THE COVERING OF SETS OF CONSTANT WIDTH

by

Michael James Cohn, B.Sc.(Hons).

Thesis submitted for the degree
of Master of Science in the
Department of Pure Mathematics,
The University of Adelaide.
Submitted September, 1971

CONTENTS

SUMMARY

In 1914, Lebesgue posed the problem of determining a closed set of least area which covers every set of diameter 1 in the plane. Although such a set has not been found, various sets with the required covering property have been discovered. Also the problem has been generalized to higher dimensions n and to measures other than n-dimensional volume.

In this thesis, the problem has been generalized further. A universal cover of order k is a collection of k bounded closed sets such that every n-dimensional set of diameter 1 can be covered by at least one of the k sets. Various examples are considered, and a number of formal results produced. In particular, it is shown that for fixed positive integers n and k there exists a universal cover of order k whose sets are optimal with respect to any one of the measures: volume, diameter or surface area.

The covering sets of Lebesgue's problem have been used successfully to solve the two and three dimensional cases of Borsuk's problem: "Can every set of diameter 1 in n-dimensional space be partitioned into n+1 sets of diameter less than 1?" Now, suppose $d_n(n+1)$ is the infimum of all numbers d such that each set of diameter 1

in n-dimensional space can be partitioned into $n+1$ sets of diameter at most d. Universal covers can be used to obtain upper bounds on the value of $d_n(n+1)$ for each integer n. The method is illustrated in the case $n=3$.

STATEMENT

This thesis contains no material which has been accepted for the award of any other degree or diploma in any University and, to the best of my knowledge, contains no material previously published or written by another person, except when due reference is made.

ACKNOWLEDGEMENTS

I offer sincere thanks to Dr. P.R. Scott, who suggested the original idea for the thesis, and provided invaluable help as a supervisor while the research was being carried out. Also, many thanks are due to Miss Joan Potter for doing an excellent job of typing the thesis.

CHAPTER I

INTRODUCTION

1.1 Lebesgue tile problem

A closed connected subset T of the Euclidean plane is called a tile if any set of diameter ⩽ 1 in the plane can be completely covered by it, i.e. if for every set A in the plane of diameter ⩽ 1 there exists a subset B of T directly congruent to A. By the diameter of a set, we mean the supremum of the distance between pairs of points of the set. In 1914, Lebesgue posed the problem of finding the tile of smallest area. Although it is easily seen that there is a positive lower bound on the area of a tile, it is by no means obvious whether there exists a tile of area equal to the greatest lower bound.

In 1920, Pál [19] reformulated the problem in the following way, restricting it to convex tiles, as we shall from this point on:

Problem. Let a_1 be the infimum of the area $a(T)$ of all convex tiles T.

Then (a) determine the value of a_1

(b) determine whether a convex tile of area a_1 exists

(c) if there exists a convex tile with area a_1, are all convex tiles with area a_1 of the form to be determined later?

The problem has been partially solved. In the next chapter we shall generalize this problem, and so we shall now look at a few known results. For a more detailed account see Meschkowski [18]. Since we defined a tile to be a closed set, we can assume that the sets to be covered are closed. Since the diameter of the convex hull of a set is equal to the diameter of the set, we can also restrict our attention to convex sets.

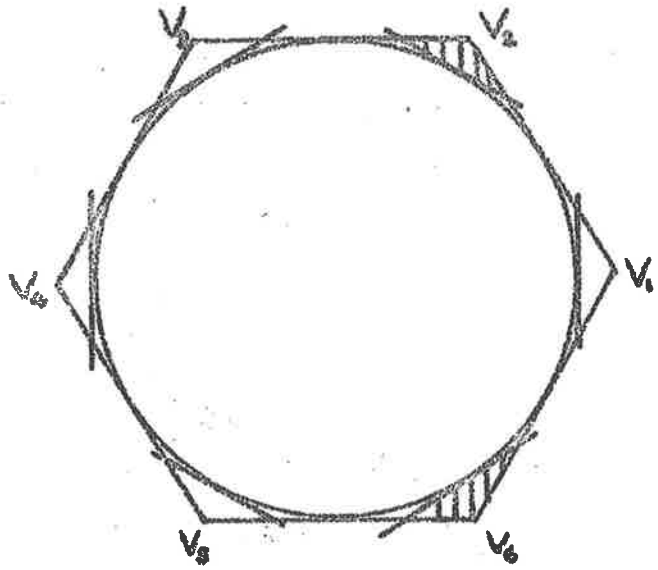
For a convex set K in E^n (Euclidean n-space) and a unit vector $\underset{\sim}{u}$, the <u>width</u> of K in the direction $\underset{\sim}{u}$ is the infimum of the distances between any pair of hyperplanes perpendicular to $\underset{\sim}{u}$ and enclosing K. The set K is said to be of <u>constant width</u> w if the width of K is w in every direction $\underset{\sim}{u}$. Any set of constant width w has diameter w. So, since every set of diameter 1 is contained in a set of constant width 1 (see Eggleston [6], p 126), we can assume that the sets to be covered are closed convex sets of constant width 1.

Firstly, let us consider examples of tiles. The smallest circular tile is the Jung's circle of radius $\frac{1}{3}\sqrt{3}$, [16]. Clearly no smaller circular region is a tile since the circumcircle of the equilateral triangle of side 1 has radius $\frac{1}{3}\sqrt{3}$. We shall denote Jung's circle by T_1; this first tile has area $a(T_1) = 1.047197....$.

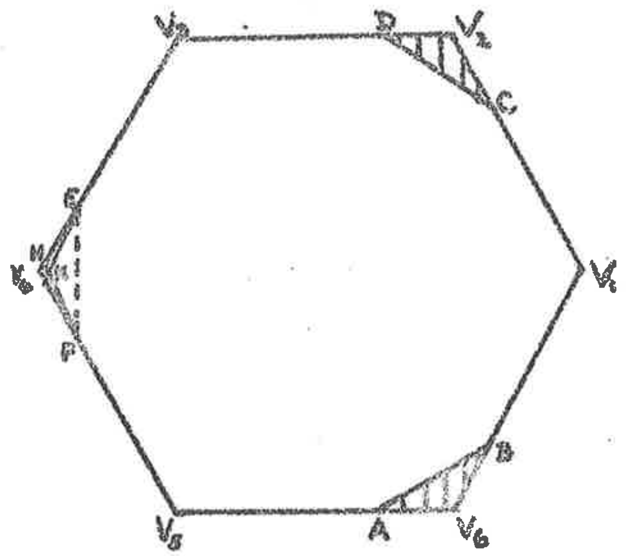
It is evident that the square of side 1 is a tile, which we shall denote by T_2. Then $a(T_2) = 1$.

Pál [19] showed that the regular hexagon of width 1 is a tile; denote it by T_3. Then $a(T_3) = \frac{1}{2}\sqrt{3} = 0.86602\ldots$.

Thus we know of three simple tiles. The two best known tiles are formed by truncation of the hexagonal tile T_3.



Suppose the vertices of the hexagonal tile are labelled $V_1, V_2, V_3, V_4, V_5, V_6$ in an anti-clockwise direction. If we draw to the incircle of the hexagon the tangents which are perpendicular to the diagonals, we obtain at each of the six vertices an isosceles triangle. Now there is a strip of width 1 between the regions at a pair of opposite vertices, and so no figure of diameter 1 can have a point in each of a pair of opposite triangles. Using the symmetry of the hexagon, without loss of generality we may remove the triangular regions at V_2 and V_6 obtaining a tile, T_4. Note that when we remove the regions, we leave the bases so that T_4 is closed. We shall always assume when we make truncations that the resulting set (or sets in later chapters) is closed unless we specifically state otherwise. On calculation of the area, we obtain $a(T_4) = 2 - \frac{2}{3}\sqrt{3} = 0.845299\ldots$. This is the best tile that Pál found. We shall at times refer to it as the Pál tile or as the truncated hexagonal tile.



We shall now consider a further improvement on this tile. This improvement is not very significant in terms of area, but is of interest since we shall use a similar technique at a later stage. Suppose we label the vertices as before, and suppose that AB, CD, EF are the tangents to the incircle near V_6, V_2 and V_4 respectively, as shown in the diagram. We have already restricted our attention to closed sets of constant width 1. Since the hexagon is of width 1, any set of constant width 1 when covered by the Pál tile will touch each edge of the hexagon at exactly one point. In particular, the line joining the two points where the set touches opposite edges will be perpendicular to those two edges. Now, the set will touch V_1V_2 and V_1V_6 at a point in the segments V_1C and V_1B respectively. Let us swing an arc FH of radius 1 and centre C, intersecting V_3V_4 at H, cutting a thin strip out of the triangle V_4EF. Similarly, we swing an arc of radius 1 and centre B intersecting the arc FH at K. Then the figure $V_1CDV_3EKFV_5AB$ forms a tile, T_5. It was discovered by R. Sprague [21] in 1936, and so we shall refer to it as the Sprague tile. Sprague calculated the area and obtained $a(T_5) = 0.844144....$.

To my knowledge, the Sprague tile is the best tile known. It is evident that the three remaining vertices of the hexagonal tile may not be removed since they are required for the Reuleaux triangle of width 1. However it may be possible to find a better tile not based on the hexagon.

To this stage our considerations have produced an upper bound for a_1, being

$$a_1 \leqslant 0.844144.... .$$

Clearly a_1 has a lower bound of zero. It is well known that of all curves of diameter 1, the circle has largest area. Since the circle must be covered by a tile, we immediately have

$$a_1 \geqslant \frac{\pi}{4} = 0.7853\ldots .$$

Pál considered the convex hull of the union of a circle of radius $\frac{1}{2}$ and the equilateral triangle of side length 1. He showed that this set has least area if the centroid of the triangle and the centre of the circle are concurrent. Since both these sets must be covered by any convex tile, and since this figure, F say, does not cover the regular pentagon of width 1 and so is not a tile, he proved

$$a_1 > a(F) = \frac{\pi}{8} + \frac{\sqrt{3}}{4}$$
$$= 0.825711\ldots .$$

Hence $\frac{\pi}{8} + \frac{\sqrt{3}}{4} = 0.825711\ldots < a_1 \leqslant 0.844144\ldots$.

So although the value of a_1 has not been determined, the value is contained in fairly narrow bounds.

Pál also proved that a tile of minimal area exists. but the problem of finding one or more minimal tiles is still open. We shall consider an extension and generalization of his proof on the existence of a tile of minimal area in Chapter III.

1.2 Extension of Lebesgue tile problem to E^3.

An analogous problem is to consider the same tile problem in E^3, where the area criterion is replaced by volume. For convenience, we shall refer to the covering sets analogous to tiles in the 2-dimensional problem as covering bodies. Hence the problem is to find a covering body of smallest volume V_1.

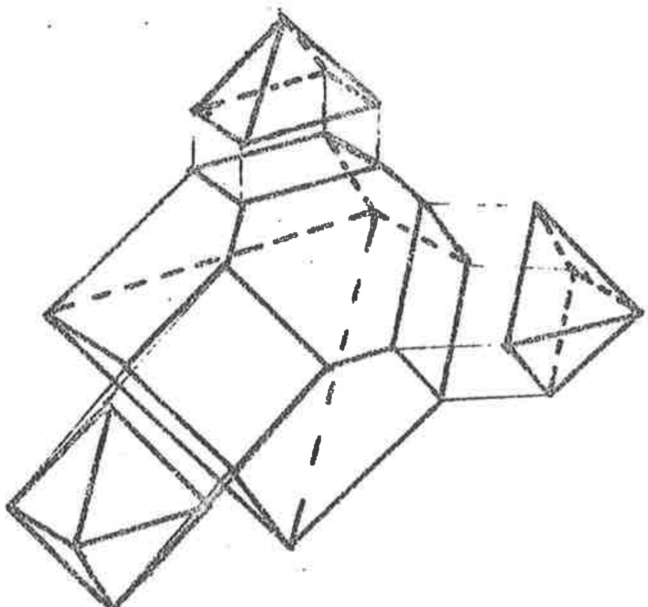
Clearly the cube of side 1 is a covering body. A slightly better covering body is the Jung's sphere of radius $\frac{1}{2}\sqrt{\frac{3}{2}}$, [15], which we shall denote by D_1. Then D_1 has volume

$$V(D_1) = \frac{\pi}{4}\sqrt{\frac{3}{2}} = 0.96191\ldots .$$

D. Gale [7] discovered that the regular octahedron of width 1 (i.e. of edge length $\sqrt{\frac{3}{2}}$) is a covering body, D_2 say.

Then $$V(D_2) = \tfrac{1}{2}\sqrt{3} = 0.86602\ldots .$$

It is possible to reduce the covering body D_2 by a method analogous to that applied by Pál to the regular hexagon.



We place planes parallel to and at a distance $\frac{1}{2}$ from each plane of symmetry of the octahedron. These cut from D_2 at each vertex a square-based pyramid with height $h = \frac{1}{2}(\sqrt{3} - 1)$. If we remove three of these pyramids in such a way that no two of the pyramids removed contain opposite vertices of the original octahedron, we obtain a new covering body D_3. Since the volume of each pyramid is $\frac{1}{6}(3\sqrt{3} - 5)$, it follows that

$$\begin{aligned} V(D_3) &= V(D_2) - \tfrac{1}{2}(3\sqrt{3} - 5) \\ &= \frac{5}{2} - \sqrt{3} \\ &= 0.76794\ldots . \end{aligned}$$

It is possible to reduce this covering body by an argument similar to that of Sprague, but this would probably result only in a small improvement.

The problem of finding a covering body of least volume V_1 remains unanswered, as for the corresponding problem in the plane, but bounds can be obtained on V_1. Clearly V_1 is greater than the volume of the sphere of radius $\frac{1}{2}$, and we can deduce

$$\frac{\pi}{6} = 0.52359\ldots < V_1 < 0.76794\ldots .$$

Certainly we know that a covering body of minimal volume exists, although as before we do not know if such a body is unique.

1.3 Further generalizations of Lebesgue tile problem

There are essentially two forms of generalization of this problem. We shall use both kinds in the following chapters.

We could generalize the problem to n-dimensional Euclidean space. Clearly, the generalized cube of side 1 is always a covering body. In E^n, the Jung's n-sphere of radius $\left(\frac{n}{2n+2}\right)^{\frac{1}{2}}$ is a covering body. Also, Gale [7] showed that each set of diameter 1 can be enclosed in a regular simplex of edge length $\left(\frac{n(n+1)}{2}\right)^{\frac{1}{2}}$. For large n, the Gale simplex is of little interest since the edge length and volume of the Gale simplex increase with n without bound. On the other hand, the radius of the Jung's sphere tends to $\frac{1}{2}\sqrt{2}$ as n increases, and the volume is

$$\frac{\pi^{\frac{n}{2}}}{2^n\Gamma(1 + \frac{1}{2}n)},$$

where Γ denotes the gamma function. This volume tends to zero as n increases.

Alternatively, we could generalize the Lebesgue tile problem by replacing the measure of volume by some other measure, where by measure we do not necessarily mean a measure in the sense of measure theory. Suitable alternative measures would be diameter or surface area.

1.4 Application to Borsuk's problem

Covering bodies have been found useful for the two and three dimensional solutions of the following problem, posed by G. Borsuk [3] in 1933:

> Problem: Is it possible to decompose every bounded set B in n-dimensional Euclidean space E^n into $n+1$ sets, each of diameter less than that of B?

The partial solutions of this problem have been well summarized by Grünbaum [9]. The following is a brief statement of the results.

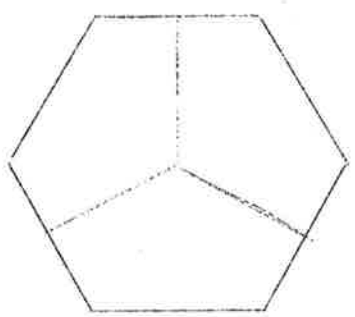
It is easily seen that without loss of generality we need consider the question only for those sets which are closed, convex and of diameter 1. Since every set in E^n of diameter 1 is contained in a set of constant width 1, it is sufficient to consider only sets of constant width 1. Hence Borsuk's problem is equivalent to the following:

> Can every closed convex set B in E^n of constant width 1 be partitioned into n+1 sets, each of diameter less than 1?

Hadwiger [10,11,12] used an analytic proof to show that the result was affirmative for sets with a smooth boundary, i.e. sets such that through every boundary point there passes exactly one support hyperplane. This result has been sharpened to some not necessarily smooth sets by Anderson-Klee [1]. However, the general problem can not be solved from Hadwiger's result by a limiting or approximation process.

A general proof for E^3 was published by Eggleston in 1955 ([4] or [5], pp 77-92). A similar proof was discovered independently by Perkal in 1947 (see the comments of Grünbaum [9], p 273). Eggleston's proof depends on the fact that two points of a convex set X of diameter d are at a distance d apart only if there exist two parallel supporting hyperplanes of X, one passing through each of the points. Due to difficulties of a topological nature, no successful extension of this method to higher dimensions has been found.

For two or three dimensions, a simple proof, giving results stronger than the affirmative solution of Borsuk's problem, has been found using covering bodies.



Gale [7] produced the following proof for the plane case. We recall that every set of diameter 1 in the plane may be covered by the regular hexagon of width 1. Hence, if we can divide the regular hexagon of width 1 into three sets of diameter at most α, it will follow that every set of diameter 1 in the plane can be divided into three sets of diameter $\leq \alpha$. To divide the tile in the required manner, we drop perpendiculars from the centre of the hexagon to three alternate sides. The three resulting pieces each have diameter $\frac{1}{2}\sqrt{3}$, being the distance between the feet of the constructed perpendiculars. Hence we deduce:

Any set in the plane of diameter ≤ 1 can be partitioned into three sets, each of diameter $\leq \frac{1}{2}\sqrt{3}$.

In addition, we note that $\frac{1}{2}\sqrt{3}$ is the best possible such number, since in a partition of the circular disc of diameter 1 into three sets, at least one of the sets has diameter $\frac{1}{2}\sqrt{3}$ or greater.

For three-dimensional Euclidean space, two similar partitions of the truncated octahedron (which we considered earlier) have been described in the literature. The diameters of the parts in Heppes' partition [14] are $\leq 0.9977\ldots$, while in that of Grünbaum [8], the diameters are $\leq 0.9887\ldots$. These dissections will be considered in more detail later. It appears evident that further truncation and meticulous partitioning of the truncated octahedron would result in an improvement in the diameters. In fact, we shall find an improvement.

Since this method of covering bodies appears to give stronger results, we now generalize Borsuk's problem in a manner suggested by Grünbaum [9]. For a set $K \subset E^n$ with diameter 1, and for a positive integer k, let $d_n(K,k)$ be the infimum of all real numbers α such that K may be partitioned into k subsets, each of diameter $\leq \alpha$. Also, for a positive integer k,

let $d_n(k) = \sup\{d_n(K,k) \mid K \subset E^n, \text{ diam } K = 1\}$.

With the definition, for the affirmative solution of Borsuk's problem, it is sufficient to obtain an affirmative solution for the following:

Is $d_n(n+1) < 1$ for every positive integer n?

However, there is also the additional problem of actually determining the value of $d_n(n+1)$, and we shall refer to this as the <u>modified Borsuk's problem</u>.

The value of $d_2(3)$ has been determined as $\frac{1}{2}\sqrt{3}$. For higher n, the value of $d_n(n+1)$ is not known. Knaster [17] mentioned the problem of determining $d_n(B^n,k)$, where B^n denotes the solid n-dimensional sphere or ball of diameter 1, but even the problem of determining $d_n(B^n,n+1)$ remains open. However, the obvious simplicial decomposition of B^n shows

$$d_n(B^n,n+1) \leqslant \begin{cases} \sqrt{\dfrac{n+1}{n+2}} & \text{for even } n \\ \sqrt{\dfrac{1}{2} + \dfrac{1}{2}\sqrt{\dfrac{n-1}{n+3}}} & \text{for odd } n. \end{cases}$$

Hadwiger [13] proved that equality holds for $n \leqslant 3$, but for $n \geqslant 4$ could establish only

$$d_n(B^n,n+1) \geqslant \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{n-1}{2n}}} .$$

Hence, in E^3 the sphere of diameter 1 can be partitioned into four sets, each of diameter

$$\sqrt{(3+\sqrt{3})/6} = 0.88807.. ,$$

and not into four sets, each of smaller diameter. No set K of diameter 1 is known in E^3 with

$$d_3(K,4) > \sqrt{(3+\sqrt{3})/6} = d_3(B^3,4).$$

We note that this is analogous to the result that there is no set K of diameter 1 in E^2 with

$$d_2(K,3) > \tfrac{1}{2}\sqrt{3} = d_2(B^2,3).$$

It has been conjectured by Hadwiger [13] and Gale [7] that $d_n(n+1) = d_n(B^n, n+1)$ for every positive integer n. Whether this is true or not, it follows from our discussion that

$$\sqrt{(3 + \sqrt{3})/6} = 0.88807\ldots \leqslant d_3(4) \leqslant 0.9887\ldots,$$

and the conjecture is that $d_3(4) = \sqrt{(3 + \sqrt{3})/6}$.

In our use of covering bodies to obtain a partial solution for the modified Borsuk's problem, we partitioned a set which covered all sets of diameter 1. We could probably obtain a better bound for $d_3(4)$ if instead we consider a collection of several sets $\{C_1, \ldots, C_k\}$ such that each set of diameter 1 is covered by at least one of the C_i. This gives us motivation for the concept of a <u>universal k-cover</u>, which we shall define in the next chapter. We shall use the k-covers to obtain a better upper bound for $d_3(4)$.

It is possible that the method of covering bodies will not lead to a solution of Borsuk's problem in dimensions greater than 3 (however, see Eggleston [5], pp 77, 91-92). However in principle at least, the method of universal k-covers can be used for any dimension.

CHAPTER II

UNIVERSAL COVERS

2.1 Definition and statement of problems

We now make a generalization of the concept of covering body.

Defn: Suppose n and k are positive integers, and that $U_k = \{C_1, C_2, \ldots, C_k\}$ is a collection of k bounded, closed convex subsets of E^n such that every subset A of E^n of diameter ≤ 1 is covered by at least one of the C_i $(1 \leq i \leq k)$, i.e. for each set $A \subset E^n$ with $\operatorname{diam} A \leq 1$, there exist an integer i with $1 \leq i \leq k$ and a set $B \subset C_i$ such that A is directly congruent to B. Then we shall call U_k a universal cover of order k in E^n, or more briefly, a k-cover in E^n.

We immediately note that each of the covering bodies of the Lebesgue tile problem in E^n forms a 1-cover. Also in the definition we need not insist that the set A is directly congruent to B, but may allow A to be congruent to B. If we do not insist on direct congruence, we may be able to find smaller sets for a k-cover since we would allow reflections to occur in the covering process. This alternative definition will be mentioned from time to time although no work will be specifically done on it. As in the case of the tile we can assume that the sets A to be covered are closed, convex and of constant width 1.

With this general definition, we shall have several questions to consider, mainly analogous to those posed by Lebesgue and Pál. However, before posing these questions, we settle some matters of notation.

We shall denote n-dimensional volume by V_n, and the surface area of a bounded closed convex set $S \subset E^n$ by

$$A_n(S) = \lim_{r\to 0+} \frac{V_n(S+B_r) - V_n(S)}{r},$$

where B_r is the sphere, centre the origin and radius r, and

$$X + Y = \{\underset{\sim}{x} + \underset{\sim}{y} \mid \underset{\sim}{x} \in X \text{ and } \underset{\sim}{y} \in Y\} \text{ for all } X,Y \subset E^n.$$

The diameter of a set $X \subset E^n$ will be denoted by $d_n(X)$ unless the dimension is clear, in which case we may use diam X. Since X is a set, this does not cause confusion with the other use of d_n used in the modification of Borsuk's problem.

For fixed positive integers n and k, suppose $U_k = \{C_1, C_2, \ldots, C_k\}$ is a k-cover in E^n. Then we define the <u>volume of the cover</u> to be

$$V_n(U_k) = \max_{1\leqslant i\leqslant k} V_n(C_i),$$

the <u>surface area of the cover</u> to be

$$A_n(U_k) = \max_{1\leqslant i\leqslant k} A_n(C_i),$$

and the <u>diameter of the cover</u> to be

$$d_n(U_k) = \max_{1 \leqslant i \leqslant k} d_n(C_i).$$

We also write

$$V_{n,k} = \inf_{U_k} V_n(U_k), \quad A_{n,k} = \inf_{U_k} A_n(U_k) \text{ and } d_{n,k} = \inf_{U_k} d_n(U_k)$$

where the infima are taken over all k-covers in E^n.

In the cases $n=2$ and $n=3$, we may sometimes shorten the notation. For $n=2$, we shall denote V_2 by a for area, and hence denote $V_{2,k}$ by a_k. A_2 may be denoted by p for perimeter, and hence $A_{2,k}$ by p_k. For $n=3$, we may denote the function V_3 by V, or the number $V_{3,k}$ by V_k, in cases where this would not cause confusion. Then our results from the Lebesgue tile problem become

$$0.825711\ldots < a_1 = V_{2,1} \leqslant 0.844144\ldots$$

and $\frac{\pi}{6} = 0.5235\ldots < V_1 = V_{3,1} < 0.7679\ldots$.

Since Barbier's theorem tells us that all sets of constant width 1 in the plane have perimeter π, and since the Pál tile has perimeter $3.38119\ldots$, we have

$$\pi = 3.14159\ldots < p_1 = A_{2,1} < 3.38119\ldots .$$

Also, by consideration of the Pál tile, we have

$$1 \leqslant d_{2,1} < \frac{2}{3}\sqrt{3} = 1.154\ldots .$$

We now pose several questions about universal covers.

<u>Question 1</u>. For given positive integers n and k, what are the values of $V_{n,k}, A_{n,k}$ and $d_{n,k}$?

Although it appears to be difficult to find actual values for these constants, we would hope to obtain close bounds on their values.

Clearly $V_{n,k}$, $A_{n,k}$ and $d_{n,k}$ are decreasing sequences in k. Now $d_{n,k}$ is bounded below by 1. Also, of all ~~curves~~ sets of constant width 1 in E^n, the sphere has the largest volume, and hence its volume is a lower bound on $V_{n,k}$ for all positive integers k. So we shall pose the following question.

<u>Question 2.</u> For a given positive integer n,

does $\lim_{k\to\infty} V_{n,k}$ = volume of the n-dimensional sphere of diameter 1,

$\lim_{k\to\infty} A_{n,k} = \sup\{A_n(S) \mid S \subset E^n,\ S \text{ closed, convex, and of constant width } 1\}$,

and $\lim_{k\to\infty} d_{n,k} = 1$?

In addition, are any of these limits attained for a positive integer k?

One important question remains:

<u>Question 3.</u> For given positive integers n and k, does there exist a k-cover U_k in E^n for which

(a) $V_n(U_k) = V_{n,k}$,

(b) $A_n(U_k) = A_{n,k}$,

(c) $d_n(U_k) = d_{n,k}$?

In each case, if there exists such an optimal k-cover,

(d) how many such k-covers exist?

(e) find such a k-cover.

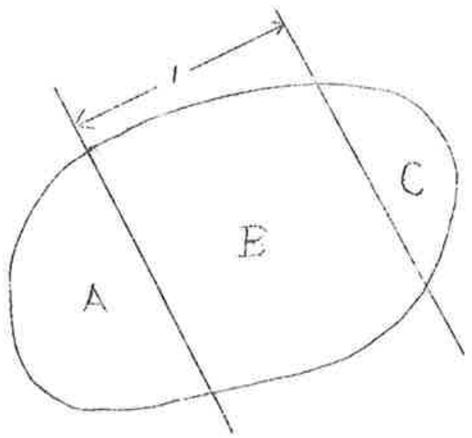
From our experience in the Lebesgue tile problem, we can expect that the actual values of $V_{n,k}$, $A_{n,k}$ and $d_{n,k}$ are difficult to find, and that it is difficult to find k-covers with these particular values, although we would expect to be able to show that such optimal k-covers exist. Of course, for the trivial case $n=1$, the line segment of length 1 is the only convex set of diameter 1 and so $V_{1,k} = d_{1,k} = 1$ and $A_{1,k} = 2$ for all positive integers k. Also for $n=1$ and k a positive integer, there is essentially only one optimal k-cover, since two k-covers would only differ in that the sets not used for covering may not be congruent. So the problem for $n=1$ is solved. Hence, from now on, we shall assume that $n \geq 2$.

