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A class of asynchronous parallel nonlinear accelerated overrelaxation methods for the nonlinear complementarity problems[1]

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Abstract

In accordance with the principle of using sufficiently the delayed information, and by making use of the nonlinear multisplitting and the nonlinear relaxation techniques, we present in this paper a class of asynchronous parallel nonlinear multisplitting accelerated overrelaxation (AOR) methods for solving the large sparse nonlinear complementarity problems on the high-speed MIMD multiprocessor systems. These new methods, in particular, include the so-called asynchronous parallel nonlinear multisplitting AOR-Newton method, the asynchronous parallel nonlinear multisplitting AOR-chord method and the asynchronous parallel nonlinear multisplitting AOR-Steffensen method. Under suitable constraints on the nonlinear multisplitting and the relaxation parameters, we establish the local convergence theory of this class of new methods when the Jacobi matrix of the involved nonlinear mapping at the solution point of the nonlinear complementarity problem is an H-matrix.

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1. Introduction

Consider the nonlinear complementarity problem

NCP(F): *Find* $x \geqslant 0$ *such that* $F(x) \geqslant 0$ *and* $x^{\mathrm{T}}F(x)=0$.

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Here, $x=(x_1,\ldots,x_n)^T$ is the unknown vector, $F:R^n\to R^n$, $F(x)=(f_1(x),\ldots,f_n(x))^T$ is a given differentiable nonlinear mapping, and the partial ordering "$\geqslant$" is understood in accordance with the elements. Since this problem has bounteous practical application backgrounds, there has been a lot of research on sequential numerical methods in the literature.

To solve this problem efficiently in the synchronous parallel computational environments, by equivalently transforming the NCP(F) into several lower-dimensional systems of nonlinear equations through the multiple nonlinear splittings of the nonlinear mapping $F:R^n\to R^n$, Bai and Wang [1] recently presented a class of synchronous parallel nonlinear multisplitting SOR (successive over-relaxation) methods by making use of the nonlinear relaxation technique. In the implementations of these methods, each processor of the multiprocessor system only need to solve a lower dimensional system of nonlinear equations. Hence, these methods have good parallelism, and they are feasible and efficient for solving the NCP(F) on the multiprocessor systems. However, since it is usually impossible to decompose the nonlinear mapping $F:R^n\to R^n$ such that the computational workload of each of the resulted sub-systems of nonlinear equations is equally distributed, the mutual waits among the processors of the multiprocessor system are then inevitable in practical implementations. To avoid the mutual waits among the processors and to rise the parallel computational efficiency of these methods, by applying the principle of using sufficiently the delayed information, and considering the concrete properties of the MIMD multiprocessor system, we propose in this paper a class of asynchronous parallel nonlinear multisplitting accelerated overrelaxation (AOR) methods for solving the NCP(F) in the asynchronous parallel computational environments. This class of methods is established by utilizing the nonlinear accelerated overrelaxation technique to the sub-systems of nonlinear equations equivalently reduced from the NCP(F). With suitable choices of its two arbitrary parameters, not only the convergence property of this class of new methods can be greatly improved, but also a series of useful asynchronous parallel nonlinear multisplitting relaxation methods, e.g., the asynchronous parallel nonlinear multisplitting Jacobi method, the asynchronous parallel nonlinear multisplitting extrapolated Jacobi method, the asynchronous parallel nonlinear multisplitting Gauss–Seidel method, the asynchronous parallel nonlinear multisplitting extrapolated Gauss–Seidel method and the asynchronous parallel nonlinear multisplitting SOR method, etc., can be generated. Moreover, besides covering the synchronous parallel nonlinear multisplitting SOR-like methods discussed in [1], the new methods can also results the asynchronous parallel nonlinear multisplitting AOR-Newton-like methods, which include the asynchronous parallel nonlinear multisplitting AOR-Newton method, the asynchronous parallel nonlinear multisplitting AOR-chord method and the asynchronous parallel nonlinear multisplitting AOR-Steffensen method. Thereby, flexible and diverse choices are afforded for solving the NCP(F) on the MIMD multiprocessor systems. Because this new class of asynchronous parallel nonlinear multisplitting relaxation methods utilizes the most currently available components of the local approximate solutions of the NCP(F) to update the corresponding components of its global approximate solution in time and the processors of the MIMD multiprocessor system dose not need any mutual wait, it has the potential of exploiting the parallel computational efficiency of the multiprocessor system as far as possible. Under similar conditions to those in [1], we establish the local convergence theories of the new asynchronous parallel nonlinear multisplitting AOR methods.

