadic frequency blocks. We introduce the notation corresponding to this decomposition in the next definition.

Definition 3.2.13. We let $\mathcal{M}_t^T[V; N_1, N_2]$ be the Fourier multiplier corresponding to the symbol

$$n \mapsto \sum_{k \in \mathbb{Z}^3} \frac{\widehat{V}(n+k)}{\langle k \rangle^2} \chi_{N_1}(k) \chi_{N_2}(k) \rho_t^T(k)^2. \tag{3.2.33}$$

In the next definition, we define our last renormalization of a stochastic object.

Definition 3.2.14. We define the correlation function on $\mathbb{T}^3$ by

$$\mathfrak{C}_t^T[N_1, N_2](y) \overset{\text{def}}{=} \sum_{k \in \mathbb{Z}^3} \frac{\chi_{N_1}(k) \chi_{N_2}(k)}{\langle k \rangle^2} \rho_t^T(k)^2 e^{i\langle k, y \rangle}. \tag{3.2.34}$$

We further define

$$:(\tau_y P_{N_1} W_t^T) P_{N_2} W_t^T: (x) \overset{\text{def}}{=} (\tau_y P_{N_1} W_t^T)(x) P_{N_2} W_t^T(x) - \mathfrak{C}_t^T[N_1, N_2](y). \tag{3.2.35}$$

Here, τ_y denotes the translation operator $\tau_y f(x) = f(x-y)$.

The next lemma relates the multiplier and correlation function from Definition 3.2.13 and Definition 3.2.14, respectively.

Lemma 3.2.15 (Physical space representation of $\mathcal{M}_t^T$)**.** For any $f \in C_x^\infty(\mathbb{T}^3)$, we have that

$$\mathcal{M}_t^T[V; N_1, N_2] f = \big(\mathfrak{C}_t^T[N_1, N_2] V \big) * f. \tag{3.2.36}$$

Proof. By definition of the multiplier $\mathcal{M}_t^T[V; N_1, N_2]$ and since

$$k \mapsto \frac{1}{\langle k \rangle^2} \chi_{N_1}(k) \chi_{N_2}(k) \rho_t^T(k)^2 \tag{3.2.37}$$

is even, the symbol in (3.2.33) is the convolution of $\widehat{V}$ with (3.2.37). As a result, the sequence $n \mapsto \mathcal{M}_t^T[V; N_1, N_2](n)$ has the inverse Fourier transform is given by

$$\Big(\sum_{k \in \mathbb{Z}^3} \frac{\chi_{N_1}(k) \chi_{N_2}(k)}{\langle k \rangle^2} \rho_t^T(k)^2 e^{i\langle k, x \rangle} \Big) V(x) = \mathfrak{C}_t^T[N_1, N_2](x) V(x).$$

□

In Lemma 3.2.5, Proposition 3.2.9, Lemma 3.2.11, and Lemma 3.2.15, we have dealt with the algebraic structure of stochastic objects. We now move from algebraic aspects towards analytic estimates. In the following lemmas, we show that several stochastic objects are well-defined and study their regularities.

Lemma 3.2.16 (Stochastic objects I)**.** For every $p \geqslant 1$, $\epsilon > 0$, and every $0 < \gamma < \min(\beta, 1)$, we have that

$$\sup_{t\geqslant 0} \Big(\mathbb{E}\Big[\| :(W_t^T)^2: \|^p_{\mathcal{C}_x^{-1-\epsilon}(\mathbb{T}^3)} \Big] \Big)^{\frac{1}{p}} \lesssim p, \tag{3.2.38}$$

$$\sup_{t\geqslant 0} \Big(\mathbb{E}\Big[\| V * :(W_t^T)^2: \|^p_{\mathcal{C}_x^{-1+\beta-\epsilon}(\mathbb{T}^3)} \Big] \Big)^{\frac{1}{p}} \lesssim p, \tag{3.2.39}$$

$$\sup_{t\geqslant 0} \Big(\mathbb{E}\Big[\| :(V * (W_t^T)^2) W_t^T: \|^p_{\mathcal{C}_x^{-\frac{3}{2}+\gamma}(\mathbb{T}^3)} \Big] \Big)^{\frac{1}{p}} \lesssim p^{\frac{3}{2}}. \tag{3.2.40}$$

Furthermore, as $t \to \infty$ and/or $T \to \infty$, the stochastic objects $:(W_t^T)^2:$, $V * :(W_t^T)^2:$, and $:(V * (W_t^T)^2) W_t^T:$ converge in their respective spaces indicated by (3.2.38)-(3.2.40).

Remark 3.2.17. The statement and proof of Lemma 3.2.16 are standard and the respective regularities can be deduced by simple "power-counting". Nevertheless, we present the proof to familiarize the reader with our set-up and as a warm-up for Lemma 3.2.20 below.

Proof. The first step in the proofs of (3.2.38)-(3.2.40) is a reduction to an estimate in $L^2(\Omega \times \mathbb{T}^3)$ using Gaussian hypercontractivity. We provide the full details of this step for (3.2.38), but will omit similar details in the remaining estimates (3.2.39)-(3.2.40).

Let $N \geqslant 1$ and let $q = q(\epsilon) \geqslant 1$ be sufficiently large. By using Hölder's inequality in $\omega \in \Omega$, it suffices to prove the estimates for $p \geqslant q$. Using Bernstein's inequality and Minkowski's integral inequality, we obtain

$$\| P_N :(W_t^T)^2: \|_{L^p_\omega \mathcal{C}_x^{-1-\epsilon}(\Omega\times\mathbb{T}^3)} \lesssim N^{-1-\frac{\epsilon}{2}} \| P_N :(W_t^T)^2: \|_{L^p_\omega L^q_x(\Omega\times\mathbb{T}^3)} \leqslant N^{-1-\frac{\epsilon}{2}} \| P_N :(W_t^T)^2: \|_{L^q_x L^p_\omega(\mathbb{T}^3\times\Omega)}.$$

By Gaussian hypercontractivity (Lemma 3.6.1), we obtain that

$$N^{-1-\frac{\epsilon}{2}} \| P_N :(W_t^T)^2: \|_{L_x^q L_\omega^p(\mathbb{T}^3\times\Omega)} \lesssim N^{-1-\frac{\epsilon}{2}} p \| P_N :(W_t^T)^2: \|_{L_x^q L_\omega^2(\mathbb{T}^3\times\Omega)}.$$

Since the distribution of $:(W_t^T)^2:$ is translation invariant, the function $x \mapsto \| :(W_t^T)^2: \|_{L_\omega^2(\Omega)}$ is constant. We can then replace $L_x^q(\mathbb{T}^3)$ by $L_x^2(\mathbb{T}^3)$ and obtain

$$\begin{aligned} N^{-1-\frac{\epsilon}{2}} p \| P_N :(W_t^T)^2: \|_{L_x^q L_\omega^2(\mathbb{T}^3\times\Omega)} &\lesssim N^{-1-\frac{\epsilon}{2}} p \| P_N :(W_t^T)^2: \|_{L_x^2 L_\omega^2(\mathbb{T}^3\times\Omega)} \\ &\lesssim N^{-\frac{\epsilon}{4}} p \| :(W_t^T)^2: \|_{L_\omega^2 H_x^{-1-\frac{\epsilon}{4}}(\Omega\times\mathbb{T}^3)}. \end{aligned}$$

In order to prove (3.2.38), it therefore remains to show uniformly in $T, t \geqslant 0$ that

$$\| :(W_t^T)^2: \|^2_{L_\omega^2 H_x^{-1-\epsilon}(\Omega\times\mathbb{T}^3)} \lesssim 1. \tag{3.2.41}$$

Using Proposition 3.2.9, the orthogonality of the iterated stochastic integrals, and Itô's isometry, we have that

$$\begin{aligned} \| :(W_t^T)^2: \|^2_{L_\omega^2 H_x^{-1-\epsilon}} &= 4 \sum_{n\in\mathbb{Z}^3} \frac{1}{\langle n\rangle^{2+2\epsilon}} \, \mathbb{E}\Big[\Big(\sum_{\substack{n_1,n_2\in\mathbb{Z}^3: \\ n_1+n_2=n}} \int_0^t \int_0^{t_1} \mathrm{d}W_{t_2}^{T,n_2} \mathrm{d}W_{t_1}^{T,n_1} \Big)^2 \Big] \\ &\lesssim \sum_{\substack{n,n_1,n_2\in\mathbb{Z}^3 \\ n_1+n_2=n}} \frac{1}{\langle n\rangle^{2+2\epsilon} \langle n_1\rangle^2 \langle n_2\rangle^2} \rho_t^T(n_1)^2 \rho_t^T(n_2)^2 \\ &\lesssim \sum_{n_1,n_2\in\mathbb{Z}^3} \frac{1}{\langle n_1+n_2\rangle^{2+2\epsilon} \langle n_1\rangle^2 \langle n_2\rangle^2} \lesssim 1. \end{aligned}$$

This completes the proof of (3.2.38). The estimate (3.2.39) can be deduced from the smoothing properties of V or by repeating the exact same argument. It remains to prove (3.2.40), which can be reduced using hypercontractivity (and the room in γ) to the estimate

$$\| :(V * (W_t^T)^2) W_t^T: \|^2_{L_\omega^2 H_x^{-\frac{3}{2}+\gamma}} \lesssim 1.$$

Using Proposition 3.2.9, the orthogonality of the iterated stochastic integrals, and Itô's isometry, we have that

$$\begin{aligned}
&\| :(V * (W_t^T)^2) W_t^T: \|^2_{L^2_\omega H_x^{-\frac{3}{2}+\gamma}} \\
&= \sum_{n \in \mathbb{Z}^3} \frac{1}{\langle n \rangle^{3-2\gamma}} \, \mathbb{E}\Big[\Big(\sum_{\pi \in S_3} \sum_{\substack{n_1, n_2, n_3 \in \mathbb{Z}^3: \\ n_1+n_2+n_3 = n}} \widehat{V}(n_{\pi(1)} + n_{\pi(2)}) \int_0^t \int_0^{t_1} \int_0^{t_2} \mathrm{d}W_{t_3}^{T,n_3} \mathrm{d}W_{t_2}^{T,n_2} \mathrm{d}W_{t_1}^{T,n_1} \Big)^2 \Big] \\
&\lesssim \sum_{n_1, n_2, n_3 \in \mathbb{Z}^3} \frac{1}{\langle n_1 + n_2 + n_3 \rangle^{3-2\gamma}} \frac{1}{\langle n_1 + n_2 \rangle^{2\beta}} \frac{1}{\langle n_1 \rangle^2 \langle n_2 \rangle^2 \langle n_3 \rangle^2}.
\end{aligned}$$

By first summing in n_3, using that $3 - 2\gamma > 1$, and then in n_1 and n_2, using $\gamma < \beta$, we obtain

$$\begin{aligned}
&\sum_{n_1, n_2, n_3 \in \mathbb{Z}^3} \frac{1}{\langle n_1 + n_2 + n_3 \rangle^{3-2\gamma}} \frac{1}{\langle n_1 + n_2 \rangle^{2\beta}} \frac{1}{\langle n_1 \rangle^2 \langle n_2 \rangle^2 \langle n_3 \rangle^2} \\
&\lesssim \sum_{n_1, n_2 \in \mathbb{Z}^3} \frac{1}{\langle n_1 + n_2 \rangle^{2+2(\beta-\gamma)}} \frac{1}{\langle n_1 \rangle^2 \langle n_2 \rangle^2} \lesssim 1.
\end{aligned}$$

□

We also record the following refinement of (3.2.40) in Lemma 3.2.16, which will be needed in the proof of Lemma 3.2.20 below.

Corollary 3.2.18. For every $0 < \gamma < \min(1, \beta)$ and any $n \in \mathbb{Z}^3$, we can control the Fourier coefficients of $:(V * (W_t^T)^2) W_t^T:$ by

$$\sup_{T, t \geqslant 0} \mathbb{E}_{\mathbb{P}} \Big| \mathcal{F}\Big(:(V * (W_t^T)^2) W_t^T: \Big)(n) \Big|^2 \lesssim \langle n \rangle^{-2\gamma}. \tag{3.2.42}$$

Proof. Arguing as in the proof of Lemma 3.2.16, it suffices to prove that

$$\sum_{\substack{n_1, n_2, n_3 \in \mathbb{Z}^3: \\ n_{123} = n}} \frac{1}{\langle n_{12} \rangle^{2\beta} \langle n_1 \rangle^2 \langle n_2 \rangle^2 \langle n_3 \rangle^2} \lesssim \frac{1}{\langle n \rangle^{2\gamma}}. \tag{3.2.43}$$

Indeed, after parametrizing the sum by n_1 and n_3, (3.2.43) follows from

$$\begin{aligned}\sum_{\substack{n_1,n_2,n_3\in\mathbb{Z}^3:\\ n_{123}=n}} \frac{1}{\langle n_{12}\rangle^{2\beta}\langle n_1\rangle^2\langle n_2\rangle^2\langle n_3\rangle^2} &= \sum_{n_1,n_3\in\mathbb{Z}^3} \frac{1}{\langle n-n_3\rangle^{2\beta}\langle n_1\rangle^2\langle n-n_1-n_3\rangle^2\langle n_3\rangle^2}\\ &\lesssim \sum_{n_3\in\mathbb{Z}^3} \frac{1}{\langle n-n_3\rangle^{1+2\beta}\langle n_3\rangle^2}\\ &\lesssim \langle n\rangle^{-2\gamma}.\end{aligned}$$

□

Lemma 3.2.19 (Stochastic objects II)**.** For any sufficiently small $\delta > 0$ and any $N_1, N_2 \geqslant 1$, it holds that

$$\sup_{T,t\geqslant 0}\Big(\mathbb{E}\Big[\sup_{y\in\mathbb{T}^3}\| :(\tau_y P_{N_1}W_t^T)P_{N_2}W_t^T: \|^p_{\mathcal{C}_x^{-1-\delta}(\mathbb{T}^3)}\Big]\Big)^{\frac{1}{p}} \lesssim \max(N_1,N_2)^{-\frac{\delta}{10}}p. \tag{3.2.44}$$

Proof. Arguing as in the proof of (3.2.38) in Lemma 3.2.16, we have that

$$\sup_{y\in\mathbb{T}^3}\Big(\mathbb{E}\Big[\| :(\tau_y P_{N_1}W_t^T)P_{N_2}W_t^T: \|^p_{\mathcal{C}_x^{-1-\delta}(\mathbb{T}^3)}\Big]\Big)^{\frac{1}{p}} \lesssim \max(N_1,N_2)^{-\frac{\delta}{2}}p. \tag{3.2.45}$$

It only remains to move the supremum in $y \in \mathbb{T}^3$ into the expectation. From a crude estimate, we have for all $y, y' \in \mathbb{T}^3$ that

$$\Big(\mathbb{E}\Big[\| :(\tau_y P_{N_1}W_t^T)P_{N_2}W_t^T: - :(\tau_{y'} P_{N_1}W_t^T)P_{N_2}W_t^T: \|^p_{\mathcal{C}_x^{-1-\delta}(\mathbb{T}^3)}\Big]\Big)^{\frac{1}{p}} \lesssim \max(N_1,N_2)^3\|y-y'\|\,p.$$

By Kolmogorov's continuity theorem (cf. [Str11, Theorem 4.3.2]), we obtain for any $0 < \alpha < 1$ that

$$\Bigg(\mathbb{E}\Bigg[\sup_{y,y'\in\mathbb{T}^3}\Bigg(\frac{\| :(\tau_y P_{N_1}W_t^T)P_{N_2}W_t^T: - :(\tau_{y'} P_{N_1}W_t^T)P_{N_2}W_t^T: \|_{\mathcal{C}_x^{-1-\delta}(\mathbb{T}^3)}}{\|y-y'\|^\alpha}\Bigg)^p\Bigg]\Bigg)^{\frac{1}{p}} \lesssim_\alpha \max(N_1,N_2)^3 p.$$

Combining this with (3.2.45) leads to the desired estimate. □

The next lemma is similar to Lemma 3.2.16, but is concerned with more complicated stochastic objects. In order to shorten the argument, we will no longer use Itô's formula to express products of stochastic integrals. Instead, we will utilize the product formula for multiple stochastic integrals from [Nua06, Proposition 1.1.3]. Before we state the lemma, we follow [BG20b, BG20a] and define

$$\mathbb{W}_t^{T,[3]} \overset{\text{def}}{=} \int_0^t (J_s^T)^2 :(V * (W_s^T)^2) W_s^T: \mathrm{d}s. \tag{3.2.46}$$

We emphasize that $\mathbb{W}_t^{T,[3]}$ contains the interaction potential V even though this is not reflected in our notation.

Lemma 3.2.20 (Stochastic objects III)**.** For every $p \geqslant 1$, $\epsilon > 0$, and every $0 < \gamma < \min(\beta, \frac{1}{2})$, we have that

$$\sup_{T,t \geqslant 0} \Big(\mathbb{E}\Big[\| \mathbb{W}_t^{T,[3]} \|^p_{\mathcal{C}_x^{\frac{1}{2}+\gamma}(\mathbb{T}^3)} \Big] \Big)^{\frac{1}{p}} \lesssim p^{\frac{3}{2}}, \tag{3.2.47}$$

$$\sup_{T,t \geqslant 0} \Big(\mathbb{E}\Big[\| (V * :(W_t^T)^2:) \mathbb{W}_t^{T,[3]} \|^p_{\mathcal{C}_x^{-1+\gamma}(\mathbb{T}^3)} \Big] \Big)^{\frac{1}{p}} \lesssim p^{\frac{5}{2}}, \tag{3.2.48}$$

$$\sup_{T,t \geqslant 0} \Big(\mathbb{E}\Big[\big\| \big(V * (W_t^T \mathbb{W}_t^{T,[3]})\big) W_t^T - \mathcal{M}_t^T \mathbb{W}_t^{T,[3]} \|^p_{\mathcal{C}_x^{-1+\gamma}(\mathbb{T}^3)} \Big] \Big)^{\frac{1}{p}} \lesssim p^{\frac{5}{2}}. \tag{3.2.49}$$

Remark 3.2.21. The analog of $(V * :(W_t^T)^2:) \mathbb{W}_t^{T,[3]}$ for the Φ^4_3-model in [BG20b] requires a further logarithmic renormalization. In our case, however, the additional smoothing from the interaction potential V eliminates the responsible logarithmic divergence.

Proof. We first prove (3.2.47), which is (by far) the easiest estimate. As in the proof of Lemma 3.2.16, we can use Gaussian hypercontractivity (Lemma 3.6.1) to reduce (3.2.48) to the estimate

$$\mathbb{E}\Big[\| \mathbb{W}_t^{T,[3]} \|^2_{H_x^{\frac{1}{2}+\gamma}(\mathbb{T}^3)} \Big] \lesssim 1. \tag{3.2.50}$$

The rest of the argument follows from Corollary 3.2.18 and a deterministic estimate. More precisely, it follows from $\|\sigma_s^T\|_{L_s^2} = 1$ that

$$\begin{aligned}
\|\mathbb{W}_t^{T,[3]}\|^2_{H_x^{\frac{1}{2}+\gamma}(\mathbb{T}^3)} &= \Big\| \int_0^t \sigma_s^T(\nabla)^2 \langle \nabla \rangle^{-\frac{3}{2}+\gamma} :(V * (W_s^T)^2) W_s^T : \mathrm{d}s \Big\|^2_{L_x^2} \\
&= \sum_{n\in\mathbb{Z}^3} \Big| \int_0^t \sigma_s^T(n)^2 \mathcal{F}\Big(\langle \nabla \rangle^{-\frac{3}{2}+\gamma} :(V * (W_s^T)^2) W_s^T : \Big)(n) \mathrm{d}s \Big|^2 \\
&\leqslant \sum_{n\in\mathbb{Z}^3} \int_0^t \sigma_s^T(n)^2 \Big| \mathcal{F}\Big(\langle \nabla \rangle^{-\frac{3}{2}+\gamma} :(V * (W_s^T)^2) W_s^T : \Big)(n) \Big|^2 \mathrm{d}s.
\end{aligned}$$

For a small $\delta > 0$, we obtain from Corollary 3.2.18 (with γ replaced by $\gamma + \delta$) that

$$\begin{aligned}
\mathbb{E}\Big[\|\mathbb{W}_t^{T,[3]}\|^2_{H_x^{\frac{1}{2}+\gamma}(\mathbb{T}^3)} \Big] &\leqslant \sum_{n\in\mathbb{Z}^3} \int_0^t \sigma_s^T(n)^2 \mathbb{E}\Big[\Big| \mathcal{F}\Big(\langle \nabla \rangle^{-\frac{3}{2}+\gamma} :(V * (W_s^T)^2) W_s^T : \Big)(n) \Big|^2 \Big] \mathrm{d}s \\
&\lesssim \sum_{n\in\mathbb{Z}^3} \int_0^t \sigma_s^T(n)^2 \frac{1}{\langle n \rangle^{3+\delta}} \mathrm{d}s \lesssim 1.
\end{aligned}$$

We now turn to the proof of (3.2.48). Using the same reductions based on Gaussian hypercontractivity as before, it suffices to prove that

$$\mathbb{E}\Big[\|(V * :(W_t^T)^2:) \mathbb{W}_t^{T,[3]}\|^2_{H_x^{-1+\gamma}(\mathbb{T}^3)} \Big] \lesssim 1. \tag{3.2.51}$$

We first rewrite $(V * :(W_t^T)^2:)(x)\mathbb{W}_t^{T,[3]}(x)$ as a product of multiple stochastic integrals instead of iterated stochastic integrals. This allows us to use the product formula from Lemma 3.6.4, which leads to a (relatively) simple expression. To simplify the notation below, we define the symmetrization of $\widehat{V}(n_1 + n_2)$ by

$$\widehat{V}_S(n_1, n_2, n_3) = \frac{1}{6} \sum_{\pi \in S_3} \widehat{V}(n_{\pi(1)} + n_{\pi(2)}).$$

From Proposition 3.2.9, (3.2.46), and the stochastic Fubini theorem (see [DZ92, Theorem 4.33]), we have that

$$
\begin{aligned}
&\mathbb{W}_t^{T,[3]}(x)\\
&= \sum_{\substack{n_1,n_2,n_3\in\mathbb{Z}^3\\ \pi\in S_3}} \frac{\widehat{V}(n_{\pi(1)}+n_{\pi(2)})}{\langle n_{123}\rangle^2} e^{i\langle n_{123},x\rangle} \int_0^t \sigma_s^T(n_{123})^2 \Big(\int_0^s \int_0^{t_1} \int_0^{t_2} \mathrm{d}W_{t_3}^{T,n_3} \mathrm{d}W_{t_2}^{T,n_2} \mathrm{d}W_{t_1}^{T,n_1} \Big) \mathrm{d}s\\
&= \sum_{n_1,n_2,n_3\in\mathbb{Z}^3} \frac{\widehat{V}_S(n_1,n_2,n_3)}{\langle n_{123}\rangle^2} e^{i\langle n_{123},x\rangle} \int_0^t \int_0^{t_1} \int_0^{t_2} \Big(\int_{\max(t_1,t_2,t_3)}^t \sigma_s^T(n_{123})^2 \mathrm{d}s \Big) \mathrm{d}W_{t_3}^{T,n_3} \mathrm{d}W_{t_2}^{T,n_2} \mathrm{d}W_{t_1}^{T,n_1}
\end{aligned}
$$

We define the symmetric function f by

$$
f(t_1,n_1,t_2,n_2,t_3,n_3;t,x) \overset{\text{def}}{=} \frac{\widehat{V}_s(n_1,n_2,n_3)}{6\langle n_{123}\rangle^2} \Big(\int_{\max(t_1,t_2,t_3)}^t \sigma_s^T(n_{123})^2 \mathrm{d}s \Big) e^{i\langle n_{123},x\rangle} 1\{0 \leqslant t_1,t_2,t_3 \leqslant t\}.
$$

where we view both $t \in \mathbb{R}_{>0}$ and $x \in \mathbb{T}^3$ as fixed parameters. Using the language from Section 3.6.1.2 and Lemma 3.6.2, we obtain that

$$
\mathbb{W}_t^{T,[3]}(x) = \mathcal{I}_3[f(\cdot;t,x)], \tag{3.2.52}
$$

where $\mathcal{I}_3$ is a multiple stochastic integral. After defining

$$
g(t_4,n_4,t_5,n_5;t,x) \overset{\text{def}}{=} \widehat{V}(n_4+n_5) e^{i\langle n_{45},x\rangle} 1\{0 \leqslant t_4,t_5 \leqslant t\},
$$

a similar but easier calculation leads to

$$
(V * :(W_t^T)^2:)(x) = \mathcal{I}_2[g(\cdot;t,x)]. \tag{3.2.53}
$$

By combining (3.2.52) and (3.2.53), we obtain that

$$
(V * :(W_t^T)^2:)(x) \mathbb{W}_t^{T,[3]}(x) = \mathcal{I}_3[f(\cdot;t,x)] \mathcal{I}_2[g(\cdot;t,x)].
$$

By using the product formula for multiple stochastic integrals (Lemma 3.6.4), we obtain that

$$
\begin{aligned}
&(V * :(W_t^T)^2:)(x) \mathbb{W}_t^{T,[3]}(x)\\
=&\mathcal{I}_5[f(\cdot;t,x)g(\cdot;t,x)] + 6\cdot \mathcal{I}_3[f(\cdot;t,x) \otimes_1 g(\cdot;t,x)] + 3\cdot \mathcal{I}_1[f(\cdot;t,x) \otimes_2 g(\cdot;t,x)].
\end{aligned}
$$

Inserting the definitions of f and g, this leads to

$$(V * :(W_t^T)^2:)(x)\mathbb{W}_t^{T,[3]}(x) = \mathcal{G}_5(t,x) + \mathcal{G}_3(t,x) + \mathcal{G}_1(t,x), \tag{3.2.54}$$

where the Gaussian chaoses $\mathcal{G}_5, \mathcal{G}_3$, and $\mathcal{G}_1$ are given by

$$\begin{aligned}
\mathcal{G}_5(t,x) &= \sum_{n_1,\ldots,n_5\in\mathbb{Z}^3} \frac{\widehat{V}(n_{12})\widehat{V}(n_{45})}{\langle n_{123}\rangle^2} e^{i\langle n_{12345},x\rangle} \int_{[0,t]^5} \Big(\int_{\max(t_1,t_2,t_3)}^t \sigma_s^T(n_{123})\mathrm{d}s \Big) \mathrm{d}W_{t_5}^{T,n_5} \ldots \mathrm{d}W_{t_1}^{T,n_1}, \\
\mathcal{G}_3(t,x) &= \sum_{n_1,\ldots,n_5\in\mathbb{Z}^3} \Bigg[\delta_{n_{35}=0} \frac{\widehat{V}_s(n_1,n_2,n_3)\widehat{V}(n_{45})}{\langle n_{123}\rangle^2 \langle n_3\rangle^2} e^{i\langle n_{124},x\rangle} \\
&\quad \times \int_{[0,t]^3} \Big(\int_0^t \int_{\max(t_1,t_2,t_3)}^t \sigma_{t_3}^T(n_3)^2 \sigma_s^T(n_{123})^2 \mathrm{d}s\mathrm{d}t_3 \Big) \mathrm{d}W_{t_4}^{T,n_4} \mathrm{d}W_{t_2}^{T,n_2} \mathrm{d}W_{t_1}^{T,n_1} \Bigg], \\
\mathcal{G}_1(t,x) &= \frac{1}{2} \sum_{n_1,\ldots,n_5\in\mathbb{Z}^3} \Bigg[\delta_{n_{24}=n_{35}=0} \frac{\widehat{V}_s(n_1,n_2,n_3)\widehat{V}(n_{45})}{\langle n_{123}\rangle^2 \langle n_2\rangle^2 \langle n_3\rangle^2} e^{i\langle n_1,x\rangle} \\
&\quad \times \int_{[0,t]} \Big(\int_0^t \int_0^t \int_{\max(t_1,t_2,t_3)}^t \sigma_{t_2}^T(n_2)^2 \sigma_{t_3}^T(n_3)^2 \sigma_s^T(n_{123})^2 \mathrm{d}s\mathrm{d}t_3\mathrm{d}t_2 \Big) \mathrm{d}W_{t_1}^{T,n_1} \Bigg]
\end{aligned}$$

Using the L^2-orthogonality of the multiple stochastic integrals together with $\|\sigma_s^T\|_{L_s^2(\mathbb{R}_{>0})} \leqslant 1$, we obtain that

$$\begin{aligned}
&\mathbb{E}\Big[\|(V * :(W_t^T)^2:)\mathbb{W}_t^{T,[3]}\|_{H_x^{-1+\gamma}}^2 \Big] \\
&\lesssim \mathbb{E}\Big[\|\mathcal{G}_5\|_{H_x^{-1+\gamma}}^2 \Big] + \mathbb{E}\Big[\|\mathcal{G}_3\|_{H_x^{-1+\gamma}}^2 \Big] + \mathbb{E}\Big[\|\mathcal{G}_1\|_{H_x^{-1+\gamma}}^2 \Big] \\
&\lesssim \sum_{n_1,n_2,n_3,n_4,n_5\in\mathbb{Z}^3} \langle n_{12345}\rangle^{-2+2\gamma} \langle n_{123}\rangle^{-4} |\widehat{V}(n_{12})|^2 |\widehat{V}(n_{45})|^2 \prod_{j=1}^5 \langle n_j\rangle^{-2}, && (3.2.55) \\
&+ \sum_{n_1,n_2,n_4\in\mathbb{Z}^3} \langle n_{124}\rangle^{-2+2\gamma} \Big(\sum_{n_3\in\mathbb{Z}^3} \langle n_{123}\rangle^{-2} \langle n_3\rangle^{-2} |\widehat{V}_s(n_1,n_2,n_3)| |\widehat{V}(n_{34})| \Big)^2 \prod_{j=1,2,4} \langle n_j\rangle^{-2} && (3.2.56) \\
&+ \sum_{n_1\in\mathbb{Z}^3} \langle n_1\rangle^{-4+2\gamma} \Big(\sum_{n_2,n_3\in\mathbb{Z}^3} \langle n_{123}\rangle^{-2} |\widehat{V}_s(n_1,n_2,n_3)| |\widehat{V}(n_{23})| \langle n_2\rangle^{-2} \langle n_3\rangle^{-2} \Big)^2. && (3.2.57)
\end{aligned}$$

The estimates of the sums (3.2.55)-(3.2.57) follow from standard arguments. We present the details for (3.2.55) and (3.2.57), but omit the details for the intermediate term (3.2.56).

We start with the estimate of (3.2.55). The interaction with n_1, n_2, n_3 at low frequency scales and n_4, n_5 at high frequency scales is worse than all other contributions, so there is a lot of room in several steps below. Using Lemma 3.6.10 for the sum in n_5, which requires $\gamma < \min(1, \beta)$, and summing in n_4, we obtain for a small $\delta > 0$ that

$$\begin{aligned}
&\sum_{n_1,n_2,n_3,n_4,n_5 \in \mathbb{Z}^3} \langle n_{12345} \rangle^{-2+2\gamma} \langle n_{123} \rangle^{-4} |\widehat{V}(n_{12})|^2 |\widehat{V}(n_{45})|^2 \prod_{j=1}^{5} \langle n_j \rangle^{-2} \\
&\lesssim \sum_{n_1,n_2,n_3,n_4 \in \mathbb{Z}^3} \langle n_{123} \rangle^{-4} \langle n_{12} \rangle^{-2\beta} \Big(\prod_{j=1}^{4} \langle n_j \rangle^{-2} \Big) \Big(\sum_{n_5 \in \mathbb{Z}^3} \langle n_{1234} + n_5 \rangle^{-2+2\gamma} \langle n_4 + n_5 \rangle^{-2\beta} \langle n_5 \rangle^{-2} \Big) \\
&\lesssim \sum_{n_1,n_2,n_3 \in \mathbb{Z}^3} \langle n_{123} \rangle^{-4} \langle n_{12} \rangle^{-2\beta} \Big(\prod_{j=1}^{3} \langle n_j \rangle^{-2} \Big) \Big(\sum_{n_4 \in \mathbb{Z}^3} \big(\langle n_{1234} \rangle^{-1-\delta} + \langle n_4 \rangle^{-1-\delta} \big) \langle n_4 \rangle^{-2} \Big) \\
&\lesssim \sum_{n_1,n_2,n_3 \in \mathbb{Z}^3} \langle n_{123} \rangle^{-4} \langle n_{12} \rangle^{-2\beta} \prod_{j=1}^{3} \langle n_j \rangle^{-2}.
\end{aligned}$$

Summing in n_3, n_2, and n_1, we obtain that

$$\sum_{n_1,n_2,n_3 \in \mathbb{Z}^3} \langle n_{123} \rangle^{-4} \langle n_{12} \rangle^{-2\beta} \prod_{j=1}^{3} \langle n_j \rangle^{-2} \lesssim \sum_{n_1,n_2 \in \mathbb{Z}^3} \langle n_{12} \rangle^{-3-2\beta} \langle n_1 \rangle^{-2} \langle n_2 \rangle^{-2} \lesssim \sum_{n_1 \in \mathbb{Z}^3} \langle n_1 \rangle^{-4} \lesssim 1.$$

We now turn to (3.2.57), which corresponds to double probabilistic resonance. We emphasize that this term would be unbounded without smoothing effect of the potential V, which is the reason for the additional renormalization in the Φ^4_3-model, see e.g. [BG20b, Lemma 24]. Using Lemma 3.6.10 for the sum in n_3, we obtain that

$$\begin{aligned}
&\sum_{n_1 \in \mathbb{Z}^3} \langle n_1 \rangle^{-4+2\gamma} \Big(\sum_{n_2,n_3 \in \mathbb{Z}^3} \langle n_{123} \rangle^{-2} |\widehat{V}_s(n_1,n_2,n_3)| |\widehat{V}(n_{23})| \langle n_2 \rangle^{-2} \langle n_3 \rangle^{-2} \Big)^2 \\
&\lesssim \sum_{n_1 \in \mathbb{Z}^3} \langle n_1 \rangle^{-4+2\gamma} \Big(\sum_{n_2,n_3 \in \mathbb{Z}^3} \langle n_{123} \rangle^{-2} \langle n_{23} \rangle^{-\beta} \langle n_2 \rangle^{-2} \langle n_3 \rangle^{-2} \Big)^2 \\
&\lesssim \sum_{n_1 \in \mathbb{Z}^3} \langle n_1 \rangle^{-4+2\gamma} \Big(\sum_{n_2 \in \mathbb{Z}^3} \big(\langle n_{12} \rangle^{-1-\beta} + \langle n_2 \rangle^{-1-\beta} \big) \langle n_2 \rangle^{-2} \Big)^2 \\
&\lesssim \sum_{n_1 \in \mathbb{Z}^3} \langle n_1 \rangle^{-4+2\gamma} \lesssim 1,
\end{aligned}$$

provided that $\gamma < 1/2$. This completes the proof of (3.2.48).

We now turn to the proof of (3.2.49). This stochastic object has a more complicated algebraic structure than the stochastic object in (3.2.48), but a similar analytic behavior. From the definition of $\mathcal{M}_t^T$, we obtain that

$$
\begin{aligned}
&\Big(V * (W_t^T \mathbb{W}_t^{T,[3]})\Big)(x) W_t^T(x) - \mathcal{M}_t^T \mathbb{W}_t^{T,[3]}(x) \\
&= \sum_{m_1,m_4,m_5 \in \mathbb{Z}^3} \widehat{V}(m_{14}) e^{i\langle m_{145}, x\rangle} \widehat{\mathbb{W}_t^{T,[3]}}(m_1) \Big(W_t^{T,m_4} W_t^{T,m_5} - \delta_{m_{45}=0} \frac{\rho_t^T(m_4)^2}{\langle m_4 \rangle^2} \Big) \\
&= \frac{1}{2} \sum_{m_1,m_4,m_5 \in \mathbb{Z}^3} \big(\widehat{V}(m_{14}) + \widehat{V}(m_{15})\big) e^{i\langle m_{145}, x\rangle} \widehat{\mathbb{W}_t^{T,[3]}}(m_1) \Big(W_t^{T,m_4} W_t^{T,m_5} - \delta_{m_{45}=0} \frac{\rho_t^T(m_4)^2}{\langle m_4 \rangle^2} \Big).
\end{aligned}
$$

Using the variable names $m_1, m_4, m_5 \in \mathbb{Z}^3$ instead of $m_1, m_2, m_3 \in \mathbb{Z}^3$ is convenient once we insert an expression for $\mathbb{W}_t^{T,[3]}$. A minor modification of the derivation of (3.2.52) shows that

$$
\widehat{\mathbb{W}_t^{T,[3]}}(m_1) = \mathcal{I}[f(\cdot; t, m_1)], \tag{3.2.58}
$$

where the symmetric function $f(\cdot; t, m_1)$ is given by

$$
\begin{aligned}
&f(t_1, n_1, t_2, n_3, t_3, n_3; t, m_1) \\
&= 1\{n_{123} = m_1\} \frac{1}{\langle n_{123} \rangle^2} \widehat{V}_S(n_1, n_2, n_3) \Big(\int_{\max(t_1,t_2,t_3)}^t \sigma_s^T(n_{123})^2 \mathrm{d}s \Big) 1\{0 \leqslant t_1, t_2, t_3 \leqslant t\}.
\end{aligned}
$$

Using Lemma 3.2.5 and Lemma 3.6.2, we obtain that

$$
W_t^{T,m_4} W_t^{T,m_5} - \delta_{m_{45}=0} \frac{\rho_t^T(m_4)^2}{\langle m_4 \rangle^2} = \mathcal{I}_2[g(\cdot; t, m_4, m_5)], \tag{3.2.59}
$$

where the symmetric function $g(\cdot; t, m_4, m_5)$ is given by

$$
g(t_4, n_4, t_5, n_5) \overset{\text{def}}{=} \frac{1}{2} \Big(1\{(n_4, n_5) = (m_4, m_5)\} + 1\{(n_4, n_5) = (m_5, m_4)\} \Big) 1\{0 \leqslant t_4, t_5 \leqslant t\}.
$$

The author believes that inserting indicators such as $1\{(n_4,n_5)=(m_4,m_5)\}$ is notationally unpleasant, but it allows us to use the multiple stochastic integrals from [Nua06] without having to "reinvent the wheel". With this notation, we obtain that

$$\begin{aligned}&\big(V*(W_t^T\mathbb{W}_t^{T,[3]})\big)(x)W_t^T(x)-\mathcal{M}_t^T\mathbb{W}_t^{T,[3]}(x)\\&=\frac{1}{2}\sum_{m_1,m_4,m_5\in\mathbb{Z}^3}e^{i\langle m_{145},x\rangle}\big(\widehat{V}(m_{14})+\widehat{V}(m_{15})\big)\cdot\mathcal{I}_3[f(\cdot;t,m_1)]\cdot\mathcal{I}_2[g(\cdot;t,m_4,m_5)].\end{aligned}$$

Using Lemma 3.6.4, we obtain that

$$\big(V*(W_t^T\mathbb{W}_t^{T,[3]})\big)(x)W_t^T(x)-\mathcal{M}_t^T\mathbb{W}_t^{T,[3]}(x)=\widetilde{\mathcal{G}}_5(t,x)+\widetilde{\mathcal{G}}_3(t,x)+\widetilde{\mathcal{G}}_1(t,x),\tag{3.2.60}$$

where the Gaussian chaoses are defined as

$$\begin{aligned}\widetilde{\mathcal{G}}_5(t,x)&=\sum_{n_1,\dots,n_5\in\mathbb{Z}^3}\frac{\widehat{V}(n_{12})\widehat{V}(n_{1234})}{\langle n_{123}\rangle^2}e^{i\langle n_{12345},x\rangle}\int_{[0,t]^5}\Big(\int_{\max(t_1,t_2,t_3)}^t\sigma_s^T(n_{123})\mathrm{d}s\Big)\mathrm{d}W_{t_5}^{T,n_5}\dots\mathrm{d}W_{t_1}^{T,n_1},\\
\widetilde{\mathcal{G}}_3(t,x)&=\frac{1}{2}\sum_{n_1,\dots,n_5\in\mathbb{Z}^3}\Bigg[\delta_{n_{35}=0}\frac{\widehat{V}_s(n_1,n_2,n_3)}{\langle n_{123}\rangle^2\langle n_3\rangle^2}\Big(\widehat{V}(n_{12})+\widehat{V}(n_{1234})\Big)e^{i\langle n_{124},x\rangle}\\
&\quad\times\int_{[0,t]^3}\Big(\int_0^t\int_{\max(t_1,t_2,t_3)}^t\sigma_{t_3}^T(n_3)^2\sigma_s^T(n_{123})^2\mathrm{d}s\mathrm{d}t_3\Big)\mathrm{d}W_{t_4}^{T,n_4}\mathrm{d}W_{t_2}^{T,n_2}\mathrm{d}W_{t_1}^{T,n_1}\Bigg],\\
\widetilde{\mathcal{G}}_1(t,x)&=\frac{1}{4}\sum_{n_1,\dots,n_5\in\mathbb{Z}^3}\Bigg[\delta_{n_{24}=n_{35}=0}\frac{\widehat{V}_s(n_1,n_2,n_3)}{\langle n_{123}\rangle^2\langle n_2\rangle^2\langle n_3\rangle^2}\Big(\widehat{V}(n_{12})+\widehat{V}(n_{13})\Big)e^{i\langle n_1,x\rangle}\\
&\quad\times\int_{[0,t]}\Big(\int_0^t\int_0^t\int_{\max(t_1,t_2,t_3)}^t\sigma_{t_2}^T(n_2)^2\sigma_{t_3}^T(n_3)^2\sigma_s^T(n_{123})^2\mathrm{d}s\mathrm{d}t_3\mathrm{d}t_2\Big)\mathrm{d}W_{t_1}^{T,n_1}\Bigg].\end{aligned}$$

This concludes the algebraic aspects of the proof of (3.2.49). Starting from (3.2.60), the analytic estimates are essentially as in the proof of the earlier estimate (3.2.48) and we omit the details. This completes the proof of the lemma. □

In the construction of the drift measure (Section 3.4), we need a renormalization of $(\langle\nabla\rangle^{-1/2}W_t^T)^n$. The term $\langle\nabla\rangle^{-1/2}W_t^T$ has regularity $0-$ and hence the n-th power is almost defined. While we could use iterated stochastic integrals to define the renormalized power, it is notationally convenient to use an equivalent definition through Hermite polynomials. This definition is also closer to the earlier literature in dispersive PDE. We recall that the Hermite polynomials $\{H_n(x,\sigma^2)\}_{n\geqslant 0}$ are defined through the generating function

$$e^{tx-\frac{1}{2}\sigma^2t^2}=\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(x,\sigma^2).$$

Definition 3.2.22. We define the renormalized n-th power by

$$:f^n:\overset{\text{def}}{=}H_n\Big(f,\ \mathbb{E}\|\langle\nabla\rangle^{-\frac{1}{2}}W_t^T\|_{L_x^2}^2\Big). \tag{3.2.61}$$

We list two basic properties of the renormalized power in the next lemma.

Lemma 3.2.23 (Stochastic objects IV)**.** We have for all $n\geqslant 1$, $p\geqslant 1$, and $\epsilon>0$ that

$$\sup_{T\geqslant 0}\Big(\mathbb{E}\Big[\|\ :(\langle\nabla\rangle^{-\frac{1}{2}}W_t^T)^n:\ \|_{C_x^{-\epsilon}(\mathbb{T}^3)}^p\Big]\Big)^{\frac{1}{p}}\lesssim_{n,\epsilon}p^{\frac{n}{2}}. \tag{3.2.62}$$

Furthermore, we have for all $f\in H_x^1(\mathbb{T}^3)$ the binomial formula

$$:(\langle\nabla\rangle^{-\frac{1}{2}}(W_t^T+f))^n:=\sum_{k=0}^{n}\binom{n}{k}:(\langle\nabla\rangle^{-\frac{1}{2}}W_t^T)^k:\ (\langle\nabla\rangle^{-\frac{1}{2}}f)^{n-k}. \tag{3.2.63}$$

Since the proof is standard, we omit the details. For similar arguments, we refer the reader to [OT20b].

3.3 Construction of the Gibbs measure

The goal of this section is to prove Theorem 3.1.3. The main ingredient is the Boué-Dupuis formula, which yields a variational formulation of the Laplace transform of $\widetilde{\mu}_T$. Our argument follows earlier work of Barashkov and Gubinelli [BG20b], but the convolution inside the nonlinearity requires additional ingredients (see Section 3.3.2 and Section 3.3.3).

3.3.1 The variational problem, uniform bounds, and their consequences

Due to the singularity of the Gibbs measure for $0 < \beta < 1/2$, which is the main statement in Theorem 3.1.5, the construction will require one final renormalization. We recall that $\lambda > 0$ denotes the coupling constant in the nonlinearity and we let $c^{T,\lambda}$ be a real-valued constant which remains to be chosen.

For the rest of this section, we let $\varphi\colon \mathcal{C}_t^0\mathcal{C}_x^{-1/2-\kappa}([0,\infty] \times \mathbb{R}) \to \mathbb{R}$ be a functional with at most linear growth. We denote the (non-renormalized) potential energy by

$$\mathcal{V}(f) \overset{\text{def}}{=} \int_{\mathbb{T}^3} (V * f^2)(x) f^2(x) \mathrm{d}x = \int_{\mathbb{T}^3 \times \mathbb{T}^3} V(x-y) f(y)^2 f(x)^2 \mathrm{d}x \mathrm{d}y. \tag{3.3.1}$$

We denote the renormalized version of $\mathcal{V}(f)$ by

$$:\mathcal{V}^{T,\lambda}(f): \overset{def}{=} \frac{\lambda}{4} \cdot \int_{\mathbb{T}^3} :(V * f^2) f^2: \mathrm{d}x + c^{T,\lambda}, \tag{3.3.2}$$

where $:(V * f^2) f^2:$ is as in Definition 3.2.6. To further simplify the notation, we denote for any $u\colon [0,\infty) \times \mathbb{T}^3 \to \mathbb{R}$ the space-time L^2-norm by

$$\|u\|_{L^2_{t,x}}^2 \overset{\text{def}}{=} \int_0^\infty \|u_t\|_{L^2_x(\mathbb{T}^3)}^2 \mathrm{d}t. \tag{3.3.3}$$

With this notation, we can now state the main estimate of this section.

Proposition 3.3.1 (Main estimate for the variational problem)**.** If the renormalization constants $c^{T,\lambda}$ are chosen appropriately, we have that

$$\begin{aligned} &\mathbb{E}_{\mathbb{P}}\Big[\varphi(W + I[u]) + :\mathcal{V}^{T,\lambda}(W_\infty^T + I_\infty^T[u]): + \frac{1}{2}\|u\|_{L^2_{t,x}}^2\Big] \\ &= \mathbb{E}_{\mathbb{P}}\Big[\Psi_\lambda^{T,\varphi}(W, I[u]) + \frac{\lambda}{4}\mathcal{V}(I_\infty^T(u)) + \frac{1}{2}\|l^T[u]\|_{L^2_{t,x}}^2\Big], \end{aligned} \tag{3.3.4}$$

where

$$l_t^T[u] \overset{def}{=} u_t + \lambda J_t^T :(V * (W_t^T)^2) W_t^T: \tag{3.3.5}$$

and

$$|\Psi^{T,\varphi}_\lambda(W,I[u])| \leqslant Q_T(W,\varphi,\lambda) + \frac{1}{2}\Big(\frac{\lambda}{4}\mathcal{V}(I^T_\infty(u)) + \frac{1}{2}\|l^T[u]\|^2_{L^2_{t,x}}\Big). \tag{3.3.6}$$

Here, $Q_T(W,\varphi,\lambda)$ satisfies for all $p \geqslant 1$ the estimate $\mathbb{E}[Q_T(W,\varphi,\lambda)^p] \lesssim_p 1$, where the implicit constant is uniform in $T \geqslant 1$.

The argument of φ in (3.3.4) is not regularized, that is, we are working with W instead of W^T. This is important to obtain control over μ_T, which is the pushforward of $\widetilde{\mu}_T$ under W_∞.

Remark 3.3.2. This is a close analog of [BG20b, Theorem 1]. Due to the smoothing effect of the interaction potential V, however, the shifted drift $l^T[u]$ is simpler. In contrast to the Φ^4_3-model, the difference $l^T(u) - u$ does not depend on u. As is evident from the proof, we have that

$$\Psi^{T,\varphi}_\lambda(W,I[u]) = \varphi(W + I[u]) + \Psi^{T,0}_\lambda(W,I[u]). \tag{3.3.7}$$

This observation will only be needed in Proposition 3.3.3 below.

We first record the following proposition, which is a direct consequence of Proposition 3.3.1 and the Boué-Dupuis formula.

Proposition 3.3.3. The measures $\widetilde{\mu}_T$ satisfy the following properties:

(i) The normalization constants $\mathcal{Z}^{T,\lambda}$ satisfy $\mathcal{Z}^{T,\lambda} \sim_\lambda 1$, i.e., they are bounded away from zero and infinity uniformly in T.

(ii) If the functional $\varphi\colon \mathcal{C}^0_t\mathcal{C}^{-1/2-\kappa}_x([0,\infty]\times\mathbb{T}^3) \to \mathbb{R}$ has at most linear growth, then

$$\sup_{T\geqslant 0}\mathbb{E}_{\widetilde{\mu}_T}\Big[\exp\big(-\varphi(W)\big)\Big] \lesssim_\varphi 1.$$

(iii) The family of measures $(\widetilde{\mu}_T)_{T\geqslant 0}$ is tight on $\mathcal{C}^0_t\mathcal{C}^{-\frac{1}{2}-\kappa}_x([0,\infty]\times\mathbb{T}^3)$.

Proof of Proposition 3.3.3: We first prove (i). From the definition of μ_T, we have that

$$\mathcal{Z}^{T,\lambda} = \mathbb{E}_{\mathbb{P}}\Big[\exp(- :\mathcal{V}^{T,\lambda}(W_\infty^T):) \Big].$$

Using the Boué-Dupuis formula and Proposition 3.3.1, we have that

$$\begin{aligned} -\log(\mathcal{Z}^{T,\lambda}) &= \inf_{u\in\mathbb{H}_a} \mathbb{E}_{\mathbb{P}}\Big[:\mathcal{V}^{T,\lambda}(W_\infty^T + I_\infty^T[u]): + \frac{1}{2}\|u\|_{L^2_{t,x}}^2 \Big] \\ &= \inf_{u\in\mathbb{H}_a} \mathbb{E}_{\mathbb{P}}\Big[\Psi_\lambda^{T,0}(W, I[u]) + \frac{\lambda}{4}\mathcal{V}(I_\infty^T(u)) + \frac{1}{2}\|l^T[u]\|_{L^2_{t,x}}^2 \Big]. \end{aligned}$$

From (3.3.6), we directly obtain that

$$-\log(\mathcal{Z}^{T,\lambda}) \geqslant -C_\lambda. \tag{3.3.8}$$

By choosing $u_t \overset{\text{def}}{=} -\lambda J_t^T :(V * (W_t^T)^2)W_t^T:$, which is equivalent to requiring $l_t^T[u] = 0$ and implies $I_t^T[u] = \mathbb{W}_t^{T,[3]}$, we obtain from Lemma 3.2.20 that

$$-\log(\mathcal{Z}^{T,\lambda}) \lesssim_\lambda 1 + \mathbb{E}_{\mathbb{P}}\Big[\mathcal{V}(\lambda \mathbb{W}_t^{T,[3]})\Big] \lesssim_\lambda 1. \tag{3.3.9}$$

By combining (3.3.8) and (3.3.9), we obtain that $\mathcal{Z}^{T,\lambda} \sim_\lambda 1$.

We now turn to (ii), which controls the Laplace transform of $\widetilde{\mu}_T$. Using the Boué-Dupuis formula and Proposition 3.3.1, we obtain that

$$-\log\Big(\mathbb{E}_{\widetilde{\mu}_T}\Big[\exp\big(-\varphi(W)\big)\Big]\Big) = \log(\mathcal{Z}^{T,\lambda}) + \inf_{u\in\mathbb{H}_a} \mathbb{E}_{\mathbb{P}}\Big[\Psi_\lambda^{T,\varphi}(W, I[u]) + \frac{\lambda}{4}\mathcal{V}(I_\infty^T(u)) + \frac{1}{2}\|l^T[u]\|_{L^2_{t,x}}^2 \Big].$$

The first summand $\log(\mathcal{Z}^{T,\lambda})$ has already been controlled. The second summand can be controlled using exactly the same estimates.

We finally prove (iii). Let $\alpha, \eta > 0$ be sufficiently small depending on κ. Since the embedding $\mathcal{C}_t^{\alpha,\eta}\mathcal{C}_x^{-\frac{1+\kappa}{2}} \hookrightarrow \mathcal{C}_t^0\mathcal{C}_x^{-\frac{1}{2}-\kappa}$ is compact (see (3.1.19) for the definition), it suffices to estimate the Laplace transform evaluated at

$$\varphi(W) = -\|W\|_{\mathcal{C}_t^{\alpha,\eta}\mathcal{C}_x^{-\frac{1+\kappa}{2}}}. \tag{3.3.10}$$

While this is not a functional on $\mathcal{C}_t^0\mathcal{C}_x^{-\frac{1}{2}-\kappa}$, we can proceed using a minor modification of the previous estimates. Using Proposition 3.3.1 and (3.3.7), it suffices to prove

$$\mathbb{E}_{\mathbb{P}}\Big[\|W\|_{\mathcal{C}_t^{\alpha,\eta}\mathcal{C}_x^{-\frac{1+\kappa}{2}}}\Big] \lesssim 1 \quad \text{and} \quad \|I_t[u]\|_{\mathcal{C}_t^{\alpha,\eta}\mathcal{C}_x^{-\frac{1+\kappa}{2}}} \lesssim \|u\|_{L^2_{t,x}}. \tag{3.3.11}$$

The first estimate follows from Kolmogorov's continuity theorem (cf. [Str11, Theorem 4.3.2]). The second estimate is deterministic and follows from Sobolev embedding and Lemma 3.6.8. □

Using Proposition 3.3.3, we easily obtain Theorem 3.1.3.

Proof of Theorem 3.1.3: The tightness is included in Proposition 3.3.3. The weak convergence of the sequence $(\mu_N)_{N\geqslant 1}$ follows from tightness and the uniqueness of weak subsequential limits (Proposition 3.6.12). □

We also record the following consequence of the proof of Proposition 3.3.1, which will play an important role in Section 3.5. The proof of this result will be postponed until Section 3.3.4.

Corollary 3.3.4 (Behavior of $c^{T,\lambda}$)**.** If $\beta > 1/2$, then we have for all $\lambda > 0$ that

$$\sup_{T\geqslant 1} |c^{T,\lambda}| \lesssim_\lambda 1. \tag{3.3.12}$$

Proposition 3.3.1 is the most challenging part in the construction of the measure and the proof will be distributed over the remainder of this subsection.

3.3.2 Visan's estimate and the cubic terms

In the variational problem, the potential energy $\mathcal{V}(I_\infty^T[u])$ appears with a favorable sign. This is crucial to control the terms in $:\mathcal{V}^{T,\lambda}(W_\infty^T + I_\infty^T[u]):$ which are cubic in $I_\infty^T[u]$ and hence cannot be controlled by the quadratic terms $\|u\|_{L^2}^2$ or $\|l^T(u)\|_{L^2}^2$. In the Φ_3^4-model, the potential energy term

$\|I_\infty^T[u]\|_{L^4}^4$ is both stronger and easier to handle. While we cannot change the strength of $\mathcal{V}(I_\infty^T[u])$, Lemma 3.3.5 solves the algebraic difficulties.

Due to the assumed lower-bound on V, we first note that

$$\|f\|_{L_x^2(\mathbb{T}^3)}^4 = \|f^2\|_{L_x^1(\mathbb{T}^3)}^2 \lesssim \int_{\mathbb{T}^3\times\mathbb{T}^3} V(x-y)f(y)^2 f(x)^2 \mathrm{d}x\mathrm{d}y = \mathcal{V}(f).$$

Since at high-frequencies the kernel of $\langle\nabla\rangle^{-\beta}$ essentially behaves like $|x-y|^{-(3-\beta)}$, we also obtain that

$$\|\langle\nabla\rangle^{-\frac{\beta}{2}}[f^2]\|_{L^2(\mathbb{T}^3)}^2 = \langle\big(\langle\nabla\rangle^{-\beta}f^2\big), f^2\rangle_{L_x^2(\mathbb{T}^3)} \lesssim \int_{\mathbb{T}^3\times\mathbb{T}^3} V(x-y)f(y)^2 f(x)^2 \mathrm{d}x\mathrm{d}y = \mathcal{V}(f). \quad (3.3.13)$$

Unfortunately, the square of f is inside the integral operator $\langle\nabla\rangle^{-\frac{\beta}{2}}$, which makes it difficult to use this estimate. The next lemma yields a much more useful lower bound on $\mathcal{V}(f)$.

Lemma 3.3.5 (Visan's estimate)**.** Let $0 < \beta < 3$ and $f \in C^\infty(\mathbb{T}^3)$. Then, it holds that

$$\|\langle\nabla\rangle^{-\frac{\beta}{4}}f\|_{L_x^4(\mathbb{T}^3)}^4 \lesssim \mathcal{V}(f). \quad (3.3.14)$$

This estimate is a minor modification of [Vis07, (5.17)] and we omit the details. We now turn to the primary application of Visan's estimate in this work.

Lemma 3.3.6 (Cubic estimate)**.** For any small $\delta > 0$ and any $\frac{1+2\delta}{2} < \theta \leqslant 1$, it holds that

$$\Big\|\langle\nabla\rangle^{\frac{1}{2}+\delta}\Big((V * f^2)f\Big)\Big\|_{L_x^1(\mathbb{T}^3)} \lesssim \mathcal{V}(f)^{\frac{1}{2}}\|f\|_{L_x^2(\mathbb{T}^3)}^{1-\theta}\|f\|_{H_x^1(\mathbb{T}^3)}^{\theta}. \quad (3.3.15)$$

Proof. We use a Littlewood-Paley decomposition to write

$$(V * f^2)f = \sum_{M,N_3} P_M\big(V * f^2\big)\cdot P_{N_3}f.$$

We first estimate the contribution for $N_3 \gtrsim M$. We have that

$$\begin{aligned}
&\sum_{M,N_3:\, N_3\gtrsim M} \|\langle\nabla\rangle^{\frac{1}{2}+\delta}\Big(P_M\big(V*f^2\big)\cdot P_{N_3}f\Big)\|_{L^1_x} \\
&\lesssim \sum_{M,N_3:\, N_3\gtrsim M} N_3^{\frac{1}{2}+\delta}\|P_M(V*f^2)\|_{L^2_x}\|P_{N_3}f\|_{L^2_x} \\
&\lesssim \Big(\sum_{M,N_3:\, N_3\gtrsim M} N_3^{\frac{1}{2}+\delta}M^{-\frac{\beta}{2}}N_3^{-\theta}\Big)\|\langle\nabla\rangle^{-\frac{\beta}{2}}f^2\|_{L^2_x}\|f\|_{L^2_x}^{1-\theta}\|f\|_{H^1_x}^{\theta} \\
&\lesssim \|\langle\nabla\rangle^{-\frac{\beta}{2}}f^2\|_{L^2_x}\|f\|_{L^2_x}^{1-\theta}\|f\|_{H^1_x}^{\theta}.
\end{aligned}$$

Due to (3.3.13), this contribution is acceptable. Next, we estimate the contribution of $N_3 \lesssim M$. We further decompose

$$f^2 = \sum_{N_1,N_2} P_{N_1}f\cdot P_{N_2}f.$$

Then, the total contribution can be bounded using Hölder's inequality and Fourier support considerations by

$$\begin{aligned}
&\sum_{\substack{N_1,N_2,N_3,M:\\ N_3\lesssim M\leqslant\max(N_1,N_2)}} \Big\|\langle\nabla\rangle^{\frac{1}{2}+\delta}\Big(P_M\big(V*(P_{N_1}f\cdot P_{N_2}f)\big)\cdot P_{N_3}f\Big)\Big\|_{L^1_x} \\
&\lesssim \sum_{\substack{N_1,N_2,N_3,M:\\ N_3\lesssim M\leqslant\max(N_1,N_2)}} M^{\frac{1}{2}+\delta}\|P_M\big(V*(P_{N_1}f\cdot P_{N_2}f)\big)\|_{L_x^{\frac{4}{3}}}\|P_{N_3}f\|_{L^4_x} \\
&\lesssim \sum_{\substack{N_1,N_2,M:\\ N_3\lesssim M\leqslant\max(N_1,N_2)}} M^{\frac{1}{2}+\delta-\beta}N_3^{\frac{\beta}{4}}\|P_{N_1}f\cdot P_{N_2}f\|_{L_x^{\frac{4}{3}}}\|P_{N_3}\langle\nabla\rangle^{-\frac{\beta}{4}}f\|_{L^4_x} \\
&\lesssim \Big(\sum_{\substack{N_1,N_2,M:\\ N_1\geqslant M,N_2}} M^{\frac{1}{2}+\delta-\frac{3\beta}{4}}N_1^{-\theta}N_2^{\frac{\beta}{4}}\Big)\|\langle\nabla\rangle^{-\frac{\beta}{4}}f\|_{L^4_x}^2\|f\|_{L^2_x}^{1-\theta}\|f\|_{H^1_x}^{\theta} \\
&\lesssim \|\langle\nabla\rangle^{-\frac{\beta}{4}}f\|_{L^4_x}^2\|f\|_{L^2_x}^{1-\theta}\|f\|_{H^1_x}^{\theta}.
\end{aligned}$$

In the last line, it is simplest to first perform the sum in N_2, then in N_1, and finally in M. □

3.3.3 A random matrix estimate and the quadratic terms

In the proof of Proposition 3.3.1, we will encounter expressions such as

$$\int_{\mathbb{T}^3} \Big(\big(V * (W_t^T I_t^T[u])\big)(x) W_t^T(x) I_t^T[u](x) - (\mathcal{M}_t^T I_t^T[u])(x) I_t^T[u](x) \Big) \mathrm{d}x. \tag{3.3.16}$$

This term no longer involves an explicit stochastic object, such as $:(W_t^T)^2:(x)$, at a single point $x \in \mathbb{T}^3$. By expanding the convolution, we can capture stochastic cancellations in terms of two spatial variables $x \in \mathbb{T}^3$ and $y \in \mathbb{T}^3$, which has already been studied in Lemma 3.2.19. The most natural way to capture stochastic cancellations in (3.3.16), however, is through random operator bounds. This is the object of the next lemma.

Proposition 3.3.7 (Random matrix estimate)**.** Let $\gamma > \max(1-\beta, 1/2)$ and let $1 \leqslant r \leqslant \infty$. We define

$$\mathrm{Op}_t^T(\gamma, r) \overset{\text{def}}{=} \sup_{\substack{f_1, f_2 : \\ \|f_1\|_{\mathbb{W}_x^{\gamma,r}(\mathbb{T}^3)} \leqslant 1, \\ \|f_2\|_{\mathbb{W}_x^{\gamma,r'}(\mathbb{T}^3)} \leqslant 1.}} \left[\int_{\mathbb{T}^3} V * (W_t^T f_1)\; W_t^T f_2 \mathrm{d}x - \int_{\mathbb{T}^3} \big(\mathcal{M}_t^T f_1\big) f_2 \mathrm{d}x \right].$$

Then, we have for all $1 \leqslant p < \infty$ that

$$\sup_{T, t \geqslant 0} \| \mathrm{Op}_t^T(\gamma, r) \|_{L^p_\omega(\Omega)} \lesssim p. \tag{3.3.17}$$

Remark 3.3.8. Aside from Fourier support considerations, the proof below mainly proceeds in physical space. If $r = 2$, an alternative approach is to view $\mathrm{Op}_t^T(\gamma, 2)$ as the operator norm of a random matrix acting on the Fourier coefficients. Using a non-trivial amount of combinatorics, one can then bound $\mathrm{Op}_t^T(\gamma, 2)$ using the moment method (see also [DNY20, Proposition 2.8]). This alternative approach is closer to the methods in the literature on random dispersive equations but more complicated. The estimate for $r \neq 2$, which is not needed in this paper, is useful in the study of the stochastic heat equation with Hartree nonlinearity.

Proof. Since this will be important in the proof, we now indicate the dependence of the multiplier on the interaction potential by writing $\mathcal{M}_t^T[V]$. We use a Littlewood-Paley decomposition of W_t^T, f_1, and f_2. We then have that

$$\begin{aligned}
&\int_{\mathbb{T}^3} V * (W_t^T f_1)\; W_t^T f_2 \mathrm{d}x - \int_{\mathbb{T}^3} \big(\mathcal{M}_t^T[V] f_1\big) f_2 \mathrm{d}x \\
&= \sum_{K_1,K_2,N_1,N_2} \Big[\int_{\mathbb{T}^3} V * (P_{N_1} W_t^T\, P_{K_1} f_1)\; P_{N_2} W_t^T P_{K_2} f_2 \mathrm{d}x - \int_{\mathbb{T}^3} \big(\mathcal{M}_t^T[V; N_1, N_2] P_{K_1} f_1\big) P_{K_2} f_2 \mathrm{d}x \Big].
\end{aligned}$$

To control this sum, we first define a frequency-localized version of $\mathrm{Op}_t^T(\gamma, r)$ by

$$\begin{aligned}
&\mathrm{Op}_t^T(r; K_1, K_2, N_1, N_2) \\
&\overset{\mathrm{def}}{=} \sup_{\substack{f_1,f_2:\\ \|f_1\|_{L_x^r}\leqslant 1,\\ \|f_2\|_{L_x^{r'}}\leqslant 1}} \Big[\int_{\mathbb{T}^3} V * (P_{N_1} W_t^T\, P_{K_1} f_1)\; P_{N_2} W_t^T P_{K_2} f_2 \mathrm{d}x - \int_{\mathbb{T}^3} \big(\mathcal{M}_t^T[V; N_1, N_2] P_{K_1} f_1\big) P_{K_2} f_2 \mathrm{d}x \Big].
\end{aligned}$$

We emphasize the change from $\mathbb{W}_x^{\gamma,r}(\mathbb{T}^3)$ to $L_x^r(\mathbb{T}^3)$, which simplifies the notation below. By proving the estimate for a slightly smaller γ, (3.3.17) reduces to

$$\sup_{T,t\geqslant 0} \| \mathrm{Op}_t^T(r; K_1, K_2, N_1, N_2)\|_{L_\omega^p(\Omega)} \lesssim p (N_1 N_2)^{-\delta} (K_1 K_2)^{\gamma}. \tag{3.3.18}$$

By using Lemma 3.2.16 and Lemma 3.2.19, it suffices to prove for a small $\delta > 0$ that

$$\begin{aligned}
\mathrm{Op}_t^T(r; K_1, K_2, N_1, N_2) &\lesssim (N_1 N_2)^{-\delta} (K_1 K_2)^{\gamma} \\
&\times \Big(1 + \|W_t^T\|^2_{\mathcal{C}_x^{-\frac{1}{2}-\delta}} + \sup_{y\in\mathbb{T}^3} \sup_{N_1,N_2} \| :(\tau_y P_{N_1} W_t^T) P_{N_2} W_t^T: \|_{\mathcal{C}_x^{-1-\delta}} \Big).
\end{aligned} \tag{3.3.19}$$

By interpolation, we can further reduce to $r = 1$ or $r = \infty$. Using the self-adjointness of the convolution with V and the multiplier $\mathcal{M}_t^T[V; N_1, N_2]$, it suffices to take $r = 1$. We now separate the cases $N_1 \sim N_2$ and $N_1 \not\sim N_2$.

Case 1: $N_1 \not\sim N_2$. This is the easier (but slightly tedious) case and it does not contain any probabilistic resonances. We note that $\mathcal{M}_t^T[V; N_1, N_2] = 0$ and hence we only need to control the convolution term. From Fourier support considerations, we also see that this term vanishes unless $\max(K_1, K_2) \gtrsim \max(N_1, N_2)$. While our conditions on f_1 and f_2 are not completely symmetric and we already used the self-adjointness to restrict to $r = 1$, we only treat the case $K_1 \gtrsim K_2$. Since our proof only relies on Hölder's inequality and Young's inequality, the case $K_1 \lesssim K_2$ can be treated similarly. We now estimate

$$\begin{aligned}
&\Big| \int_{\mathbb{T}^3} V * (P_{N_1} W_t^T P_{K_1} f_1) \; P_{N_2} W_t^T P_{K_2} f_2 \mathrm{d}x \Big| \\
&\lesssim \sum_{L \lesssim K_1} \Big| \int_{\mathbb{T}^3} P_L \Big(V * (P_{N_1} W_t^T P_{K_1} f_1) \Big) \; \widetilde{P}_L \Big(P_{N_2} W_t^T P_{K_2} f_2 \Big) \mathrm{d}x \Big| \\
&\lesssim \sum_{L \lesssim K_1} \big\| (P_L V) * (P_{N_1} W_t^T P_{K_1} f_1) \big\|_{L_x^1} \big\| \widetilde{P}_L (P_{N_2} W_t^T P_{K_2} f_2) \big\|_{L_x^\infty} \\
&\lesssim \| P_{N_1} W_t^T \|_{L_x^\infty} \| f_1 \|_{L_x^1} \sum_{L \lesssim K_1} \| P_L V \|_{L_x^1} \big\| \widetilde{P}_L (P_{N_2} W_t^T P_{K_2} f_2) \big\|_{L_x^\infty} \\
&\lesssim N_1^{\frac{1}{2}+\delta} \| W_t^T \|_{\mathcal{C}_x^{-\frac{1}{2}-\delta}} \sum_{L \lesssim K_1} L^{-\beta} \big\| \widetilde{P}_L (P_{N_2} W_t^T P_{K_2} f_2) \big\|_{L_x^\infty}.
\end{aligned}$$

We now split the last sum into the cases $L \ll N_2$ and $N_2 \lesssim L \lesssim K_1$. If $L \ll N_2$, we only obtain a non-zero contribution when $N_2 \sim K_2$. Thus, the corresponding contribution is bounded by

$$\begin{aligned}
&1\{K_2 \sim N_2\} N_1^{\frac{1}{2}+\delta} \| W_t^T \|_{\mathcal{C}_x^{-\frac{1}{2}-\delta}} \sum_{L \lesssim N_2} L^{-\beta} \big\| \widetilde{P}_L (P_{N_2} W_t^T P_{K_2} f_2) \big\|_{L_x^\infty} \\
&\lesssim 1\{K_2 \sim N_2\} N_1^{\frac{1}{2}+\delta} \| W_t^T \|_{\mathcal{C}_x^{-\frac{1}{2}-\delta}} \Big(\sum_{L \lesssim N_2} L^{-\beta} \Big) \| f_2 \|_{L_x^\infty} \| P_{N_2} W_t^T \|_{L_x^\infty} \\
&\lesssim 1\{K_2 \sim N_2\} N_1^{\frac{1}{2}+\delta} N_2^{\frac{1}{2}+\delta} \| W_t^T \|_{\mathcal{C}_x^{-\frac{1}{2}-\delta}}^2 \\
&\lesssim (N_1 N_2)^{-\delta} K_1^\gamma K_2^\gamma \| W_t^T \|_{\mathcal{C}_x^{-\frac{1}{2}-\delta}}^2.
\end{aligned}$$

In the last line, we also used $N_1 \lesssim K_1$ and $\gamma > 1/2$. If $L \gtrsim N_2$, we simply estimate

$$
\begin{aligned}
&N_1^{\frac{1}{2}+\delta} \|W_t^T\|_{\mathcal{C}_x^{-\frac{1}{2}-\delta}} \sum_{N_2 \lesssim L \lesssim K_1} L^{-\beta} \big\| \widetilde{P}_L (P_{N_2} W_t^T P_{K_2} f_2) \big\|_{L_x^\infty} \\
&\lesssim N_1^{\frac{1}{2}+\delta} \|W_t^T\|_{\mathcal{C}_x^{-\frac{1}{2}-\delta}} \Big(\sum_{N_2 \lesssim L \lesssim K_1} L^{-\beta} \Big) \|P_{N_2} W_t^T\|_{L_x^\infty} \|P_{K_2} f_2\|_{L_x^\infty} \\
&\lesssim N_1^{\frac{1}{2}+\delta} N_2^{\frac{1}{2}-\beta+\delta} \|W_t^T\|^2_{\mathcal{C}_x^{-\frac{1}{2}-\delta}} \\
&\lesssim (N_1 N_2)^{-\delta} K_1^{\gamma} \|W_t^T\|^2_{\mathcal{C}_x^{-\frac{1}{2}-\delta}},
\end{aligned}
$$

provided that $\gamma > \max(1-\beta, 1/2)$. This completes the estimate in Case 1, i.e., $N_1 \not\sim N_2$.

Case 2: $N_1 \sim N_2$. This is the more difficult case. Guided by the uncertainty principle, we decompose the interaction potential by writing $V = P_{\ll N_1} V + P_{\gtrsim N_1} V$. Using the linearity of the multiplier $\mathcal{M}_t^T[V; N_1, N_2]$ in V, we decompose

$$
\begin{aligned}
&\int_{\mathbb{T}^3} V * (P_{N_1} W_t^T P_{K_1} f_1)\ P_{N_2} W_t^T P_{K_2} f_2 \mathrm{d}x - \int_{\mathbb{T}^3} \big(\mathcal{M}_t^T[V; N_1, N_2] P_{K_1} f_1 \big) P_{K_2} f_2 \mathrm{d}x \\
&= \int_{\mathbb{T}^3} (P_{\ll N_1} V) * (P_{N_1} W_t^T P_{K_1} f_1)\ P_{N_2} W_t^T P_{K_2} f_2 \mathrm{d}x - \int_{\mathbb{T}^3} \big(\mathcal{M}_t^T[P_{\ll N_1} V; N_1, N_2] P_{K_1} f_1 \big) P_{K_2} f_2 \mathrm{d}x \\
&+ \int_{\mathbb{T}^3} (P_{\gtrsim N_1} V) * (P_{N_1} W_t^T P_{K_1} f_1)\ P_{N_2} W_t^T P_{K_2} f_2 \mathrm{d}x - \int_{\mathbb{T}^3} \big(\mathcal{M}_t^T[P_{\gtrsim N_1} V; N_1, N_2] P_{K_1} f_1 \big) P_{K_2} f_2 \mathrm{d}x.
\end{aligned}
$$

We now split the proof into two subcases corresponding to the contributions of $P_{\ll N_1} V$ and $P_{\gtrsim N_1} V$.

Case 2.a: $N_1 \sim N_2$*, contribution of* $P_{\ll N_1} V$. Similar as in Case 1, we do not rely on any cancellation between the convolution term and its renormalization. As a result, we estimates both terms separately.

We first estimate the convolution term. Due to the convolution with $P_{\ll N_1} V$, we only obtain a non-zero contribution if $N_1 \sim K_1$. Using $N_1 \sim N_2$ in the second inequality below, we obtain that

$$\begin{aligned}
&\Big| \int_{\mathbb{T}^3} (P_{\ll N_1} V) * (P_{N_1} W_t^T \, P_{K_1} f_1) \; P_{N_2} W_t^T P_{K_2} f_2 \mathrm{d}x \Big| \\
&\lesssim 1\{N_1 \sim K_1\} \| (P_{\ll N_1} V) * (P_{N_1} W_t^T \, P_{K_1} f_1) \|_{L_x^1} \| \widetilde{P}_{\ll N_1} (P_{N_2} W_t^T P_{K_2} f_2) \|_{L_x^\infty} \\
&\lesssim 1\{N_1 \sim K_1\} 1\{N_2 \sim K_2\} \| P_{N_1} W_t^T \|_{L_x^\infty} \| f_1 \|_{L_x^1} \| P_{N_2} W_t^T \|_{L_x^\infty} \| f_2 \|_{L_x^\infty} \\
&\lesssim 1\{N_1 \sim K_1\} 1\{N_2 \sim K_2\} (N_1 N_2)^{\frac{1}{2}+\delta} \| W_t^T \|_{\mathcal{C}_x^{-\frac{1}{2}-\delta}}^2 \\
&\lesssim (N_1 N_2)^{-\delta} (K_1 K_2)^{\gamma} \| W_t^T \|_{\mathcal{C}_x^{-\frac{1}{2}-\delta}}^2 .
\end{aligned}$$

Second, we turn to the multiplier term. From the definition of $\mathcal{M}_t^T[P_{\ll N_1} V; N_1, N_2]$ (see Definition 3.2.13), we see that the corresponding symbol is supported on frequencies $|n| \sim N_1$. As a result, we only obtain a non-zero contribution if $K_1 \sim K_2 \sim N_1$. Using Lemma 3.2.15, Hölder's inequality, Young's inequality, and the trivial estimate $\| \mathfrak{C}_t^T[N_1, N_2] \|_{L_x^\infty} \lesssim N_1$, we obtain

$$\begin{aligned}
&\Big| \int_{\mathbb{T}^3} \big(\mathcal{M}_t^T[P_{\ll N_1} V; N_1, N_2] P_{K_1} f_1 \big) P_{K_2} f_2 \mathrm{d}x \Big| \\
&= 1\{K_1 \sim K_2 \sim N_1\} \Big| \int_{\mathbb{T}^3} \Big((\mathfrak{C}_t^T[N_1, N_2] P_{\ll N_1} V) * P_{K_1} f_1 \Big) P_{K_2} f_2 \mathrm{d}x \Big| \\
&\lesssim 1\{K_1 \sim K_2 \sim N_1\} \| (\mathfrak{C}_t^T[N_1, N_2] P_{\ll N_1} V) * P_{K_1} f_1 \|_{L_x^1} \| P_{K_2} f_2 \|_{L_x^\infty} \\
&\lesssim 1\{K_1 \sim K_2 \sim N_1\} \| \mathfrak{C}_t^T[N_1, N_2] P_{\ll N_1} V \|_{L_x^1} \| f_1 \|_{L_x^1} \| f_2 \|_{L_x^\infty} \\
&\lesssim 1\{K_1 \sim K_2 \sim N_1\} \| \mathfrak{C}_t^T[N_1, N_2] \|_{L_x^\infty} \| V \|_{L_x^1} \\
&\lesssim 1\{K_1 \sim K_2 \sim N_1\} N_1 \lesssim (N_1 N_2)^{-\delta} (K_1 K_2)^{\gamma}.
\end{aligned}$$

This completes the estimate of the contribution from $P_{\ll N_1} V$.

Case 2.b: $N_1 \sim N_2$, contribution of $P_{\gg N_1} V$. The estimate for this case relies on the cancellation between the convolution and multiplier term, i.e., the renormalization. One important ingredient lies in the estimate $\| P_{\gg N_1} V \|_{L_x^1} \lesssim N_1^{-\beta}$, which yields an important gain.

Using the translation operator τ_y, we rewrite the convolution term as

$$\begin{aligned}
&\int_{\mathbb{T}^3} (P_{\gtrsim N_1} V) * (P_{N_1} W_t^T \, P_{K_1} f_1) \; P_{N_2} W_t^T P_{K_2} f_2 \mathrm{d}x \\
&= \int_{\mathbb{T}^3} P_{\gtrsim N_1} V(y) \bigg[\int_{\mathbb{T}^3} P_{K_1} f_1(x-y) P_{K_2} f_2(x) P_{N_1} W_t^T(x-y) P_{N_2} W_t^T(x) \mathrm{d}x \bigg] \mathrm{d}y \\
&= \int_{\mathbb{T}^3} P_{\gtrsim N_1} V(y) \bigg[\int_{\mathbb{T}^3} \big(\tau_y P_{K_1} f_1 \, P_{K_2} f_2\big)(x) \big(\tau_y P_{N_1} W_t^T \, P_{N_2} W_t^T\big)(x) \mathrm{d}x \bigg] \mathrm{d}y.
\end{aligned}$$

Using Lemma 3.2.15, we obtain that

$$\begin{aligned}
&\int_{\mathbb{T}^3} \big(\mathcal{M}_t^T[P_{\gtrsim N_1} V; N_1, N_2] P_{K_1} f_1\big) P_{K_2} f_2 \mathrm{d}x \\
&= \int_{\mathbb{T}^3} \Big(\big(\mathfrak{C}_t^T[N_1, N_2] P_{\gg N_1} V\big) * P_{K_1} f_1 \Big)(x) P_{K_2} f_2(x) \mathrm{d}x \\
&= \int_{\mathbb{T}^3} P_{\gg N_1} V(y) \bigg[\int_{\mathbb{T}^3} \big(\tau_y P_{K_1} f_1 \, P_{K_2} f_2\big)(x) \mathfrak{C}_t^T[N_1, N_2](y) \mathrm{d}x \bigg] \mathrm{d}y.
\end{aligned}$$

By recalling Definition 3.2.14 and combining both identities, we obtain that

$$\begin{aligned}
&\int_{\mathbb{T}^3} (P_{\gtrsim N_1} V) * (P_{N_1} W_t^T \, P_{K_1} f_1) \; P_{N_2} W_t^T P_{K_2} f_2 \mathrm{d}x - \int_{\mathbb{T}^3} \big(\mathcal{M}_t^T[P_{\gtrsim N_1} V; N_1, N_2] P_{K_1} f_1\big) P_{K_2} f_2 \mathrm{d}x \\
&= \int_{\mathbb{T}^3} P_{\gg N_1} V(y) \bigg[\int_{\mathbb{T}^3} \big(\tau_y P_{K_1} f_1 \, P_{K_2} f_2\big)(x) \, {:}(\tau_y P_{N_1} W_t^T) P_{N_2} W_t^T{:}(x) \mathrm{d}x \bigg] \mathrm{d}y.
\end{aligned}$$

Using that ${:}(\tau_y P_{N_1} W_t^T) P_{N_2} W_t^T{:}(x)$ is supported on frequencies $\lesssim N_1$, we obtain that

$$\begin{aligned}
&\bigg| \int_{\mathbb{T}^3} P_{\gg N_1} V(y) \bigg[\int_{\mathbb{T}^3} \big(\tau_y P_{K_1} f_1 \, P_{K_2} f_2\big)(x) \, {:}(\tau_y P_{N_1} W_t^T) P_{N_2} W_t^T{:}(x) \mathrm{d}x \bigg] \mathrm{d}y \bigg| \\
&\lesssim \| P_{\gtrsim N_1} V(y) \|_{L^1_y} \sup_{y \in \mathbb{T}^3} \Big(\sum_{L \lesssim N_1} L^{1+\delta} \| P_L \big((\tau_y P_{K_1} f_1) P_{K_2} f_2 \big) \|_{L^1_x} \Big) \sup_{y \in \mathbb{T}^3} \| {:}(\tau_y P_{N_1} W_t^T) P_{N_2} W_t^T{:} \|_{\mathcal{C}_x^{-1-\delta}} \\
&\lesssim N_1^{-\beta} \Big(\sum_{\substack{L \lesssim N_1 \\ L \lesssim \max(K_1, K_2)}} L^{1+\delta} \Big) \| f_1 \|_{L^1_x} \| f_2 \|_{L^\infty_x} \sup_{y \in \mathbb{T}^3} \| {:}(\tau_y P_{N_1} W_t^T) P_{N_2} W_t^T{:} \|_{\mathcal{C}_x^{-1-\delta}} \\
&\lesssim (N_1 N_2)^{-\delta} \max(K_1, K_2)^{\gamma} \sup_{y \in \mathbb{T}^3} \| {:}(\tau_y P_{N_1} W_t^T) P_{N_2} W_t^T{:} \|_{\mathcal{C}_x^{-1-\delta}}.
\end{aligned}$$

This completes the estimate of the contribution from $P_{\gg N_1} V$ and hence the proof of the proposition.

□

3.3.4 Proof of Proposition 3.3.1 and Corollary 3.3.4

In this subsection, we reap the benefits of our previous work and prove the main results of this section.

Proof of Proposition 3.3.1: In this proof, we treat $Q_T = Q_T(W, \varphi, \lambda)$ like an implicit constant and omit the dependence on W, φ, and λ. In particular, its precise definition may change throughout the proof.

From the quartic binomial formula (Lemma 3.2.11), it follows that

$$
\begin{aligned}
&\varphi(W + I(u)) + :\mathcal{V}^{T,\lambda}(W_\infty^T + I_\infty^T(u)): + \frac{1}{2}\|u\|_{L^2}^2 \\
&= \lambda \int_{\mathbb{T}^3} :(V * (W_\infty^T)^2)(W_\infty^T): I_\infty^T[u]\mathrm{d}x + \frac{\lambda}{4}\int_{\mathbb{T}^3} (V * (I_\infty^T[u])^2)(I_\infty^T[u])^2 \mathrm{d}x + \frac{1}{2}\|u\|_{L^2}^2 \\
&+ \frac{\lambda}{4}\int_{\mathbb{T}^3} :(V * (W_\infty^T)^2)(W_\infty^T)^2: \mathrm{d}x + c^{T,\lambda} + \varphi(W + I(u)) + \frac{\lambda}{2}\int_{\mathbb{T}^3} (V * :(W_\infty^T)^2:)(I_\infty^T[u])^2 \mathrm{d}x \\
&+ \lambda \int_{\mathbb{T}^3} \Big[(V * (W_\infty^T I_\infty^T[u]))W_\infty^T I_\infty^T[u] - (\mathcal{M}_t^T I_\infty^T[u]) I_\infty^T[u]\Big] \mathrm{d}x + \lambda \int_{\mathbb{T}^3} (V * (I_\infty^T[u])^2) I_\infty^T[u] W_\infty^T \mathrm{d}x.
\end{aligned}
$$

We have grouped the terms according to their importance and their degree in $I_\infty^T[u]$. The first line consists of the main terms, whereas the second and third line consist of less important terms of increasing degree in $I_\infty^T[u]$. We will split them further in (3.3.23)-(3.3.26) below and introduce notation for the individual terms.

Since $:(V * (W_\infty^T)^2)W_\infty^T:$ has regularity $\min(-\frac{3}{2} + \beta, -\frac{1}{2})-$ and $I_\infty^T[u]$ has regularity 1, the term

$$
\lambda \int_{\mathbb{T}^3} :(V * (W_\infty^T)^2)W_\infty^T: I_\infty^T[u]\mathrm{d}x
$$

is potentially unbounded as $T \to \infty$. As in [BG20b], we absorb it into the quadratic term $\frac{1}{2}\|u\|_{L^2}^2$. To this end, we want to remove the integral in $I_\infty^T[u]$ and obtain an expression in the drift u. From

Itô's formula, it holds that

$$\begin{aligned}
&\lambda \int_{\mathbb{T}^3} :(V * (W_\infty^T)^2) W_\infty^T \colon I_\infty^T[u] \mathrm{d}x \\
&= \lambda \int_0^T \int_{\mathbb{T}^3} :(V * (W_t^T)^2) W_t^T \colon J_t^T u_t \mathrm{d}x \mathrm{d}t + \lambda \int_0^T \int_{\mathbb{T}^3} I_t^T[u] \, \mathrm{d}(:(V * (W_t^T)^2) W_t^T :).
\end{aligned}$$

The second term is a martingale (in the upper limit of integration) and therefore has expectation equal to zero. Together with the self-adjointness of J_t, it follows that

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\Big[\lambda \int_{\mathbb{T}^3} :(V * (W_\infty^T)^2) W_\infty^T \colon I_\infty^T[u] \mathrm{d}x + \frac{1}{2}\|u\|_{L^2}^2\Big] \\
&= \mathbb{E}_{\mathbb{P}}\Big[\lambda \int_0^T \int_{\mathbb{T}^3} J_t^T\Big(:(V * (W_t^T)^2) W_t^T \colon \Big) u_t \mathrm{d}x \mathrm{d}t + \frac{1}{2}\|u\|_{L^2}^2\Big] \\
&= \mathbb{E}_{\mathbb{P}}\Big[\frac{1}{2}\Big\| l^T[u] \Big\|_{L^2}^2 - \frac{\lambda^2}{2}\Big\| J_t^T\Big(:(V * (W_t^T)) W_t^T \colon \Big)\Big\|_{L^2}^2\Big],
\end{aligned}$$

where $l^T[u]$ is as in (3.3.5). To simplify the notation, we write

$$w_t \stackrel{\text{def}}{=} l_t^T[u] = u_t + \lambda J_t^T\Big(:(V * (W_t^T)^2) W_t^T \colon \Big). \tag{3.3.20}$$

With $\mathbb{W}_t^{T,[3]}$ as in (3.2.46), it follows that

$$I_t^T[w] = I_t^T[u] + \lambda \mathbb{W}_t^{T,[3]}. \tag{3.3.21}$$

By inserting this back into the quartic binomial formula, we obtain that

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\Big[\varphi(W + I(u)) + :\mathcal{V}^{T,\lambda}(W_\infty^T + I_\infty^T[u]) \colon + \frac{1}{2}\|u\|_{L^2_{t,x}}^2\Big] \\
&= \mathbb{E}_{\mathbb{P}}\big[\mathcal{E}_0 + c^{T,\lambda}\big] + \mathbb{E}_{\mathbb{P}}\big[\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3\big] + \mathbb{E}_{\mathbb{P}}\Big[\frac{\lambda}{4}\int_{\mathbb{T}^3} (V * (I_\infty^T[w])^2)(I_\infty^T[w])^2 \mathrm{d}x + \frac{1}{2}\|w\|_{L^2_{t,x}}^2\Big].
\end{aligned} \tag{3.3.22}$$

where the "error" terms $\mathcal{E}_j$, with $j = 0, 1, 2, 3$, are given by

$$\begin{aligned}
\mathcal{E}_0 &\overset{\text{def}}{=} \frac{\lambda}{4} \int_{\mathbb{T}^3} :(V * (W_\infty^T)^2)(W_\infty^T)^2 \colon \mathrm{d}x - \frac{\lambda^2}{2} \Big\| J_t^T \big(:(V * (W_t^T)^2) W_t^T : \big) \Big\|_{L_t^2 L_x^2}^2 \\
&\quad + \frac{\lambda^3}{2} \int_{\mathbb{T}^3} (V * :(W_\infty^T)^2:)(\mathbb{W}_\infty^{T,[3]})^2 \mathrm{d}x \\
&\quad + \lambda^3 \int_{\mathbb{T}^3} \Big(V * (W_\infty^T \mathbb{W}_\infty^{T,[3]}) W_\infty^T \mathbb{W}_\infty^{T,[3]} - (\mathcal{M}_\infty^T \mathbb{W}_\infty^{T,[3]}) \mathbb{W}_\infty^{T,[3]} \Big) \mathrm{d}x,
\end{aligned} \tag{3.3.23}$$

$$\begin{aligned}
\mathcal{E}_1 &\overset{\text{def}}{=} \varphi(W + I[u]) - \lambda^2 \int_{\mathbb{T}^3} (V * :(W_\infty^T)^2:) \mathbb{W}_\infty^{T,[3]} I_\infty^T[w] \mathrm{d}x \\
&\quad - 2\lambda^2 \int_{\mathbb{T}^3} \Big(\big(V * (W_\infty^T \mathbb{W}_\infty^{T,[3]})\big) W_\infty^T - \mathcal{M}_\infty^T \mathbb{W}_\infty^{T,[3]} \Big) I_\infty^T[w] \mathrm{d}x,
\end{aligned} \tag{3.3.24}$$

$$\begin{aligned}
\mathcal{E}_2 &\overset{\text{def}}{=} \lambda \int_{\mathbb{T}^3} \Big(\big(V * (W_\infty^T I_\infty^T[w])\big) W_\infty^T I_\infty^T[w] - (\mathcal{M}_\infty^T I_\infty^T[w]) I_\infty^T[w] \Big) \mathrm{d}x \\
&\quad + \frac{\lambda}{2} \int_{\mathbb{T}^3} (V * :(W_\infty^T)^2:)(I_\infty^T[w])^2 \mathrm{d}x,
\end{aligned} \tag{3.3.25}$$

$$\begin{aligned}
\mathcal{E}_3 &\overset{\text{def}}{=} \lambda \int_{\mathbb{T}^3} \big(V * (I_\infty^T[w] - \lambda \mathbb{W}_\infty^{T,[3]})^2 \big)(I_\infty^T[w] - \lambda \mathbb{W}_\infty^{T,[3]}) W_\infty^T \mathrm{d}x \\
&\quad + \frac{\lambda}{4} \int_{\mathbb{T}^3} \Big((V * (I_\infty^T[w] - \lambda \mathbb{W}_\infty^{T,[3]})^2)(I_\infty^T[w] - \lambda \mathbb{W}_\infty^{T,[3]})^2 - (V * (I_\infty^T[w])^2)(I_\infty^T[w])^2 \Big) \mathrm{d}x.
\end{aligned} \tag{3.3.26}$$

Since $\mathcal{E}_0$ does not depend on w, we can define

$$c^{T,\lambda} \overset{\text{def}}{=} -\mathbb{E}_{\mathbb{P}}\big[\mathcal{E}_0\big]. \tag{3.3.27}$$

The behavior of $c^{T,\lambda}$ as $T \to \infty$ is irrelevant for the rest of the proof. However, it determines whether the Gibbs measure is singular or absolutely continuous with respect to the Gaussian free field (see Section 3.5). From the estimates (3.6.10) and (3.6.11), it is easy to see that

$$-Q_T + \frac{1}{2}\Big(\frac{\lambda}{4}\mathcal{V}(I_\infty^T[u]) + \frac{1}{2}\|w\|_{L_{t,x}^2}^2\Big) \leqslant \frac{\lambda}{4}\mathcal{V}(I_\infty^T[w]) + \frac{1}{2}\|w\|_{L_{t,x}^2}^2 \leqslant Q_T + 2\Big(\frac{\lambda}{4}\mathcal{V}(I_\infty^T[u]) + \frac{1}{2}\|w\|_{L_{t,x}^2}^2\Big).$$

Thus, it suffices to bound the terms in $\mathcal{E}_1, \mathcal{E}_2$, and $\mathcal{E}_3$ pointwise by

$$Q_T + \frac{1}{8}\Big(\frac{\lambda}{4}\mathcal{V}(I_\infty^T[w]) + \frac{1}{2}\|w\|_{L_{t,x}^2}^2\Big).$$

We treat the individual summands separately.

Contribution of $\mathcal{E}_1$: For the first summand in $\mathcal{E}_1$, the linear growth of φ, Sobolev embedding, a minor modification of (3.2.47), and Lemma 3.6.8 imply that

$$\begin{aligned}|\varphi(W+I[u])| &\lesssim \|W\|_{\mathcal{C}_t^0\mathcal{C}_x^{-\frac{1}{2}-\kappa}} + \|I[J_t^T(:(V*(W_t^T)^2)W_t^T:)]\|_{\mathcal{C}_t^0\mathcal{C}_x^{-\frac{1}{2}-\kappa}} + \|I[w]\|_{\mathcal{C}_t^0\mathcal{C}_x^{-\frac{1}{2}-\kappa}} \\ &\lesssim Q_T + \|I[w]\|_{C_tH_x^1} \lesssim \frac{1}{\delta}Q_T + \delta\|w\|_{L^2_{t,x}}^2.\end{aligned} \tag{3.3.28}$$

For the second summand in $\mathcal{E}_1$, we have from Lemma 3.2.20 that

$$\lambda\Big|\int_{\mathbb{T}^3}(V*:(W_\infty^T)^2:)\mathbb{W}_\infty^{T,[3]}I_\infty^T[w]\mathrm{d}x\Big| \lesssim \lambda\|(V*:(W_\infty^T)^2:)\mathbb{W}_\infty^{T,[3]}\|_{C_x^{-1}}\|I_\infty^T[w]\|_{H_x^1} \lesssim \frac{1}{\delta}Q_T + \delta\|w\|_{L^2_{t,x}}^2.$$

For the third summand in $\mathcal{E}_1$, we have from Lemma 3.2.20 and Lemma 3.6.8 that

$$\begin{aligned}&\lambda^2\int_{\mathbb{T}^3}\Big(\big(V*(W_\infty^T\mathbb{W}_\infty^{T,[3]})\big)W_\infty^T - \mathcal{M}_\infty^T\mathbb{W}_\infty^{T,[3]}\Big)I_\infty^T[w]\mathrm{d}x \\ &\lesssim \lambda^2\|\big(V*(W_\infty^T\mathbb{W}_\infty^{T,[3]})\big)W_\infty^T - \mathcal{M}_\infty^T\mathbb{W}_\infty^{T,[3]}\|_{C_x^{-1}}\|I_\infty^T[w]\|_{H_x^1} \lesssim \frac{1}{\delta}Q_T + \delta\|w\|_{L^2_{t,x}}^2.\end{aligned}$$

Contribution of $\mathcal{E}_2$: For the first summand in $\mathcal{E}_2$, the random matrix estimate (Proposition 3.3.7) implies for every $0<\gamma<\min(\beta,\frac{1}{2})$ that

$$\begin{aligned}&\lambda\Big|\int_{\mathbb{T}^3}\Big(\big(V*(W_\infty^TI_\infty^T[w])\big)W_\infty^TI_\infty^T[w] - (\mathcal{M}_\infty^TI_\infty^T[w])I_\infty^T[w]\Big)\mathrm{d}x\Big| \lesssim Q_T\|I_\infty^T[w]\|_{H_x^{1-\gamma}}^2 \\ &\lesssim \frac{1}{\delta}Q_T + \delta\big(\lambda\|I_\infty^T[w]\|_{L_x^2}^4 + \|I_\infty^T[w]\|_{H_x^1}^2\big) \lesssim \frac{1}{\delta}Q_T + \delta\big(\lambda\mathcal{V}(I_\infty^T[w]) + \|I_\infty^T[w]\|_{H_x^1}^2\big).\end{aligned}$$

The second summand in $\mathcal{E}_2$ can easily be controlled using Lemma 3.2.16.

Contribution of $\mathcal{E}_3$: We estimate the first summand in $\mathcal{E}_3$ by

$$\begin{aligned}&\lambda\Big|\int_{\mathbb{T}^3}\big(V*(I_\infty^T[w]-\lambda\mathbb{W}_\infty^{T,[3]})^2\big)(I_\infty^T[w]-\lambda\mathbb{W}_\infty^{T,[3]})W_\infty^T\mathrm{d}x\Big| \\ &\lesssim \lambda\|W_\infty^T\|_{C_x^{-\frac{1}{2}-\delta}}\Big\|\langle\nabla\rangle^{\frac{1}{2}+\delta}\Big(\big(V*(I_\infty^T[w]-\lambda\mathbb{W}_\infty^{T,[3]})^2\big)(I_\infty^T[w]-\lambda\mathbb{W}_\infty^{T,[3]})\Big)\Big\|_{L_x^1}.\end{aligned}$$

In the second factor, we bound the contribution of $(V*I_\infty^T[w]^2)I_\infty^T[w]$ using Lemma 3.3.6. In contrast, the terms containing at least one factor of $\mathbb{W}_t^{T,[3]}$ can be controlled using Lemma 3.2.20, (3.6.10) and (3.6.11). This leads to

$$
\begin{aligned}
&\|W_\infty^T\|_{C_x^{-\frac{1}{2}-\epsilon}} \Big\| \langle \nabla \rangle^{\frac{1}{2}+\delta} \Big(\big(V * (I_\infty^T[w] - \lambda \mathbb{W}_\infty^{T,[3]})^2 \big) (I_\infty^T[w] - \lambda \mathbb{W}_\infty^{T,[3]}) \Big) \Big\|_{L_x^1} \\
&\lesssim \lambda Q_T \Big(1 + \mathcal{V}(I_\infty^T[w])^{\frac{1}{2}} \|I_\infty^T[w]\|_{L^2}^{1-\theta} \|I_\infty^T[w]\|_{H_x^1}^{\theta} + \|I_\infty^T[w]\|_{H^{\frac{1}{2}+2\delta}}^2 \Big) \\
&\lesssim Q_T + \delta \Big(\lambda \mathcal{V}(I_\infty^T[w]) + \|w\|_{L_{t,x}^2}^2 \Big).
\end{aligned}
$$

The second summand in $\mathcal{E}_3$ can be controlled using the same (or simpler) arguments. □

Based on the proof of Proposition 3.3.1, we can also determine the behavior as $T \to \infty$ of the renormalization constants $c^{T,\lambda}$. In particular, we obtain a short proof of Corollary 3.3.4.

Proof of Corollary 3.3.4: We let $\beta > 1/2$ and choose any $1/2 < \gamma < \min(\beta, 1)$. Using the definition of $c^{T,\lambda}$ in (3.3.27), it remains to control the expectation of $\mathcal{E}_0$, which is defined in (3.3.23). We treat the four terms in $\mathcal{E}_0$ separately.

The first term has zero expectation by Proposition 3.2.9. For the second term, we obtain from Corollary 3.2.18 that

$$
\mathbb{E}_{\mathbb{P}} \Big[\Big\| J_t^T \big(:(V * (W_t^T)^2) W_t^T: \big) \Big\|_{L_t^2 L_x^2}^2 \Big] \lesssim \sum_{n \in \mathbb{Z}^3} \int_0^\infty \frac{\sigma_t^T(n)^2}{\langle n \rangle^2} \frac{1}{\langle n \rangle^{2\gamma}} \mathrm{d}t \lesssim \sum_{n \in \mathbb{Z}^3} \frac{1}{\langle n \rangle^{2+2\gamma}} \lesssim 1.
$$

For the third term, we obtain from Lemma 3.2.16 and Lemma 3.2.20 that

$$
\Big| \mathbb{E}_{\mathbb{P}} \Big[\int_{\mathbb{T}^3} (V * :(W_\infty^T)^2:) (\mathbb{W}_\infty^{T,[3]})^2 \mathrm{d}x \Big] \Big| \lesssim \mathbb{E}_{\mathbb{P}} \Big[\|V * :(W_\infty^T)^2\|_{\mathcal{C}_x^{-1/2}} \|\mathbb{W}_\infty^{T,[3]}\|_{\mathcal{C}_x^\gamma}^2 \Big] \lesssim 1.
$$

For the fourth term, we obtain from Lemma 3.2.20 and the random matrix estimate (Proposition 3.3.7) that

$$
\Big| \mathbb{E}_{\mathbb{P}} \Big[\int_{\mathbb{T}^3} \Big(V * (W_\infty^T \mathbb{W}_\infty^{T,[3]}) W_\infty^T \mathbb{W}_\infty^{T,[3]} - (\mathcal{M}_\infty^T \mathbb{W}_\infty^{T,[3]}) \mathbb{W}_\infty^{T,[3]} \Big) \mathrm{d}x \Big] \Big| \lesssim \mathbb{E}_{\mathbb{P}} \Big[\mathrm{Op}_\infty^T(\gamma, 2) \|\mathbb{W}_\infty^{T,[3]}\|_{H_x^\gamma}^2 \Big] \lesssim 1.
$$

This completes the argument. □

3.4 The reference and drift measures

In this section, we prove Theorem 3.1.4, which contains information regarding the reference measures. In this paper, we will use the reference measure ν_∞ to prove the singularity of the Gibbs measure (Theorem 3.1.5). In the second part of this series, the reference measures will play an essential role in the probabilistic local well-posedness theory.

As in previous sections, we replace the truncation parameter N by T. Due to its central importance, let us provide an informal description of the terms in the representation of ν_T. The first summand follows the distribution of the Gaussian free field, which has independent Fourier coefficients and regularity $-1/2-$. The second summand is a cubic Gaussian chaos with regularity $\min(1/2+\beta,1)-$. Finally, the third summand is a Gaussian chaos of order n with regularity $5/2-$. The statement of Theorem 3.1.4 is concerned with measures on $\mathcal{C}_x^{-1/2-\kappa}(\mathbb{T}^3)$. At this point, it should not be surprising to the reader that the proof mostly uses the lifted measures $\widetilde{\mu}_T$ and $\widetilde{\mu}_\infty$. We will construct a reference measure $\mathbb{Q}_T^u$ for $\widetilde{\mu}_T$, and the reference measure ν_T will be given by the pushforward of $\mathbb{Q}_T^u$ under W_∞. Since the main tool in the construction of $\mathbb{Q}_T^u$ is Girsanov's theorem, we call $\mathbb{Q}_T^u$ the drift measure. This section is a modification of the arguments in Barashkov and Gubinelli's paper [BG20a]. Since $l^T[u]$ in Proposition 3.3.1 is simpler than in the Φ_3^4-model, however, we obtain slightly stronger results. For instance, we prove L^q-bounds for the density D_T in (3.4.23), whereas the analogous density in [BG20a] only satisfies "local" L^q-bounds.

3.4.1 Construction of the drift measure

We define the forcing term

$$\Xi^T(W^T)_t \overset{\text{def}}{=} -\lambda J_t^T\Big(:(V*(W_t^T)^2)W_t^T: \Big) + J_t^T\langle\nabla\rangle^{-\frac{1}{2}} :(\langle\nabla\rangle^{-\frac{1}{2}}W_t^T)^n:, \tag{3.4.1}$$

where n is a large odd integer depending on β. The first summand in (3.4.1) is the main term. The second summand in (3.4.1) yields necessary coercivity in the proof of Lemma 3.4.3 and Proposition 3.4.7, but can be safely ignored for most of the argument. We define the drift u^T through the integral equation

$$\begin{aligned} u_t^T &= \Xi^T(W^T - I^T[u^T])_t \\ &= -\lambda J_t^T\Big(:(V * (W_t^T - I_t^T[u^T])^2)(W_t^T - I_t^T[u^T]): \Big) + J_t^T \langle \nabla \rangle^{-\frac{1}{2}} :\Big(\langle \nabla \rangle^{-\frac{1}{2}} \big(W_t^T - I_t^T[u^T] \big) \Big)^n : . \end{aligned} \tag{3.4.2}$$

We also define the drift u, which does not contain any regularization in the interaction, by

$$u_t = -\lambda J_t\Big(:(V * (W_t - I_t[u])^2)(W_t - I_t[u]): \Big) + J_t \langle \nabla \rangle^{-\frac{1}{2}} :\Big(\langle \nabla \rangle^{-\frac{1}{2}} \big(W_t - I_t[u] \big) \Big)^n : . \tag{3.4.3}$$

Using the binomial formulas (Lemma 3.2.11 and Lemma 3.2.23), we see that the integral equation has smooth coefficients on every compact subset of $[0,\infty) \times \mathbb{T}^3$. As a result, it can be solved locally in time using standard ODE-theory. Due to the polynomial nonlinearity, however, we will need to rule out finite-time blowup. To this end, we introduce the blow-up time $T_{\exp}[u^T] \in (0,\infty]$, which we will later show to be infinite almost surely with respect to both $\mathbb{P}$ and $\mathbb{Q}_T^u$. The reason is that the highest-degree term in (3.4.2), which is given by $-J_t^T \langle \nabla \rangle^{-1/2} (\langle \nabla \rangle^{-1/2} I_t^T[u^T])^n$, is defocusing. We also introduce the stopping time

$$\tau_{T,N} \overset{\text{def}}{=} \inf \Big\{ t \in [0,\infty) \colon \int_0^t \| u_s^T \|_{L_x^2}^2 \mathrm{d}s = N \Big\}. \tag{3.4.4}$$

From the integral equation, it is clear that $u_t^T(\cdot)$ is supported in frequency space on the finite set $\{ n \in \mathbb{Z}^3 \colon \|n\| \lesssim \langle t \rangle \}$. As a result, the $L_t^2 L_x^2$-norm can be used as a blow-up criterion and the solution u_t^T exists for all times $t \leqslant \tau_{T,N}$, i.e., $T_{\exp}[u^T] > \tau_{T,N}$. We then define the truncated solution by

$$u_t^{T,N} \overset{\text{def}}{=} 1\{ t \leqslant \tau_{T,N} \} \, u_t^T. \tag{3.4.5}$$

From the definition of $\tau_{T,N}$, it follows that

$$\int_0^\infty \|u_s^{T,N}\|_{L_x^2}^2 \mathrm{d}s \leqslant N.$$

Thus, $u^{T,N}$ satisfies Novikov's condition and we can define the shifted probability measure $\mathbb{Q}_{T,N}^u$ by

$$\frac{\mathrm{d}\mathbb{Q}_{T,N}^u}{\mathrm{d}\mathbb{P}} = \exp\Big(\int_0^\infty \int_{\mathbb{T}^3} u_s^{T,N} \mathrm{d}B_s - \frac{1}{2}\int_0^\infty \|u_s^{T,N}\|_{L^2}^2 \mathrm{d}s\Big). \tag{3.4.6}$$

Here, the L_x^2-pairing in the integral $\int_0^\infty \int_{\mathbb{T}^3} u_s^{T,N} \mathrm{d}B_s$ is implicit, i.e.,

$$\int_0^\infty \int_{\mathbb{T}^3} u_s^{T,N} \mathrm{d}B_s = \int_0^\infty \langle u_s^{T,N}, \mathrm{d}B_s\rangle_{L_x^2(\mathbb{T}^3)} = \sum_{\substack{n_1,n_2\in\mathbb{Z}^3:\\ n_1+n_2=0}} \int_0^\infty \widehat{u_s^{T,N}}(n_1)\, \mathrm{d}B_s^{n_2}.$$

We emphasize that the stochastic integral $\int_0^\infty \int_{\mathbb{T}^3} u_s^{T,N} \mathrm{d}B_s$ only depends on the Brownian process B through the Gaussian process W. This is important in order to view $\mathbb{Q}_{T,N}^u$ as a measure on $\mathcal{C}_t^0\mathcal{C}_x^{-1/2-\kappa}([0,\infty]\times\mathbb{T}^3)$ without changing the expression for the density. To make this direct dependence on W clear, we note that u^T and hence $\tau_{T,N}$ are functions of W^T, and hence W, directly from their definition. By using the definition of u^T, the self-adjointness of J_t^T, and $\mathrm{d}W_s^T = J_s^T \mathrm{d}B_s$, we obtain that

$$\begin{aligned}&\int_0^\infty \int_{\mathbb{T}^3} u_s^{T,N} \mathrm{d}B_s \\ &= \int_0^\infty \int_{\mathbb{T}^3} \Big(-\lambda :\!(V*(W_t^T - I_t^T[u^T])^2)(W_t^T - I_t^T[u^T])\!: + \langle\nabla\rangle^{-\frac{1}{2}} :\!\big(\langle\nabla\rangle^{-\frac{1}{2}}(W_t^T - I_t^T[u^T])\big)^n\!: \Big)\mathrm{d}W_s^T.\end{aligned}$$

The expression on the right-hand side clearly is a function of W^T and hence W. With a slight abuse of notation, we will keep writing the integral with respect to $\mathrm{d}B_s$, since it is more compact.

By Girsanov's theorem, the process

$$B_t^{u^{T,N}} \overset{\text{def}}{=} B_t - \int_0^t u_s^{T,N} \mathrm{d}s \tag{3.4.7}$$

is a cylindrical Brownian motion under $\mathbb{Q}^u_{T,N}$. In particular, the law of $B_t^{u^{T,N}}$ under $\mathbb{Q}^u_{T,N}$ coincides with the law of B_t under $\mathbb{P}$. As a consequence, the process

$$W_t^{u^{T,N}} \overset{\text{def}}{=} W_t - \int_0^t J_s u_s^{T,N} \mathrm{d}s = W_t - I_t[u^{T,N}] \tag{3.4.8}$$

satisfies

$$\operatorname{Law}_{\mathbb{Q}^u_{T,N}}(W_t^{u^{T,N}}) = \operatorname{Law}_{\mathbb{P}}(W). \tag{3.4.9}$$

To avoid confusion, let us remark on a technical detail. In the definition (3.4.8), the drift $u_s^{T,N}$ is supported on frequencies $|n| \lesssim \langle T \rangle$. The right-hand side of (3.4.8), however, does not contain a further frequency projection. In particular, W and hence $W^{u^{T,N}}$ contain arbitrarily high frequencies. This is related to the definition of the truncated Gibbs measure μ_T, where the density only depends on frequencies $\lesssim \langle T \rangle$, but whose samples contain arbitrarily high frequencies. Put differently, we regularize the interaction but not the samples themselves. To make notational matters even worse, while $W^{u^{T,N}}$ contains all frequencies, we will often work with $\rho_T(\nabla) W^{u^{T,N}}$, which only contains frequencies $\lesssim \langle T \rangle$. Similar as in Section 3.2.1, we define the truncated process $W_t^{T,u^{T,N}}$ by

$$W_t^{T,u^{T,N}} \overset{\text{def}}{=} \rho_T(\nabla) W_t^{u^{T,N}}. \tag{3.4.10}$$

Due to the integral equation (3.4.2), we have that

$$u_t^{T,N} = 1\{t \leqslant \tau_{T,N}\} \Big[-\lambda J_t^T \Big(:(V * (W_t^{T,u^{T,N}})^2) W_t^{T,u^{T,N}}: \Big) + J_t^T \langle \nabla \rangle^{-\frac{1}{2}} :\big(\langle \nabla \rangle^{-\frac{1}{2}} W_t^{T,u^{T,N}} \big)^n: \Big]. \tag{3.4.11}$$

We intend to use $\mathbb{Q}^u_{T,N}$ (and the limit as $N \to \infty$) as a reference measure for $\widetilde{\mu}_T$. Due to (3.4.9), the law of $W_t^{T,u^{T,N}}$ under $\mathbb{Q}^u_{T,N}$ does not depend on N. In our estimates of $u_t^{T,N}$ through the integral equation, it is therefore natural to view $W_t^{T,u^{T,N}}$ as given. Under this perspective, the right-hand side of (3.4.11) no longer depends on u^T and yields an explicit expression for u^T. For comparison, the corresponding equation in the Φ^4_3-model (cf. [BG20a, (14)]) is a linear integral equation. We now start to estimate the drift u^T.