In the next three chapters, we shall see some partial answers to these questions. Then we shall use the k-covers to obtain a better bound on $d_3(4)$.

2.2 Simple examples of k-covers.

As in the treatment of the Lebesgue tile problem, we shall consider a few examples first. I have not obtained many examples with the aim of lowering the upper bound on diameters or surface area, and there are possibly considerably better covers with respect to these measures than those given here.

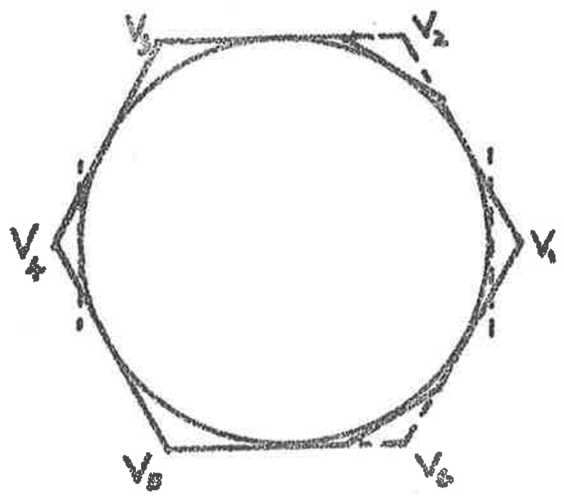
There is a general method for obtaining k-covers which we shall frequently use, although other methods may give better results at times. Suppose we have a bounded closed convex set $S \subset E^n$ which covers at least one set of constant width 1. Then if we draw two parallel hyperplanes, distance 1 apart, we divide S into three regions A, B, C, where B lies between the hyperplanes, and A and C each lie on opposite sides of the hyperplanes. We can also assume that B is closed. Then any set of diameter $\leqslant 1$ which can be covered by S can be covered by either $A \cup B$ or $B \cup C$, since the distance separating A from C is 1. Furthermore, both $A \cup B$ and $B \cup C$ are bounded, closed and convex, and so are suitable sets to form part of a universal cover. In practice, to obtain a k-cover, we apply this general method to the sets of a universal cover of lower order, making a suitable choice for the orientation and positioning of the planes.



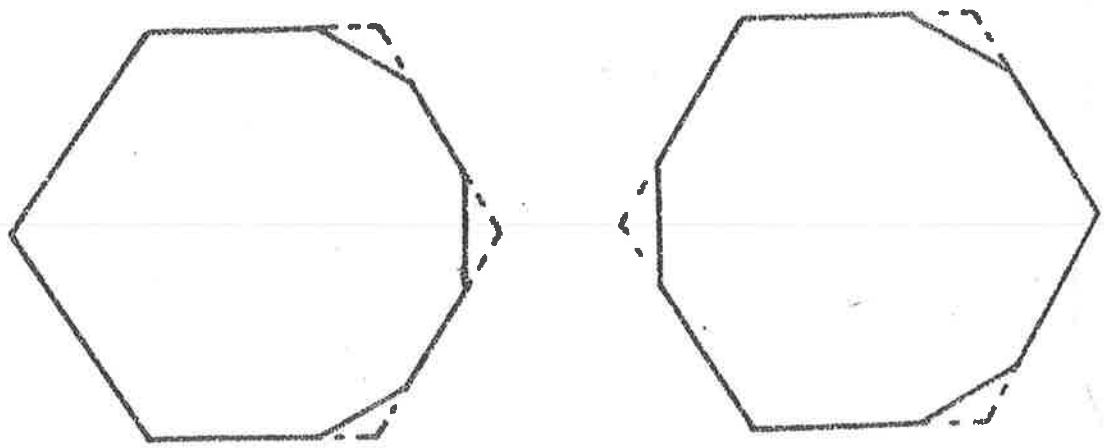
We now consider two specific examples:

A 2-cover in the plane.

We can easily form a 2-cover in the plane from the Pál tile, i.e. the truncated regular hexagon of width 1. Suppose the vertices of the hexagon are $V_1, V_2, V_3, V_4, V_5, V_6$, and that the truncations are at V_2 and V_6.



If we draw tangents to the inscribed circle of the hexagon at the points where the diagonal V_1V_4 cuts the circle, we form two isosceles triangles separated by a parallel strip of width 1. Then we see that each set of diameter 1 in the plane must be covered by one of the figures formed by truncating three corners off the regular hexagon, by using cuts tangent to the inscribed circle. This can be achieved in essentially two different ways, namely by removing vertices V_1, V_2, V_6 or by removing V_2, V_4, V_6. Hence we obtain a 2-cover.



Each set of this 2-cover U_2 has the same area, which is found to be

$$a(U_2) = 3 - \frac{5}{4}\sqrt{3}$$
$$= 0.834936\ldots .$$

The perimeter is $p(U_2) = 12 - 5\sqrt{3}$

$$= 3.33974\ldots .$$

This compares with the Pál tile T_4 which has

$$a(T_4) = 2 - \frac{2}{3}\sqrt{3} = 0.845299\ldots$$

and $$p(T_4) = 8 - \frac{8}{3}\sqrt{3} = 3.381197\ldots ,$$

and also has less area and perimeter than the Sprague tile T_5.

Since the curve of constant width 1 with greatest area is the circle, and since every curve of constant width 1 has perimeter π, we see that

$$\frac{\pi}{4} = 0.78539\ldots \leqslant a_2 = V_{2,2} \leqslant 3 - \frac{5}{4}\sqrt{3} = 0.834936\ldots$$

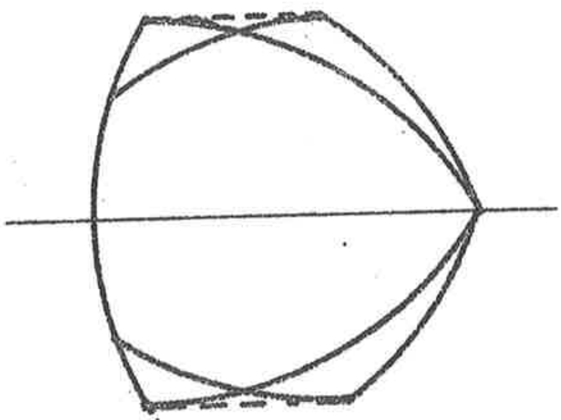
and $\pi = 3.14159\ldots \leqslant p_2 = A_{2,2} \leqslant 12 - 5\sqrt{3} = 3.33974\ldots$.

These are the best bounds we shall establish on these numbers. Pál produced a higher value for the lower bound on a_1 by considering the convex hull of the union of a Reuleaux triangle of constant width 1 and of a circle of diameter 1. He showed that this convex hull has least area when the centre of the circle and the centroid of the triangle are coincident, and then the area is

$$\frac{\pi}{8} + \frac{\sqrt{3}}{4} = 0.825711\ldots .$$

Now the Reuleaux triangle may not be covered by the same set of a 2-cover as the circle. However, if we consider the Reuleaux triangle, the regular Reuleaux pentagon and the circle, all of diameter 1, we see that at least one pair of these three sets must be covered by the same set of the 2-cover. By a similar proof to that of Pál for the case of the Reuleaux triangle and the circle, I was able to establish that the convex hull of the union of the circle of diameter 1 and of the regular Reuleaux pentagon of width 1 has least area when the centre of the circle lies at the centroid of the pentagon. I omit the proof for two reasons; firstly, it is a long, tedious and yet an easy extension of Pál's proof, and secondly, the increase in area over $\frac{\pi}{4}$ is small, since the area is just

$$\frac{\pi}{8} + \frac{5}{4}\cot\frac{2\pi}{5} = 0.7988\ldots .$$



The convex hull of the union of the Reuleaux triangle and the regular Reuleaux pentagon of width 1 has area 0.813...

when the two sets are positioned as shown in the diagram. This is the least value I have found for this area. I conjecture that it is close to the actual least value and that the least area possible is greater than 0.7988... . If this is true, then

$$a_2 \geqslant \min\{0.825711\ldots,\ 0.7988\ldots\} = 0.7988\ldots,$$

and so $0.7988\ldots \leqslant a_2 = V_{2,2} \leqslant 0.834936\ldots$.

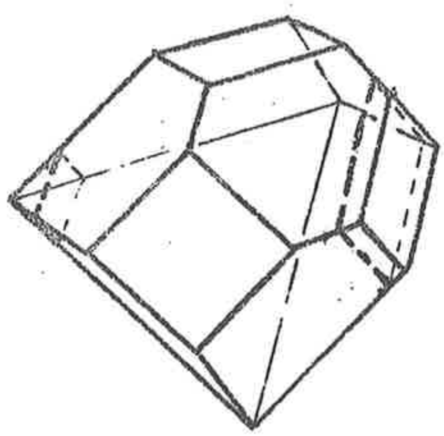
Earlier we saw that all the best known tiles were formed from the regular hexagon of width 1. We noted that there may be a better tile with respect to area which can not be obtained from the hexagonal tile. Similarly, there may also be 2-covers which can not be formed from the hexagon. Hence I tried to obtain a 2-cover using two different regular polygons of width 1, but I was unable to obtain any useful results.

A 2-cover in E^3.

In 3-dimensional space, we shall apply our general method to the truncated octahedron. This is the best covering body we obtained, although we did note that it could be improved by argument similar to that of Sprague.

In the plane we obtained a 2-cover by truncating corners, by taking tangents to the inscribed circle of the truncated hexagonal tile at two opposite and previously untruncated corners. We note that in the case of the truncated octahedron covering body, there is no pair of opposite untruncated corners, and hence this method can not be applied directly by taking tangent planes to the inscribed sphere at the vertices. However, the essential idea of our method was that there existed a strip of width 1 which did not cover the covering body, and the circle only helped us in positioning the strip. So, in the case of the truncated octahedron, we could make further truncations perpendicular to a diagonal by shifting the cutting planes towards the remaining vertex at one end of the diagonal. Alternatively, we could remove an edge. We consider a 2-cover formed by each method.

(i) Consider the case where we make a further truncation



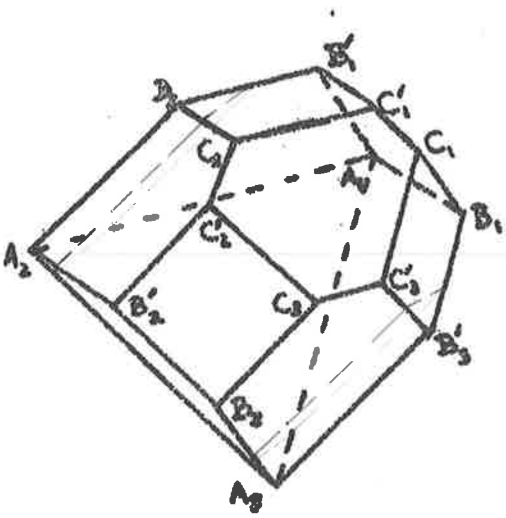
by a cut parallel to one of the previous cuts. Any set of diameter of at most 1 can be covered by the truncated octahedron when either a small square-based pyramid is removed or when a thin slice is removed at the opposite face, the pieces being removed each separated by a three-dimensional parallel strip of width 1. Clearly, the volume of the 2-cover so obtained is least when the volumes of the pyramid and of the slice being removed are equal, since one increases as the other decreases. Suppose equality occurs when the small pyramid being removed has height h. Then the volume of the pyramid is $\frac{2}{3}h^3$, and the volume of the slice is $\frac{2}{3}\{(\sqrt{3}-1-h)^3 - (\frac{1}{2}\sqrt{3}-\frac{1}{2})^3\}$. Applying Cardano's formula for the solution of a cubic equation, we find

$$h=\tfrac{1}{2}\sqrt{3}-\tfrac{1}{2} - \frac{1}{2^{\frac{4}{3}}}\{(-5+3\sqrt{3} + \sqrt{884-510\sqrt{3}})^{\frac{1}{3}} + (-5+3\sqrt{3} - \sqrt{884-510\sqrt{3}})^{\frac{1}{3}}\}$$

$$=0.305570\ldots,$$

With this choice of h, the volume removed is $0.019021\ldots$, and so the volume of the 2-cover is $0.748927\ldots$.

(ii) Now, consider the case of a 2-cover formed by truncation of edges. There are six pairs of opposite edges on the truncated octahedron. Looking at them, we see that the pairs fall into two classes.



For example, the edges A_2A_3 and C_1C_1' are opposite. Edge A_2A_3 is a long edge from the original octahedron, whereas C_1C_1' has been shortened at each end. So we shall refer to C_1C_1' as a short edge. On the other hand, the opposite edges A_2B_2' and A_1B_1 are of equal length, being shortened at only one end. We shall refer to these as medium length edges. On inspection, it is evident that any pair of opposite edges is of one of these two types. It appears that a way of obtaining a fairly good 2-cover may be to remove one of a pair of opposite edges. If we remove a pair of unequal-length opposite edges, we obtain only a small improvement in the volume of the set with the short edge removed unless we make careful allowance of the position of the cutting planes. If we remove a pair of equal-length opposite edges, i.e. medium-length edges, one at a time, by parallel cuts tangent to the inscribed sphere as shown, we obtain a 2-cover with volume

$$(142 - 64\sqrt{3} + 7\sqrt{6})/64$$
$$= 0.754612\ldots ,$$

which is larger than the volume of our previous 2-cover. However, each set of this 2-cover is obtainable from the other by a reflection, and so this gives us a lower value for the volume of a 1-cover if we allow reflection to occur in our covering process.

So we find that the first 2-cover has the lesser volume, $V_3(U_2)$ say, where

$$V_3(U_2) = 0.748927\ldots,$$

which compares with the truncated octahedron covering body which has

$$V_3(U_1) = 0.767941\ldots .$$

We note that further truncations could be applied to the sets of the 2-covers in a manner similar to either of those described above to obtain covers of higher order. In Chapter IV, we shall see a 4-cover obtained from the truncated octahedron, essentially obtained by a repetition of the method used to obtain the first 2-cover from the regular octahedron.

2.3 Formal results on k-covers.

In this section we deduce some results for k-covers in E^n where k and n can take any positive integral values.

We recall that $V_{n,k}$, $A_{n,k}$ and $d_{n,k}$ are the infima of the volume, surface area and diameter respectively taken over all k-covers in E^n. It immediately follows that $V_{n,k}$, $A_{n,k}$ and $d_{n,k}$ are decreasing sequences in k, since we can adjoin the empty set to a k-cover to form a (k+1)-cover.

So $V_{n,k} \geqslant V_{n,k+1}$, $A_{n,k} \geqslant A_{n,k+1}$ and $d_{n,k} \geqslant d_{n,k+1}$ for all positive integers n and k. We also note that a k-cover in E^{n+1} can be formed from a k-cover in E^n by taking each set of the cover in E^{n+1} to be a right prism of height 1 with cross-section congruent to the corresponding set of the k-cover in E^n. This derived k-cover in E^{n+1} has (n+1)-dimensional volume equal to the n-dimensional volume of the original k-cover in E^n. This shows that $V_{n,k}$ is a decreasing sequence in n,

i.e. $V_{n,k} \geqslant V_{n+1,k}$ for all $n,k \in Z^+$.

Even if the analogous results for surface area and diameter are true, they can not be established by this method. The obvious method of projecting the sets of a k-cover in E^{n+1} onto a hyperplane does not appear to resolve the issue or to prove any similar results.

Clearly, for all positive integers n and k, each of $V_{n,k}$, $A_{n,k}$ and $d_{n,k}$ is bounded below by zero. Hence, since for a fixed n, the sequences $\{V_{n,k}\}_{k=1,2,\ldots}$, $\{A_{n,k}\}_{k=1,2,\ldots}$ and $\{d_{n,k}\}_{k=1,2,\ldots}$ are decreasing and bounded below,

$$\lim_{k\to\infty} V_{n,k}, \quad \lim_{k\to\infty} A_{n,k} \quad \text{and} \quad \lim_{k\to\infty} d_{n,k}$$

all exist for each positive integer n.

In fact we can produce better lower bounds depending on n for these quantities. All sets in E^n of diameter $\leqslant 1$ can be covered by a set of constant width 1. But of all sets of constant width 1, the sphere has the largest volume. Hence, for any given n and $k \in Z^+$, the value of $V_{n,k}$ is bounded below by the volume of the n-dimensional sphere of diameter 1, i.e. $V_{n,k} \geq (\frac{1}{2})^n \frac{\pi^{\frac{n}{2}}}{\Gamma(1+\frac{1}{2}n)}$ for all $n,k \in Z^+$.

The corollary to the next theorem will tell us that this is the best possible such bound. In the case of surface area, the problem is not quite so simple. For the case $n=2$, surface area becomes perimeter and Barbier's theorem tells us that all curves of constant width 1 have perimeter π. In higher dimensions, there is no analogous result. However, let

$$S_n = \sup A_n(X)$$

where the supremum is taken over all closed convex sets X in E^n of constant width 1. Then, of course,

$$A_{n,k} \geq S_n \geq \text{surface area of the sphere of diameter 1},$$

i.e.

$$A_{n,k} \geq S_n \geq (\tfrac{1}{2})^{n-1} \frac{n\pi^{\frac{n}{2}}}{\Gamma(1+\frac{1}{2}n)} \quad \text{for all } n,k \in Z^+.$$

Again, we shall see that S_n is the best such bound. Clearly, we have

$$d_{n,k} \geq 1 \quad \text{for all } n,k \in Z^+,$$

and this is the best such bound as seen from the following theorem. The theorem and its corollary answer most of the questions posed as Question 2.

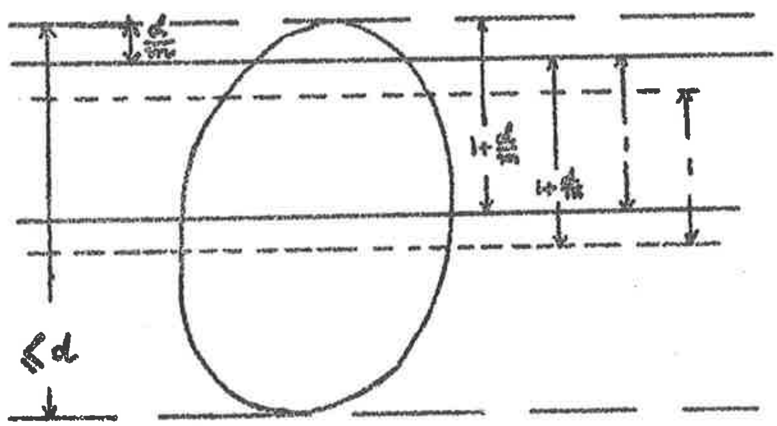
<u>Theorem 2.1</u>

$$\lim_{k\to\infty} d_{n,k} = 1 \quad \text{for all integers } n \geq 2.$$

<u>Proof</u>: The proof is in two parts. Suppose n is a fixed integer ≥ 2, throughout the proof, and suppose m is a

positive integer. Suppose U_1 is a covering body in E^n of diameter d (e.g. Jung's sphere).

(a) Choose any fixed direction, and cut closed strips of width $1 + \frac{d}{m}$ perpendicular to the given direction in such a way that each strip intersects the adjacent strip in a strip of width 1. Then each set of diameter $\leqslant 1$ can be covered by the intersection of the covering body with at least one of the strips of width $1 + \frac{d}{m}$.



Now suppose we have r strips of width $1 + \frac{d}{m}$ overlapping in this manner and covering our covering body. If we wish to guarantee that the body is covered and that we do not have an unnecessarily large number of strips, we have the inequalities

$$d + \frac{d}{m} > r\frac{d}{m} + 1 \geqslant d,$$

and hence $\quad \frac{m(d-1)}{d} \leqslant r < m+1 - \frac{m}{d},$

and so, since r is an integer

$$\frac{m(d-1)}{d} \leqslant r \leqslant m.$$

So for any given direction, the covering body can be divided into at most m overlapping pieces of width at most $1 + \frac{d}{m}$ in that direction, in such a manner that each set of diameter ≤ 1 can be covered by at least one of the pieces.

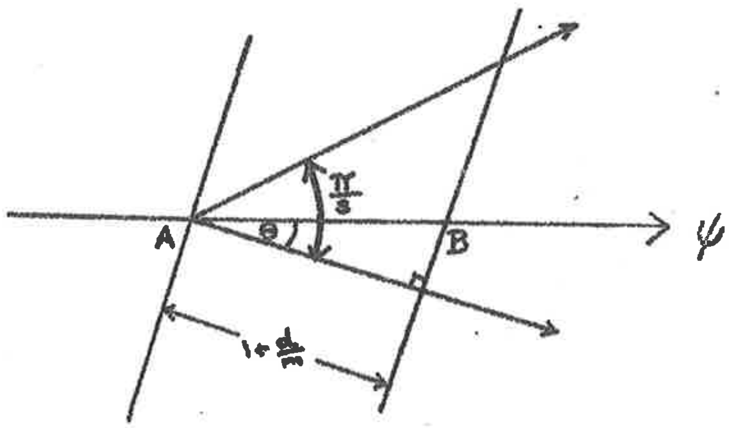
Clearly the above argument can be applied successively in many directions, since the fact that we started with a covering body was used only to prove that each set of diameter ≤ 1 could be covered by at least one of the sets produced. If we apply the argument in s different directions, we obtain a universal cover of order at most m^s.

(b) We shall use these results for a proof of our theorem. I shall give a specific argument in two dimensions and a general proof for n dimensions. The first proof is interesting since it can be used to obtain an upper bound on $d_{2,k}$ for any positive integer k, whereas the general proof does not give this extra information.

(i) Suppose $n=2$, and suppose s is an integer ≥ 2. Choose s different directions equally spaced at angular intervals of $\frac{\pi}{s}$, and apply repetitively to U_1 the process described above to obtain a universal cover of order at most m^s. Without loss of generality, assume the order is m^s, since we can adjoin sets of diameter ≤ 1 to the cover without increasing its diameter. (In fact, no sets need be adjoined if our original covering body is a set of constant width, for example the Jung's sphere, since then the order is m^s anyway.) So suppose our cover is

$$U_{m^s} = \{C_1, C_2, \ldots\ldots, C_{m^s}\}.$$

Suppose ψ describes an arbitrary direction.



Then there is at least one of the directions used in the construction of our cover which is at an angle of magnitude $\theta \leq \frac{\pi}{2s}$ to ψ. Hence the width of each set C_i in the direction ψ is at most

$$AB = (1 + \frac{d}{m}) \sec \theta$$

where $0 \leq \theta \leq \frac{\pi}{2s} \leq \frac{\pi}{4}$. Since sec x is an increasing function on $[0, \frac{\pi}{2})$, we have

width of each set C_i in the direction ψ

$$\leq (1 + \frac{d}{m}) \sec \frac{\pi}{2s} .$$

Since ψ is an arbitrary direction, and since the diameter of each C_i is equal to its maximum width, we have

$$d_2(U_{m^s}) \leq (1 + \frac{d}{m}) \sec \frac{\pi}{2s} .$$

But d_{2,m^s} is the infimum of the diameters of all m^s-covers in the plane, and so

$$1 \leqslant d_{2,m^s} \leqslant (1 + \frac{d}{m}) \sec \frac{\pi}{2s}.$$

This is true for all positive integers m, s with $s \geqslant 2$. Since $(1 + \frac{d}{m}) \sec \frac{\pi}{2s} \to 1$ as $m \to \infty$ and $s \to \infty$, and since $d_{2,k}$ is a decreasing sequence in k, we deduce that $\lim_{k \to \infty} d_{2,k} = 1$.

(ii) In the case of general n, my proof is not so explicit. It is well known that there is a countable subset of points on the surface of the n-dimensional unit sphere which is dense in the surface of the sphere. Hence there must exist a function $f_n : Z^+ \to R$ satisfying the following conditions:

(a) for any fixed integer $s \geqslant n$, there is a set of s directions in E^n such that for any arbitrary direction ψ in E^n, one of these directions is at an angle of magnitude $\leqslant f_n(s)$ to ψ, where $f_n(s) < \frac{\pi}{2}$,

and (b) $f_n(s) \to 0$ as $s \to \infty$.

Hence, by a similar argument to before, we have

$$1 \leqslant d_{n,m^s} \leqslant (1 + \frac{d}{m}) \sec(f_n(s)) \quad \text{for all integers } s \geqslant n.$$

Since $d_{n,k}$ is a decreasing sequence in k, by allowing m and s to tend to infinity, we deduce $\lim_{k \to \infty} d_{n,k} = 1$.

<u>Corollary</u>

For any integer $n \geq 2$,

$\lim_{k\to\infty} V_{n,k}$ = volume of an n-dimensional sphere of diameter 1

$$\left(= (\tfrac{1}{2})^n \frac{\pi^{\frac{n}{2}}}{\Gamma(1 + n/2)}\right),$$

and $\lim_{k\to\infty} A_{n,k} = S_n$.

<u>Proof</u>: Suppose n is a fixed positive integer. Denote the volume of an n-dimensional sphere of diameter d by $g_n(d)$. We know that

$$g_n(1) \leq V_{n,k} \qquad \text{for all } k \in Z^+,$$

and that $g_n(d) = d^n g_n(1)$ for all $d \geq 0$.

Suppose ε is an arbitrary positive number, and suppose ε_0 is a positive number which will be determined later. Then, by the theorem, there exists an integer k_0 such that

$$1 \leq d_{n,k} < 1 + \varepsilon_0 \quad \text{whenever} \quad k > k_0.$$

Hence for all $k > k_0$, each set of some k-cover in E^n can be covered by a set of constant width $1 + \varepsilon_0$. Since of all sets of constant width 1, the sphere has greatest volume it follows that

$$g_n(1) \leq V_{n,k} \leq g_n(1+\varepsilon_0) \qquad \text{for all } k > k_0,$$

i.e. $g_n(1) \leq V_{n,k} \leq (1+\varepsilon_0)^n g_n(1)$ for all $k > k_0$.

Hence $0 \leq V_{n,k} - g_n(1) \leq \{(1+\varepsilon_0)^n - 1\} g_n(1)$ for all $k > k_0$.

Since $g_n(1)$ is finite, we can choose an $\varepsilon_0 > 0$ such that

$$\{(1+\varepsilon_0)^n - 1\}\, g_n(1) < \varepsilon.$$

Therefore, with this choice for ε_0,

$$0 \leq V_{n,k} - g_n(1) < \varepsilon \quad \text{for all} \quad k > k_0,$$

and so $\lim_{k\to\infty} V_{n,k} = g_n(1)$

= volume of an n-dimensional sphere of diameter 1.

The result on the surface area $A_{n,k}$ is proved similarly. This requires the use of the result that surface area is an increasing function,
i.e. if B and C are bounded closed convex sets in E^n with $B \subset C$, then $A_n(B) \leq A_n(C)$.
(For a proof of this result, see Eggleston [6].)

In the simple cases, the corollary says that

$$\lim_{k\to\infty} V_{2,k} = \lim_{k\to\infty} a_k = \frac{\pi}{4},$$

$$\lim_{k\to\infty} A_{2,k} = \lim_{k\to\infty} p_k = \pi,$$

and

$$\lim_{k\to\infty} V_{3,k} = \lim_{k\to\infty} V_k = \frac{\pi}{6}.$$

The theorem tells us that $d_{n,k}$, and hence the diameter of a k-cover in E^n, can be made arbitrarily close to 1 by allowing k to be a sufficiently large integer. We posed the question in Question 2 whether there exists a finite integer k for which $d_{n,k} = 1$. At this stage we can deduce a part-answer to this question, namely that for n=2 or 3 and for any positive integer k, all k-covers U_k in E^n have $d_n(U_k) > 1$; later, we shall see that in fact $d_{n,k} > 1$. I also conjecture that this result is true for all integral $n \geqslant 2$.