2. Establishments of the methods

Given a positive integer α ($\alpha \leqslant n$) and a nonlinear differentiable mapping $F: R^n \to R^n$. A collection of pairs $(F^{(i)}, E_i)$ $(i=1,2,\ldots,\alpha)$ is called a nonlinear multisplitting of the nonlinear mapping $F: R^n \to R^n$ if the following three conditions are satisfied:

(a) $F^{(i)}: R^n \times R^n \to R^n$, $i=1,2,\ldots,\alpha$, are continuously differentiable mappings such that $F^{(i)}(x;x)=F(x)$ $(\forall x \in R^n)$, $i=1,2,\ldots,\alpha$;

(b) J_i, $i=1,2,\ldots,\alpha$, are nonempty subsets of the positive integer set $\{1,2,\ldots,n\}$ such that $\bigcup_{i=1}^{\alpha} J_i = \{1,2,\ldots,n\}$; and

(c) $E_i = \operatorname{diag}(e_1^{(i)}, e_2^{(i)}, \ldots, e_n^{(i)})$, $i=1,2,\ldots,\alpha$, are nonnegative diagonal matrices such that

$$\begin{cases} e_j^{(i)} = \begin{cases} e_j^{(i)} \geqslant 0, & \text{for } j \in J_i, \\ 0, & \text{otherwise,} \end{cases} & \sum_{i=1}^{\alpha} e_j^{(i)} = 1, \\ j=1,2,\ldots,n; \quad i=1,2,\ldots,\alpha. \end{cases}$$

The matrices $E_i (i=1,2,\ldots,\alpha)$ are called weighting matrices. Some concrete applicable examples of the nonlinear multisplitting $(F^{(i)}, E_i)$ $(i=1,2,\ldots,\alpha)$ of the nonlinear mapping $F: R^n \to R^n$ can be found in [4].

Assume that the referred multiprocessor system is made up of α CPUs. To set up the new asynchronous parallel nonlinear multisplitting AOR methods, we introduce the following elementary notations:

(1) for $\forall i \in \{1,2,\ldots,\alpha\}$ and $\forall p \in N_0 := \{0,1,2,\ldots\}$, $J^{(i)} = \{J_i(p)\}_{p \in N_0}$ is used to denote a sequence of subsets (may be empty) of the set J_i;

(2) for $\forall m \in \{1,2,\ldots,n\}$ and $\forall p \in N_0$, $N_m(p) := \{i \mid m \in J_i(p), \quad i=1,2,\ldots,\alpha\}$;

(3) for $\forall i \in \{1,2,\ldots,\alpha\}$, $S^{(i)} = \{s_1^{(i)}(p), s_2^{(i)}(p), \ldots, s_n^{(i)}(p)\}_{p \in N_0}$ is n infinite sequences of nonnegative integers.

$J^{(i)}$ and $S^{(i)} (i=1,2,\ldots,\alpha)$ have the following properties:

(a) for $\forall i \in \{1,2,\ldots,\alpha\}$ and $\forall m \in \{1,2,\ldots,n\}$, the set $\{p \in N_0 \mid m \in J_i(p)\}$ is infinite;

(b) for $\forall p \in N_0$, $\bigcup_{i=1}^{\alpha} J_i(p) \neq \emptyset$ (the empty set);

(c) for $\forall i \in \{1,2,\ldots,\alpha\}$, $\forall m \in \{1,2,\ldots,n\}$ and $\forall p \in N_0$, $s_m^{(i)}(p) \leqslant p$; and

(d) for $\forall i \in \{1,2,\ldots,\alpha\}$ and $\forall m \in \{1,2,\ldots,n\}$, $\lim_{p \to \infty} s_m^{(i)}(p) = \infty$.