Lemma 3.4.1. For all $1 \leqslant M \leqslant N$, all $S \geqslant 0$, and all $0 < \gamma < \min(1, \beta)$, it holds that

$$\mathbb{E}_{\mathbb{Q}^u_{T,N}}\Big[\int_0^{\tau_M \wedge S} \|u^T_s\|^2_{L^2} \mathrm{d}s\Big] \lesssim \max(S^{1-2\gamma}, 1). \tag{3.4.12}$$

In particular, it holds that

$$\mathbb{Q}^u_{T,N}\big(\tau_{T,M} \leqslant S\big) \lesssim \frac{\max(S^{1-2\gamma}, 1)}{M}. \tag{3.4.13}$$

Proof. We recall from the definition of the drift measure that

$$\mathrm{Law}_{\mathbb{Q}^u_{T,N}}(W^{u^{T,N}}) = \mathrm{Law}_{\mathbb{P}}(W) \qquad \text{and} \qquad \mathrm{Law}_{\mathbb{Q}^u_{T,N}}(W^{T,u^{T,N}}) = \mathrm{Law}_{\mathbb{P}}(W^T)$$

As a result, we obtain that

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}^u_{T,N}}\Big[\int_0^{\tau_M \wedge S} \|u^T_s\|^2_{L^2} \mathrm{d}s\Big] \\
&\leqslant \mathbb{E}_{\mathbb{P}}\Big[\int_0^S \Big\|\lambda J^T_s\big(\,:(V * (W^T_s)^2)W^T_s:\,\big) + J^T_s \langle\nabla\rangle^{-\frac{1}{2}} :\big(\langle\nabla\rangle^{-\frac{1}{2}} W^T_s\big)^n: \Big\|^2_{L^2} \mathrm{d}s\Big] \\
&\lesssim \lambda^2 \mathbb{E}_{\mathbb{P}}\Big[\int_0^S \Big\|\lambda J^T_s\big(\,:(V * (W^T_s)^2)W^T_s:\,\big)\Big\|^2_{L^2} \mathrm{d}s\Big] + \mathbb{E}\Big[\int_0^S \Big\|J^T_s \langle\nabla\rangle^{-\frac{1}{2}} :\big(\langle\nabla\rangle^{-\frac{1}{2}} W^T_s\big)^n: \Big\|^2_{L^2} \mathrm{d}s\Big].
\end{aligned}$$

For the first summand, we obtain from the definition of J^T_s and Lemma 3.2.16 that

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\Big[\int_0^S \Big\|\lambda J^T_s\big(\,:(V * (W^T_s)^2)W^T_s:\,\big)\Big\|^2_{L^2} \mathrm{d}s\Big] &\lesssim \Big(\int_0^S \langle t\rangle^{-2\gamma} \mathrm{d}t\Big) \sup_{t \geqslant 0} \mathbb{E}\big[\|\,:(V * (W^T_t)^2)W^T_t:\,\|^2_{H^{-\frac{3}{2}+\gamma}}\big] \\
&\lesssim \max(S^{1-2\gamma}, 1).
\end{aligned}$$

For the second summand, we obtain from Lemma 3.2.23 that

$$\mathbb{E}\Big[\int_0^S \Big\|J^T_s \langle\nabla\rangle^{-\frac{1}{2}} :\big(\langle\nabla\rangle^{-\frac{1}{2}} W^T_s\big)^n: \Big\|^2_{L^2} \mathrm{d}s\Big] \lesssim \Big(\int_0^S \langle t\rangle^{-4+2\epsilon} \mathrm{d}t\Big) \sup_{t \geqslant 0} \mathbb{E}\big[\|\,:(\langle\nabla\rangle^{-\frac{1}{2}} W_t)^n:\,\|^2_{H^{-\epsilon}}\big] \lesssim 1.$$

This yields the desired estimate. □

Lemma 3.4.2. For all $1 \leqslant M \leqslant N$, $1 \leqslant p < \infty$, and $\gamma < \min(1/2, \beta)$, it holds that

$$\sup_{T,t \geqslant 0} \Big(\mathbb{E}_{\mathbb{Q}^u_{T,N}}\big[\|I_t[u^{T,M}]\|^p_{\mathcal{C}^{\frac{1}{2}+\gamma}_x(\mathbb{T}^3)}\big]\Big)^{\frac{1}{p}} \lesssim_p 1.$$

Furthermore, we have that for any $0 < \alpha < 1$ and $0 < \eta < 1/2$ that

$$\sup_{T \geqslant 0} \Big(\mathbb{E}_{\mathbb{Q}^u_{T,N}} \Big[\| I[u^{T,M}] \|^p_{\mathcal{C}^{\alpha,\eta}_t \mathcal{C}^0_x([0,\infty] \times \mathbb{T}^3)} \Big] \Big)^{\frac{1}{p}} \lesssim_p 1, \tag{3.4.14}$$

where the $\mathcal{C}^{\alpha,\eta}_t \mathcal{C}^0_x$-norm is as in (3.1.19).

The proof of Lemma 3.4.2 is easier than its counterpart [BG20a, (16)] in the Φ^4_3-model, which requires a Gronwall argument. The second estimate (3.4.14) is needed for technical reasons related to tightness, and we encourage the reader to skip its proof on first reading.

Proof. The argument is similar to the proof of Lemma 3.4.1. From the definition of $u^{T,M}$ and $u^{T,N}$, we have that

$$u^{T,M}_s = 1\{ s \leqslant \tau_{T,M} \} u^{T,N}_s. \tag{3.4.15}$$

Thus, we obtain that

$$\| I_t[u^{T,M}] \|_{\mathcal{C}_x^{\frac{1}{2}+\gamma}} \leqslant \int_0^{t \wedge \tau_{T,M}} \| J_s u^{T,N}_s \|_{\mathcal{C}_x^{\frac{1}{2}+\gamma}} \mathrm{d}s \leqslant \int_0^t \| J_s u^{T,N}_s \|_{\mathcal{C}_x^{\frac{1}{2}+\gamma}}. \tag{3.4.16}$$

Using the integral equation (3.4.2) again, we obtain that

$$\begin{aligned} \| I_t[u^{T,M}] \|_{\mathcal{C}_x^{\frac{1}{2}+\gamma}} \leqslant \lambda \int_0^t & \| J_s J^T_s :(V * (W^{T,u^{T,N}}_s)^2) W^{T,u^{T,N}}_s : \|_{\mathcal{C}_x^{\frac{1}{2}+\gamma}} \mathrm{d}s \\ + \int_0^t & \| J_s J^T_s \langle \nabla \rangle^{-\frac{1}{2}} : \big(\langle \nabla \rangle^{-\frac{1}{2}} W^{T,u^{T,N}}_s \big)^n : \|_{\mathcal{C}_x^{\frac{1}{2}+\gamma}} \mathrm{d}s. \end{aligned} \tag{3.4.17}$$

Using that

$$\mathrm{Law}_{\mathbb{Q}^u_{T,N}}(W^{u^{T,N}}) = \mathrm{Law}_{\mathbb{P}}(W),$$

we obtain from Lemma 3.2.20 and Lemma 3.2.23 that

$$\begin{aligned}
&\Big(\mathbb{E}_{\mathbb{Q}^u_{T,N}}\Big[\|I_t[u^{T,M}]\|^p_{\mathcal{C}_x^{\frac{1}{2}+\gamma}}\Big]\Big)^{\frac{1}{p}} \\
&\lesssim \lambda \int_0^t \Big(\mathbb{E}_{\mathbb{P}}\|J_s J_s^T :(V*(W_s^T)^2)W_s^T : \|^p_{\mathcal{C}_x^{\frac{1}{2}+\gamma}}\Big)^{\frac{1}{p}}\mathrm{d}s + \int_0^t \Big(\mathbb{E}_{\mathbb{P}}\|J_s J_s^T \langle\nabla\rangle^{-\frac{1}{2}} :\big(\langle\nabla\rangle^{-\frac{1}{2}}W_s^T\big)^n : \|^p_{\mathcal{C}_x^{\frac{1}{2}+\gamma}}\Big)^{\frac{1}{p}}\mathrm{d}s \\
&\lesssim_p \int_0^t \langle s\rangle^{-1+\gamma-\min(1/2,\beta)+\delta}\mathrm{d}s + \int_0^t \langle s\rangle^{-3+\gamma+\delta}\mathrm{d}s \\
&\lesssim_p 1.
\end{aligned}$$

This completes the proof of the first estimate. The second estimate (3.4.14) follows from a minor modification of the proof. To simplify the notation, we set

$$A(s) \overset{\text{def}}{=} \|J_s J_s^T :(V*(W_s^{T,u^{T,N}})^2)W_s^{T,u^{T,N}} : \|_{L_x^\infty} + \|J_s J_s^T\langle\nabla\rangle^{-\frac{1}{2}} :\big(\langle\nabla\rangle^{-\frac{1}{2}}W_s^{T,u^{T,N}}\big)^n : \|_{L_x^\infty}$$

For any $K \geqslant 1$, we have from a similar argument as in (3.4.17) that

$$\begin{aligned}
\sup_{\substack{0\leqslant t'\leqslant t:\\ t,t'\sim K}} \frac{\|I_t[u^{T,M}]-I_{t'}[u^{T,M}]\|_{L_x^\infty}}{1\wedge|t-t'|^\alpha} &\lesssim \sup_{\substack{0\leqslant t'\leqslant t:\\ t,t'\sim K}} \frac{1}{1\wedge|t-t'|^\alpha}\int_{t'}^t A(s)\mathrm{d}s \\
&\lesssim \int_{s\sim K} A(s)\mathrm{d}s + \Big(\int_{s\sim K} A(s)^{\frac{1}{1-\alpha}}\mathrm{d}s\Big)^{1-\alpha}.
\end{aligned}$$

Proceeding as in the first estimate, this implies that

$$\Big(\mathbb{E}_{\mathbb{Q}^u_{T,N}}\Big[\Big(\sup_{\substack{0\leqslant t'\leqslant t:\\ t,t'\sim K}} \frac{\|I_t[u^{T,M}]-I_{t'}[u^{T,M}]\|_{L_x^\infty}}{1\wedge|t-t'|^\alpha}\Big)^p\Big]\Big)^{\frac{1}{p}} \lesssim K^{-\frac{1}{2}-\gamma}.$$

The desired estimate of the $\mathcal{C}_t^{\alpha,\eta}\mathcal{C}_x^0$-norm then follows by summing over dyadic scales and using a telescoping series if the times are not comparable. □

In Lemma 3.4.1 and Lemma 3.4.2, we controlled the process u^T with respect to the measures $\mathbb{Q}^u_{T,N}$. Unfortunately, the proof of Proposition 3.4.4 below also requires the absence of finite-time blowup for u^T with respect $\mathbb{P}$. This is the subject of the next lemma.

Lemma 3.4.3. For any $T \geqslant 1$, it holds that $T_{\exp}[u^T] = \infty$ $\mathbb{P}$-almost surely.

The proof of the analogue for the Φ^4_3-model (cf. [BG20a, Lemma 5]) extends verbatim to our situation and we omit the minor modifications. To ease the reader's mind, let us briefly explain why the same argument applies here. In most of this section, the most important term in the integral equation (3.4.2) is the first summand. It has the lowest regularity and is closely tied to the interactions in the Hamiltonian. The result of Lemma 3.4.3, however, is essentially a soft statement. If we fix a time $S \geqslant 1$ and only want to rule out $T_{\exp}[u^T] \leqslant S$, the low regularity is inessential and only leads to a loss in powers of S. The main term is then given by the (auxiliary) second summand, which is defocusing and exactly the same as in the Φ^4_3-model.

The next proposition eliminates the stopping time from our drift measures.

Proposition 3.4.4. The family of measures $(\mathbb{Q}^u_{T,N})_{T,N\geqslant 0}$ is tight on $\mathcal{C}^0_t\mathcal{C}^{-1/2-\kappa}_x([0,\infty]\times\mathbb{T}^3)$. For any fixed $T \geqslant 0$, the sequence of measures $(\mathbb{Q}^u_{T,N})_{N\geqslant 0}$ weakly converges to a measure $\mathbb{Q}^u_T$ as $N \to \infty$. For any $S \geqslant 0$, the limiting measure $\mathbb{Q}^u_T$ satisfies

$$\frac{\mathrm{d}\mathbb{Q}^u_T|_{\mathcal{F}_S}}{\mathrm{d}\mathbb{P}|_{\mathcal{F}_S}} = \exp\Big(\int_0^S \int_{\mathbb{T}^3} u^T_s \mathrm{d}B_s - \frac{1}{2}\int_0^S \|u^T_s\|^2_{L^2}\mathrm{d}s\Big). \tag{3.4.18}$$

Our argument differs from the proof of [BG20a, Lemma 7], which is the analog for the Φ^4_3-model. The argument in [BG20a] relies on Kolmogorov's extension theorem, whereas we rely on tightness and Prokhorov's theorem. This is important in the proof of Corollary 3.4.5 below, since the measures $\mathbb{Q}^u_T$ are not (completely) consistent. We also believe that this clarifies the mode of convergence. Before we begin with the proof, we state the following corollary.

Corollary 3.4.5. The measures $\mathbb{Q}^u_T$ weakly convergence to a measure $\mathbb{Q}^u_\infty$ on $\mathcal{C}^0_t\mathcal{C}^{-1/2-\kappa}_x([0,\infty]\times\mathbb{T}^3)$

as $T \to \infty$. For any $S \geqslant 0$, it holds that

$$\frac{\mathrm{d}\mathbb{Q}^u_\infty|_{\mathcal{F}_S}}{\mathrm{d}\mathbb{P}|_{\mathcal{F}_S}} = \exp\Big(\int_0^S \int_{\mathbb{T}^3} u_s \mathrm{d}B_s - \frac{1}{2}\int_0^S \|u_s\|^2_{L^2}\mathrm{d}s\Big), \tag{3.4.19}$$

where u_s is as in (3.4.3).

Proof of Proposition 3.4.4: We first prove that the family of measures $(\mathbb{Q}^u_{T,N})_{T,N\geqslant 0}$, viewed as measures for W, are tight on $\mathcal{C}^0_t\mathcal{C}^{-1/2-\kappa}_x([0,\infty]\times\mathbb{T}^3)$. From (3.4.8), we have that

$$W = W^{u^{T,N}} + I[u^{T,N}]. \tag{3.4.20}$$

Since the law of $W^{u^{T,N}}$ under $\mathbb{Q}^u_{T,N}$ agrees with the law of W under $\mathbb{P}$, an application of Kolmogorov's continuity theorem (cf. [Str11, Theorem 4.3.2]) yields for any $p \geqslant 1$, $0<\alpha<\frac{1}{2}$, and $0<\eta<\kappa/2$ that

$$\mathbb{E}_{\mathbb{Q}^u_{T,N}}\Big[\|W^{u^{T,N}}\|^p_{\mathcal{C}^{\alpha,\eta}_t\mathcal{C}^{-(1+\kappa)/2}_x}\Big] = \mathbb{E}_{\mathbb{P}}\Big[\|W\|^p_{\mathcal{C}^{\alpha,\eta}_t\mathcal{C}^{-(1+\kappa)/2}_x}\Big] \lesssim_p 1.$$

Together with (3.4.20) and Lemma 3.4.2, this implies

$$\mathbb{E}_{\mathbb{Q}^u_{T,N}}\Big[\|W\|^p_{\mathcal{C}^{\alpha,\eta}_t\mathcal{C}^{-(1+\kappa)/2}_x}\Big] \lesssim_p 1$$

Since the embedding $\mathcal{C}^{\alpha,\eta}_t\mathcal{C}^{-(1+\kappa)/2}_x \hookrightarrow \mathcal{C}^0_t\mathcal{C}^{-1/2-\kappa}_x$ is compact, this implies the tightness of the family of measures $(\mathbb{Q}^u_{T,N})_{T,N\geqslant 0}$.

By Prokhorov's theorem, a subsequence of $(\mathbb{Q}^u_{T,N})_N$ weakly converges to a measure $\mathbb{Q}^u_T$. Once we proved (3.4.18), this can be upgraded to weak convergence of the full sequence, since (3.4.18) uniquely identifies the limit. With a slight abuse of notation, we therefore ignore this distinction between a subsequence and the full sequence.

Let $S \geqslant 0$ and let $f\colon \mathcal{C}^0_t\mathcal{C}^{-1/2-\kappa}_x([0,S]\times\mathbb{T}^3) \to \mathbb{R}$ be continuous, bounded, and nonnegative. We write $f(W)$ for $f(W|_{[0,S]})$. Using the weak convergence of $\mathbb{Q}^u_{T,N}$ to $\mathbb{Q}^u_T$, we have that

$$\mathbb{E}_{\mathbb{Q}^u_T}[f(W)] = \lim_{N\to\infty}\mathbb{E}_{\mathbb{Q}^u_{T,N}}[f(W)] = \lim_{N\to\infty}\Big(\mathbb{E}_{\mathbb{Q}^u_{T,N}}[1\{\tau_{T,N}\geqslant S\}f(W)] + \mathbb{E}_{\mathbb{Q}^u_{T,N}}[1\{\tau_{T,N}< S\}f(W)]\Big).$$

Using Lemma 3.4.2, the second term is controlled by

$$\mathbb{E}_{\mathbb{Q}^u_{T,N}}[1\{\tau_{T,N} < S\} f(W)] \leqslant \|f\|_\infty \mathbb{Q}^u_{T,N}(\tau_{T,N} < S) \lesssim \|f\|_\infty \frac{\max(S^{1-2\gamma}, 1)}{N},$$

which converges to zero as $N \to \infty$. Together with the definition of $\mathbb{Q}^u_{T,N}$ and the martingale property of the Girsanov density, this implies

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}^u_T}[f(W)] &= \lim_{N\to\infty} \mathbb{E}_{\mathbb{Q}^u_{T,N}}[1\{\tau_{T,N} \geqslant S\} f(W)] \\
&= \lim_{N\to\infty} \mathbb{E}_{\mathbb{P}}\Big[f(W) 1\{\tau_{T,N} \geqslant S\} \exp\Big(\int_0^{\tau_{T,N}} u^T_s \mathrm{d}B_s - \frac{1}{2}\int_0^{\tau_{T,N}} \|u^T_s\|^2_{L^2} \mathrm{d}s \Big)\Big] \\
&= \lim_{N\to\infty} \mathbb{E}_{\mathbb{P}}\Big[f(W) 1\{\tau_{T,N} \geqslant S\} \exp\Big(\int_0^{S} u^T_s \mathrm{d}B_s - \frac{1}{2}\int_0^{S} \|u^T_s\|^2_{L^2} \mathrm{d}s \Big)\Big].
\end{aligned}$$

Using monotone convergence and Lemma 3.4.3, we obtain

$$\begin{aligned}
&\lim_{N\to\infty} \mathbb{E}_{\mathbb{P}}\Big[f(W) 1\{\tau_{T,N} \geqslant S\} \exp\Big(\int_0^{S} u^T_s \mathrm{d}B_s - \frac{1}{2}\int_0^{S} \|u^T_s\|^2_{L^2} \mathrm{d}s \Big)\Big] \\
&= \mathbb{E}_{\mathbb{P}}\Big[f(W) 1\{T_{\exp}[u^T] > S\} \exp\Big(\int_0^{S} u^T_s \mathrm{d}B_s - \frac{1}{2}\int_0^{S} \|u^T_s\|^2_{L^2} \mathrm{d}s \Big)\Big] \\
&= \mathbb{E}_{\mathbb{P}}\Big[f(W) \exp\Big(\int_0^{S} u^T_s \mathrm{d}B_s - \frac{1}{2}\int_0^{S} \|u^T_s\|^2_{L^2} \mathrm{d}s \Big)\Big].
\end{aligned}$$

□

Proof of Corollary 3.4.5: Due to Proposition 3.4.4, the family of measures $(\mathbb{Q}^u_T)_{T\geqslant 0}$ is tight. By Prokhorov's theorem, it follows that a subsequence weakly converges to a measure $\mathbb{Q}^u_\infty$. Once (3.4.19) is proven, it uniquely identifies the limit $\mathbb{Q}^u_\infty$. With a slight abuse of notation, we therefore assume as before that the whole sequence $\mathbb{Q}^u_T$ converges weakly to $\mathbb{Q}^u_\infty$.

Since $W^T_t = W_t$ and $I^T_t = I_t$ for all $0 \leqslant t \leqslant T/4$ (by our choice of ρ), it follows from the integral equation (3.4.2) that $u^T_s = u_s$ for all $0 \leqslant s \leqslant T/4$. Using (3.4.18), it follows for all $S \leqslant T/4$ that

$$\frac{\mathrm{d}\mathbb{Q}^u_T|_{\mathcal{F}_S}}{\mathrm{d}\mathbb{P}|_{\mathcal{F}_S}} = \exp\Big(\int_0^S \int_{\mathbb{T}^3} u_s \mathrm{d}B_s - \frac{1}{2}\int_0^S \|u_s\|^2_{L^2} \mathrm{d}s \Big). \tag{3.4.21}$$

The corresponding identity (3.4.19) for $\mathbb{Q}^u_\infty$ then follows by taking $T \to \infty$. □

Corollary 3.4.6. For any $T \geqslant 1$, $S \geqslant 1$, and any $0 < \gamma < \min(\beta, 1/2)$, the measure $\mathbb{Q}_T^u$ satisfies the two estimates

$$\mathbb{E}_{\mathbb{Q}_T^u}\Big[\int_0^S \|u_s^T\|_{L^2}^2 \mathrm{d}s\Big] \lesssim \max(S^{1-2\gamma}, 1),$$

$$\sup_{t\geqslant 0}\Big(\mathbb{E}_{\mathbb{Q}_T^u}\big[\|I_t[u^T]\|^p_{\mathcal{C}_x^{\frac{1}{2}+\gamma}}\big]\Big)^{\frac{1}{p}} \lesssim_p 1.$$

The corollary directly follows from Lemma 3.4.1, Lemma 3.4.2, and Proposition 3.4.4.

3.4.2 Absolutely continuity with respect to the drift measure

We recall the definition of the measure $\widetilde{\mu}_T$ from (3.2.10), which states that

$$\frac{\mathrm{d}\widetilde{\mu}_T}{\mathrm{d}\mathbb{P}} = \frac{1}{\mathcal{Z}^{T,\lambda}} \exp\Big(-:\mathcal{V}^{T,\lambda}(W_\infty^T):\Big). \tag{3.4.22}$$

Using Proposition 3.4.4, we obtain that

$$D_T \stackrel{\text{def}}{=} \frac{\mathrm{d}\widetilde{\mu}_T}{\mathrm{d}\mathbb{Q}_T^u} = \frac{\mathrm{d}\widetilde{\mu}_T}{\mathrm{d}\mathbb{P}}\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}_T^u} = \frac{1}{\mathcal{Z}^{T,\lambda}} \exp\Big(-:\mathcal{V}^{T,\lambda}(W_\infty^T): - \int_0^\infty \int_{\mathbb{T}^3} u_t^T \,\mathrm{d}B_t + \frac{1}{2}\int_0^\infty \|u_t^T\|_{L^2}^2 \mathrm{d}t\Big). \tag{3.4.23}$$

Since $\mathrm{d}B_t = \mathrm{d}B_t^{u^T} + u_t^T \mathrm{d}t$, we also obtain that

$$D_T = \frac{1}{\mathcal{Z}^{T,\lambda}} \exp\Big(-:\mathcal{V}^{T,\lambda}(W_\infty^T): - \int_0^\infty \int_{\mathbb{T}^3} u_t^T \,\mathrm{d}B_t^{u^T} - \frac{1}{2}\int_0^\infty \|u_t^T\|_{L^2}^2 \mathrm{d}t\Big). \tag{3.4.24}$$

Proposition 3.4.7 (L^q-bounds)**.** If $n \in \mathbb{N}$ in the definition of u^T is odd and sufficiently large, there exists a $q > 1$ such that

$$\sup_{T\geqslant 0} \mathbb{E}_{\mathbb{Q}_T^u}\Big[|D_T|^q\Big] \lesssim_{n,q} 1. \tag{3.4.25}$$

Remark 3.4.8. We point out two important differences between Proposition 3.4.7 and the corresponding result for the Φ^4_3-model in [BG20a, Lemma 9]. The first difference is a consequence of working with $\widetilde{\mu}_T$ instead of $\bar{\mu}_T$ as described in Section 3.2.1. Barashkov and Gubinelli define and bound the density D_T with respect to the same measure $\mathbb{Q}^u_\infty$ for all $T \geqslant 1$. In contrast, our density is defined with respect to $\mathbb{Q}^u_T$ and we make no statements about the behavior of D_T with respect to $\mathbb{Q}^u_S$ for any $S \neq T$. Since the increments of $T \mapsto \rho_T(\nabla) W_\infty$ are not independent, such a statement would be especially difficult if S and T are close. The second difference is a result of the smoothing effect of the interaction potential V. While the Hartree-nonlinearity allows us to prove the full L^q-bound (3.4.25), the corresponding result in the Φ^4_3-model requires the localizing factor $\exp(-\|W_\infty\|^n_{\mathcal{C}_x^{-1/2-\epsilon}})$.

The rest of this subsection is dedicated to the proof of the L^q-bounds (Proposition 3.4.7). Since we intend to apply the Boué-Dupuis formula to bound the density D_T in $L^q(\mathbb{Q}^u_T)$, we first study the effect of shifts in B^{u^T} on the integral equation (3.4.2). For any $w \in \mathbb{H}_a$, we define

$$\begin{aligned} u_s^{T,w} &\overset{\text{def}}{=} \Xi(W^{T,u^T} + w)_s \\ &= -\lambda :(V * (W_s^{T,u^T} + I_s^T[w])^2)(W_s^{T,u^T} + I_s^T[w]): + J_s^T \langle \nabla \rangle^{-\frac{1}{2}} :(\langle \nabla \rangle^{-\frac{1}{2}} (W_s^{T,u^T} + I_s^T[w]))^n: . \end{aligned}$$

Using the cubic binomial formula (Lemma 3.2.11), we obtain that

$$u_s^{T,w} = -\lambda J_s^T :(V * (W_s^{T,u^T})^2) W_s^{T,u^T}: + r_s^{T,w}, \tag{3.4.26}$$

where the remainder $r_s^{T,w}$ is given by

$$\begin{aligned} r_s^{T,w} = &-\lambda J_s^T \Big((V * :(W_s^{T,u^T})^2:) I_s^T[w] \Big) - 2\lambda J_s^T \Big((V * (W_s^{T,u^T} I_s^T[w])) W_s^{T,u^T} - \mathcal{M}_s^T I_s^T[w] \Big) \\ &- 2\lambda J_s^T \Big((V * (W_s^{T,u^T} I_s^T[w])) I_s^T[w] \Big) - \lambda J_s^T \Big((V * I_s^T[w]^2) W_s^{T,u^T} \Big) - \lambda J_s^T \Big((V * I_s^T[w]^2) I_s^T[w] \Big) \\ &+ J_s^T \langle \nabla \rangle^{-\frac{1}{2}} :(\langle \nabla \rangle^{-\frac{1}{2}} (W_s^{T,u^T} + I_s^T[w]))^n: . \end{aligned}$$

We also define $h^{T,w} = w + u^{T,w}$. We further decompose

$$r_s^{T,w} = \widetilde{r}_s^{T,w} + J_s^T \langle \nabla \rangle^{-\frac{1}{2}} :(\langle \nabla \rangle^{-\frac{1}{2}} (W_s^{T,u^T} + I_s^T[w]))^n: .$$

Before we begin the main argument, we prove the following auxiliary lemma.

Lemma 3.4.9 (Estimate of $\widetilde{r}_t^{T,w}$)**.** Let $\epsilon, \delta > 0$ be small absolute constants and let $n \geqslant n(\delta, \beta)$ be sufficiently large. Then, we have for all $t \geqslant 0$ that

$$\langle t \rangle^{1+\delta} \|\widetilde{r}_t^{T,w}\|_{L_x^2}^2 \lesssim_{n,\delta,\beta,\lambda} C_\epsilon Q_t(W^{T,u^T}) + \epsilon \Big(\|I_t^T[w]\|_{\mathbb{W}_x^{-\frac{1}{2},n+1}}^{n+1} + \int_0^t \|w_s\|_{L_x^2}^2 \mathrm{d}s \Big). \tag{3.4.27}$$

Remark 3.4.10. We emphasize that the implicit constant does not depend on ϵ. In the application of Lemma 3.4.9, we will choose $\epsilon > 0$ sufficiently small depending on $\delta, n, \beta, \lambda$.

Proof. In the following argument, the implicit constants are allowed to depend on n, δ, β, and λ but not on ϵ. We estimate the five terms in $\widetilde{r}_t^{T,w}$ separately and do not require any new ingredients. We only rely on Lemma 3.2.16, Proposition 3.3.7, Hölder's inequality, and Bernstein's inequality. For the first term, we have from the definition of J_t^T and Lemma 3.2.16 that

$$\begin{aligned}
\Big\| J_t^T \Big((V * :(W_t^{T,u^T})^2:) I_t^T[w] \Big) \Big\|_{L_x^2}^2 &\lesssim \langle t \rangle^{-1-2\delta} \Big\| \Big((V * :(W_t^{T,u^T})^2:) I_t^T[w] \Big) \Big\|_{H_x^{-1+\delta}}^2 \\
&\lesssim \langle t \rangle^{-1-2\delta} \Big\| V * :(W_t^{T,u^T})^2: \Big\|_{\mathcal{C}_x^{-1+2\delta}}^2 \|I_t^T[w]\|_{H_x^{1-\delta}}^2 \\
&\lesssim \langle t \rangle^{-1-2\delta} \Big\| V * :(W_t^{T,u^T})^2: \Big\|_{\mathcal{C}_x^{-1+2\delta}}^2 \|I_t^T[w]\|_{\mathbb{W}_x^{-\frac{1}{2},n+1}}^{2\delta} \|I_t^T[w]\|_{H_x^1}^{2-\delta} \\
&\lesssim \langle t \rangle^{-1-2\delta} C_\epsilon \, Q_t(W^{T,u^T}) + \langle t \rangle^{-1-2\delta} \epsilon \Big(\|I_t^T[w]\|_{\mathbb{W}_x^{-\frac{1}{2},n+1}}^{n+1} + \|I_t^T[w]\|_{H_x^1}^2 \Big).
\end{aligned}$$

For the second term, we have from duality and Proposition 3.3.7 for all $0 < \gamma < \min(\beta, 1)$ that

$$\begin{aligned}
&\|J_t \Big((V * (W_t^{T,u^T} I_t^T[w])) W_t^{T,u^T} \; - \mathcal{M}_t^T I_t^T[w] \Big) \|_{L_x^2}^2 \\
&\leqslant \langle t \rangle^{-1-2\gamma} \|J_t^T \Big((V * (W_t^{T,u^T} \; I_t^T[w])) W_t^{T,u^T} \; - \mathcal{M}_t^T I_t^T[w] \Big) \|_{H_x^{-(1-\gamma)}}^2 \\
&\lesssim \langle t \rangle^{-1-2\gamma} Q_t(W^{T,u^T}) \|I_t^T[w]\|_{H_x^{1-\gamma}}^2 \\
&\lesssim \langle t \rangle^{-1-2\gamma} C_\epsilon \, Q_t(W^{T,u^T}) + \langle t \rangle^{-1-2\gamma} \epsilon \Big(\|I_t^T[w]\|_{\mathbb{W}_x^{-\frac{1}{2},n+1}}^{n+1} + \|I_t^T[w]\|_{H_x^1}^2 \Big).
\end{aligned}$$

For the third term, we estimate

$$
\begin{aligned}
\|J_t^T\Big((V*(W_t^{T,u^T}I_t^T[w]))I_t^T[w]\Big)\|_{L_x^2}^2 &\lesssim \langle t\rangle^{-3}\|V*(W_t^{T,u^T}I_t^T[w])\|_{L_x^4}^2\|I_t^T[w]\|_{L_x^4}^2\\
&\lesssim \langle t\rangle^{-3}\|W_t^{T,u^T}\|_{L_x^\infty}^2\|I_t^T[w]\|_{L_x^4}^4\\
&\lesssim \langle t\rangle^{-3+1+2\delta}\|W_t^{T,u^T}\|_{\mathcal{C}_x^{-\frac{1}{2}-\delta}}^2\|I_t^T[w]\|_{L_x^4}^4\\
&\lesssim C_\epsilon\langle t\rangle^{-2+2\delta}\,\|W_t^{T,u^T}\|_{\mathcal{C}_x^{-\frac{1}{2}-\delta}}^{2\frac{4+\delta}{\delta}}+\epsilon\langle t\rangle^{-2+2\delta}\|I_t^T[w]\|_{L_x^4}^{4+\delta}\\
&\lesssim C_\epsilon\langle t\rangle^{-2+2\delta}Q_t(W^{T,u^T})+\epsilon\langle t\rangle^{-2+2\delta}\Big(\|I_t^T[w]\|_{\mathbb{W}_x^{-\frac{1}{2},n+1}}^{n+1}+\|I_t^T[w]]\|_{H_x^1}^2\Big).
\end{aligned}
$$

In the last line, we use [BG20a, Lemma 20].

The fourth term can be estimated exactly like the third term. To estimate the fifth term, we only rely on Hölder's inequality, Bernstein's inequality, and the Fourier support condition of $I_t^T[w]$. We have that

$$
\begin{aligned}
&\|J_t^T\Big((V*I_t^T[w]^2)I_t^T[w]\Big)\|_{L_x^2}^2 \lesssim \langle t\rangle^{-3}\|(V*I_t^T[w]^2)I_t^T[w]\|_{L_x^2}^2 \lesssim \langle t\rangle^{-3}\|I_t^T[w]\|_{L_x^6}^6 \lesssim \langle t\rangle^{-3+\frac{\delta}{2}}\|I_t^T[w]\|_{L_x^{\frac{6}{6+\delta}}}^6\\
&\lesssim \langle t\rangle^{-3+\frac{\delta}{2}}\|I_t^T[w]\|_{\mathbb{W}_x^{-\frac{1}{2},\frac{4}{\delta}}}^4\|I_t^T[w]\|_{H_x^1}^2 \lesssim \langle t\rangle^{-3+2\delta}\|I_t^T[w]\|_{\mathbb{W}_x^{-\frac{1}{2},\frac{4}{\delta}}}^{4+\delta}\|I_t^T[w]\|_{H_x^1}^{2-\delta}\\
&\lesssim C_\epsilon\langle t\rangle^{-3+2\delta}+\epsilon\langle t\rangle^{-3+2\delta}\Big(\|I_t^T[w]\|_{\mathbb{W}_x^{-\frac{1}{2},n+1}}^{n+1}+\|I_t^T[w]\|_{H_x^1}^2\Big).
\end{aligned}
$$

In the second last inequality, we used that $\|I_t^T[w]\|_{H_x^1}\lesssim\langle t\rangle^{\frac{3}{2}}\|I_t^T[w]\|_{H_x^{-1/2}}$. This completes the estimate of all five terms in $\widetilde{r}_t^{T,w}$ and hence the proof. □

Equipped with Lemma 3.4.9, we can now prove the L^q-bound for D_T.

Proof of Proposition 3.4.7: The proof splits into two steps.

Step 1: Formulation as a variational problem. In order to prove the desired estimate (3.4.25), it suffices to obtain a lower bound on $-\log \mathbb{E}_{\mathbb{Q}_T^u}[D_T^q]$. Using the Boué-Dupuis formula, we obtain

$$\begin{aligned}
&-\log \mathbb{E}_{\mathbb{Q}_T^u}[D_T^q] - q\log(\mathcal{Z}^{T,\lambda}) \\
&= -\log \mathbb{E}_{\mathbb{Q}_T^u}\Big[\exp\Big(-q\Big(:\mathcal{V}^{T,\lambda}(W_\infty^{T,u^T} + I_\infty^T[u]): - \int_0^\infty \int_{\mathbb{T}^3} u_t^T \,\mathrm{d}B_t^{u^T} - \frac{1}{2}\int_0^\infty \|u_t^T\|_{L^2}^2 \mathrm{d}t\Big)\Big)\Big] \\
&= \inf_{w\in\mathbb{H}_a} \mathbb{E}\Big[q\Big(:\mathcal{V}^{T,\lambda}(W_\infty^{T,u^T} + I_\infty^T[w] + I_\infty^T[u^{T,w}]): + \int_0^\infty \int_{\mathbb{T}^3} u_t^{T,w} \mathrm{d}B_t^{u^T} + \int_0^\infty \int_{\mathbb{T}^3} u_t^{T,w} w_t \mathrm{d}x\mathrm{d}t \\
&\qquad + \frac{1}{2}\int_0^\infty \|u_t^{T,w}\|_{L^2}^2 \mathrm{d}t\Big) + \frac{1}{2}\int_0^\infty \|w_t\|_{L^2}^2 \mathrm{d}t\Big].
\end{aligned}$$

Since $T \mapsto \int_0^T \int_{\mathbb{T}^3} u_t^{T,w} \mathrm{d}B_t^{u^T}$ is a martingale, its expectation vanishes. We now insert the change of variables $u^{T,w} = h^{T,w} - w$ into the formula above, and obtain that

$$\begin{aligned}
&-\log \mathbb{E}_{\mathbb{Q}_T^u}[D_T^q] - q\log(\mathcal{Z}^{T,\lambda}) \\
&= \inf_{w\in L^2_{t,x_a}} \mathbb{E}_{\mathbb{Q}_T^u}\Big[q\Big(:\mathcal{V}^{T,\lambda}(W_\infty^{T,u^T} + I_\infty^T[h^{T,w}]): + \frac{1}{2}\int_0^\infty \|h_t^w\|_{L^2}^2 \mathrm{d}t - \frac{1}{2}\int_0^\infty \|w_t\|_{L^2}^2 \mathrm{d}t\Big) + \frac{1}{2}\int_0^\infty \|w_t\|_{L^2}^2 \mathrm{d}t\Big] \\
&= \inf_{w\in\mathbb{H}_a} \mathbb{E}_{\mathbb{Q}_T^u}\Big[q\Big(:\mathcal{V}^{T,\lambda}(W_\infty^{T,u^T} + I_\infty^T[h^{T,w}]): + \frac{1}{2}\int_0^\infty \|h_t^w\|_{L^2}^2 \mathrm{d}t\Big) - \frac{q-1}{2}\int_0^\infty \|w_t\|_{L^2}^2 \mathrm{d}t\Big].
\end{aligned}$$

Since we want to obtain a lower bound, the most dangerous term in the expression above is $-\frac{q-1}{2}\int_0^\infty \|w_t\|_{L^2}^2 \mathrm{d}t$. Using our previous information about the variational problem (Proposition 3.3.1 and Proposition 3.3.3) and the nonnegativity of $\mathcal{V}(I_\infty^T[h^{T,w}])$, we obtain that

$$-\log \mathbb{E}_{\mathbb{Q}_T^u}[D_T^q] \geqslant -C + \inf_{w\in\mathbb{H}_a} \mathbb{E}_{\mathbb{Q}_T^u}\Big[\frac{1}{4}\int_0^\infty \|l_t^T(h^{T,w})\|_{L^2}^2 \mathrm{d}t - \frac{q-1}{2}\int_0^\infty \|w_t\|_{L^2}^2 \mathrm{d}t\Big]. \tag{3.4.28}$$

Recalling the definition of $l_t^T(h^{T,w})$ from Proposition 3.3.1 and (3.4.26), we obtain that

$$\begin{aligned}
l_t^T(h^{T,w}) &= h_t^{T,w} + \lambda J_t^T :(V * (W_t^{T,u^T})^2) W_t^{T,u^T}: \\
&= (u_t^{T,w} + w_t) + J_t^T :(V * (W_t^{T,u^T})^2) W_t^{T,u^T}: \\
&= (r_t^{T,w} + w_t).
\end{aligned}$$

Together with our previous estimate, this leads to

$$-\log \mathbb{E}_{\mathbb{Q}_T^u}[D_T^q] \geqslant -C + \inf_{w\in\mathbb{H}_a} \mathbb{E}_{\mathbb{Q}_T^u}\Big[\frac{1}{4}\int_0^\infty \|w_t + r_t^{T,w}\|_{L^2}^2 \mathrm{d}t - \frac{q-1}{2}\int_0^\infty \|w_t\|_{L^2}^2 \mathrm{d}t\Big].$$

By choosing q sufficiently close to one, it only remains to establish

$$\mathbb{E}\int_0^\infty \|w_t\|_{L^2}^2 \mathrm{d}t \lesssim 1 + \mathbb{E}\int_0^\infty \|w_t + r_t^{T,w}\|_{L^2}^2 \mathrm{d}t. \tag{3.4.29}$$

This bound is proven via a Gronwall-type argument.

Step 2: Gronwall-type argument. This step crucially relies on the smoother term in the definition of the drift (3.4.2). We essentially follow the proof of [BG20a, Lemma 11]. As in [BG20a], we introduce the auxiliary process

$$\mathrm{Aux}_s(W^{T,u^T}, w) = \sum_{i=0}^n \binom{n}{i} \langle\nabla\rangle^{-\frac{1}{2}} J_s^T \Big(:(\langle\nabla\rangle^{-\frac{1}{2}} W_s^{T,u^T})^i\!: (\langle\nabla\rangle^{-\frac{1}{2}} I_s^T[w])^{n-i}\Big). \tag{3.4.30}$$

With this notation, it holds that $r^{T,w} = \widetilde{r}^{T,w} + \mathrm{Aux}(W^{T,u^T}, w)$. We then expand

$$\begin{aligned} w_s^2 &= 2(w_s + r_s^{T,w})^2 - 4 w_s r_s^{T,w} - 2(r_s^{T,w})^2 - w_s^2 \\ &= 2(w_s + r_s^{T,w})^2 - 4 w_s \widetilde{r}_s^{T,w} - 2(r_s^w)^2 - w_s^2 - 4\,\mathrm{Aux}_s(W^{T,u^T}, w). \end{aligned} \tag{3.4.31}$$

Using Itô's integration by parts formula, we have for all $s \leqslant t$ that

$$\begin{aligned} &4\int_0^t \int_{\mathbb{T}^3} \mathrm{Aux}_s(W^{T,u^T}, w) w_s \mathrm{d}x\mathrm{d}s \\ &= 4\sum_{i=0}^n \binom{n}{i} \int_0^t \int_{\mathbb{T}^3} :(\langle\nabla\rangle^{-\frac{1}{2}} W_s^{T,u^T})^i\!:\; (\langle\nabla\rangle^{-\frac{1}{2}} I_s^T[w])^{n-i} \,(\langle\nabla\rangle^{-\frac{1}{2}} J_s^T w_s) \mathrm{d}x\mathrm{d}s \\ &= 4\sum_{i=0}^n \frac{1}{n+1-i}\binom{n}{i} \int_0^t \int_{\mathbb{T}^3} :(\langle\nabla\rangle^{-\frac{1}{2}} W_s^{T,u^T})^i\!:\; \frac{\partial}{\partial s}(\langle\nabla\rangle^{-\frac{1}{2}} I_s^T[w])^{n+1-i} \mathrm{d}x\mathrm{d}s \\ &= 4\sum_{i=0}^n \frac{1}{n+1-i}\binom{n}{i} \int_{\mathbb{T}^3} :(\langle\nabla\rangle^{-\frac{1}{2}} W_t^{T,u^T})^i\!: (\langle\nabla\rangle^{-\frac{1}{2}} I_t^T[w])^{n+1-i} \mathrm{d}x \\ &- 4\sum_{i=0}^n \frac{1}{n+1-i}\binom{n}{i} \int_0^t \int_{\mathbb{T}^3} (\langle\nabla\rangle^{-\frac{1}{2}} I_s^T[w])^{n+1-i} \mathrm{d}\Big(:(\langle\nabla\rangle^{-\frac{1}{2}} W_s^{T,u^T})^i\!: \Big). \end{aligned}$$

Due to the martingale property, the second summand has zero expectation. After setting

$$\overline{\mathrm{Aux}}_t(W^{T,u^T}, w) \stackrel{\mathrm{def}}{=} \sum_{i=0}^n \frac{1}{n+1-i}\binom{n}{i} \int_{\mathbb{T}^3} :(\langle\nabla\rangle^{-\frac{1}{2}} W_t^{T,u^T})^i\!: (\langle\nabla\rangle^{-\frac{1}{2}} I_t^T[w])^{n+1-i} \mathrm{d}x, \tag{3.4.32}$$

we obtain from (3.4.31) that

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^t \|w_s\|_{L^2}^2 \mathrm{d}s + 4\overline{\mathrm{Aux}}_t(W^{T,u^T}, w)\Big] \\
&= \mathbb{E}\Big[\int_0^t \big(2\|w_s + r_s^{T,w}\|_{L^2}^2 - 4\langle w_s, \widetilde{r}_s^{T,w}\rangle - \|w_s\|_{L^2}^2 - 2\|r_s^{T,w}\|_{L^2}^2\big)\mathrm{d}s\Big] \\
&\leqslant \mathbb{E}\Big[2\int_0^t \|w_s + r_s^{T,w}\|_{L^2}^2 \mathrm{d}s + 4\int_0^T \|\widetilde{r}_s^{T,w}\|_{L^2}^2 \mathrm{d}s\Big].
\end{aligned} \tag{3.4.33}$$

We perform the Gronwall-type argument based on the quantity $\Phi(t)$, which is defined by

$$\Phi(t) \stackrel{\text{def}}{=} \mathbb{E}\int_0^t \|w_s\|_{L^2}^2 \mathrm{d}s + \|I_t^T[w]\|^{n+1}_{\mathbb{W}_x^{-\frac{1}{2},n+1}}. \tag{3.4.34}$$

By [BG20a, Lemma 12] and (3.4.33), we have that

$$\Phi(t) \lesssim 1 + \mathbb{E}\bigg[\int_0^t \|w_s\|_{L^2}^2 \mathrm{d}s + \overline{\mathrm{Aux}}_t(W^{T,u^T}, w)\bigg] \lesssim 1 + \mathbb{E}\bigg[\int_0^t \|r_s^{T,w} + w_s\|_{L^2}^2 \mathrm{d}s + \int_0^t \|\widetilde{r}_s^{T,w}\|_{L^2}^2 \mathrm{d}s\bigg].$$

From Lemma 3.4.9, we obtain for $\epsilon, \delta > 0$ that

$$\begin{aligned}
\Phi(t) &\lesssim_\delta 1 + \mathbb{E}\bigg[\int_0^t \|r_s^{T,w} + w_s\|_{L^2}^2 \mathrm{d}s + C_\epsilon \int_0^t \langle s\rangle^{-1-\delta} Q_s(\mathbb{W}, \lambda)\mathrm{d}s\bigg] + \epsilon\int_0^t \langle s\rangle^{-1-\delta}\Phi(s)\mathrm{d}s \\
&\lesssim_\delta C_\epsilon + \mathbb{E}\bigg[\int_0^t \|r_s^{T,w} + w_s\|_{L^2}^2 \mathrm{d}s\bigg] + \epsilon \sup_{0\leqslant s\leqslant t} \Phi(s).
\end{aligned}$$

By choosing $\epsilon > 0$ sufficiently small depending on δ, this implies the desired estimate. □

3.4.3 The reference measure

Using our construction of the drift measures $\mathbb{Q}_T^u$, we now provide a short proof of Theorem 3.1.4. As in the rest of this section, we use the truncation parameter T.

Proof of Theorem 3.1.4: For any $1 \leqslant T \leqslant \infty$, we define the reference measure ν_T as

$$\nu_T \stackrel{\text{def}}{=} (W_\infty)_\# \mathbb{Q}_T^u.$$

By using the L^q-bound (Proposition 3.4.7), we have that for all Borel sets $A \subseteq \mathcal{C}_x^{-1/2-\kappa}(\mathbb{T}^3)$ that

$$\mu_T(A) = \widetilde{\mu}_T(W_\infty \in A) = \mathbb{E}_{\mathbb{Q}_T^u}\big[1\{W_\infty \in A\}\, D_T\big] \leqslant \Big(\mathbb{E}_{\mathbb{Q}_T^u}\big[D_T^q\big]\Big)^{\frac{1}{q}} \mathbb{Q}_T^u(W_\infty \in A)^{1-\frac{1}{q}} \lesssim \nu_T(A)^{1-\frac{1}{q}}.$$

This proves the first part of Theorem 3.1.4. Regarding the representation of ν_T, which forms the second part of Theorem 3.1.4, we have that

$$\begin{aligned}
&\nu_T \\
&= \mathrm{Law}_{\mathbb{Q}_T^u}(W_\infty) \\
&= \mathrm{Law}_{\mathbb{Q}_T^u}(W_\infty^u + I_\infty[u^T]) \\
&= \mathrm{Law}_{\mathbb{Q}_T^u}\Big(W_\infty^u - \lambda\rho_T(\nabla)\int_0^\infty J_s^2 :(V * (W_s^{T,u})^2)W_s^{T,u}:\mathrm{d}s + \rho_T(\nabla)\int_0^\infty \langle\nabla\rangle^{-\frac{1}{2}} J_s^2 :\big(\langle\nabla\rangle^{-\frac{1}{2}} W_s^{T,u}\big)^n:\mathrm{d}s\Big) \\
&= \mathrm{Law}_{\mathbb{P}}\Big(W_\infty - \lambda\rho_T(\nabla)\int_0^\infty J_s^2 :(V * (W_s^{T})^2)W_s^{T}:\mathrm{d}s + \rho_T(\nabla)\int_0^\infty \langle\nabla\rangle^{-\frac{1}{2}} J_s^2 :\big(\langle\nabla\rangle^{-\frac{1}{2}} W_s^{T}\big)^n:\mathrm{d}s\Big).
\end{aligned}$$

This completes the proof. □

3.5 Singularity

In this section, we prove Theorem 3.1.5. The majority of this section deals with the singularity for $0 < \beta < 1/2$. The absolute continuity for $\beta > 1/2$ will be deduced from Corollary 3.3.4 and requires no new ingredients. Theorem 3.1.5 is important for the motivation of this series of papers, since we provide the first proof of invariance for a Gibbs measure which is singular with respect to the corresponding Gaussian free field. The methods of this section, however, will not be used in the rest of this two-paper series.

We prove the singularity of the Gibbs measure μ_∞ and the Gaussian free field g through the explicit event in Proposition 3.5.1.

Proposition 3.5.1 (Singularity)**.** Let $0 < \beta < \frac{1}{2}$ and let $\delta > 0$ be sufficiently small. Then, there exists a (deterministic) sequence $(S_m)_{m=1}^{\infty} \subseteq \mathbb{R}_{>0}$ converging to infinity such that

$$\lim_{m\to\infty} \frac{1}{S_m^{1-2\beta-\delta}} \int_{\mathbb{T}^3} :(V * (\rho_{S_m}(\nabla)\phi)^2)(\rho_{S_m}(\nabla)\phi)^2: \mathrm{d}x = 0 \quad g\text{-a.s.} \tag{3.5.1}$$

and

$$\lim_{m\to\infty} \frac{1}{S_m^{1-2\beta-\delta}} \int_{\mathbb{T}^3} :(V * (\rho_{S_m}(\nabla)\phi)^2)(\rho_{S_m}(\nabla)\phi)^2: \mathrm{d}x = -\infty \quad \mu_\infty\text{-a.s.} \tag{3.5.2}$$

Here, g is the Gaussian free field, μ_∞ is the Gibbs measure, and $\phi \in \mathcal{C}_x^{-\frac{1}{2}-\kappa}(\mathbb{T}^3)$ denotes the random element.

Remark 3.5.2. In the statement of the proposition, the reader may wish to replace ϕ by W_∞, g by $\mathbb{P}$, and μ_∞ by $\widetilde{\mu}_\infty$. We choose the notation ϕ to emphasize that this is a property of g and μ_∞ only and does not rely on the stochastic control perspective. Of course, the stochastic control perspective is heavily used in the proof.

To simplify the notation, we define

$$\mathbb{W}_s^{S,3} \overset{\text{def}}{=} :(V * (W_s^S)^2)W_s^S: \qquad \text{and} \qquad \mathbb{W}_s^{S,4} \overset{\text{def}}{=} :(V * (W_s^S)^2)(W_s^S)^2: . \tag{3.5.3}$$

We note that the dependence on the interaction potential V is not reflected in this notation. We first study the behavior of the integral of $W_\infty^{S,4}$ with respect to $\mathbb{P}$. This is the easier part of the proof and the statement (3.5.1) follows from the following lemma.

Lemma 3.5.3 (Quartic power under the Gaussian free field)**.** Let $0 < \beta < 1/2$. Then, we have that

$$\sup_{S\geqslant 1} \mathbb{E}_{\mathbb{P}}\Big[\Big(\frac{1}{S^{\frac{1}{2}-\beta}} \int_{\mathbb{T}^3} \mathbb{W}_\infty^{S,4} \mathrm{d}x\Big)^2\Big] \lesssim 1. \tag{3.5.4}$$

Proof. From Proposition 3.2.9, we obtain that

$$\int_{\mathbb{T}^3} \mathbb{W}^{S,4}_\infty \mathrm{d}x = \sum_{\substack{n_1,n_2,n_3,n_4 \in \mathbb{Z}^3:\\ n_{1234}=0}} \Big(\sum_{\pi \in S_4} \widehat{V}(n_{\pi(1)} + n_{\pi(2)}) \Big) \int_0^\infty \int_0^{s_1} \int_0^{s_2} \int_0^{s_3} \mathrm{d}W^{S,n_4}_{s_4} \mathrm{d}W^{S,n_3}_{s_3} \mathrm{d}W^{S,n_2}_{s_2} \mathrm{d}W^{S,n_1}_{s_1}.$$

Since the iterated stochastic integrals are uncorrelated, we obtain that

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\Big[\Big(\int_{\mathbb{T}^3} \mathbb{W}^{S,4}_\infty \mathrm{d}x \Big)^2 \Big] \\
&\lesssim \sum_{\substack{n_1,n_2,n_3,n_4 \in \mathbb{Z}^3:\\ n_{1234}=0}} \Big(\sum_{\pi \in S_4} \widehat{V}(n_{\pi(1)} + n_{\pi(2)}) \Big)^2 \prod_{j=1}^4 \frac{\rho^S_s(n_j)^2}{\langle n_j \rangle^2} \\
&\lesssim \sum_{\substack{n_1,n_2,n_3,n_4 \in \mathbb{Z}^3:\\ n_{1234}=0}} \langle n_{12} \rangle^{-2\beta} \prod_{j=1}^4 \frac{\rho^S_s(n_j)^2}{\langle n_j \rangle^2} \\
&\lesssim \sum_{n_1,n_2,n_3 \in \mathbb{Z}^3} \langle n_{123} \rangle^{-2} \langle n_{12} \rangle^{-2\beta} \prod_{j=1}^3 \frac{\rho^S_s(n_j)^2}{\langle n_j \rangle^2}.
\end{aligned}$$

It now only remains to estimate the sum. Provided that $\beta < 1/2$, we first sum in n_3, then n_2, and finally n_1 to obtain

$$\sum_{n_1,n_2,n_3 \in \mathbb{Z}^3} \langle n_{123} \rangle^{-2} \langle n_{12} \rangle^{-2\beta} \prod_{j=1}^3 \frac{\rho^S_s(n_j)^2}{\langle n_j \rangle^2} \lesssim \sum_{n_1,n_2 \in \mathbb{Z}^3} \langle n_{12} \rangle^{-1-2\beta} \prod_{j=1}^2 \frac{\rho^S_s(n_j)^2}{\langle n_j \rangle^2} \lesssim \sum_{n_1 \in \mathbb{Z}^3} \frac{\rho^S_s(n_1)^2}{\langle n_1 \rangle^{2+2\beta}} \lesssim S^{1-2\beta}.$$

□

We now begin our study of the integral $\int_{\mathbb{T}^3} \mathbb{W}^{S,4}_\infty \mathrm{d}x$ under $\mathbb{Q}^u_\infty$. Naturally, we would like to replace (most) occurrences of W^S by $W^{S,u}$, since the law of $W^{S,u}$ under $\mathbb{Q}^u_\infty$ is explicit. This is the objective of our first (algebraic) lemma.

Lemma 3.5.4. For any $S \geqslant 1$, it holds that

$$\int_{\mathbb{T}^3} \mathbb{W}_\infty^{S,4} \mathrm{d}x = -4\lambda \int_0^\infty \int_{\mathbb{T}^3} (J_s^S \mathbb{W}_s^{S,u,3}) \cdot J_s \mathbb{W}_s^{u,3} \mathrm{d}x \mathrm{d}s \tag{3.5.5}$$

$$+ 4\int_0^\infty \int_{\mathbb{T}^3} (J_s^S \mathbb{W}_s^{S,u,3}) \mathrm{d}B_s^u - 4\lambda \sum_{j=1}^3 \int_0^\infty \int_{\mathbb{T}^3} A_s^{S,j}[u] \cdot J_s \mathbb{W}_s^{u,3} \mathrm{d}x \mathrm{d}s \tag{3.5.6}$$

$$+ 4\sum_{j=1}^3 \int_0^\infty \int_{\mathbb{T}^3} A_s^{S,j}[u] \mathrm{d}B_s^u + 4\int_0^\infty \int_{\mathbb{T}^3} \mathbb{W}_s^{S,3} \Big(J_s \langle \nabla \rangle^{-\frac{1}{2}} :\big(\langle \nabla \rangle^{-\frac{1}{2}} W_s^u\big)^n : \Big) \mathrm{d}x \mathrm{d}s, \tag{3.5.7}$$

where

$$A_s^{S,1}[u] \overset{\text{def}}{=} J_s^S \Big((V * :(W_s^{S,u})^2:) I_s^S[u] \Big) + 2J_s^S \Big(V * (W_s^{S,u} I_s^S[u]) \cdot W_s^{S,u} - \mathcal{M}_s^S I_s^S[u] \Big), \tag{3.5.8}$$

$$A_s^{S,2}[u] = J_s^S \Big((V * (I_s^S[u])^2) W_s^{S,u} \Big) + 2J_s^S \Big((V * (W_s^{S,u} I_s^S[u])) I_s^S[u] \Big), \tag{3.5.9}$$

$$A_s^{S,3}[u] = J_s^S \Big((V * I_s^S[u]^2) I_s^S[u] \Big). \tag{3.5.10}$$

Proof. Using (3.2.27) from Proposition 3.2.9 together with the integral equation for u, i.e. (3.4.3), we obtain that

$$\begin{aligned}
\int_{\mathbb{T}^3} \mathbb{W}_\infty^{S,4} \mathrm{d}x =& 4 \int_0^\infty \int_{\mathbb{T}^3} \mathbb{W}_s^{S,3} \mathrm{d}W_s^S \\
=& 4 \int_0^\infty \int_{\mathbb{T}^3} \mathbb{W}_s^{S,3} (J_s^S u_s) \mathrm{d}x \mathrm{d}s + 4\int_0^\infty \int_{\mathbb{T}^3} \mathbb{W}_s^{S,3} \mathrm{d}W_s^{S,u} \\
=& -4\lambda \int_0^\infty \int_{\mathbb{T}^3} (J_s^S \mathbb{W}_s^{S,3})(J_s \mathbb{W}_s^{u,3}) \mathrm{d}x \mathrm{d}s + 4\int_0^\infty \int_{\mathbb{T}^3} (J_s^S \mathbb{W}_s^{S,3}) \mathrm{d}B_s^u \\
&+4 \int_0^\infty \int_{\mathbb{T}^3} (J_s^S \mathbb{W}_s^{S,3})(J_s \langle \nabla \rangle^{-\frac{1}{2}} :(\langle \nabla \rangle^{-\frac{1}{2}} W_s^u)^n : \mathrm{d}x \mathrm{d}s
\end{aligned} \tag{3.5.11}$$

From the cubic binomial formula (3.2.31) and the definition of $A_s^{S,j}$, it follows that

$$J_s^S \mathbb{W}_s^{S,3} = J_s^S \mathbb{W}_s^{S,u,3} + \sum_{j=1}^3 A_s^{S,j}[u].$$

Inserting this into (3.5.11) leads to the desired identity. □

We begin by studying the right-hand side of (3.5.5), which is the main term. Our first lemma controls the expectation, which will be upgraded to a pointwise estimate later.

Lemma 3.5.5. If $0 < \beta < 1/2$ and $S \geqslant 1$ is sufficiently large, then

$$\mathbb{E}_{\mathbb{Q}^u_\infty}\Big[\int_0^\infty \int_{\mathbb{T}^3} (J^S_s \mathbb{W}^{S,u,3}_s) \cdot J_s \mathbb{W}^{u,3}_s \mathrm{d}x\mathrm{d}s\Big] \gtrsim S^{1-2\beta}. \tag{3.5.12}$$

Proof. Since the law of W^u under $\mathbb{Q}^u_\infty$ coincides with the law of W under $\mathbb{P}$, it holds that

$$\mathbb{E}_{\mathbb{Q}^u_\infty}\Big[\int_0^\infty \int_{\mathbb{T}^3} (J^S_s \mathbb{W}^{S,u,3}_s) \cdot J_s \mathbb{W}^{u,3}_s \mathrm{d}x\mathrm{d}s\Big] = \mathbb{E}_{\mathbb{P}}\Big[\int_0^\infty \int_{\mathbb{T}^3} (J^S_s \mathbb{W}^{S,3}_s) \cdot J_s \mathbb{W}^{3}_s \mathrm{d}x\mathrm{d}s\Big]. \tag{3.5.13}$$

The rest of the proof consists of a tedious but direct calculation. Using the real-valuedness of W and the stochastic integral representation (3.2.25), we have that

$$\begin{aligned}
&\int_{\mathbb{T}^3} (J^S_s \mathbb{W}^{S,3}_s) \cdot J_s \mathbb{W}^{3}_s \mathrm{d}x \\
&= \int_{\mathbb{T}^3} (J^S_s \mathbb{W}^{S,3}_s) \cdot \overline{J_s \mathbb{W}^{3}_s} \mathrm{d}x \\
&= \sum_{n\in\mathbb{Z}^3} \frac{\sigma^S_s(n)\sigma_s(n)}{\langle n\rangle^2} \sum_{\substack{n_1,n_2,n_3\in\mathbb{Z}^3,\\ m_1,m_2,m_3\in\mathbb{Z}^3\\ n_{123}=m_{123}=n}} \Bigg[\Big(\sum_{\pi\in S_3} \widehat{V}(n_{\pi(1)}+n_{\pi(2)})\Big)\Big(\sum_{\tau\in S_3} \widehat{V}(m_{\tau(1)}+m_{\tau(2)})\Big) \\
&\quad \times \Big(\int_0^s \int_0^{s_1} \int_0^{s_2} \mathrm{d}W^{S,n_3}_{s_3} \mathrm{d}W^{S,n_2}_{s_2} \mathrm{d}W^{S,n_1}_{s_1}\Big) \overline{\Big(\int_0^s \int_0^{s_1} \int_0^{s_2} \mathrm{d}W^{T,s_3}_{m_3} \mathrm{d}W^{T,s_2}_{m_2} \mathrm{d}W^{T,s_1}_{m_1}\Big)} \Bigg].
\end{aligned}$$

Taking expectations, we only obtain a non-trivial contribution for $(n_1, n_2, n_3) = (m_1, m_2, m_3)$, and it follows that

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\Big[\int_{\mathbb{T}^3}(J_s^S \mathbb{W}_s^{S,3})\cdot J_s\mathbb{W}_s^3 \mathrm{d}x\Big]\\
=&\sum_{n_1,n_2,n_3\in\mathbb{Z}^3}\Bigg[\frac{\sigma_s^S(n_{123})\sigma_s(n_{123})}{\langle n_{123}\rangle^2}\Big(\sum_{\pi\in S_3}\widehat{V}(n_{\pi(1)}+n_{\pi(2)})\Big)^2\Big(\prod_{j=1}^3\frac{1}{\langle n_j\rangle^2}\Big)\\
&\times\int_0^s\int_0^{s_1}\int_0^{s_2}\Big(\prod_{j=1}^3\big(\sigma_{s_j}(n_j)\sigma_{s_j}^S(n_j)\big)\Big)\mathrm{d}s_3\mathrm{d}s_2\mathrm{d}s_1\Bigg]\\
=&\frac{1}{6}\sum_{n_1,n_2,n_3\in\mathbb{Z}^3}\Bigg[\frac{\sigma_s^S(n_{123})\sigma_s(n_{123})}{\langle n_{123}\rangle^2}\Big(\sum_{\pi\in S_3}\widehat{V}(n_{\pi(1)}+n_{\pi(2)})\Big)^2\Big(\prod_{j=1}^3\frac{1}{\langle n_j\rangle^2}\Big)\Big(\prod_{j=1}^3\int_0^s\sigma_{s_j}(n_j)\sigma_{s_j}^S(n_j)\mathrm{d}s_j\Big)\Bigg].
\end{aligned}
$$

By recalling that $\sigma_s^S=\rho_S\cdot\sigma_s$, integrating in s, using Lemma 3.6.5, and symmetry considerations, we obtain that

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\Big[\int_0^\infty\int_{\mathbb{T}^3}(J_s^S \mathbb{W}_s^{S,3})\cdot J_s\mathbb{W}_s^3 \mathrm{d}x ds\Big]\\
&=\frac{1}{6}\sum_{n_1,n_2,n_3\in\mathbb{Z}^3}\frac{\rho_S(n_{123})}{\langle n_{123}\rangle^2}\Big(\sum_{\pi\in S_3}\widehat{V}(n_{\pi(1)}+n_{\pi(2)})\Big)^2\Big(\prod_{j=1}^3\frac{\rho_S(n_j)}{\langle n_j\rangle^2}\Big)\int_0^\infty\sigma_s(n_{123})^2\Big(\prod_{j=1}^3\rho_s(n_j)^2\Big)\mathrm{d}s\\
&\geqslant c\sum_{n_1,n_2,n_3\in\mathbb{Z}^3}\frac{\rho_S(n_{123})}{\langle n_{123}\rangle^2}\frac{1}{\langle n_{12}\rangle^{2\beta}}\Big(\prod_{j=1}^3\frac{\rho_S(n_j)}{\langle n_j\rangle^2}\Big)\int_0^\infty\sigma_s(n_{123})^2\Big(\prod_{j=1}^3\rho_s(n_j)^2\Big)\mathrm{d}s\\
&-C\sum_{n_1,n_2,n_3\in\mathbb{Z}^3}\frac{\rho_S(n_{123})}{\langle n_{123}\rangle^2}\frac{1}{\langle n_{12}\rangle^{1+2\beta}}\Big(\prod_{j=1}^3\frac{\rho_S(n_j)}{\langle n_j\rangle^2}\Big)\int_0^\infty\sigma_s(n_{123})^2\Big(\prod_{j=1}^3\rho_s(n_j)^2\Big)\mathrm{d}s,
\end{aligned}
$$

where $c,C>0$ are small and large constants depending only on V. The only difference between the two terms lies in the power of $\langle n_{12}\rangle$. The minor term can easily be estimated from above by

$$
\begin{aligned}
&\sum_{n_1,n_2,n_3\in\mathbb{Z}^3}\frac{\rho_S(n_{123})}{\langle n_{123}\rangle^2}\frac{1}{\langle n_{12}\rangle^{1+2\beta}}\Big(\prod_{j=1}^3\frac{\rho_S(n_j)}{\langle n_j\rangle^2}\Big)\int_0^\infty\sigma_s(n_{123})^2\Big(\prod_{j=1}^3\rho_s(n_j)^2\Big)\mathrm{d}s\\
&\lesssim\sum_{n_1,n_2,n_3\in\mathbb{Z}^3}\frac{1}{\langle n_{123}\rangle^2\langle n_{12}\rangle^{1+2\beta}\langle n_1\rangle^2\langle n_2\rangle^2\langle n_3\rangle^2}\\
&\lesssim 1.
\end{aligned}
$$

Using Lemma 3.6.9, the main term can be estimated from below by

$$\begin{aligned}
&\sum_{n_1,n_2,n_3\in\mathbb{Z}^3} \frac{\rho_S(n_{123})}{\langle n_{123}\rangle^2}\frac{1}{\langle n_{12}\rangle^{2\beta}}\Big(\prod_{j=1}^3 \frac{\rho_S(n_j)}{\langle n_j\rangle^2}\Big)\int_0^\infty \sigma_s(n_{123})^2\Big(\prod_{j=1}^3 \rho_s(n_j)^2\Big)\mathrm{d}s \\
&\gtrsim \sum_{\substack{n_1,n_2,n_3\in\mathbb{Z}^3:\\ |n_j-Se_j|\leqslant S/20}} \frac{1}{\langle n_{123}\rangle^2\langle n_{12}\rangle^{2\beta}\langle n_1\rangle^2\langle n_2\rangle^2\langle n_3\rangle^2} \\
&\gtrsim S^{-8-2\beta}\#\{(n_1,n_2,n_3)\in(\mathbb{Z}^3)^3\colon |n_j-Se_j|\leqslant S/20 \text{ for } j=1,2,3\} \\
&\gtrsim S^{1-2\beta}.
\end{aligned}$$

This completes the proof of the lemma. □

Before we can upgrade Lemma 3.5.5, we need the following estimate of the $A^{S,j}$.

Lemma 3.5.6. Let $0<\beta<1/2$, let $\delta>0$ sufficiently small, and let $k\geqslant 1$ be sufficiently large depending on β. For any $v\colon \mathbb{R}_{>0}\times\mathbb{T}^3\to\mathbb{R}$ and any $j=1,2,3$, it then holds that

$$\|A_s^{S,j}[v]\|_{L_x^2}^2 \lesssim \langle s\rangle^{-1-2\beta+20\delta}\Big(Q_s(\mathbb{W}^u)+\|I_s^S[v]\|^k_{\mathcal{C}_x^{-\frac{1}{2}-\delta}}+\|I_s^S[v]\|^2_{H_x^1}\Big). \tag{3.5.14}$$

Remark 3.5.7. As is clear from the proof, this estimate can be slightly refined. Ignoring δ-losses, the worst power $\langle s\rangle^{-1-2\beta}$ only occurs with $\|I_s^S[v]\|^2_{H_x^{1-\beta}}$ instead of $\|I_s^S[v]\|^2_{H_x^1}$. However, (3.5.14) is sufficient for our purposes.

Proof. We treat the estimates for $j=1,2,3$ separately. We first estimate $A_s^{S,1}$, which consists of two terms. For the first summand, we have that

$$\begin{aligned}
&\Big\| J_s^S\Big((V*:(W_s^{S,u})^2:)I_s^S[v]\Big)\Big\|^2_{L_x^2} \\
&\lesssim \langle s\rangle^{-1-2\beta+4\delta}\Big\|\Big((V*:(W_s^{S,u})^2:)I_s^S[v]\Big)\Big\|^2_{H_x^{-1+\beta-2\delta}} \\
&\lesssim \langle s\rangle^{-1-2\beta+4\delta}\|V*:(W_s^{S,u})^2:\|^2_{\mathcal{C}_x^{-1+\beta-\delta}}\|I_s^S[v]\|^2_{H_x^{1-\beta}} \\
&\lesssim \langle s\rangle^{-1-2\beta+4\delta}\|V*:(W_s^{S,u})^2:\|^2_{\mathcal{C}_x^{-1+\beta-\delta}}\|I_s^S[v]\|^{\beta}_{H_x^{-1}}\|I_s^S[v]\|^{2-\beta}_{H_x^1}.
\end{aligned}$$

Provided that $k \gtrsim \beta^{-1}$, the desired statement follows from Young's inequality. The estimate for the second summand is similar, except that in the second inequality above we use the random matrix estimate (Proposition 3.3.7) instead of Hölder's inequality.

Next, we estimate $A_s^{S,2}$. Let $\eta > 0$ remain to be chosen. Using (3.6.13) from Lemma 3.6.7, we can control the first term in $A_s^{S,2}$ by

$$\begin{aligned}
&\Big\| J_s^S \Big((V * (I_s^S[v])^2) W_s^{S,u} \Big) \Big\|_{L_x^2}^2 \\
&\lesssim \langle s \rangle^{-2+4\delta} \Big\| \langle \nabla \rangle^{-\frac{1}{2}-2\delta} \Big((V * (I_s^S[v])^2) W_s^{S,u} \Big) \Big\|_{L_x^2}^2 \\
&\lesssim \langle s \rangle^{-2+4\delta} \| W_s^{S,u} \|_{\mathcal{C}_x^{-\frac{1}{2}-\delta}}^2 \| I_s^S[v] \|_{\mathcal{C}_x^{-\frac{1}{2}-\delta}}^2 \| I_s^S[v] \|_{H_x^{1+4\delta}}^2 \\
&\lesssim \langle s \rangle^{-2+12\delta} \| W_s^{S,u} \|_{\mathcal{C}_x^{-\frac{1}{2}-\delta}}^2 \| I_s^S[v] \|_{\mathcal{C}_x^{-\frac{1}{2}-\delta}}^2 \| I_s^S[v] \|_{H_x^1}^2 \\
&\lesssim \langle s \rangle^{-2+12\delta+8\eta} \| W_s^{S,u} \|_{\mathcal{C}_x^{-\frac{1}{2}-\delta}}^2 \| I_s^S[v] \|_{\mathcal{C}_x^{-\frac{1}{2}-\delta}}^{2+\eta} \| I_s^S[v] \|_{H_x^1}^{2-\eta} \\
&\lesssim \langle s \rangle^{-2+12\delta+8\eta} \Big(\| W_s^{S,u} \|_{\mathcal{C}_x^{-\frac{1}{2}-\delta}}^{\frac{8}{\eta}} + \| I_s^S[v] \|_{\mathcal{C}_x^{-\frac{1}{2}-\delta}}^{\frac{4(2+\eta)}{\eta}} + \| I_s^S[v] \|_{H_x^1}^2 \Big).
\end{aligned}$$

After choosing $\eta = 10k^{-1}$, the desired estimate follows provided that $k \gtrsim (1/2-\beta)^{-1}$. The only difference in the estimate of the second term in $A_s^{S,2}$ is that we use (3.6.12) instead of (3.6.13).

We now turn to the estimate of $A_s^{S,3}$. Arguing exactly as in our estimate for $A_s^{S,2}$, we obtain that

$$\Big\| J_s^S \Big((V * (I_s^S[v])^2) I_s^S[v] \Big) \Big\|_{L_x^2}^2 \lesssim \langle s \rangle^{-2+12\delta+8\eta} \| I_s^S[v] \|_{\mathcal{C}_x^{-\frac{1}{2}-\delta}}^{4+\eta} \| I_s^S[v] \|_{H_x^1}^{2-\eta}.$$

Using Young's inequality, this contribution is acceptable. □

We are now ready to upgrade our bound on the expectation from Lemma 3.5.5 into a pointwise statement. The main tool will be the Boué-Dupuis formula.

Lemma 3.5.8. For any $\delta > 0$, there exists a sequence $(S_m)_{m=1}^{\infty}$ converging to infinity such that

$$\lim_{m \to \infty} \frac{1}{S_m^{1-2\beta-\delta}} \int_0^\infty \int_{\mathbb{T}^3} \Big(J_s^{S_m} W_s^{S_m,u,3} \Big) \Big(J_s W_s^{u,3} \Big) \mathrm{d}x \mathrm{d}s = \infty \qquad \mathbb{Q}_\infty^u\text{-a.s.} \tag{3.5.15}$$

Proof. Let $k \geqslant 1$ remain to be chosen. We define the auxiliary function

$$G_S = \frac{1}{S^{1-2\beta-\delta}} \int_0^\infty \int_{\mathbb{T}^3} \Big(J_s^S W_s^{S,u,3} \Big) \Big(J_s W_s^{u,3} \Big) \mathrm{d}x \mathrm{d}s + \sup_{0 \leqslant s < \infty} \| W_s^u \|^k_{\mathcal{C}_x^{-\frac{1}{2}-\delta}(\mathbb{T}^3)}. \tag{3.5.16}$$

We will now show that

$$\lim_{S \to \infty} \mathbb{E}_{\mathbb{Q}^u_\infty} \Big[e^{-G_S} \Big] = 0, \tag{3.5.17}$$

which implies the desired result. We could switch from $(W^u, \mathbb{Q}^u_\infty)$ to $(W, \mathbb{P})$, which we have done several times above. Since the $A_s^{S,j}$ in (3.5.8)-(3.5.10) are defined in terms of W^u, however, we decided not to change the measure.

We define A_s^j similar as in (3.5.8)-(3.5.10), but with J_s^S replaced by J_s, I_s^S replaced by I_s, and $W^{S,u}$ replaced by W^u. Since all our estimates for $A^{S,j}$ were uniform in $S \geqslant 1$, they also hold for A^j. Using the Boué-Dupuis formula (Theorem 3.2.1) and the cubic binomial formula, we have that

$$\begin{aligned}
&-\log \mathbb{E}_{\mathbb{Q}^u_\infty} \Big[e^{-G_S} \Big] \\
&= \inf_{v \in \mathbb{H}_a} \mathbb{E}_{\mathbb{Q}^u} \Bigg[\frac{1}{S^{1-2\beta-\delta}} \int_0^\infty \int_{\mathbb{T}^3} \Big(J_s^S \Big(:(V * (W_s^{S,u} + I_s^S[v])^2)(W_s^{S,u} + I_s^S[v]): \Big) \\
&\times J_s \Big(:(V * (W_s^u + I_s[v])^2)(W_s^u + I_s[v]): \Big) \Big) \mathrm{d}x \mathrm{d}s + \sup_{0 \leqslant s < \infty} \| W_s^u + I_s[v] \|^k_{\mathcal{C}_x^{-\frac{1}{2}-\delta}} + \frac{1}{2} \| v \|^2_{L_s^2 L_x^2} \Bigg] \\
&= \mathbb{E}_{\mathbb{Q}^u_\infty} \Bigg[\frac{1}{S^{1-2\beta-\delta}} \int_0^\infty \int_{T^3} J_s^S \Big(:(V * (W_s^{S,u})^2) W_s^{S,u}: \Big) J_s \Big(:(V * (W_s^u)^2) W_s^u: \Big) \mathrm{d}x \mathrm{d}s \Bigg] && (3.5.18) \\
&+ \inf_{v \in \mathbb{H}_a} \mathbb{E}_{\mathbb{Q}^u_\infty} \Bigg[\sup_{0 \leqslant s < \infty} \| W_s^u + I_s[v] \|^k_{\mathcal{C}_x^{-\frac{1}{2}-\delta}} + \frac{1}{2} \int_0^\infty \| v_s \|^2_{L_x^2} \mathrm{d}s && (3.5.19) \\
&+ \frac{1}{S^{1-2\beta-\delta}} \sum_{j=1}^3 \int_0^\infty \int_{\mathbb{T}^3} (J_s^S \mathbb{W}_s^{S,u,3}) A_s^j[v] \mathrm{d}x \mathrm{d}s + \frac{1}{S^{1-2\beta-\delta}} \sum_{j=1}^3 \int_0^\infty \int_{\mathbb{T}^3} (J_s \mathbb{W}_s^{u,3}) A_s^{S,j}[v] \mathrm{d}x \mathrm{d}s && (3.5.20) \\
&+ \frac{1}{S^{1-2\beta-\delta}} \sum_{i,j=1}^3 \int_0^\infty \int_{\mathbb{T}^3} A_s^{S,i}[v] A_s^j[v] \mathrm{d}x \mathrm{d}s \Bigg]. && (3.5.21)
\end{aligned}$$

The main term is given by (3.5.18). By Lemma 3.5.5, we see that (3.5.18) converges to infinity as $S \to \infty$. Thus, it remains to obtain a lower bound on the variational problem in (3.5.19)-(3.5.21). The terms in (3.5.19) are nonnegative and help with the lower bound. In contrast, the terms in

(3.5.20) and (3.5.21) are viewed as errors and will be estimated in absolute value.

Regarding (3.5.19), we briefly note that

$$\mathbb{E}_{\mathbb{Q}^u}\Big[\sup_{0\leqslant s<\infty}\|W_s^u+I_s[v]\|^k_{\mathcal{C}_x^{-\frac{1}{2}-\delta}}\Big]\geqslant \frac{1}{2}\mathbb{E}_{\mathbb{Q}^u_\infty}\Big[\sup_{0\leqslant s<\infty}\|I_s[v]\|^k_{\mathcal{C}_x^{-\frac{1}{2}-\delta}}\Big]-C.$$

In the estimates below, we will often use that $A_s^{S,j}[v]=0$ for all $s\gg S$. We begin with the first term in (3.5.20). We have that

$$\begin{aligned}
&\Big|\frac{1}{S^{1-2\beta-\delta}}\int_0^\infty\int_{\mathbb{T}^3}(J_s^S\mathbb{W}_s^{S,u,3})A_s^j[v]\mathrm{d}x\mathrm{d}t\Big| \\
&\leqslant \frac{1}{S^{1-2\beta-\delta}}\int_0^\infty 1\{s\lesssim S\}\langle s\rangle^{-\frac{1}{2}}\|J_s^S\mathbb{W}_s^{S,u,3}\|^2_{L^2}\mathrm{d}s+\frac{1}{S^{1-2\beta-\delta}}\int_0^\infty 1\{s\lesssim S\}\langle s\rangle^{\frac{1}{2}}\|A_s^j[v]\|^2_{L^2}\mathrm{d}s. \quad (3.5.22)
\end{aligned}$$

For the first term in (3.5.22), we obtain from Lemma 3.2.20 that

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}^u_\infty}\Big[\frac{1}{S^{1-2\beta-\delta}}\int_0^\infty 1\{s\lesssim S\}\langle s\rangle^{-\frac{1}{2}}\|J_s^S\mathbb{W}_s^{S,u,3}\|^2_{L^2}\mathrm{d}s\Big] \\
&\lesssim \frac{1}{S^{1-2\beta-\delta}}\int_0^\infty 1\{s\lesssim S\}\langle s\rangle^{-\frac{1}{2}-2\beta+2\delta}\mathbb{E}_{\mathbb{Q}^u}\Big[\|\mathbb{W}_s^{S,u,3}\|^2_{H_x^{-\frac{3}{2}+\beta-\delta}}\Big]\mathrm{d}s \\
&\lesssim \frac{1}{S^{1-2\beta-\delta}}\int_0^\infty 1\{s\lesssim S\}\langle s\rangle^{-\frac{1}{2}-2\beta+2\delta}\mathrm{d}s \\
&\lesssim 1.
\end{aligned} \quad (3.5.23)$$

For the second term in (3.5.22), we obtain from Lemma 3.5.6 that

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}^u_\infty}\Big[\frac{1}{S^{1-2\beta-\delta}}\int_0^\infty 1\{s\lesssim S\}\langle s\rangle^{\frac{1}{2}}\|A_s^j[v]\|^2_{L^2}\mathrm{d}s\Big] \\
&\lesssim \frac{1}{S^{1-2\beta-\delta}}\int_0^\infty 1\{s\lesssim S\}\langle s\rangle^{-\frac{1}{2}-2\beta}\mathbb{E}_{\mathbb{Q}^u_\infty}\Big[Q_s(W^u)\Big]ds \\
&+\frac{1}{S^{1-2\beta-\delta}}\mathbb{E}_{\mathbb{Q}^u}\Big[\int_0^\infty 1\{s\lesssim S\}\langle s\rangle^{-\frac{1}{2}-2\beta}\Big(\|I_s[v]\|^k_{\mathcal{C}_x^{-\frac{1}{2}-\delta}}+\|I_s[v]\|^2_{H^1_x}\Big)\mathrm{d}s\Big] \\
&\lesssim 1+S^\delta\max(S^{-\frac{1}{2}},S^{2\beta-1})\mathbb{E}_{\mathbb{Q}^u_\infty}\Big[\sup_{0\leqslant s<\infty}\Big(\|I_s[v]\|^k_{\mathcal{C}_x^{-\frac{1}{2}-\delta}}+\|I_s[v]\|^2_{H^1_x}\Big)\Big] \\
&\lesssim 1+S^\delta\max(S^{-\frac{1}{2}},S^{2\beta-1})\mathbb{E}_{\mathbb{Q}^u_\infty}\Big[\sup_{0\leqslant s<\infty}\|I_s[v]\|^k_{\mathcal{C}_x^{-\frac{1}{2}-\delta}}+\|v\|^2_{L^2_sL^2_x}\Big].
\end{aligned} \quad (3.5.24)$$

In the last line, we also Lemma 3.6.8. Since $S \to \infty$, this contribution can be absorbed in the coercive term (3.5.22). The estimate of the second summand in (3.5.20) is exactly the same.

Regarding the error terms in (3.5.21), we have that

$$\begin{aligned}
&\Big| \frac{1}{S^{1-2\beta-\delta}} \sum_{i,j=1}^{3} \int_0^\infty \int_{\mathbb{T}^3} A_s^{S,i}[v] A_s^j[v] \mathrm{d}x\mathrm{d}s \Big| \\
&\lesssim \frac{1}{S^{1-2\beta-\delta}} \sum_{j=1}^{3} \int_0^\infty 1\{s \lesssim S\} \Big(\|A_s^{S,i}[v]\|_{L_x^2}^2 + \|A_s^j[v]\|_{L_x^2}^2 \Big) \mathrm{d}x\mathrm{d}s.
\end{aligned}$$

The right-hand side can now be controlled using the same (or simpler) estimates as for the second summand in (3.5.22). This completes the proof. □

Essentially the same estimates as in the previous proof can also be used to control the minor terms in (3.5.6) and (3.5.7). We record them in the following lemma.

Lemma 3.5.9. Let $0 < \beta < 1/2$, let $\delta > 0$ and let $j = 1, 2, 3$. Then, it holds that

$$\lim_{S\to\infty} \mathbb{E}_{\mathbb{Q}_\infty^u} \Big[\Big(\frac{1}{S^{\frac{1}{2}-\beta+\delta}} \int_0^\infty \int_{\mathbb{T}^3} J_s^S W_s^{S,u,3} \mathrm{d}B_s^u \Big)^2 \Big] = 0, \tag{3.5.25}$$

$$\lim_{S\to\infty} \mathbb{E}_{\mathbb{Q}_\infty^u} \Big[\frac{1}{\max(S^{1-3\beta+\delta}, 1)} \Big| \int_0^\infty \int_{\mathbb{T}^3} A_s^{S,j}[u] \cdot J_s \mathbb{W}_s^{u,3} \mathrm{d}x\mathrm{d}s \Big| \Big] = 0, \tag{3.5.26}$$

$$\lim_{S\to\infty} \mathbb{E}_{\mathbb{Q}_\infty^u} \Big[\Big(\frac{1}{\max(S^{\frac{1}{2}-2\beta+\delta}, 1)} \int_0^\infty \int_{\mathbb{T}^3} A_s^{S,j}[u] \mathrm{d}B_s^u \Big)^2 \Big] = 0, \tag{3.5.27}$$

$$\lim_{S\to\infty} \mathbb{E}_{\mathbb{Q}_\infty^u} \Big[\frac{1}{S^\delta} \Big| \int_0^\infty \int_{\mathbb{T}^3} (J_s^S \mathbb{W}_s^{S,3}) (J_s \langle\nabla\rangle^{-\frac{1}{2}} :(\langle\nabla\rangle^{-\frac{1}{2}} W_s^u)^n:) \mathrm{d}x\mathrm{d}s \Big| \Big] = 0. \tag{3.5.28}$$

Proof. We begin with the proof of (3.5.25). Using Itô's isometry, we have that

$$\mathbb{E}_{\mathbb{Q}^u} \Big[\Big(\frac{1}{S^{\frac{1}{2}-\beta+\delta}} \int_0^\infty \int_{\mathbb{T}^3} J_s^S W_s^{S,u,3} \mathrm{d}B_s^u \Big)^2 \Big] = \frac{1}{S^{1-2\beta+2\delta}} \int_0^\infty \mathbb{E}_{\mathbb{Q}_\infty^u} \Big[\|J_s^S W_s^{S,u,3}\|_{L_x^2}^2 \Big] \mathrm{d}s.$$

Arguing essentially as in (3.5.23), we obtain that

$$\frac{1}{S^{1-2\beta+2\delta}}\int_0^\infty \mathbb{E}_{\mathbb{Q}_\infty^u}\Big[\|J_s^S W_s^{S,u,3}\|_{L_x^2}^2\Big]\mathrm{d}s \lesssim \frac{1}{S^{1-2\beta+2\delta}}\int_0^\infty 1\{s\lesssim S\}\langle s\rangle^{-2\beta+\delta}\mathrm{d}s \lesssim S^{-\delta},$$

which yields (3.5.25).

We now turn to (3.5.26). Using Lemma 3.5.6 and Corollary 3.4.6, we have for all $\epsilon > 0$ that

$$\mathbb{E}_{\mathbb{Q}_\infty^u}\Big[\|A_s^{S,j}[u]\|_{L_x^2}^2\Big] \lesssim \langle s\rangle^{-1-2\beta+20\epsilon}\big(1+\big(\langle s\rangle^{1-(\frac{1}{2}+\beta)+\epsilon}\big)^2\big) \lesssim \langle s\rangle^{-4\beta+40\epsilon}. \tag{3.5.29}$$

Using Lemma 3.2.16 and (3.5.29), we obtain that

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}_\infty^u}\bigg[\bigg|\int_0^\infty\int_{\mathbb{T}^3} A_s^{S,j}[u]\cdot J_s\mathbb{W}_s^{u,3}\mathrm{d}x\mathrm{d}s\bigg|\bigg]\\
&\lesssim \mathbb{E}_{\mathbb{Q}_\infty^u}\bigg[\int_0^\infty 1\{s\lesssim S\}\langle s\rangle^{-\beta}\|J_s\mathbb{W}_s^{u,3}\|_{L_x^2}^2\mathrm{d}s\bigg] + \mathbb{E}_{\mathbb{Q}_\infty^u}\bigg[\int_0^\infty 1\{s\lesssim S\}\langle s\rangle^{\beta}\|A_s^{S,j}[u]\|_{L_x^2}^2\mathrm{d}s\bigg]\\
&\lesssim \int_0^\infty 1\{s\lesssim S\}\langle s\rangle^{-3\beta+40\epsilon}\mathrm{d}s \lesssim S^{-\frac{\delta}{2}}\max(1,S^{1-3\beta+\delta}).
\end{aligned}$$

Next, we prove (3.5.26). Using Itô's isometry and (3.5.29), we have that

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}_\infty^u}\bigg[\bigg(\int_0^\infty\int_{\mathbb{T}^3} A_s^{S,j}[u]\mathrm{d}B_s^u\bigg)^2\bigg] &\lesssim \mathbb{E}_{\mathbb{Q}_\infty^u}\bigg[\int_0^\infty 1\{s\lesssim S\}\|A_s^{S,j}[u]\|_{L_x^2}^2\mathrm{d}s\bigg]\\
&\lesssim \int_0^\infty 1\{s\lesssim S\}\langle s\rangle^{-4\beta+40\epsilon}\mathrm{d}s\\
&\lesssim S^{-\delta}\max(S^{\frac{1}{2}-2\beta+\delta},1)^2.
\end{aligned}$$

Finally, we turn to (3.5.28), which is the most regular term. We first recall the algebraic identity $J_s^S\mathbb{W}_s^{S,3} = J^S\mathbb{W}_s^{S,u,3} + \sum_{j=1}^3 A_s^{S,j}[u]$. Then, Lemma 3.2.16 and (3.5.29) yield

$$\mathbb{E}_{\mathbb{Q}_\infty^u}\Big[\|J_s^S\mathbb{W}_s^{S,3}\|_{L_x^2}^2\Big] \lesssim \langle s\rangle^{-2\beta+2\epsilon}. \tag{3.5.30}$$

From Lemma 3.2.23, we have that

$$\mathbb{E}_{\mathbb{Q}_\infty^u}\Big[\|J_s\langle\nabla\rangle^{-\frac{1}{2}} :\!\big(\langle\nabla\rangle^{-\frac{1}{2}}W_s^u\big)^n\!: \|_{L_x^2}^2\Big] \lesssim \langle s\rangle^{-4+2\epsilon}. \tag{3.5.31}$$

By combining (3.5.30) and (3.5.31), we obtain

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}^u_\infty}\bigg[\bigg|\int_0^\infty \int_{\mathbb{T}^3} (J_s^S \mathbb{W}_s^{S,3})(J_s \langle\nabla\rangle^{-\frac{1}{2}} :(\langle\nabla\rangle^{-\frac{1}{2}} W_s^u)^n:) \mathrm{d}x\mathrm{d}s\bigg|\bigg] \\
&\lesssim \mathbb{E}_{\mathbb{Q}^u_\infty}\bigg[\int_0^\infty 1\{s \lesssim S\}\langle s\rangle^{-1} \|J_s^S \mathbb{W}_s^{S,3}\|_{L^2_x}^2 ds\bigg] + \mathbb{E}_{\mathbb{Q}^u_\infty}\bigg[\int_0^\infty \langle s\rangle \|J_s\langle\nabla\rangle^{-\frac{1}{2}} :\big(\langle\nabla\rangle^{-\frac{1}{2}} W_s^u\big)^n: \|_{L^2_x}^2 \mathrm{d}s\bigg] \\
&\lesssim \int_0^\infty \Big(\langle s\rangle^{-1-2\beta+2\epsilon} + \langle s\rangle^{-3+\epsilon}\Big)\mathrm{d}s \lesssim 1.
\end{aligned}$$

□

We are now ready to prove the main result of this section.

Proof of Proposition 3.5.1: We recall from Lemma 3.5.4 that

$$\frac{1}{S^{1-2\beta-\delta}} \int_{\mathbb{T}^3} \mathbb{W}_\infty^{S,4} \mathrm{d}x = -\frac{4\lambda}{S^{1-2\beta-\delta}} \int_0^\infty \int_{\mathbb{T}^3} (J_s^S \mathbb{W}_s^{S,u,3}) \cdot J_s \mathbb{W}_s^{u,3} \mathrm{d}x\mathrm{d}s + \mathcal{R}^S(\mathbb{W}^u, u), \tag{3.5.32}$$

where the remainder $\mathcal{R}(\mathbb{W}^u, u)$ contains the terms from (3.5.6) and (3.5.7) with an additional $S^{-1+2\beta+\delta}$. By Lemma 3.5.8, there exists a deterministic sequence S_m such that the first summand in (3.5.32) converges to $-\infty$ almost surely with respect to $\mathbb{Q}^u_\infty$. Since $0 < \beta < 1/2$, we have that

$$1 - 2\beta > \max\Big(\frac{1}{2} - \beta, 1 - 3\beta, \frac{1}{2} - 2\beta, 0\Big).$$

Using Lemma 3.5.9, this implies that the remainder $\mathcal{R}^S(\mathbb{W}^u, u)$ converges to zero in $L^1(\mathbb{Q}^u_\infty)$. By passing to a subsequence if necessary, we can assume that $\mathcal{R}^{S_m}(\mathbb{W}^u, u)$ converges to zero almost surely with respect to $\mathbb{Q}^u_\infty$. Using (3.5.32), this implies that

$$\lim_{m\to\infty} \frac{1}{S_m^{1-2\beta-\delta}} \int_{\mathbb{T}^3} \mathbb{W}_\infty^{S_m,4} \mathrm{d}x = -\infty \qquad \mathbb{Q}^u_\infty\text{-a.s.}$$

Using $\beta < 1/2$ and Lemma 3.5.3, the integral $S^{-1+2\beta+\delta} \int_{\mathbb{T}^3} \mathbb{W}_\infty^{S,4} \mathrm{d}x$ converges to zero in $L^2(\mathbb{P})$. By passing to another subsequence if necessary, we obtain that

$$\lim_{m\to\infty} \frac{1}{S_m^{1-2\beta-\delta}} \int_{\mathbb{T}^3} \mathbb{W}_\infty^{S_m,4} \mathrm{d}x = 0 \qquad \mathbb{P}\text{-a.s.}$$

Since μ_∞ is absolutely continuous with respect to $\nu_\infty = (W_\infty)_\# \mathbb{Q}^u_\infty$ and $g = \mathrm{Law}_{\mathbb{P}}(W_\infty)$, this implies (3.5.1) and (3.5.2). $\square$

Equipped with Corollary 3.3.4 and Proposition 3.5.1, we now provide a short proof of Theorem 3.1.5.

Proof of Theorem 3.1.5: If $0 < \beta < 1/2$, then the mutual singularity of the Gibbs measure μ_∞ and the Gaussian free field g directly follows from Proposition 3.5.1.

If $\beta > 1/2$, we claim that for all $p \geqslant 1$ that

$$\frac{\mathrm{d}\mu_T}{\mathrm{d}g} \in L^p(g) \tag{3.5.33}$$

with uniform bounds in $T \geqslant 1$. Since μ_T converges weakly to μ_∞, this implies the absolute continuity $\mu_\infty \ll g$.

In order to prove the claim, we recall that $\mu_T = (W_\infty)_\# \widetilde{\mu}_T$ and $g = (W_\infty)_\# \mathbb{P}$. Furthermore, we see from (3.2.10) that the density $\mathrm{d}\widetilde{\mu}_T/\mathrm{d}\mathbb{P}$ is a function of W_∞. As a result, we obtain for all $p \geqslant 1$ that

$$\int \Big(\frac{\mathrm{d}\widetilde{\mu}_T}{\mathrm{d}\mathbb{P}}\Big)^p \mathrm{d}\mathbb{P} = \int \Big(\frac{\mathrm{d}\mu_T}{\mathrm{d}g}\Big)^p \mathrm{d}g$$

Thus, it suffices to bound the density $\mathrm{d}\widetilde{\mu}_T/\mathrm{d}\mathbb{P}$ in $L^p(\mathbb{P})$. From the definition of $\widetilde{\mu}_T$ (Definition 3.2.3) and the definition of the renormalized potential energy in (3.3.2), we have that

$$\begin{aligned}
\Big(\frac{\mathrm{d}\widetilde{\mu}_T}{\mathrm{d}\mathbb{P}}\Big)^p &= \frac{1}{\big(\mathcal{Z}^{T,\lambda}\big)^p} \exp\Big(- p :\mathcal{V}^{T,\lambda}(W^T_\infty): \Big) \\
&= \frac{1}{\big(\mathcal{Z}^{T,\lambda}\big)^p} \exp\Big(- \frac{\lambda p}{4} \int_{\mathbb{T}^3} :(V * (W^T_\infty)^2)(W^T_\infty)^2: \mathrm{d}x - p c^{T,\lambda} \Big) \\
&= \frac{\mathcal{Z}^{T,p\lambda}}{\big(\mathcal{Z}^{T,\lambda}\big)^p} \exp(c^T_{p\lambda} - p c^{T,\lambda}) \cdot \frac{1}{\mathcal{Z}^{T,p\lambda}} \exp\Big(- :\mathcal{V}^{T,p\lambda}(W^T_\infty): \Big).
\end{aligned}$$

The first two factors are uniformly bounded in T by Proposition 3.3.3 and Corollary 3.3.4. The last factor is uniformly bounded in $L^1(\mathbb{P})$ for all $T \geqslant 1$ since we only replaced the coupling constant λ by $p\lambda$. This completes the proof of the claim (3.5.33).

□

3.6 Appendix

3.6.1 Probability Theory

In this section we recall two concepts from probability theory, namely, Gaussian hypercontractivity and multiple stochastic integrals.

3.6.1.1 Gaussian hypercontractivity

In several places of this paper, we reduced probabilistic L^p-bounds to probabilistic L^2-bounds using Gaussian hypercontractivity, which is closely related to logarithmic Sobolev embeddings. In the dispersive PDE community, among others, the resulting estimates are known as Wiener chaos estimates. A version of the following lemma can be found in [Sim74, Theorem I.22], [Nua06, Theorem 1.4.1], and most papers on random dispersive PDE.

Lemma 3.6.1. Let $k \geqslant 1$ and let $f\colon (\mathbb{R}_{>0} \times \mathbb{Z}^3)^k \to \mathbb{C}$ be deterministic, bounded, and measurable. For any $t \geqslant 0$, define the random variable

$$X_t = \sum_{n_1,\dots,n_k \in \mathbb{Z}^3} \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} f(t_1, n_1, \dots, t_k, n_k) \mathrm{d}W_{t_k}^{n_k} \mathrm{d}W_{t_{k-1}}^{n_{k-1}} \dots \mathrm{d}W_{t_1}^{n_1}. \tag{3.6.1}$$

Then, it holds for all $p \geqslant 2$ that

$$\|X_t\|_{L^p(\Omega)} \leqslant (p-1)^{\frac{k}{2}} \|X_t\|_{L^2(\Omega)}. \tag{3.6.2}$$

3.6.1.2 Multiple stochastic integrals

This section is based on [Nua06, Section 1.1] and we refer the reader to this excellent book for more details. Of particular importance to us is [Nua06, Example 1.1.2], which discuss the specific case of a d-dimensional Brownian motion.

We identify W^T with a Gaussian process on $\mathcal{H} = L^2(\mathbb{R}_{>0} \times \mathbb{Z}^3, \mathrm{d}t \otimes \mathrm{d}n)$, where $\mathrm{d}t$ is the Lebesgue measure and $\mathrm{d}n$ is the counting measure. For any $h \in \mathcal{H}$, we define

$$W^T[h] = \sum_{n \in \mathbb{Z}^3} \int_0^\infty h(t,n) \mathrm{d}W_t^{T,n}. \tag{3.6.3}$$

For any $h, h' \in \mathcal{H}$, we have that

$$\mathbb{E}\Big[W^T[h] W^T[h']\Big] = \sum_{n \in \mathbb{Z}^3} \int_0^\infty h(t,n) h'(t,-n) \frac{\sigma_t^T(n)^2}{\langle n \rangle^2} \mathrm{d}t. \tag{3.6.4}$$

Since we did not include a complex conjugate in the left-hand side of (3.6.4), we note that this does not yield a positive-definite bilinear form. We also did not include the weight $\rho_t^T(n)^2/\langle n \rangle^2$ in the definition of $\mathcal{H}$. Thus, the "covariance" in (3.6.4) does not coincide with the inner product on $\mathcal{H}$ and instead is only dominated by it. As is clear from [Nua06, Section 1.1], this only requires minor modifications in both the arguments and formulas.

For any $k \geqslant 1$ and any function $f \in \mathcal{H}_k = L^2\big((\mathbb{R}_{>0} \times \mathbb{Z}^3)^k, \bigotimes_{j=1}^k (\mathrm{d}t \otimes \mathrm{d}n)\big)$, the multiple stochastic integral

$$\mathcal{I}_k[f] = \sum_{n_1, \ldots, n_k \in \mathbb{Z}^3} \int_0^\infty \ldots \int_0^\infty f(t_1, n_1, \ldots, t_k, n_k) \mathrm{d}W_{t_k}^{T,n_k} \ldots \mathrm{d}W_{t_1}^{T,n_1} \tag{3.6.5}$$

can be defined as in [Nua06, Section 1.1.2]. If f is symmetric in the pairs $(t_1, n_1), (t_2, n_2), \ldots, (t_k, n_k)$, we can relate the multiple stochastic integral to an iterated stochastic integral.

Lemma 3.6.2. Let $k \geqslant 1$ and let $f \in \mathcal{H}_k$ be symmetric. Then, it holds that

$$\mathcal{I}_k[f] = k! \sum_{n_1,\dots,n_k \in \mathbb{Z}^3} \int_0^\infty \int_0^{t_1} \dots \int_0^{t_{k-1}} f(t_1, n_1, \dots, t_k, n_k) \mathrm{d}W_{t_k}^{T,n_k} \dots \mathrm{d}W_{t_1}^{T,n_1}, \tag{3.6.6}$$

where the right-hand side is understood as an iterated Itô integral.

This lemma follows from [Nua06, (1.27)] and the discussion below it. The primary reason for working with multiple stochastic integrals instead of iterated stochastic integrals is the simpler representation of their products. In order to state the product formula in Lemma 3.6.4 below, we need one further definition.

Definition 3.6.3 (Contraction)**.** Let $k, l \geqslant 1$ and let $f \in \mathcal{H}_k$ and $g \in \mathcal{H}_l$ be symmetric. For any $0 \leqslant r \leqslant \min(k, l)$, we define the contraction of r indices by

$$\begin{aligned}
&(f \otimes_r g)(t_1, n_1, \dots, t_{k+l-2r}, n_{k+l-2r}) \\
&\overset{\text{def}}{=} \sum_{m_1,\dots,m_r \in \mathbb{Z}^3} \int_0^\infty \dots \int_0^\infty \Bigg[f(t_1, n_1, \dots, t_{k-r}, n_{k-r}, s_1, m_1, \dots, s_r, m_r) \\
&\quad \times g(t_{k+1-r}, n_{k+1-r}, \dots, t_{k+l-2r}, n_{k+l-2r}, s_1, -m_1, \dots, s_r, -m_r) \prod_{j=1}^{k} \frac{\sigma_{s_j}^T (m_j)^2}{\langle m_j \rangle^2} \Bigg] \mathrm{d}s_r \dots \mathrm{d}s_1.
\end{aligned}$$

The reader should note the relationship to the covariance (3.6.4). If $f, g \in \mathcal{H} = \mathcal{H}_1$, then

$$\mathbb{E}\Big[W^T[f] W^T[g] \Big] = f \otimes_1 g.$$

A slight modification of [Nua06, Proposition 1.1.3] then yields the following result.

Lemma 3.6.4 (Product formula)**.** For any $k, l \geqslant 1$ and any symmetric $f \in \mathcal{H}_k$ and $g \in \mathcal{H}_l$, it holds that

$$\mathcal{I}_k[f] \cdot \mathcal{I}_l[g] = \sum_{r=0}^{\min(k,l)} r! \binom{k}{r} \binom{l}{r} \mathcal{I}_{k+l-2r}[f \otimes_r g]. \tag{3.6.7}$$

3.6.2 Auxiliary analytic estimates

In this section, we record several auxiliary results, which have been placed here to not interrupt the flow of the argument.

Harmonic analysis

We record a non-stationary phase argument and several standard trilinear product estimates.

Lemma 3.6.5 (Asymptotics of $\widehat{V}$)**.** There exists a constant $c = c_\beta \in \mathbb{R}$ such that

$$\Big|\widehat{V}(n) - \frac{c_\beta}{\langle n\rangle^\beta}\Big| \lesssim \frac{1}{\langle n\rangle^{\beta+1}}. \tag{3.6.8}$$

Remark 3.6.6. On the Euclidean space $\mathbb{R}^3$, instead of the periodic torus $\mathbb{T}^3$, the Fourier transform of $|x|^{\beta-3}$ is given exactly by $c_\beta|\xi|^{-\beta}$. At high frequencies, the Fourier transform $\widehat{V}$ is determined by the singularities of V, and hence the difference between $\mathbb{R}^3$ and $\mathbb{T}^3$ should not be essential. In fact, a more precise description of the asymptotics of $\widehat{V}$ is given by $c_\beta|n|^{-\beta}1\{n \neq 0\} + \mathcal{O}_M(\langle n\rangle^{-M})$, but it is easier to work with (3.6.8).

Proof. We denote by $\mathcal{F}_{\mathbb{R}^3}$ the Fourier transform on $\mathbb{R}^3$ given by

$$\mathcal{F}_{\mathbb{R}^3} f(\xi) = \int_{\mathbb{R}^3} f(x) e^{-i\langle \xi, x\rangle} \mathrm{d}x.$$

Let $\{\chi_N\}_{N\geqslant 1}$ be as in (3.1.15), which we naturally extend from $\mathbb{Z}^3$ to $\mathbb{R}^3$. Because we require additional room, we define for any $x \in \mathbb{T}^3$ and $N \geqslant 1$ the function

$$\widetilde{\chi}_N(x) \overset{\text{def}}{=} \chi_N(100x). \tag{3.6.9}$$

Let $n \in \mathbb{Z}^3 \backslash \{0\}$. Using the assumptions on the interaction potential V, we obtain that

$$\begin{aligned}
\widehat{V}(n) &= \int_{\mathbb{T}^3} V(x) e^{-i\langle n,x\rangle} \mathrm{d}x \\
&= \int_{\mathbb{T}^3} V(x) \widetilde{\chi}_1(x) e^{-i\langle n,x\rangle} \mathrm{d}x + \int_{\mathbb{T}^3} V(x)(1-\widetilde{\chi}_1(x)) e^{-i\langle n,x\rangle} \mathrm{d}x \\
&= \int_{\mathbb{R}^3} |x|^{-(3-\beta)} \widetilde{\chi}_1(x) e^{-i\langle n,x\rangle} \mathrm{d}x + \int_{\mathbb{T}^3} V(x)(1-\widetilde{\chi}_1(x)) e^{-i\langle n,x\rangle} \mathrm{d}x \\
&= \mathcal{F}_{\mathbb{R}^3}\big[|x|^{-(3-\beta)}\big](\xi) - \sum_{N \geqslant 2} \int_{\mathbb{R}^3} |x|^{-(3-\beta)} \widetilde{\chi}_N(x) e^{-i\langle n,x\rangle} \mathrm{d}x + \int_{\mathbb{T}^3} V(x)(1-\widetilde{\chi}_1(x)) e^{-i\langle n,x\rangle} \mathrm{d}x.
\end{aligned}$$

The first summand is given exactly by $c_\beta \|n\|_2^{-\beta}$. A non-stationary phase argument for the second and third term shows that they are bounded by $\mathcal{O}_M(\langle n\rangle^{-M})$ for all $M \geqslant 1$. This implies that

$$\widehat{V}(n) = c_\beta \|n\|_2^{-\beta} 1\{n \neq 0\} + \mathcal{O}_M(\langle n\rangle^{-M}).$$

Since $\|n\|_2^{-\beta} = \langle n\rangle^{-\beta} + \mathcal{O}(\langle n\rangle^{-1-\beta})$, this leads to (3.6.8). □

The following estimates are used in the paper to control several minor error terms.

Lemma 3.6.7 (Trilinear estimates)**.** For any sufficiently small $\delta > 0$, we have for all $f, g, h \in C_x^\infty(\mathbb{T}^3)$ the estimates

$$\Big\| \langle\nabla\rangle^{\frac{1}{2}+\delta} \Big(\big(V * (fg)\big)\, h \Big) \Big\|_{L_x^1} \lesssim \|f\|_{H_x^{\frac{1}{2}+2\delta}} \|g\|_{H_x^{\frac{1}{2}+2\delta}} \|h\|_{\mathcal{C}_x^{\frac{1}{2}+2\delta}}, \tag{3.6.10}$$

$$\Big\| \langle\nabla\rangle^{\frac{1}{2}+\delta} \Big(\big(V * (fg)\big)\, h \Big) \Big\|_{L_x^1} \lesssim \|f\|_{\mathcal{C}_x^{\frac{1}{2}+2\delta}} \|g\|_{H_x^{\frac{1}{2}+2\delta}} \|h\|_{H_x^{\frac{1}{2}+2\delta}}, \tag{3.6.11}$$

$$\Big\| \langle\nabla\rangle^{-\frac{1}{2}-2\delta} \Big(\big(V * (fg)\big)\, h \Big) \Big\|_{L_x^2} \lesssim \|f\|_{\mathcal{C}_x^{-\frac{1}{2}-\delta}} \Big(\|g\|_{\mathcal{C}_x^{-\frac{1}{2}-\delta}} \|h\|_{H_x^{1+4\delta}} + \|g\|_{H_x^{1+4\delta}} \|h\|_{\mathcal{C}_x^{-\frac{1}{2}-\delta}} \Big) \tag{3.6.12}$$

$$\Big\| \langle\nabla\rangle^{-\frac{1}{2}-2\delta} \Big(\big(V * (fg)\big)\, h \Big) \Big\|_{L_x^2} \lesssim \Big(\|f\|_{\mathcal{C}_x^{-\frac{1}{2}-\delta}} \|g\|_{H_x^{1+4\delta}} + \|f\|_{H_x^{1+4\delta}} \|g\|_{\mathcal{C}_x^{-\frac{1}{2}-\delta}} \Big) \|h\|_{\mathcal{C}_x^{-\frac{1}{2}-\delta}}. \tag{3.6.13}$$

These estimates are essentially an easier version of the fractional product formula. They can be proven using a paraproduct decomposition and Hölder's inequality and we omit the details. We always included δ-loss on the right-hand side of (3.6.10), so we can avoid all summability or endpoint issues. We also never rely on the smoothing effect of the interaction potential V.