Now, each set of constant width 1 in E^n must be covered by at least one of the sets of a k-cover in E^n. But sets of constant width are complete in the sense that if X is a closed convex set of constant width and $\underset{\sim}{x} \notin X$, then $\text{diam}(\{\underset{\sim}{x}\} \cup X) > \text{diam } X$. (For proof of this, see Eggleston [6], p 123.) Hence, a universal cover U_k of order k in E^n, where k is possibly an infinite cardinal number, has diameter 1 if and only if there exists a subset T of U_k which is a set of representatives of all sets of constant width 1 in the following sense:

$d_n(U_k) = 1$

<=> There exists a subset T of U_k such that

(i) if S is any closed convex set of constant width 1 in E^n, then there exists a set $W \in U_k$ such that S is directly congruent to W,

(ii) no two sets of T are directly congruent,

and (iii) every set of T is a closed convex set of constant width 1.

So, if $d_n(U_k) = 1$, it follows that k is not less than the cardinality of a set of representatives of the closed convex sets of width 1 in E^n. Since there is a regular Reuleaux r-gon of constant width for every odd integer r, it follows that the cardinality of the sets of representatives for the two dimensional case is infinite. In fact, the cardinality is at least that of the real number system, since in drawing a Reuleaux pentagon of constant width 1 with two given points distance 1 apart as vertices, two other vertices can be chosen as arbitrary points on arcs. In the three dimensional case, the cardinality is infinite since each regular Reuleaux r-gon of width 1 for odd values of r can be rotated about an axis of symmetry to give a set of constant width 1. In higher dimensions, I know of no proof that the cardinality is infinite, although I conjecture that it is. My attempts to prove the result by an inductive procedure have been unsuccessful. However, in low dimensions we can conclude the following:

Theorem 2.2

Suppose $n=2$ or 3, and that U_k is a k-cover in E^n with $d_n(U_k) = 1$. Then k is infinite.

In fact, for the case $n=2$, the conditions of the theorem imply that $k \geqslant c$, the cardinality of the real number system. I conjecture that this is true for any integer $n \geqslant 2$.

The theorem immediately gives the following corollaries. Each holds for any integer $n \geqslant 2$ for which the theorem is true.

Corollary 1

Suppose k is an integer $\geqslant 2$ and U_k is a k-cover in E^n where $n=2$ or 3. Then $d_n(U_k) > 1$.

Proof: Since $d_{n,k} \geqslant 1$ for all positive integers n and k, this is just the contrapositive of the theorem.

Corollary 2.

Suppose k is an integer $\geqslant 2$ and U_k is a k-cover in E^n where $n=2$ or 3. Then $A_n(U_k) > S_n$. In particular, for the case $n=2$,

$$p(U_k) > \pi.$$

Proof: Suppose k,n,U_k are as in the statement of the corollary. Then by the previous corollary, there is a set in U_k, C_1 say, and a closed convex set K of constant width 1 such that $d_n(C_1) > 1$ and C_1 covers K. Since $d_n(C_1) > 1$, there exists a point $\underset{\sim}{x} \in C_1 \sim K$, and hence

$$A_n(C_1) \geqslant A_n(K \cup \{\underset{\sim}{x}\}) > A_n(K) \geqslant S_n,$$

because if X and Y are bounded closed convex sets in E^n and $X \subset Y$, then either $X=Y$ or $A_n(X) = A_n(Y) = 0$ or $A_n(X) < A_n(Y)$. (For proof of this, see Eggleston [6], p 89.) So $A_n(C_1) > S_n$ and so $A_n(U_k) > S_n$.

I do not know if there is a finite k-cover in E^n $(n \geqslant 2)$ with $V_n(U_k)$ = volume of the n-dimensional sphere of diameter 1. Since all other closed convex sets of constant width 1 have volume less than that of the sphere, it may be possible that such a finite k-cover exists, although I expect otherwise.

CHAPTER III

EXISTENCE OF AN OPTIMAL k-COVER

Thus far we have considered some results concerning the values of $V_{n,k}$, $A_{n,k}$ and $d_{n,k}$ without knowing whether for each measure there is a universal cover which attains the optimal value. In this chapter, we shall prove that there is, thus answering the first part of Question 3. We now state the main result of the chapter.

Theorem 3.1

Given any positive integers n and k, and a measure (volume, surface area or diameter), then there exists at least one k-cover U_k in E^n which is optimal with respect to that measure; i.e. if the measure is volume, $V_n(U_k) = V_{n,k}$,
if the measure is surface area, $A_n(U_k) = A_{n,k}$,
and if the measure is diameter, $d_n(U_k) = d_{n,k}$.

The proof of this theorem is similar to that of Pál [19] where he proved the existence of a tile optimal with respect to area. However, several results have been proved here which Pál considered obvious. I feel that when these results are generalized to higher dimensions they can not be taken as trivial. The proof given is a proof of the existence of a k-cover in E^n with volume $V_{n,k}$, but we indicate how to overcome any extra difficulties in the proofs for the case of surface area or diameter. As with the Pál proof, this proof depends on the Blaschke Selection Theorem, and so we must first define a metric on the set of closed, bounded subsets of E^n.

Defn.

The ~~special~~ spherical neighbourhood of a point $\underset{\sim}{x}$ in E^n with radius $\varepsilon > 0$ is defined to be the set

$$N(\underset{\sim}{x},\varepsilon) = \{\underset{\sim}{y} \in E^n \mid \|\underset{\sim}{x}-\underset{\sim}{y}\| < \varepsilon\},$$

where $\|\underset{\sim}{z}\|$ denotes the Euclidean norm of $\underset{\sim}{z}$. The spherical neighbourhood of a set K in E^n with radius $\varepsilon > 0$ is defined to be

$$N(K,\varepsilon) = \bigcup_{\underset{\sim}{x} \in K} N(\underset{\sim}{x},\varepsilon).$$

From this we can define our distance function.

Defn.

Suppose K_1, K_2 are two closed bounded sets in E^n. Then the distance between K_1 and K_2 is defined to be

$$D(K_1,K_2) = \inf\{\varepsilon > 0 \mid K_1 \subset N(K_2,\varepsilon) \text{ and } K_2 \subset N(K_1,\varepsilon)\}.$$

The function D can be shown to be a metric:

i.e. If A,B,C are any closed bounded sets in E^n, then

(i) $D(A,B) \geqslant 0$ where equality holds if and only if $A=B$,

(ii) $D(A,B) = D(B,A)$,

and (iii) $D(A,B) \leqslant D(A,C) + D(C,B)$.

With this definition of distance between sets, we can define convergence of a sequence of sets in the following obvious manner.

Defn.

Suppose $\{K_i\}$ is a sequence of closed bounded subsets of E^n. If there is a closed bounded subset K of E^n such that $\lim_{i\to\infty} D(K_i, K) = 0$, then we say that the sequence $\{K_i\}$ converges to the limit set K, and write $\lim_{i\to\infty} K_i = K$.

We now state:

Blasche Selection Theorem.

Suppose $\mathcal{M}$ is an infinite collection of uniformly bounded nonempty closed convex subsets of E^n. Then $\mathcal{M}$ contains a sequence which converges to a nonempty bounded closed convex set.

(For a proof of this theorem, see a general book on convexity, e.g. Eggleston [6], Benson [2] or Valentine [23].)

In addition to the Blasche Selection Theorem, we need some lemmas. We know that the cube of side 1 is a covering body, so any good cover has volume $\leqslant 1$. Since each useful set of a k-cover covers at least one set of constant width 1, the first lemma says in effect that the sets of a good k-cover can not be long and thin.

Lemma 3.1.

Suppose n and k are positive integers, and $U_k = \{C_1, \ldots, C_k\}$ is a k-cover in E^n such that

(i) $\underset{\sim}{0} \in C_i$ for each $i=1,2,\ldots,k$,

(ii) $V_n(U_k) \leqslant 1$,

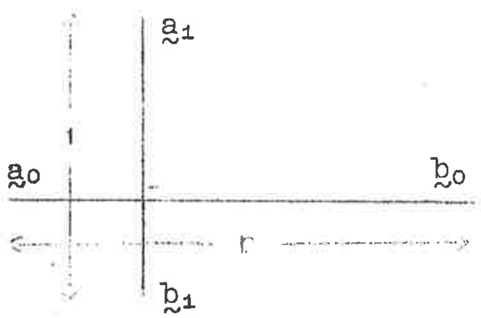
and (iii) for each $i=1,2,\ldots,k$, there is a set of constant width 1 covered by C_i.

Then each C_i $(1 \leqslant i \leqslant k)$ is a subset of the closed sphere with centre $\underset{\sim}{0}$ and radius n!

<u>Proof</u>: Suppose $n,k,U_k,C_1,\ldots,C_k$ satisfy the conditions of the lemma. Suppose i is an integer such that $1 \leqslant i \leqslant k$, and let

$$r = \sup_{\underset{\sim}{x},\underset{\sim}{y}\in C_i} \|\underset{\sim}{x}-\underset{\sim}{y}\| \quad (= \text{diam } C_i),$$

where $\|\underset{\sim}{z}\|$ denotes the Euclidean norm of $\underset{\sim}{z}$. Since C_i is bounded and closed, there exists at least one pair of points $\underset{\sim}{a}_0, \underset{\sim}{b}_0 \in C_i$ such that $\|\underset{\sim}{a}_0-\underset{\sim}{b}_0\| = r$. But we are given that C_i covers at least one set of constant width 1. Hence, since C_i is closed, there exist two points $\underset{\sim}{a}_1, \underset{\sim}{b}_1 \in C_i$ such that $\|\underset{\sim}{a}_1-\underset{\sim}{b}_1\| = 1$ and $\underset{\sim}{a}_1-\underset{\sim}{b}_1 \perp \underset{\sim}{a}_0-\underset{\sim}{b}_0$.



By induction, we obtain a sequence of points $\underset{\sim}{a}_j$ and $\underset{\sim}{b}_j \in C_i$ $(j=1,2,\ldots,n-1)$ such that $\|\underset{\sim}{a}_j-\underset{\sim}{b}_j\| = 1$ for each $j=1,2,\ldots,n-1$, and all the $\underset{\sim}{a}_j-\underset{\sim}{b}_j$ for $j=0,1,2,\ldots,n-1$ are mutually perpendicular. Then

$$\text{conv}\{\underset{\sim}{a}_0,\underset{\sim}{b}_0,\underset{\sim}{a}_1,\underset{\sim}{b}_1,\underset{\sim}{a}_2,\underset{\sim}{b}_2,\ldots,\underset{\sim}{a}_{n-1},\underset{\sim}{b}_{n-1}\} \subset C_i,$$

where conv denotes convex hull. But the volume of this convex hull is at least the volume of a generalized octahedron with perpendicular diagonals of the same length as the $\underset{\sim}{a}_j - \underset{\sim}{b}_j$ $(0 \leqslant j \leqslant n-1)$. Now the volume of the generalized octahedron

$$\{(x_1, \ldots, x_n) \mid |x_1| + \ldots + |x_n| \leqslant k\}$$

is $2^n k^n/n!$, and so the volume of the octahedron

$$\{(x_1, \ldots, x_n) \mid \tfrac{1}{r}|x_1| + (|x_2| + \ldots + |x_n|) \leqslant \tfrac{1}{2}\}$$

is $r/n!$. Hence

$$V_n(C_1) \geqslant V_n(\text{conv}\{\underset{\sim}{a}_0, \underset{\sim}{b}_0, \underset{\sim}{a}_1, \underset{\sim}{b}_1, \ldots, \underset{\sim}{a}_{n-1}, \underset{\sim}{b}_{n-1}\})$$

$$\geqslant r/n!$$

Since we are given $V_n(U_k) \leqslant 1$, we have $r/n! \leqslant 1$, and so $r \leqslant n!$ Because $\underset{\sim}{0} \in C_1$, it follows from the definition of r that

$$\|\underset{\sim}{x} - \underset{\sim}{0}\| \leqslant r \leqslant n! \quad \text{for all} \quad \underset{\sim}{x} \in C_1,$$

and hence C_1 is a subset of the closed sphere with centre $\underset{\sim}{0}$ and radius $n!$ This completes the proof of the lemma.

The importance of this lemma is that it shows that the significant sets of any k-cover in E^n which is a good cover with respect to volume can be placed inside a sphere with centre $\underset{\sim}{0}$ and radius depending only on n. In the case where we wish to optimize with respect to diameter, there is no problem since the existence of Jung's sphere tells us that there are k-covers of diameter at most $\left(\frac{n}{2n+2}\right)^{\frac{1}{2}}$. So the sets of all k-covers in E^n with diameter at most this value, and such that each set of k-cover contains the origin, are subsets of the closed sphere with centre $\underset{\sim}{O}$ and radius $\left(\frac{n}{2n+2}\right)^{\frac{1}{2}}$.

If we wish to optimize with respect to surface area, we know that the n-dimensional cube of side-length 1 has surface area $2n$, and so the following lemma is sufficient.

<u>Lemma 3.1 A</u>

Suppose n and k are positive integers, and $U_k = \{C_1, \dots, C_k\}$ is a k-cover in E^n such that

(i) $\underset{\sim}{O} \in C_i$ for each $i=1,2,\dots,k$,

(ii) $A_n(U_k) \leqslant 2n$

and (iii) for each integer i with $1 \leqslant i \leqslant k$, there is a set of constant width 1 covered by C_i.

Then each C_i is a subset of the closed sphere with centre $\underset{\sim}{O}$ and radius n!

<u>Proof</u>: Suppose $n, k, U_k, C_1, \dots, C_k$ satisfy the conditions of the lemma. Suppose i is an integer with $1 \leqslant i \leqslant k$, and let
$$r = \sup_{\underset{\sim}{x},\underset{\sim}{y} \in C_i} \|\underset{\sim}{x}-\underset{\sim}{y}\| \qquad (=\text{diam } C_i).$$

Since C_i is bounded and closed, there exist two points $\underset{\sim}{a}_0, \underset{\sim}{b}_0 \in C_i$ such that $\|\underset{\sim}{a}_0-\underset{\sim}{b}_0\| = r$. Now we project C_i onto a hyperplane parallel to the line segment $[\underset{\sim}{a}_0, \underset{\sim}{b}_0]$ by a perpendicular projection. Denote the image of C_i under this projection by P_i. Then
$$V_{n-1}(P_i) \leqslant \tfrac{1}{2} A_n(C_i) \leqslant n.$$
However, since C_i covers a set of constant width 1, P_i covers an $(n-1)$-dimensional set of constant width 1.

Since the projection was a perpendicular projection onto a hyperplane parallel to $[\underset{\sim}{a}_0, \underset{\sim}{b}_0]$, the diameter of P_1 is also r. Hence, by the argument in the proof of Lemma 3.1,

we have $$\frac{r}{(n-1)!} \leqslant V_{n-1}(P_1),$$

and so $$\frac{r}{(n-1)!} \leqslant n,$$

$$\text{i.e.} \quad r \leqslant n!$$

Therefore, by the definition of r,

$$\|\underset{\sim}{x}-\underset{\sim}{0}\| \leqslant n! \quad \text{for all} \quad \underset{\sim}{x} \in C_1,$$

i.e. C_1 is a subset of the closed sphere with centre $\underset{\sim}{0}$ and radius $n!$

The remaining lemmas concern sequences of sets converging in the sense defined earlier. The next lemma has an unavoidably long and somewhat tedious proof. However the result is essential for the proof of our main theorem.

<u>Lemma 3.2</u>

Suppose S is a simplex in E^n, and suppose $\{S_i\}$ is an infinite sequence of subsets S_i of E^n such that each S_i is obtained by a displacement of S, and all the S_i are subsets of a bounded region. Then there is a subsequence $\{S_{i_r}\}$ which converges to a limit set S_0 which is obtainable by a displacement of S.

<u>Proof</u>: Suppose S and $\{S_i\}$ are as in the statement of the lemma. Since the S_i are uniformly bounded, nonempty

closed convex subsets of E^n, we can apply the Blasche Selection Theorem. So we deduce that there is a subsequence $\{S_{i_r}\}$ of $\{S_i\}$ and a nonempty closed bounded convex set S_0 such that the subsequence $\{S_{i_r}\}$ converges to S_0. It remains to prove that S_0 is obtainable from S by a displacement. Since we are now interested only in S_0 we shall assume without loss of generality that the subsequence $\{S_{i_r}\}$ is just the original sequence, i.e. $\lim\limits_{i\to\infty} S_i = S_0$.

We now show that some further assumptions can be placed on the sequence. Without loss of generality, we can assume that $\underset{\sim}{0}$ is a vertex of S. So, suppose the vertices of S are

$$\underset{\sim}{a}_1, \underset{\sim}{a}_2, \dots, \underset{\sim}{a}_n, \underset{\sim}{a}_{n+1} = \underset{\sim}{0},$$

and the corresponding vertices of S_i are

$$\underset{\sim}{a}_{i1}, \underset{\sim}{a}_{i2}, \dots, \underset{\sim}{a}_{in}, \underset{\sim}{a}_{in+1}.$$

Consider the set of points $\{\underset{\sim}{a}_{i1} \mid i \in Z^+\}$. This is an infinite set of points contained in a bounded region, and so there is an infinite subsequence $\{S_{i_k}\}$ of our sequence $\{S_i\}$ such that $\underset{\sim}{a}_{i_k 1}$ converges to a limit, $\underset{\sim}{a}_1'$ say, as $k \to \infty$. Without loss of generality, since we are interested in the properties of S_0, we can assume that this subsequence is just the original sequence $\{S_i\}$. Proceeding inductively we can finally assume that $\lim\limits_{i\to\infty} S_i = S_0$, and that $\lim\limits_{i\to\infty} \underset{\sim}{a}_{ij}$ exists and is equal to $\underset{\sim}{a}_j'$ (say) for each

$j = 1,2,\ldots,n+1$,

$$\text{i.e. } \lim_{i\to\infty} \underset{\sim}{a}_{ij} = \underset{\sim}{a}'_j \quad \text{for} \quad j = 1,2,\ldots,n+1.$$

Without loss of generality, we can make the further assumption that $\underset{\sim}{a}'_{n+1} = \underset{\sim}{0}$. Since S_0 is closed, it follows that $\underset{\sim}{a}'_1, \underset{\sim}{a}'_2, \ldots, \underset{\sim}{a}'_{n+1} \in S_0$.

Since each simplex S_i is obtained by a displacement of S, then for each positive integer i there exist an orthogonal matrix A_i of order n with determinant 1 and a vector $\underset{\sim}{b}_i$ such that

$$\underset{\sim}{a}_{ij} = A_i \underset{\sim}{a}_j + \underset{\sim}{b}_i \quad \text{for} \quad j = 1,2,\ldots,n+1.$$

Also, since the $n+1$ vertices of a simplex are in general position, and since $\underset{\sim}{a}_{n+1} = \underset{\sim}{a}'_{n+1} = \underset{\sim}{0}$, there exists a unique $n\times n$ matrix A such that

$$\underset{\sim}{a}'_j = A\underset{\sim}{a}_j \quad \text{for} \quad j = 1,2,\ldots,n+1.$$

To complete the proof of the lemma we must show:

(i) A is the matrix of a displacement, i.e. A is orthogonal with determinant 1

(ii) $S_0 = \text{conv}\{\underset{\sim}{a}'_1, \underset{\sim}{a}'_2, \ldots, \underset{\sim}{a}'_{n+1}\}$.

(i) Since $\underset{\sim}{a}_{n+1} = \underset{\sim}{0}$, it follows immediately that

$\underset{\sim}{b}_i = \underset{\sim}{a}_{i\,n+1}$ for all $i \in Z^+$. But

$\lim\limits_{i\to\infty} \underset{\sim}{a}_{i\,n+1} = \underset{\sim}{a}'_{n+1} = \underset{\sim}{0}$, and so $\lim\limits_{i\to\infty} \underset{\sim}{b}_i = \underset{\sim}{0}$.

Now $\lim\limits_{i\to\infty} \underset{\sim}{a}_{i\,j} = \underset{\sim}{a}'_j$ for $j = 1,2,\ldots,n+1$. Hence, since $\underset{\sim}{a}_{i\,j} = A_i\underset{\sim}{a}_j + \underset{\sim}{b}_i$ and $\underset{\sim}{a}'_j = A\underset{\sim}{a}_j$, we have

$$\lim_{i\to\infty} (A_i\underset{\sim}{a}_j + \underset{\sim}{b}_i) = A\underset{\sim}{a}_j,$$

and since $\lim\limits_{i\to\infty} \underset{\sim}{b}_i = \underset{\sim}{0}$, it follows that $\lim\limits_{i\to\infty} A_i\underset{\sim}{a}_j = A\underset{\sim}{a}_j$

for $j = 1,2,\ldots,n+1$. Hence $\lim\limits_{i\to\infty} (A_i - A)\underset{\sim}{a}_j = \underset{\sim}{0}$ for

$j = 1,2,\ldots,n+1$. For each positive integer i, let $B_i = A_i - A$. Also, let D be the $n\times n$ matrix with $\underset{\sim}{a}_j$ as its j'th column. Then the condition above implies that each term of the matrix B_iD tends to zero as $i\to\infty$. Since $\underset{\sim}{a}_1,\ldots,\underset{\sim}{a}_n,\underset{\sim}{a}_{n+1} = \underset{\sim}{0}$ are in general position, it follows that $\underset{\sim}{a}_1,\ldots,\underset{\sim}{a}_n$ are linearly independent, and so D is a nonsingular matrix. Hence, since multiplication of two $n\times n$ matrices involves only a finite number of multiplications and summations (the number depending only on n), it follows that B_i tends to the zero matrix termwise as $i\to\infty$, i.e. $A_i\to A$ termwise as $i\to\infty$. Applying the same argument again, we deduce that

$$A_i^T A_i \to A^T A \text{ termwise as } i \to \infty$$

$$\text{and } \det A_i \to \det A \text{ as } i \to \infty.$$

But $A_i^T A_i = I$ and $\det A_i = 1$ for each $i \in Z^+$, since the A_i are orthogonal with determinant 1. Hence

$$A^T A = I \text{ and } \det A = 1,$$

i.e. A is orthogonal with determinant 1, and so A is the matrix of a displacement.

(ii) We now show that $S_0 = \text{conv}\{\underset{\sim}{a}_1', \ldots, \underset{\sim}{a}_{n+1}'\}$, which is sufficient to prove the result of the lemma. Now, since S_0 is convex and since $\underset{\sim}{a}_1', \ldots, \underset{\sim}{a}_{n+1}' \in S_0$, it immediately follows that

$$S_0 \supset \text{conv}\{\underset{\sim}{a}_1', \ldots, \underset{\sim}{a}_{n+1}'\}.$$

To show that $S_0 \subset \text{conv}\{\underset{\sim}{a}_1', \ldots, \underset{\sim}{a}_{n+1}'\}$, suppose $\underset{\sim}{x}$ is a point of E^n not in $\text{conv}\{\underset{\sim}{a}_1', \ldots, \underset{\sim}{a}_{n+1}'\}$. Then since $\underset{\sim}{a}_1', \ldots, \underset{\sim}{a}_{n+1}'$ are in general position, $\underset{\sim}{x}$ can be uniquely expressed in the form

$$\underset{\sim}{x} = \sum_{j=1}^{n+1} \alpha_j \underset{\sim}{a}_j'$$

where the α_j are real and $\sum_{j=1}^{n+1} \alpha_j = 1$. Suppose also that $\{\underset{\sim}{x}_i\}$ is a sequence of points converging to $\underset{\sim}{x}$. Then for each $i \in Z^+$, $\underset{\sim}{x}_i$ can be expressed uniquely in the form

$$\underset{\sim}{x}_i = \sum_{j=1}^{n+1} \alpha_{ij} \underset{\sim}{a}_{ij}$$

where the α_{ij} are real and $\sum_{j=1}^{n+1} \alpha_{ij} = 1$. Then since the α's are just barycentric coordinates of the $\underset{\sim}{x}$'s with respect to the appropriate $\underset{\sim}{a}$'s, and since $\lim_{i\to\infty} \underset{\sim}{a}_{ij} = \underset{\sim}{a}'_j$ for $j=1,2,\ldots,n+1$ and $\lim_{i\to\infty} \underset{\sim}{x}_i = \underset{\sim}{x}$, it follows by the properties of barycentric coordinates that $\lim_{i\to\infty} \alpha_{ij} = \alpha_j$ for $j=1,2,\ldots,n+1$. However, since $\underset{\sim}{x} \notin \text{conv}\{\underset{\sim}{a}'_1,\ldots,\underset{\sim}{a}'_{n+1}\}$, we deduce that at least one of the α_j's is negative, and hence for that particular j, α_{ij} is negative for all sufficiently large i, and so $\underset{\sim}{x}_i \notin S_i$ for all sufficiently large i. Hence $\underset{\sim}{x}$ is not the limit of any sequence of points $\underset{\sim}{x}_i$, each in S_i. But if $\underset{\sim}{x} \in S_0$, then if μ is a positive number, then $\underset{\sim}{x} \in N(S_i,\mu)$ for all sufficiently large i, and so $\underset{\sim}{x}$ would be the limit of a sequence of points $\underset{\sim}{x}_i$ each belonging to S_i. Hence $\underset{\sim}{x} \notin S_0$, and so

$$S_0 \subset \text{conv}\{\underset{\sim}{a}'_1,\ldots,\underset{\sim}{a}'_{n+1}\}.$$

Thus $S_0 = \text{conv}\{\underset{\sim}{a}'_1,\ldots,\underset{\sim}{a}'_{n+1}\}$, and the lemma is proved.

Lemma 3.3

Suppose $\{S_i\}$ is a sequence of nonempty closed convex uniformly bounded sets in E^n that converge to a closed convex set S, and P is a flat such that $(\text{int } S) \cap P \neq \emptyset$. Then $S_i \cap P \to S \cap P$ as $i \to \infty$.

Proof: Suppose ε is an arbitrary positive number. To prove the result, we shall show that for all sufficiently large integers i,

$$D(S_i \cap P,\ S \cap P) < \varepsilon.$$

To achieve this, we prove the existence of a positive integer n_0 such that

$$S_i \cap P \subset N(S \cap P, \tfrac{\varepsilon}{2}) \quad \text{and} \quad S \cap P \subset N(S_i \cap P, \tfrac{\varepsilon}{2})$$

whenever $i > n_0$.