For $\forall p \in N_0$, if we define

$$s(p) = \min_{\substack{1 \leqslant m \leqslant n \\ 1 \leqslant i \leqslant \alpha}} s_m^{(i)}(p),$$

then it evidently holds that $s(p) \leqslant p$ and $\lim_{p \to \infty} s(p) = \infty$.

Now, we can describe the asynchronous parallel nonlinear multisplitting AOR method for solving the NCP(F) as follows:

Method I. *Given an initial vector $x^0 \in R^n$. Suppose that we have got approximations $\{x^t\}_{t=0}^{p}$ of the solution x^* of the* NCP(F). *Then, the $(p+1)$th approximation $x^{p+1} = (x_1^{p+1}, x_2^{p+1}, \ldots, x_n^{p+1})^{\mathrm{T}}$ of x^* can be calculated element by element by*

$$x_m^{p+1} = \sum_{i=1}^{\alpha} e_m^{(i)} x_m^{p,i}, \quad m=1,2,\ldots,n, \tag{2.1}$$

where the mth component $x_m^{p,i}$ *of the local approximation* $x^{p,i}$ *is defined through the formula*

$$x_m^{p,i}=\begin{cases}\omega\hat{x}_m^{p,i}+(1-\omega)x_m^{s_m^{(i)}(p)} & \text{for } m\in J_i(p),\\ x_m^p & \text{for } m\notin J_i(p),\end{cases} \tag{2.2}$$

and the mth component $\hat{x}_m^{p,i}(m\in J_i(p))$ *of the approximation* $\hat{x}^{p,i}$ *is obtained from the nonlinear equation*

$$(f_m^{(i)}(x^{s^{(i)}(p)};\hat{u}_m^{p,i})-\hat{x}_m^{p,i})^2-f_m^{(i)}(x^{s^{(i)}(p)};\hat{u}_m^{p,i})|f_m^{(i)}(x^{s^{(i)}(p)};\hat{u}_m^{p,i})|-\hat{x}_m^{p,i}|\hat{x}_m^{p,i}|=0, \tag{2.3}$$

in which

$$\begin{aligned}&x^{s^{(i)}(p)}=(x_1^{s_1^{(i)}(p)},x_2^{s_2^{(i)}(p)},\ldots,x_n^{s_n^{(i)}(p)})^{\mathrm{T}},\\ &\hat{u}_m^{p,i}=(\tilde{x}_1^{p,i},\ldots,\tilde{x}_{m-1}^{p,i},\hat{x}_m^{p,i},x_{m+1}^{s_{m+1}^{(i)}(p)},\ldots,x_n^{s_n^{(i)}(p)})^{\mathrm{T}}\end{aligned} \tag{2.4}$$

and

$$\tilde{x}_j^{p,i}=\begin{cases}r\hat{x}_j^{p,i}+(1-r)x_j^{s_j^{(i)}(p)}, & \text{for } j\in J_i(p),\\ x_j^{s_j^{(i)}(p)}, & \text{for } j\notin J_i(p),\end{cases} \tag{2.5}$$

$$\tilde{x}^{p,i}=(\tilde{x}_1^{p,i},\tilde{x}_2^{p,i},\ldots,\tilde{x}_n^{p,i})^{\mathrm{T}}.$$

Here, $r\in[0,\infty)$ *is called a relaxation factor, while* $\omega\in(0,\infty)$ *an acceleration factor.*

In Method I, corresponding to the special choices $(0,1)$, $(0,\omega)$, $(1,1)$, $(1,\omega)$ and (ω,ω) of the involved parameter pair (r,ω), we can obtain the practical asynchronous parallel nonlinear multisplitting Jacobi, extrapolated Jacobi, Gauss–Seidel, extrapolated Gauss–Seidel and SOR methods for solving the NCP(F) on the MIMD multiprocessor systems. In particular, when

$$\begin{aligned}&J_i(p)=J_i,\quad s_m^{(i)}(p)=p,\\ &m=1,2,\ldots,n,\quad i=1,2,\ldots,\alpha,\quad p\in N_0,\end{aligned}$$

Method I reduces to a synchronous parallel nonlinear multisplitting AOR method, an extrapolated variant of the synchronous parallel nonlinear multisplitting SOR method discussed in [1], for solving the NCP(F) on the multiprocessor systems.