The integral operator and truncations

We now record two properties related to the integral operator I_t and the associated frequency truncations ρ and σ.

Lemma 3.6.8 ([BG20b, Lemma 2])**.** For any space-time function $u\colon [0,\infty) \times \mathbb{T}^3 \to \mathbb{R}$ and any $\delta > 0$, it holds that

$$\sup_{T,t\geqslant 0} \|I_t^T[u]\|_{H_x^1(\mathbb{T}^3)} \lesssim \|u\|_{L_t^2 L_x^2([0,\infty)\times\mathbb{T}^3)} \tag{3.6.14}$$

and

$$\sup_{T,t,s\geqslant 0} \|I_s^T[u] - I_t^T[u]\|^2_{H_x^{1-\delta}(\mathbb{T}^3)} \lesssim \min(s,t)^{-2\delta} \min(1, |t-s|) \|u\|^2_{L_t^2 L_x^2([0,\infty)\times\mathbb{T}^3)}. \tag{3.6.15}$$

Proof. The first estimate (3.6.14) follows directly from [BG20b, Lemma 2]. Since $I_s^T[u] - I_t^T[u]$ is supported on frequencies $\gtrsim \min(s,t)$, we have that

$$\|I_s^T[u] - I_t^T[u]\|_{H_x^{1-\delta}(\mathbb{T}^3)} \lesssim \min(t,s)^{-\delta} \|I_s^T[u] - I_t^T[u]\|_{H_x^1(\mathbb{T}^3)}.$$

The rest of the statement then again follows from [BG20b, Lemma 2]. □

The result in [BG20b] is only stated for I_t instead of I_t^T, but the same argument applies.

Lemma 3.6.9 (Well-behaved truncations)**.** If $S \geqslant 1$ and $n_1, n_2, n_3 \in \mathbb{Z}^3$ satisfy $\|n_j - Se_j\|_2 \leqslant S/20$ for all $j = 1,2,3$, where e_j is the j-th canonical basis vector, then

$$\rho_S(n_{123}) \Big(\prod_{j=1}^3 \rho_S(n_j) \Big) \int_0^\infty \sigma_s(n_{123})^2 \Big(\prod_{j=1}^3 \rho_s(n_j)^2 \Big) \mathrm{d}s \gtrsim 1. \tag{3.6.16}$$

While the proof is a bit technical and depends on the precise regions in the definition of ρ, this lemma should not be taken too seriously.

Proof. We recall the lower bound $\min(\rho(y), -\rho'(y)) \gtrsim 1$ for all $1/2 \leqslant y \leqslant 2$ from the definition of ρ. From the assumptions, we directly obtain that

$$||\|n_{123}\|_2 - \sqrt{3}S| \leqslant \frac{3}{20}S.$$

In particular, we obtain that $3/2 \cdot S \leqslant \|n_{123}\|_2 \leqslant 19/20 \cdot S$. Since $19/20 \cdot S \leqslant \|n_j\|_2 \leqslant 21/20 \cdot S$ for all $j = 1, 2, 3$, it follows that

$$\rho_S(n_{123}) \Big(\prod_{j=1}^{3} \rho_S(n_j) \Big) \gtrsim 1.$$

We estimate the integral by

$$\begin{aligned}
&\int_0^\infty \sigma_s(n_{123})^2 \Big(\prod_{j=1}^{3} \rho_s(n_j)^2 \Big) \mathrm{d}s \\
&\gtrsim \int_0^\infty \langle s \rangle^{-1} 1\Big\{ \frac{\langle s \rangle}{2} \leqslant \|n_{123}\| \leqslant 2\langle s \rangle \Big\} \Big(\prod_{j=1}^{3} 1\Big\{ \|n_j\|_2 \leqslant 2\langle s \rangle \Big\} \Big) \mathrm{d}s \\
&\gtrsim S^{-1} \Big(\int_0^\infty 1\Big\{ \frac{1}{2} \max(\|n_1\|, \|n_2\|, \|n_3\|, \|n_{123}\|) \leqslant s \leqslant 2\|n_{123}\| \Big\} \mathrm{d}s - 2 \Big) \\
&= S^{-1} \Big(\frac{3}{2} \|n_{123}\|_2 - 2 \Big) \\
&\gtrsim 1,
\end{aligned}$$

where we used that $S \geqslant 1$. □

A basic counting estimate

The following estimate has been used to control stochastic objects (see Lemma 3.2.20).

Lemma 3.6.10. Let $v, w \in \mathbb{Z}^3$ and let $\alpha, \beta > 0$ satisfy $1 < \alpha + \beta < 3$. Then,

$$\sum_{n \in \mathbb{Z}^3} \frac{1}{\langle n + v \rangle^\alpha \langle n + w \rangle^\beta \langle n \rangle^2} \lesssim \min(\langle v \rangle, \langle w \rangle)^{1 - \alpha - \beta}. \tag{3.6.17}$$

Remark 3.6.11. The estimate (3.6.17) is not sharp if v and w have different magnitudes. For our purposes, however, (3.6.17) will be sufficient.

Proof of Lemma 3.6.10: Using Young's inequality, we have that

$$\frac{1}{\langle n+v\rangle^{\alpha}\langle n+w\rangle^{\beta}} \lesssim \frac{1}{\langle n+v\rangle^{\alpha+\beta}} + \frac{1}{\langle n+w\rangle^{\alpha+\beta}}. \tag{3.6.18}$$

Using this inequality, the estimate (3.6.17) reduces to

$$\sum_{n\in\mathbb{Z}^3} \frac{1}{\langle n+v\rangle^{\alpha+\beta}\langle n\rangle^2} \lesssim \langle v\rangle^{1-\alpha-\beta}.$$

This can easily be proven by decomposing the sum into the regions $|n| \ll |v|$, $|n| \sim |v|$ and $|n| \gg |v|$. □

3.6.3 Uniqueness of weak subsequential limits

In this section, we sketch the proof of the uniqueness of weak subsequential limits of $(\mu_T)_{T\geqslant 1}$, which has been obtained in [OOT20, Proposition 6.6]. For the convenience of the reader, we present the argument from [OOT20] in our notation.

Proposition 3.6.12. The limit

$$\lim_{T\to\infty} \int \mathrm{d}\mu_T(\phi)\exp(-f(\phi)) \tag{3.6.19}$$

exists for all Lipschitz functions $f\colon \mathcal{C}_x^{-1/2-\kappa}(\mathbb{T}^3)\to\mathbb{R}$. In particular, weak subsequential limits of $(\mu_T)_{T\geqslant 1}$ are unique.

Remark 3.6.13. The only reason why Proposition 3.6.12 does not (immediately) yield the weak convergence of $(\mu_T)_{T\geqslant 1}$ is that we do not prove that the limit in (3.6.19) corresponds to the Laplace transform of a limiting measure. As described in the proof of Theorem 3.1.3, this part follows from Prokhorov's theorem.

As was observed in [OOT20], Proposition 3.6.12 follows essentially from the same estimates as in the proof of uniform bounds on the variational problem (Proposition 3.3.1).

Proof. We recall from (3.2.9) that

$$\mathrm{d}\mu_T(\phi) = \frac{1}{\mathcal{Z}^{T,\lambda}} \exp\Big(- :\mathcal{V}^{T,\lambda}(\rho_T(\nabla)\phi): \Big)\, \mathrm{d}\big((W_\infty)_\# \mathbb{P}\big)(\phi). \tag{3.6.20}$$

We now split the proof into two steps.

Step 1: Reduction. Let $k \geqslant 1$ be a large integer. In this step, we reduce the existence of the limit in (3.6.19) to the existence of the limit

$$\lim_{T\to\infty} \mathbb{E}_{\mathbb{P}}\Big[\exp\Big(-f(W_\infty) - :\mathcal{V}^{T,\lambda}(W_\infty^T): - \epsilon \|W_\infty\|^k_{\mathcal{C}_x^{-1/2-\kappa}}\Big)\Big] \tag{3.6.21}$$

for all Lipschitz functions $f\colon \mathcal{C}_x^{-1/2-\kappa}(\mathbb{T}^3) \to \mathbb{R}$ and all $\epsilon > 0$. To this end, we first note that

$$1 - \exp(-\epsilon x^k) \leqslant \epsilon x^k \leqslant k!\epsilon \exp(x)$$

for all $x \geqslant 0$. Using Proposition 3.3.3, this implies

$$\begin{aligned}
&\Big|\mathbb{E}_{\mathbb{P}}\Big[\exp\Big(-f(W_\infty) - :\mathcal{V}^{T,\lambda}(W_\infty^T): \Big)\Big] - \mathbb{E}_{\mathbb{P}}\Big[\exp\Big(-f(W_\infty) - :\mathcal{V}^{T,\lambda}(W_\infty^T): - \epsilon \|W_\infty\|^k_{\mathcal{C}_x^{-1/2-\kappa}}\Big)\Big]\Big| \\
&= \mathbb{E}_{\mathbb{P}}\Big[\Big(1 - \exp\Big(-\epsilon \|W_\infty\|^k_{\mathcal{C}_x^{-1/2-\kappa}}\Big)\Big) \exp\Big(-f(W_\infty) - :\mathcal{V}^{T,\lambda}(\rho_T(\nabla)W_\infty): \Big)\Big] \\
&\lesssim_k \epsilon \cdot \mathbb{E}_{\mathbb{P}}\Big[\exp\Big(\|W_\infty\|_{\mathcal{C}_x^{-1/2-\epsilon}} - f(W_\infty) - :\mathcal{V}^{T,\lambda}(\rho_T(\nabla)W_\infty): \Big)\Big] \\
&\lesssim_{k,\lambda,f} \epsilon.
\end{aligned}$$

Thus, the existence of the limit in (3.6.21) implies the existence of the limit

$$\lim_{T\to\infty} \mathbb{E}_{\mathbb{P}}\Big[\exp\Big(-f(W_\infty) - :\mathcal{V}^{T,\lambda}(W_\infty^T): \Big)\Big] \tag{3.6.22}$$

for all Lipschitz functions $f\colon \mathcal{C}_x^{-1/2-\kappa}(\mathbb{T}^3) \to \mathbb{R}$. By setting $f \equiv 0$, we see that (3.6.22) implies the convergence of the normalization constants $\mathcal{Z}^{T,\lambda}$ as $T \to \infty$. Since (3.6.19) and (3.6.22) only differ by a factor of $\mathcal{Z}^{T,\lambda}$, we obtain that the limit in (3.6.19) exists.

Step 2: Existence of the regularized limit (3.6.21). Using the Boué-Dupuis formula (Theorem 3.2.1) and arguing as in the derivation of (3.3.22), we have that

$$\begin{aligned} &-\log\Big(\mathbb{E}_{\mathbb{P}}\Big[\exp\Big(-f(W_\infty)- :\mathcal{V}^{T,\lambda}(W_\infty^T): -\epsilon\|W_\infty\|_{\mathcal{C}_x^{-1/2-\kappa}}^k\Big)\Big]\Big) \\ &= \inf_{w\in\mathbb{H}_a}\mathbb{E}_{\mathbb{P}}\Big[\mathcal{E}_1^T[w]+\mathcal{E}_2^T[w]+\mathcal{E}_3^T[w]+\frac{\lambda}{4}\mathcal{V}(I_\infty^T[w])+\frac{1}{2}\|w\|_{L^2_{t,x}}^2+\epsilon\|I_\infty[w]+W_\infty\|_{\mathcal{C}_x^{-1/2-\kappa}}^k\Big]. \end{aligned} \tag{3.6.23}$$

Here, $\mathcal{V}$ is as in (3.3.1) and $\mathcal{E}_1^T[w], \mathcal{E}_2^T[w]$, and $\mathcal{E}_3^T[w]$ are as in (3.3.24)-(3.3.26), but the term $\varphi(W+I[u])$ in (3.3.24) is replaced by

$$f(W_\infty + I_\infty[J_t^T :(V*(W_t^T)^2)W_t^T:] + I_\infty[w]).$$

In contrast to (3.3.24)-(3.3.26), we also reflect the dependence on T and w in our notation. To avoid confusion, we also recall that the term $\mathbb{E}_{\mathbb{P}}\big[\mathcal{E}_0 + c^{T,\lambda}\big]$ in (3.3.22) vanishes due to our choice of $c^{T,\lambda}$. Our estimates in the proof of Proposition 3.3.3 show that the infimum in (3.6.23) can be take over $w\in\mathbb{H}_a$ satisfying the additional bound

$$\mathbb{E}_{\mathbb{P}}\Big[\frac{\lambda}{4}\mathcal{V}(I_\infty^T[w])+\frac{1}{2}\|w\|_{L^2_{t,x}}^2+\epsilon\|I_\infty[w]\|_{\mathcal{C}_x^{-1/2-\kappa}}^k\Big]\lesssim_\lambda 1.$$

In order to conclude the existence of the limit (3.6.21), it therefore suffices to prove for all $T,S\geqslant 1$ the estimate

$$\begin{aligned} &\sum_{j=1}^3\Big|\mathbb{E}_{\mathbb{P}}\Big[\mathcal{E}_j^T[w]-\mathcal{E}_j^S[w]\Big]\Big| + \Big|\mathbb{E}_{\mathbb{P}}\Big[\mathcal{V}(I_\infty^T[w])-\mathcal{V}(I_\infty^S[w])\Big]\Big| \\ &\lesssim_{\lambda,\epsilon,k}\min(S,T)^{-\eta}\Big(1+\mathbb{E}_{\mathbb{P}}\Big[\frac{1}{2}\|w\|_{L^2_{t,x}}^2+\epsilon\|I_\infty[w]\|_{\mathcal{C}_x^{-1/2-\kappa}}^k\Big]\Big), \end{aligned} \tag{3.6.24}$$

where $\eta>0$ is sufficiently small. We only present the estimate (3.6.24) for $\mathcal{E}_1^T[w]-\mathcal{E}_1^S[w]$ and $\mathcal{V}(I_\infty^T[w])-\mathcal{V}(I_\infty^S[w])$, since the remaing estimates are similar.

Step 2.a: Estimate of $\mathcal{E}_1^T[w]-\mathcal{E}_1^S[w]$. For the convenience of the reader, we recall that

$$\begin{aligned}\mathcal{E}_1^T[w] &\overset{\text{def}}{=} f(W_\infty + I_\infty[J_t^T :(V*(W_t^T)^2)W_t^T:] + I_\infty[w]) - \lambda^2 \int_{\mathbb{T}^3} (V* :(W_\infty^T)^2:)\mathbb{W}_\infty^{T,[3]} I_\infty^T[w]\mathrm{d}x \\ &\quad - 2\lambda^2 \int_{\mathbb{T}^3} \Big(\big(V*(W_\infty^T \mathbb{W}_\infty^{T,[3]})\big)W_\infty^T - \mathcal{M}_\infty^T \mathbb{W}_\infty^{T,[3]}\Big) I_\infty^T[w]\mathrm{d}x.\end{aligned} \tag{3.6.25}$$

We estimate the contributions of the three terms separately. For the first summand in (3.6.25), we have that

$$\begin{aligned}&\big|f(W_\infty + I_\infty[J_t^T :(V*(W_t^T)^2)W_t^T:] + I_\infty[w]) - f(W_\infty + I_\infty[J_t^S :(V*(W_t^S)^2)W_t^S:] + I_\infty[w])\big| \\ &\leqslant \mathrm{Lip}(f)\big\|I_\infty[J_t^T :(V*(W_t^T)^2)W_t^T:] - I_\infty[J_t^S :(V*(W_t^S)^2)W_t^S:]\big\|_{\mathcal{C}_x^{-1/2-\kappa}}.\end{aligned}$$

The desired estimate then follows from a minor modification of (3.2.47).

We now turn to the second summand in (3.6.25). First, we note for any $\gamma>0$ that

$$\|I_\infty^T[w]-I_\infty^S[w]\|_{H_x^{1-\gamma}} = \|(\rho_T(\nabla)-\rho_S(\nabla))I_\infty[w]\|_{H^{1-\gamma}} \lesssim \min(S,T)^{-\gamma}\|I_\infty[w]\|_{H_x^1}. \tag{3.6.26}$$

Now, we let $0<\gamma<\gamma'<\min(1/2,\beta)$. Using (3.6.26), we obtain that

$$\begin{aligned}&\Big|\int_{\mathbb{T}^3}(V* :(W_\infty^T)^2:)\mathbb{W}_\infty^{T,[3]} I_\infty^T[w]\mathrm{d}x - \int_{\mathbb{T}^3}(V* :(W_\infty^S)^2:)\mathbb{W}_\infty^{S,[3]} I_\infty^S[w]\mathrm{d}x\Big| \\ &\lesssim \Big\|(V* :(W_\infty^T)^2:)\mathbb{W}_\infty^{T,[3]} - (V* :(W_\infty^S)^2:)\mathbb{W}_\infty^{S,[3]}\Big\|_{\mathcal{C}_x^{-1+\gamma'}} \|I_\infty^T[w]\|_{H_x^{1-\gamma}} \\ &+ \Big\|(V* :(W_\infty^S)^2:)\mathbb{W}_\infty^{S,[3]}\Big\|_{\mathcal{C}_x^{-1+\gamma'}} \|I_\infty^T[w]-I_\infty^S[w]\|_{H_x^{1-\gamma}} \\ &\lesssim \bigg(\Big\|(V* :(W_\infty^T)^2:)\mathbb{W}_\infty^{T,[3]} - (V* :(W_\infty^S)^2:)\mathbb{W}_\infty^{S,[3]}\Big\|_{\mathcal{C}_x^{-1+\gamma'}} + \min(S,T)^{-\gamma}\Big\|(V* :(W_\infty^S)^2:)\mathbb{W}_\infty^{S,[3]}\Big\|_{\mathcal{C}_x^{-1+\gamma'}}\bigg) \\ &\times \|I_\infty[w]\|_{H_x^1}.\end{aligned}$$

The desired estimate then follows from a minor modification of (3.2.47) and Lemma 3.6.8. The estimate of the third term in (3.6.25) is similar and we omit the details.

Step 2.b: Estimate of $\mathcal{V}(I^T_\infty[w]) - \mathcal{V}(I^S_\infty[w])$. Using Hölder's inequality and interpolation, it holds that

$$\|\varphi\|_{L^4_x(\mathbb{T}^3)} \lesssim \|\varphi\|^{\frac{2(1-\kappa)}{3+4\kappa}}_{\mathcal{C}_x^{-1/2-\kappa}(\mathbb{T}^3)} \|\varphi\|^{\frac{1+6\kappa}{3+4\kappa}}_{H_x^{1-\kappa}(\mathbb{T}^3)}. \tag{3.6.27}$$

Using Hölder's inequality, (3.6.27), and $I^T_t = \rho_T(\nabla) I_t$, we obtain that

$$\begin{aligned}
&\Big|\mathcal{V}(I^T_\infty[w]) - \mathcal{V}(I^S_\infty[w])\Big| \\
&\lesssim \big(\|I^T_\infty[w]\|_{L^4_x} + \|I^S_\infty[w]\|_{L^4_x}\big)^3 \|I^T_\infty[w] - I^S_\infty[w]\|_{L^4_x} \\
&\lesssim \|I_\infty[w]\|^{4\cdot\frac{2(1-\kappa)}{3+4\kappa}}_{\mathcal{C}_x^{-1/2-\kappa}} \|I_\infty[w]\|^{3\cdot\frac{1+6\kappa}{3+4\kappa}}_{H^1_x} \|I^T_\infty[w] - I^S_\infty[w]\|^{\frac{1+6\kappa}{3+4\kappa}}_{H^{1-\kappa}_x} \\
&\lesssim \min(S,T)^{-\frac{(1+6\kappa)\kappa}{3+4\kappa}} \|I_\infty[w]\|^{4\cdot\frac{2(1-\kappa)}{3+4\kappa}}_{\mathcal{C}_x^{-1/2-\kappa}} \|I_\infty[w]\|^{4\cdot\frac{1+6\kappa}{3+4\kappa}}_{H^1_x}.
\end{aligned}$$

Since $4(1+6\kappa)/(3+4\kappa) < 2$, the desired estimate follows from Lemma 3.6.8 and Young's inequality.

□

Remark 3.6.14. As seen in the proof of Proposition 3.6.12, the regularizing factor $\exp\big(-\epsilon\|W_\infty\|^k_{\mathcal{C}_x^{-1/2-\kappa}}\big)$ in (3.6.21) is needed to estimate $\mathcal{V}(I^T_\infty[w]) - \mathcal{V}(I^S_\infty[w])$.

CHAPTER 4

Invariant Gibbs measures for the three-dimensional wave equation with a Hartree nonlinearity II: Dynamics[5]

4.1 Introduction

In this chapter we deal with the dynamical aspects of Theorem 1.2.1. As a result, it is inspired by recent advances in random dispersive equations. The interest in random dispersive equations stems from their connections to several areas of research, such as analytic number theory, harmonic analysis, random matrix theory, and stochastic partial differential equations (cf. [Nah16]). In fact, much of the recent progress have been fueled through similar advances in singular stochastic partial differential equations, such as Hairer's *regularity structures* [Hai14] or Gubinelli, Imkeller, and Perkowski's *para-controlled calculus* [GIP15].

The most classical problem in random dispersive equations is the construction of invariant measures for (periodic and defocusing) nonlinear wave and Schrödinger equations. This has been an active area of research since the 1990s, and we refer the reader to Figure 4.1 for an overview of some of the most important contributions.

The first results in this direction were obtained in one-spatial dimension by Friedlander [Fri85],

[5]The contents of this chapter have been posted as a research article on ArXiv [Bri20d].

Dimension & Nonlinearity	Wave	Schrödinger
$d = 1$, $\quad |u|^{p-1}u$	[Fri85, Zhi94]	[Bou94]
$d = 2$, $\quad |u|^2 u$	[OT20b]	[Bou96]
$d = 2$, $\quad |u|^{p-1}u$		[DNY19]
$d = 3$, $\quad (|x|^{-(3-\beta)} * |u|^2) \cdot u$	$\beta > 1:$ [OOT20] $\beta > 0:$ *This thesis.*	$\beta > 2:$ [Bou97] $2 \geqslant \beta > 1 - \epsilon:$ [DNY21]. $1 - \epsilon \geqslant \beta > 0:$ Open.
$d = 3$, $\quad |u|^2 u$	Open	Open

Figure 4.1: Invariant Gibbs measures for defocusing nonlinear wave and Schrödinger equations.

Zhidkov [Zhi94] and Bourgain [Bou94]. Friedlander [Fri85] and Zhidkov [Zhi94] proved the invariance of the Gibbs measure for the one-dimensional nonlinear wave equation. Inspired by earlier work of Lebowitz, Rose, and Speer [LRS88], Bourgain [Bou94] proved the invariance of the Gibbs measure for the one-dimensional nonlinear Schrödinger equations

$$i\partial_t u + \partial_x^2 u = |u|^{p-1}u, \qquad (t,x) \in \mathbb{R} \times \mathbb{T}.$$

In this seminal paper, Bourgain introduced his famous *globalization argument*, which will be described in detail below. Even though Friedlander [Fri85], Zhidkov [Zhi94] and Bourgain [Bou94] consider random initial data (drawn from the Gibbs measure), the local theory is entirely deterministic. The reason is that the Gibbs measure is supported at spatial regularity $1/2-$, which is above the (deterministic) critical regularities $s_{\mathrm{det}} = \frac{1}{2} - \frac{1}{p}$ (cf. [CCT03]) and $s_{\mathrm{det}} = \frac{1}{2} - \frac{2}{p-1}$ for the one-dimensional wave and Schrödinger equations (in H^s), respectively.

The first result in two spatial dimensions was obtained by Bourgain in [Bou96]. He proved the invariance of the Gibbs measure for the renormalized cubic nonlinear Schrödinger equation

$$i\partial_t u + \Delta u =: |u|^2 u: \qquad (t,x) \in \mathbb{R} \times \mathbb{T}^2. \tag{4.1.1}$$

Here, $:|u|^2 u:$ denotes the Wick-ordered cubic nonlinearity. In contrast to the one-dimensional setting, the Gibbs measure is supported at spatial regularity $0-$, which is just below the (deterministic) critical regularity $s_c = 0$. To overcome this obstruction, the local theory in [Bou96] exhibits probabilistic cancellations in several multi-linear estimates. Very recently, Fan, Ou, Staffilani, and Wang [FOS21] extended Bourgain's result from the square torus $\mathbb{T}^2$ to irrational tori.

The situation for two-dimensional nonlinear wave equations is easier than for two-dimensional nonlinear Schrödinger equations. While the Gibbs measure is still supported at spatial regularity $0-$, this is partially compensated by the smoothing effect of the Duhamel integral. In [OT20b], Oh and Thomann prove the invariance of the Gibbs measure for

$$-\partial_t^2 u - u + \Delta u = :u^p: \qquad (t,x) \in \mathbb{R} \times \mathbb{T}^2,$$

where $p \geqslant 3$ is an odd integer. We emphasize that their argument for the cubic ($p = 3$) and higher-order ($p \geqslant 5$) nonlinearity is essentially identical. Due to its clear and detailed exposition, we highly recommend [OT20b] as a starting point for any beginning researcher in random dispersive equations.

In a recent work [DNY19], Deng, Nahmod, and Yue proved the invariance of the Gibbs measure for the nonlinear Schrödinger equations

$$i\partial_t u + \Delta u = :|u|^{p-1} u: \qquad (t,x) \in \mathbb{R} \times \mathbb{T}^2, \tag{4.1.2}$$

where $p \geqslant 5$ is an odd integer. In contrast to the situation for the two-dimensional nonlinear wave equations, this result is much harder than its counterpart for the cubic nonlinear Schrödinger equation (4.1.1). The main difficulty is that all high$\times$low$\times$ ... $\times$low-interactions between the random initial data with itself or smoother remainders only have spatial regularity $1/2-$, which is strictly below the (deterministic) critical regularity $s_{\mathrm{det}} = 1 - \frac{2}{p-1}$. To overcome this difficulty, Deng, Nahmod, and Yue worked with random averaging operators, which are related to the adapted linear evolutions in [Bri20a]. Their framework was recently generalized through the *theory of random tensors* [DNY20], which will be further discussed below.

Unfortunately, much less is known in three spatial dimensions. The reason is that the Gibbs measure is supported at spatial regularity $-1/2-$, which is far below the deterministic critical regularity $s_{\text{det}} = \frac{3}{2} - \frac{2}{p-1}$. In fact, the invariance of the Gibbs measure for both the cubic nonlinear wave and Schrödinger equation are famous open problems. Previous research has instead focused on simpler models, which are obtained either through additional symmetry assumptions or a mollification of the nonlinearity. In the radially-symmetric setting, the invariance of the Gibbs measure for the three-dimensional cubic wave and Schrödinger equation has been proven in [BB14b, Suz11, Xu14] and [BB14a], respectively. The radially-symmetry setting was also studied in earlier work on the two-dimensional nonlinear Schrödinger equation [Den12, Tzv06, Tzv08]. In [Bou97], Bourgain studied the defocusing and focusing three-dimensional Schrödinger equation with a Hartree nonlinearity given by

$$i\partial_t u + \Delta u = \pm :(V * |u|^2)u: \qquad (t,x) \in \mathbb{R} \times \mathbb{T}^3, \tag{4.1.3}$$

where the interaction potential V behaves like $+|x|^{-(3-\beta)}$. He proved the invariance of the Gibbs measure for $\beta > 2$, which corresponds to a relatively smooth interaction potential. In the focusing case, this is optimal (up to the endpoint $\beta = 2$), since the Gibbs measure is not normalizable for $\beta < 2$ (cf. [OOT20]). From a physical perspective, the most relevant cases are the Coulomb potential $|x|^{-1}$ (corresponding to $\beta = 2$) and the Newtonian potential $|x|^{-2}$ (corresponding to $\beta = 1$). Since the cubic nonlinear Schrödinger equation formally corresponds to (4.1.3) with the interaction potential V given by the Dirac-measure, it is also interesting (and challenging) to take β close to zero. Very recently, Deng, Nahmod, and Yue [DNY21] used random averaging operators (as in [DNY19]) to cover the regime $\beta > 1 - \epsilon$ in the defocusing case, where $\epsilon > 0$ is a small unspecified constant. As discussed in [DNY21], it is likely possible to use the more sophisticated theory of random tensors from [DNY20] to cover the regime $\beta > 1/2$. Below the threshold $\beta = 1/2$, the Gibbs measure becomes singular with respect to the Gaussian free field (see Theorem 4.1.1). Since the theory in [DNY20] is developed for Gaussian initial data, it cannot yet be used in the regime $0 < \beta < 1/2$. In fact, this is mentioned as an open problem in [DNY20, Section 9.1].

After the completion of the series [Bri20c, Bri20d], the author learned of independent work by Oh, Okamoto, and Tolomeo [OOT20]. The authors study (the stochastic analogue of) the focusing and defocusing three-dimensional nonlinear wave equation with a Hartree nonlinearity given by

$$-\partial_t^2 u - u + \Delta u = \pm\lambda :(V * u^2)u: \qquad (t,x) \in \mathbb{R} \times \mathbb{T}^3,$$

where $\lambda > 0$. The main focus of [OOT20] lies on the construction and properties of the Gibbs measures, which are discussed in Chapter 3. Regarding the dynamical results of [OOT20], the authors prove the invariance of the Gibbs measure in the following cases:

(i) focusing $(-)$: $\beta > 2$ or $\beta = 2$ in the weakly nonlinear regime.

(ii) defocusing $(+)$: $\beta > 1$.

In light of the non-normalizability of the focusing Gibbs measure for $\beta < 2$ and $\beta = 2$ in the strongly nonlinear regime (cf. [OOT20]), the result is optimal in the focusing case. In the defocusing case, however, the restriction $\beta > 1$ excludes all Gibbs measures which are singular with respect to the Gaussian free field. In contrast, Theorem 4.1.3 below covers the complete range $\beta > 0$, which includes singular Gibbs measures. In fact, this is the main motivation behind the two-paper series [Bri20c, Bri20d].

In the preceding discussion, we have seen several examples of invariant Gibbs measures supported at regularities even below the deterministic critical regularity. In [DNY19, DNY20], Deng, Nahmod,

Dimension & Nonlinearity	Wave			Schrödinger		
	s_{G}	s_{prob}	s_{det}	s_{G}	s_{prob}	s_{det}
$d=1\ ,\quad \lvert u\rvert^{p-1}u$	$\frac{1}{2}$-	$-\frac{1}{2p}$	$\frac{1}{2}-\frac{1}{p}$	$\frac{1}{2}$-	$-\frac{1}{p-1}$	$\frac{1}{2}-\frac{2}{p-1}$
$d=2,\quad \lvert u\rvert^{p-1}u$	0-	$-\frac{3}{2p}$	$1-\frac{2}{p-1}$	0-	$-\frac{1}{p-1}$	$1-\frac{2}{p-1}$
$d=3,\quad (V*\lvert u\rvert^2)\cdot u$	$-\frac{1}{2}$-	$-\min(\frac{2+\beta}{3},\frac{3}{2})$	$\max(\frac{1-2\beta}{2},0)$	$-\frac{1}{2}$-	$-\min(\frac{1+\beta}{2},1)$	$\max(\frac{1-2\beta}{2},0)$
$d=3,\quad \lvert u\rvert^2u$	$-\frac{1}{2}$-	$-\frac{2}{3}$	$\frac{1}{2}$	$-\frac{1}{2}$-	$-\frac{1}{2}$	$\frac{1}{2}$

Relevant spatial regularities for the invariance of the Gibbs measure: s_{G} (support of the Gibbs measure), s_{prob} (probabilistic scaling), s_{det} (deterministic scaling). The value of s_{prob} for power-type nonlinearities can be found in [DNY19]. The probabilistic critical regularity s_{prob} for the wave equation with a Hartree nonlinearity is a result of high$\times$high$\times$high$\rightarrow$low and (high$\times$high$\rightarrow$low)$\times$high$\rightarrow$high-interactions. For the Schrödinger equation with a Hartree nonlinearity, s_{prob} is a result of (high$\times$high$\rightarrow$high)$\times$high$\rightarrow$high and (high$\times$high$\rightarrow$low)$\times$high$\rightarrow$high-interactions.

Figure 4.2: Overview of relevant regularities.

and Yue describe a probabilistic scaling heuristic, which takes into account the expected probabilistic cancellations. We denote the critical regularity with respect to the probabilistic scaling by s_{prob} and the spatial regularity of the support of the Gibbs measure s_{G}. Based on the probabilistic scaling heuristic, we then expect probabilistic local well-posedness as long as $s_{\mathrm{G}} > s_{\mathrm{prob}}$. We record the relevant quantities for nonlinear wave and Schrödinger equations in Figure 4.2. For comparison, we also include the deterministic critical regularity s_{det}. The probabilistic scaling heuristic, however, does not address any obstructions related to the global theory, renormalizations, or measure-theoretic aspects. As a result, it does not capture some of the difficulties for dispersive equations with singular Gibbs measures, such as the cubic nonlinear wave equation in three dimensions.

Our discussion so far has been restricted to invariant Gibbs measures for nonlinear wave and Schrödinger equations. While this is the most classical problem in random dispersive equations, there exist many more active directions of research. Since a full overview of the field is well-beyond the scope of the introduction, we only mention a few directions and refer to the given references for more details.

(i) Invariance of white noise [KMV20, Oh09, QV08],

(ii) Invariant measures (at high regularity) for completely integrable equations [TV14, TV15, DTV15],

(iii) Quasi-invariant Gaussian measures for non-integrable equations [GOT18, OT20a, Tzv15],

(iv) Non-invariance methods related to scattering, solitons, and blow-up [Bri18, Bri20e, DLM20, KM19, Poc17],

(v) Wave turbulence [BGH19, CG19, CG20, DH19],

(vi) Stochastic dispersive equations [BD99, BD03, DW18, GKO18a, GKO18b].

After this overview of the relevant literature, we now turn to a more detailed description of the most relevant methods. Our discussion will be split into two parts separating the local and global aspects. As a teaser for the reader, we already mention that our contributions to the local theory will be of an intricate but technical nature, while our contributions to the global theory will be conceptual.

As mentioned above, the first local well-posedness result for dispersive equations relying on probabilistic methods was proven by Bourgain [Bou96]. He considered the renormalized cubic nonlinear

Schrödinger equation

$$\begin{cases} i\partial_t u - u + \Delta u = :|u|^2 u: \qquad (t,x) \in \mathbb{R} \times \mathbb{T}^2, \\ u|_{t=0} = \phi \end{cases} \tag{4.1.4}$$

The additional $-u$-term has been introduced for convenience, but can be easily removed through a gauge transformation. The random initial data ϕ is drawn from the corresponding Gibbs measure, which coincides with the (complex) Φ^4_2-model. Since the Φ^4_2-model is absolutely continuous with respect to the Gaussian free field and the local theory does not rely on the invariance of the Gibbs measure, we can represent ϕ through the random Fourier series

$$\phi = \sum_{n \in \mathbb{Z}^2} \frac{g_n}{\langle n \rangle} e^{i\langle n, x\rangle}. \tag{4.1.5}$$

Here, $\langle n \rangle \overset{\text{def}}{=} \sqrt{1+|n|^2}$ and $(g_n)_{n\in\mathbb{Z}^2}$ is a sequence of independent and standard complex-valued Gaussians. The independence of the Fourier coefficients, and more generally the simple structure of (4.1.5), is an essential ingredient for many arguments in [Bou96]. A direct calculation yields almost surely that $\phi \in H^s(\mathbb{T}^2)\backslash L^2(\mathbb{T}^3)$ for all $s < 0$. Since (4.1.4) is mass-critical, ϕ lives below the (deterministic) critical regularity. To overcome this obstruction, Bourgain decomposed the solution by writing

$$u(t) = e^{it(-1+\Delta)}\phi + v(t).$$

This decomposition is commonly referred to as Bourgain's trick, but is also known in the stochastic PDE literature as the Da Prato-Debussche trick [DD03]. Using this decomposition, we see that the nonlinear remainder v satisfies the evolution equation

$$i\partial_t v - v + \Delta v = :|e^{it(-1+\Delta)}\phi + v|^2 (e^{it(-1+\Delta)}\phi + v): \qquad (t,x) \in \mathbb{R} \times \mathbb{T}^2.$$

Through a combination of probabilistic and PDE arguments, Bourgain proved that the Duhamel integral

$$I\Big[:|e^{it(-1+\Delta)}\phi|^2 e^{it(-1+\Delta)}\phi: \Big]$$

lives at spatial regularity $1/2-$ (see also [CLS21]). This opens the door to a contraction argument for v at a positive (and hence sub-critical) regularity. The contraction argument requires further ingredients from random matrix theory to handle mixed terms, but can in fact be closed. We emphasize that the nonlinear remainder v is treated purely deterministically and is not shown to exhibit any random structure.

We now discuss the more recent work of Gubinelli, Koch, and Oh [GKO18a], which covers the stochastic wave equation

$$\begin{cases} -\partial_t^2 u - u + \Delta u = :u^2: + \xi \qquad (t,x) \in \mathbb{R} \times \mathbb{T}^3, \\ u[0] = 0. \end{cases}$$

Here, ξ denotes space-time white noise. Inspired by a (higher-order version of) Bourgain's trick, we decompose

$$u = \linear + \quadratic + v.$$

Here, the linear stochastic object $\linear$ solves the forced wave equation

$$(-\partial_t^2 - 1 + \Delta) \linear = \xi.$$

The black dot represents the stochastic noise ξ and the arrow represents the Duhamel integral. An elementary arguments shows that $\linear$ has spatial regularity $-1/2-$. The quadratic stochastic object $\quadratic$ is the solution of the forced wave equation

$$(-\partial_t^2 - 1 + \Delta) \quadratic = :\big(\linear\big)^2: .$$

Based on similar arguments for stochastic heat equations, one may expect that $\quadratic$ has spatial regularity $2 \cdot (-1/2-) + 1 = 0-$, where the gain of one spatial derivative comes from the Fourier multiplier $\langle \nabla \rangle^{-1}$ in the Duhamel integral. Using multilinear dispersive estimates, however, Gubinelli, Koch, and Oh proved that $\<2>$ has spatial regularity $1/2-$. Using the definition of our stochastic objects, we obtain the evolution equation

$$(-\partial_t^2 - 1 + \Delta)\, v = 2\Big(\<2> + v\Big) \cdot \<1> + \Big(\<2> + v\Big)^2$$

for the nonlinear remainder v. In the following discussion, we let $\mathbin{\textcircled{$\ll$}}$ and $\mathbin{\textcircled{$=$}}$ be the low×high and high×high-paraproducts from Definition 4.2.1. Due to low×high-interactions such as $v \mathbin{\textcircled{$\ll$}} \<1>$, we expect v to have spatial regularity at most $(-1/2-) + 1 = 1/2-$. We emphasize that, unlike high×high-interactions, the low×high-interactions are not affected by multi-linear dispersive effects. However, this implies that the spatial regularities of v and $\<1>$ do not add up to a positive number, which means that the high×high-term $v \mathbin{\textcircled{$=$}} \<1>$ cannot even be defined (without additional information on v). This problem cannot be removed through a direct higher-order expansion of u and persists through all orders of the Picard iteration scheme. Instead, Gubinelli, Koch, and Oh [GKO18a] utilize ideas from the para-controlled calculus for singular stochastic PDEs [GIP15]. We write $v = X + Y$, where X and Y solve

$$(-\partial_t^2 - 1 + \Delta)\, X = 2\Big(\<2> + X + Y\Big) \mathbin{\textcircled{$\ll$}} \<1>$$

and

$$(-\partial_t^2 - 1 + \Delta)\, Y = 2\Big(\<2> + X + Y\Big) \mathbin{\textcircled{$\geqslant$}} \<1> + \Big(\<2> + X + Y\Big)^2.$$

The para-controlled component X only has spatial regularity $1/2-$, but exhibits a random structure. In the analysis of the high×high-interactions $X \mathbin{\textcircled{$=$}} \<1>$, this random structure can be exploited through the double Duhamel trick. In contrast, Y lives at a higher spatial regularity and can be controlled through deterministic arguments. The local theory in this paper will follow a similar approach, but relies on more intricate estimates, which will be further discussed below.

After this discussion of the local theory, we now turn to the global theory. We discuss Bourgain's globalization argument [Bou94], which uses the invariance of the truncated Gibbs measures as a substitute for a conservation law. We first recall the definition of the different modes of convergence for a sequence of probability measures, which will be needed below.

Definition (Convergence of measures)**.** *Let $\mathscr{H}$ be a Hilbert space and let $B(\mathscr{H})$ be the Borel σ-algebra on $\mathscr{H}$. Furthermore, let $(\mu_N)_{N\geqslant 1}$ and μ be Borel probability measures on $\mathscr{H}$. Then, we say that*

(i) μ_N converges in total variation to μ if

$$\lim_{N\to\infty} \sup_{A\in B(\mathscr{H})} |\mu(A) - \mu_N(A)| = 0,$$

(ii) μ_N converges strongly to μ if

$$\lim_{N\to\infty} \mu_N(A) = \mu(A) \qquad \text{for all } A \in B(\mathscr{H}),$$

(iii) μ_N converges weakly to μ if

$$\lim_{N\to\infty} \mu_N(A) = \mu(A) \qquad \text{for all } A \in B(\mathscr{H}) \text{ satisfying } \mu(\partial A) = 0.$$

To isolate the key features of the argument, we switch to an abstract setting. Let $\mathscr{H}$ be a Hilbert space and let $\Phi_N\colon \mathbb{R}\times\mathscr{H} \to \mathscr{H}$ be a sequence of jointly continuous flow maps. Let μ_N be a sequence of Borel probability measures on $\mathscr{H}$. Most importantly, we assume that μ_N is invariant under Φ_N for all N, i.e.,

$$\mu_N(\Phi_N(t)^{-1}A) = \mu_N(A) \qquad \text{for all } t \in \mathbb{R} \text{ and } A \in B(\mathscr{H}).$$

In our setting, Φ_N will be the flow for a frequency-truncated nonlinear wave equation and μ_N will be the corresponding truncated Gibbs measure. Our main interest lies in the removal of the truncation, i.e., the limit of the dynamics Φ_N and measure μ_N as N tends to infinity. Let μ be a limit of the sequence μ_N, where the mode of convergence will be specified below. In order to construct the limiting dynamics on the support of μ, we need uniform bounds on Φ_N on the support of μ. At the very least, we require an estimate of the form

$$\limsup_{N\to\infty} \mu\Big(\sup_{t\in[0,1]} \|\Phi_N(t)\phi\|_{\mathscr{H}} \leqslant \epsilon^{-1} \Big) \geqslant 1 - o_\epsilon(1), \tag{4.1.6}$$

where $0 < \epsilon < 1$ and o is the small Landau symbol. Bourgain's globalization argument [Bou94] proves (4.1.6) in two steps.

In a first measure-theoretic part, we use that

$$\Big| \mu\Big(\sup_{t\in[0,1]} \|\Phi_N(t)\phi\|_{\mathscr{H}} \leqslant \epsilon^{-1} \Big) - \mu_N\Big(\sup_{t\in[0,1]} \|\Phi_N(t)\phi\|_{\mathscr{H}} \leqslant \epsilon^{-1} \Big) \Big| \leqslant \sup_{A\in B(\mathscr{H})} |\mu(A) - \mu_N(A)|.$$

As long as μ_N converges in *total variation* to μ, we can reduce (4.1.6) to

$$\limsup_{N\to\infty} \mu_N\Big(\sup_{t\in[0,1]} \|\Phi_N(t)\phi\|_{\mathscr{H}} \leqslant \epsilon^{-1} \Big) \geqslant 1 - o_\epsilon(1), \tag{4.1.7}$$

In a second dynamical part, we use the invariance of μ_N under Φ_N and the probabilistic local well-posedness. Let $J \geqslant 1$ be a large integer and define the step-size $\tau = J^{-1}$. Then,

$$\begin{aligned} \mu_N\Big(\sup_{t\in[0,1]} \|\Phi_N(t)\phi\|_{\mathscr{H}} > \epsilon^{-1} \Big) &\leqslant \sum_{j=0}^{J-1} \mu_N\Big(\sup_{t\in[j\tau,(j+1)\tau]} \|\Phi_N(t)\phi\|_{\mathscr{H}} > \epsilon^{-1} \Big) \\ &= \sum_{j=0}^{J-1} \mu_N\Big(\sup_{t\in[0,\tau]} \|\Phi_N(t)\Phi_N(j\tau)\phi\|_{\mathscr{H}} > \epsilon^{-1} \Big). \end{aligned}$$

Using the invariance of μ_N under $\Phi_N(j\tau)$, we obtain that

$$\mu_N\Big(\sup_{t\in[0,1]} \|\Phi_N(t)\phi\|_{\mathscr{H}} > \epsilon^{-1} \Big) \leqslant \tau^{-1} \mu_N\Big(\sup_{t\in[0,\tau]} \|\Phi_N(t)\phi\|_{\mathscr{H}} > \epsilon^{-1} \Big). \tag{4.1.8}$$

The right-hand side of (4.1.8) can then be controlled through an appropriate choice of τ and the local theory (as well as tail estimates for μ_N).

In (this sketch of) Bourgain's globalization argument, the convergence in total variation played an essential role. In all previous results on the invariance of (defocusing) Gibbs measures [Bou94, Bou96, Bou97, DNY19, OOT20, OT20b, Zhi94], the truncated Gibbs measures converge in total variation, so that this assumption does not pose any problems. In our case, however, the truncated Gibbs measures μ_N only converge weakly to the Gibbs measure μ. The weak mode of convergence is related to the singularity of the Gibbs measure μ with respect to the Gaussian free field $\mathscr{g}$, which requires softer arguments in the construction of μ. Using the weak convergence of μ_N to μ, we can only reduce (4.1.6) to

$$\limsup_{N\rightarrow\infty}\Big[\limsup_{M\rightarrow\infty}\mu_M\Big(\sup_{t\in[0,1]}\|\Phi_N(t)\phi\|_{\mathscr{H}}\leqslant\epsilon^{-1}\Big)\Big]\geqslant 1-o_\epsilon(1), \tag{4.1.9}$$

In (4.1.9), we will typically have $M > N$, and hence we cannot (directly) use the invariance of the truncated Gibbs measures.

In [NOR12], Nahmod, Oh, Rey-Bellet, and Staffilani prove the invariance of a Wiener measure for the periodic derivative nonlinear Schrödinger equation. The truncated Wiener measures in [NOR12] are defined using a frequency-truncation not only in the interaction but also in the Gaussian free field (cf. [NOR12, (5.13)]. As a consequence, the truncated Wiener measures only converge weakly (cf. [NOR12, Proposition 5.13]). In order to prove (4.1.9), the authors rely on the (quantitative) mutual absolute continuity of the (truncated) Wiener measure with respect to the (truncated) Gaussian free field (cf. [NOR12, (6.7)]). Unfortunately, the singularity of the Gibbs measure in this work (as stated in Theorem 4.1.1) prevents us from using a similar approach.

4.1.1 Main results and methods

Before we can state our main results, we need to define the renormalized and frequency-truncated Hamiltonians, wave equations, and Gibbs measures. For any dyadic $N \geqslant 1$, we define the renormalized and frequency-truncated potential energy by

$$\begin{aligned}
&\frac{1}{4}\int_{\mathbb{T}^3} :(V * (P_{\leqslant N}\phi)^2)(P_{\leqslant N}\phi)^2: \mathrm{d}x \\
\overset{\text{def}}{=}&\frac{1}{4}\int_{\mathbb{T}^3} \Big[(V * (P_{\leqslant N}\phi)^2)(P_{\leqslant N}\phi)^2 - 2a_N(P_{\leqslant N}\phi)^2 - 4(\mathcal{M}_N P_{\leqslant N}\phi)P_{\leqslant N}\phi + \widehat{V}(0)a_N^2 + 2b_N\Big]\mathrm{d}x + c_N.
\end{aligned}$$

Here, the renormalization constants a_N, b_N, c_N are as in Definition 3.2.6, Definition 3.2.8, and Proposition 3.3.1 in Chapter 3, but their precise values are not needed in this paper. The renormalization multiplier $\mathcal{M}_N$ is defined by

$$\widehat{\mathcal{M}_N f}(n) \overset{\text{def}}{=} \Big(\sum_{k\in\mathbb{Z}^3} \frac{\widehat{V}(n+k)}{\langle k\rangle^2}\rho_N(k)^2\Big)\widehat{f}(n), \tag{4.1.10}$$

where ρ_N is a truncation to frequencies of size $\lesssim N$. The Hamiltonian H_N is then defined as

$$H_N[\phi_0,\phi_1] \overset{\text{def}}{=} \frac{1}{2}\Big(\|\phi_0\|_{L^2}^2 + \|\langle\nabla\rangle\phi_0\|_{L^2}^2 + \|\phi_1\|_{L^2}^2\Big) + \frac{1}{4}\int_{\mathbb{T}^3} :(V * (P_{\leqslant N}\phi)^2)(P_{\leqslant N}\phi)^2: \mathrm{d}x. \tag{4.1.11}$$

The renormalized and frequency-truncated nonlinear wave equation corresponding to H_N is given by

$$\begin{cases} (-\partial_t^2 - 1 + \Delta)u = P_{\leqslant N}\Big(:(V * (P_{\leqslant N}u)^2)P_{\leqslant N}u: \Big) & (t,x)\in\mathbb{R}\times\mathbb{T}^3, \\ u|_{t=0} = \phi_0, \quad \partial_t u|_{t=0} = \phi_1, \end{cases} \tag{4.1.12}$$

where the renormalized nonlinearity is given by

$$:(V * (P_{\leqslant N}u)^2)P_{\leqslant N}u: \overset{\text{def}}{=} (V * (P_{\leqslant N}u)^2)P_{\leqslant N}u - a_N\widehat{V}(0)P_{\leqslant N}u - 2\mathcal{M}_N P_{\leqslant N}u. \tag{4.1.13}$$

For a fixed $N \geqslant 1$, the coercivity of H_N implies the global well-posedness of (4.1.12). We also define the renormalized square

$$:(P_{\leqslant N}u)^2: \overset{\text{def}}{=} (P_{\leqslant N}u)^2 - a_N, \tag{4.1.14}$$

which will simplify the notation below. The Gibbs measure $\mu_N^{\otimes}$ corresponding to H_N is given by $\mu_N^{\otimes} = \mu_N \otimes (\langle\nabla\rangle)_{\#}\mathscr{g}$, where μ_N is as in Chapter 3 and $(\langle\nabla\rangle)_{\#}\mathscr{g}$ is the pushforward of the three-dimensional Gaussian field under $\langle \nabla \rangle$. For future use, we also define

$$\mathscr{g}^{\otimes} \overset{\text{def}}{=} \mathscr{g} \otimes (\langle \nabla \rangle)_{\#} \mathscr{g}. \tag{4.1.15}$$

Before we state the properties of the truncated Gibbs measures $\mu_N^{\otimes}$, we recall the assumptions on the interaction potential from Chapter 3. In these assumptions, $0 < \beta < 3$ is a fixed parameter.

Assumptions A. *We assume that the interaction potential V satisfies*

(i) $V(x) = c_\beta |x|^{-(3-\beta)}$ *for some* $c_\beta > 0$ *and all* $x \in \mathbb{T}^3$ *satisfying* $\|x\| \leqslant 1/10$*,*

(ii) $V(x) \gtrsim_\beta 1$ *for all* $x \in \mathbb{T}^3$*,*

(iii) $V(x) = V(-x)$ *for all* $x \in \mathbb{T}^3$*,*

(iv) V *is smooth away from the origin.*

The following properties of the Gibbs measures $\mu_N^{\otimes}$ are a direct consequence of 3.1.1, which is phrased in terms of μ_N. For notational reasons related to the weak convergence instead of convergence in total variation, we use a second parameter M for the frequency-truncation. Our notation for the random variables, which is based on dots, will be discussed below the theorem.

Theorem 4.1.1 (Gibbs measures)**.** Let $\kappa > 0$ be a fixed positive parameter, let $0 < \beta < 3$ be a parameter, and let the interaction potential V be as in the Assumptions A. Then, the truncated Gibbs measures $(\mu_M^{\otimes})_{M \geqslant 1}$ weakly converge to a limiting measure $\mu_\infty^{\otimes}$ on $\mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$, which is called the Gibbs measure. If in addition $0 < \beta < \frac{1}{2}$, then the Gibbs measure $\mu_\infty^{\otimes}$ is singular with respect to the Gaussian free field $\mathscr{g}^{\otimes}$.
Furthermore, there exists a sequence of reference measures $(\nu_M^{\otimes})_{M \geqslant 1}$ on $\mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$ and an ambient probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying the following two properties:

(i) (Absolute continuity and L^q-bounds) The truncated Gibbs measure $\mu_M^{\otimes}$ is absolutely continuous with respect to the reference measure $\nu_M^{\otimes}$. More quantitatively, there exists a parameter $q > 1$ and a constant $C \geqslant 1$ independent of M such that

$$\mu_M^{\otimes}(A) \leqslant C \nu_M^{\otimes}(A)^{1-\frac{1}{q}}$$

for all Borel sets $A \subseteq \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$.

(ii) (Representation of $\nu_M^{\otimes}$) Let $\gamma = \min(1/2+\beta, 1)$. Then, there exist two random variables $\bullet, \circ_{\mathrm{M}} : (\Omega, \mathcal{F}) \to \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$ and a large integer $k = k(\beta) \geqslant 1$ satisfying for all $p \geqslant 2$ that

$$\nu_M^{\otimes} = \mathrm{Law}_{\mathbb{P}}(\bullet + \circ_{\mathrm{M}}), \quad g^{\otimes} = \mathrm{Law}_{\mathbb{P}}(\bullet), \quad \text{and} \quad \Big(\mathbb{E}_{\mathbb{P}} \| \circ_{\mathrm{M}} \|_{\mathscr{H}_x^{\gamma-\kappa}(\mathbb{T}^3)}^p\Big)^{\frac{1}{p}} \leqslant p^{\frac{k}{2}}.$$

Remark 4.1.2. After the completion of the series [Bri20c, Bri20d], the author learned of independent work by Oh, Okamoto, and Tolomeo [OOT20], which yields an analogue of Theorem 4.1.1. We refer to Remark 3.1.2 in Chapter 3 for a more detailed comparison.

We will require that the ambient probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is rich enough to contain a family of independent Brownian motions, which is clear from the definition of $(\Omega, \mathcal{F}, \mathbb{P})$ in [Bri20c] and detailed in Section 4.4.5.

Let us further explain the notation in Theorem 4.1.1. We use dots to represent the random data, since they can be used as building blocks in more complicated stochastic objects. We already saw this graphical notation in our discussion of [GKO18a] and we refer the reader to [MWX17] for a detailed discussion of similar diagrams. We use the blue dot $\bullet$ for the Gaussian random data, since it lives at low spatial regularities and is primarily viewed as a high-frequency term. We use the red dot $\circ_{\mathrm{M}}$ to denote the more regular component of the random data, since we primarily view it as a low-frequency term. Furthermore, the blue dot $\bullet$ is filled while the red dot $\circ_{\mathrm{M}}$ is not filled. The reason is that the manuscript should be accessible to colorblind readers and also readable as a black and white copy.

In the following, we often write $\blacklozenge$ for a generic element $\phi \in \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$. The purple diamond will be used as a building block for further stochastic objects. When working with the reference measure $\nu_M^{\otimes}$, we have that

$$\mathrm{Law}_{\nu_M^{\otimes}}(\blacklozenge) = \mathrm{Law}_{\mathbb{P}}(\bullet + \circ_{\mathrm{M}}).$$

Naturally, we chose the color purple since it is a mixture of blue and red. The change in shape, i.e., from a dot to a diamond, is primarily made for colorblind readers. We also only use diamonds for intrinsic objects in $\mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$, while dots are used for objects defined on the ambient probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The significance of this distinction will be discussed in Sections 4.2 and 4.3.

While Theorem 4.1.1 already contains the measure-theoretic results of the series [Bri20c, Bri20d], we now state the dynamical results.

Theorem 4.1.3 (Global well-posedness & invariance)**.** There exists a Borel-measurable set $\mathcal{S} \subseteq \mathscr{H}^{-1/2-\kappa}(\mathbb{T}^3)$ satisfying $\mu_\infty^{\otimes}(\mathcal{S}) = 1$ and such that the following two properties hold:

(i) (Global well-posedness) Let Φ_N be the flow of the renormalized and frequency-truncated wave equation (4.1.12). Then, the limit

$$\Phi_\infty[t]\blacklozenge \stackrel{\text{def}}{=} \lim_{N\to\infty} \Phi_N[t]\blacklozenge$$

exists in $\mathscr{H}^{-1/2-\kappa}(\mathbb{T}^3)$ for all $t \in \mathbb{R}$ and $\blacklozenge \in \mathcal{S}$.

(ii) (Invariance) The Gibbs measure $\mu_\infty^{\otimes}$ is invariant under Φ_∞, i.e., it holds for all $t \in \mathbb{R}$ that

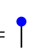
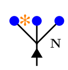
$$\Phi_\infty[t]_\# \mu_\infty^{\otimes} = \mu_\infty^{\otimes}$$

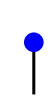
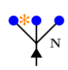
Remark 4.1.4. In the proof of Theorem 4.1.3, we restrict ourselves to the case $\beta \in (0, 1/2)$. The purpose of this restriction is purely notational. The same argument also works for $\beta \in [1/2, 3)$, as long as β in each estimate is replaced by $\min(\beta, 1/2)$.

Remark 4.1.5. While Theorem 4.1.3 shows that the limiting dynamics $\Phi_\infty[t]$ are well-defined, we do not obtain that $\Phi_\infty[t]$ satisfies the group property. The author believes that the estimates in this paper (from Sections 4.5-4.8) are strong enough to prove the group property, but the stability theory (Section 4.2.4 and Section 4.3.3) would need to be modified. Instead of working with a single flow $\Phi_N[t]$, one needs similar statements for the mixed flows $\Phi_{N_1}[t_1]\Phi_{N_2}[t_2]$. We refer the reader to [ST20] for a more detailed discussion of the group property and its relation to the recurrence properties of the flow.

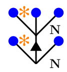
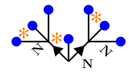
We now describe individual aspects of our argument. As in our discussion of the previous literature, we separate the local and global aspects. As mentioned above, our contributions to the local theory are of an intricate but technical nature, whereas our contributions to the global theory are conceptual.

In the local theory, we use the absolute continuity $\mu_M^{\otimes} \ll \nu_M^{\otimes}$ and the representation of $\nu_M^{\otimes}$ from Theorem 4.1.1. As a result, the reference measure $\nu_M^{\otimes}$ serves the same purposes as the Gaussian free field in earlier results on invariant Gibbs measures. We then follow the para-controlled approach of [GKO18a] and decompose the solution $u_N(t)$ of (4.1.12) as

$u_N =$  $+$  $+ X_N + Y_N,$ (4.1.16)

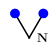
where the stochastic objects  and , the para-controlled component X_N, and the smoother nonlinear remainder Y_N are defined in Section 4.2. The smoother component $\circ_{\mathbf{M}}$ in the representation of $\nu_M^{\otimes}$ will be placed inside Y_N. In comparison to [GKO18a], however, there is an increase in the complexity of the evolution equation for Y_N. We split the terms into four different categories, which correspond to the methods used in their estimates.

- *Stochastic objects:* These terms are explicit and include

 and .

In contrast to the previous literature, we use multiple stochastic integrals for the non-resonant/resonant-decompositions, which significantly decreases the algebraic complexities. We also use counting estimates related to the dispersive symbol of the wave equation.

- *Random matrix terms:* The terms include

$\Big(V *$  $\Big) P_{\leqslant N} Y_N.$

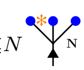
They will be controlled through a recent random matrix estimates of Deng, Nahmod, and Yue [DNY20, Proposition 2.8], which is based on the moment method.

- *Contributions of para-controlled terms:* These terms include

$$V * \Big(P_{\leqslant N}⫯ \mathbin{⊜} P_{\leqslant N} X_N\Big) P_{\leqslant N} Y_N.$$

We use the double Duhamel trick to exploit stochastic cancellations between ⫯ and X_N. In our definition of X_N, we use the paradifferential operators $⧀$ and $\boxed{⧀\&⧀}$ introduced in Section 4.2, which form a technical novelty.

- *Physical terms:* These terms include

$V * \Big(P_{\leqslant N}⫯ \cdot P_{\leqslant N} Y_N\Big) P_{\leqslant N}$  and $\Big(V * (P_{\leqslant N} Y_N)^2\Big) P_{\leqslant N} Y_N.$

The first term should be viewed as a random operator in Y_N, but is mainly treated through physical-space arguments. We believe that our approach is of independent interest, since it provides an alternative to the more Fourier-analytic estimates in [Bou96, GKO18a, DNY19, DNY20]. The second term is treated deterministically and we rely on the refined Strichartz-estimates of Klainerman and Tataru [KT99].

As we mentioned before, all stochastic objects have been based on $\bullet$ and the smoother component $\circ_{\mathbf{M}}$ is simply placed inside Y_N. This approach yields the convergence of the flows Φ_N on the support of $\mu_\infty^\otimes$ for a short time interval (see Corollary 4.2.12). The structural information in the decomposition (4.1.16), however, cannot (directly) be carried over to the support of $\mu_\infty^\otimes$, since $\bullet$ is only defined on the ambient probability space $(\Omega, \mathcal{F}, \mathbb{P})$. This defect will be addressed below, since the structural information is required for the global theory.

Remark 4.1.6. As was already mentioned in our overview of the literature, Deng, Nahmod, and Yue recently developed a theory of random tensors [DNY20], which forms a comprehensive framework for the local theory of random dispersive equations. The theory of random tensors (and its precursor [DNY19]) rely more intricately on the independence of the Fourier coefficients than the para-controlled approach. Even under the reference measure $\nu_M^\otimes$, however, the random data $\blacklozenge = \bullet + \circ_{\mathbf{M}}$ has dependent Fourier coefficient. This presents a challenge for the theory of random tensors, which was already mentioned in [DNY20, Section 9.1]. In addition, there are further technical problems related to the switch from Schrödinger to wave equations, which are described in Section 4.4.4. As a result, the author views the extension of the theory of random tensors to a local theory even for singular Gibbs measures and/or nonlinear wave equations as an interesting open problem.

After this discussion of the local theory, we turn to the global dynamics on the support of the Gibbs measure $\mu_\infty^\otimes$. As we have seen in our earlier discussion of Bourgain's globalization argument, its original version requires the convergence of the truncated Gibbs measures in total variation. Unfortunately, Theorem 4.1.1 only yields the weak convergence of the truncated Gibbs measures $\mu_M^\otimes$ to $\mu_\infty^\otimes$. We now give an informal description of our new globalization argument, but postpone a rigorous discussion until Section 4.3.

We let $T \geqslant 1$ be a large time, $B \geqslant 1$ be a large parameter describing the size of the evolution, $K \geqslant 1$ be a large frequency scale, and $\tau > 0$ be a small step-size. For any $j \geqslant 1$, we let $\mathcal{E}_K(B, j\tau) \subseteq \mathscr{H}^{-1/2-\kappa}(\mathbb{T}^3)$ be the set of initial data $\blacklozenge$ satisfying for all $t \in [0, j\tau]$ and $N \geqslant K$ that

$$\Phi_N(t)\,\blacklozenge = \overset{\blacklozenge}{|}(t) + \overset{\blacklozenge\blacklozenge\blacklozenge}{\Psi}_{N}(t) + w_N(t), \tag{4.1.17}$$

where w_N has size at most B in "structured high-regularity" norms. In our rigorous argument, B will depend on j, but we ignore this during our informal discussion. We also omit a smallness condition for the difference of $\Phi_N(t)\,\blacklozenge$ and $\Phi_K(t)\,\blacklozenge$. The goal is to prove by induction over $j \leqslant T/\tau$ that

$$\limsup_{M\to\infty} \mu_M^{\otimes}(\,\blacklozenge \in \mathcal{E}_K(B, j\tau))$$

is close to one as long as B, K, and τ are chosen appropriately. The proof relies on four separate ingredients:

(i) *(Structured local well-posedness)* This is the base case $j = 1$. Using our local theory, we only have to convert the stochastic objects in (4.1.16), which are based on $\bullet$, into stochastic objects based on $\blacklozenge$.

(ii) *(Structure and time-translation)* Using the induction hypothesis, we now assume that the probability $\mu_M^{\otimes}(\,\blacklozenge \in \mathcal{E}_K(B, (j-1)\tau))$ is close to one. In order to increase the time-interval, we let $\lozenge \overset{\text{def}}{=} \Phi_M[\tau]\,\blacklozenge$. Using the invariance of $\mu_M^{\otimes}$ under Φ_M, we obtain that

$$\mu_M^{\otimes}(\,\lozenge \in \mathcal{E}_K(B, (j-1)\tau)) = \mu_M^{\otimes}(\,\Phi_M[\tau]\,\blacklozenge \in \mathcal{E}_K(B, (j-1)\tau)) = \mu_M^{\otimes}(\,\blacklozenge \in \mathcal{E}_K(B, (j-1)\tau)),$$

which is close to one. After unpacking the definitions, we obtain information on the mixed flow $\Phi_N[t-\tau]\Phi_M[\tau]\,\blacklozenge$ for $t \in [\tau, j\tau]$. It therefore remains to analyze the difference between $\Phi_N[t-\tau]\Phi_M[\tau]\,\blacklozenge$ and $\Phi_N[t-\tau]\Phi_N[\tau]\,\blacklozenge$.

(iii) *(Structure and the cubic stochastic object)* The lowest regularity term in $\Phi_N(\tau)\,\bullet - \Phi_M(\tau)\,\bullet$ is given by a portion of the cubic stochastic object. In this step, we add the linear evolution of this portion to the mixed flow $\Phi_N[t-\tau]\Phi_M[\tau]\,\bullet$, which yields a function $\widetilde{u}_N$. It is then shown that $\widetilde{u}_N(t)$ is an approximate solution of the nonlinear wave equation (4.1.12) for $t \in [\tau, j\tau]$.

(iv) *(Stability theory)* We develop a para-controlled stability theory and construct a solution u_N close to the approximate solution $\widetilde{u}_N$, which also accounts for the remaining portion of $\Phi_N(\tau)\,\bullet - \Phi_M(\tau)\,\bullet$. Since our stability theory preserves the structure of $\widetilde{u}_N$, this yields (4.1.17) on the time-interval $[\tau, j\tau]$. Since the base case already yields the desired structure on $[0, \tau]$, this completes the induction step.

As is evident from this sketch, the proof of global well-posedness is much more involved than in Bourgain's original setting [Bou94, Bou96]. While not perfectly accurate, the author finds the following comparison with the deterministic global theory of dispersive equations illustrative. Bourgain's globalization argument [Bou94, Bou96] is the probabilistic version of a deterministic global theory using a (sub-critical) conservation law. The conservation law is replaced by the invariance, which implies that $t \mapsto \mu_N(\Phi_N(t)\phi \in \mathcal{E})$ is constant. In both cases, the global well-posedness is obtained by iterating the local well-posedness, but the estimates used in the local theory are no longer needed. In contrast, the new globalization argument is the probabilistic version of a deterministic global theory using almost conservation laws (cf. [CKS02]). The place of the almost conserved quantities is taken by the functions $t \mapsto \mu_M(\Phi_N(t)\phi \in \mathcal{E})$, which should be close to a constant function. In addition, the proof of global well-posedness often intertwines the local estimates and the choice of the almost conserved quantities. For entirely different reasons, the similarity with almost conserved quantities also appears in the globalization argument of [NOR12], which proves the invariance of a Wiener measure for the periodic derivative nonlinear Schrödinger equation. The truncated dynamics in [NOR12, (3.1)] only approximately conserve the energy (cf. [NOR12, Theorem 4.2]). Even with the same truncation parameter in the measure and the dynamics, the truncated Wiener measure is then only almost invariant (cf. [NOR12, Proof of Lemma 6.1]).

Our globalization argument for the nonlinear wave equation also differs from the globalization argument for the parabolic stochastic quantization equation as in [HM18]. While the invariant measure is singular in both situations, the dependence on the initial data in the parabolic setting is continuous even at spatial singularity $-1/2-$. The continuous dependence drastically simplifies the stability theory, which forms the most difficult part of our globalization argument.

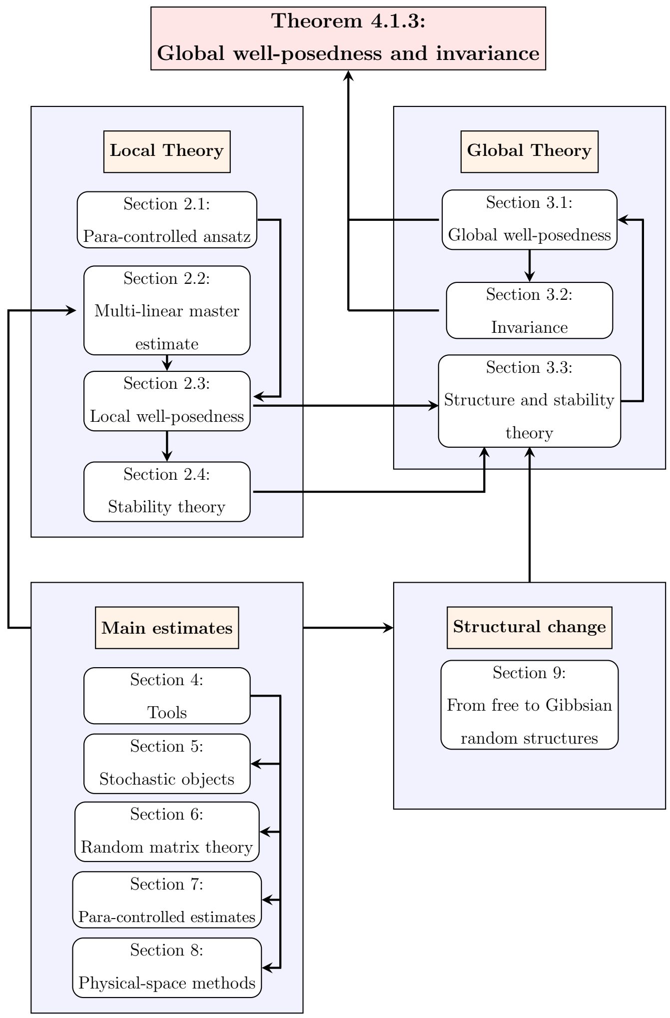
Once the global well-posedness has been proven, the proof of invariance is essentially the same as in [Bou94].

Remark 4.1.7. An argument of this length creates both mathematical challenges and different options for the exposition. The author does not claim to have found the perfect solutions or made the best expository choice in every single instance. While we postpone a more detailed discussion to Remark 4.1.6, Remark 4.2.3, Remark 4.3.4, Remark 4.4.43, Remark 4.8.2, and Remark 4.9.11, the author wanted to make this point in a central location of the paper. The author hopes that this encourages the reader to think more about our result and related open problems.

Acknowledgements: The author thanks his advisor Terence Tao for his patience and invaluable guidance. The author also thanks Nikolay Barashkov, Yu Deng, Martin Hairer, Redmond McNamara, Dana Mendelson, Andrea Nahmod, Tadahiro Oh, Felix Otto, Nikolay Tzvetkov, and Haitian Yue for helpful discussions.

4.1.2 Overview

Due to the excessive length of this chapter, we include a few suggestions for the reader. We also display the (main) relationship between the sections in Figure 4.3.



This figure illustrates the main dependencies between the different sections. The heart of the paper lies in the local and global theory (Section 4.2 and 4.3), which, as long as the reader believes certain estimates, can be read independently from the rest of the paper. A few minor dependencies between the different sections are not included in this illustration. For instance, basic properties of $\mathfrak{X}^{s,b}$-spaces, which are recalled in Section 4.4, will also be used in Section 4.2 and 4.3.

Figure 4.3: Dependencies between the different sections.

The local and global theory are described in Section 4.2 and 4.3, respectively. These sections contain the main novelties of this paper and should be interesting to most readers. As long as the reader believes several estimates, these sections are also self-contained. We therefore encourage the expert to focus on these sections.

Section 4.4 contains a collection of tools from dispersive equations, harmonic analysis, and probability theory. The reader should be familiar with the content of each subsection before moving on, but the expert should be able to only skim most content.

The Sections 4.5-4.8 contain the main technical aspects of this paper. They are concerned with separate terms in the evolution equation and rely on different methods. As a result, they can (essentially) be read independently.

In Section 4.9, we extend the multi-linear estimates from Sections 4.5-4.8, which have been phrased in terms of the Gaussian initial data $\bullet$, to random initial data $\blacklozenge$ drawn from the Gibbs measure. Each proof consists of a concatenation of previous results, and hence this section can safely be skipped on first reading.

4.1.3 Notation

We recall and introduce notation that will be used throughout the rest of the paper.

Dyadic numbers: Throughout this paper, we denote dyadic integers by K, L, M, and N. In limits or sums, such as $\lim_{M\to\infty}$ or $\sum_N$, we implicitly restrict ourselves to dyadic integers.

Parameters: We first introduce several parameters which are used in our function spaces, in the paradifferential operators, and our estimates. We fix

$$\epsilon > 0, \quad \delta_1, \delta_2 > 0, \quad \kappa > 0, \quad \eta, \eta' > 0, \quad \text{and} \quad b_+ > b > 1/2 > b_- > 0. \tag{4.1.18}$$

We use $\epsilon > 0$ in our para-differential operators, $\kappa > 0$ to capture small losses in probabilistic estimates, $\eta, \eta' > 0$ to capture gains in the highest frequency-scale, and $\delta_1, \delta_2, b_+, b, b_-$ in the definition of our function spaces. We impose the condition

$$1/2 - b_- \ll b - 1/2 \ll b_+ - 1/2 \ll \eta' \ll \eta \ll \kappa \ll \delta_2 \ll \epsilon \ll \delta_1. \tag{4.1.19}$$

In (4.1.19), the implicit constant in each "$\ll$" is allowed to depend on all parameters appearing to its right. We also define

$$s_1 = \frac{1}{2} - \delta_1 \qquad \text{and} \qquad s_2 = \frac{1}{2} + \delta_2.$$

In several statements of this paper, we will also use $0 < \zeta < 1$ and $C \geqslant 1$ as parameters. However, they may change their values between different lines and are allowed to depend on all parameters in (4.1.18).

Wave equation and flows: We denote the solution of the nonlinear wave equation (4.1.12) by $u_N(t)$. We also write

$$u_N[t] \overset{\text{def}}{=} \Big(u_N(t), \partial_t u_N(t) \Big),$$

which is standard in the literature on nonlinear wave equations. If $\blacklozenge \in \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$, we also write $\Phi_N(t)\blacklozenge$ and $\Phi_N[t]\blacklozenge$ for the solution with initial data $\blacklozenge$. When working with the flows $\Phi_N[t]$ and the Gibbs measures $\mu_M^{\otimes}$, we write $\Phi_N[t]_{\#}\mu_M^{\otimes}$ for the pushforward of $\mu_M^{\otimes}$ under $\Phi_N[t]$.

Furthermore, we denote the Duhamel integral operator of the wave equation by I. More precisely, we define

$$\mathrm{I}\big[F\big](t) \overset{\text{def}}{=} \int_0^t \frac{\sin((t-t')\langle\nabla\rangle)}{\langle\nabla\rangle} F(t')\mathrm{d}t'.$$

Fourier transform: With a slight abuse of notation, we write $\mathrm{d}x$ for the normalized Lebesgue measure on $\mathbb{T}^3 = \mathbb{R}^3/(2\pi\mathbb{Z})^3$, i.e., we require that

$$\int_{\mathbb{T}^3} 1 \mathrm{d}x = 1.$$

We then define the Fourier transform of a function $f\colon \mathbb{T}^3 \to \mathbb{C}$ by

$$\widehat{f}(n) \stackrel{\text{def}}{=} \int_{\mathbb{T}^3} f(x) e^{-inx} \mathrm{d}x. \tag{4.1.20}$$

For any $k \in \mathbb{N}$ and any $n_1, n_2, \ldots, n_k \in \mathbb{Z}^3$, we define

$$n_{12\ldots k} \stackrel{\text{def}}{=} \sum_{j=1}^{k} n_j.$$

For example, $n_{12} = n_1 + n_2$ and $n_{123} = n_1 + n_2 + n_3$.

Interaction potential: For a given interaction potential V satisfying the Assumptions A, we define

$$\widehat{V}_S(n_1, n_2, n_3) \stackrel{\text{def}}{=} \frac{1}{6} \sum_{\pi \in S_3} \widehat{V}(n_{\pi_1} + n_{\pi_2}).$$

Truncations and Littlewood-Paley operators: For each $t \geqslant 0$, we let $\rho_t\colon \mathbb{Z}^3 \to [0,1]$ be the same truncation to frequencies $n \in \mathbb{Z}^3$ satisfying $|n| \lesssim \langle t \rangle$ as in [Bri20c, Section 1.3]. For each dyadic $N \geqslant 1$, we define the Littlewood-Paley multiplier $P_{\leqslant N}$ by

$$\widehat{P_{\leqslant N} f}(n) = \rho_N(n) \widehat{f}(n).$$

We further set

$$P_1 f = P_{\leqslant 1} f \qquad \text{and} \qquad P_N f = P_{\leqslant N} f - P_{\leqslant N/2} f \quad \text{for all } N \geqslant 2.$$

The corresponding Fourier multipliers are denoted by

$$\chi(n) = \chi_1(n) = \rho_1(n) \qquad \text{and} \qquad \chi_N(n) = \rho_N(n) - \rho_{N/2}(n) \quad \text{for all } N \geqslant 2.$$

We also define fattened Littlewood-Paley multipliers by

$$\widetilde{P}_N = \sum_{N/16 \leqslant K \leqslant 16K} P_K.$$

Function spaces:

For any $s \in \mathbb{R}$, the $\mathcal{C}_x^s(\mathbb{T}^3)$-norm is defined as

$$\|f\|_{\mathcal{C}_x^s(\mathbb{T}^3)} \stackrel{\text{def}}{=} \sup_{N \geqslant 1} N^s \|P_N f\|_{L_x^\infty(\mathbb{T}^3)}. \tag{4.1.21}$$

We then define the corresponding space $\mathcal{C}_x^s(\mathbb{T}^3)$ by

$$\mathcal{C}_x^s(\mathbb{T}^3) \stackrel{\text{def}}{=} \Big\{ f \colon \mathbb{T}^3 \to \mathbb{R} \,|\, \|f\|_{\mathcal{C}_x^s} < \infty, \lim_{N \to \infty} N^s \|P_N f\|_{L_x^\infty(\mathbb{T}^3)} = 0 \Big\}. \tag{4.1.22}$$

We let $H_x^s(\mathbb{T}^3)$ be the usual L^2-based Sobolev space. more precisely, for any $f \colon \mathbb{T}^3 \to \mathbb{C}$, we define the corresponding norm by

$$\|f\|_{H_x^s(\mathbb{T}^3)} \stackrel{\text{def}}{=} \|\langle n \rangle^s \widehat{f}(n)\|_{\ell_n^2(\mathbb{Z}^3)}.$$

Furthermore, we define $\mathscr{H}_x^s(\mathbb{T}^3) \stackrel{\text{def}}{=} H_x^s(\mathbb{T}^3) \times H_x^{s-1}(\mathbb{T}^3)$. In this paper, we will also use the Bourgain spaces $\mathfrak{X}^{s,b}(\mathcal{J})$ and the low-frequency modulation space $\mathscr{L}\mathscr{M}(\mathcal{J})$, which are defined in Definition 4.4.1 and Definition 4.7.1, respectively.

4.2 Local theory

In this section, we show that the truncated and renormalized nonlinear wave equations

$$\begin{cases} (-\partial_t^2 - 1 + \Delta) u_N = P_{\leqslant N}\Big(:(V * (P_{\leqslant N} u_N)^2) P_{\leqslant N} u_N : \Big) \\ u_N[0] = \phi \end{cases} \tag{4.2.1}$$

are locally well-posed on the support of the Gibbs measures $\mu_M^\otimes$ uniformly in M. It is important in the definition of the limiting dynamics and the globalization argument that the truncation parameter N in the dynamics and the truncation parameter M in the Gibbs measure $\mu_M^\otimes$ are allowed to be different.

Due to the truncation, a soft argument based on the coercivity of the Hamiltonian shows that (4.2.1) is globally well-posed for a fixed truncation parameter N. We denote the corresponding flow by $\Phi_N(t)$.

4.2.1 Para-controlled ansatz

We now introduce our para-controlled approach. As discussed in the introduction, we will use a graphical notation for the several stochastic objects appearing in this paper. We denote the random initial data by $\blacklozenge$. In the local theory, we can work with the reference measure $\nu_M^{\otimes}$ and, more precisely, the representation of $\nu_M^{\otimes}$ with respect to the ambient measure $\mathbb{P}$.

Based on Theorem 4.1.1, we have that $\nu_M^{\otimes} = \mathrm{Law}_{\mathbb{P}}(\bullet + \circ_{\mathrm{M}})$, where $\bullet$ is the Gaussian low-regularity component and $\circ_{\mathrm{M}}$ is has regularity $\min(1/2+\beta, 1)-$. Naturally, we chose the color purple for the random initial data $\blacklozenge$ since it is a mixture of the blue and red random initial data. We emphasize that $\bullet$ and $\circ_{\mathrm{M}}$ are probabilistically dependent! Fortunately, this does not introduce any major difficulties in our treatment of the wave equation with a Hartree nonlinearity. We believe, however, that the proof of the invariance of the Gibbs measure for both the cubic wave equation and the three-dimensional Schrödinger equation with cubic or Hartree nonlinearity will require a more detailed understanding of the relationship between $\bullet$ and $\circ_{\mathrm{M}}$. This additional information is provided in Chapter 3.

Before we introduce our stochastic and para-controlled objects, we discuss the following question: Should we define our stochastic objects based on $\bullet$ or based on $\blacklozenge$? Due to the independence of the Fourier coefficient under $\mathbb{P}$ and its simple structure, it is much more convenient to work with $\bullet$. However, the decomposition $\blacklozenge = \bullet + \circ_{\mathrm{M}}$ of the samples of $\nu_M^{\otimes}$ is based on the ambient measure $\mathbb{P}$. It cannot be performed intrinsically on the samples of $\nu_M^{\otimes}$ and has *no meaning* for the Gibbs measure $\mu_M^{\otimes}$. In particular, if we want to examine the probability of an event under $\mu_M^{\otimes}$, we must phrase the event in terms of the full initial data $\blacklozenge$. Fortunately, there is a convenient solution to our conundrum: We first carry out most of our (local) analysis in terms of $\bullet$ and with respect to the ambient measure $\mathbb{P}$. Once all the estimates in terms of $\bullet$ are available, we can convert the stochastic objects and para-controlled structures from $\bullet$ into $\blacklozenge$ (see Section 4.9). Then, the absolute continuity of $\mu_M^{\otimes}$ with respect to the reference measure $\nu_M^{\otimes}$ allows us to obtain the same stochastic objects and para-controlled structures on the support of the Gibbs measure $\mu_M^{\otimes}$.

We now begin with the construction of the stochastic objects and para-controlled structures, which were briefly discussed in the introduction. We define $\stackrel{\bullet}{|}$ as the linear evolution of the random initial data $\bullet$. More precisely, $\stackrel{\bullet}{|}$ solves the evolution equation

$$(-\partial_t^2 - 1 + \Delta)\stackrel{\bullet}{|} = 0, \quad \stackrel{\bullet}{|}[0] = \bullet. \tag{4.2.2}$$

The black line in the stochastic object reflects the linear propagator of the wave equation. For future use, we define the frequency-truncated and renormalized square of $\stackrel{\bullet}{|}$ by

$$\bigvee_{N} \overset{\text{def}}{=} :(P_{\leqslant N}\stackrel{\bullet}{|})^2: . \tag{4.2.3}$$

The multiplication is reflected by the joining of the two lines and the frequency-truncation is reflected in the subscript N. We then define the renormalized nonlinearity $\ast\!\bigvee\!\!\bigvee_{N}$ by

$$\ast\!\bigvee\!\!\bigvee_{N} \overset{\text{def}}{=} P_{\leqslant N}\Big(:(V * (P_{\leqslant N}\stackrel{\bullet}{|})^2)(P_{\leqslant N}\stackrel{\bullet}{|}): \Big). \tag{4.2.4}$$

The orange asterisk reflects the convolution with the interaction potential. The color orange has no significance and we only chose it for aesthetic reasons. As before, the nonlinearity is reflected in the joining of the three lines and the truncation parameter N in the nonlinearity appears as a subscript. Finally, we define the Duhamel integral of $\overset{\bullet\ast\bullet\bullet}{\curlyvee}_{N}$ by

$$(-\partial_t^2 - 1 + \Delta)\overset{\bullet\ast\bullet\bullet}{\Psi}_{N} = \overset{\bullet\ast\bullet\bullet}{\curlyvee}_{N}, \qquad \overset{\bullet\ast\bullet\bullet}{\Psi}_{N}[0] = 0. \tag{4.2.5}$$

The line with an arrow reflects the integration in the Duhamel operator. In contrast to $\overset{\bullet}{|}$, we note that the distribution of $\overset{\bullet\ast\bullet\bullet}{\Psi}_{N}$ is not stationary in time. Naively, one may expect that $\overset{\bullet\ast\bullet\bullet}{\Psi}_{N}$ has spatial regularity $-1/2+\beta-$. Namely, one would expect spatial regularity $3\cdot(-1/2)-$ from the cube of the random initial data $\bullet$, a gain of one spatial derivative from the multiplier $\langle\nabla\rangle^{-1}$ in the Duhamel operator, and a gain of β derivatives from the convolution with the interaction potential. In Proposition 4.5.1, however, we will see that $\overset{\bullet\ast\bullet\bullet}{\Psi}_{N}$ actually has spatial regularity $\beta-$, which is half of a derivative better. The additional gain is a result of multi-linear dispersive effects. We now decompose our solution u_N by writing

$$u_N = \overset{\bullet}{|} + \overset{\bullet\ast\bullet\bullet}{\Psi}_{N} + w_N. \tag{4.2.6}$$

The remainder w_N has initial data $w_N[0] = \circ_{\mathrm{M}}$ and solves the forced nonlinear wave equation

$$\begin{aligned}
&(-\partial_t^2 - 1 + \Delta) w_N \\
=&P_{\leqslant N}\Big[:\Big(V * \Big(P_{\leqslant N}\big(\overset{\bullet}{|} + \overset{\bullet\ast\bullet\bullet}{\Psi}_{N} + w_N\big)\Big)^2\Big) P_{\leqslant N}\big(\overset{\bullet}{|} + \overset{\bullet\ast\bullet\bullet}{\Psi}_{N} + w_N\big): - \overset{\bullet\ast\bullet\bullet}{\curlyvee}_{N}\Big] \\
=&P_{\leqslant N}\Big[2\Big(V * \Big(P_{\leqslant N}\overset{\bullet}{|}\cdot P_{\leqslant N}\big(\overset{\bullet\ast\bullet\bullet}{\Psi}_{N} + w_N\big)\Big) P_{\leqslant N}\overset{\bullet}{|} - \mathcal{M}_N P_{\leqslant N}\big(\overset{\bullet\ast\bullet\bullet}{\Psi}_{N} + w_N\big)\Big) && (4.2.7) \\
&\quad + \Big(V * \Big(P_{\leqslant N}\big(\overset{\bullet\ast\bullet\bullet}{\Psi}_{N} + w_N\big)\Big)^2\Big) P_{\leqslant N}\overset{\bullet}{|} && (4.2.8) \\
&\quad + 2V * \Big(P_{\leqslant N}\overset{\bullet}{|}\cdot P_{\leqslant N}\big(\overset{\bullet\ast\bullet\bullet}{\Psi}_{N} + w_N\big)\Big) P_{\leqslant N}\big(\overset{\bullet\ast\bullet\bullet}{\Psi}_{N} + w_N\big) && (4.2.9) \\
&\quad + \Big(V * \overset{\bullet\,\bullet}{\vee}_{N}\Big) P_{\leqslant N}\big(\overset{\bullet\ast\bullet\bullet}{\Psi}_{N} + w_N\big) && (4.2.10) \\
&\quad + \Big(V * \Big(P_{\leqslant N}\big(\overset{\bullet\ast\bullet\bullet}{\Psi}_{N} + w_N\big)\Big)^2\Big) P_{\leqslant N}\big(\overset{\bullet\ast\bullet\bullet}{\Psi}_{N} + w_N\big)\Big]. && (4.2.11)
\end{aligned}$$

If we intend to construct (or control) w_N via a “direct” contraction argument, we would need the following conditions on the regularity of w_N (uniformly in N):

(i) Due to the high×high→low-interactions in factors such as $P_{\leqslant N} ⫯ \cdot P_{\leqslant N} w_N$, the regularity of w_N needs to be greater than $1/2$.

(ii) Due to “deterministic” nonlinear terms such as $(V * (P_{\leqslant N} w_N)^2) P_{\leqslant N} w_N$, the regularity of w_N needs to be greater than or equal to the deterministic critical regularity, which is given by $1/2 - \beta$.

Clearly, the first regularity condition is more restrictive. Unfortunately, the contribution of the first two summands (4.2.7) and (4.2.8) has regularity at most $1/2-$. The low×low×high-interaction gains one derivative from the multiplier $\langle \nabla \rangle^{-1}$ in the Duhamel operator, but does not experience any multi-linear dispersive effects. Thus, we are “ϵ-away” from a working contraction argument. As was observed in [GIP15, GKO18a], the term responsible for the low-regularity exhibits a para-controlled structure. Even though $P_{\leqslant N} ⫯ \cdot P_{\leqslant N} w_N$ is not well-defined for a general w_N at spatial regularity $1/2-$, we will see in Proposition 4.7.8 below that it is well-defined for a para-controlled w_N at the same regularity! We therefore decompose the solution w_N into two components: A para-controlled component X_N at regularity $1/2-$ and a smoother nonlinear remainder Y_N at a regularity greater than $1/2$.

Before we can define the decomposition, we need to introduce our para-product operators.

Definition 4.2.1 (Para-product operators)**.** Let $\epsilon > 0$ be the fixed parameter from Section 4.1.3

and let $f, g, h \colon \mathbb{T}^3 \to \mathbb{R}$. We define the low×high, high×high, and high×low-paraproducts by

$$
\begin{aligned}
f \mathbin{\textcircled{$<$}} g &\overset{\text{def}}{=} \sum_{N_1 \leqslant N_2/8} P_{N_1} f \cdot P_{N_2} g, \\
f \mathbin{\textcircled{$=$}} g &\overset{\text{def}}{=} \sum_{N_2/4 \leqslant N_1 \leqslant 4N_2} P_{N_1} f \cdot P_{N_2} g, \\
f \mathbin{\textcircled{$>$}} g &\overset{\text{def}}{=} \sum_{N_1 \geqslant 8N_2} P_{N_1} f \cdot P_{N_2} g.
\end{aligned}
$$

We also define

$$
f \mathbin{\textcircled{$\geqslant$}} g \overset{\text{def}}{=} f \mathbin{\textcircled{$>$}} g + f \mathbin{\textcircled{$=$}} g \qquad \text{and} \qquad f \mathbin{\textcircled{$\leqslant$}} g \overset{\text{def}}{=} f \mathbin{\textcircled{$<$}} g + f \mathbin{\textcircled{$=$}} g.
$$

In most of this paper, it will be convenient to replace "low" frequencies by "very low" frequencies. To this end, we define the bilinear operator

$$
f \mathbin{\textcircled{$\lessdot$}} g \overset{\text{def}}{=} \sum_{\substack{N_1, N_2 \colon \\ N_1 \leqslant N_2^{\epsilon}}} P_{N_1} f \cdot P_{N_2} g \tag{4.2.12}
$$

and the trilinear operator

$$
\boxed{\textcircled{$\lessdot$}\,\&\,\textcircled{$\lessdot$}} \Big(V * (fg) h \Big) \overset{\text{def}}{=} \sum_{\substack{N_1, N_2, N_3 \colon \\ N_1, N_2 \leqslant N_3^{\epsilon}}} V * \big(P_{N_1} f \cdot P_{N_2} g \big) P_{N_3} h. \tag{4.2.13}
$$

Furthermore, we define the negations of $\textcircled{$\lessdot$}$ and $\boxed{\textcircled{$\lessdot$}\,\&\,\textcircled{$\lessdot$}}$ by

$$
f \; \big(\neg \textcircled{$\lessdot$} \big) \; g \overset{\text{def}}{=} fg - f \mathbin{\textcircled{$\lessdot$}} g \quad \text{and} \quad \Big(\neg \boxed{\textcircled{$\lessdot$}\,\&\,\textcircled{$\lessdot$}} \Big) \Big(V * (fg) h \Big) \overset{\text{def}}{=} V * (fg) h - \boxed{\textcircled{$\lessdot$}\,\&\,\textcircled{$\lessdot$}} \Big(V * (fg) h \Big).
$$

Remark 4.2.2. The notation "$\lessdot$" is seldom used in the mathematical literature, which is precisely the reason why we use it in Definition 4.2.1. Its meaning would otherwise easily be confused with projections to $N_1 \leqslant N_2, N_1 \lesssim N_2$, or $N_1 \ll N_2$, which are again more common, but less suitable in our situation than $N_1 \leqslant N_2^{\epsilon}$. Comparing our notation for the operators $\textcircled{$\lessdot$}$ and $\boxed{\textcircled{$\lessdot$}\,\&\,\textcircled{$\lessdot$}}$, it may seem more natural to write

$$
V * (fg) \boxed{\textcircled{$\lessdot$}\,\&\,\textcircled{$\lessdot$}} h
$$

instead of (4.2.13). We found, however, that the notation in (4.2.13) is much cleaner once it is combined with the stochastic objects. We also point out that the negation of $\varolessthan$ is not $\varogreaterthan$.

We are now ready to define X_N and Y_N. We define the para-controlled component X_N by $X_N[0] = 0$ and

$$
\begin{aligned}
&(-\partial_t^2 - 1 + \Delta) X_N \\
&= P_{\leqslant N}\Bigg[2\, \boxed{\varolessthan \& \varolessthan}\Big(V * \Big(P_{\leqslant N} \overset{\bullet}{|} \cdot P_{\leqslant N}\big(\overset{\bullet\ast\bullet\,\bullet}{\Psi}_{\scriptscriptstyle N} + X_N \big)\Big) P_{\leqslant N} \overset{\bullet}{|} \Big) \\
&+ 2\Big(V * \Big(P_{\leqslant N} \overset{\bullet}{|} \cdot P_{\leqslant N}\big(Y_N\big)\Big) \varolessthan P_{\leqslant N} \overset{\bullet}{|} \Big) \\
&+ \Big(V * \Big(P_{\leqslant N}\big(\overset{\bullet\ast\bullet}{\Psi}_{\scriptscriptstyle N} + w_N \big)\Big)^2 \Big) \varolessthan P_{\leqslant N} \overset{\bullet}{|} \Bigg].
\end{aligned}
\tag{4.2.14}
$$

Remark 4.2.3. As far as the author is aware, the operator $\boxed{\varolessthan \& \varolessthan}$ has not been used in previous work on random dispersive equations. The reason for introducing the operator lies in the first term in (4.2.14), which contains $P_{\leqslant N} \overset{\bullet}{|} \cdot P_{\leqslant N} X_N$. In order to define this term (uniformly in N), the spatial regularity of X_N alone is not sufficient. It is also difficult to use the structure of X_N, since this term appears in the evolution equation for X_N (and not for Y_N), and hence one (may) run into a circular argument. By using $\boxed{\varolessthan \& \varolessthan}$, however, this problem does not occur, since we can borrow a small amount of regularity from the third argument in $\boxed{\varolessthan \& \varolessthan}\big(V * (P_{\leqslant N} \overset{\bullet}{|} \cdot P_{\leqslant N} X_N)\, P_{\leqslant N} \overset{\bullet}{|}\big)$. We mention, however, that using $\boxed{\varolessthan \& \varolessthan}$ has a small drawback, which is explained in Remark 4.9.11.

We also did not include any component of $\mathcal{M}_N P_{\leqslant N} Y_N$ in the second term of (4.2.14). It turns out that the contribution coming from the $\varolessthan$-portion of the renormalization can be controlled at regularities bigger than 1/2 and is therefore placed in the evolution equation for Y_N below.

As determined by our choice of X_N, the nonlinear remainder Y_N satisfies $Y_N[0] = \circ_{\mathrm{M}}$ and

$$
\begin{aligned}
&(-\partial_t^2 - 1 + \Delta) Y_N \\
&= 2 P_{\leqslant N} \Bigg[\Big(\big(\neg \boxed{\olessthan \,\&\, \olessthan} \big) \Big(V * \Big(P_{\leqslant N} \overset{\bullet}{|} \cdot P_{\leqslant N} \big(\overset{\bullet \ast \bullet \bullet}{\Upsilon}_{\mathrm{N}} + X_N \big) \Big) P_{\leqslant N} \overset{\bullet}{|} \Big) - \mathcal{M}_N P_{\leqslant N} \big(\overset{\bullet \ast \bullet \bullet}{\Upsilon}_{\mathrm{N}} + X_N \big) \Big) \Bigg]
\end{aligned} \tag{4.2.15}
$$

$$
+ P_{\leqslant N} \Bigg[2 \Big(V * \Big(P_{\leqslant N} \overset{\bullet}{|} \cdot P_{\leqslant N} \big(Y_N \big) \Big) \; \big(\neg \olessthan \big) \; P_{\leqslant N} \overset{\bullet}{|} \Big) - \mathcal{M}_N P_{\leqslant N} \big(Y_N \big) \Big) \tag{4.2.16}
$$

$$
+ \Big(V * \Big(P_{\leqslant N} \big(\overset{\bullet \ast \bullet \bullet}{\Upsilon}_{\mathrm{N}} + w_N \big) \Big)^2 \Big) \; \big(\neg \olessthan \big) \; P_{\leqslant N} \overset{\bullet}{|} \tag{4.2.17}
$$

$$
+ 2 V * \Big(P_{\leqslant N} \overset{\bullet}{|} \cdot P_{\leqslant N} \big(\overset{\bullet \ast \bullet \bullet}{\Upsilon}_{\mathrm{N}} + w_N \big) \Big) P_{\leqslant N} \big(\overset{\bullet \ast \bullet \bullet}{\Upsilon}_{\mathrm{N}} + w_N \big) \tag{4.2.18}
$$

$$
+ \Big(V * \overset{\bullet \; \bullet}{\vee}_{\mathrm{N}} \Big) P_{\leqslant N} \big(\overset{\bullet \ast \bullet \bullet}{\Upsilon}_{\mathrm{N}} + w_N \big) \tag{4.2.19}
$$

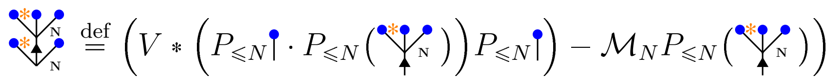
$$
+ \Big(V * \Big(P_{\leqslant N} \big(\overset{\bullet \ast \bullet \bullet}{\Upsilon}_{\mathrm{N}} + w_N \big) \Big)^2 \Big) P_{\leqslant N} \big(\overset{\bullet \ast \bullet \bullet}{\Upsilon}_{\mathrm{N}} + w_N \big) \Bigg]. \tag{4.2.20}
$$

To facilitate the analysis in the body of this paper, we further organize the terms in the evolution equation for Y_N. We write

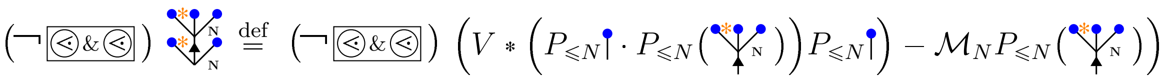
$$
(-\partial_t^2 - 1 + \Delta) Y_N = \mathbf{So} + \mathbf{CPara} + \mathbf{RMT} + \mathbf{Phy}, \tag{4.2.21}
$$

where the stochastic objects **So**, the contributions of the para-controlled terms **CPara**, the random-matrix terms **RMT**, and the physical terms **Phy** are defined as follows:

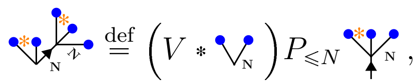
We define the individual stochastic objects by



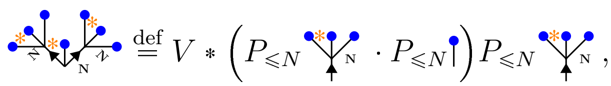
(4.2.22)



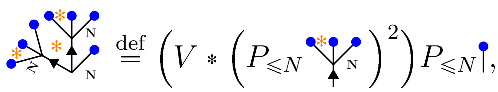
(4.2.23)



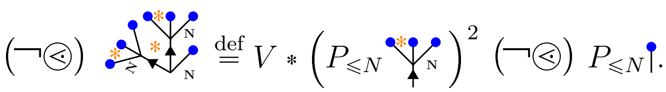
(4.2.24)



(4.2.25)

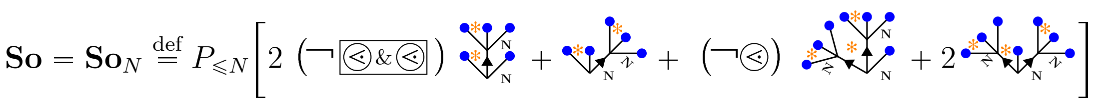


(4.2.26)



(4.2.27)

We then define

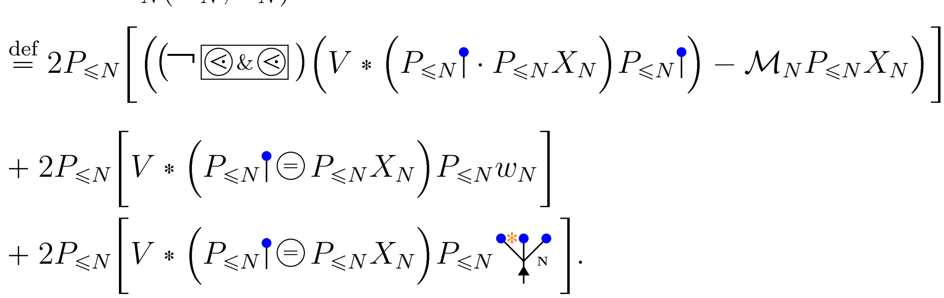


(4.2.28)

In works on singular SPDEs, such as [MWX17], the para-differential operators are usually placed at the joints of the different lines. The advantage is that it works for arbitrary "trees" and can accommodate multiple para-differential operators. Since this level of generality will not be needed here, we prefer our notation, since it is slightly easier to read.

We define

$$\mathbf{CPara} = \mathbf{CPara}_N(X_N, w_N)$$



(4.2.29)

In our analysis of **CPara**, we will use the double Duhamel-trick, i.e., we will replace X_N by the Duhamel-integral of the right-hand side in (4.2.14).

The random matrix term is defined as

$$\begin{aligned}
\mathbf{RMT} &= \mathbf{RMT}_N(Y_N, w_N) \\
&\overset{\text{def}}{=} P_{\leqslant N}\Big[\Big(V * \quad_N\Big) P_{\leqslant N} w_N\Big] && (4.2.30)\\
&+ 2P_{\leqslant N}\Big[\Big(V * \Big(P_{\leqslant N}\lin \cdot P_{\leqslant N}\big(Y_N\big)\Big)\ (\neg \circledless)\ P_{\leqslant N}\lin\Big) - \mathcal{M}_N P_{\leqslant N}\big(Y_N\big)\Big)\Big]. && (4.2.31)
\end{aligned}$$

Our reason for calling (4.2.30) the random matrix term lies in the method used in its estimate. We will view the summands as random operators in w_N and Y_N, respectively, and estimate the operator norm using the moment method (as in [DNY20, Proposition 2.8]).

Finally, we define the physical term by

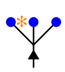
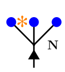
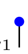
$$\begin{aligned}
\mathbf{Phy} &= \mathbf{Phy}_N(X_N, Y_N, w_N) \\
&\overset{\text{def}}{=} P_{\leqslant N}\Big[2V * \Big(P_{\leqslant N}\lin \cdot P_{\leqslant N}\cub_N\Big) P_{\leqslant N} w_N && (4.2.32)\\
&+ 2\Big(V * \Big(P_{\leqslant N}\cub_N \cdot P_{\leqslant N} w_N\Big)\ (\neg \circledless)\ P_{\leqslant N}\lin && (4.2.33)\\
&+ 2V * \Big(P_{\leqslant N}\lin \circledneq P_{\leqslant N} w_N\Big) P_{\leqslant N}\cub_N && (4.2.34)\\
&+ 2V * \Big(P_{\leqslant N}\lin \circleddash P_{\leqslant N} Y_N\Big) P_{\leqslant N}\cub_N && (4.2.35)\\
&+ 2V * \Big(P_{\leqslant N}\lin \circledneq P_{\leqslant N} w_N\Big) P_{\leqslant N} w_N && (4.2.36)\\
&+ 2V * \Big(P_{\leqslant N}\lin \circleddash P_{\leqslant N} Y_N\Big) P_{\leqslant N} w_N && (4.2.37)\\
&+ \Big(V * \Big(P_{\leqslant N} w_N\Big)^2\Big)\ (\neg \circledless)\ P_{\leqslant N}\lin && (4.2.38)\\
&+ \Big(V * \Big(P_{\leqslant N}\big(\cub_N + w_N\big)\Big)^2\Big) P_{\leqslant N}\big(\cub_N + w_N\big)\Big]. && (4.2.39)
\end{aligned}$$

Similar as for **RMT**, we call **Phy** the physical term due to the methods used in its estimate. We point out, however, that (4.2.33) and (4.2.34) are "hybrid" terms and their estimates rely on both random matrix techniques and physical methods. In the estimates of the other terms in **Phy**, we also make use of the refined Strichartz estimates by Klainerman-Tataru [KT99].

4.2.2 Multi-linear master estimate

In this subsection, we combine all multi-linear estimates from Section 4.5-4.8 into a single proposition, which we refer to as the multi-linear master estimate (Proposition 4.2.8). In particular, the multi-linear master estimate will include estimates of **So**, **CPara**, **RMT**, and **Phy**, even though the proofs of the individual estimates are quite different. Before we can state the multi-linear master estimate, however, we require additional notation. For the definition of the function spaces $\mathfrak{X}^{s,b}$ and $\mathscr{LM}$, we refer to Definition 4.4.1 and Definition 4.7.1.

Definition 4.2.4 (Types)**.** Let $\mathcal{J} \subseteq [0,\infty)$ be a bounded interval and let $\varphi\colon J \times \mathbb{T}^3 \to \mathbb{R}$. We say that φ is of type

-  if $\varphi =$ ,
-  if $\varphi =$  for some $N \geqslant 1$,
- w if $\|\varphi\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \leqslant 1$ and $\sum_{L_1 \sim L_2} \|P_{L_1}$  $\cdot P_{L_2} w\|_{L_t^2 H_x^{-4\delta_1}(\mathcal{J}\times\mathbb{T}^3)} \leqslant 1$,
- X if $\varphi = P_{\leqslant N}\,\mathrm{I}\left[1_{\mathcal{J}_0}\,\mathrm{PCtrl}(H, P_{\leqslant N}\right.$  $\left.)\right]$ for a dyadic integer $N \geqslant 1$, a sub-interval $\mathcal{J}_0 \subseteq \mathcal{J}$, and a function $H \in \mathscr{LM}(\mathcal{J}_0)$ satisfying $\|H\|_{\mathscr{LM}(\mathcal{J}_0)} \leqslant 1$,
- Y if $\|\varphi\|_{\mathfrak{X}^{s_2,b}(\mathcal{J})} \leqslant 1$.

Let $\varphi_1, \varphi_2, \varphi_3 \colon \mathcal{J} \times \mathbb{T}^3 \to \mathbb{R}$ and $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \in \Big\{ \uparrow, \Psi, w, X, Y \Big\}$. We write

$$(\varphi_1, \varphi_2) \overset{\text{type}}{=} (\mathcal{T}_1, \mathcal{T}_2)$$

if either φ_1 is of type $\mathcal{T}_1$ and φ_2 is of type $\mathcal{T}_2$ or φ_1 is of type $\mathcal{T}_2$ and φ_2 is of type $\mathcal{T}_1$. Furthermore, we write

$$(\varphi_1, \varphi_2; \varphi_3) \overset{\text{type}}{=} (\mathcal{T}_1, \mathcal{T}_2; \mathcal{T}_3)$$

if $(\varphi_1, \varphi_2) \overset{\text{type}}{=} (\mathcal{T}_1, \mathcal{T}_2)$ and φ_3 is of type $\mathcal{T}_3$.

Remark 4.2.5. The types w, X, and Y are designed for the functions w_N, X_N, and Y_N from Section 4.2.1. Our notation for the type of $(\varphi_1, \varphi_2; \varphi_3)$ respects the symmetry in the first two arguments of the nonlinearity $\big(V * (\varphi_1 \varphi_2)\big)\varphi_3$. We also mention that the types w and X implicitly depend on $\bullet$. In Section 4.9, we will therefore refer to the types w and X as $w^{\bullet}$ and $X^{\bullet}$, respectively.

In the next lemma, we show that functions of type X and Y are multiples of functions of type w. This allows us to prove several estimates for functions of type X and Y simultaneously.

Lemma 4.2.6. Let $A \geqslant 1$, $T \geqslant 1$, and let $\zeta = \zeta(\epsilon, s_1, s_2, \kappa, \eta, \eta', b_+, b) > 0$ be sufficiently small. Then, there exists a Borel set $\Theta_{\text{type}}(A, T) \subseteq \mathscr{H}_x^{-\frac{1}{2}-\kappa}(\mathbb{T}^3)$ satisfying

$$\mathbb{P}(\bullet \in \Theta_{\text{blue}}^{\text{type}}(A, T)) \geqslant 1 - \zeta^{-1} \exp(-\zeta A^{\zeta})$$

and such that the following holds for all $\bullet \in \Theta_{\text{blue}}^{\text{type}}(A, T)$: If $\varphi \colon \mathcal{J} \times \mathbb{T}^3 \to \mathbb{R}$ is of type X or Y, then $T^{-4} A^{-1} \varphi$ is of type w.

Proof. We treat the types X and Y separately. First, we assume that φ is of type X, and hence there exists a dyadic integer $N \geqslant 1$, a sub-interval $\mathcal{J}_0 \subseteq \mathcal{J}$, and a function $H \in \mathscr{LM}(\mathcal{J}_0)$ satisfying $\|H\|_{\mathscr{LM}(\mathcal{J}_0)} \leqslant 1$ such that $\varphi = P_{\leqslant N} \, \mathrm{I}\big[1_{\mathcal{J}_0} \, \mathrm{PCtrl}(H, P_{\leqslant N} \uparrow)\big]$. Using the inhomogeneous Strichartz estimate (Lemma 4.4.9) and Lemma 4.7.3, we have that

$$
\begin{aligned}
\| P_{\leqslant N} X \|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} &\lesssim \| 1_{\mathcal{J}_0} \, \mathrm{PCtrl}(H, \, P_{\leqslant N} \linear) \|_{L_t^{2b} H_x^{s_1-1}(\mathcal{J}\times\mathbb{T}^3)} \\
&\lesssim T \| H \|_{\mathscr{L}\mathscr{M}(\mathcal{J})} \| \linear \|_{L_t^\infty H_x^{s_1-1+8\epsilon}(\mathcal{J}\times\mathbb{T}^3)} \\
&\lesssim T \| \bullet \|_{\mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)}.
\end{aligned}
$$

This is bounded by TA on a set of acceptable probability. Using Proposition 4.7.8, we obtain on a set of acceptable probability that

$$
\sum_{L_1 \sim L_2} \| P_{L_1} \linear \cdot P_{L_2} \varphi \|_{L_t^2 H_x^{-4\delta_1}(\mathcal{J}\times\mathbb{T}^3)} \leqslant T^4 A \| H \|_{\mathscr{L}\mathscr{M}(\mathcal{J}_0)} \leqslant T^4 A.
$$

By combining both estimates, we see that $T^{-4}A^{-1}\varphi$ is of type w.