(i) Firstly, we show that $S_i \cap P \subset N(S \cap P, \frac{\varepsilon}{2})$ for all sufficiently large i. For suppose this is not true. Then for infinitely many positive integers i, there exist points

$$\underset{\sim}{z}_i \in (S_i \cap P) \sim N(S \cap P, \tfrac{\varepsilon}{2}).$$

Suppose $\{i_r\}$ is the sequence of such integers i. Then $\{\underset{\sim}{z}_{i_r}\}$ is an infinite sequence of points contained in a bounded region, and so it has a point of accumulation, $\underset{\sim}{z}$ say. Since $\lim_{i\to\infty} S_i = S$ and S is closed, it follows that $\underset{\sim}{z} \in S$. Now $\underset{\sim}{z}_{i_r} \in P$ for each $r \in Z^+$, and, since P is closed, we can deduce that $\underset{\sim}{z} \in P$. Therefore $\underset{\sim}{z} \in S \cap P$. Since $\underset{\sim}{z}$ is a point of accumulation of the sequence $\{\underset{\sim}{z}_{i_r}\}$, we see that

$$\underset{\sim}{z}_{i_r} \in N(S \cap P, \tfrac{\varepsilon}{2})$$

for infinitely many positive integers r. This contradicts our definition of $\underset{\sim}{z}_{i_r}$, and so our assumption is false. Hence, there exists a positive integer n_1 such that

$$S_i \cap P \subset N(S \cap P, \tfrac{\varepsilon}{2}) \quad \text{whenever} \quad i > n_1.$$

(ii) Secondly, we show that $S \cap P \subset N(S_i \cap P)$ for all sufficiently large i. For suppose this is not true. Then there exist infinitely many positive integers i_r, and points $\underset{\sim}{z}_r$, such that

$$\underset{\sim}{z}_r \in (S \cap P) \sim N(S_{i_r} \cap P, \tfrac{\varepsilon}{2}).$$

Since $\{\underset{\sim}{z}_r\}$ is an infinite sequence of points contained in the bounded set S, there exists a point of accumulation, $\underset{\sim}{z}$ say. Since $S \cap P$ is closed, $\underset{\sim}{z} \in S \cap P$. But, as S is convex and closed, and $(\text{int } S) \cap P \neq \emptyset$, it follows that $S \cap P = \overline{(\text{int } S) \cap P}$, and so there exists a point $\underset{\sim}{\omega} \in (\text{int } S) \cap P$ such that

$$\|\underset{\sim}{z} - \underset{\sim}{\omega}\| < \frac{\varepsilon}{4}\ .$$

Because $\underset{\sim}{\omega} \in (\text{int } S) \cap P$, there exist points $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_{n+1} \in S \sim P$ such that

(a) $\underset{\sim}{\omega} \in \text{int conv}\{\underset{\sim}{a}_1, \underset{\sim}{a}_2, \ldots, \underset{\sim}{a}_{n+1}\}$

and (b) $\|\underset{\sim}{a}_j - \underset{\sim}{\omega}\| < \frac{\varepsilon}{8}$ for $j = 1,2,\ldots,n+1$.

Let $\mu_j = \inf\limits_{\underset{\sim}{x} \in P} \|\underset{\sim}{a}_j - \underset{\sim}{x}\|$ for $j = 1,2,\ldots,n+1$, and let $\mu = \min\limits_{1 \leq j \leq n+1} \mu_j$. Then $0 < \mu < \frac{\varepsilon}{8}$. Now since $\lim\limits_{i \to \infty} S_i = S$, there exists a positive integer m such that

$$D(S_{i_r}, S) < \frac{\mu}{2} \text{ whenever } r > m.$$

Suppose r is an integer $> m$. Then there exist points $\underset{\sim}{a}_{r1}, \ldots, \underset{\sim}{a}_{r\,n+1} \in S_{i_r}$ such that

$$\|\underset{\sim}{a}_{rj} - \underset{\sim}{a}_j\| < \frac{\mu}{2} \text{ for } j = 1,2,\ldots,n+1.$$

Then, since S_{i_r} is convex, $\text{conv}\{\underset{\sim}{a}_{r1}, \ldots, \underset{\sim}{a}_{r\,n+1}\} \subset S_{i_r}$, and also, it follows from the definition of μ that

$$(\text{conv}\{\underset{\sim}{a}_{r1}, \ldots, \underset{\sim}{a}_{r\,n+1}\}) \cap P \neq \emptyset.$$

So suppose $y_r \in (\text{conv}\{a_{r1}, \dots, a_{r\,n+1}\}) \cup P \subset S_{i_r} \cap P$.

Then there exist nonnegative numbers $\alpha_{r1}, \dots, \alpha_{r\,n+1}$ such that

$$y_r = \sum_{j=1}^{n+1} \alpha_{rj} a_{rj} \quad \text{and} \quad \sum_{j=1}^{n+1} \alpha_{rj} = 1.$$

Then
$$\begin{aligned}
\|z_r - y_r\| &\leq \|z_r - z\| + \|z - \omega\| + \|\omega - y_r\| \\
&= \|z_r - z\| + \|z - \omega\| + \|\omega - \sum_{j=1}^{n+1} \alpha_{rj} a_{rj}\| \\
&\leq \|z_r - z\| + \|z - \omega\| + \sum_{j=1}^{n+1} \alpha_{rj} \|\omega - a_{rj}\|
\end{aligned}$$

since $\alpha_{r1}, \dots, \alpha_{r\,n+1} \geq 0$

and $\sum_{j=1}^{n+1} \alpha_{rj} = 1$

$$\begin{aligned}
&\leq \|z_r - z\| + \|z - \omega\| + \sum_{j=1}^{n+1} \alpha_{rj} \{\|\omega - a_r\| + \|a_r - a_{rj}\|\} \\
&< \frac{\mu}{2} + \frac{\varepsilon}{4} + \sum_{j=1}^{n+1} \alpha_{rj}\left(\frac{\varepsilon}{8} + \frac{\mu}{2}\right) \\
&< \frac{\varepsilon}{16} + \frac{\varepsilon}{4} + \frac{\varepsilon}{8} + \frac{\varepsilon}{16} \\
&= \frac{\varepsilon}{2}.
\end{aligned}$$

Therefore, since $y_r \in S_{i_r} \cap P$, we have

$$z_r \in N(S_{i_r} \cap P, \tfrac{\varepsilon}{2})$$

which is a contradiction to our definition of z_r. Hence our assumption is false, and so there exists a positive integer n_2 such that

$$S \cap P \subset N(S_i \cap P, \tfrac{\varepsilon}{2}) \quad \text{whenever } i > n_2.$$

(iii) Combining (i) and (ii), and letting $n_0 = \max(n_1, n_2)$, we have

$$S_i \cap P \subset N(S \cap P, \tfrac{\varepsilon}{2}) \quad \text{and} \quad S \cap P \subset N(S_i \cap P, \tfrac{\varepsilon}{2})$$

whenever $i > n_0$, and so

$$D(S_i \cap P, S \cap P) \leqslant \tfrac{\varepsilon}{2} < \varepsilon \quad \text{whenever} \quad i > n_0.$$

This completes the proof of the lemma.

Lemma 3.4

Suppose $\{S_i\}$ is a sequence of nonempty closed convex uniformly bounded sets in E^n which converges to a nonempty closed bounded convex limit set S.

Then (i) $V_n(S_i) \to V_n(S)$ as $i \to \infty$

(ii) $A_n(S_i) \to A_n(S)$ as $i \to \infty$

and (iii) $d_n(S_i) \to d_n(S)$ as $i \to \infty$.

This lemma is proved for the case of volume in three dimensional space in Benson [2], pp 142-3. His proof can be adapted for the other cases.

We are now in a position to prove our theorem, i.e. that a k-cover in E^n which is optimal with respect to a specified measure (volume, surface area or diameter) exists. The following proof holds for the case of volume; afterwards we indicate how this proof can be modified for the other measures.

Proof of theorem: Let $V_0 = \inf V_n(U)$

where the infimum is taken over all k-covers U in E^n. Then $V_0 \leqslant 1$ since the generalized cube of edge length 1 covers all sets of diameter 1. Hence, either there exists a k-cover U with $V_n(U) = V_0$, in which case there is nothing to prove, or there exists an infinite sequence $\{U_i\}$ of k-covers in E^n such that if for each $i \in Z^+$, $U_i = \{C_{i1}, C_{i2}, \ldots, C_{ik}\}$, then

(i) $V_n(U_i)$ is a decreasing sequence in i

(ii) $\lim_{i\to\infty} V_n(U_i) = V_0$

(iii) $V_n(U_i) \leqslant 1$ for all $i \in Z^+$

and (iv) for every $i \in Z^+$ and for each $j = 1,2,\ldots,k$, there exists at least one set of constant width 1 in E^n which is covered by C_{ij}.

(We note that (iv) can always be achieved by assuming that there is an integer ℓ with $1 \leqslant \ell \leqslant k$ such that $C_{i1},\ldots,C_{i\ell}$ are all distinct and $C_{i\ell}=C_{i\ell+1}=\ldots=C_{ik}$.)
By the definition of k-cover, all the sets C_{ij} are bounded, closed and convex.

There is one further assumption we can make about the sets C_{ij}. It follows from results of Steinhagen [22] and Santaló [20] that every closed convex set of constant width 1 in E^n contains a sphere with radius $R(n)$ where

$$R(n) = \begin{cases} \dfrac{(n+2)^{\frac{1}{2}}}{n+1} & \text{for even } n \\ n^{-\frac{1}{2}} & \text{for odd } n. \end{cases}$$

Suppose $\mathscr{S}$ is the closed sphere in E^n with centre $\underset{\sim}{O}$ and radius $R(n)$. Then (iv) allows us to make the extra assumption:

(v) $\mathscr{S} \subset C_{ij}$ for all $i \in Z^+$ and each $j=1,2,\ldots,k$.

Suppose i is an arbitrary positive integer. For each integer j with $1 \leqslant j \leqslant k$, let B_{ij} be the subset of E^{n+j-1} defined as

$$B_{ij} = \{(x_1,\ldots,x_{n-1},0,\ldots,0,x_{n+j-1},0,\ldots,0) \mid (x_1,\ldots,x_{n-1},x_{n+j-1}) \in C_{ij}\}.$$

Then let $B_i = \text{conv}\left(\bigcup_{j=1}^{k} B_{ij}\right)$. This defines a sequence $\{B_i\}$ of nonempty convex sets. Since each C_{ij} is closed, it follows that each B_i is closed. Also, since Lemma 3.1 tells us that the C_{ij} have a bound depending only on n, we deduce that the B_i are uniformly bounded. So, applying the Blasche Selection Theorem, we see that there is a subsequence $\{B_{i_r}\}$ of the sequence $\{B_i\}$ and a nonempty bounded convex subset B of E^{n+k-1} such that $\lim_{r\to\infty} B_{i_r} = B$. Without loss of generality, we can suppose that the subsequence $\{B_{i_r}\}$ is just the original sequence $\{B_i\}$.

Now for each integer j with $1 \leqslant j \leqslant k$, let K_j be the subspace of E^{n+k-1} with basis

$$\underset{\sim}{e}_1, \underset{\sim}{e}_2, \ldots, \underset{\sim}{e}_{n-1}, \underset{\sim}{e}_{n+j-1},$$

where $\underset{\sim}{e}_i$ is the unit vector in E^{n+k-1} with zero components except for the i'th component which is 1. Thus $K_j = \{(x_1,\ldots,x_{n-1},0,\ldots,0,x_{n+j-1},0,\ldots,0) \in E^{n+k-1}\}$. Also, for each such j, let T_j be the projection $T_j : K_j \to E^n$ defined by

$$T_j((x_1,\ldots,x_{n-1},0,\ldots,0,x_{n+j-1},0,\ldots,0)) = (x_1,\ldots,x_{n-1},x_{n+j-1}).$$

Then, let $U = \{T_1(B \cap K_1), T_2(B \cap K_2), \ldots\ldots, T_k(B \cap K_k)\}$. We shall show that U is a k-cover, optimal with respect to volume.

Since B is bounded, closed and convex, it is clear that the sets of U are bounded, closed and convex. Hence we must show that U has the required covering property and that $V_n(U) = V_0$.

However, we shall first show that $\lim_{i\to\infty} C_{ij}$ exists and equals $T_j(B \cap K_j)$ for $j = 1,2,\ldots,k$. We recall that $\lim_{i\to\infty} B_i = B$, and that K_j is a flat through $\underset{\sim}{0}$ for each integer j with $1 \leqslant j \leqslant k$. By the choice of our sequence, $\mathscr{S} \subset B$, and so $\underset{\sim}{0} \in (\text{int } B) \cap K_j$ for $j = 1,2,\ldots\ldots,k$. Hence, by Lemma 3.3, $\lim_{i\to\infty} (B_i \cap K_j)$ exists and equals $B \cap K_j$ for $j = 1,2,\ldots,k$. Since T_j is the projection of K_j onto E^n, it follows that

$$\lim_{i\to\infty} T_j(B_i \cap K_j) = T_j(B \cap K_j) \quad \text{for} \quad j = 1,2,\ldots\ldots,k,$$

$$\text{i.e.} \quad \lim_{i\to\infty} C_{ij} = T_j(B \cap K_j) \quad \text{for} \quad j = 1,2,\ldots,k,$$

since $T_j(B_i \cap K_j) = C_{ij}$ for all positive integers i and for $j = 1,2,\ldots,k$.

Now, the covering property. Since any set of diameter $\leqslant 1$ can be enclosed in a closed convex set of constant width 1, suppose X is an arbitrary closed convex subset of E^n of constant width 1. Without loss of generality, suppose also that the origin is an interior point of X. Then by the infinite pigeon hole principle, there exists an integer j for which X is covered by C_{ij} for infinitely many positive integers i. Without loss of generality, suppose $j=1$. Since we are then aiming to prove that X is covered by $T_1(B \cap K_1)$, and since B is the limit of any infinite subsequence of $\{B_i\}$, we shall assume that X is covered by C_{i1} for all $i \in Z^+$. Then for each $i \in Z^+$, there is a displacement L_i such that $L_i(X) \subset C_{i1}$. Now, suppose $\underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_n, \underset{\sim}{a}_{n+1} = \underset{\sim}{0}$ are the vertices of a simplex S ~~of width 1~~ which is a subset of X and such that no two of the edge lengths of S are equal. (There is such a simplex since X has constant width 1 and $\underset{\sim}{0} \in \operatorname{int} X$.) For each $i \in Z^+$, let $S_i = L_i(S)$, and then S_i is a simplex obtained from S by a displacement, and suppose its vertices are $\underset{\sim}{a}_1^{(i)}, \ldots, \underset{\sim}{a}_{n+1}^{(i)}$ where $\underset{\sim}{a}_j^{(i)} = L_i(\underset{\sim}{a}_j)$. Since the C_{ij} are uniformly bounded by a bound depending only on n, the sequence $\{S_i\}$ is a sequence of simplices, each obtained from S by a displacement, and all are subsets of a bounded region. So by Lemma 3.2, there exists a simplex S_0 which can be obtained by a displacement from S, and a subsequence $\{S_{i_\ell}\}$ of the sequence $\{S_i\}$ such that $\lim_{\ell\to\infty} S_{i_\ell} = S_0$. Without loss of generality, we shall suppose that this is the original sequence, for we are interested only in the limit sets involved. Hence, we have $\lim_{i\to\infty} S_i = S_0$. Since S is a simplex with no two edge lengths equal, there exists a unique displacement L such that $L(S) = S_0$. It follows that $\lim_{i\to\infty} L_i(X)$ exists and equals $L(X)$. (For suppose $\underset{\sim}{x} \in X$. Then $\underset{\sim}{x}$ is uniquely expressible in the form

$$\underset{\sim}{x} = \sum_{m=1}^{n+1} \beta_m \underset{\sim}{a}_m \quad \text{where} \quad \sum_{m=1}^{n+1} \beta_m = 1,$$

and then

$$\begin{aligned} L_i(\underset{\sim}{x}) &= \sum_{m=1}^{n+1} \beta_m L_i(\underset{\sim}{a}_m) \\ &\to \sum_{m=1}^{n+1} \beta_m L(\underset{\sim}{a}_m) \quad \text{as} \quad i\to\infty \\ &= L(\underset{\sim}{x}). \end{aligned}$$

Hence $\lim_{i\to\infty} L_i(\underset{\sim}{x})$ exists and equals $L(\underset{\sim}{x})$, and hence $\lim_{i\to\infty} L_i(X)$ exists and equals $L(X)$, since each $L_i(X)$ is bounded by a bound depending only on n.) We shall prove that $L(X) \subset T_1(B \cap K_1)$, which is a set of U, and so X is covered by U.

Now suppose $L(X) \not\subset T_1(B \cap K_1)$. Then there is a point $\underset{\sim}{x} \in L(X) \sim T_1(B \cap K_1)$. Since $T_1(B \cap K_1)$ is closed,

$$\inf_{\underset{\sim}{y} \in T_1(B\cap K_1)} \|\underset{\sim}{x} - \underset{\sim}{y}\| = \delta > 0.$$

But since $\lim_{i\to\infty} C_{i1} = T_1(B \cap K_1)$, there is an integer i_1 such that

$$D(T_1(B \cap K_1), C_{i1}) < \frac{\delta}{2} \quad \text{whenever} \quad i > i_1,$$

and since $\lim_{i\to\infty} L_i(X) = L(X)$, there is an integer i_2 such that

$$D(L(X),\ L_i(X)) < \frac{\delta}{2} \quad \text{whenever} \quad i > i_2.$$

Let $i_0 = \max(i_1, i_2)$. Then

$$D(T_1(B \cap K_1),\ C_{i1}) < \frac{\delta}{2}$$

and $\quad D(L(X),\ L_i(X)) < \frac{\delta}{2}$ whenever $i > i_0$.

Now suppose i is an arbitrary integer $> i_0$. Then there is a point $\underset{\sim}{z} \in L_i(X)$ such that

$$\|\underset{\sim}{z} - \underset{\sim}{x}\| < \frac{\delta}{2}.$$

Since $L_i(X) \subset C_{i1}$, it follows that $\underset{\sim}{z} \in C_{i1}$, and since $T_1(B \cap K_1)$ is closed, there exists a point $\underset{\sim}{\omega} \in T_1(B \cap K_1)$ such that

$$\|\underset{\sim}{z} - \underset{\sim}{\omega}\| = \inf_{\underset{\sim}{y} \in T_1(B \cap K_1)} \|\underset{\sim}{z} - \underset{\sim}{y}\|.$$

So as $D(T_1(B \cap K_1),\ C_{i1}) < \frac{\delta}{2}$, we must have

$$\|\underset{\sim}{z} - \underset{\sim}{\omega}\| < \frac{\delta}{2}.$$

Hence
$$\begin{aligned} \|\underset{\sim}{x} - \underset{\sim}{\omega}\| &\leq \|\underset{\sim}{x} - \underset{\sim}{z}\| + \|\underset{\sim}{z} - \underset{\sim}{\omega}\| \\ &< \frac{\delta}{2} + \frac{\delta}{2} \\ &= \delta. \end{aligned}$$

But since $\underset{\sim}{\omega} \in T_1(B \cap K_1)$, this is a contradiction to our definition of δ, and so $L(X) \subset T_1(B \cap K_1)$. This proves that our set U must have the required covering property.

Finally, we show that $V_n(U) = V_0$. We proved that $\lim_{i \to \infty} C_{ij} = T_j(B \cap K_j)$ for $j = 1, 2, \ldots, k$. So by Lemma 3.4,

$$\lim_{i\to\infty} V_n(C_{ij}) = V_n(T_j(B \cap K_j)) \quad \text{for} \quad j = 1,2,\ldots,k.$$

Therefore

$$\lim_{i\to\infty} \left(\max_{1\leq j\leq k} V_n(C_{ij})\right) = \max_{1\leq j\leq k} V_n(T_j(B \cap K_j)),$$

i.e.
$$\lim_{i\to\infty} V_n(U_i) = V_n(U).$$

But $\lim_{i\to\infty} V_n(U_i) = V_0$, and so $V_n(U) = V_0$. This completes the proof of the theorem for the case of volume.

The proofs in the cases of surface area and diameter are identical, with volume being replaced by the appropriate measure, since the only real differences occur when we establish a bound on the sets of a k-cover depending only on n. This has been done for the surface area case in Lemma 3.1 A and is obvious for the case of diameter.

We can now easily establish a result on the values of $A_{n,k}$ and $d_{n,k}$, providing further answers to the questions posed in Question 2 of the last chapter.

Corollary

For all positive integers k and for $n = 2$ or 3,

$$A_{n,k} > S_n \quad \text{and} \quad d_{n,k} > 1.$$

Proof: This now follows from Theorem 2.2 which says that $d_n(U_k) > 1$ for all k-covers in E^2 and E^3, and from its corollary which says $A_n(U_k) > S_n$.

Clearly, this corollary is true for all values of n for which Theorem 2.2 is true.

CHAPTER IV

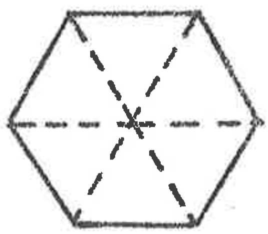
FURTHER EXAMPLES OF k-COVERS.

Earlier we saw examples of 2-covers in two and three dimensional space. These were formed from covering bodies by applying the general method of intersecting a suitable strip of width 1 with the covering body. The main problem now is to decide on a means of applying this method in several different directions simultaneously without producing a cover of large order. In the following we shall produce several k-covers in the plane and in three dimensional space by a similar application of the general method. This application can be applied profitably to covers based on the hexagon or the octahedron; we refer to it as the circle method or sphere method, depending on the dimension of the space. The method is illustrated well in the first example.

A 6-cover in the plane derived from the regular hexagon.

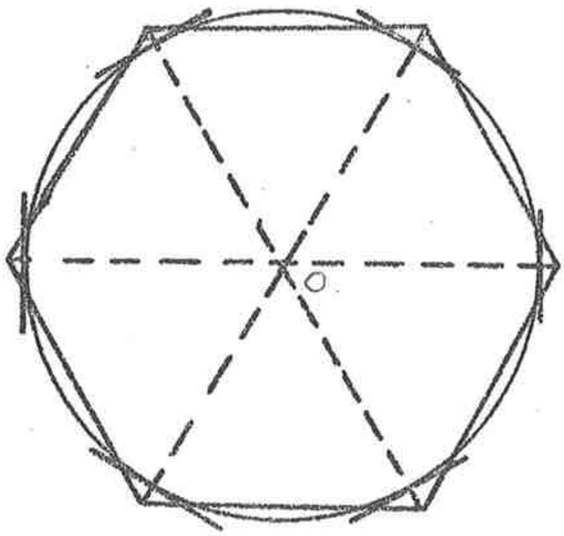
In the plane, we can apply the general method to the regular hexagon of width 1 in three different directions to obtain a 6-cover. Clearly, we could apply the method to the truncated hexagonal tile and we shall do this later.

Now, for the hexagon there are three diagonals and we try to make suitable cuts perependicular to these.

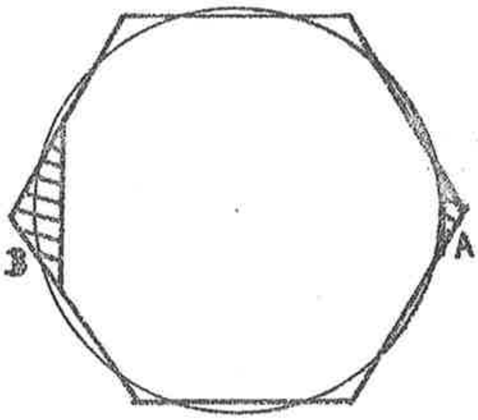


We must be careful to make use of the symmetry of the set or we may find that the order of the resulting cover is large.

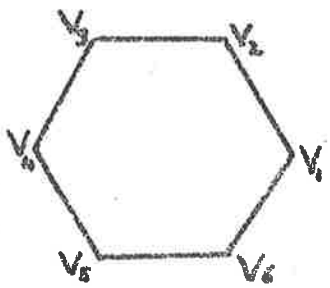
Suppose O is the centre of the hexagon, and suppose r is a number such that $\frac{1}{2} \leqslant r \leqslant \frac{1}{\sqrt{3}}$. Then we draw a circle of radius r with centre O.



The bounds on r guarantee that the circle cuts the boundary of the hexagon, and that the incircle does not lie outside the circle of radius r. Draw tangents to the circle where it intersects the diagonals of the hexagon. We refer to the regions cut from the hexagon near its vertices by these six tangents as <u>corners</u>. We say that two corners are <u>adjacent</u> if the corresponding vertices of the hexagon are adjacent, and are <u>opposite</u> if the corresponding vertices are opposite. Since $r \geqslant \frac{1}{2}$, opposite corners are separated by a strip of width at least 1. Hence a set S of diameter ≤ 1 intersects at most three corners of the hexagon. Now suppose that the set S intersects a corner A, as in the figure. Then S does not intersect the region B cut off by a line parallel to the tangent cut at A and distance 1 from the tangent. Similarly, if S intersects B, it does not intersect A. Hence we can form a 2-cover by removing the two regions A and B, one at a time. We refer to cuts B made opposite an occupied corner as <u>large cuts</u>.

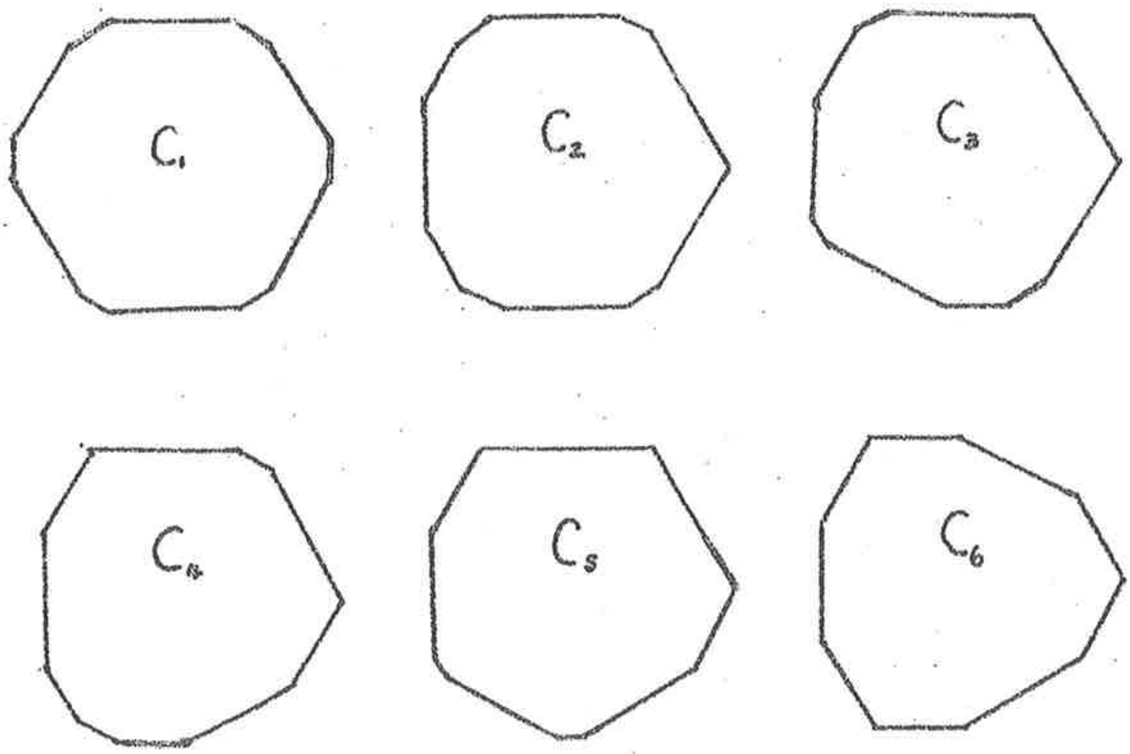


The real benefit of the method is obtained when we consider how many corners S can occupy simultaneously. Suppose we label the vertices of the hexagon consecutively $V_1, V_2, V_3, V_4, V_5, V_6$. Then the various cases are shown in the following table. If two cases are similar in that one can be obtained from the other by a rotation, only one is shown, since for our purposes they are identical. We denote the area of a corner piece (or small cut) by $S(r)$ and the area of a large cut by $L(r)$.



Case	Corners which S intersects	Cuts made at vertices: Corner cuts	Large cuts	Area removed from hexagon
C_1	-	1,2,3,4,5,6	-	6S(r)
C_2	1	2,3,5,6	4	4S(r)+L(r)
C_3	1,2	3,6	4,5	2S(r)+2L(r)
C_4	1,3	2,5	4,6	2S(r)+2L(r)
C_5	1,2,3	-	4,5,6	3L(r)
C_6	1,3,5	-	2,4,6	3L(r)

We shall denote the resulting sets by $C_1, C_2, C_3, C_4, C_5, C_6$, and then $U_6 = \{C_1, C_2, C_3, C_4, C_5, C_6\}$ is a 6-cover. The sets are shown in the following diagram.