On the other hand, in Method I, each processor can update the global approximation or any group of the components of the global approximation arbitrarily with the most currently available components of the local approximations in an elementwise manner. Hence, the communication of Method I is quite flexible, and its potential of exploiting the parallel efficiency in concrete computations is possibly high.

Note that to obtain the exact solution of the nonlinear equation (2.3) is considerably difficult in practical computations, in applications we usually make use of a known iterative procedure, for example, the Newton procedure, the chord procedure or the Steffensen procedure (see [1, 2, 8]), etc., to get an approximate solution of the nonlinear equation (2.3). This immediately leads to the

following asynchronous parallel nonlinear multisplitting AOR-Newton-like method for solving the NCP(F).

Method II. *Given an initial vector* $x^0 \in R^n$. *Suppose that we have got approximations* $\{x^t\}_{t=0}^{p}$ *of the solution* x^* *of the NCP(F). Then, the* $(p+1)$*th approximation* x^{p+1} *of* x^* *can be determined by* (2.1), (2.2), (2.5) *and*

$$x_m^{p,i} = x_m^{s_m^{(i)}(p)} + \frac{\omega}{2} \frac{g_m^{(i)}(x^{s^{(i)}(p)}; u_m^{p,i})}{\eta_{mm}^{(i)}(x^{s^{(i)}(p)}; u_m^{p,i})}; \quad m \in J_i(p), \quad i = 1, 2, \ldots, \alpha,$$

where

$$x^{s^{(i)}(p)} = (x_1^{s_1^{(i)}(p)}, x_2^{s_2^{(i)}(p)}, \ldots, x_n^{s_n^{(i)}(p)})^{\mathrm{T}},$$

$$u_m^{p,i} = (\tilde{x}_1^{p,i}, \ldots, \tilde{x}_{m-1}^{p,i}, x_m^{s_m^{(i)}(p)}, \ldots, x_n^{s_n^{(i)}(p)})^{\mathrm{T}},$$

for $\forall x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$, $\forall y = (y_1, y_2, \ldots, y_n)^{\mathrm{T}} \in R^n$,

$$g_m^{(i)}(x; y) = (f_m^{(i)}(x; y) - y_m)^2 - f_m^{(i)}(x; y)|f_m^{(i)}(x; y)| - y_m|y_m|,$$

$$\eta_{mm}^{(i)}(x; y) = f_m^{(i)}(x; y) - (1 - \operatorname{sign}(x_m))x_m + \{x_m - [1 - \operatorname{sign}(f_m^{(i)}(x; y))]f_m^{(i)}(x; y)\}H_{mm}^{(i)}(x; y),$$

$$\operatorname{sign}(x_m) = \begin{cases} 1 & \text{if } x_m > 0, \\ 0 & \text{if } x_m = 0, \\ -1 & \text{if } x_m < 0, \end{cases}$$

and $H_{mm}^{(i)}(x; y)$ *denotes the* m*th diagonal element of some matrix* $H^{(i)}(x; y) \in L(R^n)$. *Here, again,* $r \in [0, \infty)$ *is a relaxation factor and* $\omega \in (0, \infty)$ *an acceleration factor.*