Second, we assume that φ is of type Y. Then, we have that $\|\varphi\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \leqslant \|\varphi\|_{\mathfrak{X}^{s_2,b}(\mathcal{J})} \leqslant 1$. This implies

$$
\begin{aligned}
\sum_{L_1 \sim L_2} \| P_{L_1} \linear \cdot P_{L_2} \varphi \|_{L_t^2 H_x^{-4\delta_1}(\mathcal{J}\times\mathbb{T}^3)} &\lesssim T^{\frac{1}{2}} \sum_{L_1 \sim L_2} L_1^{\kappa-\delta_2} \| P_{L_1} \linear \|_{L_t^\infty \mathcal{C}_x^{-1/2-\kappa}(\mathcal{J}\times\mathbb{T}^3)} \| P_{L_2} \varphi \|_{L_t^\infty H_x^{s_2}(\mathcal{J}\times\mathbb{T}^3)} \\
&\lesssim T^{\frac{1}{2}} \| \linear \|_{L_t^\infty \mathcal{C}_x^{-1/2-\kappa}(\mathcal{J}\times\mathbb{T}^3)}
\end{aligned}
$$

As above, this is bounded by $T^{\frac{1}{2}}A$ on a set of acceptable probability. By combining both estimates, we see that $T^{-\frac{1}{2}}A^{-1}\varphi$ is of type w. □

In order to state the multi-linear master estimate, we need to introduce a multi-linear version of the renormalization in (4.1.13).

Definition 4.2.7 (Renormalization)**.** Let $\mathcal{J}$ be a compact interval, let $\varphi_1, \varphi_2, \varphi_3$ be as in Definition 4.2.4, and let $N \geqslant 1$. Furthermore, assume that

$$
(\varphi_1, \varphi_2; \varphi_3) \overset{\text{type}}{\neq} \big(\linear, \linear, \linear \big).
$$

Then, we define the renormalized and frequency-truncated nonlinearity by

$$
\begin{aligned}
&:\! V * \big(P_{\leqslant N}\varphi_1 \cdot P_{\leqslant N}\varphi_2\big) P_{\leqslant N}\varphi_3 \!: \\
&\overset{\text{def}}{=} \begin{cases}
\Big(V * \lolli\!\lolli_{N}\Big) P_{\leqslant N}\varphi_3 & \text{if } (\varphi_1,\varphi_2) \overset{\text{type}}{=} \big(\lolli, \lolli\big), \\
V * \Big(P_{\leqslant N}\lolli \cdot P_{\leqslant N}\varphi_2\Big) P_{\leqslant N}\lolli - \mathcal{M}_N P_{\leqslant N}\varphi_2 & \text{if } (\varphi_1,\varphi_3) \overset{\text{type}}{=} \big(\lolli, \lolli\big), \\
V * \Big(P_{\leqslant N}\varphi_1 \cdot P_{\leqslant N}\lolli\Big) P_{\leqslant N}\lolli - \mathcal{M}_N P_{\leqslant N}\varphi_1 & \text{if } (\varphi_2,\varphi_3) \overset{\text{type}}{=} \big(\lolli, \lolli\big), \\
V * \big(P_{\leqslant N}\varphi_1 \cdot P_{\leqslant N}\varphi_2\big) P_{\leqslant N}\varphi_3 & \text{else.}
\end{cases}
\end{aligned}
\tag{4.2.40}
$$

If $(\varphi_1,\varphi_2) \overset{\text{type}}{\neq} \big(\lolli, \lolli\big)$, we define the action of the paradifferential operators $\olessthan$ and $\boxed{\olessthan \& \olessthan}$ on the renormalized and frequency-truncated nonlinearity by

$$
\begin{aligned}
:\! V * \big(P_{\leqslant N}\varphi_1 \cdot P_{\leqslant N}\varphi_2\big) \olessthan P_{\leqslant N}\varphi_3 \!: &\overset{\text{def}}{=} V * \big(P_{\leqslant N}\varphi_1 \cdot P_{\leqslant N}\varphi_2\big) \olessthan P_{\leqslant N}\varphi_3, \\
\boxed{\olessthan \& \olessthan}\Big(:\! V * \big(P_{\leqslant N}\varphi_1 \cdot P_{\leqslant N}\varphi_2\big) P_{\leqslant N}\varphi_3 \!: \Big) &\overset{\text{def}}{=} \boxed{\olessthan \& \olessthan}\Big(V * \big(P_{\leqslant N}\varphi_1 \cdot P_{\leqslant N}\varphi_2\big) P_{\leqslant N}\varphi_3 \Big),
\end{aligned}
$$

which does not involve a renormalization. We also define the negated paradifferential operators by

$$
\begin{aligned}
&:\! V * \big(P_{\leqslant N}\varphi_1 \cdot P_{\leqslant N}\varphi_2\big) \; \big(\neg \olessthan\big) \; P_{\leqslant N}\varphi_3 \!: \\
&\overset{\text{def}}{=} :\! V * \big(P_{\leqslant N}\varphi_1 \cdot P_{\leqslant N}\varphi_2\big) P_{\leqslant N}\varphi_3 \!: - :\! V * \big(P_{\leqslant N}\varphi_1 \cdot P_{\leqslant N}\varphi_2\big) \olessthan P_{\leqslant N}\varphi_3 \!:, \\
&\big(\neg \boxed{\olessthan \& \olessthan}\big) \Big(:\! V * \big(P_{\leqslant N}\varphi_1 \cdot P_{\leqslant N}\varphi_2\big) P_{\leqslant N}\varphi_3 \!: \Big) \\
&\overset{\text{def}}{=} V * \big(P_{\leqslant N}\varphi_1 \cdot P_{\leqslant N}\varphi_2\big) P_{\leqslant N}\varphi_3 \!: - \boxed{\olessthan \& \olessthan}\Big(:\! V * \big(P_{\leqslant N}\varphi_1 \cdot P_{\leqslant N}\varphi_2\big) P_{\leqslant N}\varphi_3 \!: \Big),
\end{aligned}
$$

which contains the full renormalization.

Equipped with our notion of types and the renormalization, we can now state and prove the multi-linear master estimate.

Proposition 4.2.8 (Multi-linear master estimate)**.** Let $\zeta = \zeta(\epsilon, s_1, s_2, \kappa, \eta, \eta', b_+, b) > 0$ be sufficiently small, let $A \geqslant 1$, and let $T \geqslant 1$. Then, there exists a Borel set $\Theta^{\mathrm{ms}}_{\mathrm{blue}}(A,T) \subseteq \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$ satisfying

$$\mathbb{P}(\textcolor{blue}{\bullet} \in \Theta^{\mathrm{ms}}_{\mathrm{blue}}(A,T)) \geqslant 1 - \zeta^{-1}\exp(-\zeta A^{\zeta}) \tag{4.2.41}$$

and such that for all $\textcolor{blue}{\bullet} \in \Theta^{\mathrm{ms}}_{\mathrm{blue}}(A,T)$ the following estimates hold:

Let $\mathcal{J} \subseteq [0,T]$ be an interval and let $N \geqslant 1$. Let $\varphi_1, \varphi_2, \varphi_3 \colon \mathcal{J} \times \mathbb{T}^3 \to \mathbb{R}$ be as in Definition 4.2.4 and let

$$(\varphi_1, \varphi_2; \varphi_3) \overset{\text{type}}{\neq} \big(\overset{\textcolor{blue}{\bullet}}{|}, \overset{\textcolor{blue}{\bullet}}{|}; \overset{\textcolor{blue}{\bullet}}{|}\big), \big(\overset{\textcolor{blue}{\bullet}}{|}, w; \overset{\textcolor{blue}{\bullet}}{|}\big).$$

(i) If $(\varphi_1, \varphi_2; \varphi_3) \overset{\text{type}}{=} \big(\overset{\textcolor{blue}{\bullet}}{|}, \Psi^{\textcolor{orange}{*}}; \overset{\textcolor{blue}{\bullet}}{|}\big),\ \big(\overset{\textcolor{blue}{\bullet}}{|}, X; \overset{\textcolor{blue}{\bullet}}{|}\big)$, then

$$\Big\| \big(\neg \boxed{\textcircled{\scriptsize<} \,\&\, \textcircled{\scriptsize<}}\big) \Big(:V * \big(P_{\leqslant N}\varphi_1 \cdot P_{\leqslant N}\varphi_2\big) P_{\leqslant N}\varphi_3 : \Big) \Big\|_{\mathfrak{X}^{s_2-1, b_+-1}(\mathcal{J})} \leqslant T^{30} A.$$

(ii) If $(\varphi_1, \varphi_2; \varphi_3) \overset{\text{type}}{=} \big(\overset{\textcolor{blue}{\bullet}}{|}, Y; \overset{\textcolor{blue}{\bullet}}{|}\big)$ or $\varphi_1, \varphi_2 \overset{\text{type}}{\neq} \overset{\textcolor{blue}{\bullet}}{|}$ and $\varphi_3 \overset{\text{type}}{=} \overset{\textcolor{blue}{\bullet}}{|}$, then

$$\Big\| :V * \big(P_{\leqslant N}\varphi_1 \cdot P_{\leqslant N}\varphi_2\big) \big(\neg \textcircled{\scriptsize<}\big)\ P_{\leqslant N}\varphi_3 : \Big\|_{\mathfrak{X}^{s_2-1, b_+-1}(\mathcal{J})} \leqslant T^{30} A.$$

(iii) In all other cases,

$$\Big\| :V * \big(P_{\leqslant N}\varphi_1 \cdot P_{\leqslant N}\varphi_2\big) P_{\leqslant N}\varphi_3 : \Big\|_{\mathfrak{X}^{s_2-1, b_+-1}(\mathcal{J})} \leqslant T^{30} A.$$

Remark 4.2.9. The frequency-localized versions of each estimate in Proposition 4.2.8 gain an η'-power in the maximal frequency-scale. Furthermore, functions of the type $\Psi^{\textcolor{orange}{*}}$ can be replaced by $\Psi^{\textcolor{orange}{*}}_{\tau}$ as defined in (4.3.4). For more details on these minor modifications, we refer the reader to the proof of the individual main estimates (Section 4.5-4.8).

Proof. It suffices to prove the estimates with A on the right-hand side replaced by CA^C, where $C = C(s_1, s_2, b, b_+, \epsilon)$. Then, the desired estimate follows by replacing A with a small power of itself and adjusting the constant ζ. In the following, we freely restrict to events with acceptable probabilities.

Proof of (i)*:* If $(\varphi_1, \varphi_2; \varphi_3)$ has type

- $\big(\upharpoonleft, \curlyvee^{*}; \upharpoonleft\big)$, we use Proposition 4.5.7,
- $\big(\upharpoonleft, X; \upharpoonleft\big)$, we use Proposition 4.7.9 .

Proof of (ii)*:* If $(\varphi_1, \varphi_2; \varphi_3) \overset{\text{type}}{=} \big(\upharpoonleft, Y, \upharpoonleft\big)$, this follows from Proposition 4.6.3. Using Lemma 4.2.6, we may assume in all remaining cases that φ_1 and φ_2 have type $\curlyvee^{*}$ or w, as long as we obtain the estimate with T^{18} instead of T^{30}. If $(\varphi_1, \varphi_2; \varphi_3)$ has type

- $\big(\curlyvee^{*}, \curlyvee^{*}; \upharpoonleft\big)$, we use Lemma 4.7.4 and Proposition 4.5.10,
- $\big(w, \curlyvee^{*}; \upharpoonleft\big)$, we use Lemma 4.7.6 and Proposition 4.8.12,
- $\big(w, w; \upharpoonleft\big)$, we use Lemma 4.7.6 and Proposition 4.8.6.

Proof of (iii)*:* Using Lemma 4.2.6, we may assume that all functions φ_j are of type $\upharpoonleft$, $\curlyvee^{*}$, or w, as long as we prove the estimate with T^{18} instead of T^{30}. If no factor is of type $\upharpoonleft$, the desired estimate follows from Proposition 4.5.1 and Proposition 4.8.10. The remaining cases can be estimated as follows: If $(\varphi_1, \varphi_2; \varphi_3)$ has type

- $\big(\upharpoonleft, \upharpoonleft; \curlyvee^{*}\big)$, we use Proposition 4.5.8,
- $\big(\upharpoonleft, \upharpoonleft; w\big)$, we use Proposition 4.6.1,
- $\big(\upharpoonleft, \curlyvee^{*}; \curlyvee^{*}\big)$, we use Proposition 4.5.10,
- $\big(\upharpoonleft, \curlyvee^{*}; w\big)$, we use Proposition 4.8.12,

- $\big(\uparrow, w; \Psi\big)$, we use Lemma 4.8.8 and Proposition 4.8.12,
- $\big(\uparrow, w; w\big)$, we use Proposition 4.8.7 and Lemma 4.8.8.

□

4.2.3 Local well-posedness

In this subsection, we obtain our first local well-posedness result. It is phrased in terms of the ambient measure $\mathbb{P}$ and the random structure is based on the Gaussian initial data $\bullet$.

Proposition 4.2.10 (Structured local well-posedness w.r.t. the ambient measure)**.** Let $M \geqslant 1$, let $A \geqslant 1$, let $0 < \tau \leqslant 1$, and let $\zeta = \zeta(\epsilon, s_1, s_2, \kappa, \eta, \eta', b_+, b) > 0$ be sufficiently small. Denote by $\mathcal{L}_M^{\mathrm{amb}}(A,\tau)$ the event in the ambient space $(\Omega, \mathcal{F})$ defined by the following conditions:

(i) For any $N \geqslant 1$, the solution of (4.2.1) with initial data $\blacklozenge = \bullet + \circ_{\mathrm{M}}$ exists on $[0,\tau]$.

(ii) For all $N \geqslant 1$, there exist $w_N \in \mathscr{X}^{s_1,b}([0,\tau])$, $H_N \in \mathscr{LM}([0,\tau])$, and $Y_N \in \mathscr{X}^{s_2,b}([0,\tau])$ such that

$$\Phi_N(t)\blacklozenge = \uparrow(t) + \Psi_N(t) + w_N(t) \quad \text{and} \quad w_N(t) = P_{\leqslant N}\, \mathrm{I}\big[\mathrm{PCtrl}(H_N, P_{\leqslant N}\uparrow)\big](t) + Y_N(t)$$

for all $t \in [0,\tau]$. Furthermore, we have the bounds

$$\|w_N\|_{\mathscr{X}^{s_1,b}([0,\tau])}, \|H_N\|_{\mathscr{LM}([0,\tau])}, \|Y_N\|_{\mathscr{X}^{s_2,b}([0,\tau])} \leqslant A \quad \text{and}$$
$$\sum_{L_1 \sim L_2} \|P_{L_1}\uparrow \cdot P_{L_2} w_N\|_{L_t^2 H_x^{-4\delta_1}([0,\tau]\times\mathbb{T}^3)} \leqslant A.$$

(iii) It holds for all $N, K \geqslant 1$ that

$$\|\Phi_N[t]\blacklozenge - \Phi_K[t]\blacklozenge\|_{L_t^\infty \mathscr{H}_x^{\beta-\kappa}([0,\tau]\times\mathbb{T}^3)} \leqslant A \min(N,K)^{-\eta'}.$$

We further require that

$$\|H_N - H_K\|_{\mathscr{L}\mathscr{M}([0,\tau])},\ \ \|Y_N - Y_K\|_{\mathfrak{X}^{s_2,b}([0,\tau])} \leqslant A \min(N,K)^{-\eta'}.$$

If $A\tau^{b_+-b} \leqslant 1$, then $\mathcal{L}_M^{\mathrm{amb}}(A,\tau)$ has a high probability and it holds that

$$\mathbb{P}(\mathcal{L}_M^{\mathrm{amb}}(A,\tau)) \geqslant 1 - \zeta^{-1}\exp(-\zeta A^{\zeta}). \tag{4.2.42}$$

Remark 4.2.11. The superscript "amb" in $\mathcal{L}_M^{\mathrm{amb}}(A,\tau)$ emphasizes that the event lives in the ambient probability space. The first item (i) is only stated for expository purposes. Indeed, since (i) is a soft statement and does not contain any uniformity in the frequency-truncation parameter, it follows from the global well-posedness of (4.2.1) (which is also not uniform in N). The interesting portions of the proposition are included in (ii) and (iii), which contain uniform structural information about the solution and allow us to locally define the limiting dynamics.

By combining Theorem 4.1.1 and Proposition 4.2.10, we easily obtain the local well-posedness of the renormalized nonlinear wave equation on the support of the Gibbs measure.

Corollary 4.2.12 (Local well-posedness for Gibbsian initial data)**.** Let $0 < \tau < 1$ and let $\zeta = \zeta(\epsilon, s_1, s_2, \kappa, \eta, \eta', b_+, b) > 0$ be sufficiently small. Then, there exists a Borel set $\mathcal{L}(\tau) \subseteq \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$ such that $\Phi_N[t]\blacklozenge$ converges in $C_t^0\mathscr{H}_x^{-1/2-\kappa}([0,\tau]\times\mathbb{T}^3)$ as $N \to \infty$ and such that

$$\mu_\infty^{\otimes}(\mathcal{L}(\tau)) \geqslant 1 - \zeta^{-1}\exp(-\zeta\tau^{-\zeta}). \tag{4.2.43}$$

Corollary 4.2.12 shows that the limiting dynamics $\Phi(t)\blacklozenge = \lim_{N\to\infty}\Phi_N(t)\blacklozenge$ are locally well-defined on the support of the Gibbs measure. However, it does not contain any structural information about the solution, which will be essential in the globalization argument (Section 4.3). The main difficulty, which was described in detail in Section 4.2.1, is that the free component of the initial data $\bullet$ is only defined on the ambient space. Nevertheless, in Proposition 4.3.3 below, we obtain a structured local well-posedness theorem in terms of $\blacklozenge$.

We first use the structured local well-posedness result for the ambient measure (Proposition 4.2.10) to prove the unstructured local well-posedness for Gibbsian random data (Corollary 4.2.12). Then, we present the proof of Proposition 4.2.10.

Proof of Corollary 4.2.12: Let $M \geqslant 1$ and let A satisfy $A\tau^{b_+-b} \leqslant 1$. We define a closed set $\widetilde{\mathcal{L}}(A,\tau) \subseteq \mathscr{H}_x^{-\frac{1}{2}-\kappa}(\mathbb{T}^3)$ by requiring that $\blacklozenge \in \widetilde{\mathcal{L}}(A,\tau)$ if and only if

(a) For any $N \geqslant 1$, the solution of (4.2.1) with initial data $\blacklozenge$ exists on $[0,\tau]$.

(b) It holds for all $N, K \geqslant 1$ that

$$\|\Phi_N(t)\blacklozenge - \Phi_K(t)\blacklozenge\|_{L_t^\infty \mathscr{H}_x^{\beta-\kappa}([0,\tau]\times\mathbb{T}^3)} \leqslant A \min(N,K)^{-\eta'}.$$

It is clear from the definition that $\mathcal{L}(\tau) \subseteq \widetilde{\mathcal{L}}(A,\tau)$. We emphasize that $\widetilde{\mathcal{L}}(A,\tau)$ is defined intrinsically through $\blacklozenge$ and does not refer to the ambient probability space $(\Omega, \mathcal{F}, \mathbb{P})$. From the definition of $\mathcal{L}_M^{\mathrm{amb}}(A,\tau)$ in Proposition 4.2.10, it follows that

$$\mathcal{L}_M^{\mathrm{amb}}(A,\tau) \subseteq \{\bullet + \circ_{\mathrm{M}} \in \widetilde{\mathcal{L}}(A,\tau)\}.$$

By using Theorem 4.1.1, we have that $\mathrm{Law}_{\mathbb{P}}(\bullet + \circ_{\mathrm{M}}) = \nu_M^{\otimes}$. This yields

$$\nu_M^{\otimes}(\widetilde{\mathcal{L}}(A,\tau)) = \mathbb{P}(\bullet + \circ_{\mathrm{M}} \in \widetilde{\mathcal{L}}(A,\tau)) \geqslant \mathbb{P}(\mathcal{L}_M^{\mathrm{amb}}(A,\tau)) \geqslant 1 - \zeta^{-1}\exp(-\zeta A^{\zeta}).$$

By using the quantitative version of the absolute continuity $\mu_M^{\otimes} \ll \nu_M^{\otimes}$ in Theorem 4.1.1, we obtain that

$$\mu_M^{\otimes}(\mathscr{H}_x^{-\frac{1}{2}-\kappa}(\mathbb{T}^3)\backslash\widetilde{\mathcal{L}}(A,\tau)) \lesssim \nu_M^{\otimes}(\mathscr{H}_x^{-\frac{1}{2}-\kappa}(\mathbb{T}^3)\backslash\widetilde{\mathcal{L}}(A,\tau))^{1-\frac{1}{q}} \lesssim \zeta^{-1}\exp\big(-\zeta\big(1-q^{-1}\big)A^{\zeta}\big).$$

After adjusting the value of ζ, this yields the desired estimate (4.2.43) with $\mu_\infty^\otimes$ replaced by $\mu_M^\otimes$. Since $\widetilde{\mathcal{L}}(A,\tau)$ is closed in $\mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$ and a subsequence of $\mu_M^\otimes$ weakly converges to $\mu_\infty^\otimes$, we obtain the same probabilistic estimate for the limiting measure $\mu_\infty^\otimes$. □

Proof of Proposition 4.2.10: As discussed in Remark 4.2.11, (i) follows from a soft argument. We now turn to the proof of (ii), which is the heart of the proposition. We let $B = cA^c$, where $c = c(\epsilon, s_1, s_2, b_+, b)$ is a sufficiently small constant.

Using Theorem 4.1.1, Lemma 4.2.6, Proposition 4.2.8, and Proposition 4.5.1, we may restrict to the event

$$\begin{aligned}&\Big\{ \bullet \in \Theta^{\mathrm{ms}}_{\mathrm{blue}}(B,1) \Big\} \cap \Big\{ \bullet \in \Theta^{\mathrm{type}}_{\mathrm{blue}}(B,1) \Big\} \cap \Big\{ \| \uparrow \|_{L_t^\infty \mathcal{C}_x^{-1/2-\kappa}([0,1]\times\mathbb{T}^3)} \leqslant B \Big\} \\ &\cap \Big\{ \sup_N \| \Psi_N \|_{L_t^\infty \mathcal{C}_x^{\beta-\kappa}([0,1]\times\mathbb{T}^3)} \leqslant B \Big\} \cap \Big\{ \| \circ_{\mathbf{M}} \|_{\mathscr{H}_x^{1/2+\beta-\kappa}(\mathbb{T}^3)} \leqslant B \Big\}.\end{aligned} \tag{4.2.44}$$

We now define a map

$$\Gamma_N = (\Gamma_{N,X}, \Gamma_{N,Y}) \colon \mathfrak{X}^{s_1,b}([0,\tau]) \times \mathfrak{X}^{s_2,b}([0,\tau]) \to \mathfrak{X}^{s_1,b}([0,\tau]) \times \mathfrak{X}^{s_2,b}([0,\tau])$$

by

$$\begin{aligned}\Gamma_{N,X}(X_N,Y_N) \overset{\text{def}}{=} P_{\leqslant N}\,\mathrm{I}\Big[2\, \boxed{\circledcirc \,\&\, \circledcirc}\Big(V * \Big(P_{\leqslant N} \uparrow \cdot P_{\leqslant N}\big(\Psi_N + X_N \big)\Big) P_{\leqslant N}\uparrow \Big) \\ + 2\Big(V * \Big(P_{\leqslant N}\uparrow \cdot P_{\leqslant N}\big(Y_N\big)\Big) \circledcirc P_{\leqslant N}\uparrow \Big) + \Big(V * \Big(P_{\leqslant N}\big(\Psi_N + w_N \big)\Big)^2 \Big) \circledcirc P_{\leqslant N}\uparrow \Big]\end{aligned}$$

and

$$\begin{aligned}&\Gamma_{N,Y}(X_N,Y_N) \\ &\overset{\text{def}}{=} \mathring{\uparrow} + \mathrm{I}\big[\mathbf{So}_N + \mathbf{CPara}_N(\Gamma_{N,X}(X_N,Y_N), w_N) + \mathbf{RMT}_N(Y_N, w_N) + \mathbf{Phy}_N(X_N,Y_N,w_N)\big],\end{aligned}$$

where $w_N = X_N + Y_N$. We emphasize our use of the double Duhamel trick, which is manifested in the argument $\Gamma_{N,X}(X_N,Y_N)$ of $\mathbf{CPara}_N$. Our goal is to show that Γ_N is a contraction on a ball in $\mathfrak{X}^{s_1,b}([0,\tau]) \times \mathfrak{X}^{s_2,b}([0,\tau])$, where the radius remains to be chosen.

Using Lemma 4.7.4 and Lemma 4.7.6, it follows that there exists a (canonical) $H_N = H_N(X_N, Y_N)$ satisfying the identity

$$\Gamma_{N,X}(X_N, Y_N) = P_{\leqslant N} \, \mathrm{I}\Big[\mathrm{PCtrl}(H_N, P_{\leqslant N} \overset{\bullet}{|}) \Big]$$

and the estimate

$$\|H_N\|_{\mathscr{L}\mathscr{M}([0,\tau])} \lesssim B^2 + \|X_N\|^2_{\mathfrak{X}^{s_1,b}([0,\tau])} + \|Y_N\|^2_{\mathfrak{X}^{s_2,b}([0,\tau])}. \tag{4.2.45}$$

Using the energy estimate (Lemma 4.4.8), the inhomogeneous Strichartz estimate (Lemma 4.4.9), Lemma 4.7.3, and $s_1 - 1 + 8\epsilon < -1/2 - \kappa$, we obtain that

$$\begin{aligned}
\big\|\Gamma_{N,X}(X_N, Y_N)\big\|_{\mathfrak{X}^{s_1,b}([0,\tau])} &\lesssim \big\| \mathrm{PCtrl}(H_N, P_{\leqslant N} \overset{\bullet}{|})\big\|_{\mathfrak{X}^{s_1-1,b-1}([0,\tau])} \\
&\lesssim \big\| \mathrm{PCtrl}(H_N, P_{\leqslant N} \overset{\bullet}{|})\big\|_{L_t^{2b} H_x^{s_1-1}([0,\tau]\times\mathbb{T}^3)} \\
&\lesssim \tau^{\frac{1}{2b}} \big\| \mathrm{PCtrl}(H_N, P_{\leqslant N} \overset{\bullet}{|})\big\|_{L_t^{\infty} H_x^{s_1-1}([0,\tau]\times\mathbb{T}^3)} \\
&\lesssim \tau^{\frac{1}{2b}} \|H_N\|_{\mathscr{L}\mathscr{M}([0,\tau])} \big\| \overset{\bullet}{|} \big\|_{L_t^{\infty} H_x^{s_1-1+8\epsilon}([0,\tau]\times\mathbb{T}^3)} \\
&\lesssim \tau^{\frac{1}{2b}} B\big(B^2 + \|X_N\|^2_{\mathfrak{X}^{s_1,b}([0,\tau])} + \|Y_N\|^2_{\mathfrak{X}^{s_2,b}([0,\tau])}\big).
\end{aligned} \tag{4.2.46}$$

Using the multi-linear estimates from Proposition 4.2.8, which are available due to our restriction to the event (4.2.44), and the time-localization lemma (Lemma 4.4.3), we similarly obtain

$$\begin{aligned}
&\big\|\Gamma_{N,Y}(X_N, Y_N)\big\|_{\mathfrak{X}^{s_2,b}([0,\tau])} \\
&\lesssim \big\| \overset{\circ}{|} \big\|_{\mathfrak{X}^{s_2,b}([0,\tau])} + \big\| \mathbf{So} + \mathbf{CPara} + \mathbf{RMT} + \mathbf{Phy} \big\|_{\mathfrak{X}^{s_2-1,b-1}([0,\tau])} \\
&\lesssim B + \tau^{b_+ - b} \big\| \mathbf{So} + \mathbf{CPara} + \mathbf{RMT} + \mathbf{Phy} \big\|_{\mathfrak{X}^{s_2-1,b_+-1}([0,\tau])} \\
&\lesssim B + \tau^{b_+ - b}\big(B^3 + \|X_N\|^3_{\mathfrak{X}^{s_1,b}([0,\tau])} + \|Y_N\|^3_{\mathfrak{X}^{s_2,b}([0,\tau])}\big)
\end{aligned} \tag{4.2.47}$$

By combining (4.2.46) and (4.2.47), we obtain for a constant $C = C(\epsilon, s_1, s_2, b_+, b)$ that

$$\big\|\Gamma_N(X_N, Y_N)\big\|_{\mathfrak{X}^{s_1,b}([0,\tau])\times\mathfrak{X}^{s_2,b}([0,\tau])} \leqslant CB + C\tau^{b_+ - b}\big(B^3 + \|X_N\|^3_{\mathfrak{X}^{s_1,b}([0,\tau])} + \|Y_N\|^3_{\mathfrak{X}^{s_2,b}([0,\tau])}\big). \tag{4.2.48}$$

Since $C^4\tau^{b_+-b}B^2 \leqslant 1/100$, which follows from $\tau^{b_+-b}A \leqslant 1$ and our choice of B, we see that Γ_N maps the ball in $\mathfrak{X}^{s_1,b}([0,\tau]) \times \mathfrak{X}^{s_2,b}([0,\tau])$ of radius $2CB$ to itself. A minor modification of the above argument also yields that Γ_N is a contraction, which implies the existence of a unique fixed point (X_N, Y_N) of Γ_N satisfying

$$\|X_N\|_{\mathfrak{X}^{s_1,b}([0,\tau])}, \|Y_N\|_{\mathfrak{X}^{s_2,b}([0,\tau])} \leqslant 2CB. \tag{4.2.49}$$

Using (4.2.45), we obtain that $X_N = P_{\leqslant N}\,\mathrm{I}\big[\,\mathrm{PCtrl}(H_N, P_{\leqslant N}\overset{\bullet}{|})\big]$ with H_N satisfying $\|H_N\|_{\mathscr{LM}([0,\tau])} \lesssim B^2$. Finally, using the triangle inequality and the condition $\bullet \in \Theta^{\mathrm{type}}_{\mathrm{blue}}(B,1)$ from (4.2.44), we obtain that $w_N = X_N + Y_N$ satisfies

$$\|w_N\|_{\mathfrak{X}^{s_1,b}([0,\tau])} \leqslant 4CB \qquad \text{and} \qquad \sum_{L_1\sim L_2} \|P_{L_1}\overset{\bullet}{|} \cdot P_{L_2} w_N\|_{L^2_t H^{-4\delta_1}_x([0,\tau])} \lesssim B^2. \tag{4.2.50}$$

Using that $B = cA^c$, (4.2.49) and (4.2.50) yield the desired estimates in (ii).

We now turn to (iii). This is a notationally extremely tedious but mathematically minor modification of the arguments leading to (ii). Similar modifications are usually omitted in the literature and we only outline the argument. In the frequency-localized versions of our estimates leading to (ii), we always had an additional decaying factor $N_{\max}^{-\eta'}$, where $N_{\max}$ was the maximal frequency-scale (see Remark 4.2.9 and Sections 4.5-4.8). So far, this was only used to sum over all dyadic scales, but it also yields the smallness conditions in (iii). Indeed, one only has to apply the same estimates as above to the difference equation

$$(X_N - X_K, Y_N - Y_K) = \Gamma_N(X_N, Y_N) - \Gamma_K(X_K, Y_K).$$

□

4.2.4 Stability theory

In this subsection, we prove a stability estimate (Proposition 4.2.14) on large time-intervals. Strictly speaking, the stability estimate is part of the global instead of the local theory, but the argument is closely related to the proof of local well-posedness (Proposition 4.2.10). While the stability estimate in this section is phrased in terms of ${\color{blue}\bullet}$, it can be used to obtain a similar estimate in terms of ${\color{purple}\blacklozenge}$ (Proposition 4.3.8). This second stability estimate will then be used in the globalization argument.

In order to state the stability result, we introduce the function space $\mathfrak{X}$, which captures the admissible perturbations of the initial data.

Definition 4.2.13 (Structured perturbations)**.** Let $T \geqslant 1$, $t_0 \in [0,T]$, $N \geqslant 1$, and $K \geqslant 1$. For any ${\color{purple}\blacklozenge} \in \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$ and $Z[t_0] \in \mathscr{H}_x^{s_1}(\mathbb{T}^3)$, we define

$$
\begin{aligned}
&\|Z[t_0]\|_{\mathfrak{X}([0,T],{\color{purple}\overset{\blacklozenge}{|}};t_0,N,K)} \\
&= \inf_{Z^{\bigcirc},Z^{\square}} \max\Big(\|Z^{\square}[t_0]\|_{\mathcal{H}_x^{s_1}(\mathbb{T}^3)},\ \|Z^{\bigcirc}[t_0]\|_{\mathcal{H}_x^{s_2}(\mathbb{T}^3)},\ \sum_{L_1\sim L_2} \|P_{L_1}{\color{purple}\overset{\blacklozenge}{|}} \cdot P_{L_2} Z\|_{L_t^2 H_x^{-4\delta_1}([0,T]\times\mathbb{T}^3)}, \\
&\qquad\qquad \|\, :V * \big(P_{\leqslant N}{\color{purple}\overset{\blacklozenge}{|}} \cdot P_{\leqslant N} Z^{\bigcirc}\big)\ \big(\neg \circledless\big)\ P_{\leqslant N}{\color{purple}\overset{\blacklozenge}{|}}: \|_{\mathfrak{X}^{s_2-1,b_+-1}([0,T])}, \\
&\qquad\qquad \|\, \big(\neg \boxed{\circledless \,\&\, \circledless}\big)\ \Big(:V * \big(P_{\leqslant N}{\color{purple}\overset{\blacklozenge}{|}} \cdot P_{\leqslant N} Z^{\square}\big)\, P_{\leqslant N}{\color{purple}\overset{\blacklozenge}{|}}: \Big)\|_{\mathfrak{X}^{s_2-1,b_+-1}([0,T])}\Big),
\end{aligned}
$$

where the infimum is taken over all $Z^{\square}[t_0] \in \mathscr{H}_x^{s_1}(\mathbb{T}^3)$ and $Z^{\bigcirc}[t_0] \in \mathscr{H}_x^{s_2}(\mathbb{T}^3)$ satisfying the identity $Z[t_0] = Z^{\square}[t_0] + Z^{\bigcirc}[t_0]$ and the Fourier support condition $\operatorname{supp} \widehat{Z^{\square}}[t_0](n) \subseteq \{n \in \mathbb{Z}^3 \colon |n| \leqslant 8\max(N,K)\}$. Furthermore, we wrote $Z^{\square}$, $Z^{\bigcirc}$, and Z for the corresponding solutions to the linear wave equation.

The notation $Z_N^{\bigcirc}$ and $Z_N^{\square}$ is motivated by the paradifferential operators used in their treatment. The contributions of $Z_N^{\bigcirc}$ and $Z_N^{\square}$ are estimated using $\circledless$ and $\boxed{\circledless \& \circledless}$, respectively.

It is clear that, for a fixed parameters T, t_0, N, and K, the maximum is jointly continuous in $Z^{\bigcirc}[t_0] \in \mathscr{H}_x^{s_1}(\mathbb{T}^3)$ (satisfying the frequency-support condition), $Z^{\square}[t_0] \in \mathscr{H}_x^{s_2}(\mathbb{T}^3)$, and $\blacklozenge \in \mathscr{H}_x^{-1/2-\kappa}$. This is the primary reason for including the frequency support condition, since the sum in L_1 and L_2 would otherwise not be continuous in $Z^{\bigcirc}[t_0]$. In particular, the norm $\|Z[t_0]\|_{\mathfrak{Z}([0,T],\,\uparrow;t_0,N,K)}$ is Borel-measurable in $Z[t_0] \in \mathscr{H}_x^{s_2}(\mathbb{T}^3)$ and $\blacklozenge \in \mathscr{H}_x^{-1/2-\kappa}$.

Proposition 4.2.14 (Stability estimate)**.** Let $T \geqslant 1$, let $A \geqslant 1$, and let $\zeta(\epsilon, s_1, s_2, \kappa, \eta, \eta', b_+, b) > 0$ be sufficiently small. There exists a constant $C = C(\epsilon, s_1, s_2, b_+, b_-)$ and a Borel set $\Theta^{\mathrm{stab}}_{\mathrm{blue}}(A,T) \subseteq \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$ satisfying

$$\mathbb{P}(\bullet \in \Theta^{\mathrm{stab}}_{\mathrm{blue}}(A,T)) \geqslant 1 - \zeta^{-1}\exp(-\zeta A^{\zeta})$$

such that the following holds for all $\bullet \in \Theta^{\mathrm{stab}}_{\mathrm{blue}}(A,T)$:

Let $N \geqslant 1$, $B \geqslant 1$, $0 < \theta < 1$, $\mathcal{J} \subseteq [0,T]$ be a compact interval, and $t_0 \overset{\mathrm{def}}{=} \min \mathcal{J}$. Let $\widetilde{u}_N \colon \mathcal{J} \times \mathbb{T}^3 \to \mathbb{R}$ be an approximate solution of (4.2.1) satisfying the following assumptions.

(A1) Structure: We have the decomposition

$$\widetilde{u}_N = \uparrow + \Psi_N + \widetilde{w}_N.$$

(A2) Global bounds: It holds that

$$\|\widetilde{w}_N\|_{\mathfrak{X}^{s_1,b}(J)} \leqslant B \quad \text{and} \quad \sum_{L_1 \sim L_2} \|P_{L_1} \uparrow \cdot P_{L_2} \widetilde{w}_N\|_{L_t^2 H_x^{-4\delta_1}(\mathcal{J}\times\mathbb{T}^3)} \leqslant B.$$

(A3) Approximate solution: There exists $H_N \in \mathscr{LM}(\mathcal{J})$ and $F_N \in \mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})$ satisfying the identity

$$(-\partial_t^2 - 1 + \Delta)\widetilde{u}_N = P_{\leqslant N} :\!\big(V * (P_{\leqslant N}\widetilde{u}_N)^2\big) P_{\leqslant N}\widetilde{u}_N\!: - P_{\leqslant N}\,\mathrm{PCtrl}(H_N, P_{\leqslant N}\uparrow) - F_N$$

and the estimates

$$\|H_N\|_{\mathscr{L}\mathscr{M}(\mathcal{J})} \leqslant \theta \qquad \text{and} \qquad \|F_N\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})} \leqslant \theta.$$

Furthermore, let $Z_N[t_0] \in H_x^{s_1}(\mathbb{T}^3)$ be a perturbation satisfying the following assumption.

(A4) Structured perturbation: There exists a $K \geqslant 1$ such that

$$\|Z[t_0]\|_{\mathfrak{X}(\mathcal{J},\linear;t_0,N,K)} \leqslant \theta.$$

Finally, assume that

(A5) Parameter condition: $C\exp\Big(C(A+B)^{\frac{2}{b_+-b}}T^{\frac{40}{b_+-b}}\Big)\theta \leqslant 1.$

Then, there exists a solution $u_N\colon \mathcal{J}\times\mathbb{T}^3 \to \mathbb{R}$ of (4.2.1) satisfying the initial value condition $u_N[t_0] = \widetilde{u}_N[t_0] + Z_N[t_0]$ and the following conclusions.

(C1) Preserved structure: We have the decomposition

$$u_N = \linear + \cubic_N + w_N.$$

(C2) Closeness: The difference $u_N - \widetilde{u}_N = w_N - \widetilde{w}_N$ satisfies

$$\|u_N - \widetilde{u}_N\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \leqslant C\exp\big(C(A+B)^{\frac{2}{b_+-b}}T^{\frac{40}{b_+-b}}\big)\theta,$$

$$\sum_{L_1\sim L_2}\|P_{L_1}\linear\cdot P_{L_2}(u_N-\widetilde{u}_N)\|_{L_t^2H_x^{-4\delta_1}(\mathcal{J}\times\mathbb{T}^3)} \leqslant C\exp\big(C(A+B)^{\frac{2}{b_+-b}}T^{\frac{40}{b_+-b}}\big)\theta.$$

(C3) Preserved global bounds: It holds that

$$\|w_N\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \leqslant B_\theta \quad \text{and} \quad \sum_{L_1\sim L_2}\|P_{L_1}\linear\cdot P_{L_2}w_N\|_{L_t^2H_x^{-4\delta_1}(\mathcal{J}\times\mathbb{T}^3)} \leqslant B_\theta,$$

where $B_\theta \overset{\text{def}}{=} B + C\exp\big(C(A+B)^{\frac{2}{b_+-b}}T^{\frac{40}{b_+-b}}\big)\theta.$

As mentioned above, the proof of Proposition 4.2.14 is close to the proof of local well-posedness. The most important additional ingredient is a Gronwall-type argument in $\mathcal{X}^{s,b}$-spaces, which is slightly technical due to their non-local nature in the time-variable.

Proof. Let $N, B, \theta, \mathcal{J}, t_0, \widetilde{u}_N, \widetilde{w}_N, H_N, F_N, Z_N, Z_N^{\square}$, and $Z_N^{\bigcirc}$ be as in the statement of the proposition and assume that (A1)-(A5) are satisfied. We make the Ansatz

$$u_N(t) = \widetilde{u}_N(t) + v_N(t) + Z_N(t),$$

where the nonlinear component $v_N(t)$ will be decomposed into a para-controlled and a smoother component below. Based on the condition $u_N[t_0] = \widetilde{u}_N[t_0] + Z_N[t_0]$, we require that $v_N[t_0] = 0$. Using the assumption (A3) and that Z_N solves the linear wave equation, we obtain the evolution equation

$$\begin{aligned}(-\partial_t^2 - 1 + \Delta) v_N =& P_{\leqslant N} :\!\Big(V * \big(P_{\leqslant N}(\widetilde{u}_N + v_N + Z_N)^2\big)\Big) P_{\leqslant N}(\widetilde{u}_N + v_N + Z_N)\!: \\ &- P_{\leqslant N} :\!\big(V * (P_{\leqslant N}\widetilde{u}_N)^2\big) P_{\leqslant N}\widetilde{u}_N\!: \\ &+ P_{\leqslant N} \operatorname{PCtrl}(H_N, P_{\leqslant N} \<1>) + F_N.\end{aligned}$$

Inserting the structural assumption (A1) and using the binomial formula, we obtain that

$$\begin{aligned}(-\partial_t^2 - 1 + \Delta) v_N =& P_{\leqslant N}\Bigg[2 :\!V * \Big(P_{\leqslant N}\Big(\<1> + \<3DN> + w_N\Big) \cdot P_{\leqslant N}\Big(v_N + Z_N\Big)\Big) P_{\leqslant N}\<1>\!: \\ &+ V * \Big(\Big(P_{\leqslant N}\big(v_N + Z_N\big)\Big)^2\Big) P_{\leqslant N}\<1> \\ &+ \operatorname{PCtrl}(H_N, P_{\leqslant N}\<1>) \\ &+ 2V * \Big(P_{\leqslant N}\Big(\<1> + \<3DN> + w_N\Big) \cdot P_{\leqslant N}\Big(v_N + Z_N\Big)\Big) P_{\leqslant N}\Big(\<3DN> + w_N\Big) \\ &+ P_{\leqslant N}\Big(V * \big(:\!P_{\leqslant N}(\widetilde{u}_N + v_N + Z_N)^2\!: \big)\Big) P_{\leqslant N}(v_N + Z_N)\Bigg] + F_N.\end{aligned}$$

We then decompose $v_N = X_N + Y_N$, where X_N is the para-controlled component and Y_N is the smoother component. Since $v_N[t_0] = 0$, we impose the initial value conditions $X_N[t_0] = 0$ and

$Y_N[t_0] = 0$. Similar as in Section 4.2.1, we define X_N and Y_N through the evolution equations

$$
\begin{aligned}
(-\partial_t^2 - 1 + \Delta) X_N = P_{\leqslant N}\Big[& 2 \boxed{\varolessthan \& \varolessthan} \Big(V * \Big(P_{\leqslant N} \usebox{\Linear} \cdot P_{\leqslant N}\Big(X_N + Z_N^{\square}\Big)\Big) P_{\leqslant N} \usebox{\Linear} \Big) \\
& + 2V * \Big(P_{\leqslant N} \usebox{\Linear} \cdot P_{\leqslant N}\Big(Y_N + Z_N^{\bigcirc}\Big)\Big) \varolessthan P_{\leqslant N} \usebox{\Linear} \\
& + 2\, V * \Big(P_{\leqslant N}\Big(\usebox{\CubicN} + w_N \Big) \cdot P_{\leqslant N}\Big(X_N + Y_N + Z_N \Big)\Big) \varolessthan P_{\leqslant N} \usebox{\Linear} \\
& + V * \Big(\Big(P_{\leqslant N}(v_N + Z_N)\Big)^2 \Big) \varolessthan P_{\leqslant N} \usebox{\Linear} + \mathrm{PCtrl}(H_N, P_{\leqslant N} \usebox{\Linear}) \Big]
\end{aligned}
\tag{4.2.51}
$$

and

$$
\begin{aligned}
(-\partial_t^2 - 1 + \Delta) Y_N = P_{\leqslant N}\Big[& 2 \Big(\neg \boxed{\varolessthan \& \varolessthan} \Big) \Big(:V * \Big(P_{\leqslant N} \usebox{\Linear} \cdot P_{\leqslant N}\Big(X_N + Z_N^{\square}\Big)\Big) P_{\leqslant N} \usebox{\Linear} : \Big) \\
& + 2 : V * \Big(P_{\leqslant N} \usebox{\Linear} \cdot P_{\leqslant N}\Big(Y_N + Z_N^{\bigcirc}\Big)\Big) \Big(\neg \varolessthan \Big) P_{\leqslant N} \usebox{\Linear} : \\
& + 2\, V * \Big(P_{\leqslant N}\Big(\usebox{\CubicN} + w_N \Big) \cdot P_{\leqslant N}\Big(v_N + Z_N \Big)\Big) \Big(\neg \varolessthan \Big) P_{\leqslant N} \usebox{\Linear} \\
& + V * \Big(\Big(P_{\leqslant N}(v_N + Z_N)\Big)^2 \Big) \Big(\neg \varolessthan \Big) P_{\leqslant N} \usebox{\Linear} \\
& + 2V * \Big(P_{\leqslant N}\Big(\usebox{\Linear} + \usebox{\CubicN} + w_N \Big) \cdot P_{\leqslant N}\Big(v_N + Z_N \Big)\Big) P_{\leqslant N}\Big(\usebox{\CubicN} + w_N \Big) \\
& + \Big(V * \big(: P_{\leqslant N}(\widetilde{u}_N + v_N + Z_N)^2 : \big)\Big) P_{\leqslant N}(v_N + Z_N) \Big] + F_N .
\end{aligned}
\tag{4.2.52}
$$

Since the nonlinearity in (4.2.51) and (4.2.52) is frequency-truncated, a soft argument yields the local existence and uniqueness of X_N and Y_N in $C_t^0 \mathscr{H}_x^{s_1}$ and $C_t^0 \mathscr{H}_x^{s_2}$, respectively. Since $\mathfrak{X}^{s,b}(\mathcal{J})$ embeds into $C_t^0 \mathscr{H}_x^{s}(\mathcal{J} \times \mathbb{T}^3)$ for all $s \in \mathbb{R}$, the solutions exist as long as the restricted $\mathfrak{X}^{s_1,b}$ and $\mathfrak{X}^{s_2,b}$-norms stay bounded.

In order to prove that X_N and Y_N exist on the full interval $\mathcal{J}$ and satisfy the desired bounds, we let T_* be the maximal time of existence of X_N and Y_N on $\mathcal{J}$. We now proceed through a Gronwall-type argument in $\mathfrak{X}^{s,b}$-spaces. We first define

$$
f_N \colon [t_0, T^*) \to [0, \infty),\ t \mapsto \|X_N\|_{\mathfrak{X}^{s_1,b}([t_0,t])} + \|Y_N\|_{\mathfrak{X}^{s_2,b}([t_0,t])}.
$$

We emphasize that we neither rely on nor prove the continuity of f_N. Using Lemma 4.4.4 and Lemma 4.4.8, there exists an implicit constant $C_{\mathrm{En}} = C_{\mathrm{En}}(s_1, s_2, b)$ such that

$$g_N(t) \overset{\text{def}}{=} C_{\mathrm{En}}\Big(\big\| 1_{[t_0,t]}(-\partial_t^2 - 1 + \Delta) X_N \big\|_{\mathfrak{X}^{s_1-1,b-1}(\mathbb{R})} + \big\| 1_{[t_0,t]}(-\partial_t^2 - 1 + \Delta) Y_N \big\|_{\mathfrak{X}^{s_2-1,b-1}(\mathbb{R})} \Big)$$

satisfies $f_N(t) \leqslant g_N(t)$ for all $t \in [t_0, T_*)$. Due to Lemma 4.4.4, $g_N(t)$ is continuous. Now, let $\tau > 0$ be a step-size which remains to be chosen and assume that $t, t' \in [t_0, T_*)$ satisfy $t \leqslant t' \leqslant t + \tau$. Using Lemma 4.4.3, we obtain that for an implicit constant $C = C(s_1, s_2, b, b_+)$ that

$$\begin{aligned}
& g_N(t') \\
& \leqslant C_{\mathrm{En}}\Big(\big\| 1_{[t_0,t]}(-\partial_t^2 - 1 + \Delta) X_N \big\|_{\mathfrak{X}^{s_1-1,b-1}(\mathbb{R})} + \big\| 1_{[t_0,t]}(-\partial_t^2 - 1 + \Delta) Y_N \big\|_{\mathfrak{X}^{s_2-1,b-1}(\mathbb{R})} \Big) \\
& + C_{\mathrm{En}}\Big(\big\| 1_{(t,t']}(-\partial_t^2 - 1 + \Delta) X_N \big\|_{\mathfrak{X}^{s_1-1,b-1}(\mathbb{R})} + \big\| 1_{(t,t']}(-\partial_t^2 - 1 + \Delta) Y_N \big\|_{\mathfrak{X}^{s_2-1,b-1}(\mathbb{R})} \Big) \\
& \leqslant g_N(t) + C\tau^{b_+-b}\big(\big\| (-\partial_t^2 - 1 + \Delta) X_N \big\|_{\mathfrak{X}^{s_1-1,b_+-1}((t,t'])} + \big\| (-\partial_t^2 - 1 + \Delta) Y_N \big\|_{\mathfrak{X}^{s_2-1,b_+-1}((t,t'])} \big) \\
& \leqslant g_N(t) + C\tau^{b_+-b}\big(\big\| (-\partial_t^2 - 1 + \Delta) X_N \big\|_{\mathfrak{X}^{s_1-1,b_+-1}([t_0,t'])} + \big\| (-\partial_t^2 - 1 + \Delta) Y_N \big\|_{\mathfrak{X}^{s_2-1,b_+-1}([t_0,t'])} \big).
\end{aligned}$$

Similar as in the proof of local well-posedness (Proposition 4.2.10), we can use Lemma 4.2.6, Proposition 4.2.8, and Proposition 4.5.1 to restrict to the event

$$\begin{aligned}
& \Big\{ \bullet \in \Theta^{\mathrm{ms}}_{\mathrm{blue}}(A,T) \Big\} \cap \Big\{ \bullet \in \Theta^{\mathrm{type}}_{\mathrm{blue}}(A,T) \Big\} \cap \Big\{ \big\| \overset{\bullet}{|} \big\|_{L_t^\infty \mathcal{C}_x^{-1/2-\kappa}([0,1]\times\mathbb{T}^3)} \leqslant A \Big\} \\
& \cap \Big\{ \sup_N \big\| \Psi_N \big\|_{L_t^\infty \mathcal{C}_x^{\beta-\kappa}([0,1]\times\mathbb{T}^3)} \leqslant T^3 A \Big\}.
\end{aligned} \tag{4.2.53}$$

By combining the assumption (A2), (A3), (A4), and the multi-linear master estimate, a similar argument as in the proof of Proposition 4.2.10 yields

$$\begin{aligned}
& \tau^{b_+-b}\big(\big\| (-\partial_t^2 - 1 + \Delta) X_N \big\|_{\mathfrak{X}^{s_1-1,b_+-1}([t_0,t'])} + \big\| (-\partial_t^2 - 1 + \Delta) Y_N \big\|_{\mathfrak{X}^{s_2-1,b_+-1}([t_0,t'])} \big) \\
& \lesssim T^{30} \tau^{b_+-b} ((A+B)^2 + f_N(t')^2)(\theta + f_N(t')).
\end{aligned}$$

All together, we have proven for all $t, t' \in [t_0, T_*)$ satisfying $t \leqslant t' \leqslant t + \tau$ the estimate

$$f(t') \leqslant g(t') \leqslant g(t) + CT^{30} \tau^{b_+-b} ((A+B)^2 + f_N(t')^2)(\theta + f_N(t')).$$

Using $g(t_0) = 0$, using a continuity argument (Lemma 4.4.13), iterating the resulting bounds, and assuming the conditions

$$C(A+B)^2 e^{\frac{T}{\tau}} \theta \leqslant 1/2 \qquad \text{and} \qquad 2CT^{30}\tau^{b_+-b}((A+B)^2+6) \leqslant 1/4, \tag{4.2.54}$$

we obtain that

$$\sup_{t\in[t_0,T_*)} f(t) \leqslant \sup_{t\in[t_0,T_*)} g(t) \leqslant C(A+B)^2 e^{\frac{T}{\tau}}\theta. \tag{4.2.55}$$

Using the case of equality in the second condition in (4.2.54) as a definition for τ, the first condition follows from our assumption (A5). Recalling the definition of f, we obtain that

$$\sup_{t\in[t_0,T_*)} \Big(\|X_N\|_{\mathfrak{X}^{s_1,b}([t_0,t])} + \|Y_N\|_{\mathfrak{X}^{s_2,b}([t_0,t])} \Big) \leqslant C \exp\Big(C(A+B)^{\frac{2}{b_+-b}} T^{\frac{40}{b_+-b}} \Big)$$

This estimate rules out finite-time blowup on $\mathcal{J}$ and implies that $T_* = \sup \mathcal{J}$. Together with a soft argument, which is based on the integral equation for X_N and Y_N as well as the time-localization lemma (Lemma 4.4.3), we obtain that

$$\|X_N\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} + \|Y_N\|_{\mathfrak{X}^{s_2,b}(\mathcal{J})} \leqslant C \exp\Big(C(A+B)^{\frac{2}{b_+-b}} T^{\frac{40}{b_+-b}} \Big). \tag{4.2.56}$$

With this uniform estimates in hand, we now easily obtain the desired conclusions (C1), (C2), and (C3). In order to obtain (C1), we (are forced to) choose

$$w_N = \widetilde{w}_N + X_N + Y_N + Z_N.$$

The conclusions (C2) and (C3) follow from (A4), (4.2.56), and the condition $\bullet \in \Theta^{\text{type}}_{\text{blue}}(A,T)$ in our event (4.2.53). $\square$

4.3 Global theory

In this section, we prove the global well-posedness of the renormalized nonlinear wave equation and the invariance of the Gibbs measure. As mentioned in the introduction, the heart of this section is a new form of Bourgain's globalization argument. In Section 4.3.1, we prove the global well-posedness for Gibbsian initial data. We focus on the overall strategy and postpone several individual steps to Section 4.3.3 below. In Section 4.3.2, we prove the invariance of the Gibbs measure. Using the global well-posedness from Section 4.3.1, the proof of invariance is similar as in Bourgain's seminal paper [Bou94].

4.3.1 Global well-posedness

We now prove the (quantitative) global well-posedness of the renormalized nonlinear wave equation for Gibbsian initial data. In particular, we show that the structure

$$\Phi_N[t]\,\blacklozenge = \uparrow + [\text{tree}]_N + w_N$$

from the local theory (see Proposition 4.3.3) is preserved by the global theory. Here, the stochastic objects are defined exactly as in (4.2.2) and (4.2.5), but with $\bullet$ replaced by $\blacklozenge$.

Proposition 4.3.1 (Global well-posedness)**.** Let $A \geqslant 1$, let $T \geqslant 1$, let $C = C(\epsilon, s_1, s_2, \kappa, \eta, \eta', b_+, b)$ be sufficiently large, and let $\zeta = \zeta(\epsilon, s_1, s_2, \kappa, \eta, \eta', b_+, b) > 0$ be sufficiently small. We assume that $B, D \geqslant 1$ satisfy

$$B \geqslant B(A,T) \overset{\text{def}}{=} C\exp(C(A+T)^C) \quad \text{and} \quad D \geqslant D(A,T) \overset{\text{def}}{=} C\exp(\exp(C(A+T)^C)). \tag{4.3.1}$$

Furthermore, let $K \geqslant 1$ satisfy the condition

$$C\exp(C(A+B+T)^C)K^{-\eta'} \leqslant 1. \tag{4.3.2}$$

Then, the Borel set

$$\mathcal{E}_K(B,D,T) = \bigcap_{N\geqslant K} \Bigg(\Big\{ \blacklozenge \in \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3) \Big|\ w_N(t) = \Phi_N(t)\,\blacklozenge - \overset{\blacklozenge}{\uparrow} - \underset{\uparrow}{\overset{\blacklozenge\,*\,\blacklozenge\,\blacklozenge}{\curlyvee}}{}_{\mathrm{N}} \text{ satisfies}$$

$$\|w_N\|_{\mathfrak{X}^{s_1,b}([0,T])} \leqslant B \text{ and } \sum_{L_1\sim L_2} \|P_{L_1}\overset{\blacklozenge}{\uparrow}\cdot P_{L_2} w_N\|_{L_t^2\mathscr{H}_x^{-4\delta_1}([0,T]\times\mathbb{T}^3)} \leqslant B \Big\}$$

$$\cap \Big\{ \blacklozenge \in \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3) \Big| \|\Phi_N[t]\,\blacklozenge - \Phi_K[t]\,\blacklozenge\|_{C_t^0\mathscr{H}_x^{\beta-\kappa}([0,T]\times\mathbb{T}^3)} \leqslant DK^{-\eta'} \Big\} \Bigg)$$

satisfies the estimate

$$\inf_{M\geqslant K} \mu_M^{\otimes}(\mathcal{E}_K(B,D,T)) \geqslant 1 - T\zeta^{-1}\exp(-\zeta A^{\zeta}). \tag{4.3.3}$$

In the proof below, we need two modifications of the cubic stochastic object. We define

$$\underset{\uparrow\ \tau}{\overset{\blacklozenge\,*\,\blacklozenge\,\blacklozenge}{\curlyvee}}{}_{\mathrm{N}} \overset{\text{def}}{=} \mathrm{I}\Big[1_{[0,\tau]}(t) \overset{\blacklozenge\,*\,\blacklozenge\,\blacklozenge}{\curlyvee}{}_{\mathrm{N}} \Big] \qquad \text{and} \qquad \underset{\uparrow\ \tau}{\overset{\blacklozenge\,*\,\blacklozenge\,\blacklozenge}{\curlyvee}}{}_{\mathrm{N}\backslash\mathrm{M}} \overset{\text{def}}{=} \mathrm{I}\Big[1_{[0,\tau]}(t) \Big(\overset{\blacklozenge\,*\,\blacklozenge\,\blacklozenge}{\curlyvee}{}_{\mathrm{N}} - \overset{\blacklozenge\,*\,\blacklozenge\,\blacklozenge}{\curlyvee}{}_{\mathrm{M}} \Big) \Big]. \tag{4.3.4}$$

Proof of Proposition 4.3.1: We encourage the reader to review the informal discussion of the argument in the introduction before diving into the details of this proof.

Let $\tau \in (0,1)$ be such that $1/2 \leqslant A\tau^{b_+-b} \leqslant 1$ and $J \overset{\text{def}}{=} T/\tau \in \mathbb{N}$. We let B_j, D_j, where $1 \leqslant j \leqslant J$, be increasing sequences which remain to be chosen. We will prove below that our choice satisfies $B_j \leqslant B$ and $D_j \leqslant D$ for all $1 \leqslant j \leqslant J$. We then have that

$$\mathcal{E}_K(B_j,D_j,j\tau) = \bigcap_{N\geqslant K} \Bigg(\Big\{ \blacklozenge \in \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3) \Big|\ w_N(t) = \Phi_N(t)\,\blacklozenge - \overset{\blacklozenge}{\uparrow} - \underset{\uparrow}{\overset{\blacklozenge\,*\,\blacklozenge\,\blacklozenge}{\curlyvee}}{}_{\mathrm{N}} \text{ satisfies}$$

$$\|w_N\|_{\mathfrak{X}^{s_1,b}([0,j\tau])} \leqslant B_j \text{ and } \sum_{L_1\sim L_2} \|P_{L_1}\overset{\blacklozenge}{\uparrow}\cdot P_{L_2} w_N\|_{L_t^2\mathscr{H}_x^{-4\delta_1}([0,j\tau]\times\mathbb{T}^3)} \leqslant B_j \Big\}$$

$$\cap \Big\{ \blacklozenge \in \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3) \Big| \|\Phi_N[t]\,\blacklozenge - \Phi_K[t]\,\blacklozenge\|_{C_t^0\mathscr{H}_x^{\beta-\kappa}([0,j\tau]\times\mathbb{T}^3)} \leqslant D_jK^{-\eta'} \Big\} \Bigg).$$

We now claim for all $M \geqslant K$ that, under certain constraints on the sequences B_j and D_j detailed below,

$$\mu_M^{\otimes}\Big(\mathcal{E}_K(B_1,D_1,\tau)\Big) \geqslant 1 - \zeta^{-1}\exp(\zeta A^{\zeta}) \tag{4.3.5}$$

and

$$\mu_M^{\otimes}\Big(\mathcal{E}_K(B_j, D_j, j\tau)\Big) \geqslant \mu_M^{\otimes}\Big(\mathcal{E}_K(B_{j-1}, D_{j-1}, (j-1)\tau)\Big) - \zeta^{-1}\exp(\zeta A^{\zeta}). \tag{4.3.6}$$

We refer to (4.3.5) as the base case and to (4.3.6) as the induction step. We split the rest of the argument into several steps.

Step 1: The base case (4.3.5). We set $B_1 \overset{\text{def}}{=} A$ and $D_1 \overset{\text{def}}{=} A$. If $\mathcal{L}(A,\tau)$ is as in Proposition 4.3.3, we obtain that $\mathcal{L}(A,\tau) \subseteq \mathcal{E}_K(B_1, D_1, \tau)$. This implies

$$\mu_M^{\otimes}\Big(\mathcal{E}_K(B_1, D_1, \tau)\Big) \geqslant \mu_M^{\otimes}\Big(\mathcal{L}(A,\tau)\Big) \geqslant 1 - \zeta^{-1}\exp(\zeta A^{\zeta}).$$

Step 2: The induction step (4.3.6). We first restrict to the event

$$\mathcal{S}^{\text{gwp}}(A,T,\tau) \overset{\text{def}}{=} \mathcal{L}(A,\tau) \bigcap \mathcal{L}(A,2\tau) \bigcap \mathcal{S}^{\text{time}}(A,T,\tau) \bigcap \mathcal{S}^{\text{cub}}(A,T,\tau) \bigcap \mathcal{S}^{\text{stab}}(A,T,\tau). \tag{4.3.7}$$

Using Proposition 4.3.3, Proposition 4.3.5, Proposition 4.3.7, and Proposition 4.3.8, which also contain the definitions of the sets in (4.3.7), we obtain that

$$\mu_M^{\otimes}(\mathcal{S}^{\text{gwp}}(A,T,\tau)) \geqslant 1 - \zeta^{-1}\exp(\zeta A^{\zeta}).$$

Using the invariance of $\mu_M^{\otimes}$ under Φ_M, we also obtain that

$$\mu_M^{\otimes}\Big(\Phi_M[\tau]^{-1}\mathcal{E}_K\big(B_{j-1}, D_{j-1}, (j-1)\tau\big)\Big) = \mu_M^{\otimes}\Big(\mathcal{E}_K\big(B_{j-1}, D_{j-1}, (j-1)\tau\big)\Big).$$

In order to obtain the probabilistic estimate (4.3.6), it therefore suffices to prove the inclusion

$$\mathcal{S}^{\text{gwp}}(A,T,\tau) \bigcap \Phi_M[\tau]^{-1}\mathcal{E}_K\big(B_{j-1}, D_{j-1}, (j-1)\tau\big) \subseteq \mathcal{E}_K(B_j, D_j, j\tau). \tag{4.3.8}$$

For the rest of this proof, we assume that $\blacklozenge \in \mathcal{S}^{\text{gwp}}(A,T,\tau) \bigcap \Phi_M[\tau]^{-1}\mathcal{E}_K\big(B_{j-1}, D_{j-1}, (j-1)\tau\big)$ and $N, M \geqslant K$. To clarify the structure of the proof, we divide our argument into further substeps.

Step 2.1: Time-translation. We rephrase the condition $\diamond = \Phi_M[\tau]\blacklozenge \in \mathcal{E}_K(B_{j-1}, D_{j-1}, (j-1)\tau)$ in terms of $\blacklozenge$.

Since $\diamond \in \mathcal{E}_K(B_{j-1}, D_{j-1}, (j-1)\tau)$, we obtain for all $t \in [\tau, j\tau]$ that

$$\Phi_N(t-\tau)\Phi_M[\tau]\blacklozenge = \Phi_N(t-\tau)\diamond = \overset{\diamond}{|}(t-\tau) + \text{Y}^{\diamond * \diamond \diamond}_{N}(t-\tau) + w^{\text{grn}}_{N,M}(t-\tau),$$

where $w^{\text{grn}}_{N,M} \colon [0,(j-1)\tau] \times \mathbb{T}^3 \to \mathbb{R}$ satisfies

$$\|w^{\text{grn}}_{N,M}\|_{\mathfrak{X}^{s_1,b}([0,(j-1)\tau])} \leqslant B_{j-1} \quad \text{and} \quad \sum_{L_1 \sim L_2} \|P_{L_1}\overset{\diamond}{|} \cdot P_{L_2} w^{\text{grn}}_{N,M}\|_{L^2_t H^{-4\delta_1}_x([0,(j-1)\tau]\times\mathbb{T}^3)} \leqslant B_{j-1}.$$

The superscript "grn" emphasizes that $w^{\text{grn}}_{N,M}$ appears in the structure involving $\diamond$. Furthermore, we also have that

$$\|\Phi_N[t-\tau]\Phi_M[\tau]\blacklozenge - \Phi_K[t-\tau]\Phi_M[\tau]\blacklozenge\|_{C^0_t\mathscr{H}^{\beta-\kappa}_x([\tau,j\tau]\times\mathbb{T}^3)} \leqslant D_{j-1}K^{-\eta'}. \tag{4.3.9}$$

Since $\blacklozenge \in \mathcal{S}^{\text{time}}(A,T,\tau)$ (as in Proposition 4.3.5), it follows for all $t \in [\tau, j\tau]$ that

$$\Phi_N(t-\tau)\Phi_M[\tau]\blacklozenge = \overset{\blacklozenge}{|}(t) + \text{Y}^{\blacklozenge * \blacklozenge \blacklozenge}_{N}(t) - \text{Y}^{\blacklozenge * \blacklozenge \blacklozenge}_{N\backslash M,\,\tau}(t) + w_{N,M}(t), \tag{4.3.10}$$

where $w_{N,M} \colon [\tau, j\tau] \times \mathbb{T}^3 \to \mathbb{R}$ satisfies

$$\|w_{N,M}\|_{\mathfrak{X}^{s_1,b}([\tau,j\tau])}, \quad \sum_{L_1 \sim L_2} \|P_{L_1}\overset{\blacklozenge}{|} \cdot P_{L_2} w_{N,M}\|_{L^2_t H^{-4\delta_1}_x([\tau,j\tau]\times\mathbb{T}^3)} \leqslant T^\alpha A B_{j-1}. \tag{4.3.11}$$

Our next goal is to replace $\Phi_M[\tau]$ in (4.3.10) by $\Phi_N[\tau]$, which is done in Step 2.2 and Step 2.3.

Step 2.2: The cubic stochastic object. In this step, we correct the structure of $\Phi_N(t-\tau)\Phi_M[\tau]\blacklozenge$, as stated in (4.3.10), by adding the "partial" cubic stochastic object.

We define $\widetilde{u}_N \colon [\tau, j\tau] \times \mathbb{T}^3 \to \mathbb{R}$ by

$$\widetilde{u}_N(t) = \Phi_N(t-\tau)\Phi_M[\tau]\blacklozenge + \text{Y}^{\blacklozenge * \blacklozenge \blacklozenge}_{N\backslash M,\,\tau}(t) = \overset{\blacklozenge}{|}(t) + \text{Y}^{\blacklozenge * \blacklozenge \blacklozenge}_{N}(t) + w_{N,M}(t). \tag{4.3.12}$$

While $\widetilde{u}_N$ depends on M, this is not reflected in our notation. The reason is that, as will be shown below, $\widetilde{u}_N$ is a close approximation of $u_N(t) = \Phi_N(t)\,\blacklozenge$, which does not directly depend on M. In order to match the notation of $\widetilde{u}_N$, we also define $\widetilde{w}_N = w_{N,M}$, which leads to

$$\widetilde{u}_N(t) = \uparrow(t) + \Psi_{N}(t) + \widetilde{w}_N(t).$$

Using $\blacklozenge \in \mathcal{S}^{\mathrm{cub}}(A, T, \tau)$ (as in Proposition 4.3.7), it follows that there exist $H_N \in \mathscr{LM}([\tau, j\tau])$ and $F_N \in \mathfrak{X}^{s_2-1, b_+-1}([\tau, j\tau])$ satisfying the identity

$$\begin{aligned}&(-\partial_t^2 - 1 + \Delta)\widetilde{u}_N - P_{\leqslant N} :\!\big(V * (P_{\leqslant N}\widetilde{u}_N)^2\big)P_{\leqslant N}\widetilde{u}_N\!:\\ &= -P_{\leqslant N}\,\mathrm{PCtrl}(H_N, P_{\leqslant N}\uparrow) - F_N\end{aligned} \tag{4.3.13}$$

and the estimate

$$\|H_N\|_{\mathscr{LM}([\tau,j\tau])}, \|F_N\|_{\mathfrak{X}^{s_2-1,b_+-1}([\tau,j\tau])} \leqslant T^{4\alpha}A^4B_{j-1}^3K^{-\eta'}. \tag{4.3.14}$$

Thus, $\widetilde{u}_N$ is an approximate solution to the nonlinear wave equation on $[\tau, j\tau] \times \mathbb{T}^3$. Furthermore, it holds that

$$\|\widetilde{u}_N[t] - \Phi_N[t-\tau]\Phi_M[\tau]\,\blacklozenge\|_{C_t^0\mathscr{H}_x^{\beta-\kappa}([\tau,j\tau]\times\mathbb{T}^3)} \leqslant T^{4\alpha}A^4B_{j-1}^3K^{-\eta'}. \tag{4.3.15}$$

Step 2.3: Stability estimate. In this step, we turn the approximate solution $\widetilde{u}_N$ into an honest solution and fully correct the initial data at $t = \tau$.

We now verify the assumptions (A1)-(A5) in Proposition 4.3.8, where we replace B by $T^\alpha A B_{j-1}$ and set $\theta = T^{4\alpha}A^4B_{j-1}^3K^{-\eta'}$. The first assumption (A1) holds with $\widetilde{w}_N = w_{N,M}$ due to (4.3.12). The second assumption (A2) coincides with the bounds (4.3.11). The third assumption (A3) coincides with (4.3.13) and (4.3.14).

For the fourth assumption (A4), we rely on $\blacklozenge \in \mathcal{L}(A, \tau)$ (as in Proposition 4.3.3). First, we have that

$$\widetilde{u}_N[\tau] = \Phi_M[\tau]\,\blacklozenge + \Psi_{N\backslash M,\tau}[\tau] = \uparrow[\tau] + \Psi_{M}[\tau] + \Psi_{N\backslash M,\tau}[\tau] + w_M[\tau] = \uparrow[\tau] + \Psi_{N}[\tau] + w_M[\tau].$$

Second, we have that

$$\Phi_N[\tau]\,\blacklozenge = \overset{\blacklozenge}{\uparrow}[\tau] + \overset{\blacklozenge\ast\blacklozenge\blacklozenge}{\curlyvee}_{\scriptscriptstyle N}[\tau] + w_N[\tau].$$

Using (IV) in Proposition 4.3.3, this implies that $Z_N[\tau] \overset{\text{def}}{=} \Phi_N[\tau] - \widetilde{u}_N[\tau]$ satisfies

$$\|Z_N[\tau]\|_{\mathfrak{X}([0,T],\overset{\blacklozenge}{\uparrow};\tau,N,M)} \leqslant AT^{\alpha}K^{-\eta'}, \tag{4.3.16}$$

which yields (A4). Finally, as long as $B_j \leqslant B$, the fifth assumption (A5) follows from the parameter condition (4.3.2). Thus, the assumptions (A1)-(A5) in Proposition 4.3.8 hold. Since $\blacklozenge \in \mathcal{S}^{\text{stab}}(A,T,\tau)$, we obtain for all $t \in [\tau, j\tau]$ that

$$\Phi_N(t)\,\blacklozenge = \overset{\blacklozenge}{\uparrow}(t) + \overset{\blacklozenge\ast\blacklozenge\blacklozenge}{\curlyvee}_{\scriptscriptstyle N}(t) + w_N(t), \tag{4.3.17}$$

where the nonlinear component w_N satisfies

$$\|w_N\|_{\mathfrak{X}^{s_1,b}([\tau,j\tau]\times\mathbb{T}^3)}, \quad \sum_{L_1\sim L_2} \|P_{L_1}\overset{\blacklozenge}{\uparrow}\cdot P_{L_2}w_{N,M}\|_{L_t^2H_x^{-4\delta_1}([\tau,j\tau]\times\mathbb{T}^3)} \leqslant T^{\alpha}AB_{j-1}+1 \leqslant 2T^{\alpha}AB_{j-1}. \tag{4.3.18}$$

Furthermore,

$$\|\Phi_N[t]\,\blacklozenge - \widetilde{u}_N[t]\|_{C_t^0\mathscr{H}_x^{\beta-\kappa}([\tau,j\tau]\times\mathbb{T}^3)} \leqslant C\exp(C(A+B_{j-1}+T)^C)K^{-\eta'}. \tag{4.3.19}$$

By combining (4.3.9), (4.3.15), and (4.3.19), we obtain

$$\begin{aligned} &\|\Phi_N[t]\,\blacklozenge - \Phi_K[t-\tau]\Phi_M[\tau]\,\blacklozenge\|_{C_t^0\mathscr{H}_x^{\beta-\kappa}([\tau,j\tau]\times\mathbb{T}^3)} \\ &\leqslant \big(D_{j-1} + T^{4\alpha}A^4B_{j-1}^3 + C\exp(C(A+B_{j-1}+T)^C)\big)K^{-\eta'}. \end{aligned} \tag{4.3.20}$$

By combining the general case $N \geqslant K$ in (4.3.20) with the special case $N = K$, using the triangle inequality, and increasing C if necessary, we also obtain that

$$\begin{aligned} &\|\Phi_N[t]\,\blacklozenge - \Phi_K[t]\,\blacklozenge\|_{C_t^0\mathscr{H}_x^{\beta-\kappa}([\tau,j\tau]\times\mathbb{T}^3)} \\ &\leqslant \big(2D_{j-1} + C\exp(C(A+B_{j-1}+T)^C)\big)K^{-\eta'}. \end{aligned} \tag{4.3.21}$$

Step 2.4: Gluing. In this step, we "glue" together our information on $[0, 2\tau]$ (from local well-posedness) and $[\tau, j\tau]$ (from the previous step).

Since $\blacklozenge \in \mathcal{L}(A, 2\tau)$ (as in Proposition 4.3.3), the function w_N uniquely determined by

$$\Phi_N(t)\blacklozenge = \uparrow(t) + \overset{\blacklozenge\ast\blacklozenge\blacklozenge}{\curlyvee}_N(t) + w_N(t)$$

satisfies

$$\|w_N\|_{\mathfrak{X}^{s_1,b}([0,2\tau]\times\mathbb{T}^3)}, \quad \sum_{L_1\sim L_2} \|P_{L_1}\uparrow \cdot P_{L_2} w_N\|_{L_t^2 H_x^{-4\delta_1}([0,2\tau]\times\mathbb{T}^3)} \leqslant A.$$

Furthermore,

$$\|\Phi_N[t]\blacklozenge - \Phi_K[t]\blacklozenge\|_{C_t^0\mathscr{H}_x^{\beta-\kappa}([0,2\tau]\times\mathbb{T}^3)} \leqslant AK^{-\eta'}.$$

Together with (4.3.18), (4.3.21), and the gluing lemma (Lemma 4.4.5), which is only needed for the frequency-based $\mathfrak{X}^{s_1,b}$-space, we obtain that

$$\|w_N\|_{\mathfrak{X}^{s_1,b}([0,j\tau]\times\mathbb{T}^3)}, \quad \sum_{L_1\sim L_2} \|P_{L_1}\uparrow \cdot P_{L_2} w_N\|_{L_t^2 H_x^{-4\delta_1}([0,j\tau]\times\mathbb{T}^3)} \leqslant C\tau^{\frac{1}{2}-b} T^\alpha A B_{j-1}. \tag{4.3.22}$$

and

$$\|\Phi_N[t]\blacklozenge - \Phi_K[t]\blacklozenge\|_{C_t^0\mathscr{H}_x^{\beta-\kappa}([0,j\tau]\times\mathbb{T}^3)} \leqslant \big(2D_{j-1} + C\exp(C(A+B_{j-1}+T)^C)\big)K^{-\eta'}. \tag{4.3.23}$$

Step 2.5: Choosing B_j and D_j. Based on (4.3.22) and (4.3.23), we now define

$$B_j \overset{\text{def}}{=} C\tau^{\frac{1}{2}-b}T^\alpha A B_{j-1} \quad \text{and} \quad D_j \overset{\text{def}}{=} 2D_{j-1} + C\exp(C(A+B_{j-1}+T)^C).$$

Step 3: Finishing up. We recall that $1/2 \leqslant A\tau^{b_+-b} \leqslant 1$, $J = T/\tau \sim TA^{\frac{1}{b_+-b}}$, $B_1 = A$, and $D_1 = A$. After increasing C if necessary, we obtain that

$$B_J \leqslant C\exp(C(A+T)^C) \leqslant B \quad \text{and} \quad D_J \leqslant C\exp(C(A+B_J+T)^C) \leqslant D. \tag{4.3.24}$$

This implies $\mathcal{E}(B_J, D_J, J\tau) \subseteq \mathcal{E}_K(B, D, T)$. By iterating (4.3.6) and using the base case (4.3.5), we obtain (after decreasing ζ) that

$$\mu_M^{\otimes}(\mathcal{E}_K(B,D,T)) \geqslant \mu_M^{\otimes}(\mathcal{E}_K(B_J, D_J, J\tau)) \geqslant 1 - T\zeta^{-1}\exp(-\zeta A^\zeta).$$

This completes the proof. $\square$

In Proposition 4.3.1, we obtained a quantitative global well-posedness result. In particular, we obtained (almost) explicit bounds on the growth of w_N, which are of independent interest. In the proof of Theorem 4.1.3, however, a softer statement is sufficient, which we isolate in Corollary 4.3.2 below.

Corollary 4.3.2. Let $T \geqslant 1$, let $\theta > 0$, and $K \geqslant 1$. Then, we define a closed subset of $\mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$ by

$$\mathcal{S}_K(T,\theta) \overset{\text{def}}{=} \Big\{ \bullet \in \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3) \colon \sup_{N_1,N_2 \geqslant K} \| \Phi_{N_1}[t] \bullet - \Phi_{N_2}[t] \bullet \|_{C_t^0 \mathscr{H}_x^{\beta-\kappa}([-T,T]\times\mathbb{T}^3)} \leqslant \theta \Big\} \tag{4.3.25}$$

Furthermore, we define the event

$$\mathcal{S} \overset{\text{def}}{=} \bigcap_{T\in\mathbb{N}} \bigcap_{\theta\in\mathbb{Q}_{>0}} \bigcup_{K\geqslant 1} \mathcal{S}_K(T,\theta). \tag{4.3.26}$$

Then, it holds that

$$\lim_{K,M\to\infty} \mu_M^{\otimes}(\mathcal{S}_K(T,\theta)) = 1 \qquad \text{and} \qquad \mu_\infty^{\otimes}(\mathcal{S}) = 1. \tag{4.3.27}$$

Proof. We first prove the identity $\lim_{K,M\to\infty} \mu_M^{\otimes}(\mathcal{S}_K(T,\theta)) = 1$. Using the time-reflection symmetry, it suffices to prove the statement with $\mathcal{S}_K(T,\theta)$ replaced by

$$\mathcal{S}_K^{+}(T,\theta) \overset{\text{def}}{=} \Big\{ \bullet \in \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3) \colon \sup_{N_1,N_2 \geqslant K} \| \Phi_{N_1}[t] \bullet - \Phi_{N_2}[t] \bullet \|_{C_t^0 \mathscr{H}_x^{\beta-\kappa}([0,T]\times\mathbb{T}^3)} \leqslant \theta \Big\}.$$

For any fixed $T, A, B, D \geqslant 1$ satisfying (4.3.1) and $\theta > 0$, we have for all sufficiently large $K, L \geqslant 1$ satisfying $K \geqslant L$ that

$$\mathcal{S}_K^{+}(T,\theta) \supseteq \mathcal{E}_L(B,D,T),$$

where $\mathcal{E}_L(B,D,T)$ is as in Proposition 4.3.1. Thus,

$$\lim_{K,M\to\infty} \mu_M^{\otimes}(\mathcal{S}_K(T,\theta)) \geqslant \liminf_{M\to\infty} \mu_M^{\otimes}(\mathcal{E}_L(B,D,T)) \geqslant 1 - \zeta^{-1} T \exp(\zeta A^{\zeta}).$$

After letting $A \to \infty$, this yields the first identity in (4.3.27).

Using Theorem 4.1.1, we have that $\mu_M^{\otimes}$ converges weakly to $\mu_\infty^{\otimes}$. Since $\mathcal{S}_K(T,\theta)$ is closed, this implies

$$1 = \lim_{K,M\to\infty} \mu_M^{\otimes}(\mathcal{S}_K(T,\theta)) \leqslant \liminf_{K\to\infty} \mu_\infty^{\otimes}(\mathcal{S}_K(T,\theta)) \leqslant \mu_\infty^{\otimes}\Big(\bigcup_{K\geqslant 1} \mathcal{S}_K(T,\theta)\Big).$$

This yields the second identity in (4.3.27). □

4.3.2 Invariance

In this subsection, we complete the proof of Theorem 4.1.3. The global well-posedness follows from Corollary 4.3.2 and it remains to prove the invariance. Our argument closely resembles the proof of invariance for the one-dimensional nonlinear Schrödinger equation by Bourgain [Bou94]. The only difference is that we work with the expectation of test functions instead of probabilities of sets, since they are more convenient for weakly convergent measures.

Proof of Theorem 4.1.3: The global well-posedness follows directly from Corollary 4.3.2. Thus, it remains to prove the invariance of the Gibbs measure $\mu_\infty^{\otimes}$.

Let $t \in \mathbb{R}$ be arbitrary. In order to prove that $\Phi_\infty[t]_\# \mu_\infty^{\otimes} = \mu_\infty^{\otimes}$, it suffices to prove for all bounded Lipschitz functions $f\colon \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3) \to \mathbb{R}$ that

$$\mathbb{E}_{\mu_\infty^{\otimes}}\big[f(\Phi_\infty[t]\,\blacklozenge)\big] = \mathbb{E}_{\mu_\infty^{\otimes}}\big[f(\blacklozenge)\big]. \tag{4.3.28}$$

We first rewrite the left-hand side of (4.3.28). Using the global well-posedness and dominated convergence, we have that

$$\mathbb{E}_{\mu_\infty^{\otimes}}\big[f(\Phi_\infty[t]\,\blacklozenge)\big] = \lim_{N\to\infty} \mathbb{E}_{\mu_\infty^{\otimes}}\big[f(\Phi_N[t]\,\blacklozenge)\big].$$

Using the weak convergence of $\mu_M^{\otimes}$ to $\mu_\infty^{\otimes}$ (from Theorem 4.1.1) and the continuity of $\Phi_N[t]$ (for a fixed N), we have that

$$\lim_{N\to\infty} \mathbb{E}_{\mu_\infty^\otimes}\big[f(\Phi_N[t]\,\blacklozenge)\big] = \lim_{N\to\infty} \Big(\lim_{M\to\infty} \mathbb{E}_{\mu_M^\otimes}\big[f(\Phi_N[t]\,\blacklozenge)\big]\Big).$$

We now turn to the right-hand side of (4.3.28). Using the weak convergence of $\mu_M^\otimes$ to $\mu_\infty^\otimes$ and the invariance of $\mu_M^\otimes$ under $\Phi_M[t]$, we obtain that

$$\mathbb{E}_{\mu_\infty^\otimes}\big[f(\,\blacklozenge)\big] = \lim_{M\to\infty} \mathbb{E}_{\mu_M^\otimes}\big[f(\,\blacklozenge)\big] = \lim_{M\to\infty} \mathbb{E}_{\mu_M^\otimes}\big[f(\Phi_M[t]\,\blacklozenge)\big].$$

Combining the last three identities, we can reduce (4.3.28) to

$$\limsup_{N,M\to\infty} \Big| \mathbb{E}_{\mu_M^\otimes}\big[f(\Phi_N[t]\,\blacklozenge)\big] - \mathbb{E}_{\mu_M^\otimes}\big[f(\Phi_M[t]\,\blacklozenge)\big]\Big| = 0. \tag{4.3.29}$$

We now let $T \geqslant 1$ be such that $t \in [-T,T]$, let $\theta > 0$, and let $K \geqslant 1$. We also let $\mathcal{S}_K(T,\theta)$ be as in Corollary 4.3.2. Then, we have that

$$\begin{aligned}
&\limsup_{N,M\to\infty} \Big| \mathbb{E}_{\mu_M^\otimes}\big[f(\Phi_N[t]\,\blacklozenge)\big] - \mathbb{E}_{\mu_M^\otimes}\big[f(\Phi_M[t]\,\blacklozenge)\big]\Big| \\
\leqslant& \sup_{N,M\geqslant K} \Big| \mathbb{E}_{\mu_M^\otimes}\big[f(\Phi_N[t]\,\blacklozenge)\big] - \mathbb{E}_{\mu_M^\otimes}\big[f(\Phi_M[t]\,\blacklozenge)\big]\Big| \\
\leqslant& \sup_{N,M\geqslant K} \mathbb{E}_{\mu_M^\otimes}\Big[1\big\{\blacklozenge \in \mathcal{S}_K(T,\theta)\big\}\Big|f(\Phi_N[t]\,\blacklozenge) - f(\Phi_M[t]\,\blacklozenge)\Big|\Big] \\
&+ \sup_{N,M\geqslant K} \mathbb{E}_{\mu_M^\otimes}\Big[1\big\{\blacklozenge \notin \mathcal{S}_K(T,\theta)\big\}\Big|f(\Phi_N[t]\,\blacklozenge) - f(\Phi_M[t]\,\blacklozenge)\Big|\Big] \\
\leqslant& \operatorname{Lip}(f)\cdot\theta + 2\|f\|_\infty \sup_{M\geqslant K} \mu_M^\otimes(\mathscr{H}_x^{-1/2-\kappa}\backslash \mathcal{S}_K(T,\theta)).
\end{aligned}$$

In the last line, $\operatorname{Lip}(f)$ is the Lipschitz-constant of f and $\|f\|_\infty$ is the supremum of f. Using Corollary 4.3.2, we obtain the estimate (4.3.29) by first letting $K\to\infty$ and then letting $\theta\to 0$. □

4.3.3 Structure and stability theory

In this subsection, we provide the ingredients used in the proof of global well-posedness (Proposition 4.3.1). As described in the introduction, we will further split this subsection into four parts.

4.3.3.1 Structured local well-posedness

In Proposition 4.2.10, we obtained a structured local well-posedness result in terms of $\bullet$ and $\mathbb{P}$. In Corollary 4.2.12, we already used Proposition 4.2.10 to prove the local existence of the limiting dynamics on the support of the Gibbs measure $\mu_\infty^\otimes$, but did not obtain any structural information on the solution. We now remedy this defect and obtain a structured local well-posedness result even on the support of the Gibbs measure.

The statement of the proposition differs slightly from the earlier Proposition 4.2.10 for two reasons: First, we formulate the result closer to the assumptions in the stability theory (Proposition 4.2.14 and Proposition 4.3.8), which is useful in the globalization argument. Second, using the organization of this paper, it would be cumbersome to define the para-controlled component of $\Phi_N(t)\,\diamond$ intrinsically through $\diamond$, i.e., without relying on the ambient objects.

Proposition 4.3.3 (Structured local well-posedness w.r.t. the Gibbs measure)**.** Let $A \geqslant 1$, let $0 < \tau < 1$, let $\alpha > 0$ be a sufficiently large absolute constant, and let $\zeta = \zeta(\epsilon, s_1, s_2, \kappa, \eta, \eta', b_+, b) > 0$ be sufficiently small. We denote by $\diamond$ a generic element of $\mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$ and by $\mathcal{L}(A,\tau)$ the Borel subset of $\mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$ defined by the following conditions:

(I) For any $N \geqslant 1$, the solution of (4.2.1) with initial data $\diamond$ exists on $[-\tau, \tau]$.

(II) For all $N \geqslant 1$, there exist (a unique) $w_N \in \mathfrak{X}^{s_1,b}([0,\tau])$ such that

$$\Phi_N(t)\,\diamond = \overset{\diamond}{|}(t) + \overset{\diamond\,\ast\,\diamond\,\diamond}{\underset{\uparrow}{\curlyvee}}{}_{N}\,(t) + w_N(t).$$

Furthermore, we have the bounds

$$\|w_N\|_{\mathfrak{X}^{s_1,b}([0,\tau])} \leqslant A \quad \text{and} \quad \sum_{L_1 \sim L_2} \|P_{L_1} \overset{\diamond}{|} \cdot P_{L_2} w_N\|_{L_t^2 H_x^{-4\delta_1}([0,\tau]\times\mathbb{T}^3)} \leqslant A.$$

(III) It holds for all $N, K \geqslant 1$ that

$$\| \Phi_N[t] \blacklozenge - \Phi_K[t] \blacklozenge \|_{C_t^0 \mathscr{H}_x^{\beta-\kappa}([0,\tau]\times\mathbb{T}^3)} \leqslant A \min(N,K)^{-\eta'}.$$

(IV) It holds for all $N, K \geqslant 1$ and $T \geqslant 1$ that

$$\| w_K[\tau] \|_{\mathfrak{X}([0,T],\uparrow;\tau,N,K)} \leqslant A T^{\alpha}$$

and

$$\| w_N[\tau] - w_K[\tau] \|_{\mathfrak{X}([0,T],\uparrow;\tau,N,K)} \leqslant A T^{\alpha} \min(N,K)^{-\eta'}.$$

If $A\tau^{b_+-b} \leqslant 1$, then $\mathcal{L}(A,\tau)$ has high probability under $\mu_M^{\otimes}$ for all $M \geqslant 1$ and it holds that

$$\mu_M^{\otimes}(\mathcal{L}(A,\tau)) \geqslant 1 - \zeta^{-1}\exp(-\zeta A^{\zeta}). \tag{4.3.30}$$

Remark 4.3.4. Since we prove multilinear estimates for $\blacklozenge$ instead of $\bullet$ in Section 4.9, a different incarnation of this paper may omit Proposition 4.2.10 and instead proof Proposition 4.3.3 directly. The author believes that our approach illustrates an interesting conceptual point: The singularity of the Gibbs measure does not enter heavily into the construction of the local limiting dynamics (see Corollary 4.2.12), but does affect the global theory. We believe, however, that this would be different for the cubic nonlinear wave equation. The reason is an additional renormalization in the construction of the Φ^4_3-model (see e.g. [BG20b, Lemma 5: Step 3]).

We recall that the $\mathfrak{X}$-norm appearing in (IV) is defined in Definition 4.2.13.

Proof of Proposition 4.3.3: By using Theorem 4.1.1 and adjusting the value of ζ, it suffices to prove the probabilistic estimate (4.3.30) with the Gibbs measure $\mu_M^{\otimes}$ replaced by the reference measure $\nu_M^{\otimes}$. Using the representation of the reference measure from Theorem 4.1.1, it holds that

$$\nu_M^{\otimes} = \mathrm{Law}_{\mathbb{P}}\left(\bullet + \circ_{\mathrm{M}} \right).$$

By applying this identity to the Borel set $\mathcal{L}(A,\tau)$, we obtain that

$$\nu_M^{\otimes}(\mathcal{L}(A,\tau)) = \mathbb{P}(\bullet + \circ_{\mathbf{M}} \in \mathcal{L}(A,\tau)).$$

Let $B = cA^c \leqslant A$, where $c = c(\epsilon, s_1, s_2, \kappa, \eta, \eta', b_+, b) > 0$ is sufficiently small. Let $\mathcal{L}_M^{\mathrm{amb}}(B,\tau) \subseteq \Omega$ be as in Proposition 4.2.10. We now show that

$$\mathbb{P}\Big(\big\{ \bullet + \circ_{\mathbf{M}} \notin \mathcal{L}(A,\tau)\big\} \cap \mathcal{L}_M^{\mathrm{amb}}(B,\tau)\Big) \leqslant \frac{1}{2}\zeta^{-1}\exp(-\zeta A^{\zeta}). \tag{4.3.31}$$

The property (i) in Proposition 4.2.10 directly implies its counterpart. The main part of the argument lies in proving (II). Instead of (II), we currently only have the property

(ii): For all $N \geqslant 1$, there exist $w_N' \in \mathfrak{X}^{s_1,b}([0,\tau])$, $H_N' \in \mathscr{LM}([0,\tau])$, and $Y_N' \in \mathfrak{X}^{s_2,b}([0,\tau])$, such that for all $t \in [0,\tau]$

$$\Phi_N(t)\,\blacklozenge = \overset{\bullet}{|}(t) + \overset{\bullet\circledast\bullet\,\bullet}{\curlyvee}_{N}(t) + w_N'(t) \quad\text{and}\quad w_N'(t) = P_{\leqslant N}\,\mathrm{I}\big[\,\mathrm{PCtrl}(H_N', P_{\leqslant N}\overset{\bullet}{|})\big](t) + Y_N'(t).$$

Furthermore, we have the bounds

$$\|w_N'\|_{\mathfrak{X}^{s_1,b}([0,\tau])}, \|H_N'\|_{\mathscr{LM}([0,\tau])}, \|Y_N'\|_{\mathfrak{X}^{s_2,b}([0,\tau])} \leqslant B \quad \text{and}$$
$$\sum_{L_1\sim L_2} \|P_{L_1}\overset{\bullet}{|}\cdot P_{L_2}w_N'\|_{L_t^2 H_x^{-4\delta_1}([0,\tau]\times\mathbb{T}^3)} \leqslant B.$$

Comparing (ii) and (II), this forces us to take

$$w_N = \overset{\bullet}{|} - \overset{\blacklozenge}{|} + \overset{\bullet\circledast\bullet\,\bullet}{\curlyvee}_{N} - \overset{\blacklozenge\circledast\blacklozenge\,\blacklozenge}{\curlyvee}_{N} + w_N'. \tag{4.3.32}$$

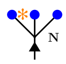
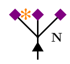
We now have to prove that the right-hand side of (4.3.32) satisfies the estimates in (II). Due to the decomposition $\blacklozenge = \bullet + \circ_{\mathbf{M}}$, we have that

$$\overset{\bullet}{|} - \overset{\blacklozenge}{|} = -\overset{\circ}{|}.$$

Using Theorem 4.1.1, we have outside a set of probability $\lesssim \exp(-cB^{\frac{2}{k}})$ under $\mathbb{P}$ that

$$\| $$  $$ \|_{\mathfrak{X}^{s_2,b}([0,\tau])} \lesssim B.$$

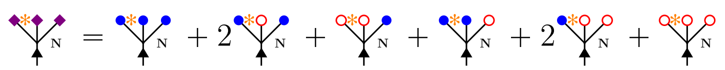
Using Proposition 4.9.1, we have outside an event with probability $\leqslant \zeta^{-1}\exp(-\zeta B^{\zeta})$ under $\mathbb{P}$ that

 $$=$$  $$+ I[\mathrm{PCtrl}(H_N^{(3)}, P_{\leqslant N}$$  $$)] + Y_N^{(3)}, \tag{4.3.33}$$

where $H_N^{(3)} \in \mathscr{LM}([0,\tau])$ and $Y_N^{(3)} \in \mathfrak{X}^{s_2,b}([0,\tau])$ satisfy

$$\|H_N^{(3)}\|_{\mathscr{LM}([0,\tau])} \leqslant B \qquad \text{and} \qquad \|Y_N^{(3)}\|_{\mathfrak{X}^{s_2,b}([0,\tau])} \leqslant B.$$

To ease the reader's mind, we mention that the proof of Proposition 4.9.1 is based on the algebraic identity



which uses mixed cubic stochastic objects. Finally, we have that

$$w'_N = P_{\leqslant N}\,\mathrm{I}\big[\,\mathrm{PCtrl}(H'_N, P_{\leqslant N}$$  $$)\big] + Y'_N$$
$$= P_{\leqslant N}\,\mathrm{I}\big[\,\mathrm{PCtrl}(H'_N, P_{\leqslant N}$$  $$)\big] - P_{\leqslant N}\,\mathrm{I}\big[\,\mathrm{PCtrl}(H'_N, P_{\leqslant N}$$  $$)\big] + Y'_N.$$

Using the inhomogeneous Strichartz estimate (Lemma 4.4.9) and Lemma 4.7.3, we have that

$$\|P_{\leqslant N}\,\mathrm{I}\big[\,\mathrm{PCtrl}(H'_N, P_{\leqslant N}$$  $$)\big]\|_{\mathfrak{X}^{s_2,b}([0,\tau])} \lesssim \|\,\mathrm{PCtrl}(H'_N, P_{\leqslant N}$$  $$)\|_{L_t^\infty H_x^{s_2-1}([0,\tau]\times\mathbb{T}^3)}$$
$$\lesssim \|H'_N\|_{\mathscr{LM}([0,\tau])}\|$$  $$\|_{L_t^\infty H_x^{s_2-1+8\epsilon}([0,\tau]\times\mathbb{T}^3)} \lesssim B^2.$$

Thus,

$$w_N = P_{\leqslant N}\,\mathrm{I}\big[\,\mathrm{PCtrl}(H_N, P_{\leqslant N}$$  $$)\big] + Y_N, \tag{4.3.34}$$

where

$$H_N = H'_N + H_N^{(3)} \qquad \text{and} \qquad Y_N = Y'_N -$$  $$+ Y_N^{(3)} - P_{\leqslant N}\,\mathrm{I}\big[\,\mathrm{PCtrl}(H'_N, P_{\leqslant N}$$  $$)\big]$$

satisfy $\|H_N\|_{\mathfrak{L}\mathcal{M}([0,\tau])}, \|Y_N\|_{\mathfrak{X}^{s_2,b}([0,\tau])} \lesssim B^2$. Using Lemma 4.9.8, we also obtain that

$$\|w_N\|_{\mathfrak{X}^{s_1,b}([0,\tau])} \lesssim B^5 \qquad \text{and} \qquad \sum_{L_1 \sim L_2} \|P_{L_1} \textcolor{purple}{\uparrow} \cdot P_{L_2} w_N\|_{L_t^2 H_x^{-4\delta_1}([0,\tau]\times\mathbb{T}^3)} \lesssim B^5.$$

Inserting our choice of B, this completes the proof of (II).

The statement (III) directly follows from (iii) in Proposition 4.2.10. It now remains to prove (IV). We emphasize that $T \geqslant 1$ is arbitrary, which will be useful in the stability theory below. We focus on the estimate for the difference, since the proof of the estimate for $w_K[\tau]$ is easier (but similar). Using Lemma 4.9.9, we may restrict to $\textcolor{blue}{\bullet} \in \Theta^{\mathrm{sp}}_{\mathrm{blue}}(B,T)$ and $\textcolor{red}{\circ} \in \Theta^{\mathrm{sp}}_{\mathrm{red}}(B,T)$. Then, we can replace the estimates in $\mathfrak{X}([0,T], \textcolor{purple}{\uparrow}; t_0, N, K)$ by estimates in $\mathfrak{X}([0,T], \textcolor{blue}{\uparrow}; t_0, N, K)$. After rearranging (4.3.34), we have that

$$w_N = P_{\leqslant N}\, \mathrm{I}\big[\,\mathrm{PCtrl}(H'_N + H^{(3)}_N, P_{\leqslant N} \textcolor{blue}{\uparrow})\big] + Y'_N - \textcolor{red}{\uparrow} + Y^{(3)}_N + P_{\leqslant N}\, \mathrm{I}\big[\,\mathrm{PCtrl}(H^{(3)}_N, P_{\leqslant N} \textcolor{red}{\uparrow})\big].$$

Thus, we obtain that

$$w_N[\tau] - w_K[\tau] = Z^{\square}_{N,K}[\tau] + Z^{\bigcirc}_{N,K}[\tau],$$

where

$$Z^{\square}_{N,K}[\tau] \overset{\mathrm{def}}{=} P_{\leqslant N}\, \mathrm{I}\big[\,\mathrm{PCtrl}(H'_N + H^{(3)}_N, P_{\leqslant N} \textcolor{blue}{\uparrow})\big][\tau] - P_{\leqslant K}\, \mathrm{I}\big[\,\mathrm{PCtrl}(H'_K + H^{(3)}_K, P_{\leqslant K} \textcolor{blue}{\uparrow})\big][\tau].$$

and

$$Z^{\bigcirc}_{N,K}[\tau] \overset{\mathrm{def}}{=} Y'_N - Y'_K + Y^{(3)}_N - Y^{(3)}_K + P_{\leqslant N}\, \mathrm{I}\big[\,\mathrm{PCtrl}(H^{(3)}_N, P_{\leqslant N} \textcolor{red}{\uparrow})\big] - P_{\leqslant K}\, \mathrm{I}\big[\,\mathrm{PCtrl}(H^{(3)}_K, P_{\leqslant K} \textcolor{red}{\uparrow})\big].$$

The desired estimate then follows from the frequency-localized version of the multi-linear master estimate (Prop 4.2.8), (iii) in Proposition 4.2.10, and Proposition 4.9.1. □

4.3.3.2 Structure and time-translation

In the globalization argument, we use the invariance of the truncated Gibbs measures under the truncated flows to transform our bounds from the time-interval $[0,(j-1)\tau]$ to the time-interval $[\tau, j\tau]$. As the reader saw in the proof of Proposition 4.3.1, however, the structural bounds are now phrased in terms of $\diamond = \Phi_M[\tau]\blacklozenge$. The next proposition translates the structural bounds back into $\blacklozenge$.

Proposition 4.3.5 (Structure and time-translation)**.** Let $A \geqslant 1$, let $T \geqslant 1$, let $0 < \tau \leqslant 1$, let $j \in \mathbb{N}$ satisfy $j\tau \leqslant T$, let $\alpha > 0$ be a sufficiently large absolute constant, and let $\zeta = \zeta(\epsilon, s_1, s_2, \kappa, \eta, \eta', b_+, b) > 0$ be sufficiently small. There exists a Borel set $\mathcal{S}^{\mathrm{time}}(A,T,\tau) \subseteq \mathcal{L}(A,\tau)$ satisfying

$$\mu_M^{\otimes}(\mathcal{S}^{\mathrm{time}}(A,T,\tau)) \geqslant 1 - \zeta^{-1}\exp(-\zeta A^{\zeta}) \tag{4.3.35}$$

for all $M \geqslant 1$ and such that the following holds for all $\blacklozenge \in \mathcal{S}^{\mathrm{time}}(A,T,\tau)$:

Let $N, K \geqslant 1$, let $B \geqslant 1$, and define $\diamond = \Phi_K[\tau]\blacklozenge$. Let $w^{\mathrm{grn}}_{N,K} \in \mathfrak{X}^{s_1,b}([0,(j-1)\tau])$ satisfy

(A1) Global structured bounds in $\diamond$:

$$\|w^{\mathrm{grn}}_{N,K}\|_{\mathfrak{X}^{s_1,b}([0,(j-1)\tau])} \leqslant B \quad \text{and} \quad \sum_{L_1\sim L_2} \|P_{L_1}\overset{\diamond}{|} \cdot P_{L_2} w^{\mathrm{grn}}_{N,K}\|_{L_t^2 H_x^{-4\delta_1}([0,(j-1)\tau]\times\mathbb{T}^3)} \leqslant B.$$

Define $w_{N,K}\colon [\tau, j\tau]\times\mathbb{T}^3 \to \mathbb{R}$ through the identity

$$\overset{\diamond}{|}(t-\tau) + \overset{\diamond\ast\diamond\diamond}{\curlyvee}_{\mathrm{N}}(t-\tau) + w^{\mathrm{grn}}_{N,K}(t-\tau) = \overset{\blacklozenge}{|}(t) + \overset{\blacklozenge\ast\blacklozenge\blacklozenge}{\curlyvee}_{\mathrm{N}}(t) - \overset{\blacklozenge\ast\blacklozenge\blacklozenge}{\curlyvee}_{\mathrm{N}\setminus\mathrm{K},\,\tau}(t) + w_{N,K}(t). \tag{4.3.36}$$

Then, we obtain the following conclusion regarding $w_{N,K}$.

(C1) Incomplete structured global bounds in $\blacklozenge$:

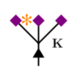
$$\|w_{N,K}\|_{\mathfrak{X}^{s_1,b}([\tau,j\tau])} \leqslant T^{\alpha}AB \quad \text{and} \quad \sum_{L_1\sim L_2} \|P_{L_1}\overset{\blacklozenge}{|} \cdot P_{L_2} w_{N,K}\|_{L_t^2 H_x^{-4\delta_1}([\tau,j\tau]\times\mathbb{T}^3)} \leqslant T^{\alpha}AB.$$

Remark 4.3.6. The superscript "grn" in $w_{N,K}^{\mathrm{grn}}$ stands for "green", which is motivated by the identity (4.3.36). We refer in the conclusion to "incomplete structured global bounds" since the right-hand side in (4.3.36) does not yet have the desired form. The partial cubic stochastic object

is subtracted from it and hence we regard the structure as incomplete.

Proof. Before we turn to the analytical and probabilistic estimates, we discuss the definition and Borel measurability of $\mathcal{S}^{\mathrm{time}}(A,T,\tau)$. We let $\mathcal{S}^{\mathrm{time}}(A,T,\tau)$ be the intersection of $\mathcal{L}(A,\tau)$ with the set of $\blacklozenge \in \mathscr{H}_x^{-1/2-\kappa}$ satisfying the implication (A1)→(C1) for all N, K, B, and $w_{N,K}^{\mathrm{grn}}$. For fixed parameters and a fixed function $w_{N,K}^{\mathrm{grn}}$, the set of $\blacklozenge \in \mathscr{H}_x^{-1/2-\kappa}$ satisfying (A1) and/or (C1) is closed and hence Borel measurable. Using a separability argument, it suffices to require the implication (A1)→(C1) for countably many $w_{N,K}^{\mathrm{grn}}$, which yields the measurability of $\mathcal{S}^{\mathrm{time}}(A,T,\tau)$.

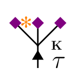
We now turn to the analytical and probabilistic estimates. If $\blacklozenge \in \mathcal{L}(A,\tau)$, it follows from (II) and (IV) from Proposition 4.3.3 that

$\lozenge =$  $[\tau] +$  $[\tau] + Z_K[\tau],$

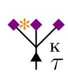
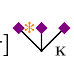
where the remainder $Z_K[\tau]$ satisfies

$$\| Z_K[\tau] \|_{\mathfrak{X}([0,T],\uparrow;\tau,N,K)} \leqslant A T^{\alpha}.$$

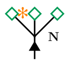
By applying the linear propagator to $\lozenge$, we obtain for all $t \geqslant \tau$ that

 $(t-\tau) =$  $(t) +$  $(t) + Z_K(t),$ (4.3.37)

where we recall from (4.3.4) that

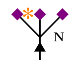
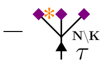
 $(t) = \mathrm{I}\Big[1_{[0,\tau]}$  $\Big](t)$

Regarding the cubic stochastic object, we have that

 $(t-\tau) = \mathrm{I}\Big[1_{[\tau,\infty)}$  $(\cdot-\tau)\Big](t)$

$= \mathrm{I}\Big[1_{[\tau,\infty)}$  $\Big](t) + \mathrm{I}\Big[1_{[\tau,\infty)}\Big($  $(\cdot-\tau) -$  $(\cdot)\Big)\Big](t)$ (4.3.38)

Combining the algebraic identity

$\mathrm{I}\Big[1_{[0,\tau]}$  $\Big](t) + \mathrm{I}\Big[1_{[\tau,\infty)}$  $\Big](t) =$  $(t) -$  (t)

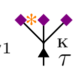
with (4.3.37) and (4.3.38), it follows that

$w_{N,K}(t) = w^{\mathrm{grn}}_{N,K}(t) + Z_K(t) + \mathrm{I}\Big[1_{[\tau,\infty)}\Big($  $(\cdot-\tau) -$  $(\cdot)\Big)\Big](t).$ (4.3.39)

Equipped with the identity (4.3.39) for $w_{N,K}$, it remains to prove the conclusion (C1) on an event satisfying (4.3.35). The second and third summand in (4.3.39) can be treated using Lemma 4.9.8, Proposition 4.9.12 (combined with (4.3.37)), and Lemma 4.9.13. Thus, it remains to prove (C1) for the first summand in (4.3.39). Using (4.3.37), we have that

$\sum_{L_1\sim L_2} \| P_{L_1}$  $(t)\cdot P_{L_2} w^{\mathrm{grn}}_{N,K}(t-\tau)\|_{L^2_t H^{-4\delta_1}_x([\tau,j\tau]\times\mathbb{T}^3)}$

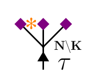
$\leqslant \sum_{L_1\sim L_2} \| P_{L_1}$  $(t)\cdot P_{L_2} w^{\mathrm{grn}}_{N,K}(t)\|_{L^2_t H^{-4\delta_1}_x([0,(j-1)\tau]\times\mathbb{T}^3)}$ (4.3.40)

$+ \sum_{L_1\sim L_2} \| P_{L_1}$  $(t)\cdot P_{L_2} w^{\mathrm{grn}}_{N,K}(t-\tau)\|_{L^2_t H^{-4\delta_1}_x([\tau,j\tau]\times\mathbb{T}^3)}$ (4.3.41)

$+ \sum_{L_1\sim L_2} \| P_{L_1} Z_K(t)\cdot P_{L_2} w^{\mathrm{grn}}_{N,K}(t-\tau)\|_{L^2_t H^{-4\delta_1}_x([\tau,j\tau]\times\mathbb{T}^3)}.$ (4.3.42)

The first term (4.3.40) can be bounded using assumption (A1). The second term (4.3.41) is bounded by Corollary 4.9.3, and the third term (4.3.42) is bounded by Lemma 4.8.8. □

4.3.3.3 Structure and the cubic stochastic object

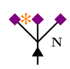
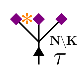
In Proposition 4.3.5 above, the right-hand side of (4.3.36) does not have the desired structure. In the next proposition, we will show that adding the "partial" cubic stochastic object  only leads to a small error in the nonlinear wave equation.