Now each of the cuts is an isosceles triangle with vertical angle 120°, which has area $\sqrt{3}\ h^2$ where h is the height of the triangle perpendicular to the odd side. For the corner cuts $h = \frac{1}{\sqrt{3}} - r$, and so $S(r) = \sqrt{3}\left(\frac{1}{\sqrt{3}} - r\right)^2$. For the large cuts, $h = \frac{1}{\sqrt{3}} - 1 + r$, and so $L(r) = \sqrt{3}\left(\frac{1}{\sqrt{3}} - 1 + r\right)^2$. In order to find the best 6-cover of this form, we must find the number r with $\frac{1}{2} \leqslant r \leqslant \frac{1}{\sqrt{3}}$ such that the area of the cover is least. Since $S(r)$ is a strictly decreasing function of r, and $L(r)$ is strictly increasing on the interval $[\frac{1}{2}, \frac{1}{\sqrt{3}}]$, the least area removed is maximized when $2S(r) = L(r)$, if this occurs in the interval $[\frac{1}{2}, \frac{1}{\sqrt{3}}]$. On solving the equation, one obtains:

$$r = \sqrt{3} + \sqrt{2} - 1 - \tfrac{2}{3}\sqrt{6}$$

$$= 0.513271....,$$

which is within our bounds for r. For this value of r,

$$S(r) = 0.007112...$$

$$\text{and}\quad L(r) = 0.014224...,$$

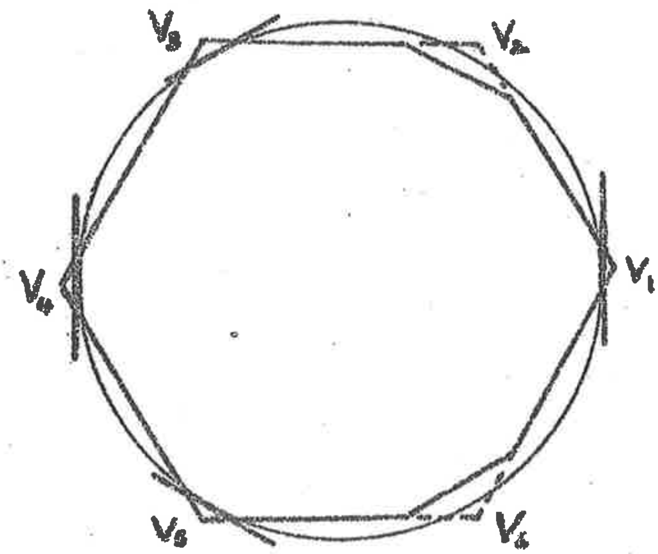
and the sets of the cover each have equal area, being

$$A(U_6) = 0.823353... .$$

A 12-cover in the plane from the truncated hexagon.

We shall now apply the same method to Pál's tile, the truncated hexagon. Again we draw tangents to the circle with centre at the point of intersection of the diagonals of the hexagon and with radius r, where $\frac{1}{2} \leq r \leq \frac{1}{\sqrt{3}}$. This time the resulting sets can be considered to have three types of cuts made on the regular hexagon, being the small corner cuts of area $S(r)$, the large cuts of $L(r)$, as before, and of course the cuts due to the truncated corners of the Pál tile. We refer to these last cuts as Pál cuts, and denote their area by P, where

$$P = \sqrt{3}\left(\frac{1}{\sqrt{3}} - \frac{1}{2}\right)^2 = 0.010362\ldots .$$



We label the vertices of the hexagon V_1, V_2, V_3, V_4, V_5 and V_6, and suppose that V_2 and V_6 are the truncated vertices of the Pál tile. As before, let us refer to the small regions "outside" the tangents as corners. Then, if S is a set of diameter $\leqslant 1$ covered by the truncated hexagon, S may intersect with corners 1,3,4 or 5, but never with both 1 and 4. We consider the various ways in which the set may intersect with corners in the following table.

Case	Corners which S intersects	Cuts made at vertices: Small	Large	Pál	Area removed from regular hexagon
C_1	-	1,3,4,5	-	2,6	4S(r)+ 2P
C_2	1	3,5	4	2,6	2S(r)+L(r)+2P
C_3	3	1,4,5	6	2	3S(r)+L(r)+ P
C_4	5	1,3,4	2	6	3S(r)+L(r)+ P
C_5	4	3,5	1	2,6	2S(r)+L(r)+2P
C_6	1,3	5	4,6	2	S(r)+2L(r)+P
C_7	1,5	3	4,2	6	S(r)+2L(r)+P
C_8	3,4	5	1,6	2	S(r)+2L(r)+P
C_9	4,5	3	1,2	6	S(r)+2L(r)+P
C_{10}	3,5	1,4	2,6	-	2S(r)+2L(r)
C_{11}	1,3,5	-	2,4,6	-	3L(r)
C_{12}	3,4,5	-	1,2,6	-	3L(r)

This list exhausts all cases, and so we obtain a 12-cover in the plane, U_{12} say. We note however that each set of the pairs (C_3, C_4), (C_6, C_7) and (C_8, C_9) is obtainable from the other by a reflection, and so if we allowed reflections to take place in our process of covering a set, we would have a 9-cover.

Now $S(r) \leqslant P \leqslant L(r)$ for $\frac{1}{2} \leqslant r \leqslant \frac{1}{\sqrt{3}}$. Also $S(r)$ is a decreasing function of r and $L(r)$ is an increasing function on the interval $[\frac{1}{2}, \frac{1}{\sqrt{3}}]$. Hence it is easily checked that the maximum area of the first nine sets is least when

$$A(C_1) = A(C_3) = A(C_4)$$

if this occurs when $\frac{1}{2} \leqslant r \leqslant \frac{1}{\sqrt{3}}$. In fact, these sets have equal volume when

$$r = \frac{9 + 2\sqrt{3}}{24}$$

$$= 0.519337\ldots$$

which is within our bounds. Then

$$S(r) = 0.005829\ldots,$$

$$L(r) = 0.016192\ldots$$

and $A(C_1) = A(C_3) = A(C_4) = 0.821982\ldots$.

For this value of r, $A(C_{10}) = A(C_1)$ and on calculation $A(C_{11}) = A(C_{12}) < A(C_1)$. Hence

$$A(U_{12}) = A(C_1) = A(C_3) = A(C_4) = A(C_{10}) = 0.821982\ldots,$$

and this is the best possible 12-cover of this form.

<u>A 12-cover in the plane from truncated hexagonal 2-cover</u>.

So far, we have applied our method for the formation of k-covers only to tiles. However there is no reason why we should not apply it to sets of a cover of order larger than 1. We recall that a 2-cover in the plane could be obtained by truncating three corners by Pál-type cuts from the regular hexagon of width 1 in two different ways; one by removing three adjacent vertices and the other by removing alternate vertices.

Suppose we label the vertices of the hexagon $V_1, V_2, V_3, V_4, V_5, V_6$. In the case where three adjacent corners have been removed, suppose V_1, V_2 and V_6 are the truncated corners. In the other case, suppose V_2, V_4 and V_6 have been removed. As before, in each case we draw a circle of radius r, $\frac{1}{2} \leqslant r \leqslant \frac{1}{\sqrt{3}}$, with centre at the centre of the regular hexagon, and take tangents to it where the diagonals of the hexagon cut it. In each case there are three small corner pieces which may be removed. The sets obtained from the set of the 2-cover with three adjacent vertices removed are shown in the first table.

Case	Corners occupied	Cuts made at vertices: Small	Large	Pál	Area removed from regular hexagon
C_1	-	3,4,5	-	1,2,6	3S(r) 3P
C_2	3	4,5	6	1,2	2S(r)+ L(r)+2P
C_3	4	3,5	1	2,6	2S(r)+ L(r)+2P
C_4	5	3,4	2	1,6	2S(r)+ L(r)+2P
C_5	3,4	5	1,6	2	S(r)+2L(r)+ P
C_6	3,5	4	2,6	1	S(r)+2L(r)+ P
C_7	4,5	3	1,2	6	S(r)+2L(r)+ P
C_8	3,4,5	-	1,2,6	-	3L(r)

The following table gives the sets obtained from the set of the 2-cover where the alternative vertices have been removed.

Case	Corners occupied	Cuts made at vertices: Small	Large	Pal	Area removed from regular hexagon
C_9	-	1,3,5	-	2,4,6	3S(r)+ 3P
C_{10}	1	3,5	4	2,6	2S(r)+ L(r)+2P
C_{11}	1,3	5	4,6	2	S(r)+2L(r)+ P
C_{12}	1,3,5	-	2,4,6	-	3L(r)

This is the list of all different cases. Hence, since we started with a 2-cover, and so any set of diameter ≤ 1 can be covered by one of the two original sets, we have produced another 12-cover in the plane. However, since each set of the pairs (C_2, C_4) and (C_5, C_7) is obtainable from the other by a reflection, we would have a 10-cover if we allowed reflections to occur in our covering process.

After using a computer to calculate the areas, I noted that all sets of the 12-cover have equal area when

$$r = \frac{9 + 2\sqrt{3}}{24}$$
$$= 0.519337\ldots .$$

Note that this is exactly the same value of r that gave the optimal 12-cover of the previous form. For this value of r,

$$S(r) = 0.005829\ldots,$$

$$L(r) = 0.016192\ldots,$$

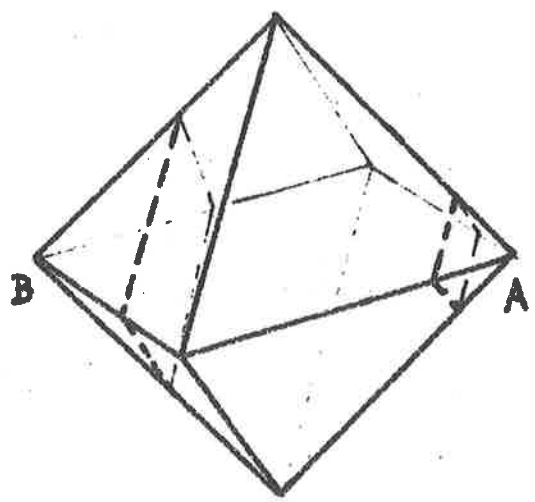
and then the cover, U'_{12} say, has area

$$A(U'_{12}) = A(C_1) = A(C_2) = \ldots = A(C_{12}) = 0.817448\ldots,$$

which is less than the area of the other 12-cover. This must be the optimal value for a 12-cover of this form since $A(C_1)$, say, is a decreasing function of r, and $A(C_8)$, say, is an increasing function on the interval $[\frac{1}{2}, \frac{1}{\sqrt{3}}]$.

A 4-cover in E^3

We can apply a very similar method in E^3 to the regular octahedron of width 1 or to the truncated octahedron covering body. Suppose we first start with the regular octahedron. In this case we draw a sphere with centre at the centre of the octahedron and radius r, $\frac{1}{2} \leq r \leq \frac{1}{2}\sqrt{3}$. Then we draw tangent planes to the sphere



where the diagonals cut its surface, cutting off pyramidal pieces at the corners. We shall refer to these pyramidal pieces as <u>corners</u>, and denote their volume by $S(r)$ (for small cut). Now each corner is a square pyramid of volume $\frac{2}{3}h^3$ where h is the height of the pyramid. Hence

$$S(r) = \frac{2}{3}(\tfrac{1}{2}\sqrt{3} - r)^3.$$

At a distance 1 from each of these tangent planes, draw a parallel plane cutting a larger pyramid from the octahedron, the volume of which we shall denote by $L(r)$. Then $L(r) = \frac{2}{3}(\frac{1}{2}\sqrt{3} - 1 + r)^3$. Now a set S of diameter $\leqslant 1$ can not intersect with a corner A, as in the diagram, and with the larger region B opposite. Also, S can intersect at most three of the six corner pieces since $r \geqslant \frac{1}{2}$. As there is a large degree of symmetry in the octahedron, we find there is only one distinct way in which S can intersect with no corner, one corner, two corners or three corners. So we obtain a 4-cover, U_4 say, the cases being listed in the table.

Case	No. of corners S intersects	No. of small cuts	No. of large cuts	Vol. removed from octahedron.
C_1	0	6	0	$6S(r)$
C_2	1	4	1	$4S(r)+ L(r)$
C_3	2	2	2	$2S(r)+2L(r)$
C_4	3	0	3	$3L(r)$

Since $S(r)$ is a decreasing function and $L(r)$ is an increasing function on $[\frac{1}{2},\frac{1}{2}\sqrt{3}]$, the volume removed is maximized if $L(r) = 2S(r)$ if this occurs for r in the interval $[\frac{1}{2},\frac{1}{2}\sqrt{3}]$. But this equation has a unique real solution for r being

$$r = \frac{3^{\frac{1}{2}}(2^{\frac{1}{3}} - 1) + 2}{2(2^{\frac{1}{3}} + 1)}$$

$$= 0.542097\ldots$$

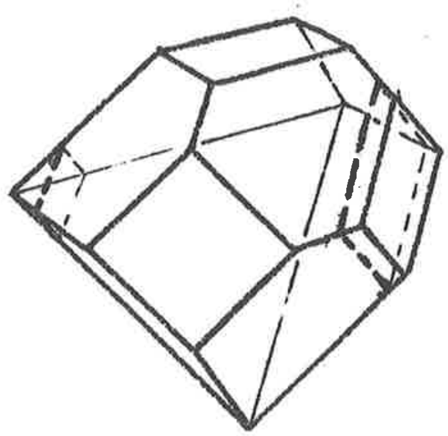
which lies in the required interval. In this case

$$S(r) = 0.022659\ldots,$$

$$L(r) = 0.045319\ldots,$$

and $V(U_4) = V(C_1) = V(C_2) = V(C_3) = V(C_4) = 0.730067\ldots$.

We can apply the same method to the truncated octahedron, except in this case there are only three corner pieces which may be occupied.



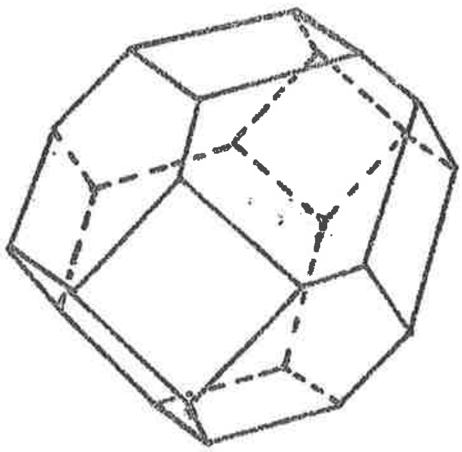
If a corner is occupied, we see that a slice can be removed near the opposite vertex, since we already have three Pál cuts of volume

$$P = \frac{2}{3}(\tfrac{1}{2}\sqrt{3} - \tfrac{1}{2})^3$$
$$= 0.032692\ldots \ .$$

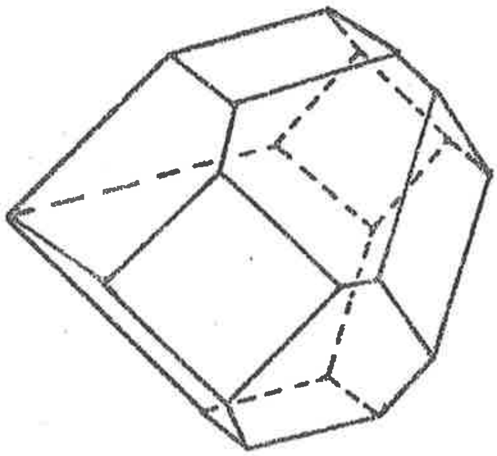
Note how these pieces being removed are of just the same form as those removed in the formation of the best 2-cover. The sphere method gives a means of making cuts in three directions simultaneously without giving a cover of too high an order. As in the case of the previous cover, there is enough symmetry to give a 4-cover, U_4' say, with one set corresponding to each of the numbers of corners which the set S of diameter ≤ 1 intersects.

Case	No. of corners S intersects	No. of cuts, Small	Large	Pal	Vol removed from reg. octahedron
C_1	0	3	0	3	$3S(r) + 3P$
C_2	1	2	1	2	$2S(r) + L(r) + 2P$
C_3	2	1	2	1	$S(r) + 2L(r) + P$
C_4	3	0	3	0	$3L(r)$

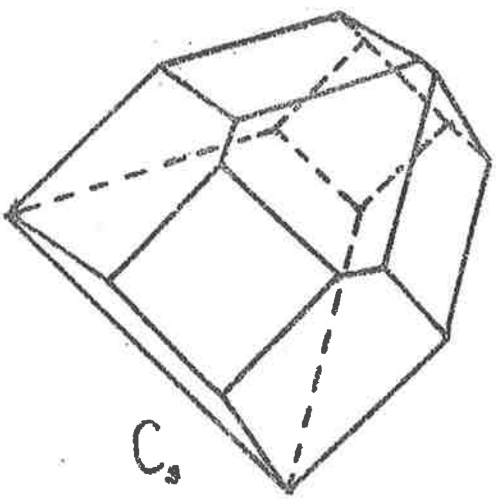
The sets are illustrated on the following page. As in the case of the 6-cover and the second 12-cover in the plane, and of the other 4-cover in E^3, this 4-cover has least volume when the four sets have equal volume. This is easily seen since $S(r)$ is decreasing and $L(r)$ is



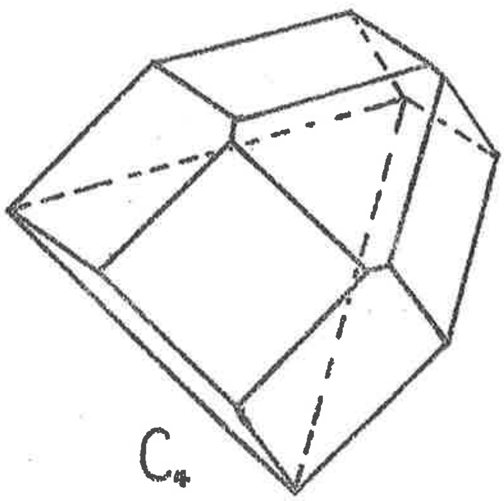
C_1



C_2



C_3



C_4

increasing on $[\frac{1}{2},\frac{1}{2}\sqrt{3}]$. Hence the least volume removed is maximized when

$$3S(r) + 3P = 2S(r) + L(r) + 2P = S(r) + 2L(r) + P = 3L(r)$$

provided this has a solution for r in the interval $[\frac{1}{2},\frac{1}{2}\sqrt{3}]$. Now the equations are equivalent to the following single equation:

$$S(r) - L(r) + P = 0,$$

i.e. $\frac{2}{3}(\frac{1}{2}\sqrt{3}-r)^3 - \frac{2}{3}(\frac{1}{2}\sqrt{3}-1+r)^3 + \frac{2}{3}(\frac{1}{2}\sqrt{3}-\frac{1}{2})^3 = 0.$

Applying Cardano's formula for the solution of a cubic equation, we obtain as the only real solution

$$r = \tfrac{1}{2} + \frac{1}{2^{\frac{5}{3}}}\left\{(-5+3\sqrt{3}+\sqrt{884-510\sqrt{3}}\,)^{\frac{1}{3}} + (-5+3\sqrt{3}-\sqrt{884-510\sqrt{3}}\,)^{\frac{1}{3}}\right\}$$

which, on evaluation using a computer, gives

$$r = 0.56045451\ldots .$$

This is certainly within our bounds on r, and yields

$$S(r) = 0.019021\ldots ,$$

$$L(r) = 0.051712\ldots ,$$

and $V(U_4') = V(C_1) = V(C_2) = V(C_3) = V(C_4)$

$$= 0.710884\ldots .$$

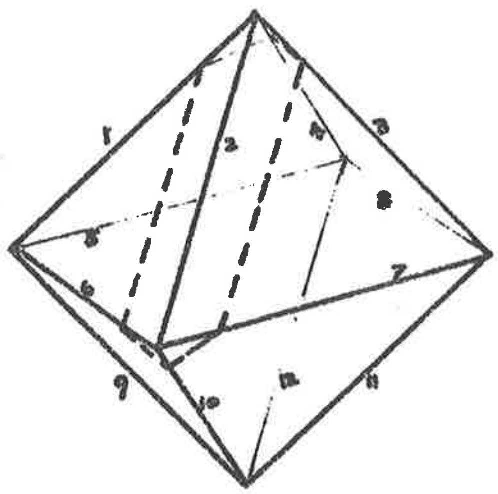
It is not surprising that the cuts made are just the same cuts as those made in the formation of the 2-cover, except here we make cuts perpendicular to each diagonal.

This second 4-cover is a better 4-cover than the first, which is not surprizing as it was formed from a better covering body. It is the best 4-cover I have found.

A 6-cover in E^3

We noted earlier that in three dimensional space we can form k-covers by removing edges rather than vertices. I describe here an attempt in this direction.

Let us label the edges of the regular octahedron of width 1 with the numbers 1 to 12 as shown in the diagram.



We remove edges by cuts which are tangent planes to the inscribed sphere, parallel to the edge being removed. For example, in the diagram edge 2 is being removed, and the cut is the tangent plane perpendicular to the line joining the mid-point of the edge to the centre of the octahedron. Now there are three planes through the centre of the octahedron with each containing four of the edges in the form of a square. Since the tangent planes cutting off opposite edges of one of these squares are distance 1 apart, we can remove either one of any pair of opposite edges of the square. Applying this argument in two directions on each of the three squares, we can remove six edges of the octahedron. The table shows the distinct cases, allowing for rotations:

Case	Edges removed from square: 1,3,11,9	2,4,12,10	5,6,7,8
C_1	1,3	2,4	5,6
C_2	"	4,12	5,6
C_3	"	"	6,7
C_4	"	"	7,8
C_5	"	"	5,8
C_6	"	12,10	5,6

This is seen to be a complete table of cases, since one can in turn remove edges from the first square, then from the second and then from the third, noting the different cases. So we obtain a 6-cover U_6 in E^3, or a 5-cover if reflection is allowed in the covering process since C_2 and C_5 are obtainable from each other by reflection.

On calculation, the volume cut off the octahedron in removing the first edge is 0.02053..., and since six edges are removed with some overlap of the removed edges, it follows that

$$V(U_6) > \tfrac{1}{2}\sqrt{3} - 6 \times 0.0206$$
$$= 0.7424\ldots .$$

Since this figure is larger than the volume of the 4-cover already obtained, and since the optimal volume of a k-cover is a decreasing function of k, this new cover provides no new insight into the value of $V_{3,6}$.

These are the interesting examples of k-covers I obtained in the plane and in E^3. I made attempts to find k-covers in the plane based on polygons other than the hexagon, or based on a combination of two or more different polygons, but I was unable to find covers with lower area than those given.

As mentioned earlier, I have done little work using the measures of diameter and surface area. However by now we have answered a large number of the questions posed in Chapter II. In Question 1, we posed the problem of determining the values of $V_{n,k}$, $A_{n,k}$ and $d_{n,k}$. The examples in this chapter give upper bounds on $V_{n,k}$ for the appropriate values of n and k.

These results are summarized in the following table.

Dimension n	Order k	Best known upper bound on $V_{n,k}$	Comment on cover determining this upper bound	Best known lower bound on $V_{n,k}$	Comment on set determining this lower bound
1	General k	1		1	
2	1	0.844144..	Sprague tile	0.825711..	Convex hull of circle and triangle
2	2	0.834936..	Hexagon less 3 corners	$\frac{\pi}{4}$ = 0.785398..	Circle Suspect $V_{2,2} > 0.7988..$ by considering circle, triangle and Reuleaux pentagon.
2	6	0.823353..	Circle method applied to hexagon	$\frac{\pi}{4}$ = 0.785398..	Circle
2	9 with reflection	0.821982..	Circle method applied to Pál's tile	$\frac{\pi}{4}$ = 0.785398..	Circle
2	10 with reflection, 12	0.817448..	Circle method on 2-cover	$\frac{\pi}{4}$ = 0.785398..	Circle
3	1	0.767941..	Truncated octahedron	$\frac{\pi}{6}$ = 0.523598..	Sphere
3	1 with reflection	0.754612	Remove one of a pair of opposite edges of truncated octahedron	$\frac{\pi}{6}$ = 0.523598	Sphere
3	2	0.748927..	Further truncation at 2 opposite vertices of truncated octahedron.	$\frac{\pi}{6}$ = 0.523598..	Sphere
3	4	0.710884..	Sphere method on truncated octahedron	$\frac{\pi}{6}$ = 0.523598..	Sphere
General n	General k	$\left(\frac{n\pi}{2n+2}\right)^{\frac{n}{2}} \frac{1}{\Gamma\left(1+\frac{n}{2}\right)}$ ($\to \frac{1}{2}\sqrt{2}$ as $n\to\infty$)	Jung's sphere	$\frac{\pi^{\frac{n}{2}}}{2^{\frac{n}{2}}\,\Gamma\left(1+\frac{n}{2}\right)}$ ($\to 0$ as $n\to\infty$)	Sphere

CHAPTER V

APPLICATION TO THE MODIFIED BORSUK'S PROBLEM

5.1 The principle involved

In Section 1.4, we saw how a covering body can be used to solve Borsuk's problem in two and three dimensions. I stated that the attempt for a solution of the modified Borsuk's problem gave motivation for the introduction of the idea of k-cover. We recall that the modified Borsuk's problem is to determine the value of $d_n(n+1)$ for each positive integer n, where $d_n(n+1)$ denotes the infimum of all real numbers d such that any set in E^n of diameter 1 can be partitioned into $n+1$ sets of diameter at most d. Now, by Theorem 2.1, for any positive integer n, $\lim_{k\to\infty} d_{n,k} = 1$. So by taking k to be a sufficiently large integer, we can find a k-cover U_k of diameter $d_n(U_k)$ arbitrarily close to 1. Then this cover may be divided into $n+1$ sets, each of diameter $\leqslant d_n(n+1) \times d_n(U_k)$. Hence, by taking k sufficiently large and then partitioning a good k-cover, we can obtain in principle an upper bound on $d_n(n+1)$ which is arbitrarily close to its actual value.

In this chapter we consider the case $n=3$. We have seen that $d_3(4)$ satisfies

$$\sqrt{(3+\sqrt{3})/6} = 0.88807\ldots \leqslant d_3(4) \leqslant 0.9887\ldots,$$

and noted a conjecture that $d_3(4)$ should equal this lower bound. We shall improve on the upper bound.

5.2 Partitioning the sets of the 4-cover into four sets.

In this section, we use the 4-cover obtained in the previous chapter to prove

$$d_3(4) \leqslant 0.97522.. \ .$$

This result, although disappointing in its magnitude, has appeal in that it is obtained using only simple geometrical ideas. In this process, we make further truncations to the individual sets of the 4-cover without producing a cover of higher order. The partitions which we shall finally obtain are in fact partitions of the derived 4-cover. Also, we notice that the construction of the 4-cover is independent of the value of the radius r of the sphere used. Hence we are free to allow r to vary within the bounds $\frac{1}{2} \leqslant r \leqslant \frac{1}{2}\sqrt{3}$. (Since we are now interested in the diameters of the sets of the partition, we do not require the value of r which gives the optimal volume.) We also note that it is sufficient to divide the surface of the sets of the 4-cover into four parts. For the radius of the sphere circumscribed about the original octahedron is $\frac{1}{2}\sqrt{3}$. Suppose that the surface is partitioned into sets of diameter at most d where $\frac{1}{2}\sqrt{3} \leqslant d < 1$. Then the convex hull of the union of each of these subsets with the centre of the circumsphere has diameter at most d. However, since $d_3(4) \geqslant 0.888...$, and $\frac{1}{2}\sqrt{3} = 0.866...$, it immediately follows that $d > \frac{1}{2}\sqrt{3}$.

In the construction of the 4-cover, we considered the number of "corner" pieces which were intersected by the set being covered. In the following, we refer to these numbers as the "number of corners occupied".

The case: 3 corners occupied.