Corresponding to different choices of the matrices $H^{(i)}(x; y)$ $(i = 1, 2, \ldots, \alpha)$ in Method II, we can derive various practical programs for the NCP(F). For example, if we take

$$H_{mm}^{(i)}(x; y) = \partial_2 f_{mm}^{(i)}(x; y), \quad m \in J_i(p), \quad i = 1, 2, \ldots, \alpha, \quad p \in N_0,$$

then the asynchronous parallel nonlinear multisplitting AOR-Newton method for the NCP(F) can be obtained, since the nonlinear equation (2.3) is approximately solved by the Newton iteration. Here, $\partial_1 f_{mm}^{(i)}(x; y)$ and $\partial_2 f_{mm}^{(i)}(x; y)$ represent the mth partial derivatives of $f_m^{(i)}(x; y)$ with respect to the variables x and y, respectively; and if we take

$$H_{mm}^{(i)}(x; y) = \frac{f_m^{(i)}(x; y + h_m^{(i)}(x; y)e_m) - f_m^{(i)}(x; y)}{h_m^{(i)}(x; y)}, \quad m \in J_i(p), \quad i = 1, 2, \ldots, \alpha, \quad p \in N_0,$$

then the asynchronous parallel nonlinear multisplitting AOR-chord method for the NCP(F) can be got, because the nonlinear equation (2.3) is approximately solved by the chord iteration. Here, $e_m = (0, \ldots, 0, 1, 0, \ldots, 0)^{\mathrm{T}} \in R^n$ represents the mth unit vector in R^n, and $h_m^{(i)}(x; y) \in R^1$ is the difference step-size. In particular, when

$$h_m^{(i)}(x; y) = f_m^{(i)}(x; y), \quad m \in J_i(p), \quad i = 1, 2, \ldots, \alpha, \quad p \in N_0,$$

this method further reduces to the asynchronous parallel nonlinear multisplitting AOR-Steffensen method for the NCP(F), as the nonlinear equation (2.3) is approximately solved by the Steffensen iteration.

Analogously, with different choices of the parameter pair (r,ω) in Method II, we can get another extensive sequence of practical asynchronous parallel nonlinear multisplitting relaxed Newton-like methods for solving the NCP(F) on the MIMD multiprocessor system. For the length of the paper, we will not enumerate them one by one, here.

3. Preliminaries

Let $A=(a_{mj})\in L(R^n)$. By $\mathrm{diag}(A)$ we denote the $n\times n$ diagonal matrix coinciding in its diagonal with $A\in L(R^n)$. For $A=(a_{mj})$, $B=(b_{mj})\in L(R^n)$, we write $A\leqslant B$ if $a_{mj}\leqslant b_{mj}$ holds for all $m,j=1,2,\ldots,n$. Calling $A\in L(R^n)$ nonnegative if $A\geqslant 0$. By $|A|=(|a_{mj}|)$ we define the absolute value of $A\in L(R^n)$; it is a nonnegative $n\times n$ matrix satisfying $|AB|\leqslant|A||B|$ for $B\in L(R^n)$. These notations can be immediately carried on the vectors in R^n. We denote by $\langle A\rangle=(\langle a_{mj}\rangle)\in L(R^n)$ the comparison matrix of $A\in L(R^n)$, where

$$\langle a_{mj}\rangle=\begin{cases}|a_{mj}| & \text{if } m=j,\\ -|a_{mj}| & \text{if } m\neq j.\end{cases}$$

We call $A=(a_{mj})\in L(R^n)$ an M-matrix if it is nonsingular with $a_{mj}\leqslant 0$ for $m\neq j$ and $A^{-1}\geqslant 0$. We call it an H-matrix if $\langle A\rangle$ is an M-matrix. Denote $D_A=\mathrm{diag}(A)$ and $B_A=D_A-A$. Then it evidently holds that $\langle A\rangle=|D_A|-|B_A|$. Moreover, if $A\in L(R^n)$ is an H-matrix, then both D_A and $|D_A|$ are nonsingular and the spectral radius of the matrix $|D_A|^{-1}|B_A|$ is less than one, i.e., $\rho(|D_A|^{-1}|B_A|)<1$. For the related properties about M-matrix and H-matrix, one can refer to [9, 4] for details.