Proposition 4.3.7 (Structure and the cubic stochastic object:)**.** Let $T \geqslant 1$, let $A \geqslant 1$, let $0 < \tau < 1$, let $\alpha > 0$ be a sufficiently large absolute constant, and let $\zeta = \zeta(\epsilon, s_1, s_2, \kappa, \eta, \eta', b_+, b) > 0$ be sufficiently small. Then, there exists a Borel set $\mathcal{S}^{\mathrm{cub}}(A,T,\tau) \subseteq \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$ satisfying

$$\mu_M^{\otimes}(\mathcal{S}^{\mathrm{cub}}(A,T,\tau)) \geqslant 1 - \zeta^{-1}\exp(-\zeta A^{\zeta})$$

for all $M \geqslant 1$ and such that the following holds for all $\blacklozenge \in \mathcal{S}^{\mathrm{cub}}(A,T,\tau)$:
Let $N, K \geqslant 1$, let $B \geqslant 1$, let $j \in \mathbb{N}$, let $\mathcal{J} = [\tau, j\tau] \subseteq [0,T]$, and let $u_{N,K} \colon \mathcal{J} \times \mathbb{T}^3 \to \mathbb{R}$. Furthermore, we make the following assumptions:

(A1) Incomplete structure: There exists a $w_{N,K}(t) \in \mathfrak{X}^{s_1,b}(\mathcal{J})$ satisfying all $t \in \mathcal{J}$ the identity

$u_{N,K}(t) =$  $(t) +$  $(t) -$  $(t) + w_{N,K}(t).$

(A2) Incomplete structured global bounds:

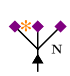
$\|w_{N,K}\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \leqslant B$ and $\sum_{L_1 \sim L_2} \| P_{L_1}$  $\cdot P_{L_2} w_{N,K} \|_{L_t^2 H_x^{-4\delta_1}(\mathcal{J}\times\mathbb{T}^3)} \leqslant B.$

We define a function $\widetilde{u}_N \colon \mathcal{J} \times \mathbb{T}^3 \to \mathbb{R}$ by

$\widetilde{u}_N(t) = u_{N,K}(t) +$  $(t).$

Then, $\widetilde{u}_N$ satisfies the following three properties.

(C1) Structure: For all $t \in \mathcal{J}$, it holds that

$\widetilde{u}_N(t) =$  $(t) +$  $(t) + \widetilde{w}_N(t),$

where $\widetilde{w}_N = w_{N,K}$.

(C2) Approximate solution: There exist $H_N \in \mathscr{LM}(\mathcal{J})$ and $F_N \in \mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})$ satisfying

$$(-\partial_t^2 - 1 + \Delta)\widetilde{u}_N - P_{\leqslant N} :\big(V * (P_{\leqslant N}\widetilde{u}_N)^2\big)P_{\leqslant N}\widetilde{u}_N:$$
$$= (-\partial_t^2 - 1 + \Delta)u_{N,K} - P_{\leqslant N} :\big(V * (P_{\leqslant N}u_{N,K})^2\big)P_{\leqslant N}u_{N,K}:$$
$$- P_{\leqslant N}\,\mathrm{PCtrl}(H_N, P_{\leqslant N}$$  $$) - F_N$$

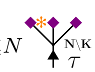
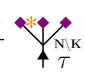
and

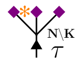
$$\|H_N\|_{\mathscr{LM}(\mathcal{J})}, \|F_N\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})} < T^\alpha A B^3 \min(N,K)^{-\eta'}.$$

(C3) Closeness: It holds that

$$\|\widetilde{u}_N[t] - u_{N,K}[t]\|_{C_t^0\mathscr{H}_x^{\beta-\kappa}(\mathcal{J}\times\mathbb{T}^3)} < T^\alpha A B^3 \min(N,K)^{-\eta'}.$$

Proof. We simply choose $\mathcal{S}^{\mathrm{cub}}(A,T,\tau)$ as the set of all $\blacklozenge \in \mathscr{H}_x^{-1/2-\kappa}$ where the implication (A1),(A2) $\to$(C1),(C2),(C3) holds for all N, K, B, j, and $w_{N,K}$. Similar as in the proof of Proposition 4.3.5, a separability argument yields the Borel measurability of $\mathcal{S}^{\mathrm{cub}}(A,T,\tau)$.
We now show that $\mathcal{S}^{\mathrm{cub}}(A,T,\tau)$ satisfies the desired probabilistic estimate. The first conclusion (C1) follows directly from the definition of $\widetilde{u}_N$. We now turn to the second conclusion, which is the main part of the argument. First, we recall that  solves the linear wave equation on $\mathcal{J} = [\tau, j\tau]$. Together with the definition of $\widetilde{u}_N$, this implies

$$(-\partial_t^2 - 1 + \Delta)\widetilde{u}_N - P_{\leqslant N} :\big(V * (P_{\leqslant N}\widetilde{u}_N)^2\big)P_{\leqslant N}\widetilde{u}_N:$$
$$- \Big((-\partial_t^2 - 1 + \Delta)u_{N,K} - P_{\leqslant N} :\big(V * (P_{\leqslant N}u_{N,K})^2\big)P_{\leqslant N}u_{N,K}: \Big)$$
$$= P_{\leqslant N} :\Big(V * \Big(P_{\leqslant N}u_{N,K} + P_{\leqslant N}$$  $$\Big)^2\Big)P_{\leqslant N}\Big(u_{N,K} + $$  $$\Big):$$
$$- P_{\leqslant N} :\big(V * (P_{\leqslant N}u_{N,K})^2\big)P_{\leqslant N}u_{N,K}: .$$

We emphasize that in the cubic stochastic object , the linear evolution  enters at a frequency $\gtrsim \min(N, K)$ in at least one of the arguments. Using the frequency-localized version of the multilinear master estimate for Gibbsian initial data (Proposition 4.9.12), we obtain the conclusion (C2).

Finally, (C3) directly follows from the frequency-localized version of Proposition 4.9.1. $\square$

4.3.3.4 Stability theory

The last ingredient for the globalization argument is a stability estimate. The proof will rely on our previous stability estimate for Gaussian random data from Proposition 4.2.14. As a result, the argument closely resembles a similar step in the local theory, where we proved Proposition 4.3.3 through Proposition 4.2.10.

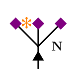
Proposition 4.3.8 (Stability estimate)**.** Let $T \geqslant 1$, let $A \geqslant 1$, let $0 < \tau \leqslant 1$, and let $\zeta = \zeta(\epsilon, s_1, s_2, \kappa, \eta, \eta', b_+, b) > 0$ be sufficiently small. There exists a constant $C = C(\epsilon, s_1, s_2, b_+, b_-)$ and a Borel set $\mathcal{S}^{\text{stab}}(A, T, \tau) \subseteq \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$ satisfying

$$\mu_M^{\otimes}(\blacklozenge \in \mathcal{S}^{\text{stab}}_{\text{blue}}(A, T, \tau)) \geqslant 1 - \zeta^{-1} \exp(-\zeta A^{\zeta}) \tag{4.3.43}$$

such that the following holds for all $\blacklozenge \in \mathcal{S}^{\text{stab}}(A, T, \tau)$:

Let $N \geqslant 1$, $B \geqslant 1$, $0 < \theta < 1$, and let $\mathcal{J} = [t_0, t_1] \subseteq [0, T]$, where $t_0, t_1 \in \tau\mathbb{Z}$. Let $\widetilde{u}_N \colon J \times \mathbb{T}^3 \to \mathbb{R}$ be an approximate solution of (4.2.1) satisfying the following assumptions.

(A1) Structure: We have the decomposition

$\widetilde{u}_N =$  $+$  $+ \widetilde{w}_N.$

(A2) Global bounds: It holds that

$$\|\widetilde{w}_N\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \leqslant B \quad \text{and} \quad \sum_{L_1 \sim L_2} \|P_{L_1}$$  $$\cdot P_{L_2} \widetilde{w}_N\|_{L_t^2 H_x^{-4\delta_1}(\mathcal{J} \times \mathbb{T}^3)} \leqslant B.$$

(A3) Approximate solution: There exists $H_N \in \mathscr{LM}(\mathcal{J})$ and $F_N \in \mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})$ satisfying the identity

$$(-\partial_t^2 - 1 + \Delta)\widetilde{u}_N = P_{\leqslant N} :\big(V * (P_{\leqslant N}\widetilde{u}_N)^2\big) P_{\leqslant N}\widetilde{u}_N \colon - P_{\leqslant N}\, \mathrm{PCtrl}(H_N, P_{\leqslant N} \uparrow) - F_N$$

and the estimates

$$\|H_N\|_{\mathscr{LM}(\mathcal{J})} < \theta \qquad \text{and} \qquad \|F_N\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})} < \theta.$$

Furthermore, let $Z_N[t_0] \in H_x^{s_1}(\mathbb{T}^3)$ be a perturbation satisfying the following assumption.

(A4) Structured perturbation: There exists a $K \geqslant 1$ such that

$$\|Z[t_0]\|_{\mathfrak{Z}(\mathcal{J}, \uparrow; t_0, N, K)} \leqslant \theta.$$

Finally, assume that

(A5) Parameter condition: $C \exp\Big(C(A + B + T)^C\Big)\theta \leqslant 1.$

Then, there exists a solution $u_N \colon J \times \mathbb{T}^3 \to \mathbb{R}$ of (4.2.1) satisfying the initial value condition $u_N[t_0] = \widetilde{u}_N[t_0] + Z_N[t_0]$ and the following conclusions.

(C1) Preserved structure: We have the decomposition

$$u_N = \uparrow + \mathrm{Y}_{N} + w_N.$$

(C2) Closeness: The difference $u_N - \widetilde{u}_N = w_N - \widetilde{w}_N$ satisfies

$$\begin{aligned} \|u_N - \widetilde{u}_N\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} &\leqslant C \exp\big(C(A+B+T)^C\big)\theta, \\ \sum_{L_1 \sim L_2} \|P_{L_1} \uparrow \cdot P_{L_2}(u_N - \widetilde{u}_N)\|_{L_t^2 H_x^{-4\delta_1}(\mathcal{J}\times\mathbb{T}^3)} &\leqslant C \exp\big(C(A+B+T)^C\big)\theta. \end{aligned}$$

(C3) Preserved global bounds: It holds that

$$\|w_N\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \leqslant B_\theta \quad \text{and} \quad \sum_{L_1 \sim L_2} \|P_{L_1} \overset{\blacklozenge}{|} \cdot P_{L_2} w_N\|_{L_t^2 H_x^{-4\delta_1}(\mathcal{J}\times\mathbb{T}^3)} \leqslant B_\theta,$$

where $B_\theta \overset{\text{def}}{=} B + C\exp\big(C(A+B+T)^C\big)\theta$.

Proof. Similar as in the proof of Proposition 4.3.7, we can define $\mathcal{S}^{\text{stab}}(A,T,\tau)$ through the implications (A1)-(A5) $\to$ (C1)-(C3) and prove its measurability using a separability argument.

It remains to prove the probabilistic estimate (4.3.43). Using Theorem 4.1.1, it suffices to prove that

$$\mathbb{P}\big(\bullet + \circ_{\mathrm{M}} \in \mathcal{S}^{\text{stab}}(A,T,\tau)\big) \geqslant 1 - \zeta^{-1}\exp(\zeta A^\zeta).$$

Using Lemma 4.2.6, Corollary 4.9.3, Proposition 4.2.14, Lemma 4.2.6, and Lemma 4.9.9, which also contain the definitions of the sites below, we may restrict to the event

$$\begin{aligned}&\Big\{\bullet \in \Theta^{\text{type}}_{\text{blue}}(A,T) \cap \Theta^{\text{stab}}_{\text{blue}}(A,T) \cap \Theta^{\text{cub}}_{\text{blue}}(A,T) \cap \Theta^{\text{sp}}_{\text{blue}}(A,T)\Big\}\\ &\cap \Big\{\circ_{\mathrm{M}} \in \Theta^{\text{type}}_{\text{red}}(A,T) \cap \Theta^{\text{sp}}_{\text{red}}(A,T)\Big\}.\end{aligned} \tag{4.3.44}$$

Our goal is to use Proposition 4.2.14 (with slightly adjusted parameters). To this end, we need to convert the assumptions (A1)-(A5) involving $\blacklozenge$ into similar statements based on $\bullet$. We let $D > 0$ be a large implicit (but absolute) constant, which may change its value between different lines. We now let $N, B, \theta, \mathcal{J}, \widetilde{u}_N, \widetilde{w}_N, H_N, F_N$, and $Z_N[t_0]$ be as in (A1)-(A5). We then define $w_N[\blacklozenge \to \bullet]$ by

$$\overset{\blacklozenge}{|} + \overset{\blacklozenge\,\blacklozenge\,\blacklozenge}{\curlyvee}_{N} = \overset{\bullet}{|} + \overset{\bullet\,\bullet\,\bullet}{\curlyvee}_{N} + w_N[\blacklozenge \to \bullet],$$

which implies

$$\widetilde{u}_N = \overset{\bullet}{|} + \overset{\bullet\,\bullet\,\bullet}{\curlyvee}_{N} + w_N[\blacklozenge \to \bullet] + \widetilde{w}_N.$$

Using Corollary 4.9.3 and Lemma 4.9.7, we obtain that

$$\|\widetilde{w}_N\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \leqslant B \quad \text{and} \quad \sum_{L_1 \sim L_2} \|P_{L_1} \textcolor{blue}{\dot{\mid}} \cdot P_{L_2} \widetilde{w}_N\|_{L_t^2 H_x^{-4\delta_1}(\mathcal{J} \times \mathbb{T}^3)} \leqslant T^\alpha A^D B$$

as well as

$$\|w_N[\textcolor{purple}{\blacklozenge} \to \textcolor{blue}{\bullet}]\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \leqslant T^\alpha A^D \quad \text{and} \quad \sum_{L_1 \sim L_2} \|P_{L_1} \textcolor{blue}{\dot{\mid}} \cdot P_{L_2} w_N[\textcolor{purple}{\blacklozenge} \to \textcolor{blue}{\bullet}]\|_{L_t^2 H_x^{-4\delta_1}(\mathcal{J} \times \mathbb{T}^3)} \leqslant T^\alpha A^D.$$

Thus, (A2) in Proposition 4.2.14 is satisfied with $B' = 2T^\alpha A^D B$. A similar argument based on Lemma 4.9.7 and Lemma 4.9.9 also yields (A3) and (A4) in Proposition 4.2.14 with $\theta' = 2T^\alpha A^D B$. Furthermore, the stronger assumption (A5) in this proposition implies (as long as C is sufficiently large) that

$$C \exp\Big(C(A + B')^{\frac{2}{b_+ - b}} T^{\frac{40}{b_+ - b}}\Big)\theta' \leqslant 1.$$

Thus, Proposition 4.2.14 implies that

$$\begin{aligned} \|u_N - \widetilde{u}_N\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} &\leqslant C \exp\big(C(A + B')^{\frac{2}{b_+ - b}} T^{\frac{40}{b_+ - b}}\big)\theta', \\ \sum_{L_1 \sim L_2} \|P_{L_1} \textcolor{blue}{\dot{\mid}} \cdot P_{L_2}(u_N - \widetilde{u}_N)\|_{L_t^2 H_x^{-4\delta_1}(\mathcal{J} \times \mathbb{T}^3)} &\leqslant C \exp\big(C(A + B')^{\frac{2}{b_+ - b}} T^{\frac{40}{b_+ - b}}\big)\theta'. \end{aligned}$$

Arguing similarly as above to replace $\textcolor{blue}{\dot{\mid}}$ by $\textcolor{purple}{\dot{\mid}}$, this proves the conclusion (C2). The conclusion (C3) then follows from the triangle inequality and assumption (A2). □

4.4 Ingredients, tools, and methods

In this section we provide tools that will be used throughout the rest of this paper. In order to make this section accessible to readers with a primary background in either dispersive or stochastic partial differential equations, our exposition will be detailed. We encourage the reader to skip sections covering areas of his or her expertise.

In Section 4.4.1, we cover $\mathfrak{X}^{s,b}$-spaces, which are also called Bourgain spaces. The $\mathfrak{X}^{s,b}$-spaces will allow us to utilize multi-linear dispersive effects. In Section 4.4.2, we present a continuity argument. In Section 4.4.3, we prove an oscillatory sum estimate for a series involving the sine-function. While the proof is standard, its relevance to dispersive equations is surprising and the cancellation was first used by Gubinelli, Koch, and Oh in [GKO18a]. In Section 4.4.4, we state several counting estimates related to the dispersive symbol of the wave equation. The counting estimate play an important role in the estimates of our stochastic objects. In Section 4.4.5, we recall elementary properties of Gaussian processes, which have been heavily used in the first part of the series [Bri20c]. In Section 4.4.6, we provide background regarding multiple stochastic integrals. This section has an algebraic flavor and the multiple stochastic integrals will be used to separate the non-resonant and resonant components of our stochastic object. In Section 4.4.7, we discuss Gaussian hypercontractivity and its implications for random matrices.

4.4.1 Bourgain spaces and transference principles

In this subsection, we recall the definitions and elementary properties of $\mathfrak{X}^{s,b}$-spaces, which are often also called Bourgain spaces. Heuristically, $\mathfrak{X}^{s,b}$-spaces contain space-time functions u which behave like solutions to the linear wave equation. This principle will be made more precise through the transference principles below. We refer the reader to [Tao06a, Section 2.6] and [ET16, Section 3.3] for a more detailed introduction.

Definition 4.4.1 ($\mathfrak{X}^{s,b}$-spaces)**.** For any $s, b \in \mathbb{R}$ and $u\colon \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R}$, we define the $\mathfrak{X}^{s,b}$-norm by

$$\|u\|_{\mathfrak{X}^{s,b}} \overset{\text{def}}{=} \|\langle n\rangle^s \langle |\lambda| - \langle n\rangle\rangle^b \widehat{u}(\lambda, n)\|_{L^2_\lambda \ell^2_n(\mathbb{R}\times\mathbb{Z}^3)}. \tag{4.4.1}$$

If $\mathcal{J} \subseteq \mathbb{R}$ is any interval, we define the restricted norm by

$$\|u\|_{\mathfrak{X}^{s,b}(\mathcal{J})} \overset{\text{def}}{=} \inf\{\|v\|_{\mathfrak{X}^{s,b}} \colon v(t,x)|_{\mathcal{J}} = u\}. \tag{4.4.2}$$

We denote the corresponding function spaces by $\mathfrak{X}^{s,b}$ and $\mathfrak{X}^{s,b}(\mathcal{J})$, respectively.

In (4.4.1), we could have used the symbol $\langle |\lambda| - |n| \rangle$ instead of $\langle |\lambda| - \langle n \rangle \rangle$. Since $\langle n \rangle = |n| + \mathcal{O}(1)$, this would yield an equivalent definition. Our first lemma shows the connection between the $\mathfrak{X}^{s,b}$-spaces and the half-wave operators.

Lemma 4.4.2 (Characterization of $\mathfrak{X}^{s,b}$)**.** Let $s, b \in \mathbb{R}$ and let $u\colon \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R}$. Then, it holds that

$$\|u\|_{\mathfrak{X}^{s,b}(\mathbb{R})} \lesssim \min_{\pm} \|\langle \nabla \rangle^s \exp(\mp it\langle \nabla \rangle) u\|_{L^2_x H^b_t(\mathbb{T}^3 \times \mathbb{R})}. \tag{4.4.3}$$

Furthermore, we have the equivalence

$$\|u\|_{\mathfrak{X}^{s,b}(\mathbb{R})} \sim \min_{\substack{u_+, u_- \in \mathfrak{X}^{s,b}(\mathbb{R}): \\ u = u_+ + u_-}} \max_{\pm} \|\langle \nabla \rangle^s \exp(\mp it\langle \nabla \rangle) u_\pm\|_{L^2_x H^b_t(\mathbb{T}^3 \times \mathbb{R})}. \tag{4.4.4}$$

Proof. Using Plancherel's identity, it holds that

$$\|\langle \nabla \rangle^s \exp(\mp it\langle \nabla \rangle) u\|_{L^2_x H^b_t(\mathbb{T}^3 \times \mathbb{R})} = \|\langle n \rangle^s \langle \pm\lambda - \langle n \rangle \rangle^b \widehat{u}(\lambda, n)\|_{L^2_\lambda \ell^2_n(\mathbb{R} \times \mathbb{Z}^3)}.$$

The first estimate (4.4.3) then follows from $||\lambda| - \langle n \rangle| \leqslant | \pm \lambda - \langle n \rangle|$. The inequality "$\lesssim$" in the identity (4.4.4) follows from the triangle inequality and (4.4.3). The inequality "$\gtrsim$" follows by defining $u_\pm$ as

$$\widehat{u}(\lambda, n) = 1\big\{ \pm \lambda \geqslant 0 \big\} \cdot \widehat{u}(\lambda, n).$$

$\square$

Our next lemma plays an important role in the local theory. It yields the required smallness of the nonlinearity on a small time-interval.

Lemma 4.4.3 (Time-localization lemma)**.** Let $-1/2 < b_1 \leqslant b_2 < 1/2$ and let $1/2 < b < 1$. Let $\psi \in \mathcal{S}(\mathbb{R})$ be a Schwartz-function and let $0 < \tau \leqslant 1$. Then, it holds for all $F \in \mathfrak{X}^{s,b_2}(\mathbb{R})$ that

$$\big\|\psi(t/\tau) F\big\|_{\mathfrak{X}^{s,b_1}(\mathbb{R})} \lesssim \tau^{b_2 - b_1} \|F\|_{\mathfrak{X}^{s,b_2}(\mathbb{R})} \qquad \text{and} \qquad \|F\|_{\mathfrak{X}^{s,b_1}([0,\tau])} \lesssim \tau^{b_2 - b_1} \|F\|_{\mathfrak{X}^{s,b_2}([0,\tau])}. \tag{4.4.5}$$

Furthermore, we have for all $u \in \mathfrak{X}^{s,b}(\mathbb{R})$ that

$$\|\psi(t/\tau)u\|_{\mathfrak{X}^{s,b}(\mathbb{R})} \lesssim \tau^{\frac{1}{2}-b}\|u\|_{\mathfrak{X}^{s,b}(\mathbb{R})}. \tag{4.4.6}$$

A proof of Lemma 4.4.3 or a similar result can be found in many textbooks on dispersive PDE, such as [Tao06a, Section 2.6] or [ET16, Section 3.3]. Since the second estimate (4.4.6) is not usually found in the literature, we present a self-contained proof.

Proof. By using duality and a composition, we may assume that $0 \leqslant b_1 \leqslant b_2 < 1/2$. Let $F_+, F_- \in \mathfrak{X}^{s,b_2}(\mathbb{R})$ satisfying $F = F_+ + F_-$. Using Lemma 4.4.2, we obtain that

$$\|\psi(t/\tau)F\|_{\mathfrak{X}^{s,b_1}(\mathbb{R})} \lesssim \max_{\pm} \|\psi(t/\tau)\langle\nabla\rangle^s \exp(\mp it\langle\nabla\rangle)F_\pm\|_{L_x^2 H_t^{b_1}(\mathbb{T}^3\times\mathbb{R})}. \tag{4.4.7}$$

Using interpolation between the $b_1 = 0$ and $b_1 = b_2$ as well as the fractional product rule (or a simple para-product estimate), one has for all $f \in H_t^{b_2}(\mathbb{R})$ the estimate

$$\|\psi(t/\tau)f\|_{H_t^{b_1}(\mathbb{R})} \lesssim \tau^{b_2-b_1}\|f\|_{H_t^{b_2}(\mathbb{R})}. \tag{4.4.8}$$

Combining (4.4.7) and (4.4.8) yields the first estimate in (4.4.5). The second estimate in (4.4.7) then follows from the first estimate and the definition of the restricted norms. Finally, the second estimate (4.4.8) follows from the same argument, except that (4.4.8) is replaced by

$$\|\psi(t/\tau)f\|_{H_t^b(\mathbb{R})} \lesssim \|\psi(t/\tau)\|_{H_t^b(\mathbb{R})}\|f\|_{H_t^b(\mathbb{R})} \lesssim \tau^{\frac{1}{2}-b}\|f\|_{H_t^b(\mathbb{R})}, \tag{4.4.9}$$

which follows from the algebra property of $H_t^b(\mathbb{R})$. □

Lemma 4.4.4 (Restricted norms and continuity)**.** Let $s \in \mathbb{R}$ and let $-1/2 < b' < 1/2$. Then, we have for any interval $\mathcal{J} \subseteq \mathbb{R}$ and any $F \in \mathfrak{X}^{s,b'}(\mathbb{R})$ that

$$\|1_{\mathcal{J}}F\|_{\mathfrak{X}^{s,b'}(\mathbb{R})} \lesssim \|F\|_{\mathfrak{X}^{s,b'}(\mathbb{R})}. \tag{4.4.10}$$

Furthermore, if $G \in \mathfrak{X}^{s,b'}(\mathcal{J})$, then

$$\|G\|_{\mathfrak{X}^{s,b'}(\mathcal{J})} \sim \|1_{\mathcal{J}} G\|_{\mathfrak{X}^{s,b'}(\mathbb{R})} \tag{4.4.11}$$

Finally, if $t_0 \overset{\text{def}}{=} \inf \mathcal{J}$, then the map

$$t \in \mathcal{J} \mapsto \|1_{[t_0,t]} G\|_{\mathfrak{X}^{s,b'}(\mathbb{R})} \tag{4.4.12}$$

is continuous.

Proof. We begin with the proof of (4.4.10). By using a similar reduction as in the proof of Lemma 4.4.3, it suffices to prove that

$$\|1_{\mathcal{J}} g(t)\|_{\mathscr{H}_t^{b'}(\mathbb{R})} \lesssim \|g(t)\|_{\mathscr{H}_t^{b'}(\mathbb{R})}. \tag{4.4.13}$$

By writing $1_{\mathcal{J}}$ as a superposition of different indicator functions, it suffices to prove the estimate for $(-\infty, a)$ and (a, ∞), where $a \in \mathbb{R}$, instead of $\mathcal{J}$. Using the time-reflection and time-translation symmetry of $H_t^{b'}(\mathbb{R})$, it suffices to prove the estimate for $\mathcal{J}$ replaced by $(0, \infty)$. Thus, it remains to prove

$$\|1_{(0,\infty)} g(t)\|_{\mathscr{H}_t^{b'}(\mathbb{R})} \lesssim \|g(t)\|_{\mathscr{H}_t^{b'}(\mathbb{R})}. \tag{4.4.14}$$

This follows from (a modification of) the fractional product rule or a simple paraproduct estimate. We now turn to the proof of (4.4.11). By the definition of the restricted norms, we clearly have the upper-bound $\|G\|_{\mathfrak{X}^{s,b'}(\mathcal{J})} \lesssim \|1_{\mathcal{J}} G\|_{\mathfrak{X}^{s,b'}(\mathbb{R})}$. Now, let $\widetilde{G} \in \mathfrak{X}^{s,b}(\mathbb{R})$ satisfies $\widetilde{G}|_{\mathcal{J}} = G$. Using (4.4.10), we obtain that

$$\|1_{\mathcal{J}} G\|_{\mathfrak{X}^{s,b'}(\mathbb{R})} = \|1_{\mathcal{J}} \widetilde{G}\|_{\mathfrak{X}^{s,b'}(\mathbb{R})} \lesssim \|\widetilde{G}\|_{\mathfrak{X}^{s,b'}(\mathbb{R})}.$$

After taking the infimum in $\widetilde{G}$, this yields the other lower-bound in (4.4.11).
Finally, we prove the continuity of (4.4.12). By a density argument, it suffices to take $G \in \mathfrak{X}^{s,1/2}(\mathbb{R})$. For any $0 < \delta < 1/2 - b$ and any $t_1, t_2 \in \mathcal{J}$, we obtain from Lemma 4.4.3 that

$$\Big| \|1_{[t_0,t_1]} G\|_{\mathfrak{X}^{s,b'}(\mathbb{R})} - \|1_{[t_0,t_2]} G\|_{\mathfrak{X}^{s,b'}(\mathbb{R})} \Big| \leqslant \|1_{(t_1,t_2]} G\|_{\mathfrak{X}^{s,b'}(\mathbb{R})} \lesssim |t_1 - t_2|^{\delta} \|G\|_{\mathfrak{X}^{s,1/2}(\mathbb{R})}.$$

This implies the Hölder-continuity. $\square$

The next gluing lemma will be used to combine $\mathfrak{X}^{s,b}$-bounds on different intervals. While such a result is trivial for purely physical function spaces, such as $L_t^q L_x^p$, it is slightly more complicated for the $\mathfrak{X}^{s,b}$-spaces, since they rely on the time-frequency variable.

Lemma 4.4.5 (Gluing lemma)**.** Let $s \in \mathbb{R}$, let $-1/2 < b' < 1/2$, let $1/2 < b < 1$, and let $\mathcal{J}, \mathcal{J}_1, \mathcal{J}_2$ be bounded intervals satisfying $\mathcal{J}_1 \cap \mathcal{J}_2 \neq \varnothing$. Then, we have for all $F\colon (\mathcal{J}_1 \cup \mathcal{J}_2) \times \mathbb{T}^3 \to \mathbb{R}$ that

$$\|F\|_{\mathfrak{X}^{s,b'}(\mathcal{J}_1 \cup \mathcal{J}_2)} \lesssim \|F\|_{\mathfrak{X}^{s,b'}(\mathcal{J}_1)} + \|F\|_{\mathfrak{X}^{s,b'}(\mathcal{J}_2)}. \tag{4.4.15}$$

Furthermore, let $\tau \overset{\text{def}}{=} |\mathcal{J}_1 \cap \mathcal{J}_2|$. Then, it holds for all $u\colon (\mathcal{J}_1 \cup \mathcal{J}_2) \times \mathbb{T}^3 \to \mathbb{R}$ that

$$\|u\|_{\mathfrak{X}^{s,b}(\mathcal{J}_1 \cup \mathcal{J}_2)} \lesssim \tau^{\frac{1}{2}-b}\big(\|u\|_{\mathfrak{X}^{s,b}(\mathcal{J}_1)} + \|u\|_{\mathfrak{X}^{s,b}(\mathcal{J}_2)}\big). \tag{4.4.16}$$

Proof. We begin with the proof of (4.4.15). Using Lemma 4.4.4, we have that

$$\begin{aligned}
\|F\|_{\mathfrak{X}^{s,b'}(\mathcal{J}_1 \cup \mathcal{J}_2)} &\lesssim \|1_{\mathcal{J}_1 \cup \mathcal{J}_2} F\|_{\mathfrak{X}^{s,b'}(\mathbb{R})} \\
&\lesssim \|1_{\mathcal{J}_1} F\|_{\mathfrak{X}^{s,b'}(\mathbb{R})} + \|1_{\mathcal{J}_2 \backslash \mathcal{J}_1} F\|_{\mathfrak{X}^{s,b'}(\mathbb{R})} \\
&\lesssim \|F\|_{\mathfrak{X}^{s,b'}(\mathcal{J}_1)} + \|F\|_{\mathfrak{X}^{s,b'}(\mathcal{J}_2 \backslash \mathcal{J}_1)} \\
&\lesssim \|F\|_{\mathfrak{X}^{s,b'}(\mathcal{J}_1)} + \|F\|_{\mathfrak{X}^{s,b'}(\mathcal{J}_2)}.
\end{aligned}$$

The proof of the second estimate (4.4.16) is similar. Instead of working with an actual indicator function, we use a smooth cut-off function on the spatial scale $\sim \tau$ and a variant of (4.4.9) instead of (4.4.14). $\square$

Our last two lemmas where concerned with the behavior of $\mathfrak{X}^{s,b}$-spaces over small or overlapping time-intervals. In this respect, the $\mathfrak{X}^{s,b}$-spaces are more complicated than purely physical function spaces. We now turn to transference principles, which do not have a direct analog in purely physical function spaces.

Lemma 4.4.6 (Linear transference principle (cf. [Tao06a, Lemma 2.9]))**.** Let $b > 1/2$, let $s \in \mathbb{R}$, and assume that the norm $\| \cdot \|_Y$ satisfies

$$\|e^{i\alpha t} e^{\pm it\langle\nabla\rangle} u_0\|_Y \leqslant C \|u_0\|_{H_x^s} \tag{4.4.17}$$

for all $\alpha \in \mathbb{R}$ and all $u_0 \in H_x^s$. Then, it holds for all $u \in \mathfrak{X}^{s,b}$ that

$$\|u\|_Y \lesssim C \|u\|_{\mathfrak{X}^{s,b}}. \tag{4.4.18}$$

The linear transference principle allows us to reduce linear estimates for functions in $\mathfrak{X}^{s,b}$-spaces to estimates for the half-wave operators.

Corollary 4.4.7. For any $b > 1/2$, $s \in \mathbb{R}$, any $4 \leqslant p \leqslant \infty$, any compact interval $J \subseteq \mathbb{R}$, and any $u \colon J \times \mathbb{T}^3 \to \mathbb{C}$, we have that

$$\|u[t]\|_{C_t^0 \mathscr{H}_x^s (J \times \mathbb{T}^3)} \lesssim \|u\|_{\mathfrak{X}^{s,b}(J)}, \tag{4.4.19}$$

$$\|\langle\nabla\rangle^{s+\frac{4}{p}-\frac{3}{2}} u(t)\|_{L_t^p L_x^p (J \times \mathbb{T}^3)} \lesssim (1+|J|)^{1/p} \|u\|_{\mathfrak{X}^{s,b}(J)}, \tag{4.4.20}$$

$$\|\langle\nabla\rangle^{s-1-} u(t)\|_{L_t^2 L_x^\infty (J \times \mathbb{T}^3)} \lesssim (1+|J|)^{1/2} \|u\|_{\mathfrak{X}^{s,b}(J)}. \tag{4.4.21}$$

The corollary follows directly from the linear transference principle (Lemma 4.4.6) and the Strichartz estimates for the linear wave equation.

The next lemma is the most basic ingredient for any contraction argument based on $\mathfrak{X}^{s,b}$-spaces.

Lemma 4.4.8 (Energy-estimate (cf. [Tao06a, Lemma 2.12] and [ET16, Lemma 3.2]))**.** Let $1/2 < b < 1$, let $s \in \mathbb{R}$, let $\mathcal{J} \subseteq \mathbb{R}$ be a compact interval, let $t_0 \in \mathcal{J}$, and let

$$(-\partial_t^2 - 1 + \Delta) u = F. \tag{4.4.22}$$

Then, it holds that

$$\|u\|_{\mathfrak{X}^{s,b}(\mathcal{J})} \lesssim \big(1+|\mathcal{J}|\big)^2 \big(\|u[t_0]\|_{\mathcal{H}_x^s} + \|F\|_{\mathfrak{X}^{s-1,b-1}(\mathcal{J})}\big). \tag{4.4.23}$$

The statement of Lemma 4.4.8 in [ET16, Tao06a] only includes intervals of size ~ 1. The more general version follows by using the triangle inequality, iterating the bound on unit intervals, and (4.4.19). The square in the pre-factor can likely be improved but is inessential in our argument, since the stability theory already loses exponential factors in the final time T.

The most important terms in the nonlinearity can only be estimated through multi-linear dispersive effects and hence require a direct analysis of the $\mathcal{X}^{s-1,b-1}$-norm. However, several more minor terms can be estimated more easily through physical methods. In order to pass back from the frequency-based $\mathcal{X}^{s-1,b-1}$-space into purely physical spaces, we provide the following inhomogeneous Strichartz estimate.

Lemma 4.4.9 (Inhomogeneous Strichartz estimate in $\mathcal{X}^{s,b}$-spaces)**.** Let $1/2 < b < 1$, let $s \in \mathbb{R}$, let $\mathcal{J} \subseteq \mathbb{R}$ be a compact interval, and let $F\colon \mathcal{J} \times \mathbb{T}^3 \to \mathbb{R}$. Then, we have the two estimates

$$\|F\|_{\mathcal{X}^{s-1,b-1}(\mathcal{J})} \lesssim \|F\|_{L_t^{2b} H_x^{s-1}(\mathcal{J}\times\mathbb{T}^3)}, \tag{4.4.24}$$

$$\|F\|_{\mathcal{X}^{s-1,b-1}(\mathcal{J})} \lesssim (1+|\mathcal{J}|)\|\langle\nabla\rangle^{s-\frac{1}{2}+\frac{2b-1}{b}s} F\|_{L_t^{4/3} L_x^{4/3}(\mathcal{J}\times\mathbb{T}^3)}. \tag{4.4.25}$$

Remark 4.4.10. For $0 \leqslant s \leqslant 1$, we will often simplify the right-hand side of (4.4.24) by using that

$$\frac{2b-1}{b} s \leqslant 4(b-1/2).$$

Proof. We first prove (4.4.24). Using (4.4.19) and duality, we have that

$$\|F\|_{\mathcal{X}^{s-1,-b}(\mathcal{J})} \lesssim \|F\|_{L_t^1 H_x^{s-1}(\mathcal{J}\times\mathbb{T}^3)}.$$

By Plancherel, we also have that

$$\|F\|_{\mathcal{X}^{s-1,0}(\mathcal{J})} \lesssim \|F\|_{L_t^2 H_x^{s-1}(\mathcal{J}\times\mathbb{T}^3)}.$$

Using interpolation, this implies (4.4.24). The proof of the second estimate (4.4.25) is similar and relies on duality, (4.4.20), Plancherel, and interpolation. □

When utilizing multilinear dispersive effects, we will often use the following lemma to estimate the $\mathfrak{X}^{s-1,b_--1}$-norm.

Lemma 4.4.11. Let $s \in \mathbb{R}$ and let $T \geqslant 1$. Let $\mathcal{A}$ be a finite index set and let $(n_\alpha)_{\alpha\in\mathcal{A}} \subseteq \mathbb{Z}^3$, $(\theta_\alpha)_{\alpha\in\mathcal{A}} \subseteq \mathbb{R}$, and $(c_\alpha)_{\alpha\in\mathcal{A}} \subseteq \mathbb{C}$. Define

$$F(t,x) \overset{\text{def}}{=} \sum_{\alpha\in\mathcal{A}} c_\alpha \exp(i\langle n_\alpha, x\rangle + it\theta_\alpha). \tag{4.4.26}$$

Then, it holds that

$$\begin{aligned} &\|F\|_{\mathfrak{X}^{s-1,b_--1}([0,T])} \\ &\lesssim T \max_{\pm} \Big\| \langle\lambda\rangle^{b_--1} \langle n\rangle^{s-1} \sum_{\alpha\in\mathcal{A}} 1\big\{n = n_\alpha\big\}\, c_\alpha \widehat{\chi}\big(T(\lambda \mp \langle n\rangle - \theta_\alpha)\big)\Big\|_{L^2_\lambda \ell^2_n(\mathbb{R}\times\mathbb{Z}^3)}. \end{aligned} \tag{4.4.27}$$

Proof. For any $G\colon \mathbb{R}\times\mathbb{T}^3 \to \mathcal{C}$, we have that

$$\begin{aligned} \|G\|_{\mathfrak{X}^{s-1,b_--1}(\mathbb{R})} &= \|\langle|\lambda| - \langle n\rangle\rangle^{b_--1}\langle n\rangle^{s-1}\widehat{G}(\lambda,n)\|_{L^2_\lambda \ell^2_n(\mathbb{R}\times\mathbb{Z}^3)} \\ &\lesssim \max_{\pm} \|\langle\lambda \pm \langle n\rangle\rangle^{b_--1}\langle n\rangle^{s-1}\widehat{G}(\lambda,n)\|_{L^2_\lambda \ell^2_n(\mathbb{R}\times\mathbb{Z}^3)} \\ &= \max_{\pm} \|\langle\lambda\rangle^{b_--1}\langle n\rangle^{s-1}\widehat{G}(\lambda \mp \langle n\rangle, n)\|_{L^2_\lambda \ell^2_n(\mathbb{R}\times\mathbb{Z}^3)}. \end{aligned}$$

We then apply this inequality to $G(t,x) = \chi(t/T)F(t,x)$. ☐

Finally, we present an estimate for the Fourier-transform of a (localized) time-integral.

Lemma 4.4.12. Let $T \geqslant 1$ and let $\lambda, \lambda_1, \lambda_2 \in \mathbb{R}$. Then, it holds that

$$\Big|\mathcal{F}_t\Big(\chi(t/T)\exp(i\lambda_1 t)\int_0^t \exp\big(i\lambda_2 t'\big)\mathrm{d}t'\Big)(\lambda)\Big| \lesssim T^2\Big(\langle\lambda-\lambda_1-\lambda_2\rangle^{-10} + \langle\lambda-\lambda_1\rangle^{-10}\Big)\langle\lambda_2\rangle^{-1}. \tag{4.4.28}$$

Furthermore, if $\mathcal{J} \subseteq [0,T]$ is an interval, then

$$\begin{aligned} &\Big|\mathcal{F}_t\Big(\chi(t/T)\exp(i\lambda_1 t)\int_0^t 1_{\mathcal{J}}(t')\exp\big(i\lambda_2 t'\big)\mathrm{d}t'\Big)(\lambda)\Big| \\ &\lesssim T^2\Big(\langle\lambda-\lambda_1-\lambda_2\rangle^{-1} + \langle\lambda-\lambda_1\rangle^{-1}\Big)\langle\lambda_2\rangle^{-1}. \end{aligned} \tag{4.4.29}$$

Proof. We first prove (4.4.28). A direct calculation yields

$$\mathcal{F}_t\Big(\chi(t/T)\exp(i\lambda_1 t)\int_0^t \exp\big(i\lambda_2 t'\big)\mathrm{d}t'\Big)(\lambda) = \frac{T}{i\lambda_2}\Big(\widehat{\chi}\big(T(\lambda-\lambda_1-\lambda_2)\big) - \widehat{\chi}\big(T(\lambda-\lambda_1)\big)\Big). \quad (4.4.30)$$

For $|\lambda_2| \gtrsim 1$, the estimate follows from the decay of $\widehat{\chi}$. For $|\lambda_2| \lesssim 1$, the estimate follows from the fundamental theorem of calculus and the decay of $\widehat{\chi}'$. We also used $T \geqslant 1$, which implies that $\langle T\cdot\rangle^{-10} \lesssim \langle\cdot\rangle^{-10}$.

We now turn to (4.4.29). Since the restriction to $\mathcal{J}$ only appears in the integral, we can replace $\mathcal{J}$ by its closure. We now let $\mathcal{J} = [t_-, t_+] \subseteq [0,T]$. By integrating the exponential, we have that

$$\int_0^t 1_{\mathcal{J}}(t')\exp\big(i\lambda_2 t'\big)\mathrm{d}t' = \frac{1}{i\lambda_2}\Big(\exp(i\lambda_2(t\wedge t_+)) - \exp(i\lambda_2(t\wedge t_-))\Big),$$

where $x \wedge y$ denotes the minimum of x and y. This implies

$$\begin{aligned}&\mathcal{F}_t\Big(\chi(t/T)\exp(i\lambda_1 t)\int_0^t 1_{\mathcal{J}}(t')\exp\big(i\lambda_2 t'\big)\mathrm{d}t'\Big)(\lambda)\\ &= \frac{1}{i\lambda_2}\int_{\mathbb{R}} \chi(t/T)\exp(i(\lambda+\lambda_1)t)\Big(\exp(i\lambda_2(t\wedge t_+)) - \exp(i\lambda_2(t\wedge t_-))\Big)\mathrm{d}t.\end{aligned}$$

The estimate then follows by distinguishing the cases $|\lambda_1| \lesssim 1$, $|\lambda_1| \gg 1 \gtrsim |\lambda_2|$, and $|\lambda_1|, |\lambda_2| \gg 1$, together with the triangle inequality and a simple integration by parts. □

4.4.2 Continuity argument

In this short subsection, we present a modification of the standard continuity argument. The modification is a result of the possibile discontinuity of $t \in [0,T] \mapsto \|u\|_{\mathfrak{X}^{s,b}([0,t])}$, where $u \in \mathfrak{X}^{s,b}([0,T])$ and $b > 1/2$. As a replacement, we will rely on the continuity statement in Lemma 4.4.4. A different approach to this problem was obtained in [Tao01, Theorem 3], which yields the quasi-continuity, and may even yield the continuity (see the discussion in [Tao01, Section 12]).

Lemma 4.4.13 (Continuity argument)**.** Let $\mathcal{J} = [t_0, t_1)$, let $f\colon \mathcal{J} \to [0,\infty)$ be a nonnegative function, and let $g\colon \mathcal{J} \to [0,\infty)$ be a continuous, nonnegative function. Let $A \geqslant 1$, $0 < \theta, \delta < 1$, and assume that

$$f(t) \leqslant g(t) \leqslant g(t_0) + \delta(A^2 + f(t)^2)(f(t) + \theta) \tag{4.4.31}$$

for all $t \in [t_0, t_1)$. Furthermore, assume that

$$g(t_0) + \delta^2 A\theta \leqslant 1 \qquad \text{and} \qquad \delta(A^2 + 6) \leqslant 1/4. \tag{4.4.32}$$

Then, it holds that

$$f(t) \leqslant g(t) \leqslant 2(g(t_0) + \delta A^2\theta)$$

for all $t \in [t_0, t_1)$.

Proof. The estimate (4.4.31) implies that

$$g(t) \leqslant g(t_0) + \delta(A^2 + g(t)^2)(g(t) + \theta)$$

for all $t \in [t_0, t_1)$. Using the condition (4.4.32), we also have that

$$g(t_0) + \delta(A^2 + 4(g(t_0) + \delta A^2\theta)^2)(g(t_0) + \delta A^2\theta + \theta) \leqslant \frac{3}{2}(g(t_0) + \delta A^2\theta).$$

Using the standard continuity method (see e.g. [Tao06a, Section 1.3]), this implies

$$g(t) \leqslant 2(g(t_0) + \delta A^2\theta)$$

for all $t \in [t_0, t_1)$. □

4.4.3 Sine-cancellation lemma

In this subsection, we prove an oscillatory sum estimate which critically relies on the fact that the sine-function is odd. The same cancellation was exploited in earlier work of Gubinelli-Koch-Oh [GKO18a, Section 4] and we present a slight generalization of their argument.

Lemma 4.4.14. Let $f\colon \mathbb{R}\times\mathbb{R}\times\mathbb{Z}^3\to\mathbb{C}$, $a\in\mathbb{Z}^3$, $T\geqslant 1$, let $\mathcal{J}\subseteq[0,T]$ be an interval, and let $A,N\geqslant 1$. Assume that $|a|\lesssim A\ll N$. Furthermore, assume that f satisfies for all $|t|,|t'|\leqslant T$ that

$$|f(t,t',n)|\leqslant A\langle n\rangle^{-3},\quad |f(t,t',n)-f(t,t',-n)|\leqslant A\langle n\rangle^{-4},\quad \text{and}\quad |\partial_{t'}f(t,t',n)|\leqslant A\langle n\rangle^{-4}.$$

Then, it holds that

$$\begin{aligned}&\sup_{\lambda\in\mathbb{R}}\sup_{|t|\leqslant T}\Big|\sum_{n\in\mathbb{Z}^3}\chi_N(n)\int_0^t 1_{\mathcal{J}}(t')\sin((t-t')\langle a+n\rangle)\cos((t-t')\langle n\rangle)\exp(i\lambda t')f(t,t',n)\mathrm{d}t'\Big|\\&\lesssim T^2A^3\log(2+N)N^{-1}.\end{aligned}\tag{4.4.33}$$

The dependence on A is not essential and can likely be improved. In all our applications of this lemma, A is negligible compared to N. We emphasize that the estimate fails if we only assume that $|f(t,t',n)|\leqslant A\langle n\rangle^{-3}$. Indeed, after removing the truncation χ_N, the corresponding sum could diverge logarithmically.

Proof. Using trigonometric identities, we have that

$$\begin{aligned}&2\sum_{n\in\mathbb{Z}^3}\chi_N(n)\int_0^t 1_{\mathcal{J}}(t')\sin((t-t')\langle a+n\rangle)\cos((t-t')\langle n\rangle)\exp(i\lambda t')f(t,t',n)\mathrm{d}t'\\&=\sum_{n\in\mathbb{Z}^3}\chi_N(n)\int_0^t 1_{\mathcal{J}}(t')\sin\big((t-t')(\langle a+n\rangle-\langle n\rangle)\big)\exp(i\lambda t')f(t,t',n)\mathrm{d}t' \qquad (4.4.34)\\&+\sum_{n\in\mathbb{Z}^3}\chi_N(n)\int_0^t 1_{\mathcal{J}}(t')\sin\big((t-t')(\langle a+n\rangle+\langle n\rangle)\big)\exp(i\lambda t')f(t,t',n)\mathrm{d}t'. \qquad (4.4.35)\end{aligned}$$

We estimate the terms (4.4.34) and (4.4.35) separately. We begin with (4.4.34), which is the more difficult term. Since $|\langle a+n\rangle-\langle n\rangle|\lesssim A$, we do not expect to gain in N through the integration in t'. Instead, we utilize a pointwise cancellation. By using the symmetry $n\leftrightarrow -n$ in the summation, we obtain

$$
\begin{aligned}
&2\Big|\sum_{n\in\mathbb{Z}^3}\chi_N(n)\sin\big((t-t')(\langle a+n\rangle-\langle n\rangle)\big)f(t,t',n)\Big| \\
&=\Big|\sum_{n\in\mathbb{Z}^3}\chi_N(n)\Big(\sin\big((t-t')(\langle n+a\rangle-\langle n\rangle)\big)f(t,t',n)+\sin\big((t-t')(\langle n-a\rangle-\langle n\rangle)\big)f(t,t',-n)\Big)\Big| \\
&\lesssim\sum_{n\in\mathbb{Z}^3}\chi_N(n)\Big|\sin\big((t-t')(\langle n+a\rangle-\langle n\rangle)\big)+\sin\big((t-t')(\langle n-a\rangle-\langle n\rangle)\big)\Big|\cdot|f(t,t',n)| \\
&+\sum_{n\in\mathbb{Z}^3}\chi_N(n)|f(t,t',n)-f(t,t',-n)|.
\end{aligned}
$$

Using the assumptions on f, the second summand is easily bounded by AN^{-1}. We now concentrate on the first summand. Using a Taylor expansion, we have that

$$
\langle n\pm a\rangle-\langle n\rangle=\pm\frac{n\cdot a}{\langle n\rangle}+\mathcal{O}\Big(A^2N^{-1}\Big). \tag{4.4.36}
$$

Using that the sine-function is odd, we obtain that

$$
\begin{aligned}
&\Big|\sin\big((t-t')(\langle n+a\rangle-\langle n\rangle)\big)+\sin\big((t-t')(\langle n-a\rangle-\langle n\rangle)\Big| \\
&=\Big|\sin\big((t-t')(\langle n+a\rangle-\langle n\rangle)\big)-\sin\big(-(t-t')(\langle n-a\rangle-\langle n\rangle)\Big| \\
&\leqslant T\Big|\langle n+a\rangle-\langle n\rangle+\langle n-a\rangle-\langle n\rangle\Big| \\
&\lesssim TA^2N^{-1}.
\end{aligned}
$$

Putting both estimates together and integrating in t', we see that the first term (4.4.34) is bounded by $T^2A^3N^{-1}$, which is acceptable.

We now turn to the estimate of (4.4.35). Since $\langle n+a\rangle+\langle n\rangle\gtrsim N$, we expect to gain a factor of

N through integration by parts. We have that

$$\begin{aligned}
&\Big| \sum_{n\in\mathbb{Z}^3} \chi_N(n) \int_0^t 1_{\mathcal{J}}(t') \sin\Big((t-t')(\langle a+n\rangle + \langle n \rangle)\Big) \exp(i\lambda t') f(t,t',n) \mathrm{d}t' \Big| \\
&\lesssim \max_{\pm} \Big| \sum_{n\in\mathbb{Z}^3} \chi_N(n) \int_0^t 1_{\mathcal{J}}(t') \exp\Big(i\lambda t' \pm it'(\langle a+n\rangle + \langle n\rangle)\Big) f(t,t',n)\mathrm{d}t' \Big| \\
&\lesssim \max_{\pm} \sum_{n\in\mathbb{Z}^3} \chi_N(n) \frac{1}{1+|\langle a+n\rangle + \langle n\rangle \pm \lambda|} \Big(\sup_{0\leqslant t'\leqslant t} \big| f(t,t',n)\big| + T \sup_{0\leqslant t'\leqslant t} \big|\partial_{t'} f(t,t',n)\big| \Big) \\
&\lesssim TAN^{-3} \max_{\pm} \sum_{n\in\mathbb{Z}^3} \chi_N(n) \frac{1}{1+|\langle a+n\rangle + \langle n\rangle \pm \lambda|}.
\end{aligned}$$

In order to finish the estimate, it only remains to prove that

$$\sum_{n\in\mathbb{Z}^3} \chi_N(n) \frac{1}{1+|\langle a+n\rangle + \langle n\rangle \pm \lambda|} \lesssim \log(2+N) N^2.$$

Since the function $x\mapsto \langle x\rangle$ is 1-Lipschitz, we can estimate the sum by an integral and obtain that

$$\sum_{n\in\mathbb{Z}^3} \chi_N(n) \frac{1}{1+|\langle a+n\rangle + \langle n\rangle \pm \lambda|} \lesssim \int_{\mathbb{R}^3} 1\{|\xi|\sim N\} \frac{1}{1+|\langle \xi+a\rangle + \langle \xi\rangle \pm \lambda|} \mathrm{d}\xi.$$

Due to the rotation invariance of the Lebesgue measure, we can then reduce to $a=(0,0,|a|)$. To estimate the integral, we first switch into polar coordinates (r,θ,φ). Since $A \ll N$, we have for fixed angles θ and φ that $r\mapsto \langle \xi+a\rangle + \langle \xi \rangle$ is bi-Lipschitz on $r\sim N$. After a further change of variables, this yields

$$\int_{\mathbb{R}^3} 1\{|\xi|\sim N\} \frac{1}{1+|\langle \xi+a\rangle + \langle \xi\rangle \pm \lambda|} \mathrm{d}\xi \lesssim N^2 \int_0^\infty 1\{r\sim N\} \frac{1}{1+|r\pm\lambda|} \mathrm{d}r \lesssim N^2 \log(2+N).$$

□

4.4.4 Counting estimates

In this subsection, we record several counting estimates. The counting estimates are the most technical part of our treatment of **So**, **CPara**, and **RMT**. Fortunately, they can be used as a black-box, and we encourage the reader to only skim this section during first reading.

Before we state our counting estimates, we discuss the main ingredients and the differences between the nonlinear wave and Schrödinger equations. In contrast to the counting estimates for the nonlinear Schrödinger equation, the counting estimates for the wave equation require no analytic number theory. The reason is that the mapping $n \mapsto \langle n \rangle$ is globally 1-Lipschitz, whereas the Lipschitz constant of $n \mapsto |n|^2$ grows linearly. This allows us to reduce all (discrete) counting estimates to estimates of the volume of (continuous) sets. More specifically, we will use that the intersection of (most) thin annuli has a smaller volume than the individual annuli.

Another difference between the wave and Schrödinger equation is related to the symmetries of the equation. The Schrödinger equation enjoys the Galilean symmetry, which is useful in obtaining "shifted" versions of several estimates. For instance, it yields that frequency-localized Strichartz estimates for the Schrödinger equation are the same for cubes centered either at or away from the origin. On the frequency-side, it is related to the Galilean transform

$$(n, \lambda) \mapsto (n - a, \lambda - 2a \cdot n + |a|^2),$$

which preserves the discrete paraboloid and plays an important role in decoupling theory (cf. [Dem20, Section 4]). It often allows us to replace conditions such as $|n| \sim N$ in counting estimates by the more general restriction $|n - a| \sim N$ for some fixed $a \in \mathbb{Z}^3$. In contrast, the Lorentzian symmetry of the wave equation on Euclidean space does not even preserve the periodicity of $u \colon \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R}$. As illustrated by the Klainerman-Tataru-Strichartz estimates (cf. [KT99] and Lemma 4.8.1), the frequency-shifted Strichartz estimates are more complicated for the wave equation than for the Schrödinger equation. As will be clear from this section, similar difficulties arise in the counting estimates.

The last difference between the Schrödinger and wave equation we mention here is a result of the multiplier $\langle \nabla \rangle^{-1}$ in the Duhamel integral for the wave equation. Together with multilinear dispersive effects, we therefore obtain two separate smoothing effects in the nonlinear wave equation, which are related to the elliptic symbol $\langle n\rangle$ and the dispersive symbol $\langle|\lambda|-|n|\rangle$. In contrast, the Schrödinger only exhibits a single smoothing effect related to the dispersive symbol $\lambda-|n|^2$. In most situations, we expect that the combined smoothing effects in the wave equation are stronger than the single smoothing effect in the Schrödinger equation. However, it may be more difficult to capture the combined smoothing effect in a single proposition, as has been done in [DNY20, Proposition 4.9] for the Schrödinger equation.

In Section 4.4.4.1, we prove basic counting estimates which form the foundation of the rest of this section. In Section 4.4.4.2-4.4.4.7, we state several cubic, quartic, quintic, and septic counting estimates. In order to not interrupt the flow of the main argument, we placed their (standard) proofs in the appendix. In Section 4.4.4.8, we present estimates for the operator norm of (deterministic) tensors. The tensor estimates are not (yet) standard in the literature on random dispersive equations, so we include their proofs in the body of the paper.

4.4.4.1 Basic counting estimates

Lemma 4.4.15 (Basic counting lemma)**.** Let $a\in\mathbb{Z}^3$, let $A,N\geqslant 1$, and assume that $|a|\sim A$. Then, it holds that

$$\sup_{m\in\mathbb{Z}}\#\Big\{n\in\mathbb{Z}^3\colon |n|\sim N, |\langle a+n\rangle\pm\langle n\rangle-m|\lesssim 1\Big\}\lesssim \min(A,N)^{-1}N^3. \tag{4.4.37}$$

We emphasize that the upper bound in (4.4.37) cannot be improved to N^2. The reason is that $|\langle a+n\rangle-\langle n\rangle|\lesssim A$, which implies that

$$\sup_{m\in\mathbb{Z}}\#\Big\{n\in\mathbb{Z}^3\colon |n|\sim N, |\langle a+n\rangle-\langle n\rangle-m|\lesssim 1\Big\}\gtrsim A^{-1}N^3.$$

As already mentioned above, the main step in the proof converts the discrete estimate (4.4.37) into a continuous analogue. After this reduction, the estimate boils down to multi-variable calculus.

Proof. Since $\langle \xi \rangle = |\xi| + \mathcal{O}(1)$, we may replace $\langle \cdot \rangle$ in (4.4.37) by $|\cdot|$ after increasing the implicit constant. Furthermore, since $\xi \mapsto \langle \xi + a \rangle \pm \langle \xi \rangle$ is globally Lipschitz, we see that the 1-neighborhood of the set on the left-hand side of (4.4.37) is contained in

$$\big\{ \xi \in \mathbb{R}^3 \colon |\xi| \sim N, ||a+\xi| \pm |\xi| - m| \lesssim 1 \big\}.$$

Since the integer vectors are 1-separated, it follows that

$$\#\big\{ n \in \mathbb{Z}^3 \colon |n| \sim N, ||a+n| \pm |n| - m| \lesssim 1 \big\} \lesssim \operatorname{Leb}\Big(\big\{ \xi \in \mathbb{R}^3 \colon |\xi| \sim N, ||a+\xi| \pm |\xi| - m| \lesssim 1 \big\} \Big).$$

We now decompose

$$\begin{aligned}
&\operatorname{Leb}\Big(\big\{ \xi \in \mathbb{R}^3 \colon |\xi| \sim N, ||a+\xi| \pm |\xi| - m| \lesssim 1 \big\} \Big) \\
&\lesssim \sum_{\substack{m_1, m_2 \in \mathbb{Z}: \\ |m_1 \pm m_2 - m| \lesssim 1}} \operatorname{Leb}\Big(\big\{ \xi \in \mathbb{R}^3 \colon |\xi| \sim N, |a+\xi| = m_1 + \mathcal{O}(1), |\xi| = m_2 + \mathcal{O}(1) \big\} \Big) \\
&\lesssim N \sup_{m_1, m_2 \in \mathbb{Z}} \operatorname{Leb}\Big(\big\{ \xi \in \mathbb{R}^3 \colon |\xi| \sim N, |a+\xi| = m_1 + \mathcal{O}(1), |\xi| = m_2 + \mathcal{O}(1) \big\} \Big).
\end{aligned}$$

In the last line, we used that there are at most $\sim N$ non-trivial choices of m_2. Once m_2 is fixed, the condition $|m_1 \pm m_2 - m| \lesssim 1$ implies that there are at most ~ 1 non-trivial choices for m_1. Thus, it remains to prove for $|m_1| \lesssim \max(A, N)$ and $|m_2| \sim N$ that

$$\operatorname{Leb}\Big(\big\{ \xi \in \mathbb{R}^3 \colon |\xi| \sim N, |a+\xi| = m_1 + \mathcal{O}(1), |\xi| = m_2 + \mathcal{O}(1) \big\} \Big) \lesssim \min(A, N)^{-1} N^2. \qquad (4.4.38)$$

Using the rotation invariance of the Lebesgue measure, we may assume that $a = |a| e_3$, i.e., a points in the direction of the z-axis. By switching into polar coordinates, we obtain that

$$\begin{aligned}
&\operatorname{Leb}\Big(\big\{ \xi \in \mathbb{R}^3 \colon |\xi| \sim N, |a+\xi| = m_1 + \mathcal{O}(1), |\xi| = m_2 + \mathcal{O}(1) \big\} \Big) \\
&\lesssim N^2 \int_0^\infty \int_0^\pi 1\big\{ r = m_2 + \mathcal{O}(1) \big\} 1\big\{ \sqrt{|a|^2 + 2r|a|\cos(\theta) + r^2} = m_1 + \mathcal{O}(1) \big\} \sin(\theta) \mathrm{d}\theta \, \mathrm{d}r.
\end{aligned}$$

The condition $\sqrt{|a|^2 + 2r|a|\cos(\theta) + r^2} = m_1 + \mathcal{O}(1)$ together with $|m_1| \lesssim \max(A, N)$ implies that

$$\cos(\theta) = 1 - \frac{(|a| + r)^2}{2|a|r} + \frac{m_1^2}{2|a|r} + \mathcal{O}\big(\max(A, N) A^{-1} N^{-1}\big). \tag{4.4.39}$$

For a fixed r, this shows that $\cos(\theta)$ is contained in an interval of size $\sim \min(A, N)^{-1}$. After a change of variables, this yields

$$\begin{aligned} &N^2 \int_0^\infty \int_0^\pi 1\big\{r = m_2 + \mathcal{O}(1)\big\} 1\Big\{\sqrt{|a|^2 + 2r|a|\cos(\theta) + r^2} = m_1 + \mathcal{O}(1)\Big\} \sin(\theta) \mathrm{d}\theta \, \mathrm{d}r \\ &\lesssim \min(A, N)^{-1} N^2 \int_0^\infty 1\big\{r = m_2 + \mathcal{O}(1)\big\} \mathrm{d}r \\ &\lesssim \min(A, N)^{-1} N^2. \end{aligned} \tag{4.4.40}$$

□

Remark 4.4.16. Our proof of the basic counting lemma (Lemma 4.4.15) easily generalizes to spatial dimensions $d \geqslant 3$. In two spatial dimensions, however, only weaker estimates are available. The reason lies in the absence of the sine-function in the area element for polar coordinates, which breaks (4.4.40). From a PDE perspective, the parallel interactions in two-dimensional wave equations are stronger than the planar interactions in three-dimensional wave equations. Ultimately, this requires a modification in the probabilistic scaling heuristic and we encourage the reader to compare [DNY19, Section 1.3.2] and [OO19, Proposition 1.5].

We now present a minor modification of the basic counting lemma (Lemma 4.4.15). The condition $|n| \sim N$ is augmented by $|n + a| \sim B$. We emphasize that the vector $a \in \mathbb{Z}^3$ in this constraint is the same vector as in the dispersive symbol.

Lemma 4.4.17 ("Two-ball" basic counting lemma)**.** Let $N, A, B \geqslant 1$. Let $a \in \mathbb{Z}^3$ satisfy $|a| \sim A$. Then, it holds that

$$\sup_{m \in \mathbb{Z}} \#\big\{n \in \mathbb{Z}^3 \colon |n| \sim N, |n + a| \sim B, |\langle a + n \rangle \pm \langle n \rangle - m| \lesssim 1\big\} \lesssim \min(A, B, N)^{-1} \min(B, N)^3. \tag{4.4.41}$$

Proof. Using the basic counting lemma (Lemma 4.4.15), we have that

$$
\begin{aligned}
&\sup_{m\in\mathbb{Z}} \#\big\{n \in \mathbb{Z}^3 \colon |n| \sim N, |n+a| \sim B, |\langle a+n\rangle \pm \langle n\rangle - m| \lesssim 1\big\} \\
&\leqslant \sup_{m\in\mathbb{Z}} \#\big\{n \in \mathbb{Z}^3 \colon |n| \sim N, |\langle a+n\rangle \pm \langle n\rangle - m| \lesssim 1\big\} \\
&\lesssim \min(A,N)^{-1} N^3.
\end{aligned}
$$

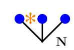
After using a change of variables $b \overset{\text{def}}{=} n + a$, we obtain similarly that

$$
\sup_{m\in\mathbb{Z}} \#\big\{n \in \mathbb{Z}^3 \colon |n| \sim N, |n+a| \sim B, |\langle a+n\rangle \pm \langle n\rangle - m| \lesssim 1\big\} \lesssim \min(A,B)^{-1} B^3.
$$

By combining both estimates we obtain (4.4.41). □

4.4.4.2 Cubic counting estimate

As mentioned in the beginning of this section, we only discuss and state the remaining counting estimates, but postpone the proofs until the appendix.

The cubic counting estimates play an important role in our analysis of the nonlinearity . In the following, we use max, med, and min for the maximum, median, and minimum of three frequency-scales.

Proposition 4.4.18 (Main cubic counting estimate)**.** Let $\pm_{123}, \pm_1, \pm_2, \pm_3 \in \{+,-\}$ and define the phase

$$
\varphi(n_1,n_2,n_3) \overset{\text{def}}{=} \pm_{123}\langle n_{123}\rangle \pm_1 \langle n_1\rangle \pm_2 \langle n_2\rangle \pm_3 \langle n_3\rangle.
$$

Let $N_1, N_2, N_3, N_{12}, N_{123} \geqslant 1$ and let $m \in \mathbb{Z}$. Then, we have the following counting estimates:

(i) In the variables n_1, n_2, and n_3, we have that

$$
\begin{aligned}
&\#\{(n_1,n_2,n_3) \colon |n_1| \sim N_1, |n_2| \sim N_2, |n_3| \sim N_3, |\varphi - m| \leqslant 1\} \\
&\lesssim \operatorname{med}(N_1,N_2,N_3)^{-1} (N_1N_2N_3)^3,
\end{aligned}
$$

(ii) In the variables n_{123}, n_1, and n_2, we have that

$$\begin{aligned}&\#\{(n_{123}, n_1, n_2)\colon |n_{123}| \sim N_{123}, |n_1| \sim N_1, |n_2| \sim N_2, |\varphi - m| \leqslant 1\}\\&\lesssim \operatorname{med}(N_{123}, N_1, N_2)^{-1}(N_{123}N_1N_2)^3.\end{aligned}$$

(iii) In the variables n_{123}, n_{12}, and n_1, we have that

$$\begin{aligned}&\#\{(n_{123}, n_{12}, n_1)\colon |n_{123}| \sim N_{123}, |n_{12}| \sim N_{12}, |n_1| \sim N_1, |\varphi - m| \leqslant 1\}\\&\lesssim \min\big(N_{12}, \max(N_{123}, N_1)\big)^{-1}(N_{123}N_{12}N_1)^3.\end{aligned}$$

(iv) In the variables n_{12}, n_1, and n_3, we have that

$$\begin{aligned}&\#\{(n_{12}, n_1, n_3)\colon |n_{12}| \sim N_{12}, |n_1| \sim N_1, |n_3| \sim N_3, |\varphi - m| \leqslant 1\}\\&\lesssim \min\big(N_{12}, \max(N_1, N_3)\big)^{-1}(N_{12}N_1N_3)^3.\end{aligned}$$

Remark 4.4.19. The four estimates in Proposition 4.4.18 are sharp. In our analysis of the cubic nonlinearity, the frequencies n_1, n_2, and n_3 represent the frequencies of the three individual factors. The frequency n_{12} appears through the convolution with the interaction potential V. Finally, the frequency n_{123}, which is the frequency of the full nonlinearity, appears through the multiplier $\langle\nabla\rangle^{-1}$ in the Duhamel integral and in estimates of the H_x^s and $\mathfrak{X}^{s,b}$-norms.

Since we postpone the proof, let us ease the reader's mind with the heuristic argument behind (i). Without the restriction due to the phase φ, the combined frequency variables (n_1, n_2, n_3) live in a set of cardinality $(N_1N_2N_3)^3$. As long as the level sets of φ have comparable cardinalities, we expect to gain a factor corresponding to the possible values of φ on the set $\{(n_1, n_2, n_3)\colon |n_1| \sim N_1, |n_2| \sim N_2, |n_3| \sim N_3\}$. Since φ is globally Lipschitz, one may ideally hope for a gain of the form $\max(N_1, N_2, N_3)$. Unfortunately, since

$$|\langle n_{123}\rangle - \langle n_1\rangle + \langle n_2\rangle + \langle n_3\rangle| \lesssim \max(N_2, N_3), \tag{4.4.42}$$

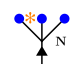
the high×low×low-interactions rule out a gain in $\max(N_1, N_2, N_3)$. As it turns out, however, our basic counting estimates allows us to obtain a gain of the form $\operatorname{med}(N_1, N_2, N_3)$, which is consistent with (4.4.42).

Proposition 4.4.20 (Cubic sum estimate)**.** Let $0 < s \leqslant 1/2$, $0 \leqslant \gamma < s+1/2$, and let $N_1, N_2, N_3 \geqslant 1$. Let the signs $\pm_{123}, \pm_1, \pm_2, \pm_3 \in \{+,-\}$ be given and define the phase

$$\varphi(n_1, n_2, n_3) \overset{\text{def}}{=} \pm_{123}\langle n_{123}\rangle \pm_1 \langle n_1\rangle \pm_2 \langle n_2\rangle \pm_3 \langle n_3\rangle. \tag{4.4.43}$$

Then, it holds that

$$\begin{aligned} &\sup_{m\in\mathbb{Z}} \sum_{n_1,n_2,n_3\in\mathbb{Z}^3} \bigg[\Big(\prod_{j=1}^{3} \chi_{N_j}(n_j) \Big) \langle n_{123}\rangle^{2(s-1)} \langle n_{12}\rangle^{-2\gamma} \Big(\prod_{j=1}^{3} \langle n_j\rangle^{-2} \Big) 1\big\{ |\varphi - m| \leqslant 1 \big\} \bigg] \\ &\lesssim \max(N_1, N_2, N_3)^{2(s-\gamma)} + \max(N_1, N_2)^{1-2\gamma} \max(N_1, N_2, N_3)^{2s-1}. \end{aligned} \tag{4.4.44}$$

Remark 4.4.21. Proposition 4.4.20 plays an essential role in proving that  has regularity $\beta-$. In that argument, we will simply set $\gamma = \beta$.

4.4.4.3 Cubic sup-counting estimates

We now present cubic counting estimates involving suprema, which will be used in the proof of the tensor estimates in Section 4.4.4.8. In turn, the tensor estimates will then be used to prove the random matrix estimates in Section 4.6.

Lemma 4.4.22 (Cubic sup-counting estimates)**.** Let $N_{123}, N_1, N_2, N_3 \geqslant 1$ and $m \in \mathbb{Z}$. Let the signs $\pm_{123}, \pm_1, \pm_2, \pm_3 \in \{+,-\}$ be given and define the phase

$$\varphi(n_1, n_2, n_3) \overset{\text{def}}{=} \pm_{123}\langle n_{123}\rangle \pm_1 \langle n_1\rangle \pm_2 \langle n_2\rangle \pm_3 \langle n_3\rangle.$$

Then, the following estimates hold:

(i) Taking the supremum in n and counting n_1, n_2, n_3, we have

$$\sup_{n \in \mathbb{Z}^3} \# \Big\{ (n_1, n_2, n_3) \colon |n_1| \sim N_1, |n_2| \sim N_2, |n_3| \sim N_3, n = n_{123}, |\varphi - m| \leqslant 1 \Big\}$$
$$\lesssim \mathrm{med}(N_1, N_2, N_3)^3 \min(N_1, N_2, N_3)^2.$$

(ii) Taking the supremum in n_2 and counting n, n_1, n_3, we have

$$\sup_{n_2 \in \mathbb{Z}^3} \# \Big\{ (n, n_1, n_3) \colon |n| \sim N_{123}, |n_1| \sim N_1, |n_3| \sim N_3, n = n_{123}, |\varphi - m| \leqslant 1 \Big\}$$
$$\lesssim \mathrm{med}(N_{123}, N_1, N_3)^3 \min(N_{123}, N_1, N_3)^2.$$

(iii) Taking the supremum in n and counting n_1, n_{12}, n_3, we have

$$\sup_{n \in \mathbb{Z}^3} \# \Big\{ (n_{12}, n_2, n_3) \colon |n_{12}| \sim N_{12}, |n_2| \sim N_2, |n_3| \sim N_3, n = n_{123}, |\varphi - m| \leqslant 1 \Big\}$$
$$\lesssim \min(N_{12}, N_1)^{-1} (N_{12} N_2)^3.$$

(iv) Taking the supremum in n_3 and counting n, n_{12}, n_2, we have

$$\sup_{n \in \mathbb{Z}^3} \# \Big\{ (n, n_{12}, n_2) \colon |n| \sim N_{123}, |n_{12}| \sim N_{12}, |n_2| \sim N_2, n = n_{123}, |\varphi - m| \leqslant 1 \Big\}$$
$$\lesssim \min(N_{12}, N_1)^{-1} (N_{12} N_2)^3.$$

4.4.4.4 Para-controlled cubic counting estimate

We now present our final cubic counting estimate. It will be used to control

$$\Big(\neg \boxed{\textcircled{$\lessdot$} \,\&\, \textcircled{$\lessdot$}} \Big) : \Big(V * \Big(P_{\leqslant N} \overset{\textcolor{blue}{\bullet}}{|} \cdot P_{\leqslant N} X_N \Big) P_{\leqslant N} \overset{\textcolor{blue}{\bullet}}{|} \Big) \colon \ ,$$

which appears in **CPara**.

Lemma 4.4.23 (Para-controlled cubic sum estimate)**.** Let $N_{123}, N_1, N_2, N_3 \geqslant 1$ and $m \in \mathbb{Z}$. Let the signs $\pm_{123}, \pm_1, \pm_2, \pm_3 \in \{+,-\}$ be given and define the phase

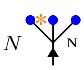
$$\varphi(n_1, n_2, n_3) \stackrel{\text{def}}{=} \pm_{123}\langle n_{123}\rangle \pm_1 \langle n_1\rangle \pm_2 \langle n_2\rangle \pm_3 \langle n_3\rangle.$$

Then, it holds that for all $0 < \gamma < \beta$ that

$$\begin{aligned} &\sup_{\substack{n_2 \in \mathbb{Z}^3: \\ |n_2| \sim N_2}} \sum_{n_1, n_3 \in \mathbb{Z}^3} \Big(\prod_{j=1,3} 1\big\{|n_j| \sim N_j\big\}\Big) \langle n_{123}\rangle^{2(s_2-1)} \langle n_{12}\rangle^{-2\beta} \langle n_1\rangle^{-2} \langle n_3\rangle^{-2}\, 1\big\{|\varphi - m| \leqslant 1\big\} \\ &\lesssim \max(N_1, N_2, N_3)^{2\delta_2} N_1^{-2\gamma} N_2^{2\gamma}. \end{aligned} \tag{4.4.45}$$

4.4.4.5 Quartic counting estimates

Our expansion of the solution u_N and **So** only contain cubic, quintic, and septic stochastic object. The quartic counting estimates will be used to control products such as

$P_{\leqslant N}$  $\cdot\, P_{\leqslant N}$  ,

which occur as factors in the physical term **Phy**. We present two estimates which control the non-resonant (Lemma 4.4.24) and resonant portions (Lemma 4.4.26) of the product, respectively. On our way to the resonant estimate, we also prove the basic resonance estimate (Lemma 4.4.25).

Lemma 4.4.24 (Non-resonant quartic sum estimate)**.** Let $s < -1/2-\eta$ and let $N_1, N_2, N_3, N_4 \geqslant 1$. Let the signs $\pm_{123}, \pm_1, \pm_2, \pm_3 \in \{+,-\}$ be given and define

$$\varphi(n_1, n_2, n_3) \stackrel{\text{def}}{=} \pm_{123}\langle n_{123}\rangle \pm_1 \langle n_1\rangle \pm_2 \langle n_2\rangle \pm_3 \langle n_3\rangle.$$

Then, it holds that

$$\begin{aligned} &\sup_{m \in \mathbb{Z}} \sum_{n_1, n_2, n_3, n_4 \in \mathbb{Z}^3} \Big(\prod_{j=1}^{4} 1\big\{|n_j| \sim N_j\big\}\Big) \langle n_{1234}\rangle^{2s} \langle n_{123}\rangle^{-2} |\widehat{V}_S(n_1, n_2, n_3)|^2 \Big(\prod_{j=1}^{4} \langle n_j\rangle^{-2}\Big) 1\big\{|\varphi - m| \leqslant 1\big\} \\ &\lesssim \max(N_1, N_2, N_3)^{-2\beta+2\eta} N_4^{-2\eta}. \end{aligned}$$

Lemma 4.4.25 (Basic resonance estimate)**.** Let $n_1, n_2 \in \mathbb{Z}^3$ be arbitrary, let $N_3 \geqslant 1$, let the signs $\pm_{123}, \pm_1, \pm_2, \pm_3 \in \{+,-\}$ be given, and define

$$\varphi(n_1,n_2,n_3) \overset{\text{def}}{=} \pm_{123}\langle n_{123}\rangle \pm_1 \langle n_1\rangle \pm_2 \langle n_2\rangle \pm_3 \langle n_3\rangle.$$

Then, it holds that

$$\sum_{m\in\mathbb{Z}} \sum_{n_3\in\mathbb{Z}^3} \langle m\rangle^{-1} 1\big\{|n_3|\sim N_3\big\}\langle n_{123}\rangle^{-1}\langle n_3\rangle^{-2} 1\big\{|\varphi - m|\leqslant 1\big\} \lesssim \log(2+N_3)\langle n_{12}\rangle^{-1}. \tag{4.4.46}$$

Lemma 4.4.26 (Resonant quartic sum estimate)**.** Let $N_1, N_2, N_3 \geqslant 1$ and let $-1/2 < s < 0$. Let the signs $\pm_{123}, \pm_1, \pm_2, \pm_3 \in \{+,-\}$ be given and define

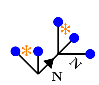
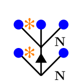
$$\varphi(n_1,n_2,n_3) \overset{\text{def}}{=} \pm_{123}\langle n_{123}\rangle \pm_1 \langle n_1\rangle \pm_2 \langle n_2\rangle \pm_3 \langle n_3\rangle.$$

Then, it holds that

$$\begin{aligned}
&\sum_{n_1,n_2\in\mathbb{Z}^3} \bigg[\Big(\prod_{j=1}^{2} 1\big\{|n_j|\sim N_j\big\} \langle n_{12}\rangle^{2s} \langle n_1\rangle^{-2}\langle n_2\rangle^{-2} \\
&\times \Big(\sum_{m\in\mathbb{Z}} \sum_{n_3\in\mathbb{Z}^3} \langle m\rangle^{-1} 1\big\{|n_3|\sim N_3\big\}\langle n_{123}\rangle^{-1}\langle n_3\rangle^{-2} 1\big\{|\varphi-m|\leqslant 1\big\}\Big)^2 \bigg] \\
&\lesssim \log(2+N_3)^2 \max(N_1,N_2)^{2s}.
\end{aligned}$$

4.4.4.6 Quintic counting estimates

In order to estimate the quintic stochastic objects

 and ,

we require quintic sum estimates. Even at the quintic level, we need to make full use of dispersive effects. This is in contrast to the septic counting effects, which only rely on dispersive effects for cubic sub-objects but do not require dispersive effects at the full septic level.

We present three separate quintic sum estimates, which correspond to zero, one, or two probabilistic resonances.

Lemma 4.4.27 (Non-resonant quintic sum estimate)**.** Let $s \leqslant 1/2-2\eta$ and $N_1, N_2, N_3, N_4, N_5 \geqslant 1$. Furthermore, we define three phase-functions by

$$\psi(n_3,n_4,n_5) \overset{\text{def}}{=} \pm_{345}\langle n_{345}\rangle \pm_3 \langle n_3\rangle \pm_4 \langle n_4\rangle \pm_5 \langle n_5\rangle,$$

$$\varphi(n_1,\ldots,n_5) \overset{\text{def}}{=} \pm_{12345}\langle n_{12345}\rangle \pm_{345} \langle n_{345}\rangle \pm_1 \langle n_1\rangle \pm_2 \langle n_2\rangle,$$

$$\widetilde{\varphi}(n_1,\ldots,n_5) \overset{\text{def}}{=} \pm_{12345}\langle n_{12345}\rangle \mp_{345} \langle n_{345}\rangle + \sum_{j=1}^{5}(\pm_j)\langle n_j\rangle.$$

Then, it holds that

$$\begin{aligned}
&\sup_{m,m'\in\mathbb{Z}} \sum_{n_1,\ldots,n_5\in\mathbb{Z}^3} \bigg[\Big(\prod_{j=1}^{5} 1\big\{|n_j|\sim N_j\big\}\Big)\langle n_{12345}\rangle^{2(s-1)}\langle n_{1345}\rangle^{-2\beta}\langle n_{345}\rangle^{-2}\langle n_{34}\rangle^{-2\beta}\Big(\prod_{j=1}^{5}\langle n_j\rangle^{-2}\Big)\\
&\times 1\big\{|\psi-m|\leqslant 1\big\}\cdot\Big(1\big\{|\varphi-m'|\leqslant 1\big\}+1\big\{|\widetilde{\varphi}-m'|\leqslant 1\big\}\Big)\bigg]\\
&\lesssim \max(N_1,N_3,N_4,N_5)^{-2\beta+4\eta}N_2^{-2\eta}.
\end{aligned}$$

Lemma 4.4.28 (Single-resonance quintic sum estimate)**.** Let $n_4, n_5 \in \mathbb{Z}^3$, $N_{45} \geqslant 1$, and $|n_{45}| \sim N_{45}$. Furthermore, let $\pm_3 \in \{+,-\}$. Then, it holds that

$$\begin{aligned}
&\sup_{m\in\mathbb{Z}^3}\sum_{n_3\in\mathbb{Z}^3}\Big[1\big\{|n_3|\sim N_3\big\}\langle n_{345}\rangle^{-1}\langle n_3\rangle^{-2}1\big\{\langle n_{345}\rangle \pm_3 \langle n_3\rangle \in [m,m+1)\big\}\Big]\\
&\lesssim N_{45}^{-1}.
\end{aligned}$$

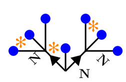
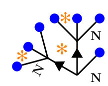
After renaming the variables, Lemma 4.4.28 is essentially the same as Lemma 4.4.25. Our reason for restating Lemma 4.4.28 is to make it easier for the reader to refer back to this section.

Lemma 4.4.29 (Double-resonance quintic sum estimate)**.** Let $N_3, N_4, N_5 \geqslant 1$ and let $\pm_3, \pm_4, \pm_5 \in \{+,-\}$. Then, it holds that

$$\begin{aligned} &\sup_{m\in\mathbb{Z}^3} \sup_{|n_5|\sim N_5} \sum_{n_3,n_4\in\mathbb{Z}^3} \Big[\Big(\prod_{j=3}^{4} 1\big\{|n_j| \sim N_j\big\}\Big) \langle n_{345}\rangle^{-1} \langle n_{45}\rangle^{-\beta} \langle n_3\rangle^{-2} \langle n_4\rangle^{-2} \\ &\times 1\big\{ \langle n_{345}\rangle \pm_3 \langle n_3\rangle \pm_4 \langle n_4\rangle \pm_5 \langle n_5\rangle \in [m, m+1) \big\} \Big] \\ &\lesssim \max(N_4, N_5)^{-\beta+\eta}. \end{aligned} \tag{4.4.47}$$

4.4.4.7 Septic counting estimates

In order to state our septic counting estimates, we need to introduce pairings, and our definition is motivated by a similar notion in [DNY19, Section 1.9]. The pairings are designed to capture the resonances in the septic stochastic objects

 and .

Definition 4.4.30 (Pairings)**.** Let $J \geqslant 1$. We call a relation $\mathscr{P} \subseteq \{1,\ldots,J\}^2$ a pairing if

(i) $\mathscr{P}$ is anti-reflexive, i.e, $(j,j) \notin \mathscr{P}$ for all $1 \leqslant j \leqslant J$,

(ii) $\mathscr{P}$ is symmetric, i.e., $(i,j) \in \mathscr{P}$ if and only if $(j,i) \in \mathscr{P}$,

(iii) $\mathscr{P}$ is univalent, i.e., for each $1 \leqslant i \leqslant J$, $(i,j) \in \mathscr{P}$ for at most one $1 \leqslant j \leqslant J$.

If $(i,j) \in \mathscr{P}$, the tuple (i,j) is called a pair (or $\mathscr{P}$-pair). If $1 \leqslant j \leqslant J$ is contained in a pair, we call j paired (or $\mathscr{P}$-paired). With a slight abuse of notation, we also write $j \in \mathscr{P}$ if j is paired. If j is not paired, we also say that j is unpaired and write $j \notin \mathscr{P}$.

Furthermore, let $\mathcal{A} = (\mathcal{A}_l)_{l=1,\ldots,L}$ be a partition of $\{1,\ldots,J\}$. We say that $\mathscr{P}$ respects $\mathcal{A}$ if $i,j \in \mathcal{A}_l$ for some $1 \leqslant l \leqslant L$ implies that $(i,j) \notin \mathscr{P}$. In other words, $\mathscr{P}$ does not pair elements of the same set inside the partition.

Finally, we call a vector $(n_1, \ldots, n_J) \in (\mathbb{Z}^3)^J$ of frequencies admissible (or $\mathscr{P}$-admissible) if $(i,j) \in \mathscr{P}$ implies that $n_i = -n_j$.

Using Definition 4.4.30, we can now state the septic sum estimate.

Lemma 4.4.31 (Septic sum estimate)**.** Let $1/2 < s < 1$ and let $N_{1234567}, N_{1234}, N_{567}, N_4 \geqslant 1$. For any $\pm_1, \pm_2, \pm_3 \in \{+,-\}$, we define the phase

$$\varphi(n_j, \pm_j \colon 1 \leqslant j \leqslant 3) \overset{\text{def}}{=} \langle n_{123} \rangle \pm_1 \langle n_1 \rangle \pm_2 \langle n_2 \rangle \pm_3 \langle n_3 \rangle.$$

Furthermore, we define

$$\Phi(n_1,n_2,n_3) = \sum_{\pm_1,\pm_2,\pm_3} \sum_{m \in \mathbb{Z}} \langle m \rangle^{-1} |\widehat{V}_S(n_1,n_2,n_3)| \langle n_{123} \rangle^{-1} \Big(\prod_{j=1}^{3} \langle n_j \rangle^{-1} \Big) 1\big\{ |\varphi - m| \leqslant 1 \big\}.$$

Finally, let $\mathscr{P}$ be a pairing of $\{1,\ldots,7\}$ which respects the partition $\{1,2,3\},\{4\},\{5,6,7\}$ and define the non-resonant frequency $n_{\mathrm{nr}} \in \mathbb{Z}^3$ by

$$n_{\mathrm{nr}} \overset{\text{def}}{=} \sum_{j \notin \mathscr{P}} n_j.$$

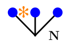
Then, it holds that

$$\begin{aligned}
&\sum_{(n_j)_{j\notin\mathscr{P}}} \langle n_{\mathrm{nr}} \rangle^{2(s-1)} \bigg(\sideset{}{^*}\sum_{(n_j)_{j\in\mathscr{P}}} 1\big\{ |n_{1234567}| \sim N_{1234567} \big\} 1\big\{ |n_{1234}| \sim N_{1234} \big\} 1\big\{ |n_{567}| \sim N_{567} \big\} 1\big\{ |n_4| \sim N_4 \big\} \\
&\times |\widehat{V}(n_{1234})| \Phi(n_1,n_2,n_3) \langle n_4 \rangle^{-1} \Phi(n_5,n_6,n_7) \bigg)^2 \\
&\lesssim \log(2+N_4)^2 \Big(N_{1234567}^{2(s-\frac{1}{2})} N_{567}^{-2(\beta-\eta)} + N_{1234567}^{-2(1-s-\eta)} \Big) N_{1234}^{-2\beta},
\end{aligned}$$

where $\sum^*_{(n_j)_{j\in\mathscr{P}}}$ denotes the sum over admissible frequencies.

While the septic sum estimate (Lemma 4.4.31) may appear complicated, its proof is much easier than the cubic sum estimate (Lemma 4.4.20) or the quintic sum estimate (Lemma 4.4.27). The reason is that we do not rely on dispersive effects at the (full) septic level, and only use the dispersive effects in the cubic stochastic sub-objects.

4.4.4.8 Tensor estimates

The counting estimates from Section 4.4.4.2-4.4.4.7 will be combined with Wiener chaos estimates to control stochastic objects such as . The estimates of the random matrix terms will follow a similar spirit. However, the Wiener chaos estimates will be replaced by the moment method (see Proposition 4.4.50) and the counting estimates will be replaced by deterministic tensor estimates. The tensor estimates, which partially rely on the counting estimates, are the main goal of this subsection.

We first recall the tensor notation from [DNY20, Section 2.1].

Definition 4.4.32 (Tensors and tensor norms)**.** Let $\mathcal{J} \subseteq \mathbb{N}_0$ be a finite set. A tensor $h = h_{n_{\mathcal{J}}}$ is a function from $(\mathbb{Z}^3)^{|\mathcal{J}|}$ into $\mathbb{C}$, where the input variables are given by $n_{\mathcal{J}}$. A partition of $\mathcal{J}$ is a pair of sets $(\mathcal{A}, \mathcal{B})$ such that $\mathcal{A} \bigcup \mathcal{B} = \mathcal{J}$ and $\mathcal{A} \bigcap \mathcal{B} = \varnothing$. For any partition $(\mathcal{A}, \mathcal{B})$, we define the tensor norm

$$\|h\|^2_{n_{\mathcal{A}} \to n_{\mathcal{B}}} = \sup \Big\{ \sum_{n_{\mathcal{B}}} \Big| \sum_{n_{\mathcal{A}}} h_{n_{\mathcal{J}}} z_{n_{\mathcal{A}}} \Big|^2 : \sum_{n_{\mathcal{A}}} |z_{n_{\mathcal{A}}}|^2 = 1 \Big\}. \tag{4.4.48}$$

For example, if $h = h_{nn_1n_2n_3}$, then

$$\|h\|^2_{n_1n_2n_3 \to n} = \sup \Big\{ \sum_{n \in \mathbb{Z}^3} \Big| \sum_{n_1, n_2, n_3 \in \mathbb{Z}^3} h_{nn_1n_2n_3} z_{n_1n_2n_3} \Big|^2 : \sum_{n_1, n_2, n_3 \in \mathbb{Z}^3} |z_{n_1n_2n_3}|^2 = 1 \Big\}.$$

Lemma 4.4.33 (First deterministic tensor estimate)**.** Let $s < 1/2 + \beta - 2\delta_1 - 6\eta$, $N_1, N_2, N_3, N_{12}, N_{123} \geqslant 1$, $m \in \mathbb{Z}$, and $\pm_1, \pm_2, \pm_3, \pm_{123} \in \{+, -\}$. Define the phase-function φ by

$$\varphi(n_1, n_2, n_3) \overset{\text{def}}{=} \pm_{123} \langle n_{123} \rangle \pm_1 \langle n_1 \rangle \pm_2 \langle n_2 \rangle \pm_3 \langle n_3 \rangle.$$

and the truncated tensor h by

$$\begin{aligned} h_{nn_1n_2n_3} \overset{\text{def}}{=} & \chi_{N_{123}}(N_{123}) \chi_{N_{12}}(n_{12}) \Big(\prod_{j=1}^{3} \rho_{\leqslant N}(n_j) \chi_{N_j}(n_j) \Big) \\ & 1\big\{ n = n_{123} \big\} 1\{ |\varphi - m| \leqslant 1 \} \langle n \rangle^{s-1} \widehat{V}(n_{12}) \langle n_1 \rangle^{-1} \langle n_2 \rangle^{-1} \langle n_3 \rangle^{-s_1}. \end{aligned} \tag{4.4.49}$$

Then, we have the estimate

$$\max\left(\|h\|_{n_1n_2n_3\to n}, \|h\|_{n_3\to nn_1n_2}, \|h\|_{n_1n_3\to nn_2}, \|h\|_{n_2n_3\to nn_1}\right) \lesssim \max(N_1, N_2, N_3)^{-\eta}. \tag{4.4.50}$$

Remark 4.4.34. The first deterministic tensor estimate (Lemma 4.4.33) is the main ingredient in the estimate of

$$w_N \mapsto \Big(V * \overset{\bullet\;\bullet}{\vee}_{N}\Big) P_{\leqslant N} w_N,$$

which is the first term in **RMT**. In contrast to the second tensor estimate below, we only impose $s < 1/2 + \beta$ instead of $s < 1/2$ (up to small corrections). The reason is that both instances of $\overset{\bullet}{|}$ are part of the convolution with V.

Proof. The main ingredients are Schur's test and the sup-counting estimate (Lemma 4.4.22).

Step 1: $\|h\|_{n_1n_2n_3\to n}$. Due to the symmetry $n_1 \leftrightarrow n_2$, we may assume that $N_1 \geqslant N_2$. Using Schur's test, we have that

$$\begin{aligned}
&\|h\|^2_{n_1n_2n_3\to n}\\
&\lesssim N_{123}^{2(s-1)} N_{12}^{-2\beta} N_1^{-2} N_2^{-2} N_3^{-2s_1}\\
&\times \sup_{n\in\mathbb{Z}^3} \sum_{n_1,n_2,n_3\in\mathbb{Z}^3} \Big(\prod_{j=1}^{3} 1\big\{|n_j| \sim N_k\big\}\Big) 1\big\{|n_{12}| \sim N_{12}|\big\} 1\big\{|n| \sim N_{123}\big\} 1\big\{n = n_{123}\big\} 1\big\{|\varphi - m| \leqslant 1\big\}\\
&\times \sup_{n_1,n_2,n_3\in\mathbb{Z}^3} \sum_{n\in\mathbb{Z}^3} \Big(\prod_{j=1}^{3} 1\big\{|n_j| \sim N_k\big\}\Big) 1\big\{|n_{12}| \sim N_{12}|\big\} 1\big\{|n| \sim N_{123}\big\} 1\big\{n = n_{123}\big\} 1\big\{|\varphi - m| \leqslant 1\big\}
\end{aligned}$$

Since n is uniquely determined by n_1, n_2, and n_3, the last factor can easily be bounded by one. By using (iii) in Lemma 4.4.22 and $\max(N_{12}, N_2) \lesssim \max(N_1, N_2) = N_1$, we obtain that

$$\begin{aligned}
\|h\|^2_{n_1n_2n_3\to n} &\lesssim N_{123}^{2(s-1)} N_{12}^{-2\beta} N_1^{-2} N_2^{-2} N_3^{-2s_1} \max(N_{12}, N_2) N_{12}^2 N_2^2\\
&\lesssim N_{123}^{2(s-1)} N_{12}^{2-2\beta} N_1^{-1} N_3^{-2s_1}\\
&\lesssim N_{123}^{2(s-1)} N_{12}^{1-2\beta+2\eta} N_1^{-2\eta} N_3^{-2s_1}.
\end{aligned}$$

Furthermore, we have that $N_{12} \lesssim \max(N_{123}, N_3) \lesssim N_{123} \cdot N_3$. Inserting this into the last inequality yields

$$\|h\|^2_{n_1 n_2 n_3 \to n} \lesssim N_{123}^{2s-1-2\beta+2\eta} N_1^{-2\eta} N_3^{1-2s_1-2\beta+2\eta} \lesssim (N_1 N_3)^{-2\eta}.$$

Step 2: $\|h\|_{n_3 \to n_1 n_2 n}$. The argument follow Step 1 nearly verbatim, except that we use (iv) in Lemma 4.4.22 instead of (iii).

Step 3: $\|h\|_{n_1 n_3 \to n_2 n}$. In this step, we ignore the dispersive effects, i.e., we simply bound

$$1\Big\{|\varphi - m| \leqslant 1\Big\} \leqslant 1.$$

By increasing s if necessary, we may assume $s \geqslant 1/2$. Using Schur's test and a simple volume argument, we have that

$$\begin{aligned}
&\|h\|^2_{n_1 n_3 \to n_2 n} \\
&\lesssim N_{123}^{2(s-1)} N_{12}^{-2\beta} N_1^{-2} N_2^{-2} N_3^{-2s_1} \\
&\times \sup_{n, n_2 \in \mathbb{Z}^3} \sum_{n_1, n_3 \in \mathbb{Z}^3} \Big(\prod_{j=1}^{3} 1\big\{|n_j| \sim N_k\big\}\Big) 1\big\{|n_{12}| \sim N_{12}|\big\} 1\big\{|n| \sim N_{123}\big\} 1\big\{n = n_{123}\big\} \\
&\times \sup_{n_1, n_3 \in \mathbb{Z}^3} \sum_{n_2, n \in \mathbb{Z}^3} \Big(\prod_{j=1}^{3} 1\big\{|n_j| \sim N_k\big\}\Big) 1\big\{|n_{12}| \sim N_{12}|\big\} 1\big\{|n| \sim N_{123}\big\} 1\big\{n = n_{123}\big\} \\
&\lesssim N_{123}^{2(s-1)} N_{12}^{-2\beta} N_1^{-2} N_2^{-2} N_3^{-2s_1} \min(N_1, N_{12}, N_3)^3 \min(N_2, N_{12}, N_{123})^3 \\
&\lesssim N_{123}^{2(s-1)} N_{12}^{-2\beta} N_1^{-2} N_2^{-2} N_3^{-2s_1} N_1^{2-2\eta} N_{12}^{1+4\eta-2s_1} N_3^{2s_1-2\eta} N_2^{2-2\eta} N_{12}^{2s-1+2\eta} N_{123}^{2(s-1)} \\
&\lesssim N_{12}^{2s-1-2\beta+2\delta_1+6\eta} (N_1 N_2 N_3)^{-2\eta}.
\end{aligned}$$

In the second last inequality, we used $s \geqslant 1/2$. Since $2s - 1 - 2\beta + 2\delta_1 + 6\eta \leqslant 0$, this is acceptable.

Step 4: $\|h\|_{n_2 n_3 \to n_1 n}$. Due to the symmetry $n_1 \leftrightarrow n_2$, the estimate follows from Step 3. □

We now turn to the second tensor estimate.

Lemma 4.4.35 (Second deterministic tensor estimate)**.** Let $s < 1/2 - \eta$, $N_1, N_2, N_3, N_{12}, N_{123} \geqslant 1$, $m \in \mathbb{Z}$, and $\pm_1, \pm_2, \pm_3, \pm_{123} \in \{+,-\}$. Define the phase-function φ by

$$\varphi(n_1,n_2,n_3) \overset{\text{def}}{=} \pm_{123}\langle n_{123}\rangle \pm_1 \langle n_1\rangle \pm_2 \langle n_2\rangle \pm_3 \langle n_3\rangle$$

and the truncated tensor h by

$$\begin{aligned} h_{nn_1n_2n_3} \overset{\text{def}}{=} & \chi_{N_{123}}(N_{123})\chi_{N_{12}}(n_{12})\Big(\prod_{j=1}^{3} \rho_{\leqslant N}(n_j)\chi_{N_j}(n_j)\Big) \\ & 1\big\{n = n_{123}\big\}1\{|\varphi - m| \leqslant 1\}\langle n\rangle^{s-1}\widehat{V}(n_{12})\langle n_1\rangle^{-1}\langle n_2\rangle^{-s_2}\langle n_3\rangle^{-1}. \end{aligned} \tag{4.4.51}$$

Then, we have the estimate

$$\max\Big(\|h\|_{n_1n_2n_3\to n}, \|h\|_{n_2\to nn_1n_3}, \|h\|_{n_2n_3\to nn_1}, \|h\|_{n_1n_2\to nn_3}\Big) \lesssim N_{12}^{-\beta}\max(N_1,N_2,N_3)^{-\eta}. \tag{4.4.52}$$

Remark 4.4.36. Lemma 4.4.35 is the main ingredient in the estimate of

$$Y_N \mapsto :V * \Big(P_{\leqslant N}\stackrel{\bullet}{|} \cdot P_{\leqslant N}\big(Y_N\big)\Big)\ \big(\neg\olessthan\big)\ P_{\leqslant N}\stackrel{\bullet}{|}\Big):\ ,$$

which is the second term in **RMT**.

Proof. The argument is similar to the proof of Lemma 4.4.33.

Step 1: $\|h\|_{n_1n_2n_3\to n}$. Using Schur's test, we have that

$$\begin{aligned} \|h\|_{n_1n_2n_3\to n}^2 \lesssim\ & N_{123}^{2(s-1)}N_{12}^{-2\beta}N_1^{-2}N_2^{-2s_2}N_3^{-2} \\ & \times \sup_{|n|\sim N_{123}} \sum_{n_1,n_2,n_3\in\mathbb{Z}^3}\Big(\prod_{j=1}^{3}1\big\{|n_j|\sim N_k\big\}\Big)1\big\{n=n_{123}\big\}1\big\{|\varphi-m|\leqslant 1\big\} \\ & \times \sup_{n_1,n_2,n_3\in\mathbb{Z}^3}\sum_{n\in\mathbb{Z}^3}\Big(\prod_{j=1}^{3}1\big\{|n_j|\sim N_k\big\}\Big)1\big\{n=n_{123}\big\}1\big\{|\varphi-m|\leqslant 1\big\}. \end{aligned}$$

The last factor is easily bounded by one, since n is uniquely determined by n_1, n_2, and n_3. By using (i) in Lemma 4.4.22 and $s_2 \leqslant 1$, we obtain that

$$\begin{aligned}
\|h\|^2_{n_1n_2n_3\to n} &\lesssim N_{123}^{2(s-1)} N_{12}^{-2\beta} \operatorname{med}(N_1,N_2,N_3)^3 \min(N_1,N_2,N_3)^2 N_1^{-2} N_2^{-2s_2} N_3^{-2} \\
&\lesssim N_{123}^{2(s-1)} N_{12}^{-2\beta} \max(N_1,N_2,N_3)^{-2s_2} \operatorname{med}(N_1,N_2,N_3)^{3-2} \min(N_1,N_2,N_3)^{2-2} \\
&\lesssim N_{123}^{2(s-1)} N_{12}^{-2\beta} \max(N_1,N_2,N_3)^{1-2s_2}.
\end{aligned}$$

This is acceptable since $s \leqslant 1$ and $\eta \ll \delta_2$.

Step 2: $\|h\|_{n_2\to n_1n_3n}$. This argument is similar to Step 1, but the roles of n_2 and n are reversed. Using Schur's test, we obtain that

$$\begin{aligned}
\|h\|^2_{n_2\to n_1n_3n} \lesssim\ & N_{123}^{2(s-1)} N_{12}^{-2\beta} N_1^{-2} N_2^{-2s_2} N_3^{-2} \\
&\times \sup_{|n_2|\sim N_2} \sum_{n_1,n_3,n\in\mathbb{Z}^3} \Big(\prod_{j=1}^3 1\big\{|n_j|\sim N_k\big\}\Big) 1\big\{|n|\sim N_{123}\big\} 1\big\{n=n_{123}\big\} 1\big\{|\varphi-m|\leqslant 1\big\} \\
&\times \sup_{n_1,n_3,n\in\mathbb{Z}^3} \sum_{n_2\in\mathbb{Z}^3} \Big(\prod_{j=1}^3 1\big\{|n_j|\sim N_k\big\}\Big) 1\big\{|n|\sim N_{123}\big\} 1\big\{n=n_{123}\big\} 1\big\{|\varphi-m|\leqslant 1\big\}.
\end{aligned}$$

As before, the last factor is easily bounded by one. By using (ii) in Lemma 4.4.22 and $2(s-1) \geqslant -2$, we obtain that

$$\begin{aligned}
\|h\|^2_{n_2\to n_1n_3n} &\lesssim N_{123}^{2(s-1)} N_{12}^{-2\beta} \operatorname{med}(N_{123},N_1,N_3)^3 \min(N_{123},N_1,N_3)^2 N_1^{-2} N_2^{-2s_2} N_3^{-2} \\
&\lesssim N_{12}^{-2\beta} N_2^{-2s_2} \max(N_{123},N_1,N_3)^{2s-1} \\
&\lesssim N_{12}^{-2\beta} N_2^{-2s_2} \max(N_1,N_2,N_3)^{-2\eta}.
\end{aligned}$$

In the last line, we used that $s < 1/2 - \eta$.

Step 3: $\|h\|_{n_1n_2\to n_3n}$. In this step, we ignore the dispersive effects, i.e., we simply bound

$$1\big\{|\varphi-m|\leqslant 1\big\} \leqslant 1.$$

Using Schur's test and a simple volume bound, we obtain that

$$\begin{aligned}
\|h\|_{n_1n_2\to n_3n}^2 \lesssim\, & N_{123}^{2(s-1)} N_{12}^{-2\beta} N_1^{-2} N_2^{-2s_2} N_3^{-2} \\
& \times \sup_{n_3,n\in\mathbb{Z}^3} \sum_{n_1,n_2\in\mathbb{Z}^3} \Big(\prod_{j=1}^3 1\big\{|n_j|\sim N_k\big\}\Big) 1\big\{|n|\sim N_{123}\big\} 1\big\{n=n_{123}\big\} \\
& \times \sup_{n_1,n_2\in\mathbb{Z}^3} \sum_{n_3,n\in\mathbb{Z}^3} \Big(\prod_{j=1}^3 1\big\{|n_j|\sim N_k\big\}\Big) 1\big\{|n|\sim N_{123}\big\} 1\big\{n=n_{123}\big\} \\
\lesssim\, & N_{123}^{2(s-1)} N_{12}^{-2\beta} N_1^{-2} N_2^{-2s_2} N_3^{-2} \min(N_1,N_2)^3 \min(N_3,N_{123})^3 \\
\lesssim\, & N_{123}^{2(s-1)} N_{12}^{-2\beta} N_1^{-2} N_2^{-2s_2} N_3^{-2} N_1^{2-2\eta} N_2^{1+2\eta} N_3^{2-2\eta} N_{123}^{1+2\eta} \\
\lesssim\, & N_{12}^{-2\beta} \max(N_1,N_2,N_3)^{-2\eta}.
\end{aligned}$$

Step 4: $\|h\|_{n_2n_3\to n_1n}$ Arguing exactly as in Step 3, we obtain that

$$\begin{aligned}
\|h\|_{n_2n_3\to n_1n}^2 &\lesssim N_{123}^{2(s-1)} N_{12}^{-2\beta} N_1^{-2} N_2^{-2s_2} N_3^{-2} \min(N_2,N_3)^3 \min(N_1,N_{123})^3 \\
&\lesssim N_{12}^{-2\beta} \max(N_1,N_2,N_3)^{-2\eta}.
\end{aligned}$$

□

4.4.5 Gaussian processes

We briefly review the notation from the stochastic control perspective of Chapter 3, which was used in the proof of Theorem 4.1.1. In comparison with the first part of this series, however, we change the notation for the stochastic time variable. We use $\mathfrak{s}$, which is a calligraphic "s", to denote the time-variable in the stochastic control perspective. While the chosen font in $\mathfrak{s}$ may be slightly unusual, we hope that this prevents any confusion with the time-variable t in the nonlinear wave equation.

We let $(B_{\mathfrak{s}}^n)_{n\in\mathbb{Z}^3\backslash\{0\}}$ be a sequence of standard complex Brownian motions such that $B_{\mathfrak{s}}^{-n} = \overline{B_{\mathfrak{s}}^n}$ and $B_{\mathfrak{s}}^n, B_{\mathfrak{s}}^m$ are independent for $n \neq \pm m$. We let $B_{\mathfrak{s}}^0$ be a standard real-valued Brownian motion independent of $(B^n_{\mathfrak{s}})_{n\in\mathbb{Z}^3\setminus\{0\}}$. Furthermore, we let $B_{\mathfrak{s}}(\cdot)$ be the Gaussian process with Fourier coefficients $(B^n_{\mathfrak{s}})_{n\in\mathbb{Z}^3}$, i.e.,

$$B_{\mathfrak{s}}(x) \overset{\text{def}}{=} \sum_{n\in\mathbb{Z}^3} e^{i\langle n,x\rangle} B^n_{\mathfrak{s}}.$$

For every $\mathfrak{s} \geqslant 0$, the Gaussian process formally satisfies $\mathbb{E}[B_{\mathfrak{s}}(x)B_{\mathfrak{s}}(y)] = \mathfrak{s}\cdot\delta(x-y)$ and hence $B_{\mathfrak{s}}(\cdot)$ is a scalar multiple of spatial white noise. We also let $(\mathcal{F}_{\mathfrak{s}})_{\mathfrak{s}\geqslant 0}$ be the filtration corresponding to the family of Gaussian processes $(B^n_{\mathfrak{s}})_{\mathfrak{s}\geqslant 0}$.

The Gaussian free field g, however, has covariance $(1-\Delta)^{-1}$. To this end, we now introduce the Gaussian process $W_{\mathfrak{s}}(x)$. We let $\sigma_{\mathfrak{s}}(\xi) = \big(\frac{\mathrm{d}}{\mathrm{d}\mathfrak{s}}\rho^2_{\mathfrak{s}}(\xi)\big)^{1/2}$, where $\rho_{\mathfrak{s}}$ is the frequency-truncation from Section 4.1.3. For any $n\in\mathbb{Z}^3$, we then define

$$W^n_{\mathfrak{s}} \overset{\text{def}}{=} \int_0^{\mathfrak{s}} \frac{\sigma_{\mathfrak{s}'}(n)}{\langle n\rangle}\,\mathrm{d}B^n_{\mathfrak{s}'}\,. \tag{4.4.53}$$

We note that $W^n_{\mathfrak{s}}$ is a complex Gaussian random variable with variance $\rho^2_{\mathfrak{s}}(n)/\langle n\rangle^2$. We finally set

$$W_{\mathfrak{s}}(x) \overset{\text{def}}{=} \sum_{n\in\mathbb{Z}^3} e^{i\langle n,x\rangle} W^n_{\mathfrak{s}}. \tag{4.4.54}$$

Since the Gaussian random data $\bullet \in \mathscr{H}^{-1/2-\kappa}_x(\mathbb{T}^3)$ in Theorem 4.1.1 is a tuple of the initial data and initial velocity, we now let $(B^{\cos}, W^{\cos})$ and $(B^{\sin}, W^{\sin})$ be two independent copies of (B, W). Using this notation, we then take

$$\bullet = \Big(W^{\cos}_\infty(x), \langle\nabla\rangle W^{\sin}_\infty(x)\Big)\,. \tag{4.4.55}$$

Using (4.4.55), we can represent the linear evolution as

$$\dot{|}(t) = \cos(t\langle\nabla\rangle)W^{\cos}_\infty + \sin(t\langle\nabla\rangle)W^{\sin}_\infty,$$

which also motivates our notation.

4.4.6 Multiple stochastic integrals

In this section, we recall several definitions and results related to multiple stochastic integrals. A similar but shorter section already appeared in the appendix of the first paper of this series [Bri20c]. A more detailed introduction can be found in the excellent textbook [Nua06]. The usefulness of this section is best illustrated by Proposition 4.4.44 below.