We shall consider the case "3 corners occupied" first for the following reasons. For $r=\frac{1}{2}$, the set is identical to the truncated octahedron covering body, and hence the similar dissections of Grünbaum and Heppes suggest an obvious means of dissecting the set. Also, if we obtain a good result for the case when $r=\frac{1}{2}$, we obtain a new upper bound on $d_3(4)$.

We recall that this set of the 4-cover is an octahedron with square-based pyramids removed from three adjacent vertices. We use the notation which Grünbaum used in the publication of his dissection. Let the three remaining vertices be labelled A_1, A_2 and A_3, and the other vertices as in the Schlegel diagram.

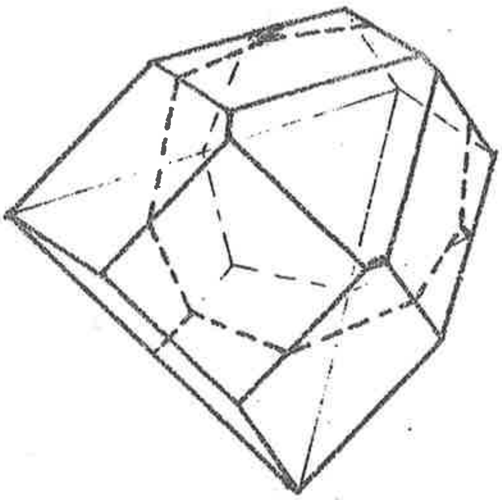
The dissection which we consider is shown in detail in the figure. Now, for $i=1,2,3$,

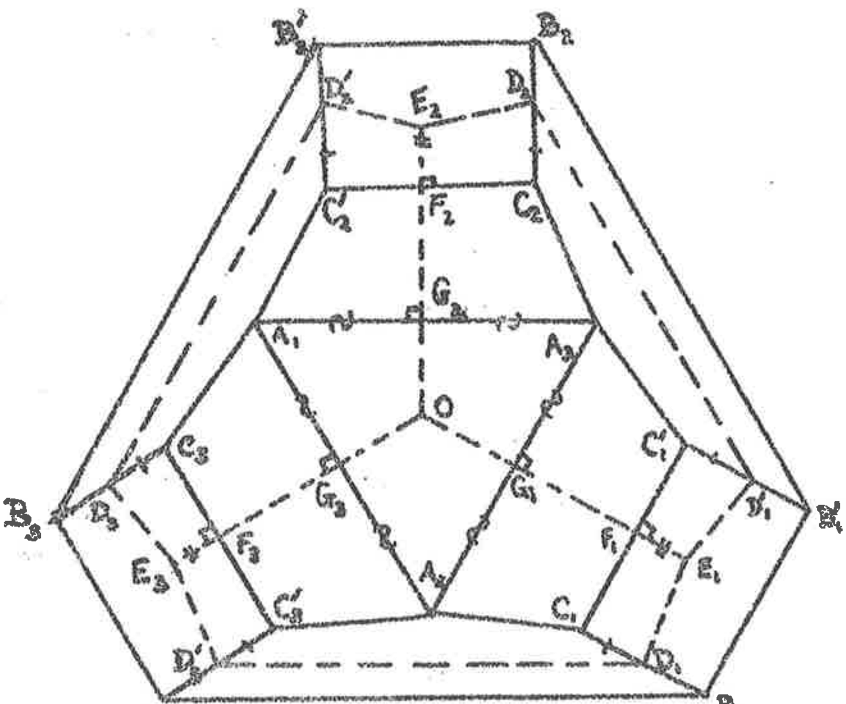
let $\quad E_i = \alpha F_i + (1-\alpha)\frac{B_i+B_i'}{2}$,

$\quad\quad D_i = \beta C_i + (1-\beta)B_i$

and $\quad D_i' = \beta C_i' + (1-\beta)B_i'$

where $0 \leqslant \alpha \leqslant 1$ and $0 \leqslant \beta \leqslant 1$. The three pieces of the




B'2
B2
D'2
E2
D2
C'2
F2
C2
G2
A1
A2
O
C3
C'1
B3
D3
G3
G1
D'1
B'1
F3
F1
E3
E1
C'3
C1
D'3
D1
B'3
B1

dissection meeting at O are identical, while the fourth piece does not contain a vertex of the original octahedron and so is quite distinct. The diameters of each piece are functions of α and β. To find the best dissection of this form, we seek the values of α and β which give the least maximum diameter of the pieces. Since this is difficult, I used a computer scan to find good dissections of this form. However the dissections obtained are not necessarily optimal.

From simple geometric considerations, one of the following pairs must determine the diameter of each piece. Where a distance occurs twice, I give only one case.

3 similar pieces:

(O,E_1), (O,D_3'), (O,A_2), (O,C_3'), (G_1,E_3), (G_1,F_3), (G_1,D_3'), (G_1,C_3'), (F_1,F_3), (F_1,E_3), (F_1,D_3'), (E_1,A_2), (F_1,A_2), (E_1,C_3') or (D_3',A_2).

Other piece:

(E_1,E_3), (E_3,D_2), (E_2,B_3'), (D_2,D_3') or (D_2,B_3').

Having decided on possible critical distances, we introduce coordinates. For simplicity, we suppose that the centre of the circumcircle of the regular octahedron lies at the origin, and that A_1,A_2 and A_3 lie on the X,Y and Z axes respectively. Then the coordinates needed are listed below:

A_2 $(0,-\frac{1}{2}\sqrt{3},0)$

B_3' $(a,0,r)$

C_3' $(0,-a,r)$

F_1 $\left(r,-\frac{a}{2},-\frac{a}{2}\right)$

F_3 $\left(\frac{a}{2},-\frac{a}{2},r\right)$

G_1 $\left(0,-\frac{1}{4}\sqrt{3},-\frac{1}{4}\sqrt{3}\right)$

0 $\left(-\frac{1}{6}\sqrt{3},-\frac{1}{6}\sqrt{3},-\frac{1}{6}\sqrt{3}\right)$

D_2 $((1-\beta)a,r,-\beta a)$

D_3' $((1-\beta)a,-\beta a,r)$

$$E_1 \quad \left(r,\ -\frac{\alpha a}{2}+(1-\alpha)\frac{a}{2},\ -\frac{\alpha a}{2}+(1-\alpha)\frac{a}{2}\right)$$

$$E_2 \quad \left(-\frac{\alpha a}{2}+(1-\alpha)\frac{a}{2},r,-\frac{\alpha a}{2}+(1-\alpha)\frac{a}{2}\right)$$

$$E_3 \quad \left(-\frac{\alpha a}{2}+(1-\alpha)\frac{a}{2},-\frac{\alpha a}{2}+(1-\alpha)\frac{a}{2},r\right)$$

where $a = \frac{1}{2}\sqrt{3} - 1 + r$.

Using this information and a computer, I found the following dissections. For each value of r, the diameters of the different pieces are equal in at least the first five decimal places, and this is the value given. In each case I state one of the edges which determine the diameter of each set; these are not unique.

r	α	β	Critical edges 3 pieces	 Other piece	Diameter of pieces
0.5	0.88576..	0.47196..	(G_1,E_3)	(E_3,D_2)	0.98772..
0.51	0.87328..	0.47015..	(G_1,E_3)	(E_3,D_2)	0.97841..
0.52	0.86121..	0.46864..	(G_1,E_3)	(E_3,D_2)	0.96913..
0.53	0.84952..	0.46740..	(G_1,E_3)	(E_3,D_2)	0.95988..
0.56045..	0.81603..	0.46509..	(G_1,E_3)	(E_3,D_2)	0.93191..

The last value of r is that for the 4-cover of optimal volume. The other values of r extend over the range in which we shall be interested when we consider dissections of each set of the 4-cover for the same value of r. However, note the value for the case $r=\frac{1}{2}$. This case is just the truncated octahedron covering body, and so

$$\sqrt{(3+\sqrt{3})/6} = 0.88807.. \leq d_3(4) \leq 0.98772.. .$$

This compares with Grünbaum's upper bound of 0.9887.. and Heppes' figure of 0.9977, both of which were obtained by a similar dissection.

We need not consider any better dissection of this set of the 4-cover; for this dissection divides this set into pieces of sufficiently low diameter to prove the result on the upper bound of $d_3(4)$ which we shall obtain from the 4-cover. We shall see that the value of this bound is determined by the cases "No corner occupied" and "1 corner occupied", for which I have not been able to find as good a dissection. However, in Section 5.3, we shall see a method how this dissection can be improved.

The case: No corner occupied.

We next consider the case "no corners occupied". This set has symmetry like that of the case just considered. Also, it covers the sphere and therefore is likely to give a high value for the diameter of the sets of a dissection.

Clearly, we would hope to use the symmetry of the set in finding a dissection. We could use the same dissection as that for the truncated octahedron covering body. However, we note that none of the critical points in the dissection of that cover have been removed and so no improvement is possible by this dissection. Therefore, in order to obtain a better dissection, we use the fact that all corners have been removed. There are two obvious forms for the dissection as shown in the diagrams.

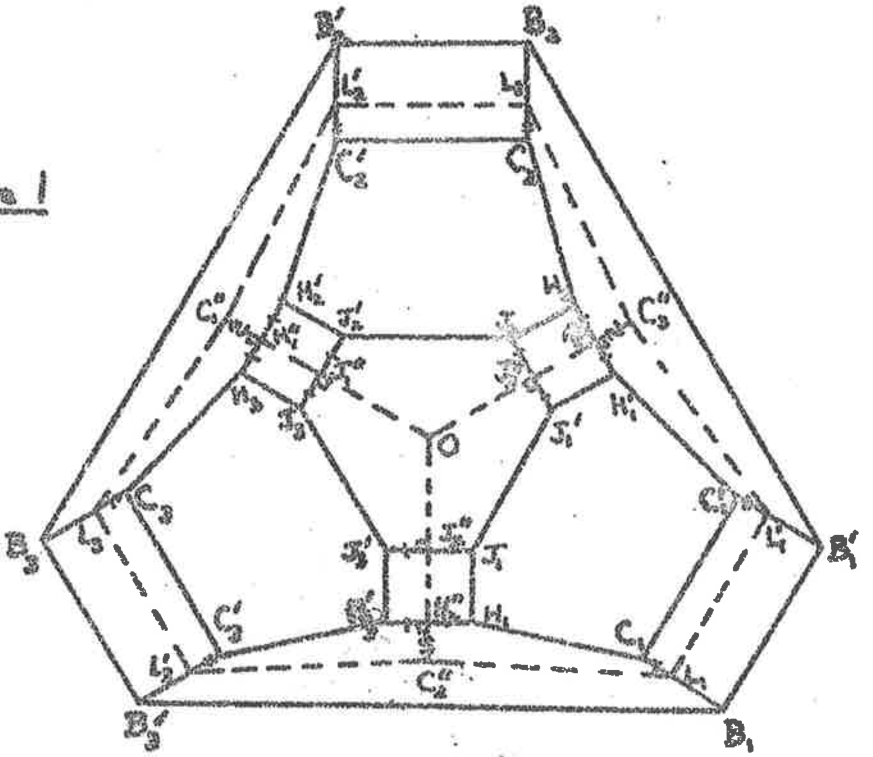
Where points are common with those in the previous set, the notation is the same. The square formed on removing A_1 has been labelled $H_2'J_2'J_3H_3$, on removing A_2 as $H_3'J_3'J_1H_1$ and on removing A_3 as $H_1'J_1'J_2H_2$. Each of these cuts is made at distance $1-r$ from the centre of the regular octahedron, and these are the so-called "small cuts".

In the first dissection, $H_1'' H_2''$ is fairly long, having length

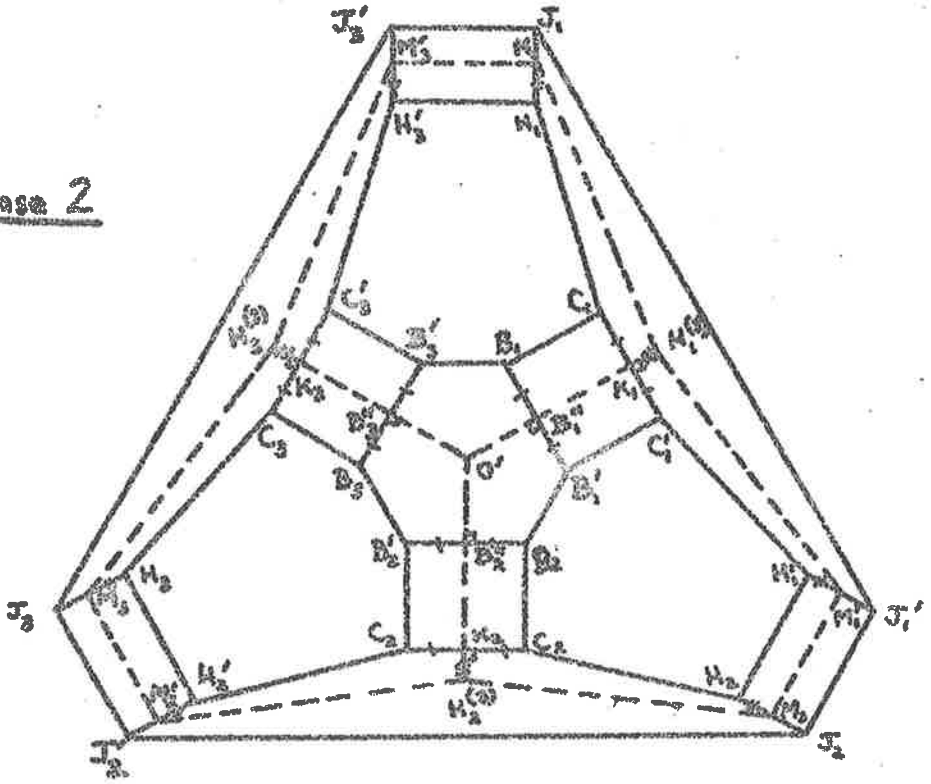
$$\begin{aligned} &0.96592\ldots \quad \text{if} \quad r = 0.5, \\ &0.98713\ldots \quad \text{if} \quad r = 0.53 \\ \text{and} \quad &1.00867\ldots \quad \text{if} \quad r = 0.560454\ldots \,. \end{aligned}$$

As a result, the other dissection gives lower values, so we shall use it.

Case 1



Case 2



The second dissection gives three similar pieces, all meeting at O', and one other piece. By simple geometric considerations, the diameters are determined by one of the following pairs of points if

$$0.5 \leqslant r \leqslant 0.560454\ldots .$$

3 similar pieces:

(O',M_3'), $(O',H_3^{(3)})$, (O',C_1), (O',H_3'), (B_1'',H_3'), (B_1'',M_3'), $(B_1'',H_3^{(3)})$, (B_1'',K_3), (K_1,M_3'), (K_1,C_3'), $(K_1,H_3^{(3)})$, (K_1,K_3), $(H_3^{(3)},B_1)$, $(H^{(3)},C_1)$ or (M_3',B_1).

Other piece:

$(H_1^{(3)},H_3^{(3)})$, $(H_1^{(3)},M_3)$, $(H_1^{(3)},J_3)$ or (M_1',M_3).

We now introduce the necessary parameters. We suppose that
$$M_i = \alpha J_i + (1-\alpha)H_i,$$
$$M_i' = \alpha J_i' + (1-\alpha)H_i'$$
and
$$H_i^{(3)} = \beta K_i + (1-\beta)\,\frac{J_i+J_i'}{2} \quad \text{for} \quad i=1,2,3,$$
where $0 \leqslant \alpha \leqslant 1$ and $0 \leqslant \beta \leqslant 1$. Then the necessary coordinates are:

$O'\ (\frac{\sqrt{3}}{6}, \frac{\sqrt{3}}{6}, \frac{\sqrt{3}}{6})$	$B_1(\frac{1}{2},0,\frac{1}{2}(\sqrt{3}-1))$
$B_1''\ (\frac{1}{2},\frac{1}{4}(\sqrt{3}-1),\frac{1}{4}(\sqrt{3}-1))$	$C_1(\frac{1}{2},-\frac{1}{2}(\sqrt{3}-1),0)$
$C_3'\ (0,-\frac{1}{2}(\sqrt{3}-1),\frac{1}{2})$	$H_3'(0,-r,\frac{1}{2}\sqrt{3}-r)$
$J_3\ (-r,-(\frac{1}{2}\sqrt{3}-r),0)$	$K_1(\frac{1}{2},-\frac{1}{4}(\sqrt{3}-1),-\frac{1}{4}(\sqrt{3}-1))$

$K_3(-\frac{1}{4}(\sqrt{3}-1),\ -\frac{1}{4}(\sqrt{3}-1),\frac{1}{2})$ $\qquad$ $H_1^{(3)}(\frac{\beta}{2},\frac{1}{4}(\beta-\sqrt{3}),\frac{1}{4}(\beta-\sqrt{3}))$

$H_3^{(3)}(\frac{1}{4}(\beta-\sqrt{3}),\frac{1}{4}(\beta-\sqrt{3}),\frac{\beta}{2})$

$M_1'((1-\alpha)(\frac{1}{2}\sqrt{3}-r),-\alpha(\frac{1}{2}\sqrt{3}-r),-r)$

$M_3(-r,-\alpha(\frac{1}{2}\sqrt{3}-r),(1-\alpha)(\frac{1}{2}\sqrt{3}-r))$

$M_3'(-\alpha(\frac{1}{2}\sqrt{3}-r),-r,(1-\alpha)(\frac{1}{2}\sqrt{3}-r))$.

The following dissections were found using a computer. In each case, the diameters of each set of the dissection agree to at least five decimal places, so I shall give the figure only once. I also state one edge which determines the diameter of each piece.

r	α	β	Critical edges 3 pieces	Critical edges Other piece	Diameter
0.5	0.5	0.5	(B_1'',M_3)	(M_1',M_3)	0.96592..
0.51	0.50459..	0.44444..	(B_1'',M_3)	(M_1',M_3)	0.97068..
0.52	0.50926..	0.41444..	(B_1'',M_3)	(M_1',M_3)	0.97553..
0.53	0.51402..	0.39894..	(B_1'',M_3)	(M_1,M_3)	0.98047..
0.56045..	0.53402..	0.35894..	(B_1'',M_3)	(M_1',M_3)	0.99709..

We note that for the case $r = 0.560454...$, the diameter of the sets is larger than that given by Grünbaum's dissection. Hence this dissection is useful only for values of r close to 0.5. Unfortunately the values of the diameter are fairly high. This is not surprising since this set covers the sphere which is conjectured to be the set which has the worst dissection.

Now the critical points of the dissection are B_1'', B_2'', B_3'', M_1, M_1', M_2, M_2', M_3 and M_3'. Of these points we can remove at least one of the B_i'''s by an argument similar to that of Sprague. Because of the similarity to the "3 corners occupied" case, we can apply a similar procedure to that we shall see in Section 5.3. This gives two different ways to round edges containing critical points. I tried both methods, but neither gave a significant improvement. Hence we omit the details.

The given dissection is the best dissection I could find for values of r close to 0.5. Later we shall see values of the diameter for other values of r.

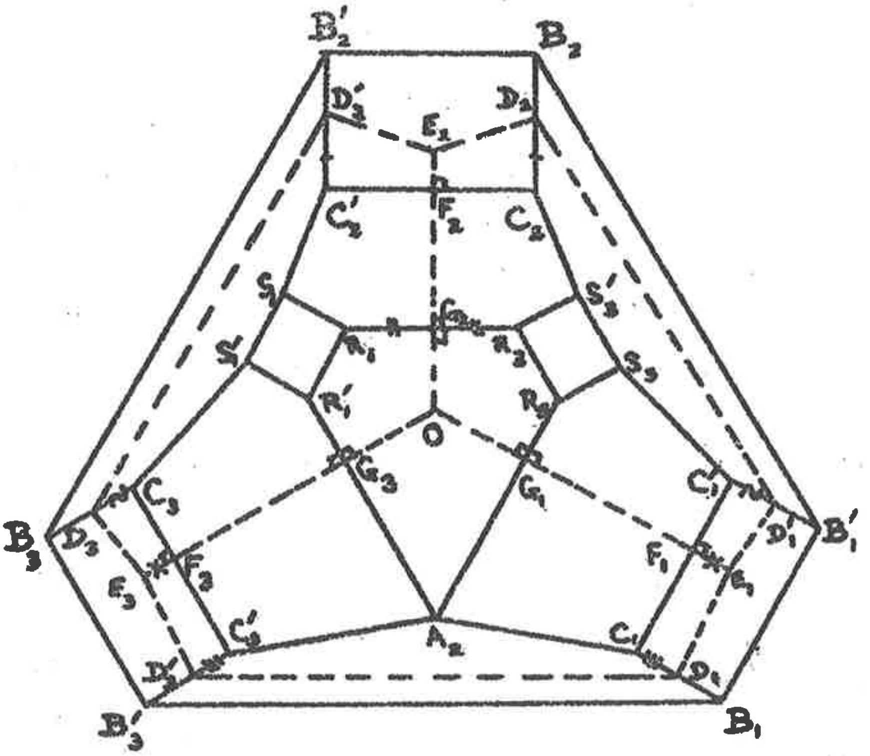
<u>The case: 1 corner occupied</u>

We now consider the case "1 corner occupied". In the two previous cases we had a degree of rotational symmetry. In this case, we have only symmetry about a plane and we make use of this in our dissection. The figure and dissection are shown in detail in the Schlegel diagram. The square faces $B_1B_1'C_1'C_1$ and $B_3B_3'C_3'C_3$ are the faces formed by Pál cuts. $B_2B_2'C_2'C_2$ is a face formed by large cuts, and the faces $R_1R_1'S_1'S_1$ and $R_3R_3'S_3'S_3$ are formed by small cuts. The notation is similar to that in the previous cases.

Suppose that

$$O = \alpha G_2 + (1-\alpha)A_2 \quad \text{where} \quad 0 \leqslant \alpha \leqslant 1.$$

Then $\quad G_1 = \beta R_3 + (1-\beta)A_2 \quad \text{where} \quad \beta = \frac{3\sqrt{3}}{8r}\alpha.$



We could impose the extra condition that $0 \leqslant \beta \leqslant 1$. However this is not necessary, since if this condition is not satisfied, the point so defined lies just off the surface of the cover. Similarly,

$$G_3 = \beta R_1' + (1-\beta)A_2.$$

Further, suppose that

$$E_i = F_i + \delta(B_i - C_i) \quad \text{for} \quad i=1,3 \text{ where } 0\leqslant\delta\leqslant 1$$

$$\text{and} \quad E_2 = F_2 + \phi(B_2 - C_2) \quad \text{where} \quad 0\leqslant\phi\leqslant 1.$$

Also, for parameters ϵ, θ and ψ satisfying

$$0\leqslant\epsilon\leqslant 1, \ 0\leqslant\theta\leqslant 1 \ \text{and} \ 0\leqslant\psi\leqslant 1,$$

let

$$D_1 = \epsilon B_1 + (1-\epsilon)C_1,$$

$$D_3' = \epsilon B_3' + (1-\epsilon)C_3',$$

$$D_1' = \theta B_1' + (1-\theta)C_1',$$

$$D_3 = \theta B_3 + (1-\theta)C_3',$$

$$D_2 = \psi B_2 + (1-\psi)C_2$$

and $$D_2' = \psi B_2' + (1-\psi)C_2'.$$

The dissection yields four pieces. Two of these are obtained from each other by a reflection and so have the same diameter. By simple geometric considerations, the diameter of each piece is determined by one of the following pairs.

<u>2 congruent pieces</u>:

(O,E_1), (O,D_1'), (O,D_2), (O,E_2), (O,F_2), (O,C_2), (O,C_1'), (G_1',E_2), (G_1,F_2), (G_1,D_2), (G_1,C_2), (G_1,D_1), (G_1,E_1), (F_1,G_2), (F_1,F_2), (F_1,R_3'), (E_1,R_3'), (E_1,G_2), (D_1',G_2), (D_1',F_2), (D_2,R_3) or (F_1,E_2).

<u>Piece containing the vertex A_2</u>:

(O,E_1), (O,D_1), (O,A_2), (O,C_1), (G_3,E_1), (G_3,D_1), (G_3,C_1), (F_1,F_3), (E_3,A_2) or (D_1,A_2).

<u>Other piece</u>:

(D_1',D_3), (D_1',B_3), (D_1',E_3), (D_1',D_3'), (D_1',D_2'), (D_1,D_2'), (D_1,E_2), (D_1,B_2'), (D_2,B_3'), (E_2,B_3'), (E_1,B_3) or (E_1,B_2').

The following coordinates are used.

$A_2(0,-\frac{1}{2}\sqrt{3},0)$	$B_2'(0,1-r,\frac{1}{2}\sqrt{3}-1+r)$
$B_3(0,\frac{1}{2}(\sqrt{3}-1),\frac{1}{2})$	$B_3'(\frac{1}{2}(\sqrt{3}-1),0,\frac{1}{2})$
$C_1(\frac{1}{2},-\frac{1}{2}(\sqrt{3}-1),0)$	$C_1'(\frac{1}{2},0,-\frac{1}{2}(\sqrt{3}-1))$
$C_2(0,1-r,-(\frac{1}{2}\sqrt{3}-1+r))$	$R_3(0,-(\frac{1}{2}\sqrt{3}-r),-r)$
$R_3'(-(\frac{1}{2}\sqrt{3}-r),0,-r)$	$O(-\frac{\alpha\sqrt{3}}{4},-\frac{(1-\alpha)\sqrt{3}}{4},-\frac{\alpha\sqrt{3}}{4})$
$G_1(0,-\frac{1}{2}\sqrt{3}+\beta r,-\beta r)$	$G_2(-\frac{1}{4}\sqrt{3},0,-\frac{1}{4}\sqrt{3})$

$G_3(-\beta r, -\frac{1}{2}\sqrt{3}+\beta r, 0)$

$F_1(\frac{1}{2}, -\frac{1}{2}\gamma(\sqrt{3}-1), -\frac{1}{2}(1-\gamma)(\sqrt{3}-1))$

$F_2(-\frac{1}{2}(\frac{1}{2}\sqrt{3}-1+r), 1-r, -\frac{1}{2}(\frac{1}{2}\sqrt{3}-1+r)$

$F_3(-\frac{1}{2}(1-\gamma)(\sqrt{3}-1), -\frac{1}{2}\gamma(\sqrt{3}-1), \frac{1}{2})$

$E_1(\frac{1}{2}, \delta(1-r)-\frac{1}{2}\gamma(\sqrt{3}-1), \delta(1-r)-\frac{1}{2}(1-\gamma)(\sqrt{3}-1))$

$E_2(\frac{1}{2}(2\phi-1)(\frac{1}{2}\sqrt{3}-1+r), 1-r, \frac{1}{2}(2\phi-1)(\frac{1}{2}\sqrt{3}-1+r))$

$E_3(\delta(1-r)-\frac{1}{2}(1-\gamma)(\sqrt{3}-1), \delta(1-r)-\frac{1}{2}\gamma(\sqrt{3}-1), \frac{1}{2})$

$D_1(\frac{1}{2}, -\frac{1}{2}(1-\epsilon)(\sqrt{3}-1), \frac{1}{2}\epsilon(\sqrt{3}-1))$

$D_1'(\frac{1}{2}, \frac{1}{2}\theta(\sqrt{3}-1), -\frac{1}{2}(1-\theta)(\sqrt{3}-1))$

$D_2(\psi(\frac{1}{2}\sqrt{3}-1+r), 1-r, -(1-\psi)(\frac{1}{2}\sqrt{3}-1+r))$

$D_2'(-(1-\psi)(\frac{1}{2}\sqrt{3}-1+r), 1-r, \psi(\frac{1}{2}\sqrt{3}-1+r))$

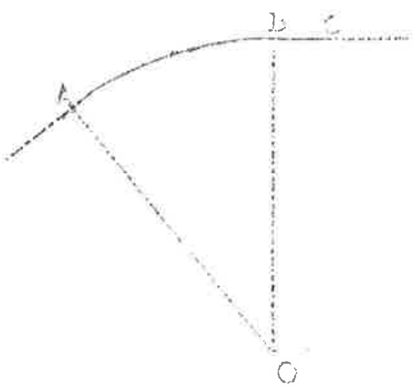
$D_3(-\frac{1}{2}(1-\theta)(\sqrt{3}-1), \frac{1}{2}\theta(\sqrt{3}-1), \frac{1}{2})$

$D_3'(\frac{1}{2}\epsilon(\sqrt{3}-1), -\frac{1}{2}(1-\epsilon)(\sqrt{3}-1), \frac{1}{2})$.