Given a matrix $A\in L(R^n)$. For $i=1,2,\ldots,\alpha$, let $\mathscr{D}_{A,i}=\mathrm{diag}(A)$, $\mathscr{L}_{A,i}\in L(R^n)$ be strictly lower triangular matrices, $\mathscr{V}_{A,i}\in L(R^n)$ be general matrices and $E_i\in L(R^n)$ be nonnegative diagonal matrices such that

(i) $A=\mathscr{D}_{A,i}-\mathscr{L}_{A,i}-\mathscr{V}_{A,i}$, $i=1,2,\ldots,\alpha$;
(ii) $\mathscr{D}_{A,i}$, $i=1,2,\ldots,\alpha$, are nonsingular; and
(iii) $\sum_{i=1}^{\alpha}E_i=I$ (the $n\times n$ identity matrix).

Then, the collection of triples $(\mathscr{D}_{A,i}-\mathscr{L}_{A,i},\mathscr{V}_{A,i},E_i)$ $(i=1,2,\ldots,\alpha)$ is called a multisplitting of the matrix $A\in L(R^n)$. For details about this concept, one can refer to [1–4] and [10].

Consider a nonlinear differentiable mapping $F:R^n\to R^n$ and its nonlinear multisplitting $(F^{(i)},E_i)$ $(i=1,2,\ldots,\alpha)$. For each $i\in\{1,2,\ldots,\alpha\}$, by $\partial_1F^{(i)}$ and $\partial_2F^{(i)}$ we represent the partial derivatives of the nonlinear mapping $F^{(i)}:R^n\times R^n\to R^n$ with respect to its first and second variables, respectively. Obviously, it holds that for all $x,y\in R^n$,

$$\partial_1F^{(i)}(x;y)=(\partial_1f_{mj}^{(i)}(x;y))=\left(\frac{\partial f_m^{(i)}(x;y)}{\partial x_j}\right)\in L(R^n),$$

$$\partial_2F^{(i)}(x;y)=(\partial_2f_{mj}^{(i)}(x;y))=\left(\frac{\partial f_m^{(i)}(x;y)}{\partial y_j}\right)\in L(R^n)$$

and

$$F'(x)=\partial_1 F^{(i)}(x;x)+\partial_2 F^{(i)}(x;x)\in L(R^n).$$

Now, for an $x^*\in R^n$, by introducing matrices

$$\begin{cases}\mathscr{D}_{F,i}=\operatorname{diag}(\partial_2 F^{(i)}(x^*;x^*))=\operatorname{diag}(\partial_2 f_{11}^{(i)}(x^*;x^*),\dots,\partial_2 f_{nn}^{(i)}(x^*;x^*)),\\ \mathscr{L}_{F,i}=(L_{m,j}^{(f,i)})\in L(R^n),\quad L_{m,j}^{(f,i)}=\begin{cases}-\partial_2 f_{mj}^{(i)}(x^*;x^*) & \text{for } m>j \text{ and } m,j\in J_i,\\ 0 & \text{otherwise,}\end{cases}\\ \mathscr{U}_{F,i}=(U_{m,j}^{(f,i)})\in L(R^n),\quad U_{m,j}^{(f,i)}=\begin{cases}0 & \text{for } m>j \text{ and } m,j\in J_i,\\ 0 & \text{for } m=j,\\ -\partial_2 f_{mj}^{(i)}(x^*;x^*) & \text{otherwise,}\end{cases}\\ \mathscr{V}_{F,i}=(V_{m,j}^{(f,i)})\in L(R^n),\quad V_{m,j}^{(f,i)}=U_{m,j}^{(f,i)}-\partial_1 f_{mj}^{(i)}(x^*;x^*);\\ m,j=1,2,\dots,n;\quad i=1,2,\dots,\alpha,\end{cases}\tag{3.1}$$

we easily have

$$F'(x^*)=\mathscr{D}_{F,i}-\mathscr{L}_{F,i}-\mathscr{V}_{F,i},\quad i=1,2,\dots,\alpha.$$

If the matrices $\mathscr{D}_{F,i}$ $(i=1,2,\dots,\alpha)$ are nonsingular, then $(\mathscr{D}_{F,i}-\mathscr{L}_{F,i},\mathscr{V}_{F,i},E_i)$ $(i=1,2,\dots,\alpha)$ clearly forms a multisplitting of the matrix $F'(x^*)\in L(R^n)$.