We define a Borel measure λ on $\mathbb{R}_{\geqslant 0} \times \mathbb{Z}^3$ by

$$\mathrm{d}\lambda(\mathfrak{s}, n) = \frac{\sigma_{\mathfrak{s}}^2(n)}{\langle n \rangle^2} \mathrm{d}\mathfrak{s}\mathrm{d}n,$$

where $\mathrm{d}\mathfrak{s}$ is the Lebesgue measure and $\mathrm{d}n$ is the counting measure on $\mathbb{Z}^3$. We define the corresponding inner product by

$$\langle f, g \rangle = \sum_{n \in \mathbb{Z}^3} \int_0^\infty f(\mathfrak{s}, n) \overline{g(\mathfrak{s}, n)} \frac{\sigma_{\mathfrak{s}}^2(n)}{\langle n \rangle^2} \mathrm{d}\mathfrak{s}. \tag{4.4.56}$$

For any $f \in L^2(\mathbb{R}_{\geqslant 0} \times \mathbb{Z}^3, \mathrm{d}\lambda)$, we define

$$W[f] = \sum_{n \in \mathbb{Z}^3} \int_0^\infty f(\mathfrak{s}, n) \mathrm{d}W_{\mathfrak{s}}^n.$$

The inner integral can be understood as an Itô-integral. Then, we can identify W with the family of complex-valued Gaussian random variables

$$W = \{W[f] \colon f \in L^2(\mathbb{R}_{\geqslant 0} \times \mathbb{Z}^3, \mathrm{d}\lambda)\}.$$

For any $f \in L^2(\mathbb{R}_{\geqslant 0} \times \mathbb{Z}^3, \mathrm{d}\lambda)$, we define the reflection operator $\mathcal{R}$ by

$$\mathcal{R}f(\mathfrak{s}, n) \stackrel{\text{def}}{=} \overline{f(\mathfrak{s}, -n)}.$$

Clearly, $\mathcal{R}$ is a real-linear isometry. Using Itô's isometry, a short calculation yields that

$$\mathbb{E}\big[W[f]\overline{W[g]}\big] = \langle f, g \rangle \quad \text{and} \quad \mathbb{E}\big[W[f]W[g]\big] = \langle f, \mathcal{R}g \rangle.$$

Since this will be important below, we note that the second identity reads

$$\mathbb{E}\big[W[f]W[g]\big] = \sum_{n\in\mathbb{Z}^3} \int_0^\infty f(\mathfrak{s}, n) g(\mathfrak{s}, -n) \frac{\sigma_{\mathfrak{s}}^2(n)}{\langle n \rangle^2} \mathrm{d}\mathfrak{s}. \tag{4.4.57}$$

To emphasize the integral character of $W[f]$, we now write

$$\mathcal{I}_1[f] \stackrel{\text{def}}{=} W[f].$$

In this notation, it becomes evident that we have been working with single-variable stochastic calculus. In order to express the resonances in our stochastic objects, it is more natural to work with multi-variable stochastic calculus. For $k \geqslant 1$, we define the measure λ_k on $(\mathbb{R}_{\geqslant 0} \times \mathbb{Z}^3)^k$ by

$$\lambda_k \stackrel{\text{def}}{=} \lambda \otimes \ldots \otimes \lambda.$$

To simplify the notation, we set $\mathcal{H}_k \stackrel{\text{def}}{=} L^2((\mathbb{R} \times \mathbb{Z}^3)^k, \mathrm{d}\lambda_k)$. For any $f \in \mathcal{H}_k$, the multiple stochastic integral $\mathcal{I}_k[f]$ can then be constructed as in [Nua06, Section 1.1.2]. We only recall the basic ingredients and refer to [Nua06] for more details.

We denote by $\mathcal{E}_k$ the set of elementary functions of the form

$$f(\mathfrak{s}_1, n_1, \ldots, \mathfrak{s}_k, n_k) = \sum_{l_1,\ldots,l_k=1}^{L} a_{l_1,\ldots,l_k} 1_{A_{l_1}\times\ldots\times A_{l_k}}(\mathfrak{s}_1, n_1, \ldots, \mathfrak{s}_k, n_k),$$

where $A_1, \ldots, A_L$ are pairwise disjoint sets with finite measure under λ_k and $a_{l_1,\ldots,l_k}$ vanishes if the indices $l_1, \ldots, l_k$ are not pairwise distinct. For an elementary function, we define the multiple stochastic integral by

$$\mathcal{I}_k[f] \stackrel{\text{def}}{=} \sum_{l_1,\ldots,l_k=1}^{L} a_{l_1,\ldots,l_k} \prod_{j=1}^{k} W[A_{l_j}]. \tag{4.4.58}$$

Furthermore, we define the symmetrization of f by

$$\widetilde{f}(\mathfrak{s}_1, n_1, \ldots, \mathfrak{s}_k, n_k) = \frac{1}{k!} \sum_{\pi\in S_k} f\big(\mathfrak{s}_{\pi(1)}, n_{\pi(1)}, \ldots, \mathfrak{s}_{\pi(k)}, n_{\pi(k)}\big). \tag{4.4.59}$$

Lemma 4.4.37 (Basic properties)**.** For any $k, l \geqslant 1$, $f \in \mathcal{E}_k$, and $g \in \mathcal{E}_l$, it holds that:

(i) $\mathcal{I}_k$ is linear.

(ii) The integral is invariant under symmetrization, i.e., $\mathcal{I}_k[f] = \mathcal{I}_k[\widetilde{f}]$.

(iii) We have the Itô-isometry formula

$$\mathbb{E}\big[\mathcal{I}_k[f] \cdot \overline{\mathcal{I}_l[g]}\big] = \delta_{kl} k! \int \widetilde{f}\, \overline{\widetilde{g}} \,\mathrm{d}\lambda_k.$$

(iv) We have the formula for the expectation

$$\begin{aligned}&\mathbb{E}\big[\mathcal{I}_k[f] \cdot \mathcal{I}_l[g]\big]\\ &= \delta_{kl} k! \sum_{n_1,\ldots,n_k} \int_0^\infty \ldots \int_0^\infty \widetilde{f}(s_1, n_1, \ldots, s_k, n_k) \cdot \widetilde{g}(s_1, -n_1, \ldots, s_k, -n_k) \Big(\prod_{j=1}^k \frac{\sigma_{s_j}^2(n_j)}{\langle n_j \rangle^2} \Big) \mathrm{d}s_k \ldots \mathrm{d}s_1.\end{aligned}$$

Proof. Except for a minor extension from the real-valued to the complex-valued setting, the proof can be found on [Nua06, p.9]. □

Using the density argument from [Nua06, p.10], we can extend $\mathcal{I}_k$ from elementary functions to $\mathcal{H}_k$. In particular, for any fixed $m_1, \ldots, m_k \in \mathbb{Z}^3$, we have that

$$\prod_{j=1}^k \delta_{n_j = m_j} \in \mathcal{H}_k$$

and we can write

$$\int_{[0,\infty)^k} \mathrm{d}W_{s_k}^{m_k} \ldots \mathrm{d}W_{s_1}^{m_1} \stackrel{\text{def}}{=} \mathcal{I}_k \Big[\prod_{j=1}^k \delta_{n_j = m_j} \Big]. \tag{4.4.60}$$

We vehemently emphasize that the stochastic integral (4.4.60) does not coincide with the product $\prod_{j=1}^k W_\infty^{m_j}$. Instead, as will be clear from the product formula (Lemma 4.4.40) below, the stochastic integral (4.4.60) only contains the non-resonant portion of this product.

If $f = f(n_1, \ldots, n_k)$ does not depend on the stochastic-time variables $\mathfrak{s}_1, \ldots, \mathfrak{s}_k$, the linearity of the multiple stochastic integral $\mathcal{I}_k$ and (4.4.60) naturally imply that

$$\mathcal{I}_k[f] = \sum_{n_1,\ldots,n_k \in \mathbb{Z}^3} f(n_1, \ldots, n_k) \int_{[0,\infty)^k} \mathrm{d}W_{\mathfrak{s}_k}^{n_k} \ldots \mathrm{d}W_{\mathfrak{s}_1}^{n_1}. \tag{4.4.61}$$

Using (iii) in Lemma 4.4.37, it follows that

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_{[0,\infty)^m} \mathrm{d}W_{\mathfrak{s}_k}^{n_k} \ldots \mathrm{d}W_{\mathfrak{s}_1}^{n_1}\Big) \cdot \overline{\Big(\int_{[0,\infty)^k} \mathrm{d}W_{\mathfrak{s}_k}^{m_k} \ldots \mathrm{d}W_{\mathfrak{s}_1}^{m_1}\Big)}\Big] \\
&= 1\big\{n \text{ is a permutation of } m\big\} \int_{[0,\infty)^m} \prod_{j=1}^{m} \frac{\sigma_{\mathfrak{s}_j}^2(n_j)}{\langle n_j \rangle^2} \mathrm{d}\mathfrak{s}_k \ldots \mathrm{d}\mathfrak{s}_1 \\
&= 1\big\{n \text{ is a permutation of } m\big\} \prod_{j=1}^{m} \langle n_j \rangle^{-2}.
\end{aligned}$$

Up to permutations, the family of multiple stochastic integrals (4.4.60) is therefore orthogonal. Naturally, a similar formula holds without the complex conjugate. More generally, if f depends on the stochastic time-variables $\mathfrak{s}_1, \ldots, \mathfrak{s}_k$, we have that

$$\mathcal{I}_k[f] = \sum_{n_1,\ldots,n_k \in \mathbb{Z}^3} \int_{[0,\infty)^k} f(\mathfrak{s}_1, n_1, \ldots, \mathfrak{s}_k, n_k) \mathrm{d}W_{\mathfrak{s}_k}^{n_k} \ldots \mathrm{d}W_{\mathfrak{s}_1}^{n_1}. \tag{4.4.62}$$

Here, the summands on the right-hand side are understood as multiple stochastic integrals with fixed $n_1, \ldots, n_k$ (by inserting an indicator as in (4.4.60)). As is shown in the next lemma, this notation is consistent with iterated Itô-integrals.

Lemma 4.4.38. Let $k \geqslant 1$ and let $f \in \mathcal{H}_k$ be symmetric. Then, it holds that

$$\mathcal{I}_k[f] = k! \sum_{n_1,\ldots,n_k \in \mathbb{Z}^3} \int_0^\infty \int_0^{\mathfrak{s}_1} \ldots \int_0^{\mathfrak{s}_{k-1}} f(\mathfrak{s}_1, n_1, \ldots, \mathfrak{s}_k, n_k) \mathrm{d}W_{\mathfrak{s}_k}^{n_k} \ldots \mathrm{d}W_{\mathfrak{s}_1}^{n_1}, \tag{4.4.63}$$

where the right-hand side is understood as an iterated Itô integral.

This follows from the discussion of [Nua06, (1.27)]. As a consequence of this lemma, we could also work with iterated Itô-integrals instead of multiple stochastic integrals. While the iterated

Itô-integrals are more natural whenever martingale properties are utilized, the multiple stochastic integrals have a much simpler product formula, which simplifies many of our computations.

Before we can state the product formula, we need to define the contraction.

Definition 4.4.39 (Contraction)**.** Let $k, l \geqslant 1$, let $f \in \mathcal{H}_k$, and let $g \in \mathcal{H}_l$. For any $0 \leqslant r \leqslant \min(k, l)$, we define the contraction of r indices by

$$
\begin{aligned}
&(f \otimes_r g)(\mathfrak{s}_1, n_1, \ldots, \mathfrak{s}_{k+l-2r}, n_{k+l-2r}) \\
&\overset{\text{def}}{=} \sum_{m_1,\ldots,m_r \in \mathbb{Z}^3} \int_0^\infty \cdots \int_0^\infty \Bigg[f(\mathfrak{s}_1, n_1, \ldots, \mathfrak{s}_{k-r}, n_{k-r}, \mathfrak{r}_1, m_1, \ldots, \mathfrak{r}_r, m_r) \\
&\quad \times g(\mathfrak{s}_{k+1-r}, n_{k+1-r}, \ldots, \mathfrak{s}_{k+l-2r}, n_{k+l-2r}, \mathfrak{r}_1, -m_1, \ldots, \mathfrak{r}_r, -m_r) \prod_{j=1}^{k} \frac{\sigma_{\mathfrak{r}_j}^2(m_j)}{\langle m_j \rangle^2} \Bigg] \mathrm{d}\mathfrak{r}_r \ldots \mathrm{d}\mathfrak{r}_1.
\end{aligned}
$$

We note that even if $f \in \mathcal{H}_k$ and let $g \in \mathcal{H}_l$ are both symmetric, the contraction $f \otimes_r g$ may not be symmetric. The reader should note the similarity of the contraction with the formula for the expectation in (iv) of Lemma 4.4.37, which is no coincidence. If $f, g \in \mathscr{H}_1$, then

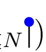
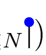
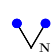
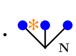
$$
\mathbb{E}\Big[\mathcal{I}_1[f] \cdot \mathcal{I}_1[g] \Big] = f \otimes_1 g. \tag{4.4.64}
$$

Thus, $f \otimes_1 g$ describes the (full) resonance portion of the product $\mathcal{I}_1[f] \cdot \mathcal{I}_1[g]$. The product formula is a (major) generalization of this simple fact.

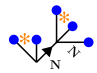
Lemma 4.4.40 (Product formula for multiple stochastic integrals (cf. [Nua06, Prop 1.1.3]))**.** Let $k, l \geqslant 1$ and let $f \in \mathcal{H}_k$ and $g \in \mathcal{H}_l$ be symmetric. Then, it holds that

$$
\mathcal{I}_k[f] \cdot \mathcal{I}_l[g] = \sum_{r=0}^{\min(k,l)} r! \binom{k}{r} \binom{l}{r} \mathcal{I}_{k+l-2r}[f \otimes_r g]. \tag{4.4.65}
$$

Using the product formula (Lemma 4.4.40), we can compute the non-resonant, partially resonant, and fully resonant portions of products such as

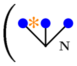
$(P_{\leqslant N}$  $)(t,x) \cdot (P_{\leqslant N}$  $)(t,x)$ and  $(t,x) \cdot$  (t,x).

Once the Duhamel operator occurs in the expression, however, we also need to consider two different physical times t and t'. For instance, in our estimate of the quintic stochastic object

,

we need to control

$$\Big(V * \text{ } (t,x)\Big) \cdot \Big(P_{\leqslant N} \sin((t-t')\langle\nabla\rangle)\langle\nabla\rangle^{-1}\Big)\Big(\text{ } (t',x)\Big)$$

In order to consider two different physical times t and t', we need to consider multiple stochastic integrals with respect to two different (correlated) Gaussian processes, which we abstractly denote by W^a and W^b. We will assume that $\mathrm{Law}_{\mathbb{P}}(W^a) = \mathrm{Law}_{\mathbb{P}}(W^b) = \mathrm{Law}_{\mathbb{P}}(W)$. Regarding the relationship between the different Gaussian processes W^a and W^b, we assume that $W^{a,n}$ and $W^{b,m}$ are independent for $m \neq \pm n$. Furthermore, let $\mathfrak{C}\colon \mathbb{Z}^3 \to [-1,1]$ be an even function. We assume that

$$\mathbb{E}\Big[W^{(a),n}_{\mathfrak{s}_1} W^{(b),m}_{\mathfrak{s}_2}\Big] = \delta_{n=-m}\, \mathfrak{C}(n) \int_0^{\mathfrak{s}_1 \wedge \mathfrak{s}_2} \frac{\sigma^2_{\mathfrak{s}}(n)}{\langle n \rangle^2} \mathrm{d}\mathfrak{s} \tag{4.4.66}$$

and

$$\mathbb{E}\Big[W^{(a),n}_{\mathfrak{s}_1} \overline{W^{(b),m}_{\mathfrak{s}_2}}\Big] = \delta_{n=m}\, \mathfrak{C}(n) \int_0^{\mathfrak{s}_1 \wedge \mathfrak{s}_2} \frac{\sigma^2_{\mathfrak{s}}(n)}{\langle n \rangle^2} \mathrm{d}\mathfrak{s} \tag{4.4.67}$$

Thus, $\mathfrak{C}$ is the (appropriately normalized) correlation of W^a and W^b. We can then set up the theory of multiple stochastic integrals with respect to a mixture of W^a and W^b as before. In order to fit this theory into the same framework as in [Nua06], one only has to replace $\mathbb{R}\times\mathbb{Z}^3$ by $\mathbb{R}\times\mathbb{Z}^3\times\{a,b\}$. A short calculation shows for any bounded and compactly supported $f,g\colon \mathbb{R}\times\mathbb{Z}^3\times\{a,b\} \to \mathbb{C}$ that

$$\begin{aligned} &\mathbb{E}\Big[\Big(\sum_{\iota=a,b}\sum_{n\in\mathbb{Z}^3}\int_0^\infty f(\mathfrak{s},n,\iota)\mathrm{d}W^{(\iota),n}_{\mathfrak{s}}\Big)\Big(\sum_{\iota=a,b}\sum_{n\in\mathbb{Z}^3}\int_0^\infty g(\mathfrak{s},n,\iota)\mathrm{d}W^{(\iota),n}_{\mathfrak{s}}\Big)\Big] \\ &= \sum_{\iota,\iota'=a,b}\sum_{n\in\mathbb{Z}^3}\Big(1\big\{\iota=\iota'\big\} + \mathfrak{C}(n)1\big\{\iota\neq\iota'\big\}\Big)\int_0^\infty f(\mathfrak{s},n,\iota)\cdot g(\mathfrak{s},-n,\iota')\frac{\sigma^2_{\mathfrak{s}}(n)}{\langle n\rangle^2}\mathrm{d}\mathfrak{s}. \end{aligned} \tag{4.4.68}$$

and

$$\begin{aligned}&\mathbb{E}\Big[\Big(\sum_{\iota=a,b}\sum_{n\in\mathbb{Z}^3}\int_0^\infty f(\mathfrak{s},n,\iota)\mathrm{d}W_{\mathfrak{s}}^{(\iota),n}\Big)\overline{\Big(\sum_{\iota=a,b}\sum_{n\in\mathbb{Z}^3}\int_0^\infty g(\mathfrak{s},n,\iota)\mathrm{d}W_{\mathfrak{s}}^{(\iota),n}\Big)}\Big]\\&=\sum_{\iota,\iota'=a,b}\sum_{n\in\mathbb{Z}^3}\Big(1\{\iota=\iota'\}+\mathfrak{C}(n)1\{\iota\neq\iota'\}\Big)\int_0^\infty f(\mathfrak{s},n,\iota)\cdot\overline{g(\mathfrak{s},n,\iota')}\frac{\sigma_{\mathfrak{s}}^2(n)}{\langle n\rangle^2}\mathrm{d}\mathfrak{s}.\end{aligned}\tag{4.4.69}$$

The sesquilinear form in (4.4.69), viewed as a function in f and g, is no longer positive definite. For instance, if $W^{(a)}=-W^{(b)}$, and hence $\mathfrak{C}=-1$, $f=g$, and $f(\mathfrak{s},n,a)=f(\mathfrak{s},n,b)$ for all $\mathfrak{s}\in\mathbb{R}_{\geqslant 0}$ and $n\in\mathbb{Z}^3$, it vanishes identically. Nevertheless, due to the condition $|\mathfrak{C}|\leqslant 1$ imposed on the correlation function $\mathfrak{C}$, it is bounded by (a scalar multiple of) the inner product

$$\sum_{\iota=a,b}\sum_{n\in\mathbb{Z}^3}\int_0^\infty f(\mathfrak{s},n,\iota)\cdot\overline{g(\mathfrak{s},n,\iota)}\frac{\sigma_{\mathfrak{s}}^2(n)}{\langle n\rangle^2}\mathrm{d}\mathfrak{s}.$$

After defining a measure $\widetilde{\lambda}$ on $\mathbb{R}\times\mathbb{Z}^3\times\{a,b\}$ by $\mathrm{d}\widetilde{\lambda}=\mathrm{d}\lambda\mathrm{d}\iota$, where $\mathrm{d}\iota$ is the integration with respect to the counting measure on $\{a,b\}$, this allows us to construct multiple stochastic integrals for functions in

$$L^2((\mathbb{R}\times\mathbb{Z}^3\times\{a,b\})^k,\widetilde{\lambda}_k).$$

Similar as in (4.4.60), this allows us to define mixed multiple stochastic integrals such as

$$\int_{[0,\infty)^3}\mathrm{d}W_{\mathfrak{s}_3}^{(a),n_3}\mathrm{d}W_{\mathfrak{s}_2}^{(a),n_2}\mathrm{d}W_{\mathfrak{s}_1}^{(b),n_1}.\tag{4.4.70}$$

Unfortunately, the general theory now becomes notationally cumbersome. We therefore decided to only state the much simpler special case of the product formula needed in this paper.

Lemma 4.4.41 (Quadratic-Cubic product formula)**.** Let $f\colon(\mathbb{Z}^3)^2\to\mathbb{C}$ and let $g\colon(\mathbb{Z}^3)^3\to\mathbb{C}$. We assume that g is symmetric but do not require any symmetry of f. Then, it holds that

$$\begin{aligned}
&\Big(\sum_{n_1,n_2\in\mathbb{Z}^3} f(n_1,n_2) \int_{[0,\infty)^2} \mathrm{d}W^{(a),n_2}_{s_2} \mathrm{d}W^{(a),n_1}_{s_1} \Big) \times \Big(\sum_{n_3,n_4,n_5\in\mathbb{Z}^3} g(n_3,n_4,n_5) \int_{[0,\infty)^3} \mathrm{d}W^{(b),n_5}_{s_5} \mathrm{d}W^{(b),n_4}_{s_4} \mathrm{d}W^{(b),n_3}_{s_3} \Big) \\
&= \sum_{n_1,n_2,n_3,n_4,n_5\in\mathbb{Z}^3} f(n_1,n_2) g(n_3,n_4,n_5) \int_{[0,\infty)^5} \mathrm{d}W^{(b),n_5}_{s_5} \mathrm{d}W^{(b),n_4}_{s_4} \mathrm{d}W^{(b),n_3}_{s_3} \mathrm{d}W^{(a),n_2}_{s_2} \mathrm{d}W^{(a),n_1}_{s_1} \\
&+ 3 \sum_{n_2,n_4,n_5\in\mathbb{Z}^3} \Big(\sum_{n_1\in\mathbb{Z}^3} f(n_1,n_2) g(-n_1,n_4,n_5) \frac{\mathfrak{C}(n_1)}{\langle n_1\rangle^2} \Big) \int_{[0,\infty)^3} \mathrm{d}W^{(b),n_5}_{s_5} \mathrm{d}W^{(b),n_4}_{s_4} \mathrm{d}W^{(a),n_2}_{s_2} \\
&+ 3 \sum_{n_1,n_4,n_5\in\mathbb{Z}^3} \Big(\sum_{n_2\in\mathbb{Z}^3} f(n_1,n_2) g(-n_2,n_4,n_5) \frac{\mathfrak{C}(n_2)}{\langle n_2\rangle^2} \Big) \int_{[0,\infty)^3} \mathrm{d}W^{(b),n_5}_{s_5} \mathrm{d}W^{(b),n_4}_{s_4} \mathrm{d}W^{(a),n_1}_{s_1} \\
&+ 6 \sum_{n_5\in\mathbb{Z}^3} \Big(\sum_{n_1,n_2\in\mathbb{Z}^3} f(n_1,n_2) g(-n_1,-n_2,n_5) \frac{\mathfrak{C}(n_1)\mathfrak{C}(n_2)}{\langle n_1\rangle^2 \langle n_2\rangle^2} \Big) \int_0^\infty \mathrm{d}W^{(b),n_5}_{s_5}.
\end{aligned}$$

Remark 4.4.42. Instead of working with the product $f(n_1,n_2)g(n_3,n_4,n_5)$, the formula has a natural extension to functions $h(n_1,\ldots,n_5)$ which are symmetric in n_3, n_4, and n_5. To this end, one only has to decompose

$$h(n_1,n_2,n_3,n_4,n_5) = \sum_{m_1,m_2\in\mathbb{Z}^3} 1\big\{(n_1,n_2)=(m_1,m_2)\big\} \cdot h(m_1,m_2,n_3,n_4,n_5).$$

We can then apply Lemma 4.4.41 to the individual summands.

Remark 4.4.43. While the formula in Lemma 4.4.41 is complicated, it is still an order of magnitude easier than working with products of Gaussians directly. If the reader is not convinced, we encourage him to work out (by hand) the corresponding resonant/non-resonant decomposition of

$$\begin{aligned}
&\Big(\sum_{n_1,n_2\in\mathbb{Z}^3} f(n_1,n_2) \Big(G^{(a)}_{n_1} \cdot G^{(a)}_{n_2} - \frac{\delta_{n_{12}=0}}{\langle n_1\rangle^2} \Big) \Big) \Big) \\
&\times \Big(\sum_{n_3,n_4,n_5\in\mathbb{Z}^3} g(n_3,n_4,n_5) \Big(G^{(b)}_{n_3} \cdot G^{(b)}_{n_4} \cdot G^{(b)}_{n_5} - \frac{\delta_{n_{34}=0}}{\langle n_3\rangle^2} G^{(b)}_{n_5} - \frac{\delta_{n_{35}=0}}{\langle n_3\rangle^2} G^{(b)}_{n_4} - \frac{\delta_{n_{45}=0}}{\langle n_4\rangle^2} G^{(b)}_{n_3} \Big) \Big),
\end{aligned}$$

where $G^{(\iota)} = W^{(\iota)}_\infty$ for $\iota = a, b$ are (correlated) families of Gaussian random variables.

After establishing the important definitions and properties of multiple stochastic integrals, it only remains to connect them with our stochastic objects. Let $W_{\mathfrak{s}}^{(\cos),n}$ and $W_{\mathfrak{s}}^{(\sin),n}$ be the Gaussian processes defined in Section 4.4.5. We recall that the linear evolution of the random initial data $\bullet$ is given by

$$\begin{aligned}\overset{\bullet}{|}(t) &= \sum_{n\in\mathbb{Z}^3} \big(\cos(t\langle n\rangle) W_{\infty}^{(\cos),n} + \sin(t\langle n\rangle) W_{\infty}^{(\sin),n} \big) \exp(i\langle n, x\rangle) \\ &= \sum_{n\in\mathbb{Z}^3} \Big(\int_0^\infty \mathrm{d}\big(\cos(t\langle n\rangle) W_{\mathfrak{s}}^{(\cos),n} + \sin(t\langle n\rangle) W_{\mathfrak{s}}^{(\sin),n} \big)\Big) \exp(i\langle n, x\rangle).\end{aligned} \tag{4.4.71}$$

In order to obtain a similar expression for the stochastic objects $\overset{\bullet\ \bullet}{\vee}_{N}$ and $\overset{\bullet\,*\,\bullet\ \bullet}{\vee}_{N}$, we define for any $k \geqslant 1$ and $n_1, \ldots, n_k \in \mathbb{Z}^3$ the multiple stochastic integral

$$\begin{aligned}\mathcal{I}_k[t, n_1, \ldots, n_k] \overset{\text{def}}{=} \int_{[0,\infty)^k} &\mathrm{d}\big(\cos(t\langle n_k\rangle) W_{\mathfrak{s}_k}^{(\cos),n_k} + \sin(t\langle n\rangle) W_{\mathfrak{s}_k}^{(\sin),n_k} \big) \ldots \\ &\mathrm{d}\big(\cos(t\langle n_1\rangle) W_{\mathfrak{s}_1}^{(\cos),n_1} + \sin(t\langle n_1\rangle) W_{\mathfrak{s}_1}^{(\sin),n_1} \big).\end{aligned} \tag{4.4.72}$$

In the proof of multi-linear dispersive estimates, it is essential to separate the time-variable t from the randomness. To this end, we define the Gaussian processes

$$W_{\mathfrak{s}}^{(\pm),n} \overset{\text{def}}{=} W_{\mathfrak{s}}^{(\cos),n} \pm W_{\mathfrak{s}}^{(\sin),n}. \tag{4.4.73}$$

Similar as in (4.4.72), we define for any $k \geqslant 1$, any $\pm_1, \ldots, \pm_k \in \{+, -\}$, and any $n_1, \ldots, n_k \in \mathbb{Z}^3$ the multiple stochastic integral

$$\mathcal{I}_k[n_j; \pm_j \colon 1 \leqslant j \leqslant k] \overset{\text{def}}{=} \int_{[0,\infty)^k} \mathrm{d}W_{\mathfrak{s}_k}^{(\pm_k),n_k} \ldots \mathrm{d}W_{\mathfrak{s}_1}^{(\pm_1),n_1}. \tag{4.4.74}$$

It then follows that there exists coefficients $c \colon \{+,-\}^k \to \mathbb{C}$ depending only on the signs such that

$$\mathcal{I}_k[t, n_1, \ldots, n_k] = \sum_{\pm_1, \ldots, \pm_k} c(\pm_1, \ldots, \pm_k) \Big(\prod_{j=1}^k \exp(\pm_j it\langle n_j\rangle) \Big) \mathcal{I}_k[n_j; \pm_j \colon 1 \leqslant j \leqslant k]. \tag{4.4.75}$$

For convenience, we also define the normalized multiple stochastic integrals by

$$\widetilde{\mathcal{I}}_k[n_j; \pm_j \colon 1 \leqslant j \leqslant k] = \Big(\prod_{j=1}^{k} \langle n_j \rangle \Big) \cdot \mathcal{I}_k[n_j; \pm_j \colon 1 \leqslant j \leqslant k] \tag{4.4.76}$$

We close this subsection with the following stochastic representation, which expresses the quadratic and cubic stochastic objects through multiple stochastic integrals.

Proposition 4.4.44. Let $t \in \mathbb{R}$ and $N \geqslant 1$. Then, we have for all $n_1, n_2 \in \mathbb{Z}^3$ that

$$\widehat{\,\bullet\!|\,}(t, n_1) \cdot \widehat{\,\bullet\!|\,}(t, n_2) - \frac{1}{\langle n_{12} \rangle^2} \delta_{n_{12}=0} = \mathcal{I}_2[t, n_1, n_2]. \tag{4.4.77}$$

Furthermore, it holds that

$$\vee_{N}(t, x) = \sum_{n_1, n_2 \in \mathbb{Z}^3} \Big(\prod_{j=1}^{2} \rho_N(n_j) \Big) \mathcal{I}_2[t, n_1, n_2], \tag{4.4.78}$$

$$\vee\!\!\vee_{N}(t, x) = \sum_{n_1, n_2, n_3 \in \mathbb{Z}^3} \Big(\prod_{j=1}^{3} \rho_N(n_j) \Big) \widehat{V}(n_{12})\, \mathcal{I}_3[t, n_1, n_2, n_3]. \tag{4.4.79}$$

Proof. This follows from [Bri20c, Lemma 2.5 and Proposition 2.9], Lemma 4.4.38, and that the distribution of

$$(\mathfrak{s}, n) \mapsto \cos(t\langle n \rangle) W_{\mathfrak{s}}^{(\cos), n} + \sin(t\langle n \rangle) W_{\mathfrak{s}}^{(\sin), n}$$

is the same for all $t \in \mathbb{R}$. □

4.4.7 Gaussian hypercontractivity and the moment method

In this section, we first review Gaussian hypercontractivity and its consequences. To help the reader with a primary background in dispersive equations, let us first illustrate this phenomenon through a basic example. Let Z_σ be a Gaussian random variable with mean zero and variance σ^2. Using the exact formula for the moments of a Gaussian, we have for all $m \geqslant 1$ that

$$\mathbb{E}\big[Z_\sigma^2\big] = \sigma^2 \qquad \text{and} \qquad \mathbb{E}\big[Z_\sigma^{2m}\big] = \frac{(2m)!}{2^m m!} \cdot \sigma^{2m}.$$

A simple estimate now yields that

$$\Big(\mathbb{E}\big[Z_\sigma^{2m}\big]\Big)^{\frac{1}{2m}} \leqslant \Big(\frac{(2m)^{2m}}{2^m (m/e)^m}\Big)^{\frac{1}{2m}} \cdot \sigma = \sqrt{2em}\Big(\mathbb{E}\big[Z_\sigma^2\big]\Big)^{\frac{1}{2}}.$$

Using Hölder's inequality, we obtain for all $p \geqslant 2$ that

$$\|Z_\sigma\|_{L^p_\omega} \lesssim \sqrt{p}\|Z_\sigma\|_{L^2_\omega}. \tag{4.4.80}$$

Thus, higher L^p_ω-norms of Gaussians can be controlled through the lower L^2_ω-norm. The *"hyper"* in Gaussian hypercontractivity refers exactly to this gain of integrability. While (4.4.80) is not too interesting by itself, its significance lies in its generalizations to polynomials in infinitely many Gaussians! Furthermore, Gaussian hypercontractivity has connections to many different inequalities in analysis and probability theory, such as logarithmic Sobolev inequalities.

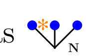
Our first proposition is also known as a Wiener chaos estimate. A version of this proposition can be found in [Sim74, Theorem I.22] or [Nua06, Theorem 1.4.1].

Proposition 4.4.45 (Gaussian hypercontractivity)**.** Let $k \geqslant 1$, let $\pm_1, \ldots, \pm_k \in \{+,-\}$, and let $a\colon (\mathbb{Z}^3)^k \to \mathbb{C}$ be a discrete function with finite support. Define the k-th order Gaussian chaos $\mathscr{G}_k$ by

$$\mathscr{G}_k \overset{\text{def}}{=} \sum_{n_1,\ldots,n_k \in \mathbb{Z}^3} a(n_1,\ldots,n_k)\mathcal{I}_k[\pm_j, n_j \colon 1 \leqslant j \leqslant k]. \tag{4.4.81}$$

Then, it holds for all $p \geqslant 2$ that

$$\|\mathscr{G}_k\|_{L^p_\omega(\mathbb{P})} \lesssim p^{\frac{k}{2}} \|\mathscr{G}_k\|_{L^2_\omega(\mathbb{P})}. \tag{4.4.82}$$

Proposition 4.4.45 will play an important role in the estimates of stochastic objects such as .
While Proposition 4.4.45 bounds the moments of the Gaussian chaos, the reader may prefer or be more familiar with a bound on probabilistic tails. As the next lemma shows, the two viewpoints are equivalent.

Lemma 4.4.46 (Moments and tails)**.** Let Z be a random variable and let $\gamma > 0$. Then, the following properties are equivalent, where the parameter $K_1, K_2 > 0$ appearing below differ from each other by at most a constant factor depending only on γ.

(i) The tails of Z satisfy for all $\lambda \geqslant 0$ the inequality

$$\mathbb{P}(|Z| \geqslant \lambda) \leqslant 2 \exp\big(- (\lambda/K_1)^\gamma \big).$$

(ii) The moments of Z satisfy for all $p \geqslant 2$ the inequality

$$\|Z\|_{L^p} \leqslant K_2 p^{\frac{1}{\gamma}}.$$

The lemma is an easy generalization of [Ver18, Proposition 2.5.2 or Proposition 2.7.1]. As we have seen above, a Gaussian random variable corresponds to $\gamma = 2$. It is convenient to capture the size of K_2 in Lemma 4.4.46 (and hence K_1) through a norm.

Definition 4.4.47. Let $\gamma > 0$ and let Z be a random variable. We define

$$\|Z\|_{\Psi_\gamma} = \sup_{p \geqslant 2} p^{-\frac{1}{\gamma}} \|Z\|_{L^p_\omega}.$$

For more information regarding the Ψ_γ-norms, we refer the reader to the excellent textbook [Ver18]. The next lemma shows that the Ψ_γ-norm is well-behaved under taking maxima of several random variables.

Lemma 4.4.48 (Maxima and the Ψ_γ-norm)**.** Let $\gamma > 0$, let $J \in \mathbb{N}$, and let $Z_1, \ldots, Z_J$ be random variables on the same probability space. Then, it holds that

$$\| \max(Z_1, \ldots, Z_J) \|_{\Psi_\gamma} \leqslant e \log(2 + J)^{\frac{1}{\gamma}} \max_{j=1,\ldots,J} \|Z_j\|_{\Psi_\gamma}.$$

While this is only a minor generalization of [Ver18, Exercise 2.5.10], we include the short proof.

Proof. Let $p \geqslant 2$. For any $r \geqslant p$, it follows from the embedding $\ell_j^r \hookrightarrow \ell_j^\infty$ and Hölder's inequality that

$$\| \max(Z_1, \ldots, Z_J) \|_{L_\omega^p} \leqslant \| Z_j \|_{L_\omega^p \ell_j^\infty} \leqslant \| Z_j \|_{L_\omega^p \ell_j^r} \leqslant \| Z_j \|_{L_\omega^r \ell_j^r} \leqslant J^{\frac{1}{r}} r^{\frac{1}{\gamma}} \max_{j=1,\ldots,J} \| Z_j \|_{\Psi_\gamma}.$$

Then, we choose $r = \log(2+J)p$, which yields the desired estimate. □

We now turn to a combination of Gaussian hypercontractivity and the moment method, which will be essential to our treatment of the random matrix terms **RMT**. The following proposition, which is easy-to-use, general, and essentially sharp, was recently obtained by Deng, Nahmod, and Yue in [DNY20, Proposition 2.8]. Before we state the estimate, we need the following definition, which relies on the tensor notation from Definition 4.4.32.

Definition 4.4.49 (Contracted random tensor)**.** Let $\mathcal{J} \subseteq \mathbb{N}_0$, let $(\pm_j)_{j \in \mathcal{J}}$ be given, and let $N_{\max} \geqslant 1$. Let $h = h_{n_\mathcal{J}}$ be a tensor and assume that all vectors in the support of h satisfy $\| n_\mathcal{J} \| \leqslant N_{\max}$. Let $\mathcal{S} \subseteq \mathcal{J}$ and define $k \overset{\text{def}}{=} \#\mathcal{S}$. We then define the contracted random tensor $h_c = (h_c)_{n_{\mathcal{J} \backslash \mathcal{S}}}$ by

$$h_c(n_i \colon i \notin \mathcal{S}) \overset{\text{def}}{=} \sum_{(n_j)_{j \in \mathcal{S}}} h(n_\mathcal{J}) \cdot \widetilde{\mathcal{I}}_k[\pm_j, n_j \colon j \in \mathcal{S}], \tag{4.4.83}$$

where the normalized multiple stochastic integrals are as in (4.4.76).

In the next proposition, we use the tensor norms from Definition 4.4.32.

Proposition 4.4.50 ([DNY20, Proposition 2.8, Proposition 4.14])**.** Let $\mathcal{J}, \mathcal{S}, N_{\max}, h, h_c$, and k be as in Definition 4.4.49. Let $\mathcal{A}, \mathcal{B}$ be a partition of $\{1, \ldots, J\} \backslash \mathcal{S}$. Then, we have for all $p \geqslant 2$ and $\theta > 0$ that

$$\| \| h_c \|_{n_\mathcal{A} \to n_\mathcal{B}} \|_{L_\omega^p(\mathbb{P})} \lesssim_\theta N_{\max}^\theta \Big(\max_{\mathcal{X}, \mathcal{Y}} \| h \|_{n_\mathcal{X} \to n_\mathcal{Y}} \Big) p^{\frac{k}{2}}, \tag{4.4.84}$$

where the maximum is taken over all sets $\mathcal{X}, \mathcal{Y}$ which satisfy $\mathcal{A} \subseteq \mathcal{X}$, $\mathcal{B} \subseteq \mathcal{Y}$, and form a partition of $\mathcal{J}$.

In [DNY20], the proposition is stated in terms of non-resonant products of Gaussians instead of multiple stochastic integrals. Furthermore, the probabilistic estimate is stated in terms of the tail-behavior instead of the moment growth. Both of these modifications can be obtained easily by replacing the large deviation estimate [DNY20, Lemma 4.4] in the proof by Proposition 4.4.45. We often simply refer to Proposition 4.4.50 as the moment method, since it is the main ingredient of the proof (cf. [DNY20]). While the full generality of Proposition 4.4.50 is needed in [DNY20], we will only rely on the following special case.

Example 4.4.51. Let $\pm_1, \pm_2 \in \{+,-\}$, let $h = h(n,n_1,n_2,n_3)$ be a tensor and assume that $\|(n,n_1,n_2,n_3)\| \lesssim N_{\max}$ on the support of h. Define the contracted random tensor h_c by

$$h_c(n,n_3) \overset{\text{def}}{=} \sum_{n_1,n_2\in\mathbb{Z}^3} h(n,n_1,n_2,n_3)\cdot \mathcal{I}_2[\pm_j, n_j \colon j=1,2]. \tag{4.4.85}$$

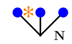
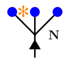
Then, we have for all $p \geqslant 2$ and $\theta > 0$ that

$$\Big\| \|h_c\|_{n_3\rightarrow n} \Big\|_{L^p_\omega} \lesssim_\theta N_{\max}^\theta \max\big(\|h\|_{n_1n_2n_3\rightarrow n}, \|h\|_{n_3\rightarrow nn_1n_2}, \|h\|_{n_1n_3\rightarrow nn_2}, \|h\|_{n_2n_3\rightarrow nn_1}\big)\cdot p.$$

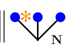
4.5 Explicit stochastic objects

In this section, we estimate the stochastic objects appearing in the expansion of u_N and in the evolution equations for X_N and Y_N. The analysis of explicit stochastic objects is necessary for both dispersive and parabolic equations. We refer the interested reader to the treatment of the cubic stochastic heat equation in [CC18, Hai16] and the quadratic stochastic wave equation in [GKO18a] for illustrative examples. While the algebraic aspects are similar in dispersive and parabolic settings, the analytic aspects are quite different. In the parabolic setting, the regularity of stochastic objects can be determined through simple "power-counting". In contrast, the optimal estimates in the dispersive setting require more complicated multi-linear dispersive estimates. We remind the reader that, as explained in Remark 4.1.4, we restrict ourselves to $0 < \beta < 1/2$.

4.5.1 Cubic stochastic objects

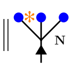
In this subsection, we analyze the cubic stochastic object  and the corresponding solution to the forced wave equation . Ignoring the smoother component $\circ_{\mathrm{M}}$ of the initial data, they correspond to the first Picard iterate of (4.2.1).

Proposition 4.5.1 (Cubic stochastic objects)**.** Let $T \geqslant 1$ and let $s < \beta - \eta$. Then, it holds that

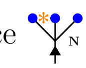
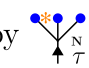
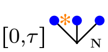


$$\Big\| \sup_{N \geqslant 1} \| \cdot \|_{\mathfrak{X}^{s-1,b_+-1}([0,T])} \Big\|_{L^p_\omega(\mathbb{P})} \lesssim T^2 p^{\frac{3}{2}}. \tag{4.5.1}$$

Furthermore, we have that



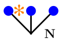
$$\Big\| \sup_{N \geqslant 1} \| \cdot \|_{C_t^0 \mathcal{C}_x^s([0,T] \times \mathbb{T}^3)} \Big\|_{L^p_\omega(\mathbb{P})} \lesssim T^2 p^{\frac{3}{2}}. \tag{4.5.2}$$

In the frequency-localized version of (4.5.1) and (4.5.2), which is detailed in the proof, we gain an η'-power of the maximal frequency-scale. Furthermore, we can replace  by  $= \mathrm{I}\big[1_{[0,\tau]}$  $\big]$.

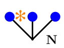
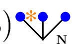
Remark 4.5.2. We recall that the parameter T is important for the globalization argument, but does not enter into the local well-posedness theory. In order to achieve smallness on a short interval, we will instead use the time-localization lemma (Lemma 4.4.3) and $b_+ > b$.

Proof. We first prove (4.5.1), which forms the main part of the argument. In the end, we follow a standard and short argument to show that (4.5.1), Gaussian hypercontractivity, and translation invariance imply (4.5.2). To simplify the notation, we set $N_{\max} = \max(N_1, N_2, N_3)$. In this argument, we rely on multiple stochastic integrals. Recalling the multiple stochastic integrals from

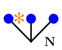
(4.4.72) and the stochastic representation formula (Proposition 4.4.44), we have that


$$(t,x) = \sum_{n_1,n_2,n_3\in\mathbb{Z}^3} \rho_N(n_{123})\Big(\prod_{j=1}^3 \rho_N(n_j)\Big)\widehat{V}(n_{12})\exp\big(i\langle n_{123},x\rangle\big)\mathcal{I}_3[t,n_1,n_2,n_3]$$
$$= \sum_{\pm_1,\pm_2,\pm_3}\sum_{n_1,n_2,n_3\in\mathbb{Z}^3}\Big[c(\pm_j\colon 1\leqslant j\leqslant 3)\Big(\prod_{j=1}^3\rho_N(n_j)\Big)\widehat{V}(n_{12})\exp\big(i\langle n_{123},x\rangle\big)$$
$$\times\Big(\prod_{j=1}^3\exp(\pm_j it\langle n_j\rangle)\Big)\mathcal{I}_3[\pm_j,n_j\colon 1\leqslant j\leqslant 3]\Big],$$

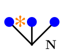
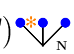
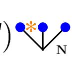
where $c(\pm_j\colon 1\leqslant j\leqslant 3)$ are deterministic coefficients. Using a Littlewood-Paley decomposition, we obtain that


$$= \sum_{\pm_1,\pm_2,\pm_3}\sum_{N_1,N_2,N_3\geqslant 1} c(\pm_j\colon 1\leqslant j\leqslant 3)$$

$$[\pm_j,N_j\colon 1\leqslant j\leqslant 3],$$

where

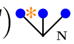

$$[\pm_j,N_j\colon 1\leqslant j\leqslant 3](t,x) \overset{\text{def}}{=} \sum_{n_1,n_2,n_3\in\mathbb{Z}^3}\Big[\rho_N(n_{123})\Big(\prod_{j=1}^3\rho_N(n_j)\chi_{N_j}(n_j)\Big)\widehat{V}(n_{12})$$
$$\times\exp\big(i\langle n_{123},x\rangle\big)\Big(\prod_{j=1}^3\exp(\pm_j it\langle n_j\rangle)\Big)\mathcal{I}_3[\pm_j,n_j\colon 1\leqslant j\leqslant 3]\Big].$$

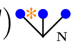
We estimate each dyadic block separately. We first prove the desired estimate for b_- instead of b_+ and then later upgrade the estimate. Using Minkowski's integral inequality and Gaussian hypercontractivity (Proposition 4.4.45), we obtain that


$$\Big\|\Big\| \;\cdot\; [\pm_j,N_j\colon 1\leqslant j\leqslant 3]\Big\|_{\mathfrak{X}^{s-1,b_--1}([0,T])}\Big\|_{L^p_\omega}$$

$$\lesssim \max_{\pm_{123}}\Big\|\mathcal{F}_{t,x}\Big(\chi(t/T)\;\cdot\;[\pm_j,N_j\colon 1\leqslant j\leqslant 3](t,x)\Big)(\lambda\mp_{123}\langle n\rangle,n)\|_{L^p_\omega L^2_\lambda\ell^2_n(\Omega\times\mathbb{R}\times\mathbb{T}^3)}$$

$$\lesssim p^{\frac{3}{2}}\max_{\pm_{123}}\Big\|\mathcal{F}_{t,x}\Big(\chi(t/T)\;\cdot\;[\pm_j,N_j\colon 1\leqslant j\leqslant 3](t,x)\Big)(\lambda\mp_{123}\langle n\rangle,n)\|_{L^2_\omega L^2_\lambda\ell^2_n(\Omega\times\mathbb{R}\times\mathbb{T}^3)}. \tag{4.5.3}$$

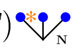
For a fixed sign $\pm_{123}$, we define the phase φ by

$$\varphi(n_1,n_2,n_3) \overset{\text{def}}{=} \pm_{123}\langle n_{123}\rangle \pm_1 \langle n_1\rangle \pm_2\langle n_2\rangle \pm_3\langle n_3\rangle.$$

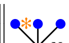
Using the definition of φ, we can write the space-time Fourier transform of a dyadic piece in the cubic stochastic object $\chi(t/T)$ as

$$\mathcal{F}_{t,x}\Big(\chi(t/T)$$  $$[\pm_j, N_j \colon 1 \leqslant j \leqslant 3](t,x)\Big)(\lambda \mp_{123} \langle n\rangle, n)$$
$$= T \sum_{n_1,n_2,n_3 \in \mathbb{Z}^3} \Big[1\big\{n = n_{123}\big\} \rho_N(n_{123}) \Big(\prod_{j=1}^{3} \rho_N(n_j) \chi_{N_j}(n_j) \Big) \widehat{V}(n_{12}) \qquad (4.5.4)$$
$$\times\, \widehat{\chi}\big(T(\lambda - \varphi(n_1,n_2,n_3))\big) \mathcal{I}_3[\pm_j, n_j \colon 1 \leqslant j \leqslant 3] \Big].$$

Using the orthogonality of the multiple stochastic integrals and the decay of $\widehat{\chi}$, we obtain that

$$\Big\| \mathcal{F}_{t,x}\Big(\chi(t/T)$$  $$[\pm_j, N_j \colon 1 \leqslant j \leqslant 3](t,x)\Big)(\lambda \mp_{123} \langle n\rangle, n) \Big\|^2_{L^2_\omega L^2_\lambda \ell^2_n(\Omega \times \mathbb{R} \times \mathbb{T}^3)}$$
$$\lesssim T^2 N_1^{-2} N_2^{-2} N_3^{-2} \sum_{n_1,n_2,n_3 \in \mathbb{Z}^3} \Big[\Big(\prod_{j=1}^{3} \chi_{N_j}(n_j) \Big) \langle n_{123} \rangle^{2(s-1)} |\widehat{V}(n_{12})|^2$$
$$\times \int_{\mathbb{R}} \mathrm{d}\lambda\, \langle \lambda \rangle^{2(b_- - 1)} |\widehat{\chi}\big(T(\lambda - \varphi(n_1,n_2,n_3))\big)|^2 \Big]$$
$$\lesssim T^2 N_1^{-2} N_2^{-2} N_3^{-2} \sum_{n_1,n_2,n_3 \in \mathbb{Z}^3} \Big(\prod_{j=1}^{3} \chi_{N_j}(n_j) \Big) \langle n_{123} \rangle^{2(s-1)} |\widehat{V}(n_{12})|^2 \langle \varphi(n_1,n_2,n_3) \rangle^{2(b_- - 1)}$$
$$\lesssim T^2 N_1^{-2} N_2^{-2} N_3^{-2} \sup_{m \in \mathbb{Z}^3} \sum_{n_1,n_2,n_3 \in \mathbb{Z}^3} \Big(\prod_{j=1}^{3} \chi_{N_j}(n_j) \Big) \langle n_{123} \rangle^{2(s-1)} |\widehat{V}(n_{12})|^2 1\big\{ |\varphi - m| \leqslant 1 \big\}.$$

Combining this with (4.5.3) and using the cubic sum estimate (Proposition 4.4.20), we obtain that

$$\Big\| \Big\|$$  $$[\pm_j, N_j \colon 1 \leqslant j \leqslant 3] \Big\|_{\mathfrak{X}^{s-1,b_- -1}([0,T])} \Big\|_{L^p_\omega} \lesssim T p^{\frac{3}{2}} N_{\max}^{s-\beta}.$$

Since there are at most $\lesssim \log(10 + N_{\max})$ non-trivial choices for N, we obtain from Lemma 4.4.48 that

$$\Big\| \sup_{N \geqslant 1} \Big\|$$  $$[\pm_j, N_j \colon 1 \leqslant j \leqslant 3] \Big\|_{\mathfrak{X}^{s-1,b_- -1}([0,T])} \Big\|_{L^p_\omega} \qquad (4.5.5)$$
$$\lesssim T \log\log(10 + N_{\max})^2 N_{\max}^{s-\beta} p^{\frac{3}{2}}.$$

After summing over the dyadic scales, (4.5.5) almost implies (4.5.1) except that b_- needs to be replaced by b_+. To achieve this, we utilize the room of the estimate (4.5.5) in the maximal frequency scale. Using Plancherel's theorem, Minkowski's integral inequality, and Gaussian hypercontractivity, we have that

$$
\begin{aligned}
&\Big\| \sup_{N\geqslant 1} \Big\| \Psi_N [\pm_j, N_j \colon 1 \leqslant j \leqslant 3] \Big\|_{\mathfrak{X}^{0,0}([0,T])} \Big\|_{L^p_\omega} \\
&\lesssim \log\log(10+N_{\max})^2 \sup_N \Big\| 1\big\{0 \leqslant t \leqslant T\big\} \Psi_N [\pm_j, N_j \colon 1 \leqslant j \leqslant 3] \|_{L^p_\omega L^2_t L^2_x} \\
&\lesssim T^{\frac{1}{2}} \log\log(10+N_{\max})^2 p^{\frac{3}{2}} \Big(\sum_{n_1,n_2,n_3 \in \mathbb{Z}^3} \prod_{j=1}^{3} \big(\chi_{N_j}(n_j) \langle n_j \rangle^{-2} \big) \Big)^{\frac{1}{2}} \\
&\lesssim T^{\frac{1}{2}} \log\log(10+N_{\max})^2 N_{\max}^{\frac{3}{2}} p^{\frac{3}{2}}.
\end{aligned}
$$

By interpolating this estimate with (4.5.5), we obtain that

$$
\begin{aligned}
&\Big\| \sup_{N\geqslant 1} \Big\| \Psi_N [\pm_j, N_j \colon 1 \leqslant j \leqslant 3] \Big\|_{\mathfrak{X}^{s-1,b_+-1}([0,T])} \Big\|_{L^p_\omega} \\
&\lesssim T \log\log(10+N_{\max})^2 N_{\max}^{s-\beta+4(b_+-b_-)} p^{\frac{3}{2}} \qquad (4.5.6) \\
&\lesssim T N_{\max}^{s-\beta+5(b_+-b_-)} p^{\frac{3}{2}}.
\end{aligned}
$$

After summing over the dyadic scales, this finally yields (4.5.1). We prove the second estimate (4.5.2) using the (frequency-localized version of the) first estimate. We present the details of the (standard) argument, but skip similar steps in subsequent proofs. Using the energy estimate (Lemma 4.4.8) and the (frequency-localized version of the) first estimate (4.5.1), we obtain that

$$
\Big\| \sup_{N\geqslant 1} \Big\| \Psi^{\uparrow}_N [\pm_j, N_j \colon 1 \leqslant j \leqslant 3] \Big\|_{L^\infty_t H^s_x} \Big\|_{L^p_\omega} \lesssim (1+T) N_{\max}^{s-\beta+5(b_+-b_-)} p^{\frac{3}{2}}. \qquad (4.5.7)
$$

For any $2 \leqslant q \leqslant p$, we have from Sobolev embedding (in space-time), Minkowski's integral inequality, and Gaussian hypercontractivity that

$$
\begin{aligned}
&\Big\| \mathbb{Y}^{*}_{N} [\pm_j, N_j \colon 1 \leqslant j \leqslant 3] \Big\|_{L^p_\omega L^\infty_t \mathcal{C}^s_x} \\
&\lesssim N_{\max}^{\frac{4}{q}} \Big\| \langle \nabla \rangle^s \mathbb{Y}^{*}_{N} [\pm_j, N_j \colon 1 \leqslant j \leqslant 3] \Big\|_{L^p_\omega L^q_t L^q_x} \\
&\lesssim N_{\max}^{\frac{4}{q}} \Big\| \langle \nabla \rangle^s \mathbb{Y}^{*}_{N} [\pm_j, N_j \colon 1 \leqslant j \leqslant 3] \Big\|_{L^q_t L^q_x L^p_\omega} \\
&\lesssim N_{\max}^{\frac{4}{q}} p^{\frac{3}{2}} \Big\| \langle \nabla \rangle^s \mathbb{Y}^{*}_{N} [\pm_j, N_j \colon 1 \leqslant j \leqslant 3] \Big\|_{L^q_t L^q_x L^2_\omega}. \qquad (4.5.8)
\end{aligned}
$$

For a fixed $t \in \mathbb{R}$, the distribution of $\langle \nabla \rangle^s \mathbb{Y}^{*}_{N} [\pm_j, N_j \colon 1 \leqslant j \leqslant 3](t,x)$ is translation invariant. Thus, we can replace the L^q_x-norm in (4.5.8) by the L^2_x-norm. Using Minkowski's integral inequality and (4.5.7) then yields

$$
\begin{aligned}
&\Big\| \mathbb{Y}^{*}_{N} [\pm_j, N_j \colon 1 \leqslant j \leqslant 3] \Big\|_{L^p_\omega L^\infty_t \mathcal{C}^s_x} \lesssim N_{\max}^{\frac{4}{q}} p^{\frac{3}{2}} \Big\| \langle \nabla \rangle^s \mathbb{Y}^{*}_{N} [\pm_j, N_j \colon 1 \leqslant j \leqslant 3] \Big\|_{L^2_\omega L^q_t L^2_x} \\
&\lesssim T^{1+\frac{1}{q}} N_{\max}^{s-\beta+5(b_+-b_-)+\frac{4}{q}} p^{\frac{3}{2}}.
\end{aligned}
$$

By choosing $q = q(b_+, b_-)$ sufficiently large and then summing over dyadic scales, this proves (4.5.2) for $p \gtrsim_{b_+,b_-} 1$. The smaller values of p can be handled by using Hölder's inequality in ω.

Finally, the statement for $\mathbb{Y}^{*}_{N}$ replaced by $\mathbb{Y}^{*}_{N,\tau}$ follows from the boundedness of $1_{[0,\tau]}(t)$ on $\mathfrak{X}^{s_2-1,b_+-1}$, which was proven in Lemma 4.4.4. $\square$

4.5.2 Quartic stochastic objects

The expansion $u_N = \overset{\bullet}{|} + \mathbb{Y}^{*}_{N} + w_N$ or the explicit stochastic objects in **So** only contain linear, cubic, quintic, or septic stochastic objects. However, the physical terms **Phy** contain terms such as

$$
V * \Big(P_{\leqslant N} \overset{\bullet}{|} \cdot P_{\leqslant N} \mathbb{Y}^{*}_{N} \Big) P_{\leqslant N} w_N \qquad \text{or} \qquad V * \Big(P_{\leqslant N} \overset{\bullet}{|} \,\textcircled{$\neq$}\, P_{\leqslant N} w_N \Big) P_{\leqslant N} \mathbb{Y}^{*}_{N}.
$$

Since we treat $w_N \in \mathfrak{X}^{s_1,b}$ using deterministic methods, they can be viewed as quartic expressions in the random initial data $\bullet$. Furthermore, due to the convolution with the interaction potential V in the second term, we also have to understand the product of [linear tree] and [cubic tree]_{N} at two different spatial points.

Proposition 4.5.3. Let $N_{123}, N_4 \geqslant 1$. Then, we have for all $s < -1/2 - \eta$ and all $T \geqslant 1$ that

$$\begin{aligned}&\Big\| \sup_{N\geqslant 1} \sup_{y \in \mathbb{T}^3} \Big\| \Big(P_{N_{123}} P_{\leqslant N} \text{[cubic tree]}_{N} (t, x-y) \Big) \cdot P_{N_4} P_{\leqslant N} \text{[linear tree]}(t,x) \Big\|_{C_t^0 \mathcal{C}_x^s([0,T]\times\mathbb{T}^3)} \Big\|_{L^p_\omega(\mathbb{P})} \\ &\lesssim T^3 p^2 \max(N_{123}, N_4)^{-\frac{\eta}{2}} N_4^{\kappa}.\end{aligned} \tag{4.5.9}$$

If $N_{123} \sim N_4$, we have for all $s < -1/2 + \beta - 2\eta$ that

$$\begin{aligned}&\Big\| \sup_{N\geqslant 1} \sup_{y \in \mathbb{T}^3} \Big\| \Big(P_{N_{123}} P_{\leqslant N} \text{[cubic tree]}_{N} (t, x-y) \Big) \cdot P_{N_4} P_{\leqslant N} \text{[linear tree]}(t,x) \Big\|_{C_t^0 \mathcal{C}_x^s([0,T]\times\mathbb{T}^3)} \Big\|_{L^p_\omega(\mathbb{P})} \\ &\lesssim T^3 p^2 N_4^{\kappa}.\end{aligned} \tag{4.5.10}$$

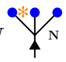
Finally, without the shift in $y \in \mathbb{T}^3$, we have for $s < -1/2 - \eta$ that

$$\begin{aligned}&\Big\| \sup_{N\geqslant 1} \Big\| \Big(P_{N_{123}} P_{\leqslant N} \text{[cubic tree]}_{N} (t, x) \Big) \cdot P_{N_4} P_{\leqslant N} \text{[linear tree]}(t,x) \Big\|_{C_t^0 \mathcal{C}_x^s([0,T]\times\mathbb{T}^3)} \Big\|_{L^p_\omega(\mathbb{P})} \\ &\lesssim T^3 p^2 \max(N_{123}, N_4)^{-\frac{\eta}{10}}.\end{aligned} \tag{4.5.11}$$

Remark 4.5.4. In the fully frequency-localized version of Proposition 4.5.3, which is detailed in the proof, we gain an η'-power of the maximal frequency-scale. As in Proposition 4.5.1, we may also replace [cubic tree]_{N} by $\text{[cubic tree]}_{N,\tau} = \mathrm{I}\big[1_{[0,\tau]} \text{[cubic nonlinearity]}_{N}\big]$.

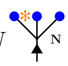
Remark 4.5.5. We recall that η is much smaller than κ and hence the right-hand sides of (4.5.9) and (4.5.10) diverge as $N_4 \to \infty$. The third estimate (4.5.11) is quite delicate and requires the sine-cancellation lemma. A similar estimate is not available for the partially shifted process and it is likely that at least a logarithmic loss is necessary in (4.5.9) and (4.5.10) as N_4 tends to infinity.

Proof. We prove (4.5.9) and (4.5.10) simultaneously. The third estimate (4.5.11) will mainly utilize the same estimates, but also requires the sine-cancellation lemma (Lemma 4.4.14). Using the representation based on multiple stochastic integrals (Proposition 4.4.44), we have that

$$\Big(P_{N_{123}} P_{\leqslant N}$$  $$(t, x-y)\Big) \cdot P_{N_4} P_{\leqslant N}$$  $$(t,x)$$

$$= \sum_{N_1,N_2,N_3 \geqslant 1} \sum_{n_1,n_2,n_3,n_4 \in \mathbb{Z}^3} \rho_N^2(n_{123}) \chi_{N_{123}}(n_{123}) \Big(\prod_{j=1}^{4} \rho_N(n_j) \chi_{N_j}(n_j) \Big) \widehat{V}_S(n_1,n_2,n_3)$$
$$\times \langle n_{123} \rangle^{-1} \exp\Big(i \langle n_{1234}, x \rangle - i \langle n_{123}, y \rangle \Big) \Big(\int_0^t \sin((t-t')\langle n_{123} \rangle)\, \mathcal{I}_3[t'; n_1, n_2, n_3] \cdot \mathcal{I}_1[t; n_4]\, \mathrm{d}t' \Big).$$

Using the product formula for multiple stochastic integrals, we obtain that

$$\Big(P_{N_{123}} P_{\leqslant N}$$  $$(t, x-y)\Big) \cdot P_{N_4} P_{\leqslant N}$$  $$(t,x)$$

$$= \sum_{N_1,N_2,N_3 \geqslant 1} \mathscr{G}^{(4)}(t,x,y;N_\ast) + \sum_{N_1,N_2,N_3 \geqslant 1} \mathscr{G}^{(2)}(t,x,y;N_\ast),$$

where the dependence on $N_{123}, N_1, N_2, N_3, N_4$ is indicated by $N_\ast$ and the quartic and quadratic Gaussian chaoses are given by

$$\mathscr{G}^{(4)}(t,x,y;N_\ast)$$
$$= \sum_{\pm_1,\pm_2,\pm_3,\pm_4} \sum_{n_1,n_2,n_3,n_4 \in \mathbb{Z}^3} \Big[c(\pm_j \colon 1 \leqslant j \leqslant 4) \rho_N^2(n_{123}) \rho_N(n_4) \chi_{N_{123}}(n_{123})$$
$$\times \Big(\prod_{j=1}^{4} \rho_{\leqslant N}(n_j) \chi_{N_j}(n_j) \Big) \widehat{V}_S(n_1,n_2,n_3) \langle n_{123} \rangle^{-1} \exp\Big(i \langle n_{1234}, x \rangle - i \langle n_{123}, y \rangle \Big)$$
$$\times \exp(\pm_4 i t \langle n_4 \rangle) \Big(\int_0^t \sin((t-t')\langle n_{123} \rangle) \Big(\prod_{j=1}^{3} \exp(\pm_j i t' \langle n_j \rangle) \Big)\, \mathrm{d}t' \Big) \mathcal{I}_4(\pm_j, n_j \colon 1 \leqslant j \leqslant 4) \Big]$$

and

$$
\begin{aligned}
&\mathscr{G}^{(2)}(t,x,y;N_*)\\
&= 3 \sum_{\pm_1,\pm_2} \sum_{n_1,n_2\in\mathbb{Z}^3} \bigg[c(\pm_1,\pm_2) \Big(\prod_{j=1}^{2} \rho_N(n_j)\chi_{N_j}(n_j) \Big) \exp\Big(i\langle n_{12}, x\rangle \Big)\\
&\times \bigg(\sum_{n_3\in\mathbb{Z}^3} \Big[\rho_N^2(n_{123})\rho_N^2(n_3)\chi_{N_{123}}(n_{123})\chi_{N_3}(n_3)\chi_{N_4}(n_3)\langle n_{123}\rangle^{-1}\langle n_3\rangle^{-2}\widehat{V}_S(n_1,n_2,n_3)\\
&\times \exp\big(-i\langle n_{123}, y\rangle\big) \int_0^t \sin((t-t')\langle n_{123}\rangle)\cos((t-t')\langle n_3\rangle) \prod_{j=1}^{2} \exp(\pm_j it'\langle n_j\rangle)\, \mathrm{d}t' \Big] \bigg)\\
&\times \mathcal{I}_2(\pm_j, n_j \colon j=1,2) \bigg].
\end{aligned}
$$

The quartic Gaussian chaos $\mathscr{G}^{(4)}$ and quadratic Gaussian chaoses $\mathscr{G}^{(2)}$ contain the resonant and non-resonant terms of the product, respectively. We estimate both terms separately.

The non-resonant term $\mathscr{G}^{(4)}$: We first let $s < -1/2 - \eta$. Using Gaussian hypercontractivity and standard reductions (see e.g. the proof of Proposition 4.5.1), it suffices to estimate the $L_t^\infty L_\omega^2 H_x^s$-norm instead of the $L_\omega^p L_t^\infty \mathcal{C}_x^s$-norm. Let the phase-function φ be as in (4.4.43). Using the orthogonality of the multiple stochastic integrals, we have for a fixed $t \in [0,T]$ that

$$
\begin{aligned}
&\big\| \mathscr{G}^{(4)}(t,x,y;N_*)) \big\|_{L_\omega^2 H_x^s}^2\\
&\lesssim \sum_{\pm_1,\pm_2,\pm_3} \sum_{n_1,n_2,n_3,n_4\in\mathbb{Z}^3} \bigg[\chi_{N_{123}}(n_{123}) \Big(\prod_{j=1}^{4} \chi_{N_j}(n_j) \Big) |\widehat{V}_S(n_1,n_2,n_3)|^2 \langle n_{1234}\rangle^{2s} \langle n_{123}\rangle^{-2} \Big(\prod_{j=1}^{4} \langle n_j\rangle^{-2} \Big)\\
&\times \Big| \int_0^t \sin((t-t')\langle n_{123}\rangle) \big(\prod_{j=1}^{3} \exp(\pm_j it'\langle n_j\rangle) \big)\, \mathrm{d}t' \Big|^2 \bigg]\\
&\lesssim (1+T)^2 \sum_{\pm_1,\pm_2,\pm_3} \sum_{n_1,n_2,n_3,n_4\in\mathbb{Z}^3} \sum_{m\in\mathbb{Z}} \bigg[\langle m\rangle^{-2} \chi_{N_{123}}(n_{123}) \Big(\prod_{j=1}^{4} \chi_{N_j}(n_j) \Big) |\widehat{V}_S(n_1,n_2,n_3)|^2 \langle n_{1234}\rangle^{2s}\\
&\times \langle n_{123}\rangle^{-2} \Big(\prod_{j=1}^{4} \langle n_j\rangle^{-2} \Big) 1\big\{ |\varphi - m| \leqslant 1 \big\} \bigg]
\end{aligned}
$$

$$\lesssim T^2 \sup_{m\in\mathbb{Z}} \sum_{\pm_1,\pm_2,\pm_3} \sum_{n_1,n_2,n_3,n_4\in\mathbb{Z}^3} \Big[\chi_{N_{123}}(n_{123}) \Big(\prod_{j=1}^4 \chi_{N_j}(n_j) \Big) |\widehat{V}_S(n_1,n_2,n_3)|^2 \langle n_{1234}\rangle^{2s}$$
$$\times \langle n_{123}\rangle^{-2} \Big(\prod_{j=1}^4 \langle n_j \rangle^{-2} \Big) 1\big\{ |\varphi - m| \leqslant 1 \big\} \Big].$$

Using the non-resonant quartic sum estimate (Lemma 4.4.24), it follows that

$$\big\| \mathscr{G}^{(4)}(t,x,y;N_{123},N_1,N_2,N_3,N_4)) \big\|^2_{L^2_\omega H^s_x} \lesssim T^2 \max(N_1,N_2,N_3)^{-2\beta+2\eta} N_4^{-2\eta}.$$

This yields (4.5.9) for the non-resonant component. If $N_{123} \sim N_4$, then $\max(N_1,N_2,N_3) \gtrsim N_4$, and hence we can raise the value of s by $\beta-\eta$. Thus, we also obtain (4.5.10) for the non-resonant component. Even when $y \neq 0$, our estimate for the non-resonant component does not exhibit any growth in N_4, and hence it also yields (4.5.11) for the non-resonant component.

The resonant term $\mathscr{G}^{(2)}$: This term exhibits a higher spatial regularity and we let $-1/2 < s < 0$. Using Gaussian hypercontractivity and standard reductions (see e.g. the proof of Proposition 4.5.1), it suffices to estimate the $L^\infty_t L^2_\omega H^s_x$-norm instead of the $L^p_\omega L^\infty_t \mathcal{C}^s_x$-norm. Using the orthogonality of the multiple stochastic integrals, we have that

$$\begin{aligned} &\big\| \mathscr{G}^{(2)}(t,x,y;N_*) \big\|^2_{L^2_\omega H^s_x} \\ &\lesssim \sum_{\pm_1,\pm_2} \sum_{n_1,n_2\in\mathbb{Z}^3} \Big[\Big(\prod_{j=1}^2 \chi_{N_j}(n_j) \Big) \langle n_{12}\rangle^{2s} \langle n_1\rangle^{-2} \langle n_2\rangle^{-2} \\ &\times \Big| \sum_{n_3\in\mathbb{Z}^3} \Big[\rho_N^2(n_{123}) \rho_N^2(n_3) \chi_{N_{123}}(n_{123}) \chi_{N_3}(n_3) \chi_{N_4}(n_3) \langle n_{123}\rangle^{-1} \langle n_3\rangle^{-2} \widehat{V}_S(n_1,n_2,n_3) \\ &\times \exp\big(-i\langle n_{123}, y\rangle \big) \int_0^t \sin((t-t')\langle n_{123}\rangle) \cos((t-t')\langle n_3\rangle) \prod_{j=1}^2 \exp(\pm_j i t' \langle n_j \rangle\rangle)\, \mathrm{d}t' \Big] \Big|^2 \Big]. \end{aligned} \tag{4.5.12}$$

We now present two estimates of (4.5.12). The first estimate will yield (4.5.9) and (4.5.10). The second estimate is restricted to the case $y=0$ and yields, combined with the first estimate, (4.5.11).

After computing the integral in t' and decomposing according to the dispersive symbol, we obtain from Cauchy-Schwarz that

$$\begin{aligned}(4.5.12) \lesssim T^2 1\big\{N_3 \sim N_4\big\} \sum_{n_1,n_2\in\mathbb{Z}^3} \Bigg[\Big(\prod_{j=1}^{2} \chi_{N_j}(n_j) \Big) \langle n_{12}\rangle^{2s} \langle n_1\rangle^{-2} \langle n_2\rangle^{-2} \\ \times \Big(\sum_{m\in\mathbb{Z}} \sum_{n_3\in\mathbb{Z}^3} \langle m\rangle^{-1} \chi_{N_3}(n_3) |\widehat{V}(n_1,n_2,n_3)| \langle n_{123}\rangle^{-1} \langle n_3\rangle^{-2} 1\big\{|\varphi - m| \leqslant 1\big\} \Big)^2 \Bigg].\end{aligned}$$

Using the resonant quartic sum estimate (Lemma 4.4.26), this implies that

$$(4.5.12) \lesssim T^2 1\big\{N_3 \sim N_4\big\} \log(2+N_4)^2 \max(N_1,N_2)^{2s}.$$

This clearly implies (4.5.9) and (4.5.10). Except for the logarithmic divergence in N_4 (and hence N_3), it also implies (4.5.11). We now need to restrict to $y = 0$ and we may assume that $N_1, N_2 \ll N_3$. For fixed $n_1, n_2 \in \mathbb{Z}^3$, we can apply the sine-cancellation lemma (Lemma 4.4.14) with $A = \max(N_1, N_2)$ and

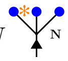
$$\begin{aligned}&f(t,t',n_3) \\ &\overset{\text{def}}{=} \rho_N^2(n_{123}) \rho_N^2(n_3) \chi_{N_{123}}(n_{123}) \chi_{N_3}(n_3) \chi_{N_4}(n_3) \langle n_{123}\rangle^{-1} \langle n_3\rangle^{-2} \widehat{V}_S(n_1,n_2,n_3) \prod_{j=1}^{2} \exp(\pm_j it'\langle n_j\rangle)\end{aligned}$$

This yields

$$\begin{aligned}&(4.5.12)\big|_{y=0} \\ &\lesssim T^4 1\big\{N_3 \sim N_4\big\} \max(N_1,N_2)^8 N_3^{-2} \sum_{n_1,n_2\in\mathbb{Z}^3} \langle n_{12}\rangle^{2s} \Big(\prod_{j=1}^{2} 1\big\{|n_j| \sim N_j\big\} \langle n_j\rangle^{-2} \Big) \\ &\lesssim T^4 \max(N_1,N_2)^{10} N_3^{-2}.\end{aligned}$$

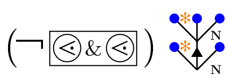
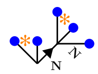
By combining our two estimates of $(4.5.12)\big|_{y=0}$ we arrive at (4.5.11).

□

Remark 4.5.6. As we have seen in the proof of Proposition 4.5.3, the (probabilistic) resonant portion of $P_{\leqslant N}$  $\cdot P_{\leqslant N}$  has spatial regularity $0-$, which is better than the sum of the individual spatial regularities. As a result, the probabilistic resonances between linear and cubic stochastic objects in Section 4.5.4 are relatively harmless.

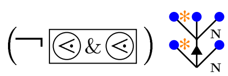
4.5.3 Quintic stochastic objects

In this subsection, we control the quintic stochastic objects in **So**, i.e.,

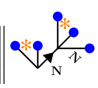
 and .

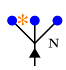
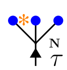
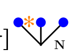
Since **So** is part of the evolution equation for the smoother nonlinear remainder Y_N, the quintic stochastic objects have to be controlled at regularity $s_2 - 1$.

Proposition 4.5.7 (First quintic stochastic object)**.** For any $T \geqslant 1$ and any $p \geqslant 2$, it holds that

$$\Big\| \sup_{N \geqslant 1} \Big\|$$  $$\Big\|_{\mathfrak{X}^{s_2-1,b_+-1}([0,T])} \Big\|_{L^p_\omega(\Omega)} \lesssim T^2 p^{\frac{5}{2}}. \tag{4.5.13}$$

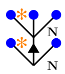
Proposition 4.5.8 (Second quintic stochastic object)**.** For any $T \geqslant 1$ and any $p \geqslant 2$, it holds that

$$\Big\| \sup_{N \geqslant 1} \Big\|$$  $$\Big\|_{\mathfrak{X}^{s_2-1,b_+-1}([0,T])} \Big\|_{L^p_\omega(\Omega)} \lesssim T^2 p^{\frac{5}{2}}. \tag{4.5.14}$$

Remark 4.5.9. In the frequency-localized versions of Proposition 4.5.7 and Proposition 4.5.8, which are detailed in the proof, we gain an η'-power of the maximal frequency-scale. As in Proposition 4.5.1, we may also replace  by  $= \mathrm{I}\big[1_{[0,\tau]}$  $\big]$. We will not further comment on these minor modifications.

Proof of Proposition 4.5.7: Throughout the proof, we ignore the supremum in $N \geqslant 1$ and only prove a uniform estimate for a fixed N. Using the frequency-localized estimates below and the same argument as in the proof of Proposition 4.5.1, we can insert the supremum in N at the end of the proof.

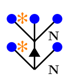
We first obtain a representation of the quintic stochastic object using multiple stochastic integral. Using (4.2.23) and Proposition 4.4.44, we have that

$$\Big(\neg \boxed{\lessdot \,\&\, \lessdot}\Big)$$  $$(t,x)$$

$$= \sum_{\substack{N_{345}, N_1, \ldots, N_5 : \\ \max(N_1, N_{345}) > N_2^{\epsilon}}} \sum_{n_1, \ldots, n_5 \in \mathbb{Z}^3} \Bigg[\rho_N^2(n_{345}) \chi_{N_{345}}(n_{345}) \Big(\prod_{j=1}^{5} \rho_N(n_j) \chi_{N_j}(n_j) \Big) \widehat{V}(n_{1345}) \widehat{V}_S(n_3, n_4, n_5)$$

$$\times \langle n_{345} \rangle^{-1} \exp\big(i \langle n_{12345}, x \rangle \big) \mathcal{I}_2[t, n_1, n_2] \Big(\int_0^t \sin\big((t-t') \langle n_{123} \rangle \big) \mathcal{I}_3[t', n_3, n_4, n_5] \mathrm{d}t' \Big) \Bigg].$$

Using the product formula for mixed multiple stochastic integrals (Proposition 4.4.44 and Lemma 4.4.41), we obtain that

$$\Big(\neg \boxed{\lessdot \,\&\, \lessdot}\Big)$$  $$(t,x) = \sum_{\substack{N_{345}, N_1, \ldots, N_5 : \\ \max(N_1, N_{345}) > N_2^{\epsilon}}} \Big(\mathscr{G}^{(5)} + \mathscr{G}^{(3)} + \widetilde{\mathscr{G}}^{(3)} + \mathscr{G}^{(1)} \Big)(t, x; N_*), \qquad (4.5.15)$$

where the dependence on $N_{345}, N_1, \ldots, N_5$ is indicated by N_* and the quintic, cubic, and linear Gaussian chaoses are defined as follows. The quintic chaos is given by

$$\mathscr{G}^{(5)}(t, x; N_*)$$

$$\overset{\text{def}}{=} \sum_{\pm_1, \ldots, \pm_5} c(\pm_j : 1 \leqslant j \leqslant 5) \sum_{n_1, \ldots, n_5 \in \mathbb{Z}^3} \Bigg[\rho_N^2(n_{345}) \chi_{N_{345}}(n_{345}) \Big(\prod_{j=1}^{5} \rho_N(n_j) \chi_{N_j}(n_j) \Big) \widehat{V}(n_{1345})$$

$$\times \widehat{V}_S(n_3, n_4, n_5) \langle n_{345} \rangle^{-1} \exp\big(i \langle n_{12345}, x \rangle \big) \Big(\prod_{j=1}^{2} \exp\big(\pm_j \, it \langle n_j \rangle \big) \Big)$$

$$\times \Big(\int_0^t \sin\big((t-t') \langle n_{123} \rangle \big) \prod_{j=3}^{5} \exp\big(\pm_j \, it' \langle n_j \rangle \big) \mathrm{d}t' \Big) \mathcal{I}_5[\pm_j, n_j : 1 \leqslant j \leqslant 5] \Bigg].$$

The two cubic Gaussian chaoses are given by

$$
\begin{aligned}
&\mathscr{G}^{(3)}(t,x;N_*)\\
&\overset{\text{def}}{=} \sum_{\pm_2,\pm_4,\pm_5} c(\pm_2,\pm_4,\pm_5) \sum_{n_2,n_4,n_5\in\mathbb{Z}^3} \Bigg[\Big(\prod_{j=2,4,5} \rho_N(n_j)\chi_{N_j}(n_j)\Big) \widehat{V}(n_{45}) \exp\big(i\langle n_{245},x\rangle\big)\\
&\times \sum_{n_3\in\mathbb{Z}^3} \Bigg(\rho_N^2(n_3)\rho_N^2(n_{345})\chi_{N_{345}}(n_{345})\chi_{N_1}(n_3)\chi_{N_3}(n_3)\widehat{V}_S(n_3,n_4,n_5)\langle n_{345}\rangle^{-1}\langle n_3\rangle^{-2}\exp\big(\pm_2 it\langle n_2\rangle\big)\\
&\times \int_0^t \sin\big((t-t')\langle n_{345}\rangle\big)\cos\big((t-t')\langle n_3\rangle\big) \prod_{j=4,5}\exp\big(\pm_j it'\langle n_j\rangle\big)\mathrm{d}t' \Bigg) \mathcal{I}_3[\pm_j,n_j\colon j=2,4,5]\Bigg]
\end{aligned}
$$

and

$$
\begin{aligned}
&\widetilde{\mathscr{G}}^{(3)}(t,x;N_*)\\
&\overset{\text{def}}{=} \sum_{\pm_1,\pm_4,\pm_5} c(\pm_1,\pm_4,\pm_5) \sum_{n_1,n_4,n_5\in\mathbb{Z}^3} \Bigg[\Big(\prod_{j=1,4,5} \rho_N(n_j)\chi_{N_j}(n_j)\Big) \exp\big(i\langle n_{145},x\rangle\big) \sum_{n_3\in\mathbb{Z}^3} \Bigg(\rho_N^2(n_3)\\
&\times \rho_N^2(n_{345})\chi_{N_{345}}(n_{345})\chi_{N_1}(n_3)\chi_{N_3}(n_3)\widehat{V}_S(n_3,n_4,n_5)\widehat{V}(n_{1345})\langle n_{345}\rangle^{-1}\langle n_3\rangle^{-2}\exp\big(\pm_1 it\langle n_1\rangle\big)\\
&\times \int_0^t \sin\big((t-t')\langle n_{345}\rangle\big)\cos\big((t-t')\langle n_3\rangle\big) \prod_{j=4,5}\exp\big(\pm_j it'\langle n_j\rangle\big)\mathrm{d}t' \Bigg) \mathcal{I}_3[\pm_j,n_j\colon j=1,4,5]\Bigg].
\end{aligned}
$$

Finally, the linear Gaussian chaos (or simply Gaussian) is given by

$$
\begin{aligned}
&\mathscr{G}^{(1)}(t,x;N_*)\\
&\overset{\text{def}}{=} \sum_{\pm_5} c(\pm_5) \sum_{n_5\in\mathbb{Z}^3} \rho_N(n_5)\chi_{N_5}(n_5)\exp\big(i\langle n_5,x\rangle\big) \sum_{n_3,n_4\in\mathbb{Z}^4} \Bigg[\rho_N^2(n_{345})\rho_N^2(n_3)\rho_N^2(n_4)\chi_{N_{345}}(n_{345})\\
&\times \chi_{N_1}(n_3)\chi_{N_3}(n_3)\chi_{N_2}(n_4)\chi_{N_4}(n_4)\widehat{V}_S(n_3,n_4,n_5)\widehat{V}(n_{45})\langle n_{345}\rangle^{-1}\langle n_3\rangle^{-2}\langle n_4\rangle^{-2}\\
&\times \int_0^t \sin\big((t-t')\langle n_{345}\rangle\big)\cos\big((t-t')\langle n_3\rangle\big)\cos\big((t-t')\langle n_4\rangle\big)\exp\big(\pm_5 it'\langle n_5\rangle\big)\mathrm{d}t'\Bigg]\mathcal{I}_1[\pm_5,n_5].
\end{aligned}
$$

Each of the frequency-localized Gaussian chaoses in (4.5.15) is now estimated separately. We encourage the reader to concentrate on the estimates for $\mathscr{G}^{(5)}$ and $\mathscr{G}^{(1)}$, which already contain all ideas and ingredients.

The non-resonant term $\mathscr{G}^{(5)}$:

Let $s = 1/2 - \eta$. We will first estimate the $\mathfrak{X}^{s-1,b_--1}$-norm of a dyadic piece and then use the condition $\max(N_1, N_{345}) > N_2^\epsilon$ to increase the value of s. Using Gaussian hypercontractivity (Proposition 4.4.45), the orthogonality of multiple stochastic integrals, and Lemma 4.4.12, we obtain that

$$\begin{aligned}
&\Big\| \| \mathscr{G}^{(5)}(t,x;N_*) \|_{\mathfrak{X}^{s-1,b_--1}([0,T])} \Big\|_{L^p_\omega}^2 \\
&\lesssim \max_{\pm_{12345}} \Big\| \| \langle \lambda \rangle^{b_--1} \langle n \rangle^{s-1} \mathcal{F}_{t,x}\big(\chi(t/T)\mathscr{G}^{(5)}(t,x;N_*)\big)(\lambda \mp_{12345} \langle n \rangle, n) \|_{L^2_\lambda \ell^2_n(\mathbb{R}\times\mathbb{Z}^3)} \Big\|_{L^p_\omega}^2 \\
&\lesssim p^5 \max_{\pm_{12345}} \Big\| \| \langle \lambda \rangle^{b_--1} \langle n \rangle^{s-1} \mathcal{F}_{t,x}\big(\chi(t/T)\mathscr{G}^{(5)}(t,x;N_*)\big)(\lambda \mp_{12345} \langle n \rangle, n) \|_{L^2_\lambda \ell^2_n(\mathbb{R}\times\mathbb{Z}^3)} \Big\|_{L^2_\omega}^2 \\
&\lesssim T^2 p^5 \max_{\substack{\pm_{12345},\pm_{345},\\ \pm_1,\ldots,\pm_5}} \sum_{n_1,\ldots,n_5\in\mathbb{Z}^3} \Bigg[\chi_{N_{345}}(n_{345}) \Big(\prod_{j=1}^5 \chi_{N_j}(n_j) \Big) \langle n_{12345} \rangle^{2(s-1)} \langle n_{345} \rangle^{-2} \\
&\times |\widehat{V}(n_{1345})|^2 |\widehat{V}_S(n_3,n_4,n_5)|^2 \Big(\prod_{j=1}^5 \langle n_j \rangle^{-2} \Big) \big(1 + | \pm_{345} \langle n_{345} \rangle \pm_3 \langle n_3 \rangle \pm_4 \langle n_4 \rangle \pm_5 \langle n_5 \rangle | \big)^{-2} \\
&\times \int_{\mathbb{R}} \langle \lambda \rangle^{2(b_--1)} \Bigg(1 + \min\Big(\big| \lambda - (\pm_{12345} \langle n_{12345} \rangle \pm_{345} \langle n_{345} \rangle \pm_1 \langle n_1 \rangle \pm_2 \langle n_2 \rangle) \big|, \\
&\qquad\qquad \big| \lambda - (\pm_{12345} \langle n_{12345} \rangle \mp_{345} \langle n_{345} \rangle + \sum_{j=1}^5 (\pm_j) \langle n_j \rangle \big| \Big) \Bigg)^{-2} \mathrm{d}\lambda \Bigg].
\end{aligned} \tag{4.5.16}$$

To break down this long formula, we define the phase-functions

$$\begin{aligned}
\psi(n_3,n_4,n_5) &\stackrel{\text{def}}{=} \pm_{345} \langle n_{345} \rangle \pm_3 \langle n_3 \rangle \pm_4 \langle n_4 \rangle \pm_5 \langle n_5 \rangle, \\
\varphi(n_1,\ldots,n_5) &\stackrel{\text{def}}{=} \pm_{12345} \langle n_{12345} \rangle \pm_{345} \langle n_{345} \rangle \pm_1 \langle n_1 \rangle \pm_2 \langle n_2 \rangle, \\
\widetilde{\varphi}(n_1,\ldots,n_5) &\stackrel{\text{def}}{=} \pm_{12345} \langle n_{12345} \rangle \mp_{345} \langle n_{345} \rangle + \sum_{j=1}^5 (\pm_j) \langle n_j \rangle.
\end{aligned}$$

Integrating in λ and decomposing according to the value of the phases, we obtain that

$$
\begin{aligned}
(4.5.16) \lesssim T^2 p^5 \log(2+\max(N_1,\dots,N_5)) &\max_{\substack{\pm_{12345},\pm_{345},\\ \pm_1,\dots,\pm_5}} \sup_{m,m'\in\mathbb{Z}} \sum_{n_1,\dots,n_5\in\mathbb{Z}^3} \Bigg[\chi_{N_{345}}(n_{345}) \\
&\times \Big(\prod_{j=1}^{5}\chi_{N_j}(n_j)\Big)\langle n_{12345}\rangle^{2(s-1)}\langle n_{345}\rangle^{-2}|\widehat{V}(n_{1345})|^2|\widehat{V}_S(n_3,n_4,n_5)|^2\Big(\prod_{j=1}^{5}\langle n_j\rangle^{-2}\Big) \\
&\times 1\big\{|\psi-m|\leqslant 1\big\}\Big(1\big\{|\varphi-m'|\leqslant 1\big\}+1\big\{|\widetilde{\varphi}-m'|\leqslant 1\big\}\Big)\Bigg].
\end{aligned}
$$

Using the non-resonant quintic sum estimate (Lemma 4.4.27), we finally obtain that

$$
\Big\| \|\mathscr{G}^{(5)}(t,x;N_*)\|_{\mathfrak{X}^{s-1,b_--1}([0,T])}\Big\|_{L^p_\omega} \lesssim T p^{\frac{5}{2}} \max(N_1,N_3,N_4,N_5)^{-\beta+\eta} N_2^{-\eta}. \tag{4.5.17}
$$

Due to the operator $\big(\neg\,\boxed{\circledless \,\&\, \circledless}\big)$, we have that

$$
\max(N_1,N_3,N_4,N_5) \gtrsim \max(N_1,N_2,N_3,N_4,N_5)^{\epsilon}.
$$

Thus, (4.5.17) implies

$$
\Big\| \|\mathscr{G}^{(5)}(t,x;N_*)\|_{\mathfrak{X}^{s_2-1,b_--1}([0,T])}\Big\|_{L^p_\omega} \lesssim T p^{\frac{5}{2}} \max(N_1,N_2,N_3,N_4,N_5)^{\delta_2+3\eta-\epsilon\beta},
$$

which is acceptable.

Single-resonance term $\mathscr{G}^{(3)}$:

This term only yields a non-trivial contribution if $N_1 \sim N_3$. In particular, $\max(N_1,N_{345}) > N_2^{\epsilon}$ implies that $\max(N_3,N_4,N_5) \gtrsim N_2^{\epsilon}$. Using the inhomogeneous Strichartz estimate (Lemma 4.4.9) and Gaussian hypercontractivity, we have that

$$
\begin{aligned}
\Big\| \|\mathscr{G}^{(3)}(t,x;N_*)\|_{\mathfrak{X}^{s_2-1,b_--1}([0,T])}\Big\|_{L^p_\omega} &\lesssim \Big\| \|\mathscr{G}^{(3)}(t,x;N_*)\|_{L_t^{2b_+}H_x^{s_2-1}([0,T]\times\mathbb{T}^3)}\Big\|_{L^p_\omega} \\
&\lesssim T^{\frac{1}{2}} \Big\| \|\mathscr{G}^{(3)}(t,x;N_*)\|_{L_t^2 H_x^{s_2-1}([0,T]\times\mathbb{T}^3)}\Big\|_{L^p_\omega} \\
&\lesssim T p^{\frac{3}{2}} \sup_{t\in[0,T]} \Big\| \|\mathscr{G}^{(3)}(t,x;N_*)\|_{H_x^{s_2-1}(\mathbb{T}^3)}\Big\|_{L^2_\omega}.
\end{aligned} \tag{4.5.18}
$$

Using the orthogonality of the multiple stochastic integrals, we have that

$$\begin{aligned}&\sup_{t\in[0,T]} \Big\| \|\mathscr{G}^{(3)}(t,x;N_*)\|_{H_x^{s_2-1}(\mathbb{T}^3)} \Big\|_{L^2_\omega}^2 \\ &\lesssim N_{45}^{-2\beta} N_2^{-2} N_4^{-2} N_5^{-2} \sum_{n_2,n_4,n_5\in\mathbb{Z}^3} \chi_{N_{45}}(n_{45}) \Big(\prod_{j=2,4,5} \chi_{N_j}(n_j) \Big) \langle n_{245}\rangle^{2(s_2-1)} \mathcal{S}(n_2,n_4,n_5;t,N_*)^2,\end{aligned} \tag{4.5.19}$$

where

$$\begin{aligned}&\mathcal{S}(n_2,n_4,n_5;t,N_*) \\ &\overset{\text{def}}{=} \Big| \sum_{n_3\in\mathbb{Z}^3} \Big[\rho_N^2(n_3)\rho_N^2(n_{345})\chi_{N_{345}}(n_{345})\chi_{N_1}(n_3)\chi_{N_3}(n_3)\widehat{V}_S(n_3,n_4,n_5)\langle n_{345}\rangle^{-1}\langle n_3\rangle^{-2} \\ &\times \exp\big(\pm_2 it\langle n_2\rangle\big) \int_0^t \sin\big((t-t')\langle n_{345}\rangle\big)\cos\big((t-t')\langle n_3\rangle\big) \prod_{j=4,5} \exp\big(\pm_j it'\langle n_j\rangle\big)\mathrm{d}t' \Big]\Big|.\end{aligned}$$

Define the phase-function φ by

$$\varphi(n_3,n_4,n_5) \overset{\text{def}}{=} \langle n_{345}\rangle \pm_3 \langle n_3\rangle \pm_4 \langle n_4\rangle \pm_5 \langle n_5\rangle. \tag{4.5.20}$$

By performing the integral, using the triangle-inequality, expanding the square, and using Lemma 4.4.25, we obtain that

$$\begin{aligned}&\mathcal{S}(n_2,n_4,n_5;t,N_*)^2 \\ &\lesssim T^2 \max_{\pm_3,\pm_4,\pm_5} \Big(\sum_{m\in\mathbb{Z}} \sum_{n_3\in\mathbb{Z}^3} \langle m\rangle^{-1} \chi_{N_3}(n_3)\langle n_{345}\rangle^{-1}\langle n_3\rangle^{-2} 1\big\{|\varphi-m|\leqslant 1\big\}\Big)^2 \\ &\lesssim T^2 \log(2+\max(N_3,N_4,N_5)) \Big(\max_{\pm_3,\pm_4,\pm_5} \sup_{m\in\mathbb{Z}} \sum_{n_3\in\mathbb{Z}^3} \chi_{N_3}(n_3)\langle n_{345}\rangle^{-1}\langle n_3\rangle^{-2} 1\big\{|\varphi-m|\leqslant 1\big\}\Big) \\ &\times \Big(\max_{\pm_3,\pm_4,\pm_5} \sum_{m\in\mathbb{Z}} \sum_{n_3\in\mathbb{Z}^3} \langle m\rangle^{-1} \chi_{N_3}(n_3)\langle n_{345}\rangle^{-1}\langle n_3\rangle^{-2} 1\big\{|\varphi-m|\leqslant 1\big\}\Big) \\ &\lesssim T^2 \log(2+\max(N_3,N_4,N_5)) \langle n_{45}\rangle^{-1} \\ &\times \max_{\pm_3,\pm_4,\pm_5} \sum_{m\in\mathbb{Z}} \sum_{n_3\in\mathbb{Z}^3} \langle m\rangle^{-1} \chi_{N_3}(n_3)\langle n_{345}\rangle^{-1}\langle n_3\rangle^{-2} 1\big\{|\varphi-m|\leqslant 1\big\}.\end{aligned}$$

By inserting this into (4.5.19) and summing in $n_2 \in \mathbb{Z}^3$ first, we obtain

$$
\begin{aligned}
&\sup_{t\in[0,T]} \Big\| \|\mathscr{G}^{(3)}(t,x;N_*)\|_{H_x^{s_2-1}(\mathbb{T}^3)} \Big\|_{L^2_\omega}^2 \\
&\lesssim T^2 \log(2+\max(N_3,N_4,N_5)) \Big(\prod_{j=2}^5 N_j^{-2} \Big) \\
&\times \max_{\pm_3,\pm_4,\pm_5} \sum_{m\in\mathbb{Z}} \sum_{n_2,n_3,n_4,n_5\in\mathbb{Z}^3} \Big[\langle m\rangle^{-1} \Big(\prod_{j=2}^5 \chi_{N_j}(n_j) \Big) \langle n_{245}\rangle^{2(s_2-1)} \langle n_{345}\rangle^{-1} \langle n_{45}\rangle^{-1-2\beta} 1\big\{ |\varphi - m| \leqslant 1 \big\} \Big] \\
&\lesssim T^2 \log(2+\max(N_3,N_4,N_5)) N_2^{2s_2-1} \Big(\prod_{j=3}^5 N_j^{-2} \Big) \\
&\times \max_{\pm_3,\pm_4,\pm_5} \sum_{m\in\mathbb{Z}} \sum_{n_3,n_4,n_5\in\mathbb{Z}^3} \Big[\langle m\rangle^{-1} \Big(\prod_{j=3}^5 \chi_{N_j}(n_j) \Big) \langle n_{345}\rangle^{-1} \langle n_{45}\rangle^{-1-2\beta} 1\big\{ |\varphi - m| \leqslant 1 \big\} \Big] \\
&\lesssim T^2 \log(2+\max(N_3,N_4,N_5)) N_2^{2s_2-1} \max(N_4,N_5)^{-2\beta}.
\end{aligned}
$$

In the last line, we have used the cubic sum estimate (Proposition 4.4.20). In total, this yields

$$
\sup_{t\in[0,T]} \Big\| \|\mathscr{G}^{(3)}(t,x;N_*)\|_{H_x^{s_2-1}(\mathbb{T}^3)} \Big\|_{L^2_\omega} \lesssim T \log(2+\max(N_3,N_4,N_5)) N_2^{s_2-\frac{1}{2}} \max(N_4,N_5)^{-\beta}. \quad (4.5.21)
$$

Recalling that $\max(N_3,N_4,N_5) > N_2^{\epsilon}$, we are only missing decay in N_3. By using the sine-cancellation lemma (Lemma 4.4.14) to estimate $\mathcal{S}(n_2,n_4,n_5;t,N_*)$, we easily obtain that

$$
\sup_{t\in[0,T]} \Big\| \|\mathscr{G}^{(3)}(t,x;N_*)\|_{H_x^{s_2-1}(\mathbb{T}^3)} \Big\|_{L^2_\omega} \lesssim T^2 N_2^{s_2-\frac{1}{2}} \max(N_4,N_5)^5 N_3^{-1}. \quad (4.5.22)
$$

After combining (4.5.21), (4.5.22), and the condition $\max(N_3,N_4,N_5) > N_2^{\epsilon}$, we obtain an acceptable estimate.

Single-resonance term $\widetilde{\mathscr{G}}^{(3)}$: This term can be controlled through similar (or simpler) arguments than $\mathscr{G}^{(3)}$ and we omit the details.

Double-resonance term $\mathscr{G}^{(1)}$:

This term only yields a non-trivial contribution when $N_1 \sim N_3$ and $N_2 \sim N_4$. We note that the sum in $n_3 \in \mathbb{Z}^3$ may appear to diverge logarithmically (once the dyadic localization is removed). However, the sine-function in the Duhamel integral yields additional cancellation, which was first observed by Gubinelli, Koch, and Oh in [GKO18a] and generalized slightly in Lemma 4.4.14.

Using the inhomogeneous Strichartz estimate (Lemma 4.4.9), it follows that

$$\begin{aligned} \|\mathscr{G}^{(1)}(t,x;N_*)\|_{\mathfrak{X}^{s_2-1,b_--1}([0,T])} &\lesssim \|\mathscr{G}^{(1)}(t,x;N_*)\|_{L_t^{2b_+}H_x^{s_2-1}([0,T]\times\mathbb{T}^3)} \\ &\lesssim T^{\frac{1}{2}}\|\mathscr{G}^{(1)}(t,x;N_*)\|_{L_t^2H_x^{s_2-1}([0,T]\times\mathbb{T}^3)}. \end{aligned}$$

Using Gaussian hypercontractivity (Proposition 4.4.45) and the orthogonality of multiple stochastic integrals, we obtain that

$$\begin{aligned} &T\Big\|\|\mathscr{G}^{(1)}(t,x;N_*)\|_{L_t^2H_x^{s_2-1}([0,T]\times\mathbb{T}^3)}\Big\|_{L_\omega^p}^2 \\ &\lesssim Tp\Big\|\|\mathscr{G}^{(1)}(t,x;N_*)\|_{L_t^2H_x^{s_2-1}([0,T]\times\mathbb{T}^3)}\Big\|_{L_\omega^2}^2 \\ &\lesssim T^2p\sup_{t\in[0,T]}\sum_{n_5\in\mathbb{Z}^3}\chi_{N_5}(n_5)\langle n_5\rangle^{2(s_2-1)-2}\mathcal{S}(n_5;t,N_*)^2 \end{aligned} \tag{4.5.23}$$

where

$$\begin{aligned} \mathcal{S}(n_5;t,N_*) \overset{\text{def}}{=} \Bigg| \sum_{n_3,n_4\in\mathbb{Z}^4} \Bigg[&\rho_N^2(n_{345})\rho_N^2(n_3)\rho_N^2(n_4)\chi_{N_{345}}(n_{345})\chi_{N_1}(n_3)\chi_{N_3}(n_3)\chi_{N_2}(n_4)\chi_{N_4}(n_4) \\ &\times \widehat{V}_S(n_3,n_4,n_5)\widehat{V}(n_{45})\langle n_{345}\rangle^{-1}\langle n_3\rangle^{-2}\langle n_4\rangle^{-2} \\ &\times \int_0^t \sin\big((t-t')\langle n_{345}\rangle\big)\cos\big((t-t')\langle n_3\rangle\big)\cos\big((t-t')\langle n_4\rangle\big)\exp\big(\pm_5\, it'\langle n_5\rangle\big)\mathrm{d}t'\Bigg]\Bigg|. \end{aligned}$$

We now present two different estimates of $\mathcal{S}(n_5;t,N_*)$. The first (and main) estimates almost yields control over $\mathscr{G}^{(1)}$, but exhibits a logarithmic divergence in N_3. The second estimates exhibits polynomial growth in N_4 and N_5, but yields the desired decay in N_3.

Using that $|\widehat{V}(n_{45})| \lesssim \langle n_{45} \rangle^{-\beta}$ and the crude estimate $|\widehat{V}_S(n_3, n_4, n_5)| \lesssim 1$, we obtain that

$$\begin{aligned}
\mathcal{S}(n_5; t, N_*) &\lesssim N_{345}^{-1} N_3^{-2} N_4^{-2} \sum_{n_3, n_4 \in \mathbb{Z}^3} \bigg[1\Big\{ |n_3| \sim N_3,\ |n_4| \sim N_4,\ |n_{345}| \sim N_{345} \Big\} \langle n_{45} \rangle^{-\beta} \\
&\quad \times \Big| \int_0^t \sin\big((t-t')\langle n_{345} \rangle \big) \cos\big((t-t')\langle n_3 \rangle \big) \cos\big((t-t')\langle n_4 \rangle \big) \exp\big(\pm_5 it' \langle n_5 \rangle \big) \mathrm{d}t' \Big| \bigg] \\
&\lesssim T \log(2 + \max(N_3, N_4, N_5)) N_{345}^{-1} N_3^{-2} N_4^{-2} \\
&\quad \times \max_{\pm_3, \pm_4, \pm_5} \sup_{m \in \mathbb{Z}} \sum_{n_3, n_4 \in \mathbb{Z}^3} \bigg[1\Big\{ |n_3| \sim N_3,\ |n_4| \sim N_4,\ |n_{345}| \sim N_{345} \Big\} \langle n_{45} \rangle^{-\beta} 1\Big\{ |\varphi - m| \leqslant 1 \Big\} \bigg],
\end{aligned}$$

where the phase-function φ is given by

$$\varphi(n_3, n_4, n_5) \stackrel{\text{def}}{=} \langle n_{345} \rangle \pm_3 \langle n_3 \rangle \pm_4 \langle n_4 \rangle \pm_5 \langle n_5 \rangle.$$

Using the counting estimate from Lemma 4.4.29, it follows that

$$\mathcal{S}(n_5; t, N_*) \lesssim T \log(2 + \max(N_3, N_4, N_5)) \max(N_4, N_5)^{-\beta + \eta}. \tag{4.5.24}$$

Alternatively, it follows from the sine-cancellation lemma (Lemma 4.4.14) with $A = N_4^2 N_5^2$, say, that

$$\mathcal{S}(n_5; t, N_*) \lesssim T^2 N_3^{-1} N_4^5 N_5^2. \tag{4.5.25}$$

By combining (4.5.23), (4.5.24), and (4.5.25), it follows that

$$\begin{aligned}
&T^{\frac{1}{2}} \Big\| \| \mathscr{G}^{(1)}(t, x; N_*) \|_{L_t^2 H_x^{s_2 - 1}([0,T] \times \mathbb{T}^3)} \Big\|_{L_\omega^p} \\
&\lesssim T^3 p^{\frac{1}{2}} \log(2 + \max(N_3, N_4, N_5)) N_5^{s_2 - \frac{1}{2}} \min\Big(N_4^{-\beta}, N_5^{-\beta}, N_3^{-1} N_4^5 N_5^5 \Big) \\
&\lesssim T^3 p^{\frac{1}{2}} N_5^{s_2 - \frac{1}{2} - \beta + 20\eta} \max(N_3, N_4, N_5)^{-\eta} \\
&\lesssim T^3 p^{\frac{1}{2}} \max(N_3, N_4, N_5)^{-\eta}.
\end{aligned}$$

This contribution is acceptable.

□

Proof of Proposition 4.5.8: This estimate is similar (but easier) than Proposition 4.5.7 and we therefore omit the details. Instead of gaining additional regularity through the para-differential operator as in Proposition 4.5.8, we simply use interaction potential V and the crude inequality

$$\langle n_{12} \rangle^{-2\beta} \lesssim \langle n_{12} \rangle^{-2\gamma} \lesssim \langle n_{12345} \rangle^{-2\gamma} \langle n_{345} \rangle^{2\gamma}$$

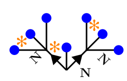
for $0 < \gamma < \beta$. $\square$

4.5.4 Septic stochastic objects

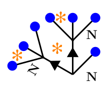
The next proposition controls the third and fourth term in **So**, i.e., in (4.2.28).

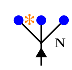
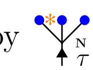
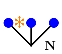
Proposition 4.5.10 (Septic stochastic objects)**.** Let $T \geqslant 1$ and $p \geqslant 1$. Then, it holds that

$$\Big\| \sup_{N \geqslant 1} \Big\| \quad \Big\|_{\mathfrak{X}^{s_2 -1, b_+ -1}([0,T])} \Big\|_{L^p_\omega(\mathbb{P})} \lesssim T^4 p^{7/2}, \tag{4.5.26}$$



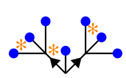
$$\Big\| \sup_{N \geqslant 1} \Big\| \; (\neg \circledless) \quad \Big\|_{\mathfrak{X}^{s_2 -1, b_+ -1}([0,T])} \Big\|_{L^p_\omega(\mathbb{P})} \lesssim T^4 p^{7/2}. \tag{4.5.27}$$



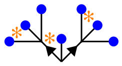
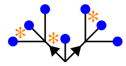
Remark 4.5.11. In the frequency-localized version of Proposition 4.5.10, we gain an η'-power of the maximal frequency-scale. As in Proposition 4.5.1, we may also replace  by  $=$ $\mathrm{I}\left[1_{[0,\tau]}\right.$  $\left.\right]$. We will not further comment on these minor modifications.

Proof. We only prove (4.5.26). The second estimate (4.5.27) follows from similar (but slightly simpler) arguments. To simplify the notation, we formally set $N = \infty$. The same argument also yields the estimate for the supremum over N. Using the inhomogeneous Strichartz estimate (Lemma 4.4.9) and Gaussian hypercontractivity (Proposition 4.4.45), it suffices to prove that

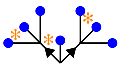
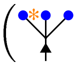
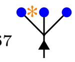
$$\sup_{t \in [0,T]} \Big\| \Big\| \quad \Big\|_{H^{s_2-1}_x(\mathbb{T}^3)} \Big\|_{L^2_\omega(\mathbb{P})} \lesssim T^3. \tag{4.5.28}$$



Using a Littlewood-Paley decomposition, we write

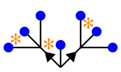
 $$= \sum_{N_{1234567}, N_{1234}, N_4, N_{567}}$$  $$[N_{1234567}, N_{1234}, N_4, N_{567}],$$

where

 $$[N_{1234567}, N_{1234}, N_4, N_{567}] \overset{\text{def}}{=} P_{N_{1234567}}\Bigg[(P_{N_{1234}}\widehat{V}) * \Big($$  $$\cdot P_{N_4}$$  $$\Big) P_{N_{567}}$$  $$\Bigg]. \tag{4.5.29}$$

We now present two separate estimates of (4.5.29). The first estimate, which is the main part of the argument, almost yields (4.5.28), but contains a logarithmic divergence in N_4. The second (short) estimate exhibits polynomial decay in N_4, and is only used to remove this logarithmic divergence.

Main estimate: Using the stochastic representation of the cubic nonlinearity (Proposition 4.4.44) and (4.4.76), we obtain that

 $$[N_{1234567}, N_{1234}, N_4, N_{567}]$$

$$\begin{aligned}
&= \sum_{n_1,\ldots,n_7 \in \mathbb{Z}^3} \sum_{\pm_1,\ldots,\pm_7} \Bigg[\chi_{N_{1234567}}(n_{1234567}) \chi_{N_{1234}}(n_{1234}) \chi_{N_4}(n_4) \chi_{N_{567}}(n_{567}) \widehat{V}(n_{1234}) \\
&\times \Phi(t, n_j, \pm_j \colon 1 \leqslant j \leqslant 3) e^{\pm it\langle n_4 \rangle} \frac{1}{\langle n_4 \rangle} \Phi(t, n_j, \pm_j \colon 5 \leqslant j \leqslant 7) \exp\big(i\langle n_{1234567}, x\rangle\big) \\
&\times \widetilde{\mathcal{I}}_3[n_j, \pm_j \colon 1 \leqslant j \leqslant 3] \widetilde{\mathcal{I}}_1[n_4, \pm_4] \widetilde{\mathcal{I}}_3[n_j, \pm_j \colon 5 \leqslant j \leqslant 7] \Bigg].
\end{aligned} \tag{4.5.30}$$

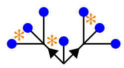
Here, the amplitude Φ is given by

$$\begin{aligned}
&\Phi(t, n_j, \pm_j \colon 1 \leqslant j \leqslant 3) \\
&\overset{\text{def}}{=} \langle n_{123} \rangle^{-1} \widehat{V}_S(n_1, n_2, n_3) \Big(\prod_{j=1}^{3} \langle n_j \rangle^{-1} \Big) \Big(\int_0^t \sin\big((t-t')\langle n_{123} \rangle \prod_{j=1}^{3} \exp\big(\pm_j\, it' \langle n_j \rangle\big) \mathrm{d}t' \Big).
\end{aligned}$$

Comparing with $\Phi(n_1, n_2, n_3)$ as in Lemma 4.4.31, we have that

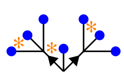
$$\sup_{t\in[0,T]} |\Phi(t, n_j, \pm_j \colon 1 \leqslant j \leqslant 3)| \lesssim T\Phi(n_1, n_2, n_3). \tag{4.5.31}$$

We now rely on the notation from Definition 4.4.30 and Lemma 4.4.31. Using the product formula for multiple stochastic integrals twice (Lemma 4.4.40), the orthogonality of multiple stochastic integrals, and (4.5.31), we obtain that



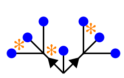
$$\begin{aligned}&\sup_{t\in[0,T]} \Big\| \,\cdot\, [N_{1234567}, N_{1234}, N_4, N_{567}] \Big\|^2_{L^2_\omega H^{s_2-1}_x(\Omega\times\mathbb{T}^3)} \\ &\lesssim T^4 \sum_{\mathscr{P}} \sum_{(n_j)_{j\notin\mathscr{P}}} \langle n_{\mathrm{nr}}\rangle^{2(s_2-1)} \Bigg(\sum^{*}_{(n_j)_{j\in\mathscr{P}}} 1\big\{|n_{1234567}| \sim N_{1234567}\big\} 1\big\{|n_{1234}| \sim N_{1234}\big\} 1\big\{|n_{567}| \sim N_{567}\big\} \\ &\times 1\big\{|n_4| \sim N_4\big\} |\widehat{V}(n_{1234})| \Phi(n_1, n_2, n_3) \langle n_4\rangle^{-1} \Phi(n_5, n_6, n_7) \Bigg)^2 .\end{aligned}$$

The sum in $\mathscr{P}$ is taken over all pairings which respect the partition $\{1,2,3\}, \{4\}, \{5,6,7\}$. For a similar argument, we refer the reader to [DNY19, Lemma 4.1]. Using Lemma 4.4.31, it follows that



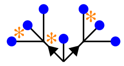
$$\begin{aligned}&\sup_{t\in[0,T]} \Big\| \,\cdot\, [N_{1234567}, N_{1234}, N_4, N_{567}] \Big\|_{L^2_\omega H^{s_2-1}_x(\Omega\times\mathbb{T}^3)} \\ &\lesssim T^2 \log(2+N_4) \Big(N_{1234567}^{(s_2-\frac{1}{2})} N_{567}^{-(\beta-\eta)} + N_{1234567}^{-(1-s_2-\eta)} \Big) N_{1234}^{-\beta}.\end{aligned} \tag{4.5.32}$$

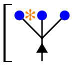
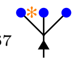
Since $N_{1234567} \lesssim \max(N_{1234}, N_{567})$ and $N_{1234567} \sim N_{567}$ if $N_{1234} \ll N_{567}$, we obtain that



$$\begin{aligned}&\sup_{t\in[0,T]} \Big\| \,\cdot\, [N_{1234567}, N_{1234}, N_4, N_{567}] \Big\|_{L^2_\omega H^{s_2-1}_x(\Omega\times\mathbb{T}^3)} \\ &\lesssim T^2 \log(2+N_4) \max(N_{1234567}, N_{1234}, N_{567})^{-(\beta-\eta-\delta_2)}.\end{aligned} \tag{4.5.33}$$

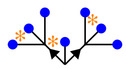
Removing the logarithmic divergence in N_4: Using Proposition 4.5.1 and (4.5.11) from Proposition 4.5.3, we obtain that

$$\sup_{t\in[0,T]} \Big\|$$  $$[N_{1234567}, N_{1234}, N_4, N_{567}] \Big\|_{L^2_\omega H^{s_2-1}_x(\Omega\times\mathbb{T}^3)}$$

$$\lesssim \Big\| P_{N_{1234}} \Big[$$  $$\cdot P_{N_4} \Big] \Big\|_{L^4_\omega L^\infty_t L^2_x(\Omega\times[0,T]\times\mathbb{T}^3)} \Big\| P_{N_{567}}$$  $$\Big\|_{L^4_\omega L^\infty_t L^\infty_x(\Omega\times[0,T]\times\mathbb{T}^3)} \qquad (4.5.34)$$

$$\lesssim T^5 N_{1234} N_4^{-\frac{\eta}{10}}.$$

By combing (4.5.33) and (4.5.34), we obtain that

$$\sup_{t\in[0,T]} \Big\|$$  $$[N_{1234567}, N_{1234}, N_4, N_{567}] \Big\|_{L^2_\omega H^{s_2-1}_x(\Omega\times\mathbb{T}^3)} \qquad (4.5.35)$$

$$\lesssim T^3 N_4^{-\eta^2} \max(N_{1234567}, N_{1234}, N_{567})^{-(\beta-\eta-2\delta_2)}.$$

After summing over the dyadic scales, this yields (4.5.26). □

4.6 Random matrix theory estimates

In this section, we control the random matrix terms **RMT**. Techniques from random matrix theory, such as the moment method, were first applied to dispersive equations in Bourgain's seminal paper [Bou96]. Over the last decade, they have become an indispensable tool in the study of dispersive PDE and we refer the interested reader to [Bou97, CG19, DH19, DNY19, FOS21, GKO18a, Ric16]. Very recently, Deng, Nahmod, and Yue [DNY20, Proposition 2.8] obtained an easy-to-use, general, and essentially sharp random matrix estimate, which is proved using the moment method. We have previously recalled their estimate in Proposition 4.4.50. The proofs of Proposition 4.6.1 and Proposition 4.6.3 combine their random matrix estimate with the counting estimates in Section 4.4.4.