Using a computer, I found that the diameter of the sets of the dissection is determined by the two congruent pieces and by the piece containing the vertex A_2. The diameters are then determined by the lengths of the segments (E_1,G_2) or (E_2,G_1) and (G_3,E_1) respectively. For each r in the range $0.5 \leqslant r \leqslant 0.560454...$, it appears that the best dissection is obtained with these lengths approximately 0.988. Hence, to improve on Grünbaum's result, we need to remove some critical points.

We see that the face $B_1B_1'B_2B_2'B_3B_3'$ is parallel to and distance 1 from the face $A_2R_3R_3'R_1R_1'$ which contains the G_i's. Now, any closed set of constant width 1, when covered by this set of the 4-cover, touches each of these two faces at exactly one point with the line joining these two perpendicular to the faces. So we can apply a similar rounding argument to that used when we constructed the Sprague tile. Hence we can remove the edges containing G_1, G_2 and G_3 by spherical and cylindrical cuts. Thus we remove three critical points. Of course, A_2, R_3, R_3', R_1 and R_1' are also removed by these cuts, but we shall consider these points to be remaining unless they appear critical. We now use the same dissection as that we have just considered. Hence, we have a dissection, which, although it is not a dissection of the original set of the 4-cover, is a dissection of a set which covers the same sets of constant width 1.

In our dissection, some of the edges are curved. So it appears that a critical distance may involve a point in the middle of a curve. The following lemma, when appropriately extended, implies that such points need not be considered. We use $|XY|$ to denote the distance between two points X and Y.



Lemma

Consider the curve ABC in the plane shown in the diagram where

(i) AB is a circular arc of radius 1 with centre O .

(ii) BC is a tangent to the circular arc at B

(iii) $\angle AOB$ is no larger than $\frac{\pi}{2}$ radians and

(iv) $|AC| \leqslant 1$.

Suppose P is a point on the same side of arc ABC as O (i.e. the curve appears convex from P), and that

$$|AP| \leqslant 1 \quad \text{and} \quad |CP| \leqslant 1.$$

Then if S is a point on ABC, then

$$|SP| \leqslant \max\{|AP|,|CP|\}.$$

The truth of this lemma is easily checked using the following steps:

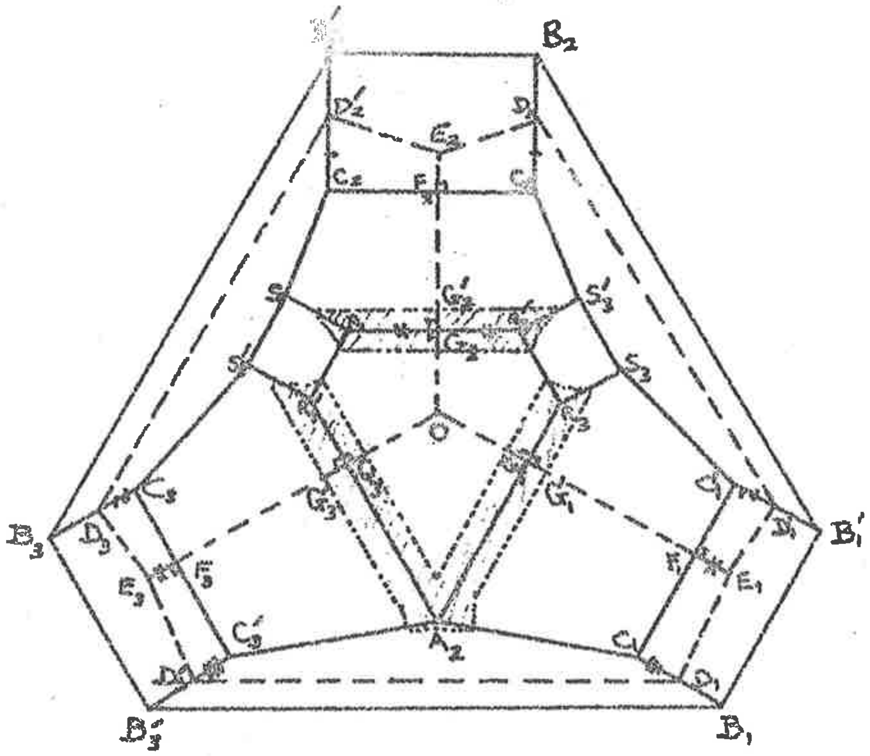
(1) Show that if S is a point on AB, then
$|SP| \leqslant \max\{|AP|,|BP|\}$.

(2) Clearly, if S is a point on BC, then
$|SP| \leqslant \max\{|BP|,|CP|\}$.

(3) Show that $|BP|$ is bounded by $|AP|$ and $|CP|$.

We now revert to consideration of our dissection. It is shown in detail in the Schlegel diagram. The regions removed have been indicated by shading.
Suppose G_1' is the point on the face $A_2C_1C_1'S_3R_3$ near G_1 where the dissection passes from the curved surface onto the flat face. Also, suppose G_2' and G_3' are points on the faces $R_3'S_3'C_2C_2'S_1R_1$ and $A_2C_3'C_3S_1'R_1'$ respectively defined similarly.

In our dissection, we can consider the curves like $G_2'O$ to be like the curve AC in the lemma. By the lemma, we need only consider the endpoints of the curve, provided the figure obtained for the diameter of each set is at most



1, which is the only case in which we are interested anyway. At times, the length of the line BC in the lemma may be zero, but this does not affect the truth of the result. So we can replace all mention of G_1, G_2 and G_3 in our previous argument by G_1', G_2' and G_3' respectively. The coordinates of these points are:

$$G_1'(\tfrac{1}{2}\sqrt{3} + 2t,\ \tfrac{1}{4}\sqrt{3} + t,\ \tfrac{1}{4}\sqrt{3} + t)$$

$$G_2'(-\tfrac{1}{4}\sqrt{3} + \zeta(\tfrac{1}{2} - \tfrac{r}{2}), \zeta(1-r), -\tfrac{1}{4}\sqrt{3} + \zeta(\tfrac{1}{2} - \tfrac{r}{2}))$$

$$G_3'(\tfrac{1}{4}\sqrt{3} + t,\ \tfrac{1}{4}\sqrt{3} + t,\ \tfrac{1}{2}\sqrt{3} + 2t)$$

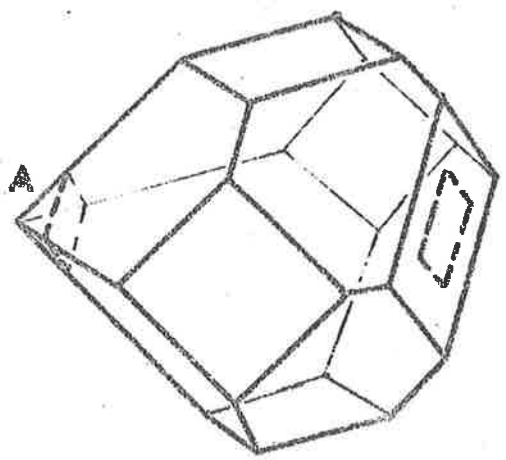
where ζ is the smaller solution of the equation

$$\tfrac{5}{4}(1-r)^2 z^2 - \tfrac{1}{2}(1-r)(3 + \sqrt{3})z + (\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}(1-r) + \tfrac{5}{4}(1-r)^2) = 0,$$

and, letting $a = \tfrac{1}{2}(\sqrt{3}-1)$,

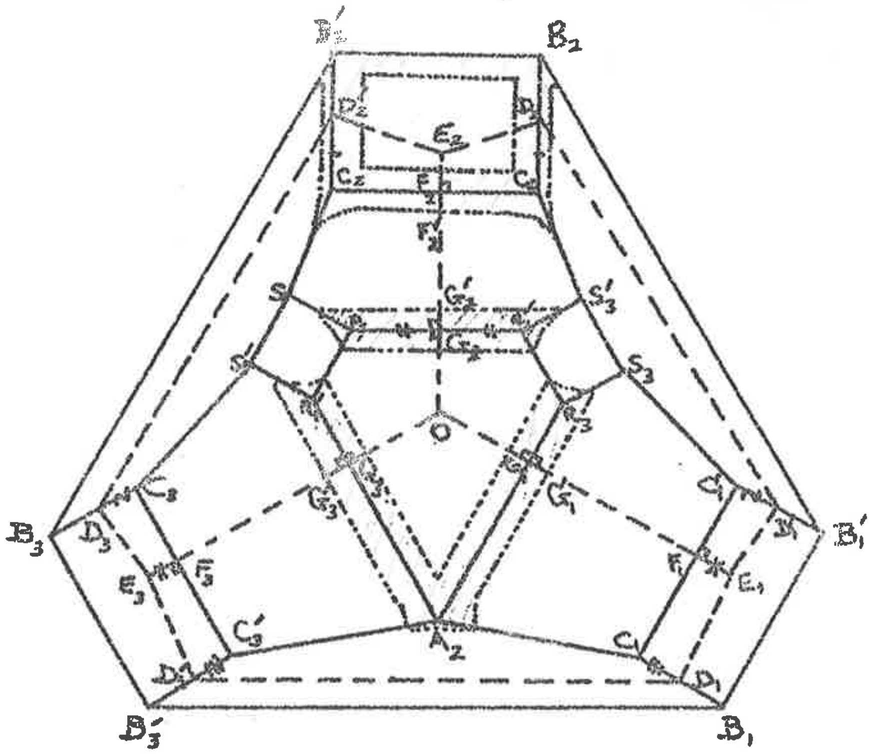
$$t = \frac{-a - \sqrt{6-8a^2}}{6}.$$

This modification results only in small improvements and the critical lengths become (F_1, F_2) and (G_3', E_1). Of these critical points we can remove F_2 by the following rounding argument.



In the construction of the 4-cover, we considered "corners" as the pieces of the octahedron beyond tangent planes to the sphere of radius r. Now a "corner" is a square-based pyramid. Suppose S is a closed set of constant width 1 which when covered by the octahedron intersects the corner A. Then no point of S can have distance more than 1 from the base of the pyramid. Hence in this set of the 1-cover, we can round the edges of the large square opposite the occupied corner.

The argument allows us to remove F_2. In addition, we can remove D_2 and D_2', but we shall do so only if they appear to be critical points of the dissection. Hence we do not change the coordinates of D_2 and D_2'.



Also we assume that E_2 lies on the original face, although a better dissection may exist with E_2 on the curved surface near F_2. Hence, the only new point we consider is F_2', as shown in the diagram, since by our lemma this is the only new point which may determine a diameter. Now F_2' has coordinates (x,y,z) where

$$z=x,\ y = 2x + \tfrac{1}{2}\sqrt{3},$$

and x is the larger root of the equation

$$2(x+\tfrac{1}{2}d)^2 + (2x+\tfrac{1}{2}\sqrt{3}+r)^2 = 1$$

where $d = \tfrac{1}{2}\sqrt{3}-r$.

Replacing all mention in the computer program of F_2 by F_2', I obtained the following dissections. In each case the diameters of the resulting sets are equal to at least five decimal places.

r	0.5	0.51	0.52	0.53	0.56045..
α	0.66828..	0.65953..	0.64897..	0.64247..	0.63081..
δ	0.10000..	0.11063..	0.13653..	0.12568..	0.14378..
ϵ	0.48683..	0.47251..	0.50839..	0.42701..	0.34181..
θ	0.58850..	0.58927..	0.54034..	0.61325..	0.59552..
ψ	0.43571..	0.45927..	0.48214..	0.52165..	0.60691..
Critical	edges				
2 pieces	(G_1', E_2)	(G_1', E_2)	(G_1', E_2)	(G_1', E_2)	(F_1, F_2')
Piece with A_2	(G_3', E_1)	(G_3', E_1)	(G_3', E_1)	(G_3', E_1)	(G_3', E_1)
Other piece	(D_1', E_3)	(D_1', E_2)	(D_1', E_3)	(D_1', E_2)	(D_1', E_3)
Diameter	0.98623..	0.98057..	0.97470..	0.96947..	0.96263..

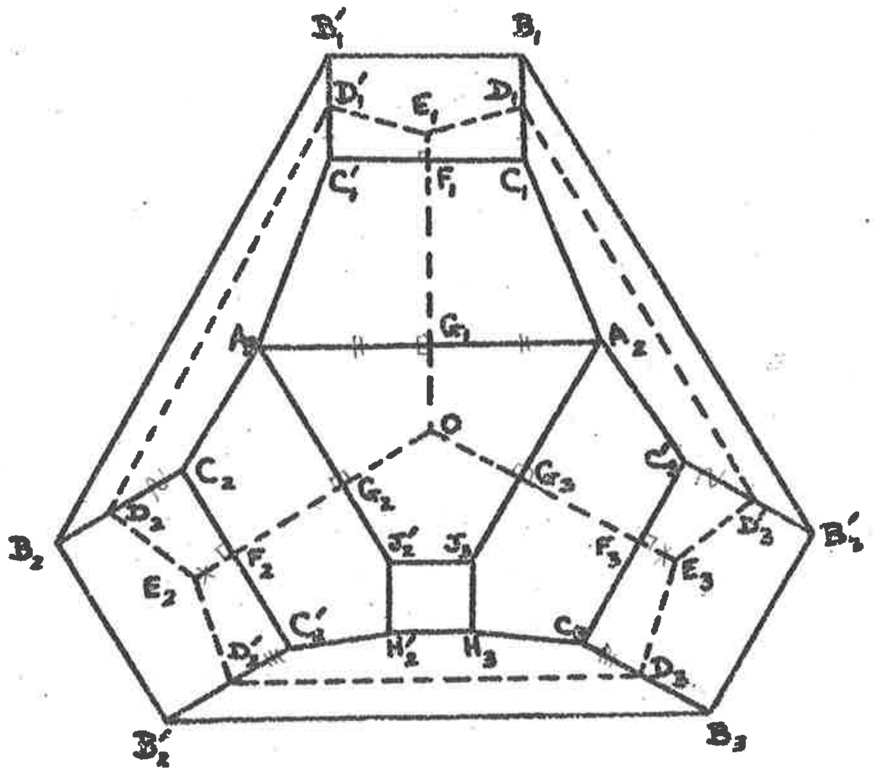
These are the best dissections of this set (or strictly of a set covering the same sets) of the 4-cover that I have found. Unfortunately, I can see no way of obtaining significant improvement.

The case: 2 corners occupied.

Finally, we consider the case "2 corners occupied". As in the case "1 corner occupied", we have symmetry only about a plane. The figure and the dissection we shall consider are shown in detail in the Schlegel diagram. The square face $B_1B_1'C_1'C_1$ is a face formed by a Pál cut. The faces $R_2S_2S_2'R_2'$ and $R_3S_3S_3'R_3'$ are both formed by large cuts, and the face $H_3J_3J_2'H_2'$ by a small cut. The notation is similar to that used in the previous cases.

As usual, we introduce parameters to describe the points determining the dissection. Suppose

$$O = \frac{\alpha}{2}(J_2'+J_3) + (1-\alpha)G_1 \quad \text{where} \quad 0\leqslant\alpha\leqslant 1.$$



We could impose extra restrictions on α by imposing the conditions that G_2 and F_2 lie in the closed line segments A_3J_2' and C_2C_2' respectively. However these conditions are not necessary. Further, supposing that $\beta,\gamma,\delta,\epsilon$ and θ are all parameters between 0 and 1,

let
$$E_1 = \frac{\beta}{2}(B_1+B_1') + (1-\beta)F_1,$$
$$E_2 = F_2 + \gamma(B_2-C_2),$$
$$E_3 = F_3 + \gamma(B_3-C_3),$$
$$D_2 = (1-\delta)C_2 + \delta B_2,$$
$$D_3' = (1-\delta)C_3' + \delta B_3',$$
$$D_1 = (1-\epsilon)C_1 + \epsilon B_1,$$
$$D_1' = (1-\epsilon)C_1' + \epsilon B_1',$$
$$D_2' = (1-\theta)C_2' + \theta B_2'$$
and
$$D_3 = (1-\theta)C_3 + \theta B_3.$$

As with the dissections in the case "1 corner occupied", there are two pieces of this dissection obtainable from each other by a reflection and two other incongruent pieces. By the usual considerations, the diameters of each set are determined by one of the following pairs.

<u>2 congruent pieces</u>:

(O,E_1), (O,D_1'), (O,D_2), (O,E_2), (O,C_1'), (O,A_3), (G_1,C_2), (G_1,F_2), (G_1,E_2), (G_1,D_2), (G_1,D_1'), (G_1,E_1), (F_1,E_2), (F_1,F_2), (F_1,G_2), (E_1,G_2), (E_1,A_3), (E_1,F_2), (D_1',G_2), (D_1',A_3), (D_2,A_3), (D_2,G_2), (E_2,A_3), (E_2,G_2), (E_2,A_3), (E_2,G_2) or (F_2,A_3).

<u>Other base piece</u> (i.e. other piece containing O):

(O,C_2), (O,D_2'), (G_2,E_3), (G_2,F_3), (G_2,D_3), (G_2,C_3) or (F_2,F_3).

<u>Other piece</u>:

(E_1,D_3), (E_1,B_3), (E_1,E_3), (E_1,D_2), (D_1,D_2'), (D_1,E_2), (D_1',R_3), (D_3',D_2), (D_3',B_2), (E_3,B_2), (E_3,B_1'), (E_3,D_1'), (E_3,D_2), (D_3,B_1') or (E_2,E_3).

The following coordinates are used.

$A_3(0,0,-\frac{1}{2}\sqrt{3})$

$B_1'(\frac{1}{2},\frac{1}{2}(\sqrt{3}-1),0)$

$B_2(\frac{1}{2}\sqrt{3}-1+r,1-r,0)$

$B_2'(0,1-r,\frac{1}{2}\sqrt{3}-1+r)$

$B_3(0,\frac{1}{2}\sqrt{3}-1+r,1-r)$

$B_3'(\frac{1}{2}\sqrt{3}-1+r,0,1-r)$

$C_1'(\frac{1}{2},0,-\frac{1}{2}(\sqrt{3}-1))$

$C_2(0,1-r,-(\frac{1}{2}\sqrt{3}-1+r))$

$C_2'(-(\frac{1}{2}\sqrt{3}-1+r),1-r,0)$

$C_3(-(\frac{1}{2}\sqrt{3}-1+r),0,1-r)$

$C_3'(0,-(\frac{1}{2}\sqrt{3}-1+r),1-r)$

$H_3(-r,0,-(\frac{1}{2}\sqrt{3}-r))$

$J_3(-r,-(\frac{1}{2}\sqrt{3}-r),0)$

$G_1(0,-\frac{1}{4}\sqrt{3},-\frac{1}{4}\sqrt{3})$

$O\ (-\alpha r,\frac{1}{2}\alpha r-\frac{1}{4}\sqrt{3},\frac{1}{2}\alpha r-\frac{1}{4}\sqrt{3})$

$G_2(r(t-1),0,(t-\frac{3}{2})r+\frac{1}{2}$

$F_1(\frac{1}{2}, -\frac{1}{4}(\sqrt{3}-1), -\frac{1}{4}(\sqrt{3}-1))$

$F_2(-(1-u)(\frac{1}{2}\sqrt{3}-1+r), 1-r, -u(\frac{1}{2}\sqrt{3}-1+r))$

$F_3(-u(\frac{1}{2}\sqrt{3}-1+r), -(1-u)(\frac{1}{2}\sqrt{3}-1+r), 1-r)$

$E_1(\frac{1}{2}, \frac{1}{4}(2\beta-1)(\sqrt{3}-1), \frac{1}{4}(2\beta-1)(\sqrt{3}-1))$

$E_2((u+\gamma-1)(\frac{1}{2}\sqrt{3}-1+r), 1-r, (\gamma-u)(\frac{1}{2}\sqrt{3}-1+r))$

$E_3((\gamma-u)(\frac{1}{2}\sqrt{3}-1+r), (u+\gamma-1)(\frac{1}{2}\sqrt{3}-1+r), 1-r)$

$D_1(\frac{1}{2}, \frac{1}{2}(\epsilon-1)(\sqrt{3}-1), \frac{1}{2}\epsilon(\sqrt{3}-1))$

$D_1'(\frac{1}{2}, \frac{1}{2}\epsilon(\sqrt{3}-1), \frac{1}{2}(\epsilon-1)(\sqrt{3}-1))$

$D_2(\delta(\frac{1}{2}\sqrt{3}-1+r), 1-r, (\delta-1)(\frac{1}{2}\sqrt{3}-1+r))$

$D_2'((\theta-1)(\frac{1}{2}\sqrt{3}-1+r), 1-r, \theta(\frac{1}{2}\sqrt{3}-1+r))$

$D_3((\theta-1)(\frac{1}{2}\sqrt{3}-1+r), \theta(\frac{1}{2}\sqrt{3}-1+r), 1-r)$

$D_3'(\delta(\frac{1}{2}\sqrt{3}-1+r), (\delta-1)(\frac{1}{2}\sqrt{3}-1+r), 1-r)$

where

$$t = 1 - \frac{3\alpha}{4} - \frac{\sqrt{3}}{8r}$$

and

$$u = \frac{1}{2} + \frac{\frac{1}{2}\sqrt{3} + (t-1)r}{\frac{1}{2}\sqrt{3} - 1 + r} .$$

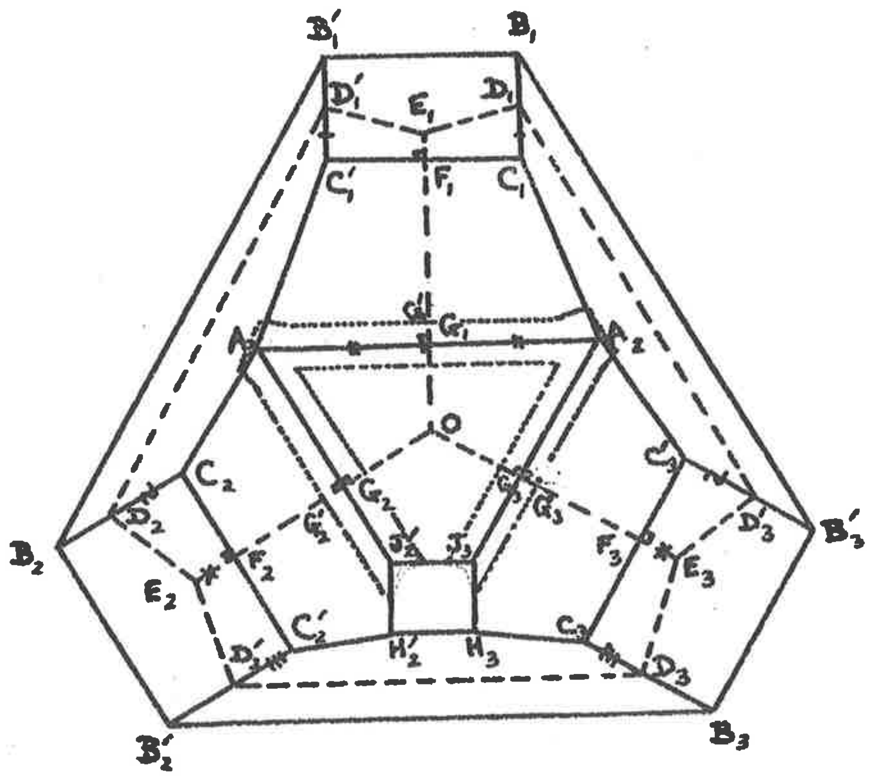
Using a computer, I obtained the dissections as shown below.

r	Diameter
0.5	0.98770..
0.51	0.98397..
0.52	0.97632..
0.53	0.97279..
0.560454..	0.96232..

However, we shall obtain better dissections than these. The critical points in the dissections are G_2, E_3, D_1', E_1, D_3

and similar points. Of these, G_2 and D_3 can be removed by rounding arguments. D_3 is removed by using the fact that the opposite corner is occupied. However, we get a sufficiently good result for our purposes by removing G_2 only. This is because the diameter of the final partitioning of the sets of the modified 4-cover is determined by the cases "No corner occupied" and "1 corner occupied".

Since the faces $B_1B_1'B_2B_2'B_3B_3'$ and $A_2A_3J_3J_2'$ are distance 1 apart and parallel, we can round the edges of the face $A_2A_3J_3J_2'$, in particular removing G_1, G_2 and G_3.



In fact, we also remove A_2, A_3, J_3 and J_2', but we need not bother about changing the coordinates of the points unless they appear critical in our modified dissection. Suppose G_1', G_2' and G_3' are the points near G_1, G_2 and G_3 respectively where the dissection line passes from the curved surface to the face containing C_1C_1', C_2C_2' and C_3C_3' respectively. The lemma now allows us to replace all mention of G_1, G_2 and G_3 in the argument to this stage by G_1', G_2' and G_3'. The new coordinates required are:

$$G_1'(2y + \tfrac{1}{2}\sqrt{3}, y, y)$$

where

$$y = \frac{-c - \sqrt{6-8c^2}}{6}, \quad c = \tfrac{1}{2}(\sqrt{3}-1)$$

and $G_2'(r(t-1)+x+\tfrac{1}{4}\sqrt{3},\ 2x+\tfrac{1}{2}\sqrt{3},\ (t-\tfrac{3}{2})r+\tfrac{1}{2}+x+\tfrac{1}{4}\sqrt{3})$

where

$$x = \frac{-a - \sqrt{6-8a^2}}{6}, \quad a = \tfrac{1}{2}\sqrt{3}-1+r, \quad \text{and } t \text{ is}$$

defined as before.

The following dissections were obtained. We shall consider other values of r later. In each case, the diameters of each set are equal to at least five decimal places.

r	0.5	0.51	0.52	0.53	0.56045..
α	0.57735..	0.52267..	0.49474..	0.45666..	0.37637..
β	0.12764..	0.18179..	0.16786..	0.18873..	0.21536..
γ	0.14665..	0.12720..	0.16451..	0.18409..	0.21236..
δ	0.48864..	0.47906..	0.47141..	0.45862..	0.47734..
ϵ	0.44605..	0.57074..	0.50629..	0.51463..	0.50111..
θ	0.49176..	0.44497..	0.47760..	0.46438..	0.47955..
Critical edges					
2 pieces	(G_1', E_2)	(D_1', G_2')	(E_1, G_2')	(E_1, G_2')	(E_1, G_2')
Other base piece	(G_2', E_3)	(G_2', E_3)	(G_2', E_3)	(G_2', E_3)	(G_2', D_3)
Other piece	(D_1, E_2)	(D_1', E_3)	(D_3, E_1)	(D_3', D_2)	(D_3, E_1)
Diameter	0.98769..	0.98013..	0.97355..	0.96788..	0.95507..

<u>Combination of the four dissections.</u>

We now have divided into four pieces each set of a 4-cover derived from the 4-cover given in the last chapter. If for a particular value of r $(\frac{1}{2} \leqslant r \leqslant \frac{1}{2}\sqrt{3})$, each set can be divided into four pieces of diameter $\leqslant x$ (say), then $d_3(4) \leqslant x$. The table shows the diameters of interest as calculated by the program. Now the diameters of the dissections in the cases "No corners occupied" and "3 corners occupied" are monotonic functions of r (as I conjecture are also the other two diameters). Hence, for $r = 0.519$ and $r = 0.5191$, the diameters in the case "No corner occupied" could read 0.97516... . The computer appears to have found a local minimum for the diameter. The results are illustrated on the graph.

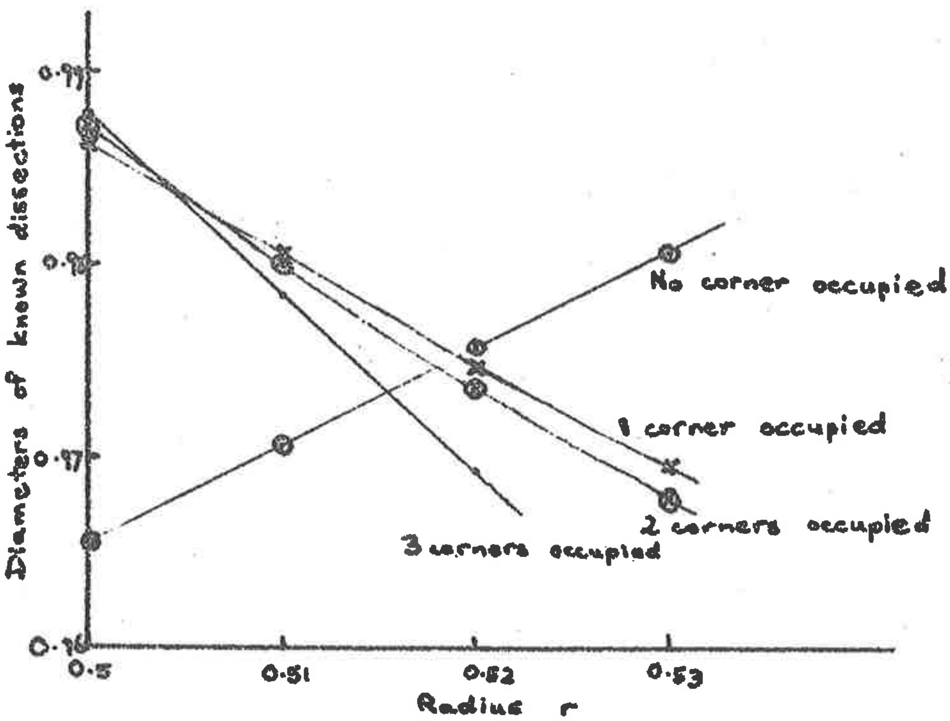


Table: Diameters of known dissections

r	Case: No. of corners occupied			
	0	1	2	3
0.5	0.96592..	0.98623..	0.98769..	0.98772..
0.51	0.97068..	0.98057..	0.98013..	0.97841..
0.52	0.97553..	0.97470..	0.97355..	0.96913..
0.53	0.98047..	0.96947..	0.96788..	0.95988..
0.5604..	0.98772..	0.96263..	0.95507..	0.93191..
0.511	0.97279..	0.97987..	0.97898..	0.97748..
0.512	0.97166..	0.97916..	0.97851..	0.97655..
0.513	0.97226..	0.97876..	0.97779..	0.97562..
0.514	0.97261..	0.97814..	0.97724..	0.97469..
0.515	0.97310..	0.97756..	0.97630..	0.97377..
0.516	0.97360..	0.97702..	0.97567..	0.97284..
0.517	0.97415..	0.97653..	0.97549..	0.97191..
0.518	0.97465..	0.97596..	0.97447..	0.97099..
0.519	0.97518..	0.97536..	0.97388..	0.97006..
0.5191	0.97522..	0.97530..	0.97395..	0.96997..
0.5192	0.97516..	0.97522..	0.97369..	0.96987..
0.5193	0.97525..	0.97519..	0.97360..	0.96978..
0.5194	0.97525..	0.97515..	0.97354..	0.96969..
0.5195	0.97540..	0.97509..	0.97365..	0.96960..
0.5196	0.97540..	0.97501..	0.97351..	0.96950..
0.5197	0.97543..	0.97495..	0.97348..	0.96941..
0.5198	0.97544..	0.97489..	0.97340..	0.96932..
0.5199	0.97560..	0.97484..	0.97330..	0.96922..

Looking at the case $r = 0.5192$, we see that

$$d_3(4) \leqslant 0.97522\ldots \;.$$

This compares with the result proved earlier that $d_3(4) \leqslant 0.98772\ldots$. We could improve on this new result by meticulous partitioning, but the improvement would only be slight. The table below gives the values of the parameters for each partition in the case $r = 0.5192$. Also, it shows the critical distances in the order they appear in the earlier tables.

	No. of corners occupied			
	0	1	2	3
Parameters	α=0.50896.. β=0.47006..	α=0.65017.. δ=0.13212.. ϵ=0.49221.. θ=0.55091.. ϕ=0.14931.. ψ=0.45676..	α=0.49600.. β=0.16807.. γ=0.15187.. δ=0.46107.. ϵ=0.54290.. θ=0.47807..	α=0.86213.. β=0.46875..
Critical edges	(B_1'', M_3) (M_1', M_3)	(G_1', E_2) (G_3', E_1) (D_1', E_3)	(E_1, G_2') (G_2', E_3) (D_1', E_3)	(G_1, E_3) (E_3, D_2)

The result overall is disappointing. It should be noted that the dissections given may not be the best dissections of their form. Even with a computer it is impossible to undertake a thorough scanning process when six variables are involved. Hence any of the programs may have produced a result which is a good approximation to a local minimum but not a good approximation to the actual minimum diameter.

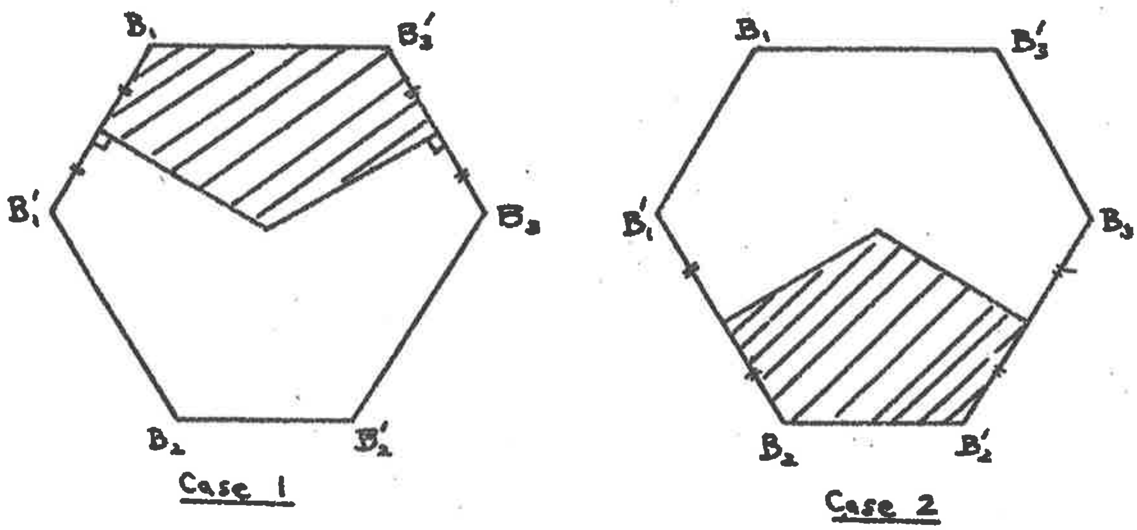
From the graph, we see that an improvement in the case "No corner occupied" would give an improvement in the diameter. However, even by applying rounding techniques and using the symmetry of the set, I could obtain no significant improvement in this case. For any value of $r(\frac{1}{2} \leq r \leq \frac{1}{2}\sqrt{3})$, the diameter of the sets of a partition of this set is at least 0.96592.. . This critical distance occurs twelve times in the set and I can see no way of eliminating it from the sets of a partition. Hence the 4-cover (with modification) can not give a result better than 0.96592.. . To obtain a better result by the method of k-covers, either a cover of larger order or a completely different cover is needed. We shall look at an attempt using a different cover in Section 5.4.

In an attempt to obtain a better result, I tried to adjoin another set to the 4-cover to obtain a 5-cover which could be dissected to give a lower diameter. In particular, I tried to adjoin a sphere of diameter slightly greater than 1 to the 4-cover, so that many of the sets covered by the set of the 4-cover where no "corners" are occupied would be covered by this sphere. Hopefully, this would allow further truncation of this troublesome set of the 4-cover, and hence yield a smaller bound on $d_3(4)$. My attempts were unsuccessful.

5.3 Dissection of a 1-cover

Using the 4-cover, we have shown that $d_3(4) \leq 0.97522..$. It is an obvious question to ask whether there exists a dissection of a 1-cover which gives as good a result. So we now consider modifications to our dissection of the truncated octahedron covering body discussed in the previous section as the case "3 corners occupied" with $r=\frac{1}{2}$.

Suppose the dissection is labelled as before. The critical points of the dissection are G_1, G_2, G_3 and $D_1, D_1', D_2, D_2', D_3, D_3'$. We can improve the dissection by removing the G_i's from the covering set, hence producing smaller covering bodies. For suppose S is a closed set of constant width 1 covered by the truncated octahedron. Since the faces A_1, A_2, A_3 and $B_1B_1'B_2B_2'B_3B_3'$ are distance 1 apart and parallel, S touches each at exactly one point and the line joining these two points is normal to the two faces. Then, using the symmetry of the truncated octahedron, we can make either one of the following assumptions. We can assume that S touches the face $B_1B_1'B_2B_2'B_3B_3'$ in the region shaded in the figure labelled Case 1. Alternatively, S touches the face in the region shaded in the figure labelled Case 2.



I tried both assumptions separately, and obtained a better dissection using Case 2, so we shall look at that case here.

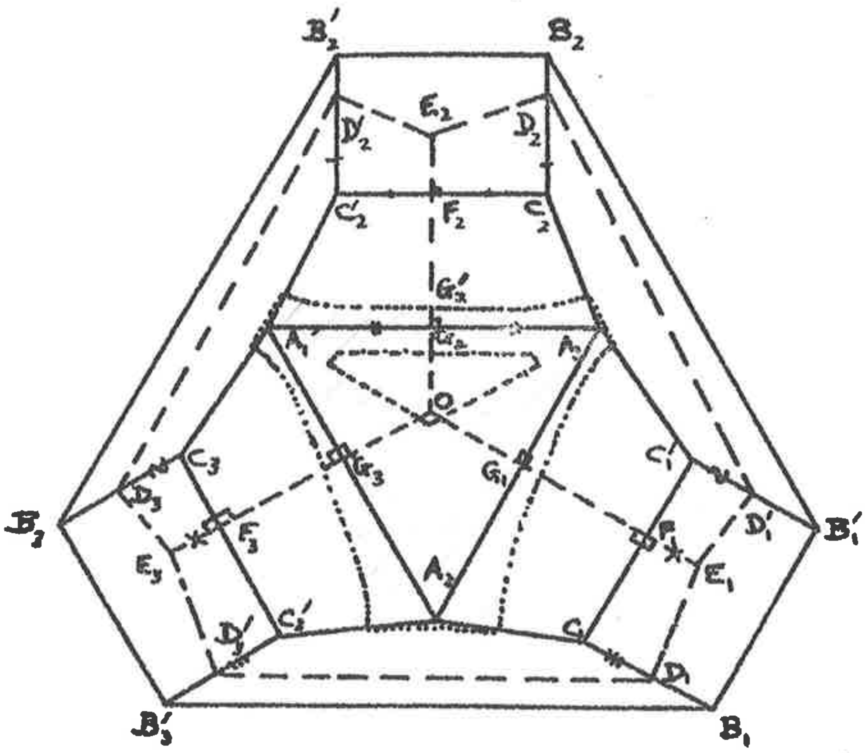
We now assume that any set of constant width 1, when covered by the truncated octahedron, intersects the shaded region in the figure labelled Case 2. It follows that no point of the covering body with distance more than 1 from this region need be included in the cover. Hence we can round the edges of the face $A_1A_2A_3$ and, in particular, remove G_1, G_2 and G_3. Hence we assume that we have removed all points that we can from the covering body by this argument using spherical and cylindrical cuts. This gives a better covering body than the truncated octahedron. We do not calculate its volume here, essentially because of the difficulties involved, but also because the improvement in volume is probably not large.

We label the vertices of the covering body as before. Several of its vertices have been removed. However, in our calculations, we assume that they are still present since no vertex of the truncated octahedron is a critical point in the previous dissection. If one appears to be critical in our new dissection, we can then allow for its new position. Hence for our calculation of diameters, we assume that only G_1, G_2 and G_3 have been removed. In the diagram, the region removed has been shaded.

The dissection we use is shown in detail in the diagram. We introduce parameters. The point O is no longer fixed, being

$$O = (1-\gamma)A_2 + \gamma\,\frac{A_1+A_3}{2}$$

where $0\leqslant\gamma\leqslant1$.



Suppose G_2' is the point on the face $A_1C_2'C_2A_3$ where the dissection line on the new covering body passes from that face onto the curved surface. Suppose G_1' and G_3' are points near G_1 and G_3 similarly defined. As before, let

$$E_2 = \alpha F_2 + (1-\alpha)\,\frac{B_2+B_2'}{2},$$

$$D_2 = \beta C_2 + (1-\beta)B_2$$

and $$D_2' = \beta C_2' + (1-\beta)B_2'$$

where $0\leqslant\alpha\leqslant1$ and $0\leqslant\beta\leqslant1$. In addition, for parameters

θ,ϕ and ψ between 0 and 1,

let $E_i = F_i + (1-\theta)(B_i - C_i)$ for $i=1,3$

$D_1 = \phi C_1 + (1-\phi)B_1$,

$D_3' = \phi C_3' + (1-\phi)B_3'$,

$D_1' = \psi C_1' + (1-\psi)B_1'$

and $D_3 = \psi C_3 + (1-\psi)B_3$.

The parameters $\alpha,\beta,\gamma,\theta,\phi$ and ψ determine the dissection.

Two of the sets of the dissection are obtainable from one another by a reflection. By simple geometric considerations including use of the lemma proved in the previous section, one of the following pairs must determine the diameter of each piece.

<u>2 similar pieces</u>:

$(0,E_2)$, $(0,E_1)$, $(0,D_2)$, $(0,D_1')$, $(0,A_3)$, $(0,C_1')$, $(0,C_2)$, (G_1',E_2), (G_2',E_1), (G_2',F_1), (G_1',D_2), (G_2',D_1'), (G_1',C_3'), (F_1,F_2), (F_1,E_2), (F_2,E_1), (F_1,D_2), (F_2,D_1'), (E_2,A_3), (E_1,A_3), (F_1,A_3), (E_2,C_1'), (E_1,C_2), (D_2,A_3) or (D_1',A_3).

<u>Piece containing the vertex A_2</u>:

$(0,E_1)$, $(0,D_3')$, $(0,A_2)$, $(0,C_3')$, (G_1',E_3), (G_1',F_3), (G_1',D_3'), (F_1,F_3), (F_1,D_3'), (F_1,E_3), (E_1,A_2), (F_1,A_2), (E_1,C_3') or (D_3',A_2).

<u>Other piece</u>:

(E_1,E_2), (E_1,E_3), (E_3,D_2), (E_2,D_3'), (E_1,B_2'), (E_1,B_3), (E_2,B_3'), (D_2,D_3'), (D_1',D_3), (D_1,B_2'), (D_1,B_3), (D_2,B_3'), (D_1',B_2') or (D_1',B_3).

We now introduce coordinates. The following are needed:

$A_2(0,-\frac{1}{2}\sqrt{3},0)$

$B_2'(0,\ \frac{1}{2}\ ,\ a)$

$B_3'(a,\ 0,\ \frac{1}{2})$

$C_2(0,\ \frac{1}{2},-a)$

$F_2(-\frac{1}{2}a,\frac{1}{2},-\frac{1}{2}a)$

$D_2((1-\beta)a,\frac{1}{2},-\beta a)$

$F_1(\frac{1}{2},-\epsilon a,-(1-\epsilon)a)$

$E_1(\frac{1}{2},(1-\epsilon-\theta)a,(\epsilon-\theta)a)$

$D_1(\frac{1}{2},-\phi a,(1-\phi)a)$

$D_1'(\frac{1}{2},(1-\psi)a,-\psi a)$

$A_3(0,0,-\frac{1}{2}\sqrt{3})$

$B_3(0,\ a,\ \frac{1}{2})$

$C_1'(\frac{1}{2},\ 0,-a)$

$C_3'(0,-a,\ \frac{1}{2})$

$E_2(\frac{1}{2}(1-2\alpha)a,\frac{1}{2},\frac{1}{2}(1-2\alpha)a)$

$O(-\frac{1}{4}\gamma\sqrt{3},-\frac{1}{2}(1-\gamma)\sqrt{3},-\frac{1}{4}\gamma\sqrt{3})$

$F_3(-(1-\epsilon)a,-\epsilon a,\frac{1}{2})$

$E_3((\epsilon-\theta)a,(1-\epsilon-\theta)a,\frac{1}{2})$

$D_3'((1-\phi)a,-\phi a,\frac{1}{2})$

$D_3(-\psi a,(1-\psi)a,\frac{1}{2})$

where

$a = \frac{1}{2}(\sqrt{3}-1)$

and

$\delta = 1 - \frac{3}{4}\gamma,$ $\qquad \epsilon = \dfrac{2\sqrt{3}\delta-1}{4a}$.

Also, suppose x is the smaller solution of the equation

$$6z^2 + 2az + \frac{3a^2}{2} = 1, \quad \text{where } a \text{ is as above.}$$

Then $G_2' = (x,2x + \frac{1}{2}\sqrt{3},x)$.

Finally, we determine the coordinates of G_1'. There are two possibilities. Either the curved surface at G_1' is spherical or it is cylindrical, depending on the value of γ. Now,

$$G_1' = (0,-\tfrac{1}{2}\delta\sqrt{3},-\tfrac{1}{2}(1-\delta)\sqrt{3}) + \rho(\tfrac{1}{2},\tfrac{1}{4},\tfrac{1}{4})$$

where ρ is yet to be determined. If the cut is spherical, then ρ is the smaller solution of the equation

$$9z^2 - 14\sqrt{3}z + 12(3\delta^2-3\delta+1) = 0.$$

If the cut is cylindrical, then ρ is the smaller solution of the equation

$$105z^2 + 12(3\delta-16)\sqrt{3}z + 108\delta^2 = 0.$$

To decide whether the spherical or the cylindrical cut applies, one simply takes the cut with the lower value of ρ.

Using a computer I found a dissection of the new covering body into four sets of diameter 0.98006... . The value of the parameters then are:

$$\alpha=0.63139.. ,$$
$$\beta=0.31575.. ,$$
$$\gamma=0.68205.. ,$$
$$\theta=0.91435.. ,$$
$$\phi=0.42166..$$
and $\psi=0.49931..$.

A critical edge of the two similar pieces is (G_2', E_1), of the piece containing A_2 is (F_1, F_3), and of the other piece (E_3, D_2).

Hence, using a 1-cover, we have shown that

$$d_3(4) \leqslant 0.98006... .$$

We had previously dissected a 1-cover into four sets of diameter 0.98772... . This new result is the best result I was able to find using a 1-cover. It compares with the bound

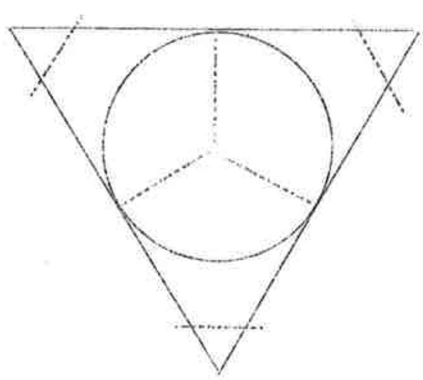
$$d_3(4) \leqslant 0.97522...$$

found using the 4-cover.

5.4 Use of a different k-cover

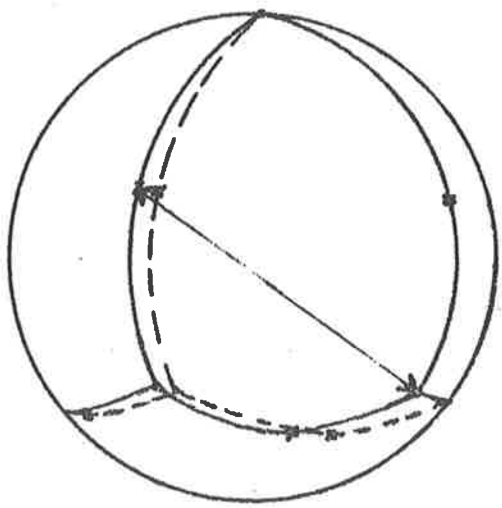
Unfortunately the 4-cover has not produced a large reduction in the upper bound on $d_3(4)$. So we shall try another approach to the problem. Instead of starting with a given k-cover, we shall try to obtain a k-cover which gives a good dissection. Since the method of universal covers is so effective in the case $n=2$, we look in detail at the dissection involved.

When a circle is partitioned into three sets in the best manner, the circumference is divided into three equal-length arcs. The points determining the diameters are the endpoints of the arcs and are equally spaced around the circle. Suppose we draw tangents to the circle at three points. Then we obtain an equilateral triangle which happens to be a tile. In fact, we can remove the corners of the triangle, as shown, provided we do not make the resulting set smaller than the regular hexagon of width 1. Provided we

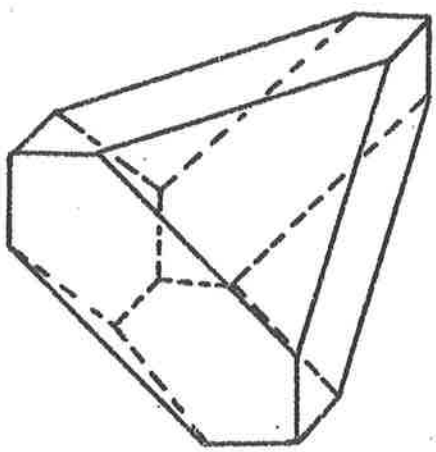


remove sufficiently large regions at the corners, the dissection induced by the circle proves that $d_2(3)=\frac{1}{2}\sqrt{3}$.

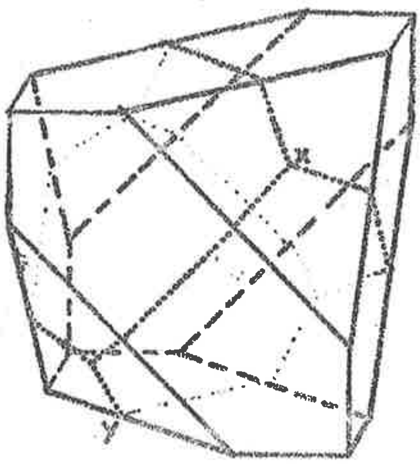
We try a similar approach in three dimensional space. The best partition of the sphere is the obvious



simplical partition. The diameters are determined by segments with one endpoint at a vertex of the curvilinear triangle on the surface, and the other at the midpoint of the opposite arc of the triangle. Now there are four such vertices and six such midpoints. Suppose we draw tangent to the sphere at the points. The tangent planes at the six midpoints determine a cube of width 1, and the four other planes truncate four corners to give a figure as shown in the diagram.



We observe that no two opposite corners of the cube have been removed. This suggests an obvious method for forming a universal cover by removing one of each pair of opposite corners by making cuts at tangent planes to the sphere. This method produces a 3-cover of volume 0.83012..., so the cover gives no new insight into the value of $V_{3,3}$ since it is higher than the volume of the truncated octahedron.



The dissection of the set formed by taking tangent planes to the sphere is shown. It is the obvious partition induced by the sphere and gives four congruent pieces. The diameter of each set of the dissection is easily calculated. It is determined by segments like XY in the diagram, where X is the centroid of a triangular face and Y is the midpoint of an intersection line of a triangular face with a face of the cube. Then the diameter of the sets of the partition is

$$\sqrt{\frac{8+\sqrt{3}}{12}} = 0.90055\ldots .$$

This figure is encouraging since we have partitioned a set containing the sphere. However, I could not dissect the other sets of the 3-cover in a satisfactory manner, and so this attempt proves unsuccessful.

5.3 Conclusion

We introduced the idea of k-cover to obtain an improved upper bound on $d_n(n+1)$, especially for $n=3$. Using a 1-cover, we have shown

$$\sqrt{(3+\sqrt{3})/6} = 0.88807.. \leq d_3(4) \leq 0.98006\ldots .$$

By using a 4-cover, we have been able to obtain the better upper bound of

$$d_3(4) \leqslant 0.97522\ldots .$$

The result is disappointing in its magnitude. However, it is pleasing that it is obtained from simple geometrical ideas. Our efforts indicate that although the method of k-covers works in principle, a large amount of work is required to obtain good results.

BIBLIOGRAPHY

[1] Anderson, R.D. and Klee, V.R. Jr. "Convex functions and upper semi-continuous collections", Duke Math. J. 19 (1952), 349-357.

[2] Benson, R.V. "Euclidean geometry and convexity", McGraw-Hill, New York, 1966.

[3] Borsuk, K. "Drei Sätze über die n-dimensionale euklidische Sphäre", Fund. Math. 20 (1933), 177-190.

[4] Eggleston, H.G. "Covering a three-dimensional set with sets of smaller diameter", J. London Math. Soc. 30 (1955), 11-24.

[5] Eggleston, H.G. "Problems in Euclidean space: Application of convexity", Pergamon, New York, 1957.

[6] Eggleston, H.G. "Convexity", Cambridge Univ. Press, Cambridge, 1958.

[7] Gale, D. "On inscribing n-dimensional sets in a regular n-simplex", Proc. Amer. Math. Soc. 4 (1953), 222-225.

[8] Grünbaum, B. "A simple proof of Borsuk's conjecture in three dimensions", Proc. Cambridge Philos. Soc. 53 (1957), 776-778.

[9] Grünbaum, B. "Borsuk's problem and related questions", Amer. Math. Soc., Proc. Symp. Pure Math. 7 (1963), 271-284.

[10] Hadwiger, H. "Überdeckung einer Menge durch Mengen kleineren Durchmessers", Comment. Math. Helv. 18 (1945/46), 73-75.

[11] Hadwiger, H. "Mitteilung betreffend meine Note: Überdeckung einer Menge durch Mengen kleineren Durchmessers", Comment. Math. Helv. 19 (1946/47), 72-73.

[12] Hadwiger, H. "Über die Zerstückung eines Eikörpers", Math. Z. 51 (1947), 161-165.

[13] Hadwiger, H. "Von der Zerlegung der Kugel in kleinere Teile", Gaz. Mat. Lisboa 15 (1954), 1-3.

[14] Heppes, A. "On the splitting of point sets in three space into the union of sets of smaller diameter", Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 7 (1957), 413-416. (Hungarian.)

[15] Jung, H.W.E. "Über die kleinste Kugel, die eine räumliche Figur einschliesst", (Crelles) J. Reine Angew. Math. 123 (1901), 241-257.

[16] Jung, H.W.E. "Über den kleinsten Kreis, der eine ebene Figur einschliesst", (Crelles) J. Reine Angew. Math. 137 (1910), 310-313.

[17] Knaster, B. "Ein Zerlegungssatz uber unikohärente Kontinua", Verh. Internat. Math.-Kongr., Zürich, 1932, Vol.2, pp 193-194.

[18] Meschkowski, H. "Unsolved and unsolvable problems in geometry" (English translation by Jane A.C. Burlak), Oliver and Boyd, 1966; German original published by Friedr. Vieweg and Sohn, Brunswick, 1960.

[19] Pál, J. "Über ein elementares Variationsproblem", Danske Vid. Selsk. Math.-Fys Medd. 3 (1920), no.2.

[20] Santalo, L.A. "Convex regions on the n-dimensional spherical surface", Ann. of Math. (2) 47 (1946), 448-459.

[21] Sprague, R. "Über ein elementares Variationsproblem", Mat. Tideskrift (1936), 96-99.

[22] Steinhagen, P. "Über die grösste Kugel in einer konvexen Punktmenge", Abh. Math. Sem. Hamburg Univ. 1 (1922), 15-26.

[23] Valentine, F.A. "Convex sets", McGraw-Hill, New York, 1964.