Proposition 4.6.1 (First RMT estimate)**.** Let $T \geqslant 1$ and let $p \geqslant 1$. Then, it holds that

$$\Big\| \sup_{N\geqslant 1} \sup_{\mathcal{J}\subseteq[0,T]} \sup_{\|w\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})}\leqslant 1} \Big\| (V * $$ $$) \cdot P_{\leqslant N} w \Big\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})} \Big\|_{L^p_\omega(\mathbb{P})} \lesssim Tp. \qquad (4.6.1)$$

Remark 4.6.2. This proposition controls the first term in **RMT**, i.e., in (4.2.30). In the frequency-localized version of (4.6.1), which is detailed in the proof, we gain an η'-power in the maximal frequency-scale.

Proof. The arguments splits into two steps: First, we bring (4.6.1) into a random matrix form. Then, we prove a random matrix estimate using the moment method (Proposition 4.4.50).

Step 1: The random matrix form. By definition of the restricted norms, it holds that

$$\begin{aligned} &\sup_{\mathcal{J}\subseteq[0,T]} \sup_{\|w\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})}\leqslant 1} \Big\| (V * \overset{\bullet\,\bullet}{\vee}_{N}) \cdot P_{\leqslant N} w \Big\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})} \\ &\leqslant \sup_{\|w\|_{\mathfrak{X}^{s_1,b}(\mathbb{R})}\leqslant 1} \Big\| \chi(t/T)(V * \overset{\bullet\,\bullet}{\vee}_{N}) \cdot P_{\leqslant N} w \Big\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathbb{R})}. \end{aligned} \tag{4.6.2}$$

We bound the right-hand side of (4.6.2) with b_+ replaced by b_-. Using the frequency-localized estimate in the arguments below and a similar reduction as in the proof of Proposition 4.5.1, we can then upgrade the value from b_- to b_+. Let $w \in \mathfrak{X}^{s_1,b}(\mathbb{R})$ satisfy $\|w\|_{\mathfrak{X}^{s_1,b}(\mathbb{R})} \leqslant 1$. We define $w_\pm \in \mathfrak{X}^{s_1,b}(\mathbb{R})$ by

$$\widehat{w}_\pm(\lambda,n) \overset{\text{def}}{=} 1\big\{ \pm\lambda \geqslant 0\big\} \widehat{w}(\lambda,n).$$

Then, it holds that $w = w_+ + w_-$ and

$$\|w\|_{\mathfrak{X}^{s_1,b}(\mathbb{R})} \sim \max_{\pm} \|\langle n\rangle^{s_1}\langle\lambda\rangle^{b}\widehat{w}_\pm(\lambda\pm\langle n\rangle, n)\|_{L^2_\lambda \ell^2_n(\mathbb{R}\times\mathbb{T}^3)}.$$

Using this decomposition of w and the stochastic representation of the renormalized square, we obtain that the nonlinearity is given by

$$
\begin{aligned}
&(V * \bigvee_{\scriptscriptstyle N}) \cdot P_{\leqslant N} w \\
&= \sum_{\pm_1,\pm_2,\pm_3} \sum_{N_1,N_2,N_3} \sum_{n_1,n_2,n_3 \in \mathbb{Z}^3} \Bigg[c(\pm_1,\pm_2) \Big(\prod_{j=1}^{3} \rho_N(n_j) \chi_{N_j}(n_j) \Big) \widehat{V}(n_{12}) \mathcal{I}_2[\pm_j, n_j \colon j=1,2] \\
&\times \Big(\prod_{j=1}^{2} \exp\big(\pm_j \, it\langle n_j \rangle \big) \Big) \widehat{w}_{\pm_3}(t,n_3) \exp\big(i \langle n_{123}, x \rangle \big) \Bigg] \\
&= \sum_{\pm_1,\pm_2,\pm_3} \sum_{N_1,N_2,N_3} \int_{\mathbb{R}} \mathrm{d}\lambda_3 \sum_{n_1,n_2,n_3 \in \mathbb{Z}^3} \Bigg[c(\pm_1,\pm_2) \Big(\prod_{j=1}^{3} \rho_N(n_j) \chi_{N_j}(n_j) \Big) \widehat{V}(n_{12}) \mathcal{I}_2[\pm_j, n_j \colon j=1,2] \\
&\times \exp\big(it\lambda_3 \big) \Big(\prod_{j=1}^{3} \exp\big(\pm_j \, it\langle n_j \rangle \big) \Big) \widehat{w}_{\pm_3}(\lambda_3 \pm_3 \langle n_3 \rangle, n_3) \exp\big(i \langle n_{123}, x \rangle \big) \Bigg].
\end{aligned}
$$

To simplify the notation, we define the phase-function $\varphi \colon (\mathbb{Z}^3)^3 \to \mathbb{R}$ by

$$
\varphi(n_1,n_2,n_3) = \pm_{123} \langle n_{123} \rangle \pm_1 \langle n_1 \rangle \pm_2 \langle n_2 \rangle \pm_3 \langle n_3 \rangle. \tag{4.6.3}
$$

The space-time Fourier transform of the time-truncated nonlinearity is therefore given by

$$
\begin{aligned}
&\mathcal{F}\Big(\chi(\cdot/T)(V * \bigvee_{\scriptscriptstyle N}) \cdot P_{\leqslant N} w \Big)(\lambda \pm_{123} \langle n \rangle, n) \\
&= T \sum_{\pm_1,\pm_2,\pm_3} \sum_{N_1,N_2,N_3} \int_{\mathbb{R}} \mathrm{d}\lambda_3 \sum_{n_1,n_2,n_3 \in \mathbb{Z}^3} \Bigg[c(\pm_1,\pm_2) 1\big\{ n = n_{123} \big\} \widehat{\chi}\big(T(\lambda - \lambda_3 - \varphi(n_1,n_2,n_3)) \big) \\
&\times \Big(\prod_{j=1}^{3} \rho_N(n_j) \chi_{N_j}(n_j) \Big) \widehat{V}(n_{12}) \mathcal{I}_2[\pm_j, n_j \colon j=1,2] \widehat{w}_{\pm_3}(\lambda_3 \pm_3 \langle n_3 \rangle, n_3) \Bigg].
\end{aligned} \tag{4.6.4}
$$

To simplify the following notation, we emphasize the dependence on the frequency-scales N_1, N_2, N_3 by writing N_* and omit the dependence on $\pm_{123}, \pm_1, \pm_2, \pm_3$, and T from our notation. We define the tensor $h(n,n_1,n_2,n_3;\lambda,\lambda_3,N_*)$ by

$$
\begin{aligned}
h(n,n_1,n_2,n_3;\lambda,\lambda_3,N_*) &\overset{\text{def}}{=} T c(\pm_1,\pm_2) 1\big\{ n = n_{123} \big\} \widehat{\chi}\big(T(\lambda - \lambda_3 - \varphi(n_1,n_2,n_3)) \big) \\
&\times \Big(\prod_{j=1}^{3} \rho_N(n_j) \chi_{N_j}(n_j) \Big) \widehat{V}(n_{12}) \langle n \rangle^{s_2 - 1} \langle n_1 \rangle^{-1} \langle n_2 \rangle^{-1} \langle n_3 \rangle^{-s_1}.
\end{aligned} \tag{4.6.5}
$$

Furthermore, we define the contracted random tensor $h_c(n, n_3; \lambda, \lambda_3)$ by

$$h_c(n, n_3; \lambda, \lambda_3, N_*) = \sum_{n_1, n_2 \in \mathbb{Z}^3} h(n, n_1, n_2, n_3; \lambda, \lambda_3, N_*) \cdot \widetilde{\mathcal{I}}_2[\pm_j, n_j \colon j = 1, 2]. \tag{4.6.6}$$

By combining our previous expression of the nonlinearity (4.6.4) with the definition (4.6.6), we obtain that

$$\begin{aligned}
&\mathcal{F}\Big(\chi(\cdot/T)(V * \curlyvee_{N}) \cdot P_{\leqslant N} w\Big)(\lambda \pm_{123} \langle n \rangle, n) \\
&= \langle n \rangle^{-(s_2 - 1)} \sum_{\pm_1, \pm_2, \pm_3} \sum_{N_1, N_2, N_3} \int_{\mathbb{R}} \mathrm{d}\lambda_3 \sum_{n_3 \in \mathbb{Z}^3} h_c(n, n_3; \lambda, \lambda_3, N_*) \langle n_3 \rangle^{s_1} \widehat{w}_{\pm_3}(\lambda_3 \pm_3 \langle n_3 \rangle, n_3).
\end{aligned}$$

We estimate each combination of signs and each dyadic block separately. Using the tensor norms from Definition 4.4.32, the contribution to the $\mathfrak{X}^{s_2 - 1, b_- - 1}$-norm is bounded by

$$\begin{aligned}
&\Big\| \langle \lambda \rangle^{b_- - 1} \int_{\mathbb{R}} \mathrm{d}\lambda_3 \sum_{n_3 \in \mathbb{Z}^3} h_c(n, n_3; \lambda, \lambda_3, N_*) \langle n_3 \rangle^{s_1} \widehat{w}_{\pm_3}(\lambda_3 \pm_3 \langle n_3 \rangle, n_3) \Big\|_{L^2_\lambda \ell^2_n(\mathbb{R} \times \mathbb{T}^3)} \\
&\lesssim \Big\| \langle \lambda \rangle^{b_- - 1} \langle \lambda_3 \rangle^{-b} \| h_c(n, n_3; \lambda, \lambda_3, N_*) \|_{n_3 \to n} \Big\|_{L^2_\lambda L^2_{\lambda_3}(\mathbb{R} \times \mathbb{R})} \cdot \| w \|_{\mathfrak{X}^{s_1, b}(\mathbb{R})}.
\end{aligned}$$

In order to control the operator norm in (4.6.2), it therefore remains to prove that

$$\Big\| \Big\| \langle \lambda \rangle^{b_- - 1} \langle \lambda_3 \rangle^{-b} \| h_c(n, n_3; \lambda, \lambda_3, N_*) \|_{n_3 \to n} \Big\|_{L^2_\lambda L^2_{\lambda_3}(\mathbb{R} \times \mathbb{R})} \Big\|_{L^p_\omega(\mathbb{P})} \lesssim T \max(N_1, N_2, N_3)^{-\frac{\eta}{2}} p. \tag{4.6.7}$$

Step 2: Proof of the random matrix estimate (4.6.7)*.* Using Minkowski's integral inequality, we have that

$$\begin{aligned}
&\Big\| \Big\| \langle \lambda \rangle^{b_- - 1} \langle \lambda_3 \rangle^{-b} \| h_c(n, n_3; \lambda, \lambda_3, N_*) \|_{n_3 \to n} \Big\|_{L^2_\lambda L^2_{\lambda_3}(\mathbb{R} \times \mathbb{R})} \Big\|_{L^p_\omega(\mathbb{P})} \\
&\leqslant \Big\| \langle \lambda \rangle^{b_- - 1} \langle \lambda_3 \rangle^{-b} \Big\| \| h_c(n, n_3; \lambda, \lambda_3, N_*) \|_{n_3 \to n} \Big\|_{L^p_\omega(\mathbb{P})} \Big\|_{L^2_\lambda L^2_{\lambda_3}(\mathbb{R} \times \mathbb{R})} \\
&\leqslant \Big\| \langle \lambda \rangle^{b_- - 1} \langle \lambda_3 \rangle^{-b} \Big\|_{L^2_\lambda L^2_{\lambda_3}(\mathbb{R} \times \mathbb{R})} \cdot \sup_{\lambda, \lambda_3 \in \mathbb{R}} \Big\| \| h_c(n, n_3; \lambda, \lambda_3, N_*) \|_{n_3 \to n} \Big\|_{L^p_\omega(\mathbb{P})} \\
&\lesssim \sup_{\lambda, \lambda_3 \in \mathbb{R}} \Big\| \| h_c(n, n_3; \lambda, \lambda_3, N_*) \|_{n_3 \to n} \Big\|_{L^p_\omega(\mathbb{P})}.
\end{aligned}$$

We emphasize that the supremum over $\lambda, \lambda_3 \in \mathbb{R}$ is outside of the $L^p_\omega(\mathbb{P})$-norm. Using the moment method (Proposition 4.4.50), it holds that

$$\begin{aligned}
&\sup_{\lambda,\lambda_3\in\mathbb{R}} \Big\| \|h_c(n,n_3;\lambda,\lambda_3,N_*)\|_{n_3\to n}\Big\|_{L^p_\omega(\mathbb{P})} \\
&\lesssim \max(N_1,N_2,N_3)^{\frac{\eta}{2}} \sup_{\lambda,\lambda_3\in\mathbb{R}} \max\Big(\|h(\cdot;\lambda,\lambda_3,N_*)\|_{n_1n_2n_3\to n}, \|h(\cdot;\lambda,\lambda_3,N_*)\|_{n_3\to nn_1n_2}, \\
&\qquad \|h(\cdot;\lambda,\lambda_3,N_*)\|_{n_1n_3\to nn_2}, \|h(\cdot;\lambda,\lambda_3,N_*)\|_{n_2n_3\to nn_1}\Big) p.
\end{aligned}$$

In order to estimate the tensor norms of $h(\cdot;\lambda,\lambda_3,N_*)$, we further decompose it according to the value of the phase-function φ. For any $m \in \mathbb{Z}$, we define

$$\begin{aligned}
\widetilde{h}(n,n_1,n_2,n_3;m,N_*) &\overset{\text{def}}{=} T 1\big\{n = n_{123}\big\} 1\big\{|\varphi(n_1,n_2,n_3)) - m| \leqslant 1\big\} \Big(\prod_{j=1}^{3} \rho_N(n_j)\chi_{N_j}(n_j)\Big) \\
&\qquad \times |\widehat{V}(n_{12})| \langle n\rangle^{s_2-1}\langle n_1\rangle^{-1}\langle n_2\rangle^{-1}\langle n_3\rangle^{-s_1}.
\end{aligned}$$

Using the definition of h in (4.6.5) and the decay of $\widehat{\chi}$, we obtain that

$$\begin{aligned}
&|h(n,n_1,n_2,n_3;\lambda,\lambda_3,N_*)| \\
&\lesssim \sum_{m\in\mathbb{Z}} |\widehat{\chi}\big(T(\lambda-\lambda_3-\varphi(n_1,n_2,n_3))\big)| \, 1\big\{|\varphi(n_1,n_2,n_3)-m| \leqslant 1\big\} \, \widetilde{h}(n,n_1,n_2,n_3;m,N_*) \\
&\lesssim \sum_{m\in\mathbb{Z}} \langle \lambda_3 - \lambda - m\rangle^{-2} \, \widetilde{h}(n,n_1,n_2,n_3;m,N_*).
\end{aligned}$$

Using the triangle inequality for the tensor norms and the first deterministic tensor estimate (Lemma 4.4.33), it follows that

$$\begin{aligned}
&\max(N_1,N_2,N_3)^{\frac{\eta}{2}} \sup_{\lambda,\lambda_3\in\mathbb{R}} \max\Big(\|h(\cdot;\lambda,\lambda_3,N_*)\|_{n_1n_2n_3\to n}, \|h(\cdot;\lambda,\lambda_3,N_*)\|_{n_3\to nn_1n_2}, \\
&\quad \|h(\cdot;\lambda,\lambda_3,N_*)\|_{n_1n_3\to nn_2}, \|h(\cdot;\lambda,\lambda_3,N_*)\|_{n_2n_3\to nn_1}\Big) \\
&\lesssim \max(N_1,N_2,N_3)^{\frac{\eta}{2}} \sup_{m\in\mathbb{Z}} \max\Big(\|\widetilde{h}(\cdot;m,N_*)\|_{n_1n_2n_3\to n}, \|\widetilde{h}(\cdot;m,N_*)\|_{n_3\to nn_1n_2}, \\
&\quad \|\widetilde{h}(\cdot;m,N_*)\|_{n_1n_3\to nn_2}, \|\widetilde{h}(\cdot;m,N_*)\|_{n_2n_3\to nn_1}\Big) \\
&\lesssim T \max(N_1,N_2,N_3)^{-\frac{\eta}{2}}.
\end{aligned}$$

□

Proposition 4.6.3 (Second RMT estimate)**.** Let $T \geqslant 1$ and let $p \geqslant 1$. Then, it holds that

$$\Big\| \sup_{N\geqslant 1} \sup_{\mathcal{J}\subseteq[0,T]} \sup_{\|Y\|_{\mathfrak{X}^{s_2,b}(\mathcal{J})}\leqslant 1} \Big\| :V * \big(P_{\leqslant N}\dot{|} \cdot P_{\leqslant N}Y\big) \ \big(\neg\varolessthan\big)\ P_{\leqslant N}\dot{|}: \Big\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})} \Big\|_{L^p_\omega(\mathbb{P})} \lesssim Tp. \tag{4.6.8}$$

Remark 4.6.4. This proposition controls the second term in **RMT**, i.e., in (4.2.30). In the frequency-localized version of (4.6.8), which is detailed in the proof, we gain an η'-power in the maximal frequency-scale.

Proof. Due to the operator $\big(\neg\varolessthan\big)$, the renormalization $\mathcal{M}_N P_{\leqslant N}Y$ does not just cancel the probabilistic resonances between the two factors of $\dot{|}$ in

$$V * \big(P_{\leqslant N}\dot{|} \cdot P_{\leqslant N}Y\big) \ \big(\neg\varolessthan\big)\ P_{\leqslant N}\dot{|}.$$

As a result, we need to decompose $\mathcal{M}_N = \mathcal{M}_N^{\varolessthan} + \mathcal{M}_N^{\neg\varolessthan}$, where the symbols corresponding to the multipliers are given by

$$\begin{aligned} m_N^{\varolessthan}(n) &\overset{\text{def}}{=} \sum_{L,K\colon L\leqslant K^\epsilon} \frac{\widehat{V}(n+k)}{\langle k\rangle^2} \chi_L(n+k)\chi_K(k)\rho_N^2(k), \\ m_N^{\neg\varolessthan}(n) &\overset{\text{def}}{=} \sum_{L,K\colon L> K^\epsilon} \frac{\widehat{V}(n+k)}{\langle k\rangle^2} \chi_L(n+k)\chi_K(k)\rho_N^2(k). \end{aligned}$$

The random operator

$$V * \big(P_{\leqslant N}\dot{|} \cdot P_{\leqslant N}Y\big) \ \big(\neg\varolessthan\big)\ P_{\leqslant N}\dot{|} - \mathcal{M}_N^{\neg\varolessthan} P_{\leqslant N}Y$$

can then be controlled using the same argument as in the proof of Proposition 4.6.1, except that we use Lemma 4.4.35 instead of Lemma 4.4.33. Thus, it only remains to show that

$$\|\mathcal{M}_N^{\varolessthan} P_{\leqslant N}Y\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})} \lesssim T\|Y\|_{\mathfrak{X}^{s_2,b}(\mathcal{J})}. \tag{4.6.9}$$

The estimate (4.6.9) has a lot of room and can be established through the following simple argument. On the support of the summand in the definition of $m_N^{\circledless}$, it holds that $|n+k| \lesssim |k|^\epsilon$. Using only that $\widehat{V}$ is bounded, this implies that

$$|m_N^{\circledless}(n)| \lesssim \sum_{K \geqslant 1} \sum_{k \in \mathbb{Z}^3} K^{-2} 1\{|n+k| \lesssim K^\epsilon\} \lesssim \sum_{K \geqslant 1} K^{-2+3\epsilon} \lesssim 1.$$

Thus, the symbol $m_N^{\circledless}(n)$ is uniformly bounded and hence the corresponding multiplier $\mathcal{M}_N^{\circledless}$ is bounded on each Sobolev-space $H_x^s(\mathbb{T}^3)$. Using the Strichartz estimates (Corollary 4.4.7 and Lemma 4.4.9), we obtain that

$$\begin{aligned} \|\mathcal{M}_N^{\circledless} P_{\leqslant N} Y\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})} &\lesssim \|\mathcal{M}_N^{\circledless} P_{\leqslant N} Y\|_{L_t^{2b_+} H_x^{s_2-1}(\mathcal{J}\times\mathbb{T}^3)} \\ &\lesssim (1+|\mathcal{J}|) \|Y\|_{L_t^\infty H_x^{s_2-1}(\mathcal{J}\times\mathbb{T}^3)} \lesssim (1+|\mathcal{J}|)\|Y\|_{\mathfrak{X}^{s_2,b}(\mathcal{J})}. \end{aligned}$$

□

4.7 Para-controlled estimates

The main goal of this section is to estimate the terms in **CPara**. We remind the reader that the para-controlled approach to stochastic partial differential equations was introduced in the seminal paper of Gubinelli, Imkeller, and Perkowski [GIP15] and first applied to dispersive equations by Gubinelli, Koch, and Oh in [GKO18a].

The following definitions of the low-frequency modulation space $\mathscr{L}\mathscr{M}$ and the para-controlled structure PCtrl are following similar ideas as the framework in [GKO18a].

Definition 4.7.1 (Low-frequency modulation space)**.** Let $H = \{H(t,x;K)\}_{K\geqslant 1}\}$ be a family of space-time functions from $\mathbb{R}\times\mathbb{T}^3$ into $\mathbb{C}$ satisfying

$$\operatorname{supp}(\widehat{H}(t,x;K)) \subseteq \{k \in \mathbb{Z}^3 \colon |k| \leqslant 8K^\epsilon\}. \tag{4.7.1}$$

We define the low-frequency modulation norm by

$$\|H\|_{\mathscr{LM}(\mathbb{R})} \overset{\text{def}}{=} \sup_{K\geqslant 1} K^{-4\epsilon} \|\widehat{H}(\lambda,k;K)\|_{\ell_k^\infty L_\lambda^1(\mathbb{Z}^3\times\mathbb{R})}. \tag{4.7.2}$$

We define the corresponding low-frequency modulation space $\mathscr{LM}(\mathbb{R})$ by

$$\mathscr{LM}(\mathbb{R}) = \big\{ H \colon \|H\|_{\mathscr{LM}(\mathbb{R})} < \infty \big\}. \tag{4.7.3}$$

Furthermore, let $\mathcal{J} \subseteq \mathbb{R}$ be a time-interval and let $H = \{H(t,x;K)\}_{K\geqslant 1}\}$ be a family of space-time functions from $\mathcal{J} \times \mathbb{T}^3$ into $\mathbb{R}$ satisfying (4.7.1). Similar as in the definition of $\mathfrak{X}^{s,b}$-spaces, we define the restricted norm by

$$\|H\|_{\mathscr{LM}(\mathcal{J})} = \inf \big\{ \|H'\|_{\mathscr{LM}(\mathbb{R})} \colon H'(t) = H(t) \quad \text{for all } t \in \mathcal{J} \big\}. \tag{4.7.4}$$

The corresponding time-restricted low-frequency modulation space $\mathscr{LM}(\mathcal{J})$ can then be defined as in (4.7.2) after replacing the norm.

Definition 4.7.2 (Para-controlled)**.** Let $\mathcal{J} \subseteq \mathbb{R}$ be an interval, let $\phi\colon \mathcal{J}\times\mathbb{T}^3 \to \mathbb{C}$ be a distribution, and let H be as in Definition 4.7.1. Then, we define

$$\mathrm{PCtrl}(H,\phi)(t,x) = \sum_{K\geqslant 1} H(t,x;K)(P_K\phi)(t,x). \tag{4.7.5}$$

If $H \in \mathscr{LM}(\mathbb{R})$, we have that

$$\begin{aligned} &\mathrm{PCtrl}(H,\phi)(t,x) \\ &= \sum_{K\geqslant 1} \sum_{k_1\in\mathbb{Z}^3} \int_{\mathbb{R}} \mathrm{d}\lambda_1 \widehat{H}(\lambda_1,k_1;K) \Big(\exp\big(i\lambda_1 t\big) \sum_{k_2\in\mathbb{Z}^3} \chi_K(k_2) \widehat{\phi}(t,k_2) \exp\big(i\langle k_{12},x\rangle\big)\Big). \end{aligned} \tag{4.7.6}$$

The expression (4.7.6) will be used in all of our estimates involving PCtrl. The sum in k_1, the integral in λ_1, and the pre-factor $\widehat{H}(\lambda_1,k_1;K)$ will be inessential. The main step will consist of estimates for

$$\exp\big(i\lambda_1 t\big) \sum_{k_2\in\mathbb{Z}^3} \chi_K(k_2) \widehat{\phi}(t,k_2) \exp\big(i\langle k_{12},x\rangle\big),$$

which essentially behaves like $P_K\phi(t,x)$. For most purposes, the reader may simply think of $\mathrm{PCtrl}(H,\phi)$ as ϕ.

Lemma 4.7.3 (Basic mapping properties of PCtrl)**.** For any $s \in \mathbb{R}$, any interval $\mathcal{J} \subseteq \mathbb{R}$, any $\phi \in L_t^\infty H_x^s(\mathcal{J} \times \mathbb{T}^3)$, and any $H \in \mathscr{LM}(\mathcal{J})$, we have

$$\| \mathrm{PCtrl}(H,\phi)\|_{L_t^\infty H_x^{s-8\epsilon}(\mathcal{J}\times\mathbb{T}^3)} \lesssim \|H\|_{\mathscr{LM}(\mathcal{J})} \|\phi\|_{L_t^\infty H_x^s(\mathcal{J}\times\mathbb{T}^3)}. \tag{4.7.7}$$

Proof. We treat each dyadic piece in PCtrl separately. Using the Fourier support condition (4.7.1), we have that

$$\begin{aligned}
\|H(t,x;K)(P_K\phi)(t,x)\|_{H_x^{s-8\epsilon}(\mathbb{T}^3)} &= \Big\| \sum_{k_1,k_2\in\mathbb{Z}^3} \chi_K(k_2)\widehat{H}(t,k_1;K)\widehat{\phi}(t,k_2)\exp\big(i\langle k_{12},x\rangle\big)\Big\|_{H_x^{s-8\epsilon}(\mathbb{T}^3)} \\
&\lesssim \sum_{k_1\in\mathbb{Z}^3} |\widehat{H}(t,k_1;K)| \Big\| \sum_{k_2\in\mathbb{Z}^3} \chi_K(k_2)\widehat{\phi}(t,k_2)\exp\big(i\langle k_{12},x\rangle\big)\Big\|_{H_x^{s-8\epsilon}(\mathbb{T}^3)} \\
&\lesssim K^{-8\epsilon}\Big(\sum_{k_1\in\mathbb{Z}^3} |\widehat{H}(t,k_1;K)|\Big)\|\phi(t)\|_{H_x^s(\mathbb{T}^3)} \\
&\lesssim K^{-\epsilon}\|H\|_{\mathscr{LM}(\mathcal{J})}\|\phi(t)\|_{H_x^s(\mathbb{T}^3)}.
\end{aligned}$$

The desired estimate follows after summing in K. ☐

In the next two lemmas, we show that the terms appearing in the evolution equation (4.2.14) for X_N fit into our para-controlled framework.

Lemma 4.7.4. Let $\mathcal{J} \subseteq \mathbb{R}$ be an interval and let $f, g \in \mathfrak{X}^{-1,b}(\mathcal{J})$. Then, there exists a (canonical) $H \in \mathscr{LM}(\mathcal{J})$ satisfying

$$\boxed{\circledless \,\&\, \circledless}\Big(V * (f\, g)\phi\Big) = \mathrm{PCtrl}(H,\phi) \tag{4.7.8}$$

for all space-time distributions $\phi\colon \mathcal{J} \times \mathbb{T}^3 \to \mathcal{C}$. Furthermore, it holds that

$$\|H\|_{\mathscr{LM}(\mathcal{J})} \lesssim \|f\|_{\mathfrak{X}^{-1,b}(\mathcal{J})} \cdot \|g\|_{\mathfrak{X}^{-1,b}(\mathcal{J})}. \tag{4.7.9}$$

Remark 4.7.5. Due to the overlaps in the support of the Littlewood-Paley multipliers χ_K, the low-frequency modulation $H \in \mathscr{LM}(\mathcal{J})$ is not quite unique. As will be clear from the proof, however, there is a canonical choice. This canonical choice is also bilinear in f and g.

Proof. Using the definition of the restricted norms, it suffices to treat the case $\mathcal{J} = \mathbb{R}$. We have that

$$
\begin{aligned}
&\boxed{\olessthan \,\&\, \olessthan}\Big(V * (f\, g)\phi\Big)(t,x) \\
&= \sum_{\substack{N_1,N_2,K:\\ N_1,N_2 \leqslant K^\epsilon}} \sum_{n_1,n_2,k\in\mathbb{Z}^3} \chi_{N_1}(n_1)\chi_{N_2}(n_2)\chi_K(k)\widehat{V}(n_{12})\widehat{f}(t,n_1)\widehat{g}(t,n_2)\widehat{\phi}(t,k)\exp\Big(i\langle n_{12}+k,x\rangle\Big) \\
&= \mathrm{PCtrl}(H,\phi)(t,x),
\end{aligned}
$$

where

$$
\widehat{H}(t,k_1;K) = \sum_{\substack{N_1,N_2:\\ N_1,N_2\leqslant K^\epsilon}} \sum_{\substack{n_1,n_2\in\mathbb{Z}^3:\\ n_{12}=k_1}} \chi_{N_1}(n_1)\chi_{N_2}(n_2)\widehat{V}(n_{12})\widehat{f}(t,n_1)\widehat{g}(t,n_2) \tag{4.7.10}
$$

It therefore remains to show $H \in \mathscr{LM}(\mathbb{R})$ and the estimate (4.7.9). The Fourier support condition (4.7.1) is a consequence of the multiplier $\chi_{N_1}(n_1)\chi_{N_2}(n_2)$ in (4.7.10). To see the estimate (4.7.9), we first note that

$$
\widehat{H}(\lambda,k_1;K) = \sum_{\substack{N_1,N_2:\\ N_1,N_2\leqslant K^\epsilon}} \sum_{\substack{n_1,n_2\in\mathbb{Z}^3:\\ n_{12}=k_1}} \chi_{N_1}(n_1)\chi_{N_2}(n_2)\widehat{V}(n_{12})\Big(\widehat{f}(\cdot,n_1) * \widehat{g}(\cdot,n_2)\Big)(\lambda).
$$

Using Young's convolution inequality and Cauchy-Schwarz, we obtain that

$$
\begin{aligned}
&\|\widehat{H}(\lambda, k_1; K)\|_{L^1_\lambda(\mathbb{R})} \\
&\lesssim \sum_{\substack{N_1,N_2:\\ N_1,N_2\leqslant K^\epsilon}} \sum_{\substack{n_1,n_2\in\mathbb{Z}^3:\\ n_{12}=k_1}} \chi_{N_1}(n_1)\chi_{N_2}(n_2)|\widehat{V}(n_{12})|\|\widehat{f}(\lambda,n_1)\|_{L^1_\lambda(\mathbb{R})}\|\widehat{g}(\lambda,n_2)\|_{L^1_\lambda(\mathbb{R})} \\
&\lesssim \sum_{\substack{n_1,n_2\in\mathbb{Z}^3:\\ n_{12}=k_1}} 1\big\{|n_1|,|n_2|\lesssim K^\epsilon\big\}\|\langle|\lambda|-\langle n_1\rangle\rangle^b\widehat{f}(\lambda,n_1)\|_{L^2_\lambda(\mathbb{R})}\|\langle|\lambda|-\langle n_2\rangle\rangle^b\widehat{g}(\lambda,n_2)\|_{L^2_\lambda(\mathbb{R})} \\
&\lesssim \Big(\sum_{n_1\in\mathbb{Z}^3} 1\big\{|n_1|\lesssim K^\epsilon\big\}\|\langle|\lambda|-\langle n_1\rangle\rangle^b\widehat{f}(\lambda,n_1)\|^2_{L^2_\lambda(\mathbb{R})}\Big)^{\frac{1}{2}} \\
&\quad\times\Big(\sum_{n_2\in\mathbb{Z}^3} 1\big\{|n_2|\lesssim K^\epsilon\big\}\|\langle|\lambda|-\langle n_2\rangle\rangle^b\widehat{g}(\lambda,n_2)\|^2_{L^2_\lambda(\mathbb{R})}\Big)^{\frac{1}{2}} \\
&\lesssim K^{-2\epsilon}\|f\|_{\mathfrak{X}^{-1,b}(\mathcal{J})}\cdot\|g\|_{\mathfrak{X}^{-1,b}(\mathcal{J})}.
\end{aligned}
$$

The desired estimate (4.7.9) now follows after taking the supremum in $K\geqslant 1$ and $k_1\in\mathbb{Z}^3$. □

Lemma 4.7.6. Let $\mathcal{J}\subseteq\mathbb{R}$ be an interval, let $s\in[-1,1]$, let $f\in\mathfrak{X}^{-s,b}(\mathcal{J})$, and let $g\in\mathfrak{X}^{s,b}$. Then, there exists a (canonical) $H\in\mathscr{LM}(\mathcal{J})$ satisfying

$$V*(f\,g)\circledcirc\phi=\mathrm{PCtrl}(H,\phi) \tag{4.7.11}$$

for all space-time distributions $\phi\colon J\times\mathbb{T}^3\to\mathbb{C}$. Furthermore, it holds that

$$\|H\|_{\mathscr{LM}(\mathcal{J})}\lesssim\|f\|_{\mathfrak{X}^{-s,b}(\mathcal{J})}\cdot\|g\|_{\mathfrak{X}^{s,b}(\mathcal{J})}. \tag{4.7.12}$$

Remark 4.7.7. We emphasize that Lemma 4.7.6 fails if we replace the assumptions by $f,g\in\mathfrak{X}^{-1,b}(\mathcal{J})$ as in Lemma 4.7.4. The reason is that the product $f\cdot g$ inside the convolution with the interaction potential V is not even well-defined.

Proof. The argument is similar to the proof of Lemma 4.7.4. As before, it suffices to treat the case

$\mathcal{J} = \mathbb{R}$. A direct calculation yields the identity (4.7.11) with

$$H(t,k_1;K) = \sum_{K_1 \leqslant K^\epsilon} \chi_{K_1}(k_1)\widehat{V}(k_1) \sum_{\substack{n_1,n_2\in\mathbb{Z}^3:\\ n_{12}=k_1}} \widehat{f}(t,n_1)\widehat{g}(t,n_2). \tag{4.7.13}$$

Using Young's convolution inequality and Cauchy-Schwarz, we obtain that

$$\begin{aligned}
&\|\widehat{H}(\lambda,k_1;K)\|_{L^1_\lambda(\mathbb{R})} \\
&\lesssim \sum_{\substack{n_1,n_2\in\mathbb{Z}^3:\\ n_{12}=k_1}} \|\widehat{f}(\lambda,n_1)\|_{L^1_\lambda(\mathbb{R})}\|\widehat{g}(\lambda,n_2)\|_{L^1_\lambda(\mathbb{R})} \\
&\lesssim \Big(\sum_{n_1\in\mathbb{Z}^3} \langle n_1\rangle^{-2s}\|\langle|\lambda|-\langle n_1\rangle\rangle^b \widehat{f}(\lambda,n_1)\|^2_{L^2_\lambda(\mathbb{R})}\Big)^{\frac{1}{2}} \Big(\sum_{n_2\in\mathbb{Z}^3} \langle n_2-k_1\rangle^{2s}\|\langle|\lambda|-\langle n_2\rangle\rangle^b \widehat{g}(\lambda,n_2)\|^2_{L^2_\lambda(\mathbb{R})}\Big)^{\frac{1}{2}}.
\end{aligned}$$

Using that $\langle n_2-k_1\rangle \lesssim \langle k_1\rangle + \langle n_2\rangle \lesssim K^\epsilon\langle n_2\rangle$, we obtain the estimate (4.7.12). □

4.7.1 Quadratic para-controlled estimate

In this subsection, we show that $P_{\leqslant N}X_N \circleddash P_{\leqslant N}\stackrel{\bullet}{\mid}$ is well-defined uniformly in N even though the sum of the individual spatial regularities is negative. Together with Lemma 4.8.8, this will control the second and third term in **Phy**, i.e.,

$$V * \Big(P_{\leqslant N}X_N \circleddash P_{\leqslant N}\stackrel{\bullet}{\mid}\Big)\cdot P_{\leqslant N}\overset{\bullet\bullet\bullet}{\Psi}_{N} \quad \text{and} \quad V * \Big(P_{\leqslant N}X_N \circleddash P_{\leqslant N}\stackrel{\bullet}{\mid}\Big)\cdot P_{\leqslant N}w_N.$$

Proposition 4.7.8 (Quadratic para-controlled object)**.** Let $T \geqslant 1$. For any $s < -2\eta - 10\epsilon$ and $p \geqslant 2$, we have that

$$\begin{aligned}
&\sum_{L_1\sim L_2} L_1^{2\eta}\Big\| \sup_{N\geqslant 1}\sup_{\mathcal{J}\subseteq[0,T]}\sup_{\|H\|_{\mathscr{L}\mathscr{M}(\mathcal{J})}\leqslant 1} \Big\|\big(P_{L_1}P_{\leqslant N}\,\mathrm{I}\big)\Big[1_{\mathcal{J}}\,\mathrm{PCtrl}(H,P_{\leqslant N}\stackrel{\bullet}{\mid})\Big]\cdot P_{L_2}\stackrel{\bullet}{\mid}\Big\|_{L^\infty_t\mathcal{C}^s_x([0,T]\times\mathbb{T}^3)}\Big\|_{L^p_\omega(\mathbb{P})} \\
&\lesssim T^3 p,
\end{aligned}$$

where the supremum in $\mathcal{J}$ is taken only over intervals.

Proof. The supremum in N can be handled through the decay in the frequency-localized version below and we omit it throughout the proof. Using the definition of the $\mathscr{L}\mathscr{M}(\mathcal{J})$-norm, we may take the supremum over $H \in \mathscr{L}\mathscr{M}(\mathbb{R})$ with norm bounded by one. By inserting the expansion (4.7.6), we obtain that

$$
\begin{aligned}
&\big(P_{L_1} P_{\leqslant N} \,\mathrm{I}\big)\Big[1_{\mathcal{J}} \,\mathrm{PCtrl}(H, P_{\leqslant N} \linear)\Big](t,x) \cdot P_{L_2} \linear(t,x) \\
&= \sum_{N_1} \sum_{k_1 \in \mathbb{Z}^3} \int_{\mathbb{R}} \mathrm{d}\lambda_1 \widehat{H}(\lambda_1, k_1; N_1) \sum_{n_1, n_2 \in \mathbb{Z}^3} \Big[\rho_N(k_1 + n_1) \chi_{L_1}(n_1 + k_1) \rho_N(n_1) \chi_{N_1}(n_1) \chi_{L_2}(n_2) \\
&\times \widehat{\linear}(t, n_2) \bigg(\int_0^t 1_{\mathcal{J}}(t') \frac{\sin((t-t')\langle k_1 + n_1 \rangle)}{\langle k_1 + n_1 \rangle} \exp\big(i\lambda_1 t'\big) \widehat{\linear}(t', n_1) \mathrm{d}t' \bigg) \exp\big(i \langle n_{12} + k_1, x \rangle\big) \Big].
\end{aligned}
$$

Due to the definition of $\mathscr{L}\mathscr{M}$, we only obtain a non-trivial contribution if $N_1 \sim L_1 \sim L_2$. Using the triangle inequality, it follows that

$$
\begin{aligned}
&\sup_{\mathcal{J} \subseteq [0,T]} \sup_{\|H\|_{\mathscr{L}\mathscr{M}(\mathbb{R})} \leqslant 1} \Big\| \big(P_{L_1} P_{\leqslant N} \,\mathrm{I}\big)\Big[1_{\mathcal{J}} \,\mathrm{PCtrl}(H, P_{\leqslant N} \linear)\Big] \cdot P_{L_2} \linear \Big\|_{L_t^\infty \mathcal{C}_x^s([0,T] \times \mathbb{T}^3)} \\
&\lesssim \sum_{N_1} N_1^{7\epsilon} \sup_{\mathcal{J} \subseteq [0,T]} \sup_{\substack{k_1 \in \mathbb{Z}^3: \\ |k_1| \leqslant 8 N_1^\epsilon}} \sup_{\lambda_1 \in \mathbb{R}} \Big\| \sum_{n_1, n_2 \in \mathbb{Z}^3} \Big[\rho_N(k_1 + n_1) \chi_{L_1}(n_1 + k_1) \rho_N(n_1) \chi_{N_1}(n_1) \chi_{L_2}(n_2) \\
&\times \exp\big(i \langle n_{12} + k_1, x \rangle\big) \widehat{\linear}(t, n_2) \bigg(\int_0^t 1_{\mathcal{J}}(t') \frac{\sin((t-t')\langle k_1 + n_1 \rangle)}{\langle k_1 + n_1 \rangle} \exp\big(i\lambda_1 t'\big) \widehat{\linear}(t', n_1) \mathrm{d}t' \bigg) \Big] \Big\|_{L_t^\infty \mathcal{C}_x^s([0,T] \times \mathbb{T}^3)}.
\end{aligned}
$$

To obtain the desired estimate, it suffices to prove for all $N_1 \sim L_1 \sim L_2$ that

$$
\begin{aligned}
&\Big\| \sup_{\mathcal{J} \subseteq [0,T]} \sup_{\substack{k_1 \in \mathbb{Z}^3: \\ |k_1| \leqslant 8 N_1^\epsilon}} \sup_{\lambda_1 \in \mathbb{R}} \Big\| \sum_{n_1, n_2 \in \mathbb{Z}^3} \Big[\rho_N(k_1 + n_1) \chi_{L_1}(n_1 + k_1) \\
&\times \rho_N(n_1) \chi_{N_1}(n_1) \chi_{L_2}(n_2) \exp\big(i \langle n_{12} + k_1, x \rangle\big) \widehat{\linear}(t, n_2) \\
&\times \bigg(\int_0^t 1_{\mathcal{J}}(t') \frac{\sin((t-t')\langle k_1 + n_1 \rangle)}{\langle k_1 + n_1 \rangle} \exp\big(i\lambda_1 t'\big) \widehat{\linear}(t', n_1) \mathrm{d}t' \bigg) \Big] \Big\|_{L_t^\infty \mathcal{C}_x^s([0,T] \times \mathbb{T}^3)} \Big\|_{L_\omega^p(\Omega)} \\
&\lesssim T^3 N_1^{-2\eta - 9\epsilon}.
\end{aligned}
\tag{4.7.14}
$$

We claim that instead of (4.7.14), it suffices to prove the simpler estimate

$$
\begin{aligned}
&\sup_{t\in[0,T]} \sup_{\mathcal{J}\subseteq[0,T]} \sup_{\substack{k_1\in\mathbb{Z}^3:\\ |k_1|\leqslant 8N_1^\epsilon}} \sup_{\lambda_1\in\mathbb{R}} \Bigg\| \sum_{n_1,n_2\in\mathbb{Z}^3} \Bigg[\rho_N(k_1+n_1)\chi_{L_1}(n_1+k_1) \\
&\times \rho_N(n_1)\chi_{N_1}(n_1)\chi_{L_2}(n_2)\exp\big(i\langle n_{12}+k_1,x\rangle\big)\widehat{\mid}(t,n_2) \\
&\times \Bigg(\int_0^t 1_{\mathcal{J}}(t') \frac{\sin((t-t')\langle k_1+n_1\rangle)}{\langle k_1+n_1\rangle} \exp\big(i\lambda_1 t'\big)\widehat{\mid}(t',n_1)\mathrm{d}t' \Bigg) \Bigg] \Bigg\|_{L^2_\omega H^s_x} \\
&\lesssim T^2 N_1^{-2\eta-10\epsilon} p.
\end{aligned}
\tag{4.7.15}
$$

The reduction of (4.7.14) to (4.7.15) is standard and we only sketch the argument. The supremum in k_1 can easily be moved outside the moment by using Lemma 4.4.48 and accepting a logarithmic loss in N_1. To deal with the supremum in $\lambda_1 \in \mathbb{R}$, we treat two separate cases. Using the Lipschitz estimate $|\exp(i\lambda_1 t') - \exp(i\widetilde{\lambda}_1 t')| \lesssim |t'||\lambda_1 - \widetilde{\lambda}_1|$, the supremum over $|\lambda_1| \lesssim N_1^{10}$ can easily be replaced by the supremum over a grid on $[-N_1^{10}, N_1^{10}]$ with mesh size $\sim N_1^{-10}$. The discrete supremum can then be moved outside the probabilistic moment using Lemma 4.4.48. For $|\lambda_1| \gtrsim N_1^{10}$, a simple integration by parts gains a factor of $|\lambda_1|^{-1}$ and we can proceed using crude estimates. The supremum over $t \in [0,T]$ and $\mathcal{J} \subseteq [0,T]$, which is parametrized by its two endpoints, can be moved outside of the probabilistic moment using the first part of the argument for λ_1. Finally, Gaussian hypercontractivity allows us to replace $L^p_\omega \mathcal{C}^s_x$ by $L^2_\omega H^s_x$.

We now turn to the proof of the simpler estimate (4.7.15). Using the product formula for multiple stochastic integrals, we have that

$$
\begin{aligned}
&\sum_{n_1,n_2\in\mathbb{Z}^3} \Bigg[\rho_N(k_1+n_1)\chi_{L_1}(n_1+k_1)\rho_N(n_1)\chi_{N_1}(n_1)\chi_{L_2}(n_2)\exp\big(i\langle n_{12}+k_1,x\rangle\big) \\
&\times \widehat{\mid}(t,n_2) \Bigg(\int_0^t 1_{\mathcal{J}}(t') \frac{\sin((t-t')\langle k_1+n_1\rangle)}{\langle k_1+n_1\rangle} \exp\big(i\lambda_1 t'\big)\widehat{\mid}(t',n_1)\mathrm{d}t' \Bigg) \Bigg] \\
&= \mathscr{G}^{(2)}(t,x) + \mathscr{G}^{(0)}(t,x),
\end{aligned}
$$

where the Gaussian chaoses $\mathscr{G}^{(2)}$ and $\mathscr{G}^{(0)}$ are given by

$$\begin{aligned}
\mathscr{G}^{(2)}(t,x) \overset{\text{def}}{=} \sum_{\pm_1,\pm_2} \sum_{n_1,n_2\in\mathbb{Z}^3} \Big[& c(\pm_1,\pm_2)\rho_N(k_1+n_1)\chi_{L_1}(n_1+k_1)\rho_N(n_1)\chi_{N_1}(n_1)\chi_{L_2}(n_2) \\
& \times \bigg(\int_0^t 1_{\mathcal{J}}(t') \frac{\sin((t-t')\langle k_1+n_1\rangle)}{\langle k_1+n_1\rangle} \exp\big(i\lambda_1 t' \pm_1 it'\langle n_1\rangle \pm_2 it'\langle n_2\rangle\big)\mathrm{d}t' \bigg) \\
& \exp\big(i\langle n_{12}+k_1, x\rangle\big)\mathcal{I}_2[\pm_j, n_j\colon j=1,2]\Big], \\
\mathscr{G}^{(0)}(t,x) \overset{\text{def}}{=} \exp(i\langle k_1, x\rangle) \sum_{n_1\in\mathbb{Z}^3} \Big[& \rho_N(k_1+n_1)\chi_{L_1}(n_1+k_1)\rho_N(n_1)\chi_{N_1}(n_1)\chi_{L_2}(n_1)\frac{1}{\langle n_1+k_1\rangle\langle n_1\rangle^2} \\
& \times \bigg(\int_0^t 1_{\mathcal{J}}(t')\sin((t-t')\langle k_1+n_1\rangle)\cos((t-t')\langle n_1\rangle)\exp\big(i\lambda_1 t'\big)\mathrm{d}t' \bigg)\Big].
\end{aligned}$$

The quadratic Gaussian chaos $\mathscr{G}^{(2)}$ is the non-resonant part and the constant "Gaussian chaos" $\mathscr{G}^{(0)}$ is the resonant part. We now treat both components separately.

Contribution of the quadratic Gaussian chaos $\mathscr{G}^{(2)}$: Using the orthogonality of the multiple stochastic integrals and taking absolute values inside the t'-integral, we have that

$$\begin{aligned}
&\|\mathscr{G}^{(2)}(t,x)\|^2_{L^2_\omega H^s_x(\Omega\times\mathbb{T}^3)} \\
&\lesssim T^2 \sum_{n_1,n_2\in\mathbb{Z}^3} \chi_{N_1}(n_1)\chi_{L_2}(n_2)\langle k_1+n_{12}\rangle^{2s}\langle k_1+n_1\rangle^{-2}\langle n_1\rangle^{-2}\langle n_2\rangle^{-2} \\
&\lesssim T^2 N_1^{-6} \sum_{n_1\in\mathbb{Z}^3} \chi_{N_1}(n_1)\chi_{L_2}(n_2)\langle k_1+n_{12}\rangle^{2s} \\
&\lesssim T^2 N_1^{-4\eta-20\epsilon},
\end{aligned}$$

which is acceptable.

Contribution of the constant "Gaussian chaos" $\mathscr{G}^{(0)}$: Using the sine-cancellation lemma (Lemma

4.4.14), we have that

$$
\begin{aligned}
&\|\mathscr{G}^{(0)}(t,x)\|_{H^s_x(\mathbb{T}^3)} \\
&\lesssim \Bigg| \sum_{n_1\in\mathbb{Z}^3} \Bigg[\rho_N(k_1+n_1)\chi_{L_1}(n_1+k_1)\rho_N(n_1)\chi_{N_1}(n_1)\chi_{L_2}(n_1) \frac{1}{\langle n_1+k_1\rangle\langle n_1\rangle^2} \\
&\times \bigg(\int_0^t 1_{\mathcal{J}}(t') \sin((t-t')\langle k_1+n_1\rangle)\cos((t-t')\langle n_1\rangle)\exp\big(i\lambda_1 t'\big)\mathrm{d}t' \bigg) \Bigg] \Bigg| \\
&\lesssim N_1^{-1+3\epsilon},
\end{aligned}
$$

which is also acceptable. □

4.7.2 Cubic para-controlled estimate

In this subsection, we control the cubic para-controlled object, i.e., the first summand in the definition of **CPara** in (4.2.29).

Proposition 4.7.9. Let $T \geqslant 1$. For any interval $\mathcal{J} \subseteq [0,T]$, any $\phi\colon [0,T]\times\mathbb{T}^3 \to \mathbb{C}$, and $H \in \mathscr{LM}(\mathcal{J})$, we define

$$
\begin{aligned}
&\mathrm{PCtrl}^{(3)}_N(H,\phi;\mathcal{J}) \\
&\overset{\mathrm{def}}{=} \big(\neg \boxed{\circledless \,\&\, \circledless}\big)\ \Big(V * \Big(\big(P^2_{\leqslant N}\,\mathrm{I}\big)\Big[1_{\mathcal{J}}\,\mathrm{PCtrl}(H,\phi)\Big]\cdot\phi\Big)\cdot\phi\Big) - \mathcal{M}_N P^2_{\leqslant N}\,\mathrm{I}\Big[1_{\mathcal{J}}\,\mathrm{PCtrl}(H,\phi)\Big].
\end{aligned}
$$

Then, it holds that for all $p \geqslant 2$ that

$$
\Big\| \sup_{N\geqslant 1}\ \sup_{\mathcal{J}\subseteq[0,T]}\ \sup_{\|H\|_{\mathscr{LM}(\mathcal{J})}\leqslant 1} \Big\| \mathrm{PCtrl}^{(3)}_N(H, P_{\leqslant N}\overset{\bullet}{|};\mathcal{J}) \Big\|_{\mathfrak{X}^{s_2-1,b_+-1}([0,T])} \Big\|_{L^p_\omega(\mathbb{P})} \lesssim T^3 p^{\frac{3}{2}},
$$

where the supremum in $\mathcal{J}$ is only taken over intervals.

Remark 4.7.10. The notation $\mathrm{PCtrl}^{(3)}_N(H, P_{\leqslant N}\overset{\bullet}{|};\mathcal{J})$ will only be used in Proposition 4.7.9 and its proof. The frequency-localized version of Proposition 4.7.9 also gains an η'-power in the maximal frequency-scale.

Proof. As before, we ignore the supremum in N, which can be easily handled through the decay in the frequency-localized version below. Using the decay in the frequency-localized version and a crude estimate, we can also replace the $\mathfrak{X}^{s_2,b_+-1}$-norm by the $\mathfrak{X}^{s_2,b_--1}$-norm. Using the definition of the restricted norms, it suffices to consider $H \in \mathscr{LM}(\mathbb{R})$ with $\|H\|_{\mathscr{LM}(\mathbb{R})} \leqslant 1$. In order to use a Littlewood-Paley decomposition, we need to break up the multiplier $\mathcal{M}_N$. We define $\mathcal{M}_N[N_1, N_2, N_3]$ as the multiplier with the symbol

$$m_N[N_1, N_2, N_3](n_2) = \sum_{k\in\mathbb{Z}^3} \frac{\widehat{V}(k+n_2)}{\langle k\rangle^2} \rho_N^2(k)\chi_{N_1}(k)\chi_{N_2}(n_2)\chi_{N_3}(k). \tag{4.7.16}$$

We note that $\mathcal{M}_N[N_1, N_2, N_3]$ is only non-zero when $N_1 \sim N_3$, and hence, in particular, when $N_1 > N_3^\epsilon$. We now face a notational nuisance; namely, that both PCtrl and $\boxed{\lessdot \& \lessdot}$ contain frequency-projections. To this end, we use N_2 and N_2' for the respective frequency-scales, but encourage the reader to mentally set $N_2 = N_2'$. It then follows that

$$\begin{aligned}
&\mathrm{PCtrl}^{(3)}_N(H, P_{\leqslant N}\uparrow; \mathcal{J})\\
&= \sum_{\substack{N_1,N_2',N_3:\\ \max(N_1,N_2')>N_3^\epsilon}} \Big[V * \Big(P_{N_1}P_{\leqslant N}\uparrow \cdot P_{N_2'}P^2_{\leqslant N}\,\mathrm{I}\Big[1_{\mathcal{J}}\,\mathrm{PCtrl}(H, P_{\leqslant N}\uparrow)\Big]\Big) \cdot P_{N_3}P_{\leqslant N}\uparrow \\
&\quad - \mathcal{M}_N[N_1, N_2', N_3]P^2_{\leqslant N}\,\mathrm{I}\Big[1_{\mathcal{J}}\,\mathrm{PCtrl}(H, P_{\leqslant N}\uparrow)\Big]\Big].
\end{aligned} \tag{4.7.17}$$

Using the stochastic representation formula (4.4.77) in Proposition 4.4.44 and the expansion (4.7.6), we obtain that

$$
\begin{aligned}
&\mathrm{PCtrl}^{(3)}_N(H, P_{\leqslant N}\overset{\color{blue}\bullet}{|}\,; \mathcal{J})(t,x) \\
&= \sum_{\substack{N_1,N_2,N_2',N_3:\\ \max(N_1,N_2')>N_3,\\ N_2\sim N_2'}} \sum_{k_2\in\mathbb{Z}^3} \int_{\mathbb{R}} \mathrm{d}\lambda_2\ \widehat{H}(\lambda_2,k_2;N_2) \sum_{n_1,n_2,n_3\in\mathbb{Z}^3} \Big[\rho_N^2(n_2+k_2)\chi_{N_2'}(n_2+k_2) \\
&\times \Big(\prod_{j=1}^3 \rho_N(n_j)\chi_{N_j}(n_j)\Big) \widehat{V}(n_{12}+k_2) \Big(\int_0^t 1_{\mathcal{J}}(t') \frac{\sin((t-t')\langle n_2+k_2\rangle)}{\langle n_2+k_2\rangle} \exp\big(it'\lambda_2\big)\mathcal{I}_1[t',n_2]\mathrm{d}t'\Big) \\
&\times \exp\big(i\langle n_{123}+k_2,x\rangle\big)\mathcal{I}_2[t,n_1,n_3]\Big].
\end{aligned}
$$

Using the product formula for multiple stochastic integrals, we can decompose the inner sum in n_1, n_2, and n_3 as

$$
\begin{aligned}
&\sum_{n_1,n_2,n_3\in\mathbb{Z}^3} \Big[\rho_N^2(n_2+k_2)\chi_{N_2'}(n_2+k_2) \Big(\prod_{j=1}^3 \rho_N(n_j)\chi_{N_j}(n_j)\Big) \widehat{V}(n_{12}+k_2)\mathcal{I}_2[t,n_1,n_3] \\
&\times \Big(\int_0^t 1_{\mathcal{J}}(t') \frac{\sin((t-t')\langle n_2+k_2\rangle)}{\langle n_2+k_2\rangle} \exp\big(it'\lambda_2\big)\mathcal{I}_1[t',n_2]\mathrm{d}t'\Big) \exp\big(i\langle n_{123}+k_2,x\rangle\big)\Big] \\
&= \mathscr{G}^{(3)}(t,x;\lambda_2,k_2,\mathcal{J},N_*) + \mathscr{G}^{(1)}(t,x;\lambda_2,k_2,\mathcal{J},N_*) + \widetilde{\mathscr{G}}^{(1)}(t,x;\lambda_2,k_2,\mathcal{J},N_*),
\end{aligned}
$$

where the cubic and linear Gaussian chaoses are given by

$$
\begin{aligned}
\mathscr{G}^{(3)}(t,x) &= \sum_{\pm_1,\pm_2,\pm_3} c(\pm_j\colon 1\leqslant j\leqslant 3) \sum_{n_1,n_2,n_3\in\mathbb{Z}^3} \Big[\rho_N^2(n_2+k_2)\chi_{N_2'}(n_2+k_2) \Big(\prod_{j=1}^3 \rho_N(n_j)\chi_{N_j}(n_j)\Big) \\
&\times \widehat{V}(n_{12}+k_2) \Big(\int_0^t 1_{\mathcal{J}}(t') \frac{\sin((t-t')\langle n_2+k_2\rangle)}{\langle n_2+k_2\rangle} \exp\big(it'\lambda_2 \pm_2 it'\langle n_2\rangle\big)\mathrm{d}t'\Big) \\
&\times \exp(\pm_1 it\langle n_1\rangle \pm_3 it\langle n_3\rangle) \exp\big(i\langle n_{123}+k_2,x\rangle\big)\mathcal{I}_3[\pm_j, n_j\colon 1\leqslant j\leqslant 3]\Big], \\
\mathscr{G}^{(1)}(t,x) &= \sum_{n_3\in\mathbb{Z}^3} \rho_N(n_3)\chi_{N_3}(n_3) \exp\big(\langle n_3+k_2,x\rangle\big) \sum_{n_1\in\mathbb{Z}^3} \Big[\rho_N^2(n_2+k_2)\chi_{N_2'}(n_2+k_2)\rho_N^2(n_2) \\
&\times \chi_{N_1}(n_2)\chi_{N_2}(n_2) \Big(\int_0^t 1_{\mathcal{J}}(t') \sin((t-t')\langle n_2+k_2\rangle)\cos((t-t')\langle n_2\rangle) \exp\big(it'\lambda_2\big)\mathrm{d}t'\Big) \\
&\times \widehat{V}(k_2)\langle n_2+k_2\rangle^{-1}\langle n_2\rangle^{-2}\Big]\ \mathcal{I}_1[t;n_3],
\end{aligned}
$$

$$\widetilde{\mathcal{G}}^{(1)}(t,x) = \sum_{n_1\in\mathbb{Z}^3} \rho_N(n_1)\chi_{N_1}(n_1)\exp\big(\langle n_1+k_2,x\rangle\big) \sum_{n_1\in\mathbb{Z}^3}\Bigg[\rho_N^2(n_2+k_2)\chi_{N_2'}(n_2+k_2)\rho_N^2(n_2)$$
$$\times \chi_{N_2}(n_2)\chi_{N_3}(n_2)\Big(\int_0^t 1_{\mathcal{J}}(t')\sin((t-t')\langle n_2+k_2\rangle)\cos((t-t')\langle n_2\rangle)\exp\big(it'\lambda_2)\mathrm{d}t'\Big)$$
$$\times \widehat{V}(n_{12}+k_2)\langle n_2+k_2\rangle^{-1}\langle n_2\rangle^{-2}\Bigg]\,\mathcal{I}_1[t;n_1].$$

We refer to $\mathcal{G}^{(3)}$ as the non-resonant term and to $\mathcal{G}^{(1)}$ and $\widetilde{\mathcal{G}}^{(1)}$ as the resonant terms. Using the triangle inequality and $\|H\|_{\mathcal{LM}(\mathbb{R})}\leqslant 1$, we obtain that

$$\Big\|\,\mathrm{PCtrl}_N^{(3)}(H,P_{\leqslant N}\,\rceil;\mathcal{J})\Big\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})}$$
$$\lesssim \sum_{\substack{N_1,N_2,N_2',N_3:\\ \max(N_1,N_2')>N_3,\\ N_2\sim N_2'}} N_2^{7\epsilon}\sup_{\lambda_2\in\mathbb{R}}\sup_{\substack{k_2\in\mathbb{Z}^3:\\ |k_2|\lesssim N_2^{\epsilon}}}\Big(\|\mathcal{G}^{(3)}(\cdot;\lambda_2,k_2,\mathcal{J},N_*)\|_{\mathfrak{X}^{s_2-1,b_--1}([0,T])}$$
$$+\|\mathcal{G}^{(1)}(\cdot;\lambda_2,k_2,\mathcal{J},N_*)\|_{\mathfrak{X}^{s_2-1,b_--1}([0,T])}+\|\widetilde{\mathcal{G}}^{(1)}(\cdot;\lambda_2,k_2,\mathcal{J},N_*)\|_{\mathfrak{X}^{s_2-1,b_--1}([0,T])}\Big).$$

We now use Gaussian hypercontractivity and a similar reduction as in the proof of Proposition 4.7.8 to move the supremum outside the probabilistic moments. Then, it remains to show for all frequency scales N_1, N_2, and N_3 satisfying $\max(N_1,N_2)>N_3^{\epsilon}$ that

$$\sup_{\lambda_2\in\mathbb{R}}\sup_{\substack{k_2\in\mathbb{Z}^3:\\ |k_2|\lesssim N_2^{\epsilon}}}\Big\|\|\mathcal{G}^{(3)}(\cdot;\lambda_2,k_2,\mathcal{J},N_*)\|_{\mathfrak{X}^{s_2-1,b_--1}([0,T])}+\|\mathcal{G}^{(1)}(\cdot;\lambda_2,k_2,\mathcal{J},N_*)\|_{\mathfrak{X}^{s_2-1,b_--1}([0,T])}$$
$$+\|\widetilde{\mathcal{G}}^{(1)}(\cdot;\lambda_2,k_2,\mathcal{J},N_*)\|_{\mathfrak{X}^{s_2-1,b_--1}([0,T])}\Big\|_{L^2_\omega}$$
$$\lesssim T^2\max(N_1,N_2,N_3)^{-\eta}.$$

We treat the estimates for the non-resonant and resonant components separately.

Contribution of the non-resonant terms: To estimate the $\mathfrak{X}^{s_2-1,b_--1}$-norm, we calculate the space-time Fourier transform of $\chi(t/T)\mathscr{G}^{(3)}(t,x;\lambda_2,k_2,\mathcal{J},N_*)$. We have that

$$
\begin{aligned}
&\mathcal{F}_{t,x}\Big(\chi(t/T)\mathscr{G}^{(3)}(t,x;\lambda_2,k_2,\mathcal{J},N_*)\Big)(\lambda\mp\langle n\rangle,n)\\
&=\sum_{\pm_1,\pm_2,\pm_3} c(\pm_j\colon 1\leqslant j\leqslant 3)\sum_{n_1,n_2,n_3\in\mathbb{Z}^3}\bigg[1\big\{n=n_{123}+k_2\big\}\rho_N^2(n_2+k_2)\Big(\prod_{j=1}^3\rho_N(n_j)\chi_{N_j}(n_j)\Big)\\
&\times\chi_{N_2'}(n_2+k_2)\widehat{V}(n_{12}+k_2)\mathcal{I}_3[\pm_j,n_j\colon 1\leqslant j\leqslant 3]\\
&\times\mathcal{F}_t\bigg(\exp(\pm_1 it\langle n_1\rangle\pm_3 it\langle n_3\rangle)\int_0^t 1_{\mathcal{J}}(t')\frac{\sin((t-t')\langle n_2+k_2\rangle)}{\langle n_2+k_2\rangle}\exp\big(it'\lambda_2\pm_2 it'\langle n_2\rangle)\mathrm{d}t'\bigg)(\lambda\mp\langle n\rangle)\bigg].
\end{aligned}
$$

Using the orthogonality of the multiple stochastic integrals and Lemma 4.4.12 to estimate the Fourier transform of the time-integral, we obtain that

$$
\begin{aligned}
&\Big\|\|\mathscr{G}^{(3)}\|_{\mathfrak{X}^{s_2-1,b_--1}([0,T])}\Big\|_{L^2_\omega}^2\\
&\lesssim\max_{\pm}\Big\|\langle\lambda\rangle^{b_--1}\langle n\rangle^{s_2-1}\mathcal{F}_{t,x}\Big(\chi(t/T)\mathscr{G}^{(3)}(t,x;\lambda_2,k_2,\mathcal{J},N_*)\Big)(\lambda\mp\langle n\rangle,n)\|_{L^2_\lambda\ell^2_n(\mathbb{R}\times\mathbb{Z}^3)}\Big\|_{L^2_\omega}^2\\
&\lesssim T^4\max_{\pm,\pm_1,\pm_2,\pm_3}\max_{\iota_2=-1,0,1}\int_{\mathbb{R}}\mathrm{d}\lambda\,\langle\lambda\rangle^{2(b_--1)}\sum_{n_1,n_2,n_3\in\mathbb{Z}^3}\bigg[\Big(\prod_{j=1}^3\chi_{N_j}(n_j)\Big)\langle n_{123}+k_2\rangle^{2(s_2-1)}\langle n_{12}+k_2\rangle^{-2\beta}\\
&\times\langle n_1\rangle^{-2}\langle n_2\rangle^{-4}\langle n_3\rangle^{-2}\Big(1+\big|\lambda-\lambda_3-\big(\pm\langle n_{123}+k_2\rangle\pm_1\langle n_1\rangle\pm_2\langle n_2\rangle+\iota_2\langle n_2+k_2\rangle\pm_3\langle n_3\rangle\big|\Big)^{-2}\bigg]\\
&\lesssim T^4N_2^{-1+5\epsilon}\max_{\pm,\pm_1,\pm_3}\sup_{\substack{n_2\in\mathbb{Z}^3\colon\\|n_2|\sim N_2}}\sup_{m\in\mathbb{Z}^3}\sum_{n_1,n_3\in\mathbb{Z}^3}\bigg[\Big(\prod_{j=1,3}\chi_{N_j}(n_j)\Big)\langle n_{123}\rangle^{2(s_2-1)}\langle n_{12}\rangle^{-2\beta}\langle n_1\rangle^{-2}\langle n_3\rangle^{-2}\\
&\times 1\big\{\pm\langle n_{123}\rangle\pm_1\langle n_1\rangle\pm_3\langle n_3\rangle\in[m,m+1)\big\}\bigg]\\
&\lesssim T^4\max(N_1,N_2,N_3)^{2\delta_2}N_1^{-2\epsilon}N_2^{-1+7\epsilon}.
\end{aligned}
$$

In the last line, we used Lemma 4.4.23 with $\gamma=\epsilon$. Since $\max(N_1,N_2)>N_3^\epsilon$ and δ_2 is much smaller than ϵ^2, this contribution is acceptable.

Contribution of the resonant terms: We only estimate $\mathscr{G}^{(1)}$. Due to the factor $\widehat{V}(n_{12}+k_2)$, a simpler but similar argument also controls $\widetilde{\mathscr{G}}^{(1)}$.

Using the inhomogeneous Strichartz estimate (Lemma 4.4.9), we have that

$$\|\mathscr{G}^{(1)}\|_{\mathfrak{X}^{s_2-1,b_--1}([0,T])} \lesssim \|\mathscr{G}^{(1)}\|_{L_t^{2b+}H_x^{s_2-1}([0,T]\times\mathbb{T}^3)} \lesssim T^{\frac{1}{2}}\|\mathscr{G}^{(1)}\|_{L_t^2H_x^{s_2-1}([0,T]\times\mathbb{T}^3)}.$$

Using Fubini's theorem and the sin-cancellation lemma (Lemma 4.4.14), this yields

$$\begin{aligned}
&\Big\|\|\mathscr{G}^{(1)}\|_{\mathfrak{X}^{s_2-1,b_--1}([0,T])}\Big\|_{L^2_\omega}^2 \\
&\lesssim T^2 \sup_{t\in[0,T]} \Big\|\|\mathscr{G}^{(1)}\|_{H_x^{s_2-1}(\mathbb{T}^3)}\Big\|_{L^2_\omega}^2 \\
&\lesssim T^2 \sum_{n_3\in\mathbb{Z}^3} \chi_{N_3}(n_3)\langle n_3+k_2\rangle^{2(s_2-1)}\langle n_3\rangle^{-2}\Big| \sum_{n_2\in\mathbb{Z}^3} \rho_N^2(n_2+k_2)\rho_N^2(n_2)\chi_{N_1}(n_2)\chi_{N_2}(n_2)\chi_{N_2'}(n_2+k_2) \\
&\times \langle n_2+k_2\rangle^{-1}\langle n_2\rangle^{-2}\Big(\int_0^t 1_{\mathcal{J}}(t')\sin((t-t')\langle n_2+k_2\rangle)\cos((t-t')\langle n_2\rangle)\exp\big(it'\lambda_2)\mathrm{d}t'\Big)\Big|^2 \\
&\lesssim T^4\, 1\big\{N_1\sim N_2\big\} N_1^{-2+6\epsilon}\langle k_2\rangle^{2(1-s_2)} \sum_{n_3\in\mathbb{Z}^3}\chi_{N_3}(n_3)\langle n_3\rangle^{2(s_2-1)}\langle n_3\rangle^{-2} \\
&\lesssim T^4\, 1\big\{N_1\sim N_2\big\} N_1^{-2+8\epsilon}N_3^{2\delta_2}.
\end{aligned}$$

Since $\max(N_1,N_2)\gtrsim N_3^{\epsilon}$ and δ_2 is much smaller than ϵ, this contribution is acceptable.

□

4.8 Physical-space methods

In this section, we estimate the terms in **Phy**. The main ingredients are para-product decompositions and Strichartz estimates. In Section 4.8.1, we recall the refined Strichartz estimates for the wave equation by Klainerman and Tataru [KT99]. In Section 4.8.2, we use the Klainerman-Tataru-Strichartz estimate to control several terms in **Phy**. The remaining terms in **Phy** are estimated in Section 4.8.3, which also requires estimates on the quartic stochastic object from Section 4.5.2.

4.8.1 Klainerman-Tataru-Strichartz estimates

We first recall the refined (linear) Strichartz estimate from [KT99, (A.59)].

Lemma 4.8.1 (Klainerman-Tataru-Strichartz estimates)**.** Let $\mathcal{J}$ be a compact interval. Let Q be a box of sidelength $\sim M$ at a distance $\sim N$ from the origin. Let P_Q be the corresponding Fourier truncation operator and let $2 \leqslant p, q < \infty$ satisfy the sharp wave-admissibility condition $1/q + 1/p = 1/2$. Then,

$$\|P_Q u\|_{L_t^q L_x^p(\mathcal{J}\times\mathbb{T}^3)} \lesssim (1+|\mathcal{J}|)^{\frac{1}{q}} \left(\frac{M}{N}\right)^{\frac{1}{2}-\frac{1}{p}} N^{\frac{3}{2}-\frac{1}{q}-\frac{3}{p}} \|P_Q u\|_{\mathfrak{X}^{0,b}(\mathcal{J})}. \tag{4.8.1}$$

Remark 4.8.2. The factor $N^{\frac{3}{2}-\frac{1}{q}-\frac{3}{p}}$ is the same as in the standard deterministic Strichartz estimate. The gain from the stronger localization in frequency space is described by the factor $(M/N)^{\frac{1}{2}-\frac{1}{p}}$. Naturally, there is no gain when $p = 2$.
We emphasize that (4.8.1) has a more complicated dependence on M and N than the corresponding result for the Schrödinger equation. In the Schrödinger setting, the frequency-localized Strichartz estimates for the operator P_Q and the standard Littlewood-Paley operators $P_{\leqslant M}$ are equivalent, which follows from the Galilean symmetry. This difference between the Schrödinger and wave equation already played a role in our counting estimates (Section 4.4.4).

Corollary 4.8.3. Let $\mathcal{J}$ be a compact interval. Let Q be a box of sidelength $\sim M$ at a distance $\sim N$ from the origin. Let P_Q be the corresponding Fourier truncation operator and let $q \geqslant 4$. Then, it holds that

$$\|P_Q u\|_{L_t^q L_x^q(\mathcal{J}\times\mathbb{T}^3)} \lesssim (1+|\mathcal{J}|)^{\frac{1}{q}} M^{\frac{3}{2}-\frac{5}{q}} N^{\frac{1}{q}} \|P_Q u\|_{\mathfrak{X}^{0,b}(\mathcal{J})}. \tag{4.8.2}$$

Proof. This follows by combining Lemma 4.8.1 (with $q = p = 4$) and the Bernstein inequality

$$\|P_Q u\|_{L_t^\infty L_x^\infty(\mathcal{J}\times\mathbb{T}^3)} \lesssim M^{\frac{3}{2}} \|P_Q u\|_{\mathfrak{X}^{0,b}(\mathcal{J})}.$$

□

We now state a bilinear version of the Klainerman-Tataru-Strichartz estimate, which is a consequence of Lemma 4.8.1 (cf. [KT99, Theorem 4 and 5]). However, since we only require a special case, we provide a self-contained proof.

Lemma 4.8.4 (Bilinear Klainerman-Tataru-Strichartz estimate)**.** Let $T \geqslant 1$, $q \geqslant 4$, let $\gamma < 3 - 10/q$ and let $N_1, N_2 \geqslant 1$. Then, it holds that

$$\|\langle \nabla \rangle^{-\gamma} \big(P_{N_1} f \cdot P_{N_2} g \big) \|_{L_t^{\frac{q}{2}} L_x^{\frac{q}{2}}([0,T]\times \mathbb{T}^3)} \lesssim T^{\frac{2}{q}} \max(N_1, N_2)^{3-2s_1-\frac{8}{q}-\gamma} \|f\|_{\mathfrak{X}^{s_1,b}([0,T])} \|g\|_{\mathfrak{X}^{s_1,b}([0,T])}.$$

In particular,

$$\sum_{N_1,N_2} \|P_{N_1} f \cdot P_{N_2} g\|_{L_t^2 H_x^{-4\delta_1}([0,T]\times\mathbb{T}^3)} \lesssim T^{\frac{1}{2}} \|f\|_{\mathfrak{X}^{s_1,b}([0,T])} \|g\|_{\mathfrak{X}^{s_1,b}([0,T])}.$$

Furthermore, if $N_{12} \geqslant 1$, then

$$\begin{aligned} &\|(P_{N_{12}} V) * \big(P_{N_1} f \cdot P_{N_2} g \big) \|_{L_t^2 L_x^2([0,T]\times\mathbb{T}^3)} \\ &\lesssim T^{\frac{1}{2}} N_{12}^{\frac{1}{2}-\beta-2\delta_1} \max(N_1,N_2)^{-\frac{1}{2}+4\delta_1} \|f\|_{\mathfrak{X}^{s_1,b}([0,T])} \|g\|_{\mathfrak{X}^{s_1,b}([0,T])}. \end{aligned}$$

Remark 4.8.5. Bilinear Strichartz estimates are also important in the random data theory for nonlinear Schrödinger equations in [BOP15a, BOP19a]. In the proof of Proposition 4.8.10 below, we will only require the case $q = 4+$ and the reader may simply think of q as four.

Proof. We begin with the first estimate, which is the main part of the argument. Using the definition of the restricted $\mathfrak{X}^{s,b}$-spaces, we may replace $\|f\|_{\mathfrak{X}^{s_1,b}([0,T])}$ and $\|g\|_{\mathfrak{X}^{s_1,b}([0,T])}$ by $\|f\|_{\mathfrak{X}^{s_1,b}(\mathbb{R})}$ and $\|g\|_{\mathfrak{X}^{s_1,b}(\mathbb{R})}$, respectively. The proof relies on the linear Klainerman-Tataru-Strichartz estimate (Corollary 4.8.3) and box localization. We decompose

$$\|\langle \nabla \rangle^{-\gamma} \big(P_{N_1} f \cdot P_{N_2} g \big) \|_{L_t^{\frac{q}{2}} L_x^{\frac{q}{2}}([0,T]\times \mathbb{T}^3)} \lesssim \sum_{\substack{N_{12}: \\ N_{12} \lesssim \max(N_1,N_2)}} N_{12}^{-\gamma} \|P_{N_{12}} \big(P_{N_1} f \cdot P_{N_2} g \big) \|_{L_t^{\frac{q}{2}} L_x^{\frac{q}{2}}([0,T]\times \mathbb{T}^3)}.$$

If $N_1 \not\sim N_2$, then $N_{12} \sim \max(N_1, N_2)$ and the desired estimate follows from Hölder's inequality and the $L_t^q L_x^q$-estimate from Corollary 4.8.3 with $M \sim N$. Thus, it remains to treat the case $N_1 \sim N_2$. Let $\mathcal{Q} = \mathcal{Q}(N_1, N_{12})$ be a cover of the dyadic annulus at distance $\sim N_1$ by finitely overlapping cubes of diameter $\sim N_{12}$. From Fourier support considerations and Lemma 4.8.1, it follows that

$$
\begin{aligned}
&\big\| P_{N_{12}}\big(P_{N_1} f \cdot P_{N_2} g\big)\big\|_{L_t^{\frac{q}{2}} L_x^{\frac{q}{2}}([0,T]\times\mathbb{T}^3)} \\
&\lesssim \sum_{\substack{Q_1,Q_2\in\mathcal{Q}:\\ d(Q_1,Q_2)\lesssim N_{12}}} \big\| P_{Q_1} P_{N_1} f \cdot P_{Q_2} P_{N_2} g \big\|_{L_t^{\frac{q}{2}} L_x^{\frac{q}{2}}([0,T]\times\mathbb{T}^3)} \\
&\lesssim \sum_{\substack{Q_1,Q_2\in\mathcal{Q}:\\ d(Q_1,Q_2)\lesssim N_{12}}} \| P_{Q_1} P_{N_1} f\|_{L_t^q L_x^q([0,T]\times\mathbb{T}^3)} \| P_{Q_2} P_{N_2} g\|_{L_t^q L_x^q([0,T]\times\mathbb{T}^3)} \\
&\lesssim T^{\frac{2}{q}} N_{12}^{3-\frac{10}{q}} N_1^{\frac{2}{q}-2s_1} \sum_{\substack{Q_1,Q_2\in\mathcal{Q}:\\ d(Q_1,Q_2)\lesssim N_{12}}} \| P_{Q_1} P_{N_1} f\|_{\mathfrak{X}^{s_1,b}(\mathbb{R})} \| P_{Q_2} P_{N_2} g\|_{\mathfrak{X}^{s_1,b}(\mathbb{R})} \\
&\lesssim T^{\frac{2}{q}} N_{12}^{3-\frac{10}{q}} N_1^{\frac{2}{q}-2s_1} \Big(\sum_{\substack{Q_1,Q_2\in\mathcal{Q}:\\ d(Q_1,Q_2)\lesssim N_{12}}} \| P_{Q_1} P_{N_1} f\|_{\mathfrak{X}^{s_1,b}(\mathbb{R})}^2 \Big)^{\frac{1}{2}} \Big(\sum_{\substack{Q_1,Q_2\in\mathcal{Q}:\\ d(Q_1,Q_2)\lesssim N_{12}}} \| P_{Q_2} P_{N_2} g\|_{\mathfrak{X}^{s_1,b}(\mathbb{R})}^2 \Big)^{\frac{1}{2}} \\
&\lesssim T^{\frac{2}{q}} N_{12}^{3-\frac{10}{q}} N_1^{\frac{2}{q}-2s_1} \| f\|_{\mathfrak{X}^{s_1,b}(\mathbb{R})} \| g\|_{\mathfrak{X}^{s_1,b}(\mathbb{R})}.
\end{aligned}
$$

The desired result then follows by using the upper bound $\gamma < 3 - \frac{10}{q}$ and summing in N_{12}.

We now turn to the second estimate. After estimating

$$
\big\|(P_{N_{12}} V) * \big(P_{N_1} f \cdot P_{N_2} g\big)\big\|_{L_t^2 L_x^2([0,T]\times\mathbb{T}^3)} \lesssim N_{12}^{\frac{1}{2}-\beta-2\delta_1} \big\| \langle \nabla \rangle^{-\frac{1}{2}+2\delta_1} \big(P_{N_1} f \cdot P_{N_2} g\big)\big\|_{L_t^2 L_x^2([0,T]\times\mathbb{T}^3)},
$$

the result follows from the first estimate. □

4.8.2 Physical terms

In this subsection, we use the Klainerman-Tataru-Strichartz estimate and a para-product decomposition to control several terms in **Phy**.

Proposition 4.8.6. Let $\mathcal{J}$ be a bounded interval and let $f, g \in \mathfrak{X}^{s_1,b}(\mathcal{J})$. Then, it holds that

$$\begin{aligned}
&\sup_{N \geqslant 1} \Big\| V * \big(P_{\leqslant N} f \cdot P_{\leqslant N} g\big) \ \big(\neg \textcircled{$<$}\big) \ P_{\leqslant N} \overset{\bullet}{|} \Big\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})} \\
&\lesssim (1 + |\mathcal{J}|)^2 \|f\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \|g\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \|\overset{\bullet}{|}\|_{L_t^\infty \mathcal{C}_x^{-1/2-\kappa}(\mathcal{J}\times\mathbb{T}^3)}
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{N \geqslant 1} \Big\| \ \big(\neg \boxed{\textcircled{$<$} \,\&\, \textcircled{$<$}}\big) \ \Big(V * \big(P_{\leqslant N} f \cdot P_{\leqslant N} g\big) \ P_{\leqslant N} \overset{\bullet}{|} \Big) \Big\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})} \\
&\lesssim (1 + |\mathcal{J}|)^2 \|f\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \|g\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \|\overset{\bullet}{|}\|_{L_t^\infty \mathcal{C}_x^{-1/2-\kappa}(\mathcal{J}\times\mathbb{T}^3)}
\end{aligned}$$

In the frequency-localized versions of the two estimates, which are detailed in the proof, we gain an η'-power in the maximal frequency-scale.

Proof. After using a Littlewood-Paley decomposition, we obtain

$$\begin{aligned}
&\Big\| V * \big(P_{\leqslant N} f \cdot P_{\leqslant N} g\big) \ \big(\neg \textcircled{$<$}\big) \ P_{\leqslant N} \overset{\bullet}{|} \Big\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})} \\
+&\Big\| \ \big(\neg \boxed{\textcircled{$<$} \,\&\, \textcircled{$<$}}\big) \ \Big(V * \big(P_{\leqslant N} f \cdot P_{\leqslant N} g\big) \ P_{\leqslant N} \overset{\bullet}{|} \Big) \Big\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})} \\
\lesssim &\sum_{\substack{N_1,N_2,N_3,N_{12}: \\ \max(N_1,N_2) \gtrsim N_3^\epsilon}} \Big\| (P_{N_{12}} V) * \big(P_{\leqslant N} P_{N_1} f \cdot P_{\leqslant N} P_{N_2} g\big) P_{\leqslant N} P_{N_3} \overset{\bullet}{|} \Big\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})},
\end{aligned}$$

where we also used that $N_{12} \lesssim \max(N_1, N_2)$. We estimate each dyadic piece separately and distinguish two cases:

Case 1: $N_{12} \not\sim N_3$. Using the inhomogeneous Strichartz estimate (Lemma 4.4.9) and Lemma 4.8.4, we obtain that

$$
\begin{aligned}
&\Big\| (P_{N_{12}} V) * \big(P_{\leqslant N} P_{N_1} f \cdot P_{\leqslant N} P_{N_2} g \big) P_{\leqslant N} P_{N_3} \lollipop \Big\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})} \\
&\lesssim \Big\| (P_{N_{12}} V) * \big(P_{\leqslant N} P_{N_1} f \cdot P_{\leqslant N} P_{N_2} g \big) P_{\leqslant N} P_{N_3} \lollipop \Big\|_{L_t^{2b_+} H_x^{s_2-1}(\mathcal{J}\times\mathbb{T}^3)} \\
&\lesssim (1+|\mathcal{J}|)^{\frac{1}{2}} \max(N_{12},N_3)^{s_2-1} \Big\| (P_{N_{12}} V) * \big(P_{\leqslant N} P_{N_1} f \cdot P_{\leqslant N} P_{N_2} g \big) \Big\|_{L_t^2 L_x^2(\mathcal{J}\times\mathbb{T}^3)} \\
&\times \Big\| P_{\leqslant N} P_{N_3} \lollipop \Big\|_{L_t^\infty L_x^\infty(\mathcal{J}\times\mathbb{T}^3)} \\
&\lesssim (1+|\mathcal{J}|) \max(N_{12},N_3)^{s_2-1} N_{12}^{\frac{1}{2}-\beta+2\delta_1} \max(N_1,N_2)^{-\frac{1}{2}+4\delta_1} N_3^{\frac{1}{2}+\kappa} \\
&\times \|f\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \|g\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \|\lollipop\|_{L_t^\infty \mathcal{C}_x^{-1/2-\kappa}(\mathcal{J}\times\mathbb{T}^3)}.
\end{aligned}
$$

Since $\max(N_1,N_2) \geqslant N_3^\epsilon$, we can bound the pre-factor by

$$
\begin{aligned}
&\max(N_{12},N_3)^{s_2-1} N_{12}^{\frac{1}{2}-\beta+2\delta_1} \max(N_1,N_2)^{-\frac{1}{2}+4\delta_1} N_3^{\frac{1}{2}+\kappa} \lesssim \max(N_1,N_2)^{-\beta+6\delta_1} N_3^{\delta_2+\kappa} \\
&\lesssim \max(N_1,N_2,N_3)^{-2\eta}.
\end{aligned}
$$

Case 2: $N_{12} \sim N_3$. By symmetry, we can assume that $N_1 \geqslant N_2$. Furthermore, we have that $N_3 \sim N_{12} \lesssim N_1$. Using the inhomogeneous Strichartz estimate (Lemma 4.4.9), we obtain that

$$
\begin{aligned}
&\Big\| (P_{N_{12}} V) * \big(P_{\leqslant N} P_{N_1} f \cdot P_{\leqslant N} P_{N_2} g \big) P_{\leqslant N} P_{N_3} \lollipop \Big\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})} \\
&\lesssim (1+|\mathcal{J}|) \Big\| \langle \nabla \rangle^{s_2-\frac{1}{2}+4(b_+-\frac{1}{2})} \Big((P_{N_{12}} V) * \big(P_{\leqslant N} P_{N_1} f \cdot P_{\leqslant N} P_{N_2} g \big) P_{\leqslant N} P_{N_3} \lollipop \Big) \Big\|_{L_t^{\frac{4}{3}} L_x^{\frac{4}{3}}(\mathcal{J}\times\mathbb{T}^3)} \\
&\lesssim (1+|\mathcal{J}|)^{\frac{3}{2}} N_3^{s_2-\frac{1}{2}+4(b_+-\frac{1}{2})-\beta} \Big\| P_{N_1} f \Big\|_{L_t^\infty L_x^2(\mathcal{J}\times\mathbb{T}^3)} \Big\| P_{N_2} g \Big\|_{L_t^4 L_x^4(\mathcal{J}\times\mathbb{T}^3)} \Big\| P_{N_3} \lollipop \Big\|_{L_t^\infty L_x^\infty(\mathcal{J}\times\mathbb{T}^3)} \\
&\lesssim (1+|\mathcal{J}|)^2 N_1^{-s_1} N_2^{\frac{1}{2}-s_1} N_3^{s_2-\frac{1}{2}+4(b_+-\frac{1}{2})-\beta+\frac{1}{2}+\kappa} \|f\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \|g\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \|\lollipop\|_{L_t^\infty \mathcal{C}_x^{-1/2-\kappa}(\mathcal{J}\times\mathbb{T}^3)}.
\end{aligned}
$$

Since $N_2, N_3 \geqslant 1$, the pre-factor can be bounded by

$$
N_1^{-s_1} N_2^{\frac{1}{2}-s_1} N_3^{s_2-\frac{1}{2}+4(b_+-\frac{1}{2})-\beta+\frac{1}{2}+\kappa} \lesssim N_1^{1-2s_1+s_2-\frac{1}{2}+4(b_+-\frac{1}{2})-\beta+\kappa} = N_1^{2\delta_1+\delta_2+4(b_+-\frac{1}{2})+\kappa-\beta},
$$

which is acceptable. □

Proposition 4.8.7. Let $T \geqslant 1$, let $J \subseteq [0,T]$ be an interval, and let $f, g \colon J \times \mathbb{T}^3 \to \mathbb{R}$. Then, it holds that

$$\sup_{N \geqslant 1} \Big\| V * \Big(P_{\leqslant N} \overset{\bullet}{\mid} \circledneq P_{\leqslant N} f \Big) P_{\leqslant N} g \Big\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})} \lesssim (1+|\mathcal{J}|)^2 \| \overset{\bullet}{\mid} \|_{L_t^\infty \mathcal{C}^{-\frac{1}{2}-\kappa}(\mathcal{J}\times\mathbb{T}^3)} \|f\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \|g\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})}.$$

In the frequency-localized version of this estimate, which is detailed in the proof, we gain an η'-power in the maximal frequency-scale.

Proof. By using a Littlewood-Paley decomposition and the definitions of $\circledneq$, we have that

$$V * \Big(P_{\leqslant N} \overset{\bullet}{\mid} \circledneq P_{\leqslant N} f \Big) P_{\leqslant N} g = \sum_{\substack{N_1, N_2, N_3 \colon \\ N_1 \not\sim N_2}} V * \Big(P_{\leqslant N} P_{N_1} \overset{\bullet}{\mid} \cdot P_{\leqslant N} P_{N_2} f \Big) P_{\leqslant N} P_{N_3} g.$$

We treat each dyadic block separately and distinguish two cases.

Case 1: $N_1 \gg N_2, N_3$. Using the inhomogeneous Strichartz estimate (Lemma 4.4.9), we have that

$$\begin{aligned}
&\Big\| V * \Big(P_{\leqslant N} P_{N_1} \overset{\bullet}{\mid} \cdot P_{\leqslant N} P_{N_2} f \Big) P_{\leqslant N} P_{N_3} g \Big\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})} \\
&\lesssim \Big\| V * \Big(P_{\leqslant N} P_{N_1} \overset{\bullet}{\mid} \cdot P_{\leqslant N} P_{N_2} f \Big) P_{\leqslant N} P_{N_3} g \Big\|_{L_t^{2b_+} H_x^{s_2-1}(\mathcal{J}\times\mathbb{T}^3)} \\
&\lesssim (1+|\mathcal{J}|)^{\frac{1}{2}} N_1^{s_2-1-\beta} \| P_{N_1} \overset{\bullet}{\mid} \|_{L_t^\infty L_x^\infty(\mathcal{J}\times\mathbb{T}^3)} \| P_{N_1} f \|_{L_t^4 L_x^4(\mathcal{J}\times\mathbb{T}^3)} \| P_{N_2} g \|_{L_t^4 L_x^4(\mathcal{J}\times\mathbb{T}^3)} \\
&\lesssim (1+|\mathcal{J}|) N_1^{s_2-1-\beta+\frac{1}{2}+\kappa} N_2^{\frac{1}{2}-s_1} N_3^{\frac{1}{2}-s_1} \| \overset{\bullet}{\mid} \|_{L_t^\infty \mathcal{C}_x^{-\frac{1}{2}-\kappa}(\mathcal{J}\times\mathbb{T}^3)} \|f\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \|g\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})}.
\end{aligned}$$

Since $N_2, N_3 \ll N_1$, the pre-factor can be bounded by

$$N_1^{s_2-1-\beta+\frac{1}{2}+\kappa} N_2^{\frac{1}{2}-s_1} N_3^{\frac{1}{2}-s_1} \lesssim N_1^{2\delta_1+\delta_2+\kappa-\beta},$$

which is acceptable.

Case 2.a: $N_1 \ll N_2$, $N_3 \lesssim N_2$. Using the inhomogeneous Strichartz estimate (Lemma 4.4.9), we have that

$$
\begin{aligned}
&\Big\| V * \Big(P_{\leqslant N} P_{N_1} \lollipop \cdot P_{\leqslant N} P_{N_2} f \Big) P_{\leqslant N} P_{N_3} g \Big\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})} \\
&\lesssim (1+|\mathcal{J}|) \Big\| \langle \nabla \rangle^{s_2-\frac{1}{2}+4(b_+-1)} \Big(V * \Big(P_{\leqslant N} P_{N_1} \lollipop \cdot P_{\leqslant N} P_{N_2} f \Big) P_{\leqslant N} P_{N_3} g \Big) \Big\|_{L_t^{\frac{4}{3}} L_x^{\frac{4}{3}}(\mathcal{J}\times\mathbb{T}^3)} \\
&\lesssim (1+|\mathcal{J}|) N_2^{s_2-\frac{1}{2}+4(b_+-1)} \| V * \big(P_{\leqslant N} P_{N_1} \lollipop \cdot P_{\leqslant N} P_{N_2} f \big) \|_{L_t^2 L_x^2(\mathcal{J}\times\mathbb{T}^3)} \| P_{\leqslant N} P_{N_3} g \|_{L_t^4 L_x^4(\mathcal{J}\times\mathbb{T}^3)} \\
&\lesssim (1+|\mathcal{J}|) N_2^{s_2-\frac{1}{2}+4(b_+-1)-\beta} \| \lollipop \|_{L_t^\infty L_x^\infty(\mathcal{J}\times\mathbb{T}^3)} \| P_{N_2} f \|_{L_t^2 L_x^2(\mathcal{J}\times\mathbb{T}^3)} \| P_{N_3} g \|_{L_t^4 L_x^4(\mathcal{J}\times\mathbb{T}^3)} \\
&\lesssim (1+|\mathcal{J}|)^2 N_1^{\frac{1}{2}+\kappa} N_2^{s_2-\frac{1}{2}+4(b_+-1)-\beta-s_1} N_3^{\frac{1}{2}-s_1} \| \lollipop \|_{L_t^\infty \mathcal{C}_x^{-\frac{1}{2}-\kappa}(\mathcal{J}\times\mathbb{T}^3)} \| f \|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \| g \|_{\mathfrak{X}^{s_1,b}(\mathcal{J})}.
\end{aligned}
$$

The pre-factor can now be bounded as before.

Case 2.b: $N_1 \ll N_2$, $N_2 \ll N_3$. Using the inhomogeneous Strichartz estimate (Lemma 4.4.9), we have that

$$
\begin{aligned}
&\Big\| V * \Big(P_{\leqslant N} P_{N_1} \lollipop \cdot P_{\leqslant N} P_{N_2} f \Big) P_{\leqslant N} P_{N_3} g \Big\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})} \\
&\lesssim (1+|\mathcal{J}|) \Big\| \langle \nabla \rangle^{s_2-\frac{1}{2}+4(b_+-1)} \Big(V * \Big(P_{\leqslant N} P_{N_1} \lollipop \cdot P_{\leqslant N} P_{N_2} f \Big) P_{\leqslant N} P_{N_3} g \Big) \Big\|_{L_t^{\frac{4}{3}} L_x^{\frac{4}{3}}(\mathcal{J}\times\mathbb{T}^3)} \\
&\lesssim (1+|\mathcal{J}|) \max(N_1,N_2)^{-\beta} N_3^{s_2-\frac{1}{2}+4(b_+-1)} \| \lollipop \|_{L_t^\infty L_x^\infty(\mathcal{J}\times\mathbb{T}^3)} \| P_{N_2} f \|_{L_t^4 L_x^4(\mathcal{J}\times\mathbb{T}^3)} \| P_{N_3} g \|_{L_t^2 L_x^2(\mathcal{J}\times\mathbb{T}^3)} \\
&\lesssim (1+|\mathcal{J}|)^2 \max(N_1,N_2)^{-\beta} N_1^{\frac{1}{2}+\kappa} N_2^{\frac{1}{2}-s_1} N_3^{s_2-\frac{1}{2}+4(b_+-1)-s_1} \| \lollipop \|_{L_t^\infty \mathcal{C}_x^{-\frac{1}{2}-\kappa}(\mathcal{J}\times\mathbb{T}^3)} \| f \|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \| g \|_{\mathfrak{X}^{s_1,b}(\mathcal{J})}.
\end{aligned}
$$

The pre-factor can now be bounded by

$$
\max(N_1,N_2)^{-\beta} N_1^{\frac{1}{2}+\kappa} N_2^{\frac{1}{2}-s_1} N_3^{s_2-\frac{1}{2}+4(b_+-1)-s_1} \lesssim N_1^{\frac{1}{2}+\kappa-\beta} N_2^{\delta_1} N_3^{-\frac{1}{2}+\delta_1+\delta_2+4(b_+-1)} \lesssim N_3^{2\delta_1+\delta_2+\kappa-\beta},
$$

which is acceptable.

$\square$

Lemma 4.8.8 (Bilinear physical estimate)**.** Let $\mathcal{J} \subseteq \mathbb{R}$ be a bounded interval. If $\Psi, f \colon \mathcal{J}\times\mathbb{T}^3 \to \mathbb{C}$, then

$$
\| (V * \Psi) f \|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J}\times\mathbb{T}^3)} \lesssim (1+|\mathcal{J}|)^{\frac{3}{2}} \| \Psi \|_{L_t^2 H_x^{-4\delta_1}(\mathcal{J}\times\mathbb{T}^3)} \min\big(\| f \|_{L_t^\infty \mathcal{C}_x^{\beta-\kappa}(\mathcal{J}\times\mathbb{T}^3)}, \| f \|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \big).
$$

In the frequency-localized version of this estimate we also gain an η'-power in the maximal frequency-scale.

Lemma 4.8.8 can be combined with our bound on $\text{(lollipop)} \circleddash w_N$ in the stability theory (see Section 4.3.3). In the local theory, its primary application is isolated in the following corollary.

Corollary 4.8.9. Let $\mathcal{J} \subseteq \mathbb{R}$ be a bounded interval and let $w, Y \colon \mathcal{J} \times \mathbb{T}^3 \to \mathbb{R}$. Then, we have uniformly in $N \geqslant 1$ that

$$
\begin{aligned}
&\Big\| V * \Big(P_{\leqslant N} \text{(lollipop)} \circleddash P_{\leqslant N} Y \Big) P_{\leqslant N} \text{(tree)}_N \Big\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})} \\
&\lesssim (1+|\mathcal{J}|)^2 \| \text{(lollipop)} \|_{L_t^\infty \mathcal{C}_x^{-\frac{1}{2}+\kappa}(\mathcal{J}\times\mathbb{T}^3)} \|Y\|_{\mathfrak{X}^{s_2,b}(\mathcal{J})} \Big\| \text{(tree)}_N \Big\|_{L_t^\infty \mathcal{C}_x^{\beta-\kappa}(\mathcal{J}\times\mathbb{T}^3)}, \\
&\Big\| V * \Big(P_{\leqslant N} \text{(lollipop)} \circleddash P_{\leqslant N} Y \Big) P_{\leqslant N} w \Big\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})} \\
&\lesssim (1+|\mathcal{J}|)^2 \| \text{(lollipop)} \|_{L_t^\infty \mathcal{C}_x^{-\frac{1}{2}+\kappa}(\mathcal{J}\times\mathbb{T}^3)} \|Y\|_{\mathfrak{X}^{s_2,b}(\mathcal{J})} \|w\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})}.
\end{aligned}
$$

Proof of Corollary 4.8.9: We have that

$$
\begin{aligned}
\| P_{\leqslant N} \text{(lollipop)} \circleddash P_{\leqslant N} Y \|_{L_t^2 H_x^{-4\delta_1}(\mathcal{J}\times\mathbb{T}^3)} &\lesssim |\mathcal{J}|^{\frac{1}{2}} \| \text{(lollipop)} \|_{L_t^\infty \mathcal{C}_x^{-\frac{1}{2}-\kappa}(\mathcal{J}\times\mathbb{T}^3)} \|Y\|_{L_t^\infty H_x^{s_2}(\mathcal{J}\times\mathbb{T}^3)} \\
&\lesssim |\mathcal{J}|^{\frac{1}{2}} \| \text{(lollipop)} \|_{L_t^\infty \mathcal{C}_x^{-\frac{1}{2}-\kappa}(\mathcal{J}\times\mathbb{T}^3)} \|Y\|_{\mathfrak{X}^{s_2,b}(\mathcal{J})}.
\end{aligned}
$$

Together with Lemma 4.8.9, this implies the corollary. □

Proof of Lemma 4.8.8: Let $0 \leqslant \theta \ll \beta$ remain to be chosen. Using the inhomogeneous Strichartz estimate and (a weaker version of) the fractional product rule, we have that

$$
\begin{aligned}
&\| (V * \Psi) f \|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J}\times\mathbb{T}^3)} \\
&\lesssim (1+|\mathcal{J}|) \| \langle \nabla \rangle^{s_2-\frac{1}{2}+4(b_+-\frac{1}{2})} \big((V*\Psi) f \big) \|_{L_t^{\frac{4}{3}} L_x^{\frac{4}{3}}(\mathcal{J}\times\mathbb{T}^3)} \\
&\lesssim (1+|\mathcal{J}|) \| \langle \nabla \rangle^{s_2-\frac{1}{2}+4(b_+-\frac{1}{2})} (V*\Psi) \|_{L_t^2 L_x^{\frac{4}{2-\theta}}(\mathcal{J}\times\mathbb{T}^3)} \| \langle \nabla \rangle^{s_2-\frac{1}{2}+4(b_+-\frac{1}{2})} f \|_{L_t^4 L_x^{\frac{4}{1+\theta}}(\mathcal{J}\times\mathbb{T}^3)}.
\end{aligned}
$$

Using Sobolev embedding, the first factor is bounded by

$$\begin{aligned}\|\langle\nabla\rangle^{s_2-\frac{1}{2}+4(b_+-\frac{1}{2})}\big(V*\Psi\big)\|_{L_t^2 L_x^{\frac{4}{2-\theta}}(\mathcal{J}\times\mathbb{T}^3)} &\lesssim \|\langle\nabla\rangle^{s_2-\frac{1}{2}+4(b_+-\frac{1}{2})+\frac{3\theta}{4}-\beta}\Psi\|_{L_t^2 L_x^2(\mathcal{J}\times\mathbb{T}^3)} \\ &\lesssim \|\Psi\|_{L_t^2 H_x^{-4\delta_1}(\mathcal{J}\times\mathbb{T}^3)}.\end{aligned}$$

Thus, it remains to present two different estimates of the second factor. By simply choosing $\theta=0$, we see that

$$\|\langle\nabla\rangle^{s_2-\frac{1}{2}+4(b_+-\frac{1}{2})}f\|_{L_t^4 L_x^4(\mathcal{J}\times\mathbb{T}^3)} \lesssim (1+|\mathcal{J}|)^{\frac{1}{4}}\|f\|_{L_t^\infty \mathcal{C}_x^{\beta-\kappa}(\mathcal{J}\times\mathbb{T}^3)},$$

which yields the first term in the minimum. Using Hölder's inequality in time and Strichartz estimates, we also have that

$$\begin{aligned}\|\langle\nabla\rangle^{s_2-\frac{1}{2}+4(b_+-\frac{1}{2})}f\|_{L_t^4 L_x^{\frac{4}{1+\theta}}(\mathcal{J}\times\mathbb{T}^3)} &\lesssim (1+|\mathcal{J}|)^{\frac{\theta}{4}}\|\langle\nabla\rangle^{s_2-\frac{1}{2}+4(b_+-\frac{1}{2})}f\|_{L_t^{\frac{4}{1-\theta}} L_x^{\frac{4}{1+\theta}}(\mathcal{J}\times\mathbb{T}^3)} \\ &\lesssim (1+|\mathcal{J}|)^{\frac{1}{4}}\|f\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})},\end{aligned}$$

provided that

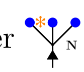
$$s_2-\frac{1}{2}+4(b_+-\frac{1}{2})+\frac{3}{2}-\frac{1-\theta}{4}-3\frac{1+\theta}{4}\leqslant s_1.$$

The last condition can be satisfied by choosing $\theta=4\delta_1$, which also satisfies $\theta\ll\beta$. □

Proposition 4.8.10. Let $\mathcal{J}\subseteq\mathbb{R}$ be a bounded interval and let $f,g,h\colon J\times\mathbb{T}^3$. Then, it holds that

$$\begin{aligned}&\sup_{N\geqslant 1}\Big\|V*\Big(P_{\leqslant N}f\cdot P_{\leqslant N}g\Big)P_{\leqslant N}h\Big\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})} \\ &\lesssim (1+|\mathcal{J}|)^2\prod_{\varphi=f,g,h}\min\Big(\|\varphi\|_{L_t^\infty\mathcal{C}_x^{\beta-\kappa}(\mathcal{J}\times\mathbb{T}^3)},\|\varphi\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})}\Big).\end{aligned} \tag{4.8.3}$$

In the frequency-localized version of this estimate we also gain an η'-power in the maximal frequency-scale.

Remark 4.8.11. In applications of Lemma 4.8.10, we will choose f,g, and h as either , which is contained in $L_t^\infty\mathcal{C}_x^{\beta-\kappa}$, or w_N, which is contained in $\mathfrak{X}^{s_1,b}$.

Proof. Since the proof is relatively standard, we only present the argument when all functions f, g, and h are placed in the same space. The intermediate cases follow from a combination of our arguments below.

Estimate for $L_t^\infty \mathcal{C}_x^{\beta-\kappa}$: Using the inhomogeneous Strichartz estimate (Lemma 4.4.9) and $s_2 \leqslant 1$, we have that

$$\Big\| V * \Big(P_{\leqslant N} f \cdot P_{\leqslant N} g\Big) P_{\leqslant N} h \Big\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})} \lesssim \Big\| V * \Big(P_{\leqslant N} f \cdot P_{\leqslant N} g\Big) P_{\leqslant N} h \Big\|_{L_t^{2b_+} L_x^2(\mathcal{J}\times\mathbb{T}^3)}$$
$$\lesssim (1+|\mathcal{J}|) \prod_{\varphi=f,g,h} \|\varphi\|_{L_t^\infty L_x^\infty(\mathcal{J}\times\mathbb{T}^3)} \lesssim (1+|\mathcal{J}|) \prod_{\varphi=f,g,h} \|\varphi\|_{L_t^\infty \mathcal{C}_x^{\beta-\kappa}(\mathcal{J}\times\mathbb{T}^3)}.$$

Estimate for $\mathfrak{X}^{s_1,b}(\mathcal{J})$: Let $0<\theta \ll 1$ remain to be chosen. Using the inhomogeneous Strichartz estimate (Lemma 4.4.9), we have that

$$\Big\| V * \Big(P_{\leqslant N} f \cdot P_{\leqslant N} g\Big) P_{\leqslant N} h \Big\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})}$$
$$\lesssim (1+|\mathcal{J}|) \Big\| \langle \nabla \rangle^{s_2-\frac{1}{2}+4(b_+-\frac{1}{2})} \Big(V * \Big(P_{\leqslant N} f \cdot P_{\leqslant N} g\Big) P_{\leqslant N} h \Big) \Big\|_{L_t^{\frac{4}{3}} L_x^{\frac{4}{3}}(\mathcal{J}\times\mathbb{T}^3)}$$
$$\lesssim (1+|\mathcal{J}|) \Big\| \langle \nabla \rangle^{s_2-\frac{1}{2}+4(b_+-\frac{1}{2})} \Big(V * \Big(P_{\leqslant N} f \cdot P_{\leqslant N} g\Big)\Big) \Big\|_{L_t^{\frac{4}{2-\theta}} L_x^{\frac{4}{2-\theta}}(\mathcal{J}\times\mathbb{T}^3)}$$
$$\times \Big\| \langle \nabla \rangle^{s_2-\frac{1}{2}+4(b_+-\frac{1}{2})} h \Big\|_{L_t^{\frac{4}{1+\theta}} L_x^{\frac{4}{1+\theta}}(\mathcal{J}\times\mathbb{T}^3)}.$$

Using Lemma 4.8.4, the first term is bounded by $(1+|\mathcal{J}|)^{\frac{2-\theta}{4}} \|f\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \|g\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})}$ as long as

$$2\delta_1 + \delta_2 + 4\Big(b_+ - \frac{1}{2}\Big) + \theta < \beta. \tag{4.8.4}$$

Using Hölder's inequality in the time-variable and the linear Strichartz estimate, we have that

$$\Big\| \langle \nabla \rangle^{s_2-\frac{1}{2}+4(b_+-\frac{1}{2})} h \Big\|_{L_t^{\frac{4}{1+\theta}} L_x^{\frac{4}{1+\theta}}(\mathcal{J}\times\mathbb{T}^3)} \lesssim (1+|\mathcal{J}|)^{\frac{\theta}{2}} \Big\| \langle \nabla \rangle^{s_2-\frac{1}{2}+4(b_+-\frac{1}{2})} h \Big\|_{L_t^{\frac{4}{1-\theta}} L_x^{\frac{4}{1+\theta}}(\mathcal{J}\times\mathbb{T}^3)}$$
$$\lesssim (1+|\mathcal{J}|)^{\frac{1+\theta}{4}} \|h\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})},$$

provided that

$$\frac{\theta}{2} > \delta_1 + \delta_2 + 4\big(b_+ - \frac{1}{2}\big). \tag{4.8.5}$$

In order to satisfy both conditions (4.8.4) and (4.8.5), we can choose $\theta = 4\delta_1$. $\square$

4.8.3 Hybrid physical-RMT terms

In this subsection, we estimate the remaining terms in **Phy**. Our estimates will be phrased as bounds on the operator norm of certain random operators. In contrast to Proposition 4.6.1 and Proposition 4.6.3, however, we will not need the moment method (from [DNY20]). Instead, we will rely on Strichartz estimates and the estimates for the quartic stochastic object from Section 4.5.2.

Proposition 4.8.12. Let $T \geqslant 1$ and $p \geqslant 1$. Then, we have the following three estimates:

$$\Big\| \sup_{N\geqslant 1} \sup_{\mathcal{J}\subseteq[0,T]} \sup_{\|w\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})}\leqslant 1} \Big\| V * \Big(P_{\leqslant N} \overset{\bullet}{|} \cdot P_{\leqslant N} \overset{\bullet\ast\bullet\bullet}{\curlyvee}_{N} \Big) P_{\leqslant N} w \Big\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})} \Big\|_{L^p_\omega(\mathbb{P})} \lesssim T^3 p^2, \tag{4.8.6}$$

$$\Big\| \sup_{N\geqslant 1} \sup_{\mathcal{J}\subseteq[0,T]} \sup_{\|w\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})}\leqslant 1} \Big\| V * \Big(P_{\leqslant N} \overset{\bullet}{|} \circledneq P_{\leqslant N} w \Big) P_{\leqslant N} \overset{\bullet\ast\bullet\bullet}{\curlyvee}_{N} \Big\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})} \Big\|_{L^p_\omega(\mathbb{P})} \lesssim T^3 p^2, \tag{4.8.7}$$

$$\Big\| \sup_{N\geqslant 1} \sup_{\mathcal{J}\subseteq[0,T]} \sup_{\|w\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})}\leqslant 1} \Big\| V * \Big(P_{\leqslant N} \overset{\bullet\ast\bullet\bullet}{\curlyvee}_{N} \cdot P_{\leqslant N} w \Big) \big(\neg \circledless\big) \; P_{\leqslant N} \overset{\bullet}{|} \Big\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})} \Big\|_{L^p_\omega(\mathbb{P})} \tag{4.8.8}$$

$$\lesssim T p^2.$$

Remark 4.8.13. In the frequency-localized versions of (4.8.6), (4.8.7), and (4.8.8), we also gain an η'-power of the maximal frequency-scale. Similar as in Proposition 4.5.3 and Remark 4.5.4, we may also replace $\overset{\bullet\ast\bullet\bullet}{\curlyvee}_{N}$ by $\overset{\bullet\ast\bullet\bullet}{\curlyvee}_{N,\mathcal{T}}$.

Proof. We first prove (4.8.6), which is the easiest part. Using the inhomogeneous Strichartz estimate (Lemma 4.4.9), $s_2 - 1 < -s_1$, and the (dual of) the fractional product rule, we have that

$$\begin{aligned}
&\Big\| V * \Big(P_{\leqslant N} \overset{\bullet}{|} \cdot P_{\leqslant N} \overset{\bullet \ast \bullet \bullet}{\curlyvee}_{\mathrm{N}} \Big) P_{\leqslant N} w \Big\|_{\mathfrak{X}^{s_2 - 1, b_+ - 1}(\mathcal{J})} \\
&\lesssim \Big\| V * \Big(P_{\leqslant N} \overset{\bullet}{|} \cdot P_{\leqslant N} \overset{\bullet \ast \bullet \bullet}{\curlyvee}_{\mathrm{N}} \Big) P_{\leqslant N} w \Big\|_{L_t^{2b_+} H_x^{s_2 - 1}(\mathcal{J})} \\
&\lesssim \Big\| V * \Big(P_{\leqslant N} \overset{\bullet}{|} \cdot P_{\leqslant N} \overset{\bullet \ast \bullet \bullet}{\curlyvee}_{\mathrm{N}} \Big) P_{\leqslant N} w \Big\|_{L_t^{2b_+} H_x^{-s_1}(\mathcal{J})} \\
&\lesssim T \Big\| V * \Big(P_{\leqslant N} \overset{\bullet}{|} \cdot P_{\leqslant N} \overset{\bullet \ast \bullet \bullet}{\curlyvee}_{\mathrm{N}} \Big) \Big\|_{L_t^\infty \mathcal{C}_x^{-s_1 + \eta}([0,T] \times \mathbb{T}^3)} \| w \|_{L_t^\infty H_x^{s_1}(\mathcal{J} \times \mathbb{T}^3)}.
\end{aligned}$$

Using (4.5.11) in Proposition 4.5.3, this implies (4.8.6).

We now turn to (4.8.7) and (4.8.8), which are more difficult. The main step consists of the following estimate: For any $M_1, N_1, K_1, K_2 \geqslant 1$, we have that

$$\begin{aligned}
&\Big\| \sup_{N \geqslant 1} \sup_{t \in [0,T]} \sup_{\|f\|_{H_x^{s_1}}, \|g\|_{H_x^{s_1}} \leqslant 1} \Big| \int_{\mathbb{T}^3} V * \big(P_{M_1} P_{\leqslant N} \overset{\bullet \ast \bullet \bullet}{\curlyvee}_{\mathrm{N}} \cdot P_{K_1} P_{\leqslant N} f \big) P_{N_1} P_{\leqslant N} \overset{\bullet}{|} \cdot P_{K_2} P_{\leqslant N} g \, \mathrm{d}x \Big| \Big\|_{L_\omega^p(\mathbb{P})} \\
&\lesssim T^3 \max(K_1, K_2, N_1, M_1)^{-\eta} \Big(1 + 1\big\{ N_1 \sim K_2 \big\} M_1^{-\beta + \kappa + \eta} K_1^{-s_1 + \eta} N_1^{\frac{1}{2} + \kappa - s_1} \Big) p^2.
\end{aligned} \tag{4.8.9}$$

For notational convenience, we now omit the multiplier $P_{\leqslant N}$. As will be evident from the proof, the same argument applies (uniformly in N) with the multiplier. The proof of (4.8.9) splits into two cases. The impatient reader may wish to skim ahead to Case 2.b, which contains the most interesting part of the argument.

Case 1: $M_1 \not\sim N_1$. From Fourier support considerations, it follows that $\max(K_1, K_2) \gtrsim \max(N_1, M_1)$.

Then, we estimate the integral in (4.8.9) by

$$
\begin{aligned}
&\Big| \int_{\mathbb{T}^3} V * \big(P_{M_1} \treeN \cdot P_{K_1} f \big) P_{N_1} \lolli \cdot P_{K_2} g \mathrm{d}x \Big| \\
&\lesssim \sum_{L \lesssim \max(N_1,K_2)} \Big| \int_{\mathbb{T}^3} (P_L V) * \big(P_{M_1} \treeN \cdot P_{K_1} f \big) \cdot \widetilde{P}_L \big(P_{N_1} \lolli \cdot P_{K_2} g \big) \mathrm{d}x \Big| \\
&\lesssim \sum_{L \lesssim \max(N_1,K_2)} \| P_L V \|_{L^1_x} \Big\| P_{M_1} \treeN \cdot P_{K_1} f \Big\|_{L^2_x} \Big\| \widetilde{P}_L \big(P_{N_1} \lolli \cdot P_{K_2} g \big) \mathrm{d}x \Big\|_{L^2_x} \\
&\lesssim M_1^{-\beta+\kappa} K_1^{-s_1} \Big\| \treeN \Big\|_{\mathcal{C}_x^{\beta-\kappa}} \sum_{L \lesssim \max(N_1,K_2)} L^{-\beta} \Big\| \widetilde{P}_L \big(P_{N_1} \lolli \cdot P_{K_2} g \big) \mathrm{d}x \Big\|_{L^2_x}. \qquad (4.8.10)
\end{aligned}
$$

We now further split the argument into two subcases.

Case 1.a: $M_1 \not\sim N_1$, $K_2 \not\sim N_1$. Then, we only obtain a non-trivial contribution if $L \sim \max(N_1, K_2)$. Using $\max(K_1, K_2) \gtrsim \max(M_1, N_1) \geqslant N_1$, we obtain that

$$
\begin{aligned}
(4.8.10) &\lesssim M_1^{-\beta+\kappa} K_1^{-s_1} \max(K_2, N_1)^{-\beta} K_2^{-s_1} N_1^{\frac{1}{2}+\kappa} \Big\| \treeN \Big\|_{\mathcal{C}_x^{\beta-\kappa}} \big\| \lolli \big\|_{\mathcal{C}_x^{-\frac{1}{2}-\kappa}} \\
&\lesssim M_1^{-\beta+\kappa} K_1^{-\eta} K_2^{-\eta} N_1^{\frac{1}{2}+\kappa+\eta-\beta-s_1} \Big\| \treeN \Big\|_{\mathcal{C}_x^{\beta-\kappa}} \big\| \lolli \big\|_{\mathcal{C}_x^{-\frac{1}{2}-\kappa}}.
\end{aligned}
$$

The pre-factor is bounded by $(M_1 K_1 K_2 N_1)^{-\eta}$, which is acceptable.

Case 1.b: $M_1 \not\sim N_1$, $K_2 \sim N_1$. In this case, the worst case corresponds to $L \sim 1$. Using only Hölder's inequality, we obtain that

$$
(4.8.10) \lesssim 1\big\{ K_2 \sim N_1 \big\} M_1^{-\beta+\kappa} K_1^{-s_1} N_1^{\frac{1}{2}+\kappa-s_1} \Big\| \treeN \Big\|_{\mathcal{C}_x^{\beta-\kappa}} \big\| \lolli \big\|_{\mathcal{C}_x^{-\frac{1}{2}-\kappa}}.
$$

This case is responsible for the second summand in (4.8.9).

Case 2: $M_1 \sim N_1$. This case is more delicate and requires the estimates on the quartic stochastic objects from Section 4.5.2. Inspired by the uncertainty principle, we decompose

$$
\begin{aligned}
&\Big| \int_{\mathbb{T}^3} V * \big(P_{M_1} \cubicN \cdot P_{K_1} f\big) P_{N_1} \linear \cdot P_{K_2} g \mathrm{d}x \Big| \\
&\leqslant \Big| \int_{\mathbb{T}^3} (P_{\ll N_1} V) * \big(P_{M_1} \cubicN \cdot P_{K_1} f\big) P_{N_1} \linear \cdot P_{K_2} g \mathrm{d}x \Big| \\
&+ \Big| \int_{\mathbb{T}^3} (P_{\gtrsim N_1} V) * \big(P_{M_1} \cubicN \cdot P_{K_1} f\big) P_{N_1} \linear \cdot P_{K_2} g \mathrm{d}x \Big|.
\end{aligned}
$$

We estimate both terms separately and hence divide the argument into two subcases.

Case 2.a: $M_1 \sim N_1$, contribution of $P_{\ll N_1} V$. For this term, we only obtain a non-trivial contribution if $K_1 \sim K_2 \sim N_1$. Using Hölder's inequality and Young's convolution inequality, we obtain that

$$
\begin{aligned}
&\Big| \int_{\mathbb{T}^3} (P_{\ll N_1} V) * \big(P_{M_1} \cubicN \cdot P_{K_1} f\big) P_{N_1} \linear \cdot P_{K_2} g \mathrm{d}x \Big| \\
&\lesssim 1\big\{K_1 \sim K_2 \sim M_1 \sim N_1\big\} \| P_{\ll N_1} V \|_{L^1_x} \Big\| P_{M_1} \cubicN \Big\|_{L^\infty_x} \| P_{K_1} f \|_{L^2_x} \big\| P_{N_1} \linear \big\|_{L^\infty_x} \| P_{K_2} g \|_{L^2_x} \\
&\lesssim 1\big\{K_1 \sim K_2 \sim M_1 \sim N_1\big\} N_1^{\frac{1}{2}+2\kappa-\beta-2s_1} \Big\| \cubicN \Big\|_{\mathcal{C}^{\beta-\kappa}_x} \big\| \linear \big\|_{\mathcal{C}_x^{-\frac{1}{2}-\kappa}}.
\end{aligned}
$$

The pre-factor is easily bounded by (and generally much smaller than) $(M_1 K_1 K_2 N_1)^{-\eta}$.

Case 2.b: $M_1 \sim N_1$, contribution of $P_{\gtrsim N_1} V$. By expanding the convolution with the interaction potential, we obtain that

$$
\begin{aligned}
&\Big| \int_{\mathbb{T}^3} (P_{\gtrsim N_1} V) * \big(P_{M_1} \cubicN \cdot P_{K_1} f\big) P_{N_1} \linear \cdot P_{K_2} g \mathrm{d}x \Big| \\
&\leqslant \int_{\mathbb{T}^3} |P_{\gtrsim N_1} V(y)| \Big| \int_{\mathbb{T}^3} \big(P_{K_1} f(x-y) \cdot P_{K_2} g(x)\big) \cdot \Big(P_{M_1} \cubicN (t, x-y) \cdot P_{N_1} \linear(t,x)\Big) \mathrm{d}x \Big| \mathrm{d}y \\
&\lesssim \| P_{\gtrsim N_1} V(y) \|_{L^1_y} \cdot \sup_{y \in \mathbb{T}^3} \| \langle \nabla_x \rangle^{\frac{1}{2}-\beta+2\kappa} \big(P_{K_1} f(x-y) \cdot P_{K_2} g(x)\big) \|_{L^1_x} \\
&\times \sup_{y \in \mathbb{T}^3} \Big\| P_{M_1} \cubicN (t, x-y) \cdot P_{N_1} \linear(t,x) \Big\|_{\mathcal{C}_x^{-\frac{1}{2}+\beta-\kappa}}
\end{aligned}
$$

$$\lesssim N_1^{-\beta} K_1^{-\eta} K_2^{-\eta} \sup_{y \in \mathbb{T}^3} \Big\| P_{M_1} Y^{*}_{\mathrm{N}} (t, x-y) \cdot P_{N_1} \upharpoonleft (t,x) \Big\|_{\mathcal{C}_x^{-\frac{1}{2}+\beta-\kappa}}.$$

Using Proposition 4.5.3, this contribution is acceptable. We note that the pre-factor $N_1^{-\beta}$ is essential, since Proposition 4.5.3 is not uniformly bounded over all frequency scales.

By combining Case 1 and Case 2, we have finished the proof of (4.8.9). It remains to show that (4.8.9) implies (4.8.7) and (4.8.8). To simplify the notation, we denote the expression inside the L^p_ω-norm in (4.8.9) by

$$\mathcal{A}(K_1, K_2, M_1, N_1) \overset{\text{def}}{=} \sup_{t \in [0,T]} \sup_{\|f\|_{H_x^{s_1}}, \|g\|_{H_x^{s_1}} \leqslant 1} \Big| \int_{\mathbb{T}^3} V * \big(P_{M_1} Y^{*}_{\mathrm{N}} \cdot P_{K_1} f \big) P_{N_1} \upharpoonleft \cdot P_{K_2} g \mathrm{d}x \Big|. \qquad (4.8.11)$$

To see (4.8.7), we use the self-adjointness of V, duality, and $s_1 < 1 - s_2$, which leads to

$$\begin{aligned} &\Big\| V * \Big(\upharpoonleft \textcircled{$\neq$} w \Big) Y^{*}_{\mathrm{N}} \Big\|_{H_x^{s_2-1}} \\ &\leqslant \sum_{K_1, K_2, M_1, N_1} 1\big\{ K_2 \not\sim N_1 \big\} \Big\| P_{K_1} \Big(V * \big(P_{N_1} \upharpoonleft \cdot P_{K_2} w \big) P_{M_1} Y^{*}_{\mathrm{N}} \Big) \Big\|_{H_x^{s_2-1}} \\ &\lesssim \Big(\sum_{K_1, K_2, M_1, N_1} 1\big\{ K_2 \not\sim N_1 \big\} \mathcal{A}(K_1, K_2, M_1, N_1) \Big) \|w\|_{H_x^{s_1}}. \end{aligned}$$

After using the inhomogeneous Strichartz estimate and (4.8.9), this completes the argument.

Finally, we turn to (4.8.8). Using duality, we have that

$$\begin{aligned} &\Big\| V * \Big(Y^{*}_{\mathrm{N}} \cdot w \Big) \; (\neg \textcircled{$<$}) \; \upharpoonleft \Big\|_{H_x^{s_2-1}} \\ &\leqslant \sum_{K_1, K_2, M_1, N_1} 1\big\{ \max(M_1, K_1) \geqslant N_1^{\epsilon} \big\} \Big\| P_{K_2} \Big(V * \Big(P_{M_1} Y^{*}_{\mathrm{N}} \cdot P_{K_1} w \Big) P_{N_1} \upharpoonleft \Big) \Big\|_{H_x^{s_2-1}} \\ &\lesssim \sum_{K_1, K_2, M_1, N_1} 1\big\{ \max(M_1, K_1) \geqslant N_1^{\epsilon} \big\} K_2^{s_1+s_2-1} \Big\| P_{K_2} \Big(V * \Big(P_{M_1} Y^{*}_{\mathrm{N}} \cdot P_{K_1} w \Big) P_{N_1} \upharpoonleft \Big) \Big\|_{H_x^{-s_1}} \\ &\lesssim \sum_{K_1, K_2, M_1, N_1} 1\big\{ \max(M_1, K_1) \geqslant N_1^{\epsilon} \big\} K_2^{s_1+s_2-1} \mathcal{A}(K_1, K_2, M_1, N_1) \|w\|_{H_x^{s_1}}. \end{aligned}$$

We now note that $\max(M_1, K_1) \geqslant N_1^{\epsilon}$ implies

$$1\big\{N_1 \sim K_2\big\} M_1^{-\beta+\kappa+\eta} K_1^{-s_1+\eta} K_2^{s_1+s_2-1} N_1^{\frac{1}{2}+\kappa-s_1} \lesssim N_1^{-\epsilon \min(\beta-\kappa-\eta, 1/2-\delta_1-\eta)} N_1^{\kappa+\delta_2} \lesssim 1.$$

In the last inequality, we used the parameter conditions (4.1.19). We also emphasize that the factor $K_2^{s_1+s_2-1}$ is essential for this inequality. Using inhomogeneous Strichartz estimate and (4.8.9), we then obtain the desired estimate. $\square$

4.9 From free to Gibbsian random structures

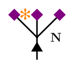
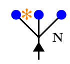
In the previous four sections, we proved several estimates for stochastic objects, random matrices, and para-controlled structures based on $\bullet$. In Section 4.2, these estimates were used to prove the local convergence of the truncated dynamics as N tends to infinity. Unfortunately, the object $\bullet$ only exists on the ambient probability space and the global theory requires (intrinsic) estimates for $\blacklozenge$ with respect to the Gibbs measure. If the desired estimate does not rely on the invariance of $\mu_M^{\otimes}$ under the nonlinear flow, however, we can use Theorem 4.1.1 to replace the Gibbs measure $\mu_M^{\otimes}$ by the reference measure $\nu_M^{\otimes}$. In particular, this works for stochastic objects only depending on the linear evolution of $\blacklozenge$, such as $\overset{\blacklozenge}{|}$ or $\underset{N}{\blacklozenge\!\!\overset{*}{\blacklozenge}\!\!\blacklozenge}$. Once we are working with the reference measure $\nu_M^{\otimes}$, we can then use that

$$\nu_M^{\otimes} = \mathrm{Law}_{\mathbb{P}}\big(\bullet + \circ_{\mathrm{M}}\big).$$

Since $\circ_{\mathrm{M}}$ has spatial regularity $1/2 + \beta-$, we expect that our estimates for $\bullet$ will imply the same estimates for $\blacklozenge$. As a result, this section contains no inherently new estimates and only combines our previous bounds.

4.9.1 The Gibbsian cubic stochastic object

This subsection should be seen as a warm-up for Section 4.9.2 below. We explore the relationship between the two cubic stochastic objects

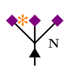
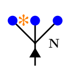
 and  .

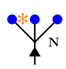
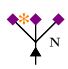
This is already sufficient for the structured local well-posedness in Proposition 4.3.3 on the support of the Gibbs measure. It will also be needed in the proof of several propositions and lemmas in Section 4.9.3 below.

Proposition 4.9.1. Let $A \geqslant 1$, let $T \geqslant 1$, and let $\zeta = \zeta(\epsilon, s_1, s_2, \kappa, \eta, \eta', b_+, b) > 0$ be sufficiently small. There exist two Borel sets $\Theta^{\mathrm{cub}}_{\mathrm{blue}}(A,T), \Theta^{\mathrm{cub}}_{\mathrm{red}}(A,T) \subseteq \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$ satisfying

$$\mathbb{P}\Big(\bullet \in \Theta^{\mathrm{cub}}_{\mathrm{blue}}(A,T) \text{ and } \circ_{\mathrm{M}}\ \Theta^{\mathrm{cub}}_{\mathrm{red}}(A,T) \Big) \geqslant 1 - \zeta^{-1} \exp(\zeta A^{\zeta})$$

for all $M \geqslant 1$ and such that the following holds for all $\bullet \in \Theta^{\mathrm{cub}}_{\mathrm{blue}}(A,T)$ and $\circ_{\mathrm{M}}\ \Theta^{\mathrm{cub}}_{\mathrm{red}}(A,T)$: For all $N \geqslant 1$, there exist $H_N[\blacklozenge \to \bullet], H_N[\bullet \to \blacklozenge] \in \mathscr{L}\mathscr{M}([0,T])$ and $Y_N[\blacklozenge \to \bullet], Y_N[\bullet \to \blacklozenge] \in \mathfrak{X}^{s_2,b}([0,T])$ satisfying the identities

 $=$  $+ P_{\leqslant N}\, \mathrm{I}\Big[\mathrm{PCtrl}\Big(H_N[\blacklozenge \to \bullet], P_{\leqslant N} \overset{\bullet}{\uparrow} \Big)\Big] + Y_N[\blacklozenge \to \bullet],$

 $=$  $+ P_{\leqslant N}\, \mathrm{I}\Big[\mathrm{PCtrl}\Big(H_N[\bullet \to \blacklozenge], P_{\leqslant N} \overset{\blacklozenge}{\uparrow} \Big)\Big] + Y_N[\bullet \to \blacklozenge].$

and the estimates

$$\| H_N[\bullet \to \blacklozenge] \|_{\mathscr{L}\mathscr{M}([0,T])}, \| H_N[\blacklozenge \to \bullet] \|_{\mathscr{L}\mathscr{M}([0,T])} \leqslant T^2 A$$

and

$$\| Y_N[\bullet \to \blacklozenge] \|_{\mathfrak{X}^{s_2,b}([0,T])}, \| Y_N[\blacklozenge \to \bullet] \|_{\mathfrak{X}^{s_2,b}([0,T])} \leqslant T^3 A.$$

Furthermore, in the frequency-localized version of this estimate, we gain an η'-power of the maximal frequency-scale.

Remark 4.9.2. The results in Proposition 4.9.1 do not yield a bound on $\overset{\blacklozenge\ast\blacklozenge\blacklozenge}{\curlyvee}_{\mathrm{N}}$ in $L_t^\infty \mathcal{C}_x^{\beta-\kappa}$, since $\mathfrak{X}^{s_2,b}$ does not embed into $L_t^\infty \mathcal{C}_x^{\beta-\kappa}$ and we do not state any additional information on Y_N. However, such an estimate is possible and only requires the translation invariance of the law of $(\bullet, \circ_{\mathrm{M}})$, which is a consequence of [Bri20c, Theorem 1.4].

Before we start with the proof of Proposition 4.9.1, we record and prove the following corollary.

Corollary 4.9.3. Let $A \geqslant 1$, let $T \geqslant 1$, let $\alpha > 0$ be a large absolute constant, and let $\zeta = \zeta(\epsilon, s_1, s_2, \kappa, \eta, \eta', b_+, b) > 0$ be sufficiently small. Then, there exists a Borel set $\Theta_{\mathrm{pur}}^{\mathrm{bil}}(A,T) \subseteq \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$ satisfying

$$\mu_M^{\otimes}\Big(\Theta_{\mathrm{pur}}^{\mathrm{bil}}(A,T)\Big), \nu_M^{\otimes}\Big(\Theta_{\mathrm{pur}}^{\mathrm{bil}}(A,T)\Big) \geqslant 1 - \zeta^{-1}\exp(\zeta A^{\zeta}) \tag{4.9.1}$$

for all $M \geqslant 1$ and such that the following holds for all $\blacklozenge \in \Theta_{\mathrm{pur}}^{\mathrm{bil}}(A,T)$:

For all intervals $\mathcal{J} \subseteq [0,T]$ and $w \in \mathfrak{X}^{s_1,b}(\mathcal{J})$, it holds that

$$\sum_{L_1,L_2} \Big\| P_{L_1} \overset{\blacklozenge\ast\blacklozenge\blacklozenge}{\curlyvee}_{\mathrm{N}} \cdot P_{L_2} w \Big\|_{L_t^2 H_x^{-4\delta_1}(\mathcal{J}\times\mathbb{T}^3)} \leqslant T^\alpha A \|w\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})}. \tag{4.9.2}$$

Proof of Corollary 4.9.3: We simply define $\Theta_{\mathrm{pur}}^{\mathrm{bil}}(A,T)$ as set the of initial data $\blacklozenge \in \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$ where (4.9.2) holds for a countable but dense subset of $\mathfrak{X}^{s_1,b}(\mathbb{R})$, which is Borel measurable, and it remains to prove the probabilistic estimate (4.9.1). Using Theorem 4.1.1, it suffices to prove that

$$\mathbb{P}\Big(\bullet + \circ_{\mathrm{M}} \in \Theta_{\mathrm{pur}}^{\mathrm{bil}}(A,T)\Big) \geqslant 1 - \zeta^{-1}\exp(\zeta A^{\zeta}).$$

This follows directly from Proposition 4.5.1, Lemma 4.8.4, and Proposition 4.9.1. □

We now turn to the proof of Proposition 4.9.1. The argument relies on the multi-linearity of the stochastic objects in the initial data. In order to use the decomposition of $\blacklozenge$, we define mixed cubic stochastic objects. In Section 4.3.1, we defined stochastic objects in $\blacklozenge$ instead of $\bullet$, which had the exact same renormalization constants and multipliers. In the proof of Proposition 4.9.1, we also work with stochastic objects that contain a mixture of both $\bullet$ and $\circ = \circ_{\mathrm{M}}$. In this case, only factors of $\bullet$ require a renormalization. The renormalized mixed stochastic objects are then defined by

$$\mathrm{Y}^{\bullet\ast\circ\bullet}_{N} \overset{\mathrm{def}}{=} P_{\leqslant N}\Big[V * \Big(P_{\leqslant N}\overset{\bullet}{|} \cdot P_{\leqslant N}\overset{\circ}{|} \Big) \cdot P_{\leqslant N}\overset{\bullet}{|} - \mathcal{M}_N P_{\leqslant N}\overset{\circ}{|} \Big],$$

$$\mathrm{Y}^{\bullet\ast\bullet\circ}_{N} \overset{\mathrm{def}}{=} P_{\leqslant N}\Big[\Big(V * \mathrm{V}^{\bullet\bullet}_{N} \Big) \cdot P_{\leqslant N}\overset{\circ}{|} \Big],$$

$$\mathrm{Y}^{\circ\ast\circ\bullet}_{N} \overset{\mathrm{def}}{=} P_{\leqslant N}\Big[V * \Big(P_{\leqslant N}\overset{\circ}{|} \cdot P_{\leqslant N}\overset{\circ}{|} \Big) \cdot P_{\leqslant N}\overset{\bullet}{|} \Big],$$

$$\mathrm{Y}^{\bullet\ast\circ\circ}_{N} \overset{\mathrm{def}}{=} P_{\leqslant N}\Big[V * \Big(P_{\leqslant N}\overset{\bullet}{|} \cdot P_{\leqslant N}\overset{\circ}{|} \Big) \cdot P_{\leqslant N}\overset{\circ}{|} \Big],$$

$$\mathrm{Y}^{\circ\ast\circ\circ}_{N} \overset{\mathrm{def}}{=} P_{\leqslant N}\Big[V * \Big(P_{\leqslant N}\overset{\circ}{|} \cdot P_{\leqslant N}\overset{\circ}{|} \Big) \cdot P_{\leqslant N}\overset{\circ}{|} \Big].$$

Furthermore, we define the solution to the nonlinear wave equation with forcing term $\mathrm{Y}^{\bullet\ast\circ\bullet}_{N}$ by

$$(-\partial_t^2 - 1 + \Delta)\, \mathrm{Y}^{\bullet\ast\circ\bullet}_{N,\uparrow} = \mathrm{Y}^{\bullet\ast\circ\bullet}_{N}, \qquad \mathrm{Y}^{\bullet\ast\circ\bullet}_{N,\uparrow}[0] = 0.$$

The solutions for the other forcing terms above are defined similarly. Using these definitions, we obtain that identity

$$\mathrm{Y}^{\blacklozenge\ast\blacklozenge\blacklozenge}_{N,\uparrow} = \mathrm{Y}^{\bullet\ast\bullet\bullet}_{N,\uparrow} + 2\,\mathrm{Y}^{\bullet\ast\circ\bullet}_{N,\uparrow} + \mathrm{Y}^{\circ\ast\circ\bullet}_{N,\uparrow} + \mathrm{Y}^{\bullet\ast\bullet\circ}_{N,\uparrow} + 2\,\mathrm{Y}^{\bullet\ast\circ\circ}_{N,\uparrow} + \mathrm{Y}^{\circ\ast\circ\circ}_{N,\uparrow}. \tag{4.9.3}$$

Using this identity, the proof of Proposition 4.9.1 is now split into two lemmas.

Lemma 4.9.4. Let $A \geqslant 1$, let $T \geqslant 1$, and let $\zeta = \zeta(\epsilon, s_1, s_2, \kappa, \eta, \eta', b_+, b) > 0$ be sufficiently small. Then, there exists two Borel sets $\Theta^{\mathrm{cub},(1)}_{\mathrm{blue}}(A,T), \Theta^{\mathrm{cub},(1)}_{\mathrm{red}}(A,T) \subseteq \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$ satisfying

$$\mathbb{P}\Big(\bullet \in \Theta^{\mathrm{cub},(1)}_{\mathrm{blue}}(A,T) \text{ and } \circ_{\mathrm{M}} \in \Theta^{\mathrm{cub},(1)}_{\mathrm{red}}(A,T) \Big) \geqslant 1 - \zeta^{-1}\exp(\zeta A^{\zeta}) \tag{4.9.4}$$

for all $M \geqslant 1$ and such that the following holds for all $\bullet \in \Theta^{\mathrm{cub},(1)}_{\mathrm{blue}}(A,T)$ and $\circ_{\mathrm{M}} \in \Theta^{\mathrm{cub},(1)}_{\mathrm{red}}(A,T)$:

For all $N \geqslant 1$, there exists a $H_N \in \mathscr{LM}([0,T])$ satisfying the identity

$$2\,(\circledless)\,\Upsilon_N^{\bullet\,*\circ\,\bullet} + (\circledless)\,\Upsilon_N^{\circ*\circ\,\bullet} = P_{\leqslant N}\, \mathrm{I}[\mathrm{PCtrl}(H_N, P_{\leqslant N} \mathbf{\dagger}^{\bullet})] \tag{4.9.5}$$

and the estimate

$$\|H_N\|_{\mathscr{LM}([0,T])} \leqslant T^2 A.$$

Furthermore, the difference $H_N - H_K$ gains an η'-power of $\min(N,K)$.

Proof. From Lemma 4.7.6, it follows that there exists a (canonical) random variable $H_N \colon \Omega \to \mathscr{LM}([0,T])$ such that

$$2\,(\circledless)\,\Upsilon_N^{\bullet\,*\circ\,\bullet} + (\circledless)\,\Upsilon_N^{\circ*\circ\,\bullet} = P_{\leqslant N}\, \mathrm{I}[\mathrm{PCtrl}(H_N, P_{\leqslant N} \mathbf{\dagger}^{\bullet})]$$

and

$$\begin{aligned} \|H_N\|_{\mathscr{LM}[0,T]} &\lesssim \Big(\| \mathbf{\dagger}^{\bullet} \|_{\mathfrak{X}^{-s_2,b}([0,T])} + \| \mathbf{\dagger}^{\circ} \|_{\mathfrak{X}^{-s_2,b}([0,T])} \Big) \| \mathbf{\dagger}^{\circ} \|_{\mathfrak{X}^{s_2,b}([0,T])} \\ &\lesssim T^2 \Big(\| \bullet \|_{\mathscr{H}_x^{-s_2}(\mathbb{T}^3)} + \| \circ \|_{\mathscr{H}_x^{s_2}(\mathbb{T}^3)} \Big) \cdot \| \circ \|_{\mathscr{H}_x^{s_2}(\mathbb{T}^3)} \end{aligned}$$

The estimate for H_N then follows from elementary properties of $\bullet$ and the high-regularity bound for $\circ$ in Theorem 4.1.1. □

Lemma 4.9.5. Let $A \geqslant 1$, let $T \geqslant 1$, and let $\zeta = \zeta(\epsilon, s_1, s_2, \kappa, \eta, \eta', b_+, b) > 0$ be sufficiently small. Then, there exists two Borel sets $\Theta^{\mathrm{cub},(2)}_{\mathrm{blue}}(A,T), \Theta^{\mathrm{cub},(2)}_{\mathrm{red}}(A,T) \subseteq \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$ satisfying

$$\mathbb{P}\Big(\bullet \in \Theta^{\mathrm{cub},(2)}_{\mathrm{blue}}(A,T) \text{ and } \circ_{\mathrm{M}} \in \Theta^{\mathrm{cub},(2)}_{\mathrm{red}}(A,T) \Big) \geqslant 1 - \zeta^{-1}\exp(\zeta A^{\zeta}) \tag{4.9.6}$$

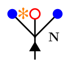
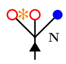
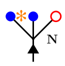
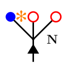
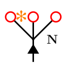
for all $M \geqslant 1$ and such that the following holds for all $\bullet \in \Theta^{\mathrm{cub},(2)}_{\mathrm{blue}}(A,T)$ and $\circ_{\mathrm{M}} \in \Theta^{\mathrm{cub},(2)}_{\mathrm{red}}(A,T)$:

For all $N \geqslant 1$, we have that

$$\begin{aligned} \max\Big(&\Big\| (\neg\circledless)\, \Upsilon_N^{\circ*\circ\,\bullet} \Big\|_{\mathfrak{X}^{s_2,b}([0,T])}, \Big\| (\neg\circledless)\, \Upsilon_N^{\circ*\circ\,\bullet} \Big\|_{\mathfrak{X}^{s_2,b}([0,T])}, \Big\| \Upsilon_N^{\bullet\bullet\,\circ} \Big\|_{\mathfrak{X}^{s_2,b}([0,T])}, \Big\| \Upsilon_N^{\bullet*\circ\,\circ} \Big\|_{\mathfrak{X}^{s_2,b}([0,T])}, \\ &\Big\| \Upsilon_N^{\circ*\circ\,\circ} \Big\|_{\mathfrak{X}^{s_2,b}([0,T])} \Big) \leqslant T^3 A. \end{aligned}$$

Furthermore, the difference of the cubic stochastic objects with two parameters N and K gains an η'-power of $\min(N, K)$.

Proof. This follows from our previous estimates for $\bullet$ from Section 4.5-4.8 and the high-regularity bound for $\circ$ in Theorem 4.1.1. More precisely, we estimate the $L^p_\omega \mathfrak{X}^{s_2,b}$-norm of

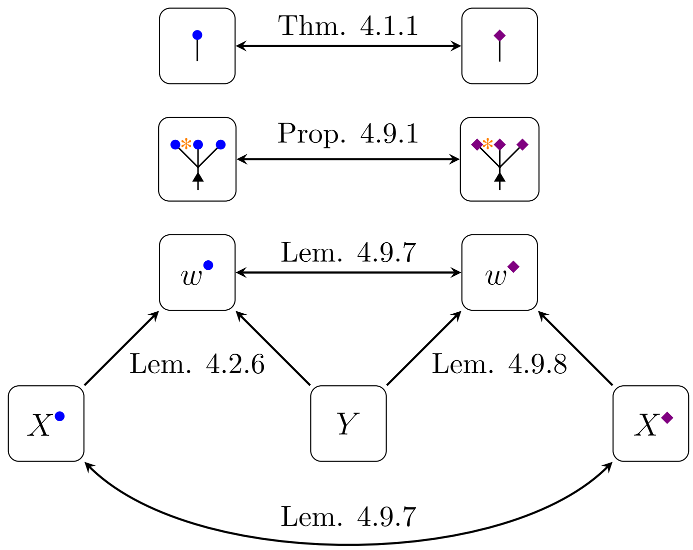
- $(\neg \circledless)$  by $T^2 p^{\frac{2+k}{2}}$ through Proposition 4.6.3,
- $(\neg \circledless)$  by $T^3 p^{\frac{1+2k}{2}}$ through Proposition 4.8.6,
-  by $T^2 p^{\frac{2+k}{2}}$ through Proposition 4.6.1,
-  by $T^3 p^{\frac{1+2k}{2}}$ through Proposition 4.8.7 and Corollary 4.8.9,
-  by $T p^{\frac{3k}{2}}$ through Proposition 4.8.10

□

Proof of Proposition 4.9.1: The first algebraic identity and related estimates follow directly from (4.9.3), Lemma 4.9.4 and Lemma 4.9.5. By using $\diamond - \bullet = \circ$ and the high regularity bound for $\circ$, we obtain the second identity and the related estimates from the first identity. □

4.9.2 Comparing random structures in Gibbsian and Gaussian initial data

In Definition 4.2.4, we introduced the types of functions occurring in our multi-linear master estimate for $\bullet$ (Proposition 4.2.8). The types w and X in Definition 4.2.4 implicitly depend on $\bullet$ and, as already mentioned in Remark 4.2.5, we now refer to type w and X as type $w^{\bullet}$ and $X^{\bullet}$, respectively. We now introduce a similar notation for the generic initial data $\diamond$. In order to orient the reader, we include an overview of the different types and their relationship in Figure 4.4.

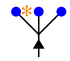
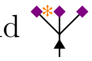


We display the relationship between the different types of functions used in this paper. The equivalence "↔" means that both types agree modulo scalar multiples and/or terms further down in the hierarchy. The implication "→" means that, up to scalar multiples, the left type forms a sub-class of the right type.

Figure 4.4: Relationship between the different types.

Definition 4.9.6 (Purple types)**.** Let $\mathcal{J} \subseteq [0,\infty)$ be a bounded interval and let $\varphi\colon J \times \mathbb{T}^3 \to \mathbb{R}$. We say that φ is of type

- $\purpleLinear$ if $\varphi = \purpleLinear$,
- $\purpleCubic$ if $\varphi = \purpleCubic_N$ for some $N \geqslant 1$,
- $w^{\blacklozenge}$ if $\|\varphi\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \leqslant 1$ and $\sum_{L_1 \sim L_2} \|P_{L_1} \purpleLinear \cdot P_{L_2} w\|_{L_t^2 H_x^{-4\delta_1}(\mathcal{J}\times\mathbb{T}^3)} \leqslant 1$ for all $N \geqslant 1$,
- $X^{\blacklozenge}$ if $\varphi = P_{\leqslant N}\, \mathrm{I}\big[1_{\mathcal{J}_0}\, \mathrm{PCtrl}(H, P_{\leqslant N} \purpleLinear)\big]$ for a dyadic integer $N \geqslant 1$, a subinterval $\mathcal{J}_0 \subseteq \mathcal{J}$, and a function $H \in \mathscr{LM}(\mathcal{J}_0)$ satisfying $\|H\|_{\mathscr{LM}(\mathcal{J}_0)} \leqslant 1$.

Since the type Y in Definition 4.2.4 does not depend on the stochastic object, its meaning remains unchanged. In Proposition 4.9.1, we have already seen that the types  and  only differ by functions of type $X^{\bullet}$ and Y (or $X^{\blacklozenge}$ and Y). In the next lemma, we clarify the relationship between the types $w^{\bullet}$ and $w^{\blacklozenge}$ as well as $X^{\bullet}$ and $X^{\blacklozenge}$.

Lemma 4.9.7 (The equivalences $w^{\bullet} \leftrightarrow w^{\blacklozenge}$ and $X^{\bullet} \leftrightarrow X^{\blacklozenge}$)**.** Let $A \geqslant 1$, let $T \geqslant 1$, and let $\zeta = \zeta(\epsilon, s_1, s_2, \kappa, \eta, \eta', b_+, b) > 0$ be sufficiently small. Then, there exists a Borel set $\Theta^{\text{type}}_{\text{red}}(A,T) \subseteq \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$ such that

$$\mathbb{P}(\, \circ_{\mathbf{M}} \in \Theta^{\text{type}}_{\text{red}}(A,T)) \geqslant 1 - \zeta^{-1}\exp(-\zeta A^{\zeta})$$

and such that the following holds for $\blacklozenge = \bullet + \circ_{\mathbf{M}}$:

- The types $w^{\bullet}$ and $w^{\blacklozenge}$ are equivalent up to multiplication by a scalar $\lambda \in \mathbb{R}_{>0}$ satisfying $\lambda, \lambda^{-1} \leqslant T^2 A$.
- The types $X^{\bullet}$ and $X^{\blacklozenge}$ are equivalent up to addition/subtraction of a function in $\mathfrak{X}^{s_2,b}$ with norm $\leqslant TA$.

Proof. We will prove the desired statement on the event

$$\Theta^{\text{type}}_{\text{red}}(A,T) = \Big\{ \phi \in \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3) \colon \|\phi\|_{\mathscr{H}_x^{\frac{1}{2}+\beta-\kappa}(\mathbb{T}^3)} \leqslant cA \Big\},$$

where $c = c(\epsilon, s_1, s_2, b)$ is a small constant. Based on Theorem 4.1.1, this event has an acceptable probability.

We start with the statement regarding the types $w^{\bullet}$ and $w^{\blacklozenge}$. Let $\varphi \in \mathfrak{X}^{s_1,b}(\mathcal{J})$ satisfy $\|\varphi\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \leqslant 1$, which holds for φ of type either $w^{\bullet}$ or $w^{\blacklozenge}$. For any $L \geqslant 1$, we have that

$$\begin{aligned}
&\Big| \sum_{L_1 \sim L_2} \|P_{L_1} \overset{\blacklozenge}{|} \cdot P_{L_2}\varphi\|_{L_t^2 H_x^{-4\delta_1}(\mathcal{J}\times\mathbb{T}^3)} - \sum_{L_1 \sim L_2} \|P_{L_1} \overset{\bullet}{|} \cdot P_{L_2}\varphi\|_{L_t^2 H_x^{-4\delta_1}(\mathcal{J}\times\mathbb{T}^3)} \Big| \\
&\leqslant \sum_{L_1 \sim L_2} \|P_{L_1} \overset{\circ}{|} \cdot P_{L_2}\varphi\|_{L_t^2 H_x^{-4\delta_1}(\mathcal{J}\times\mathbb{T}^3)}
\end{aligned}$$

Using Lemma 4.8.4, it follows that

$$\sum_{L_1 \sim L_2} \| P_{L_1} \overset{\circ}{|} \cdot P_{L_2}\varphi \|_{L_t^2 H_x^{-4\delta_1}(\mathcal{J}\times\mathbb{T}^3)} \lesssim T^{\frac{1}{2}} \Big(\sum_{L_1} L_1^{3-s_1-\left(\frac{1}{2}+\beta-\kappa\right)-2} \Big) \| \overset{\circ}{|} \|_{\mathfrak{X}^{\frac{1}{2}+\beta-\kappa,b}(\mathcal{J})} \|\varphi\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})}$$

$$\lesssim T^{\frac{3}{2}} \| \circ_{\mathrm{M}} \|_{\mathscr{H}_x^{\frac{1}{2}+\beta-\kappa}(\mathbb{T}^3)} \|\varphi\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})} \leqslant \frac{1}{2} T^{\frac{3}{2}} A \|\varphi\|_{\mathfrak{X}^{s_1,b}(\mathcal{J})}.$$

This yields the stated equivalence of the types $w^{\bullet}$ and $w^{\blacklozenge}$.

We now turn to the statement regarding the types $X^{\bullet}$ and $X^{\blacklozenge}$. For any $H \in \mathscr{LM}(\mathcal{J})$, it holds that

$$\| P_{\leqslant N} \mathrm{I}\big[1_{\mathcal{J}_0} \operatorname{PCtrl}(H, P_{\leqslant N} \overset{\blacklozenge}{|}) \big] - P_{\leqslant N} \mathrm{I}\big[1_{\mathcal{J}_0} \operatorname{PCtrl}(H, P_{\leqslant N} \overset{\bullet}{|}) \big] \|_{\mathfrak{X}^{s_2,b}(\mathcal{J})}$$

$$\lesssim \| P_{\leqslant N} \mathrm{I}\big[1_{\mathcal{J}_0} \operatorname{PCtrl}(H, P_{\leqslant N} \overset{\circ}{|}) \big] \|_{\mathfrak{X}^{s_2,b}(\mathcal{J})}.$$

Using Lemma 4.4.8, Lemma 4.4.9, and Lemma 4.7.3, we have that

$$\| P_{\leqslant N} \mathrm{I}\big[1_{\mathcal{J}_0} \operatorname{PCtrl}(H, P_{\leqslant N} \overset{\circ}{|}) \big] \|_{\mathfrak{X}^{s_2,b}(\mathcal{J})} \lesssim T \| PCtrl(H, P_{\leqslant N} \overset{\circ}{|}) \|_{L_t^\infty H_x^{s_2-1}(\mathcal{J}_0\times\mathbb{T}^3)}$$

$$\lesssim T \|H\|_{\mathscr{LM}(\mathcal{J}_0)} \| \overset{\circ}{|} \|_{L_t^\infty H_x^{s_2-1+8\epsilon}(\mathcal{J}\times\mathbb{T}^3)} \lesssim T \| \circ_{\mathrm{M}} \|_{\mathscr{H}_x^{\frac{1}{2}+\beta-\kappa}(\mathbb{T}^3)} \leqslant \frac{1}{2} T A.$$

This yields the desired estimate. □

Lemma 4.9.8 (The implication $X^{\blacklozenge}, Y \to w^{\blacklozenge}$)**.** Let $\zeta = \zeta(\epsilon, s_1, s_2, \kappa, \eta, \eta', b_+, b) > 0$ be sufficiently small, let $A \geqslant 1$, and let $T \geqslant 1$. Then, there exists a Borel set $\Theta_{\mathrm{pur}}^{\mathrm{type}}(A,T) \subseteq \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$ satisfying

$$\mu_M^{\otimes}\Big(\Theta_{\mathrm{pur}}^{\mathrm{type}}(A,T)\Big), \nu_M^{\otimes}\Big(\Theta_{\mathrm{pur}}^{\mathrm{type}}(A,T)\Big) \geqslant 1 - \zeta^{-1}\exp(\zeta A^{\zeta}) \tag{4.9.7}$$

for all $M \geqslant 1$ and such that the following holds for all $\blacklozenge \in \Theta_{\mathrm{pur}}^{\mathrm{type}}(A,T)$: If φ is of type $X^{\blacklozenge}$ or Y, the scalar multiple $T^{-7}A^{-1}\varphi$ is of type $w^{\blacklozenge}$.

Proof. Using a separability argument, we can define $\Theta_{\mathrm{pur}}^{\mathrm{type}}(A,T)$ through countably many bounds of the same form as in the definition of the type $w^{\blacklozenge}$. We first note that, after adjusting ζ, we can replace A^{-1} in the conclusion by A^{-3}. Using Theorem 4.1.1, it suffices to prove that

$$\mathbb{P}\big(\bullet + \circ_{\mathrm{M}} \in \Theta_{\mathrm{pur}}^{\mathrm{type}}(A,T)\Big) \geqslant 1 - \zeta^{-1}\exp(\zeta A^{\zeta}).$$

Thus, we may restrict both $\bullet$ and $\circ_{\mathrm{M}}$ to sets with acceptable probabilities under $\mathbb{P}$. After these preparations, we now start with the main part of the argument.

First, we let w be of type Y. Using Lemma 4.2.6, it follows that $T^{-4}A\varphi$ is of type $w^{\bullet}$. Using Lemma 4.9.7, it follows that $T^{-6}A^{-2}\varphi$ is of type $w^{\blacklozenge}$.

Now, let φ be of type $X^{\blacklozenge}$. Using Lemma 4.9.7 and the first step in this proof, we can assume that φ is of type $X^{\bullet}$. Using Lemma 4.2.6, $T^{-4}A^{-1}\varphi$ is of type $w^{\bullet}$. Finally, using Lemma 4.9.7 again, we obtain that $T^{-6}A^{-2}\varphi$ is of type $w^{\blacklozenge}$. $\square$

In Definition 4.2.13 above, we introduced the function $\mathfrak{X}$-norms, which are used to quantify structured perturbations of the initial data. We now prove the equivalence of the $\mathfrak{X}([0,T], \upharpoonleft; t_0, N, K)$ and $\mathfrak{X}([0,T], \upharpoonleft; t_0, N, K)$-norms, which is similar to the statements in Lemma 4.9.7 and Lemma 4.9.8.

Lemma 4.9.9 (Equivalence of the blue and purple structured perturbations)**.** Let $A \geqslant 1$, let $\alpha > 0$ be a sufficiently large absolute constant, and let $\zeta = \zeta(\epsilon, s_1, s_2, \kappa, \eta, \eta', b_+, b) > 0$ be sufficiently small. Then, there exist Borel sets $\Theta^{\mathrm{sp}}_{\mathrm{blue}}(A), \Theta^{\mathrm{sp}}_{\mathrm{red}}(A) \subseteq \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$ satisfying

$$\mathbb{P}\big(\bullet \in \Theta^{\mathrm{sp}}_{\mathrm{blue}}(A),\ \circ_{\mathrm{M}} \in \Theta^{\mathrm{sp}}_{\mathrm{red}}(A)\big) \geqslant 1 - \zeta^{-1}\exp(-\zeta A^{\zeta}). \tag{4.9.8}$$

and such that the following holds on this event:

For all $T \geqslant 1$, $t_0 \in [0,T]$, $N, K \geqslant 1$, and $Z[t_0] \in \mathscr{H}_x^{s_1}(\mathbb{T}^3)$, we have that

$$T^{-\alpha}A^{-1}\|Z[t_0]\|_{\mathfrak{X}([0,T],\upharpoonleft;t_0,N,K)} \leqslant \|Z[t_0]\|_{\mathfrak{X}([0,T],\upharpoonleft;t_0,N,K)} \leqslant T^{\alpha}A\|Z[t_0]\|_{\mathfrak{X}([0,T],\upharpoonleft;t_0,N,K)} \tag{4.9.9}$$

Proof. It suffices to prove the estimate (4.9.9) for events $\Theta^{\mathrm{sp}}_{\mathrm{blue}}(A,T)$ and $\Theta^{\mathrm{sp}}_{\mathrm{red}}(A,T)$ satisfying the probabilistic estimate (4.9.8), as long as the lower bound in (4.9.8) does not depend on T. We can then simply take the intersection of $\Theta^{\mathrm{sp}}_{\mathrm{blue}}(T\cdot A, T)$ and $\Theta^{\mathrm{sp}}_{\mathrm{red}}(T\cdot A, T)$ over all integer times and increase α by one.

After using Lemma 4.9.7 to compare the high×high-interaction terms (involving $L_1 \sim L_2$), it remains to prove that

$$\Big| \Big\| \big(\neg \boxed{\textcircled{$<$}\,\&\,\textcircled{$<$}}\big) \Big(:V * \big(P_{\leqslant N}\overset{\blacklozenge}{|} \cdot P_{\leqslant N} Z_N^{\square}\big)\, P_{\leqslant N}\overset{\blacklozenge}{|}: \Big\|_{\mathfrak{X}^{s_2-1,b_+-1}([0,T])}\Big)$$

$$- \Big\| \big(\neg \boxed{\textcircled{$<$}\,\&\,\textcircled{$<$}}\big) \Big(:V * \big(P_{\leqslant N}\overset{\bullet}{|} \cdot P_{\leqslant N} Z_N^{\square}\big)\, P_{\leqslant N}\overset{\bullet}{|}: \Big)\Big\|_{\mathfrak{X}^{s_2-1,b_+-1}([0,T])}\Big)\Big|$$

$$\lesssim T^\alpha A\Big(\| \| Z^{\square}[t_0] \|_{\mathscr{H}_x^{s_1}} + \sum_{L_1 \sim L_2} \| P_{L_1}\overset{\bullet}{|} \cdot P_{L_2} Z \|_{L_t^2 H_x^{-4\delta_1}([0,T]\times \mathbb{T}^3)}\Big)$$

and

$$\Big| \Big\| :V * \big(P_{\leqslant N}\overset{\blacklozenge}{|} \cdot P_{\leqslant N} Z_N^{\bigcirc}\big)\, \big(\neg \textcircled{$<$}\big)\, P_{\leqslant N}\overset{\blacklozenge}{|}: \Big\|_{\mathfrak{X}^{s_2-1,b_+-1}([0,T])}$$

$$- \Big\| :V * \big(P_{\leqslant N}\overset{\bullet}{|} \cdot P_{\leqslant N} Z_N^{\bigcirc}\big)\, \big(\neg \textcircled{$<$}\big)\, P_{\leqslant N}\overset{\bullet}{|}: \Big\|_{\mathfrak{X}^{s_2-1,b_+-1}([0,T])}\Big|$$

$$\lesssim T^\alpha A \| Z^{\bigcirc}[t_0] \|_{\mathscr{H}_x^{s_2}}.$$

Regarding the first estimate, we have that

$$\Big| \Big\| \big(\neg \boxed{\textcircled{$<$}\,\&\,\textcircled{$<$}}\big) \Big(:V * \big(P_{\leqslant N}\overset{\blacklozenge}{|} \cdot P_{\leqslant N} Z_N^{\square}\big)\, P_{\leqslant N}\overset{\blacklozenge}{|}: \Big)\Big\|_{\mathfrak{X}^{s_2-1,b_+-1}([0,T])}$$

$$- \Big\| \big(\neg \boxed{\textcircled{$<$}\,\&\,\textcircled{$<$}}\big) \Big(:V * \big(P_{\leqslant N}\overset{\bullet}{|} \cdot P_{\leqslant N} Z_N^{\square}\big)\, P_{\leqslant N}\overset{\bullet}{|}: \Big)\Big\|_{\mathfrak{X}^{s_2-1,b_+-1}([0,T])}\Big|$$

$$\lesssim \Big\| \big(\neg \boxed{\textcircled{$<$}\,\&\,\textcircled{$<$}}\big) \Big(V * \big(P_{\leqslant N}\overset{\circ}{|} \cdot P_{\leqslant N} Z_N^{\square}\big)\, P_{\leqslant N}\overset{\bullet}{|}\Big)\Big\|_{\mathfrak{X}^{s_2-1,b_+-1}([0,T])} \tag{4.9.10}$$

$$+ \Big\| \big(\neg \boxed{\textcircled{$<$}\,\&\,\textcircled{$<$}}\big) \Big(V * \big(P_{\leqslant N}\overset{\bullet}{|} \cdot P_{\leqslant N} Z_N^{\square}\big)\, P_{\leqslant N}\overset{\circ}{|}\Big)\Big\|_{\mathfrak{X}^{s_2-1,b_+-1}([0,T])} \tag{4.9.11}$$

$$+ \Big\| \big(\neg \boxed{\textcircled{$<$}\,\&\,\textcircled{$<$}}\big) \Big(V * \big(P_{\leqslant N}\overset{\circ}{|} \cdot P_{\leqslant N} Z_N^{\square}\big)\, P_{\leqslant N}\overset{\circ}{|}\Big)\Big\|_{\mathfrak{X}^{s_2-1,b_+-1}([0,T])}. \tag{4.9.12}$$

We can then control

- (4.9.10) through Proposition 4.8.6,
- (4.9.11) through Proposition 4.8.7 and Lemma 4.8.8,
- (4.9.11) through Proposition 4.8.10.

The proof of the second estimate is similar, except that we use Corollary 4.8.9 instead of Lemma 4.8.8.

$\square$

4.9.3 Multi-linear master estimate for Gibbsian initial data

In this subsection, we prove a version of the multi-linear master estimate for Gaussian data (Proposition 4.2.8) for the purple types (Definition 4.9.6) instead of the blue types (Definition 4.2.4). Since we will only need this estimate in Proposition 4.3.5 and Proposition 4.3.7, which do not involve contraction or continuity arguments, we can be less precise than in the multi-linear master estimate for Gaussian data and simply capture the size of the forcing term in the following norm.

Definition 4.9.10. Let $N \geqslant 1$, let $\mathcal{J} \subseteq \mathbb{R}$ be a compact interval, and let $R, \varphi \colon \mathcal{J} \times \mathbb{T}^3 \to \mathbb{R}$. Then, we define

$$\|R\|_{\mathscr{NL}_N(J,\varphi)} \overset{\text{def}}{=} \inf \Big\{ \|H\|_{\mathscr{LM}(\mathcal{J})} + \|F\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})} \colon R = P_{\leqslant N} \operatorname{PCtrl}[H, P_{\leqslant N}\varphi] + F \text{ on } \mathcal{J} \times \mathbb{T}^3 \Big\}.$$

Remark 4.9.11 (Drawback of $\boxed{\circledless \& \circledless}$)**.** As mentioned above, the $\mathscr{NL}_N(J,\varphi)$-norm is less precise than our estimates in Section 4.2.1, since it does not give an explicit description of the low-frequency modulation H. This allows us to circumvent a technical problem which the author was unable to resolve. In Proposition 4.5.7, we proved that

$$\big(\neg \boxed{\circledless \& \circledless}\big) \Big(:V * \Big(P_{\leqslant N} \,\rotatebox{0}{$\mid$} \cdot P_{\leqslant N} \Psi_N \Big) P_{\leqslant N} \,\rotatebox{0}{$\mid$} : \Big)$$

lives in $\mathfrak{X}^{s_2-1,b_+-1}$. One may therefore expect that

$$\left(\neg\,\boxed{\circledless\,\&\,\circledless}\right)\Big(:V*\Big(P_{\leqslant N}{\color{purple}\uparrow}\cdot P_{\leqslant N}\,\mathbf{Y}_{N}\Big)\,P_{\leqslant N}{\color{purple}\uparrow}:\Big)$$

also lives in $\mathfrak{X}^{s_2-1,b_+-1}$. However, after using Proposition 4.9.1, we would need an estimate for

$$\left(\neg\,\boxed{\circledless\,\&\,\circledless}\right)\Big(:V*\Big(P_{\leqslant N}{\color{blue}\uparrow}\cdot P_{\leqslant N}Y_N\Big)\,P_{\leqslant N}{\color{blue}\uparrow}:\Big)$$

in $\mathfrak{X}^{s_2-1,b_+-1}$. Unfortunately, this is not covered by Proposition 4.6.3. In fact, without any additional assumptions on Y_N other than bounds in $\mathfrak{X}^{s_2,b}$, the high×high→low-interactions in $P_{\leqslant N}{\color{blue}\uparrow}\cdot P_{\leqslant N}Y_N$ rule out this estimate.

Equipped with the $\mathcal{N}\mathcal{L}$-norm, we now turn to the master estimate for Gibbsian initial data.

Proposition 4.9.12 (Multi-linear master estimate for Gibbsian initial data)**.** Let $A\geqslant 1$, let $T\geqslant 1$, let $\alpha>0$ be a sufficiently large absolute constant, and let $\zeta=\zeta(\epsilon,s_1,s_2,\kappa,\eta,\eta',b_+,b)>0$ be sufficiently small. Then, there exists a Borel set $\Theta^{\mathrm{ms}}_{\mathrm{pur}}(A,T)\subseteq\mathscr{H}_x^{-1/2-\kappa}$ satisfying

$$\mu_M^{\otimes}({\color{purple}\blacklozenge}\in\Theta^{\mathrm{ms}}_{\mathrm{pur}}(A,T))\geqslant 1-\zeta^{-1}\exp(-\zeta A^{\zeta}) \tag{4.9.13}$$

for all $M\geqslant 1$ and such that the following estimates hold for all ${\color{purple}\blacklozenge}\in\Theta^{\mathrm{ms}}_{\mathrm{pur}}(A,T)$:
Let $\mathcal{J}\subseteq[0,T]$ be an interval and let $N\geqslant 1$. Let $\varphi_1,\varphi_2,\varphi_3\colon\mathcal{J}\times\mathbb{T}^3\to\mathbb{R}$ be as in Definition 4.9.6 and let

$$(\varphi_1,\varphi_2;\varphi_3)\overset{\text{type}}{\neq}\big({\color{purple}\uparrow},{\color{purple}\uparrow};{\color{purple}\uparrow}\big),\ \big({\color{purple}\uparrow},w^{{\color{purple}\blacklozenge}};{\color{purple}\uparrow}\big).$$

(i) If $\varphi_3\overset{\text{type}}{=}{\color{purple}\uparrow}$, then

$$\Big\|P_{\leqslant N}\Big(:V*\big(P_{\leqslant N}\varphi_1\cdot P_{\leqslant N}\varphi_2\big)P_{\leqslant N}\varphi_3:\Big)\Big\|_{\mathcal{N}\mathcal{L}_N\left(\mathcal{J},P_{\leqslant N}{\color{purple}\uparrow}\right)}\leqslant T^{\alpha}A.$$

(ii) In all other cases,

$$\Big\|:V*\big(P_{\leqslant N}\varphi_1\cdot P_{\leqslant N}\varphi_2\big)P_{\leqslant N}\varphi_3:\Big\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})}\leqslant T^{\alpha}A.$$

Proof. While the proof requires no new ingredients, it relies on several earlier results. For the advantage of the reader, we break up the proof into several steps.

Step 1: Definition of $\Theta^{\mathrm{ms}}_{\mathrm{pur}}(A,T)$ and its Borel measurability. Using the definition of the time-restricted norms, we see that the statement for all intervals $\mathcal{J} \subseteq [0,T]$ is equivalent to the statement for only $\mathcal{J} = [0,T]$. Thus, we may simply choose $\Theta^{\mathrm{ms}}_{\mathrm{pur}}(A,T)$ as the set where (i) and (ii) hold for all $N \geqslant 1$. To see that this leads to a Borel measurable set, we note that both $\mathscr{L}\mathscr{M}([0,T])$ and $\mathfrak{X}^{s_2,b}([0,T])$ are separable. For a fixed $N \geqslant 1$, we also have that the functions

$$(\varphi_1,\varphi_2,\varphi_3) \mapsto \Big\| P_{\leqslant N}\Big(:V * \big(P_{\leqslant N}\varphi_1 \cdot P_{\leqslant N}\varphi_2\big) P_{\leqslant N}\varphi_3 : \Big) \Big\|_{\mathcal{N}\mathscr{L}_N\left(\mathcal{J}, P_{\leqslant N} \blacklozenge\right)}$$

and

$$(\varphi_1,\varphi_2,\varphi_3) \mapsto \Big\| :V * \big(P_{\leqslant N}\varphi_1 \cdot P_{\leqslant N}\varphi_2\big) P_{\leqslant N}\varphi_3 : \Big\|_{\mathfrak{X}^{s_2-1,b_+-1}(\mathcal{J})}$$

are continuous w.r.t. the $C^0_t H^{-1/2-\kappa}_x([0,T] \times \mathbb{T}^3)$-norm. Thus, we can represent $\Theta^{\mathrm{ms}}_{\mathrm{pur}}(A,T)$ through countably many constraints of the same form as in (i) and (ii), and hence as a countable intersection of closed sets. In particular, $\Theta^{\mathrm{ms}}_{\mathrm{pur}}(A,T)$ is Borel measurable.

Step 2: Reductions. It therefore remains to show the probabilistic estimate (4.9.13). Using the absolute continuity and representation of the reference measures from Theorem 4.1.1, it suffices to prove that

$$\mathbb{P}(\bullet + \circ_{\mathbf{M}} \in \Theta^{\mathrm{ms}}_{\mathrm{pur}}(A,T)) \geqslant 1 - \zeta^{-1}\exp(-\zeta A^{\zeta})$$

for all $M \geqslant 1$. Furthermore, we can replace the upper bound $T^{\alpha}A$ in (i) and (ii) by $CT^{\alpha}A^{C}$, where $C = C(\epsilon, s_1, s_2, \kappa, \eta, \eta', b_+, b) \geqslant 1$. After the estimate has been proven, this can then be repaired by adjusting A and ζ. Using Lemma 4.2.6, Proposition 4.2.8, Corollary 4.9.3, Lemma 4.9.7, and

Lemma 4.9.8, we may restrict to the event

$$\Big\{ {\color{blue}\bullet} \in \Theta^{\mathrm{ms}}_{\mathrm{blue}}(A,T) \cap \Theta^{\mathrm{type}}_{\mathrm{blue}}(A,T) \cap \Theta^{\mathrm{cub}}_{\mathrm{blue}}(A,T) \Big\} \cap \Big\{ {\color{red}\circ_{\mathrm{M}}} \in \Theta^{\mathrm{type}}_{\mathrm{red}}(A,T) \cap \Theta^{\mathrm{cub}}_{\mathrm{red}}(A,T) \Big\}$$
$$\cap \Big\{ {\color{blue}\bullet} + {\color{red}\circ_{\mathrm{M}}} \in \Theta^{\mathrm{type}}_{\mathrm{pur}}(A,T) \Big\}.$$

Step 3: Multi-linear estimates. The estimates for $\varphi_3 \overset{\mathrm{type}}{\neq} {\color{purple}\overset{\blacklozenge}{|}}$ follow directly from the multi-linear master estimate for ${\color{blue}\bullet}$ and the equivalence of the types in Corollary 4.9.3, Lemma 4.9.7, and Lemma 4.9.8. It then remains to treat the case $\varphi_3 \overset{\mathrm{type}}{=} {\color{purple}\overset{\blacklozenge}{|}}$. We further separate the proof of the estimates into two cases.

Step 3.1: $\varphi_1, \varphi_2 \overset{type}{\neq} {\color{purple}\overset{\blacklozenge}{|}}$. We first remind the reader that in this case the nonlinearity does not require a renormalization. We then decompose

$$\begin{aligned}
&P_{\leqslant N}\Big(V * \big(P_{\leqslant N}\varphi_1 \cdot P_{\leqslant N}\varphi_2 \big) P_{\leqslant N} {\color{purple}\overset{\blacklozenge}{|}} \Big) \\
=&P_{\leqslant N}\Big(V * \big(P_{\leqslant N}\varphi_1 \cdot P_{\leqslant N}\varphi_2 \big) \circledless P_{\leqslant N} {\color{purple}\overset{\blacklozenge}{|}} \Big) \\
+&P_{\leqslant N}\Big(V * \big(P_{\leqslant N}\varphi_1 \cdot P_{\leqslant N}\varphi_2 \big) \,(\neg \circledless)\, P_{\leqslant N} {\color{blue}\overset{\bullet}{|}} \Big) \\
+&P_{\leqslant N}\Big(V * \big(P_{\leqslant N}\varphi_1 \cdot P_{\leqslant N}\varphi_2 \big) \,(\neg \circledless)\, P_{\leqslant N} {\color{red}\overset{\circ}{|}} \Big).
\end{aligned}$$

Using Lemma 4.7.6, the first term is of the form $P_{\leqslant N}\,\mathrm{PCtrl}(H_N, P_{\leqslant N} {\color{purple}\overset{\blacklozenge}{|}})$ with $\|H_N\|_{\mathscr{LM}([0,T])} \lesssim T^\alpha A^2$. The second and third term can be controlled through the multi-linear master estimate for Gaussian random data.

Step 3.2: $\varphi_1, \varphi_3 \overset{type}{=} {\color{purple}\overset{\blacklozenge}{|}}$, $\varphi_2 \overset{type}{\neq} {\color{purple}\overset{\blacklozenge}{|}}$. Using the equivalence of types (as in Corollary 4.9.3 and Lemma 4.9.7) together with the previous cases, it suffices to treat

$$\varphi_1, \varphi_3 \overset{\mathrm{type}}{=} {\color{blue}\overset{\bullet}{|}}, \ \varphi_2 \overset{\mathrm{type}}{=} \overset{{\color{blue}\bullet}\,{\color{blue}\bullet}\,{\color{blue}\bullet}}{\Psi}, \ X^{\color{blue}\bullet}, \ Y.$$

We decompose the nonlinearity

$$V * \big(P_{\leqslant N}\overset{\bullet}{|} \cdot P_{\leqslant N}\varphi_2\big)P_{\leqslant N}\overset{\bullet}{|}$$

using $\boxed{\textcircled{$<$}\&\textcircled{$<$}}$ if $\varphi_2 \overset{\text{type}}{=} \overset{\bullet\,*\,\bullet\,\bullet}{\curlyvee}$, $X^{\bullet}$ and using $\textcircled{$<$}$ if $\varphi_2 \overset{\text{type}}{=} Y$. Then, the bound follows from the multi-linear master estimate for Gaussian initial data, Lemma 4.7.4, and Lemma 4.7.6. □

In Definition 4.2.13, we also introduced a structured perturbation of the initial data, which we briefly examined in Lemma 4.9.9 above. While the multi-linear estimate does not apply to the type $\big(\overset{\blacklozenge}{|}, w^{\blacklozenge}; \overset{\blacklozenge}{|}\big)$, we now obtain a multi-linear estimate if the second argument is a linear evolution with initial data as in Definition 4.2.13. Since the definition has been tailored towards this estimate, the prove will be easy and short.

Lemma 4.9.13 (Multi-linear estimate for the structured perturbation)**.** Let $A \geqslant 1$, let $T \geqslant 1$, let $\alpha > 0$ be a sufficiently large absolute constant, and let $\zeta = \zeta(\epsilon, s_1, s_2, \kappa, \eta, \eta', b_+, b) > 0$ be sufficiently small. Then, there exists a Borel set $\Theta^{\mathrm{sp}}_{\mathrm{pur}}(A,T) \subseteq \mathscr{H}_x^{-1/2-\kappa}$ satisfying

$$\mu_M^{\otimes}(\blacklozenge \in \Theta^{\mathrm{sp}}_{\mathrm{pur}}(A,T)) \geqslant 1 - \zeta^{-1}\exp(-\zeta A^{\zeta}) \tag{4.9.14}$$

for all $M \geqslant 1$ and such that the following estimates hold for all $\blacklozenge \in \Theta^{\mathrm{sp}}_{\mathrm{pur}}(A,T)$:
Let $N, K \geqslant 1$, let $t_0 \in [0,T]$, let $Z[t_0] \in \mathscr{H}_x^{-1/2-\kappa}(\mathbb{T}^3)$, and let $Z(t)$ be the corresponding solution to the linear wave equation. Then, it holds that

$$\Big\| P_{\leqslant N}\Big[:V * \big(P_{\leqslant N}\overset{\blacklozenge}{|} \cdot P_{\leqslant N}Z\big)P_{\leqslant N}\overset{\blacklozenge}{|}: \Big]\Big\|_{\mathcal{N}\mathscr{L}_N\left([0,T],\, P_{\leqslant N}\overset{\blacklozenge}{|}\right)} \leqslant T^{\alpha} A \|Z[t_0]\|_{\mathfrak{X}([0,T],\overset{\blacklozenge}{|};t_0,N,K)}.$$

Proof. Let $Z^{\Box}[t_0]$ and $Z^{\bigcirc}[t_0]$ be as in Definition 4.2.13. Then, we can decompose

$$
\begin{aligned}
& P_{\leqslant N}\Big[:V * \big(P_{\leqslant N}\text{\scriptsize{♦|}} \cdot P_{\leqslant N} Z\big) P_{\leqslant N}\text{\scriptsize{♦|}}: \Big] \\
=& \boxed{\circledless \& \circledless}\Big(P_{\leqslant N}\Big[V * \big(P_{\leqslant N}\text{\scriptsize{♦|}} \cdot P_{\leqslant N} Z^{\Box}\big) P_{\leqslant N}\text{\scriptsize{♦|}}\Big]\Big) \\
+& \big(\neg \boxed{\circledless \& \circledless}\big)\ \Big(P_{\leqslant N}\Big[:V * \big(P_{\leqslant N}\text{\scriptsize{♦|}} \cdot P_{\leqslant N} Z^{\Box}\big) P_{\leqslant N}\text{\scriptsize{♦|}}: \Big]\Big) \\
+& P_{\leqslant N}\Big[:V * \big(P_{\leqslant N}\text{\scriptsize{♦|}} \cdot P_{\leqslant N} Z^{\bigcirc}\big) \circledless\, P_{\leqslant N}\text{\scriptsize{♦|}}: \Big] \\
+& P_{\leqslant N}\Big[V * \big(P_{\leqslant N}\text{\scriptsize{♦|}} \cdot P_{\leqslant N} Z^{\bigcirc}\big)\ \big(\neg \circledless\big)\ P_{\leqslant N}\text{\scriptsize{♦|}}\Big].
\end{aligned}
$$

The estimate then directly follows from Definition 4.2.13, Lemma 4.7.4, and Lemma 4.7.6. □

4.10 Appendix: Proofs of counting estimates

4.10.1 Cubic counting estimate

We start with the proof of the cubic counting estimate.

Proof of Proposition 4.4.18: We separately prove the four counting estimates (i)-(iv).

Proof of (i)*:* By symmetry, we can assume that $N_1 \geqslant N_2 \geqslant N_3$. Using the basic counting estimate to perform the sum in $n_2 \in \mathbb{Z}^3$, we obtain that

$$
\begin{aligned}
& \#\{(n_1, n_2, n_3) \colon |n_1| \sim N_1, |n_2| \sim N_2, |n_3| \sim N_3, |\varphi - m| \leqslant 1\} \\
\lesssim & \sum_{n_1, n_3 \in \mathbb{Z}^3} \Big(\prod_{j=1,3} 1\big\{|n_j| \sim N_j\big\}\Big) \min\big(\langle n_{13}\rangle, N_2\big)^{-1} N_2^3 \\
\lesssim & N_1^2 N_2^3 N_3^3 + N_1^3 N_2^2 N_3^3 \\
\lesssim & N_2^{-1} (N_1 N_2 N_3)^3,
\end{aligned}
$$

which is acceptable.

Proof of (ii)*:* We emphasize that n_{123} is viewed as a free variable. In the variables (n_{123}, n_1, n_2), the phase takes the form

$$\varphi = \pm_{123}\langle n_{123}\rangle \pm_1 \langle n_1\rangle \pm_2 \langle n_2\rangle \pm_3 \langle n_{123} - n_1 - n_2\rangle.$$

After changing $(n_1, n_2) \to (-n_1, -n_2)$, we obtain the same form as in (i) and hence the desired estimate.

Proof of (iii)*:* In the variables (n_{123}, n_{12}, n_1), the phase takes the form

$$\varphi = \pm_{123}\langle n_{123}\rangle \pm_1 \langle n_1\rangle \pm_2 \langle n_{12} - n_1\rangle \pm_3 \langle n_{123} - n_{12}\rangle.$$

By first summing in n_1 and using the basic counting lemma, we gain a factor of $\min(N_1, N_{12})$. Alternatively, by first summing in n_{123} and using the basic counting lemma, we gain a factor of $\min(N_{123}, N_{12})$. By combining both estimates, we gain a factor of

$$\max\big(\min(N_1, N_{12}), \min(N_{123}, N_{12})\big) = \min\big(N_{12}, \max(N_{123}, N_1)\big).$$

While not part of the proof, we also remark that

$$\Big|\langle n_{123}\rangle + \langle n_1\rangle - \langle n_{12} - n_1\rangle - \langle n_{123} - n_{12}\rangle\big| \lesssim N_{12}.$$

This shows that we cannot gain a factor of the form $\mathrm{med}(N_{123}, N_{12}, N_1)$.

Proof of (iv)*:* In the variables (n_{12}, n_1, n_3), the phase takes the form

$$\varphi = \pm_{123}\langle n_{12} + n_3\rangle \pm_1 \langle n_1\rangle \pm_2 \langle n_{12} - n_1\rangle \pm_3 \langle n_3\rangle.$$

By first summing in n_1 and using the basic counting lemma, we gain a factor of $\min(N_{12}, N_1)$. Alternatively, by first summing in n_3 and using the basic counting lemma, we gain a factor of $\min(N_{12}, N_3)$. By combining both estimates, this completes the argument. The same obstruction as described in (iii) shows that the estimate is sharp. $\square$

We now use the cubic counting estimate to prove the cubic sum estimate.

Proof of Proposition 4.4.20: Due to the symmetry $n_1 \leftrightarrow n_2$, we may assume that $N_1 \geqslant N_2$. To simplify the notation, we set

$$\begin{aligned}\mathcal{C}(m) &= \mathcal{C}(N_1, N_2, N_3, N_{12}, N_{123}, m)\\ &= \Big\{(n_1, n_2, n_3) \in (\mathbb{Z}^3)^3 \colon |n_j| \sim N_j, 1 \leqslant j \leqslant 3, |n_{12}| \sim N_{12}, |n_{123}| \sim N_{123}, |\varphi - m| \leqslant 1\Big\}.\end{aligned}$$

We then have that

$$\begin{aligned}&\sum_{n_1,n_2,n_3 \in \mathbb{Z}^3} \Big[\Big(\prod_{j=1}^{3} \chi_{N_j}(n_j)\Big)\langle n_{123}\rangle^{2(s-1)} \langle n_{12}\rangle^{-2\gamma} \Big(\prod_{j=1}^{3} \langle n_j\rangle^{-2}\Big) 1\big\{|\varphi - m| \leqslant 1\big\}\Big]\\ &\lesssim \sum_{N_{123}, N_{12}} N_{123}^{2(s-1)} N_{12}^{-2\gamma} \Big(\prod_{j=1}^{3} N_j^{-2}\Big) \# \mathcal{C}(m).\end{aligned} \tag{4.10.1}$$

To obtain the optimal estimate, we unfortunately need to distinguish five cases, which we listed in Figure 4.5. Case 1 and 2 distinguish between the high×high and high×low-interactions in the first two factors. This distinction is necessary to utilize the gain in N_{12}. The subcases mostly deal with the relation between N_{12} and N_3, which is important to use the gain in N_{123}.

Case 1.a: $N_1 \sim N_2$, $N_1 \ll N_3$. In this case, $N_{123} \sim N_3$. Using (iv) in Proposition 4.4.18, the contribution is bounded by

$$\sum_{\substack{N_{12}: \\ N_{12} \lesssim N_1}} N_{12}^{-2\gamma} N_1^{-4} N_3^{2s-4} \# \mathcal{C}(m) \lesssim \sum_{\substack{N_{12}: \\ N_{12} \lesssim N_1}} N_{12}^{2-2\gamma} N_1^{-1} N_3^{2s-1} \lesssim N_1^{1-2\gamma} N_3^{2s-1},$$

which is acceptable. In performing the sum, we used that $\gamma < 1$.

Case 1.b.i: $N_1 \sim N_2$, $N_1 \gtrsim N_3$, $N_3 \ll N_{12}$. In this case, $N_{123} \sim N_{12}$. Using (iv) in Proposition 4.4.18, the contribution is bounded by

$$\sum_{\substack{N_{12}:\\ N_3 \ll N_{12} \lesssim N_1}} N_{12}^{2s-2-2\gamma} N_1^{-4} N_3^{-2} \#\mathcal{C}(m) \lesssim \sum_{\substack{N_{12}:\\ N_3 \ll N_{12} \lesssim N_1}} N_{12}^{2s-2\gamma} N_1^{-1} N_3 \lesssim \sum_{\substack{N_{12}:\\ N_{12} \lesssim N_1}} N_{12}^{2s-2\gamma+1} N_1^{-1} \lesssim N_1^{2(s-\gamma)},$$

which is acceptable. In performing the sum, we used that $\gamma < s + 1/2$.

Case 1.b.i: $N_1 \sim N_2$, $N_1 \gtrsim N_3$, $N_3 \gtrsim N_{12}$. We note that $N_{123} \lesssim \max(N_{12}, N_3) \lesssim N_3$. Using (iii) in Proposition 4.4.18, the contribution is bounded by

$$\begin{aligned}
&\sum_{\substack{N_{12}, N_{123}:\\ N_{12}, N_{123} \lesssim N_3}} N_{123}^{2s-2} N_{12}^{-2\gamma} N_1^{-4} N_3^{-2} \#\mathcal{C}(m) \lesssim \sum_{\substack{N_{12}, N_{123}:\\ N_{12}, N_{123} \lesssim N_3}} \min(N_{123}, N_{12})^{-1} N_{123}^{2s+1} N_{12}^{3-2\gamma} N_1^{-1} N_3^{-2} \\
&\lesssim N_1^{-1} N_3^{2s-2\gamma+1} \lesssim N_1^{2(s-\gamma)},
\end{aligned}$$

which is acceptable. In the last inequality, we used again that $\gamma < s + 1/2$.

Case 2.a: $N_1 \gg N_2$, $N_1 \not\sim N_3$. In this case, $N_{12} \sim N_1$ and $N_{123} \sim \max(N_1, N_3)$. Using (i) in Proposition 4.4.18, the contribution is bounded by

$$\begin{aligned}
&\max(N_1, N_3)^{2s-2} N_1^{-2-2\gamma} N_2^{-2} N_3^{-2} \#\mathcal{C}(m) \lesssim \max(N_1, N_3)^{2s-2} \min(N_1, N_3)^{-1} N_1^{1-2\gamma} N_2 N_3 \\
&\lesssim \max(N_1, N_3)^{2s-2} \min(N_1, N_3)^{-1} N_1^{2-2\gamma} N_3 = \max(N_1, N_3)^{2s-1} N_1^{1-2\gamma}.
\end{aligned}$$

The restriction $s \leqslant 1/2$ is not strictly necessary for the statement of the proposition, but ensures that the first factor does not grow in N_1 or N_3, which is essential in applications.

Case	$N_1 \leftrightarrow N_2$	$N_1 \leftrightarrow N_3$	$N_3 \leftrightarrow N_{12}$	Basic counting estimate
1.a	$N_1 \sim N_2$	$N_1 \ll N_3$		(iv)
1.b.i	$N_1 \sim N_2$	$N_1 \gtrsim N_3$	$N_3 \ll N_{12}$	(iv)
1.b.ii	$N_1 \sim N_2$	$N_1 \gtrsim N_3$	$N_3 \gtrsim N_{12}$	(iii)
2.a	$N_1 \gg N_2$	$N_1 \not\sim N_3$		(i)
2.b	$N_1 \gg N_2$	$N_1 \sim N_3$		(ii)

Figure 4.5: Case distinction in the proof of Proposition 4.4.20.

Case 2.a: $N_1 \gg N_2$, $N_1 \sim N_3$. In this case, $N_{12} \sim N_1$. Using (ii) in Proposition 4.4.18, the contribution is bounded by

$$\sum_{\substack{N_{123}:\\ N_{123} \lesssim N_1}} N_{123}^{2s-2} N_1^{-4-2\gamma} N_2^{-2} \#\mathcal{C}(m) \lesssim \sum_{\substack{N_{123}:\\ N_{123} \lesssim N_1}} N_{123}^{2s} N_1^{-1-2\gamma} N_2 \lesssim N_1^{2s-2\gamma},$$

which is acceptable. In performing the sum, we used that $s > 0$. □

4.10.2 Cubic sup-counting estimates

Proof of Lemma 4.4.22: We prove the four estimates separately.

Proof of (i)*:* By symmetry, we can assume without loss of generality that $N_1 \geqslant N_2 \geqslant N_3$. Using the basic counting estimate in $n_2 \in \mathbb{Z}^3$, we have that

$$\begin{aligned}
&\#\Big\{(n_1, n_2, n_3) \colon |n_1| \sim N_1, |n_2| \sim N_2, |n_3| \sim N_3, n = n_{123}, |\varphi - m| \leqslant 1\Big\} \\
&\lesssim \#\Big\{(n_2, n_3) \colon |n_2| \sim N_2, |n_3| \sim N_3, | \pm_{123} \langle n \rangle \pm_1 \langle n - n_{23} \rangle \pm_2 \langle n_2 \rangle \pm_3 \langle n_3 \rangle - m| \leqslant 1\Big\} \\
&\lesssim \sum_{n_3 \in \mathbb{Z}^3} 1\big\{|n_3| \sim N_3\big\} \min\big(\langle n - n_3 \rangle, N_2\big)^{-1} N_2^3 \\
&\lesssim N_2^3 N_3^2.
\end{aligned}$$

Proof of (ii)*:* The proof is essentially the same as the proof of (i) and we omit the details.

Proof of (iii)*:* Using the basic counting estimate in $n_2 \in \mathbb{Z}^3$, we have that

$$
\begin{aligned}
&\#\Big\{(n_{12}, n_2, n_3) \colon |n_{12}| \sim N_{12}, |n_2| \sim N_2, |n_3| \sim N_3, n = n_{123}, |\varphi - m| \leqslant 1\Big\} \\
&\lesssim \#\Big\{(n_{12}, n_2) \colon |n_{12}| \sim N_{12}, |n_2| \sim N_2, | \pm_{123} \langle n \rangle \pm_1 \langle n_{12} - n_2 \rangle \pm_2 \langle n_2 \rangle \pm_3 \langle n - n_{12} \rangle - m| \leqslant 1\Big\} \\
&\lesssim \sum_{n_{12} \in \mathbb{Z}^3} 1\big\{|n_{12}| \sim N_{12}\big\} \min(N_{12}, N_2)^{-1} N_2^3 \\
&\lesssim \min(N_{12}, N_2)^{-1} N_{12}^3 N_2^3.
\end{aligned}
$$

Proof of (iv)*:* The proof is essentially the same as the proof of (iii) and we omit the details.

□

4.10.3 Para-controlled cubic counting estimates

Proof of Lemma 4.4.23: To simplify the notation, we set $N_{\max} = \max(N_1, N_2, N_3)$. For $0 < \gamma < \beta$, we have that

$$\langle n_{12} \rangle^{-2\beta} \lesssim \langle n_{12} \rangle^{-2\gamma} \lesssim \langle n_1 \rangle^{-2\gamma} \langle n_2 \rangle^{2\gamma}.$$

Together with (ii) from Lemma 4.4.22, this yields

$$
\begin{aligned}
&\sum_{n_1, n_3 \in \mathbb{Z}^3} \Big(\prod_{j=1,3} 1\big\{|n_j| \sim N_j\big\}\Big) \langle n_{123} \rangle^{2(s_2-1)} \langle n_{12} \rangle^{-2\beta} \langle n_1 \rangle^{-2} \langle n_3 \rangle^{-2} \, 1\big\{|\varphi - m| \leqslant 1\big\} \\
&\lesssim N_1^{-2-2\gamma} N_2^{2\gamma} N_3^{-2} \sum_{N_{123}} N_{123}^{2(s_2-1)} \#\big\{(n_1, n_3) \colon |n_{123}| \sim N_{123}, |n_1| \sim N_1, |n_3| \sim N_3, |\varphi - m| \leqslant 1\big\} \\
&\lesssim N_1^{-2-2\gamma} N_2^{2\gamma} N_3^{-2} \sum_{\substack{N_{123}\colon \\ |N_{123}| \lesssim N_{\max}}} N_{123}^{2(s_2-1)} \operatorname{med}\big(N_{123}, N_1, N_3\big)^3 \min\big(N_{123}, N_1, N_3\big)^2.
\end{aligned}
$$

Using that $\operatorname{med}\big(N_{123}, N_1, N_3\big)^3 \min\big(N_{123}, N_1, N_3\big)^2 \lesssim N_{123} N_1^2 N_3^2$, we obtain that

$$
\begin{aligned}
&N_1^{-2-2\gamma} N_2^{2\gamma} N_3^{-2} \sum_{\substack{N_{123}:\\ |N_{123}| \lesssim N_{\max}}} N_{123}^{2(s_2-1)} \operatorname{med}\big(N_{123}, N_1, N_3\big)^3 \min\big(N_{123}, N_1, N_3\big)^2 \\
&\lesssim N_1^{-2\gamma} N_2^{2\gamma} \sum_{\substack{N_{123}:\\ |N_{123}| \lesssim N_{\max}}} N_{123}^{2s_2-1} \lesssim N_{\max}^{2\delta_2} N_1^{-2\gamma} N_2^{2\gamma}.
\end{aligned}
$$

□

4.10.4 Quartic counting estimate

Proof of Lemma 4.4.24: Using the upper bound on s, we can first sum in $n_4 \in \mathbb{Z}^3$ and obtain that

$$
\begin{aligned}
&\sum_{n_1,n_2,n_3,n_4 \in \mathbb{Z}^3} \Big(\prod_{j=1}^4 1\big\{|n_j| \sim N_j\big\}\Big) \langle n_{1234}\rangle^{2s} \langle n_{123}\rangle^{-2} |\widehat{V}_S(n_1,n_2,n_3)|^2 \Big(\prod_{j=1}^4 \langle n_j\rangle^{-2}\Big) 1\big\{|\varphi - m| \leqslant 1\big\} \\
&\lesssim N_4^{-2\eta} \sum_{n_1,n_2,n_3 \in \mathbb{Z}^3} \Big(\prod_{j=1}^3 1\big\{|n_j| \sim N_j\big\}\Big) \langle n_{123}\rangle^{-2} |\widehat{V}_S(n_1,n_2,n_3)|^2 \Big(\prod_{j=1}^3 \langle n_j\rangle^{-2}\Big) 1\big\{|\varphi - m| \leqslant 1\big\}.
\end{aligned}
$$

The remaining sum in n_1, n_2, and n_3 can then be estimated using Proposition 4.4.20, which yields the desired estimate. □

After the proof of the non-resonant quartic sum estimate (Lemma 4.4.24), we now turn to the resonant quartic sum estimate. We begin with the basic resonance estimate (Lemma 4.4.25), which forms the main part of the proof.

Proof of Lemma 4.4.25: Since $n_1, n_2 \in \mathbb{Z}^3$ are fixed and the phase φ is globally Lipschitz, there are at most $\sim N_1$ non-trivial choices of $m \in \mathbb{Z}$. Due to the log-factor in (4.4.46), it suffices to prove

$$
\sup_{m \in \mathbb{Z}} \sum_{n_3 \in \mathbb{Z}^3} 1\big\{|n_3| \sim N_3\big\} \langle n_{123}\rangle^{-1} \langle n_3\rangle^{-2} 1\big\{|\varphi - m| \leqslant 1\big\} \lesssim \langle n_{12}\rangle^{-1}.
$$

By inserting an additional dyadic localization, we obtain that

$$\begin{aligned}&\sum_{n_3\in\mathbb{Z}^3} 1\big\{|n_3|\sim N_3\big\}\langle n_{123}\rangle^{-1}\langle n_3\rangle^{-2}1\big\{|\varphi-m|\leqslant 1\big\}\\&\leqslant N_3^{-2}\sum_{N_{123}\geqslant 1} N_{123}^{-1}\sum_{n_3\in\mathbb{Z}^3} 1\big\{|n_{123}|\sim N_{123}\big\}1\big\{|n_3|\sim N_3\big\}1\big\{|\varphi-m|\leqslant 1\big\}.\end{aligned}\tag{4.10.2}$$

To simplify the notation, we write N_{12} for the dyadic scale of $n_{12}\in\mathbb{Z}^3$. Using Lemma 4.4.17, we have that

$$\begin{aligned}&N_{123}^{-1}N_3^{-2}\sum_{n_3\in\mathbb{Z}^3} 1\big\{|n_{123}|\sim N_{123}\big\}1\big\{|n_3|\sim N_3\big\}1\big\{|\varphi-m|\leqslant 1\big\}\\&\lesssim N_{123}^{-1}N_3^{-2}\min(N_{123},N_{12},N_3)^{-1}\min(N_{123},N_3)^3.\end{aligned}$$

We now separate the contributions of the three cases $N_{123}\ll N_3$, $N_{123}\sim N_3$, $N_{123}\gg N_3$. In the following, we implicitly restrict the sum over N_{123} to values which are consistent with $|n_{123}|\sim N_{123}$, $|n_{12}|\sim N_{12}$, and $|n_3|\sim N_3$ for some $n_1,n_2,n_3\in\mathbb{Z}^3$.

If $N_{123}\ll N_3$, then $N_{12}\sim N_3$. Thus,

$$\sum_{N_{123}\ll N_3} N_{123}^{-1}N_3^{-2}\min(N_{123},N_{12},N_3)^{-1}\min(N_{123},N_3)^3\lesssim 1\big\{N_{12}\sim N_3\big\}\sum_{N_{123}\ll N_3} N_{123}N_3^{-2}\lesssim N_{12}^{-1}.$$

If $N_{123}\sim N_3$, then $N_{12}\lesssim N_{123}\sim N_3$. Thus,

$$\sum_{N_{123}\sim N_3} N_{123}^{-1}N_3^{-2}\min(N_{123},N_{12},N_3)^{-1}\min(N_{123},N_3)^3\sim N_{12}^{-1}.$$

Finally, if $N_{123}\gg N_3$, then $N_{123}\sim N_{12}\gg N_3$. Thus,

$$\sum_{N_{123}\gg N_3} N_{123}^{-1}N_3^{-2}\min(N_{123},N_{12},N_3)^{-1}\min(N_{123},N_3)^3=N_{12}^{-1}N_3^{-2}N_3^{-1}N_3^3\sim N_{12}^{-1}.$$

This completes the proof. □

The resonant quartic sum estimate (Lemma 4.4.26) is now an easy consequence of the basic resonance estimate (Lemma 4.4.25).

Proof of Lemma 4.4.26: Using Lemma 4.4.25, we have that

$$\begin{aligned}
&\sum_{n_1,n_2\in\mathbb{Z}^3}\Big[\Big(\prod_{j=1}^{2}1\big\{|n_j|\sim N_j\big\}\langle n_{12}\rangle^{2s}\langle n_1\rangle^{-2}\langle n_2\rangle^{-2}\\
&\times\Big(\sum_{m\in\mathbb{Z}}\sum_{n_3\in\mathbb{Z}^3}\langle m\rangle^{-1}1\big\{|n_3|\sim N_3\big\}\langle n_{123}\rangle^{-1}\langle n_3\rangle^{-2}1\big\{|\varphi-m|\leqslant 1\big\}\Big)^2\Big]\\
&\lesssim \log(2+N_3)^2\sum_{n_1,n_2\in\mathbb{Z}^3}\Big[\Big(\prod_{j=1}^{2}1\big\{|n_j|\sim N_j\big\}\langle n_{12}\rangle^{2s-2}\langle n_1\rangle^{-2}\langle n_2\rangle^{-2}\Big]\\
&\lesssim \log(2+N_3)^2\max(N_1,N_2)^{2s}.
\end{aligned}$$

□

4.10.5 Quintic counting estimates

Before we turn to the proof of the non-resonant quintic counting estimate, we isolate a helpful auxiliary lemma.

Lemma 4.10.1 (Frequency-scale estimate)**.** Let $N_1, N_2, N_{1345}, N_{12345}$ be frequency scales which can be achieved by frequencies $n_1,\dots,n_5\in\mathbb{Z}^3$, i.e., satisfying

$$1\big\{|n_1|\sim N_1\big\}\cdot 1\big\{|n_2|\sim N_2\big\}\cdot 1\big\{|n_{1345}|\sim N_{1345}\big\}\cdot 1\big\{|n_{12345}|\sim N_{12345}\big\}\not\equiv 0.$$

Then, it holds that

$$\frac{\min(N_2,N_{12345})^2\min(N_1,N_{1345})}{\min(N_{12345},N_{1345},N_2)}\lesssim N_2\cdot N_{12345}.$$

Proof. By using the properties of min and max, we have that

$$\frac{\min(N_2,N_{12345})\min(N_1,N_{1345})}{\min(N_{12345},N_{1345},N_2)}\lesssim\frac{\min(N_2,N_{12345})N_{1345}}{\min(N_{12345},N_{1345},N_2)}\lesssim\max\big(\min(N_2,N_{12345}),N_{1345}\big).$$

Since $N_{1345}\lesssim\max(N_2,N_{12345})$, this yields

$$\frac{\min(N_2,N_{12345})^2\min(N_1,N_{1345})}{\min(N_{12345},N_{1345},N_2)}\lesssim\min(N_2,N_{12345})\cdot\max(N_2,N_{12345})=N_2N_{12345}.$$

□

Proof of Lemma 4.4.27: Let $m, m' \in \mathbb{Z}$ be arbitrary. We introduce N_{12345} and N_{1345} to further decompose according to the size of n_{12345} and n_{1345}. Using the two-ball basic counting lemma (Lemma 4.4.17) for the sum in $n_2 \in \mathbb{Z}^3$ and summing in $n_1 \in \mathbb{Z}^3$ directly, we obtain that

$$\begin{aligned}
&\sum_{n_1,\dots,n_5 \in \mathbb{Z}^3} \bigg[\Big(\prod_{j=1}^{5} 1\big\{|n_j| \sim N_j\big\}\Big) 1\big\{|n_{12345}| \sim N_{12345}\big\} 1\big\{|n_{1345}| \sim N_{1345}\big\} \\
&\quad \times \langle n_{12345}\rangle^{2(s-1)} \langle n_{1345}\rangle^{-2\beta} \langle n_{345}\rangle^{-2} \langle n_{34}\rangle^{-2\beta} \Big(\prod_{j=1}^{5} \langle n_j\rangle^{-2} \Big) \\
&\quad \times 1\big\{|\psi - m| \leqslant 1\big\} \cdot \Big(1\big\{|\varphi - m'| \leqslant 1\big\} + 1\big\{|\widetilde{\varphi} - m'| \leqslant 1\big\} \Big) \bigg] \\
&\lesssim N_{12345}^{2(s-1)} N_{1345}^{-2\beta} \min(N_{12345}, N_{1345}, N_2)^{-1} \min(N_2, N_{12345})^3 \prod_{j=1}^{5} N_j^{-2} \\
&\times \sum_{n_1,n_3,n_4,n_5 \in \mathbb{Z}^3} \Big(\prod_{j=1,3,4,5} 1\big\{|n_j| \sim N_j\big\}\Big) 1\big\{|n_{1345}| \sim N_{1345}\big\} \langle n_{345}\rangle^{-2} \langle n_{34}\rangle^{-2\beta} 1\big\{|\psi - m| \leqslant 1\big\} \\
&\lesssim N_{12345}^{2(s-1)} N_{1345}^{-2\beta} \min(N_{12345}, N_{1345}, N_2)^{-1} \min(N_2, N_{12345})^3 \min(N_1, N_{1345})^3 \prod_{j=1}^{5} N_j^{-2} \\
&\times \sum_{n_3,n_4,n_5 \in \mathbb{Z}^3} \Big(\prod_{j=3}^{5} 1\big\{|n_j| \sim N_j\big\}\Big) \langle n_{345}\rangle^{-2} \langle n_{34}\rangle^{-2\beta} 1\big\{|\psi - m| \leqslant 1\big\}.
\end{aligned}$$

Using Proposition 4.4.20 with $s = 0$ and $\gamma = \beta$ to bound the remaining sum in n_3, n_4, and n_5, we obtain a bound of the total contribution by

$$N_{12345}^{2(s-1)} (N_1 N_2)^{-2} \frac{\min(N_2, N_{12345})^3 \min(N_1, N_{1345})^3}{\min(N_{12345}, N_{1345}, N_2)} (N_{1345} \max(N_3, N_4, N_5))^{-2\beta}.$$

As long as the contribution is non-trivial, it holds that $N_{1345} \max(N_3, N_4, N_5) \gtrsim \max(N_1, N_3, N_4, N_5)$. Thus, it remains to prove that

$$N_{12345}^{2(s-1)} (N_1 N_2)^{-2} \frac{\min(N_2, N_{12345})^3 \min(N_1, N_{1345})^3}{\min(N_{12345}, N_{1345}, N_2)} \lesssim N_2^{-2\eta},$$

which follows from a short calculation. Indeed, using Lemma 4.10.1, we can estimate the left-hand side by

$$\begin{aligned}
&N_{12345}^{2(s-1)}(N_1N_2)^{-2}\frac{\min(N_2,N_{12345})^3\min(N_1,N_{1345})^3}{\min(N_{12345},N_{1345},N_2)}\\
&\lesssim N_{12345}^{2s-1}\min(N_2,N_{12345})\min(N_1,N_{1345})^2N_1^{-2}N_2^{-1}\\
&\lesssim N_{12345}^{2s-1+2\eta}N_2^{-2\eta}.
\end{aligned}$$

Due to our condition on s, this is acceptable. □

We now prove the double-resonance quintic counting estimate.

Proof of Lemma 4.4.29: We also use a dyadic localization to $|n_{345}| \sim N_{345}$ and $|n_{45}| \sim N_{45}$. By paying a factor of $\log(2+\max(N_4,N_5))^2$, it suffices to estimate the maximum over N_{345}, N_{45} instead of the sum. We do not require a logarithmic loss in N_3, since $N_3 \gg N_4, N_5$ implies that there are only ~ 1 non-trivial choices for N_{345}. We first sum in $n_3 \in \mathbb{Z}^3$ using the two-ball basic counting lemma (Lemma 4.4.17). We then sum in $n_4 \in \mathbb{Z}^3$ using only the dyadic constraint. This yields

$$\begin{aligned}
&N_3^{-2}N_4^{-2}\sup_{m\in\mathbb{Z}^3}\sup_{|n_5|\sim N_5}\sum_{n_3,n_4\in\mathbb{Z}^3}\Big[\Big(\prod_{j=3}^4 1\big\{|n_j|\sim N_j\big\}\Big)1\big\{|n_{345}|\sim N_{345}\big\}1\big\{|n_{45}|\sim N_{45}\big\}\langle n_{345}\rangle^{-1}\langle n_{45}\rangle^{-\beta}\\
&\times 1\big\{\langle n_{345}\rangle \pm_3 \langle n_3\rangle \pm_4 \langle n_4\rangle \pm_5 \langle n_5\rangle \in [m,m+1)\big\}\Big]\\
&\lesssim \min(N_{345},N_{45},N_3)^{-1}\min(N_3,N_{345})^3N_{345}^{-1}N_{45}^{-\beta}N_3^{-2}N_4^{-2}\sum_{n_4\in\mathbb{Z}^3}1\big\{|n_4|\sim N_4\big\}1\big\{|n_{345}|\sim N_{345}\big\}\\
&\lesssim \min(N_{345},N_{45},N_3)^{-1}\min(N_3,N_{345})^3\min(N_4,N_{45})^3N_{345}^{-1}N_{45}^{-\beta}N_3^{-2}N_4^{-2}.
\end{aligned}$$

Using a minor variant of Lemma 4.10.1, this contribution is bounded by

$$N_{45}^{-\beta}N_3^{-1}N_4^{-2}\min(N_3,N_{345})\min(N_4,N_{45})^2 \lesssim \max(N_4,N_{45})^{-\beta} \lesssim \max(N_4,N_5)^{-\beta}.$$

□

4.10.6 Septic counting estimates

Proof of Lemma 4.4.31: Using the decay of $\widehat{V}$, it suffices to prove

$$\begin{aligned}
&\sum_{(n_j)_{j\notin\mathcal{P}}} \langle n_{\mathrm{nr}}\rangle^{2(s-1)} \Big(\sum_{(n_j)_{j\in\mathcal{P}}}^{*} 1\big\{|n_{1234567}| \sim N_{1234567}\big\} 1\big\{|n_{567}| \sim N_{567}\big\} 1\big\{|n_4| \sim N_4\big\} \\
&\times \Phi(n_1,n_2,n_3)\langle n_4\rangle^{-1}\Phi(n_5,n_6,n_7)\Big)^2 \qquad (4.10.3)\\
&\lesssim \log(2+N_4)^2\Big(N_{1234567}^{2(s-\frac{1}{2})} N_{567}^{-2(\beta-\eta)} + N_{1234567}^{-2(1-s+\eta)}\Big).
\end{aligned}$$

The argument relies on two of our previous estimates. Using the cubic sum estimate (Proposition 4.4.20), we have that for all $N_{123} \geqslant 1$ that

$$\sum_{n_1,n_2,n_3\in\mathbb{Z}^3} 1\big\{|n_{123}| \sim N_{123}\big\}\Big(\prod_{j=1}^{3}\langle n_j\rangle^{\frac{\eta}{3}}\Big)\Phi^2(n_1,n_2,n_3) \lesssim N_{123}^{-2(\beta-\eta)}. \qquad (4.10.4)$$

Using the basic resonance estimate (Lemma 4.4.25), we have for all $N_3 \geqslant 1$ that

$$\sum_{n_3\in\mathbb{Z}^3} 1\big\{|n_3| \sim N_3\big\}\langle n_3\rangle^{-1}\Phi(n_1,n_2,n_3) \lesssim \log(2+N_3)\langle n_{12}\rangle^{-1}\langle n_1\rangle^{-1}\langle n_2\rangle^{-1}. \qquad (4.10.5)$$

Using the symmetry of Φ, it remains to consider the following three cases.

Case 1: $j = 4$ is unpaired. By first using Cauchy-Schwarz, summing in n_4, and then using (4.10.4), we obtain that

$$\begin{aligned}
&\sum_{(n_j)_{j\notin\mathcal{P}}} \langle n_{\mathrm{nr}}\rangle^{2(s-1)} \Big(\sum_{(n_j)_{j\in\mathcal{P}}}^{*} 1\big\{|n_{1234567}| \sim N_{1234567}\big\} 1\big\{|n_{567}| \sim N_{567}\big\} 1\big\{|n_4| \sim N_4\big\} \\
&\times \Phi(n_1,n_2,n_3)\langle n_4\rangle^{-1}\Phi(n_5,n_6,n_7)\Big)^2 \\
&\lesssim \sum_{(n_j)_{j\notin\mathcal{P}}} \Big[1\big\{|n_{\mathrm{nr}}| \sim N_{1234567}\big\}\langle n_{\mathrm{nr}}\rangle^{2(s-1)}\langle n_4\rangle^{-2}\Big(\sum_{(n_j)_{j\in\mathcal{P}}}^{*}\Phi(n_1,n_2,n_3)^2\Big) \\
&\times \Big(\sum_{(n_j)_{j\in\mathcal{P}}}^{*} 1\big\{n_{567} \sim N_{567}\big\}\Phi(n_5,n_6,n_7)^2\Big)\Big]
\end{aligned}$$

$$\lesssim N_{1234567}^{2(s-\frac{1}{2})} \sum_{(n_j)_{j\notin \mathscr{P} \wedge j\neq 4}} \Big(\sum_{(n_j)_{j\in\mathscr{P}}}^{*} \Phi(n_1,n_2,n_3)^2 \Big) \Big(\sum_{(n_j)_{j\in\mathscr{P}}}^{*} 1\big\{ n_{567} \sim N_{567} \big\} \Phi(n_5,n_6,n_7)^2 \Big) \Big]$$

$$= N_{1234567}^{2(s-\frac{1}{2})} \Big(\sum_{n_1,n_2,n_3\in\mathbb{Z}^3} \Phi(n_1,n_2,n_3)^2 \Big) \Big(\sum_{n_5,n_6,n_7\in\mathbb{Z}^3} 1\big\{ n_{567} \sim N_{567} \big\} \Phi(n_5,n_6,n_7)^2 \Big) \Big]$$

$$\lesssim N_{1234567}^{2(s-\frac{1}{2})} N_{567}^{-2(\beta-\eta)}.$$

This contribution is acceptable.

Case 2: $(3,4) \in \mathscr{P}$. We let $\mathscr{P}'$ be the pairing on $\{1,2,5,6,7\}$ obtained by removing the pair $(3,4)$ from $\mathscr{P}$. We also understand the condition $j \notin \mathscr{P}'$ as a subset of $\{1,2,5,6,7\}$. By first using (4.10.5) and then Cauchy-Schwarz, we have that

$$\sum_{(n_j)_{j\notin\mathscr{P}}} \langle n_{\mathrm{nr}} \rangle^{2(s-1)} \Big(\sum_{(n_j)_{j\in\mathscr{P}}}^{*} 1\big\{ |n_{1234567}| \sim N_{1234567} \big\} 1\big\{ |n_{567}| \sim N_{567} \big\} 1\big\{ |n_4| \sim N_4 \big\}$$

$$\times \Phi(n_1,n_2,n_3) \langle n_4 \rangle^{-1} \Phi(n_5,n_6,n_7) \Big)^2$$

$$\lesssim \log(2+N_4)^2 N_{1234567}^{2(s-1+\eta)}$$

$$\times \sum_{(n_j)_{j\notin\mathscr{P}'}} \langle n_{\mathrm{nr}} \rangle^{-2\eta} \Big(\sum_{(n_j)_{j\in\mathscr{P}'}}^{*} 1\big\{ |n_{567}| \sim N_{567} \big\} \langle n_{12} \rangle^{-1} \langle n_1 \rangle^{-1} \langle n_2 \rangle^{-1} \Phi(n_5,n_6,n_7) \Big)^2$$

$$\lesssim \log(2+N_4)^2 N_{1234567}^{2(s-1+\eta)} \sum_{(n_j)_{j\notin\mathscr{P}'}} \Big[\Big(\sum_{(n_j)_{j\in\mathscr{P}'}}^{*} \langle n_{\mathrm{nr}} \rangle^{-2\eta} \Big(\prod_{j\in\mathscr{P}'} \langle n_j \rangle^{-\frac{\eta}{6}} \Big) \langle n_{12} \rangle^{-2} \langle n_1 \rangle^{-2} \langle n_2 \rangle^{-2} \Big)$$

$$\times \Big(\sum_{(n_j)_{j\in\mathscr{P}'}}^{*} 1\big\{ |n_{567}| \sim N_{567} \big\} \Big(\prod_{j\in\mathscr{P}'} \langle n_j \rangle^{\frac{\eta}{6}} \Big) \Phi(n_5,n_6,n_7)^2 \Big) \Big].$$

We then use a direct calculation to bound the first inner factor and to estimate the sum in n_5, n_6, and n_7. The total contribution is bounded by $\log(2+N_4)^2 N_{1234567}^{2(s-1+\eta)} N_{567}^{-2(\beta-\eta)}$, which is bounded by $\log(2+N_4)^2 N_{1234567}^{2(s-1+\eta)}$ and hence is acceptable.

Case 3: $(4,5) \in \mathscr{P}$. We let $\mathscr{P}'$ be the pairing on $\{1,2,3,6,7\}$ obtained by removing the pair $(4,5)$ from $\mathscr{P}$. We also understand the condition $j \notin \mathscr{P}'$ as a subset of $\{1,2,3,6,7\}$. By first using

(4.10.5) and then Cauchy-Schwarz, we have that

$$\sum_{(n_j)_{j\notin\mathscr{P}}} \langle n_{\mathrm{nr}}\rangle^{2(s-1)} \bigg(\sum^{*}_{(n_j)_{j\in\mathscr{P}}} 1\big\{|n_{1234567}| \sim N_{1234567}\big\} 1\big\{|n_{567}| \sim N_{567}\big\} 1\big\{|n_4| \sim N_4\big\}$$

$$\times \Phi(n_1,n_2,n_3)\langle n_4\rangle^{-1}\Phi(n_5,n_6,n_7)\bigg)^2$$

$$\lesssim \log(2+N_4)^2 N_{1234567}^{2(s-1+\eta)} \sum_{(n_j)_{j\notin\mathscr{P}'}} \langle n_{\mathrm{nr}}\rangle^{-2\eta} \Big(\sum^{*}_{(n_j)_{j\in\mathscr{P}'}} \Phi(n_1,n_2,n_3)\langle n_{67}\rangle^{-1}\langle n_6\rangle^{-1}\langle n_7\rangle^{-1}\Big)^2.$$

Arguing similarly as in Case 2, we obtain an upper bound by $\log(2+N_4)^2 N_{1234567}^{2(s-1+\eta)}$. While this bound does not contain the gain in N_{567}, it is still acceptable. □

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