Correspondingly, if we again introduce α nonlinear mappings $G^{(i)}:R^n\times R^n\to R^n$ $(i=1,2,\dots,\alpha)$ in accordance with

$$\begin{cases}G^{(i)}(x;y)^{\mathrm{T}}=(g_1^{(i)}(x;y),g_2^{(i)}(x;y),\dots,g_n^{(i)}(x;y)),\\ g_j^{(i)}(x;y)=(f_j^{(i)}(x;y)-y_j)^2-f_j^{(i)}(x;y)|f_j^{(i)}(x;y)|-y_j|y_j|,\\ j=1,2,\dots,n;\quad i=1,2,\dots,\alpha,\end{cases}\tag{3.2}$$

then $G^{(i)}:R^n\times R^n\to R^n$ $(i=1,2,\dots,\alpha)$ are clearly continuously differentiable, and satisfy

$$G^{(i)}(x;x)=G(x),\quad \forall x\in R^n,\quad i=1,2,\dots,\alpha,$$

with $G:R^n\to R^n$ being defined by

$$\begin{aligned}&G(x)^{\mathrm{T}}=(g_1(x),g_2(x),\dots,g_n(x)),\\ &g_j(x)=(f_j(x)-x_j)^2-f_j(x)|f_j(x)|-x_j|x_j|,\quad j=1,2,\dots,n.\end{aligned}\tag{3.3}$$

That is to say, the collection of pairs $(G^{(i)},E_i)$ $(i=1,2,\dots,\alpha)$ naturally forms a nonlinear multisplitting of the nonlinear mapping $G:R^n\to R^n$. Note that $G:R^n\to R^n$ is also differentiable due to the differentiability of the nonlinear mapping $F:R^n\to R^n$.

Now, let $x^*\in R^n$ be a solution of the NCP(F) and $(\mathscr{D}_{G,i}-\mathscr{L}_{G,i},\mathscr{V}_{G,i},E_i)$ $(i=1,2,\dots,\alpha)$ be a multisplitting of the matrix $G'(x^*)\in L(R^n)$. From [1] we immediately know that

$$\begin{aligned}&\mathscr{D}_{G,i}=-2(\mathscr{D}\mathscr{D}_{F,i}+\mathscr{D}_F)\\ &\mathscr{L}_{G,i}=-2\mathscr{D}\mathscr{L}_{F,i}\qquad (i=1,2,\dots,\alpha),\\ &\mathscr{V}_{G,i}=-2\mathscr{D}\mathscr{V}_{F,i}\end{aligned}$$

where

$$\mathscr{D}_F = \operatorname{diag}(f_1(x^*), f_2(x^*), \ldots, f_n(x^*)),$$
$$\mathscr{D} = \operatorname{diag}(x_1^*, x_2^*, \ldots, x_n^*),$$

and

$$G'(x^*) = -2(\mathscr{D}F'(x^*) + \mathscr{D}_F).$$

With the above preparations, we are now ready to set up the local convergence theories of Methods I and II.

4. Local convergence theories

In this section, under suitable conditions we will demonstrate local convergence theorems for Methods I and II by making use of Theorems 1 and 2 in [2], respectively.

Theorem 4.1. *Let $x^* \in R^n$ be a solution of the* NCP(F). *Let $(F^{(i)}, E_i)$ $(i=1,2,\ldots,\alpha)$ be a nonlinear multisplitting of the nonlinear mapping $F: R^n \to R^n$, and $F^{(i)}: R^n \times R^n \to R^n$ $(i=1,2,\ldots,\alpha)$ be continuously differentiable in a neighborhood of $(x^*;x^*)$. Assume that $F'(x^*) \in L(R^n)$ is an H-matrix, and $(\mathscr{D}_{F,i} - \mathscr{L}_{F,i}, \mathscr{V}_{F,i}, E_i)$ $(i=1,2,\ldots,\alpha)$ is its multisplitting satisfying*

$$D := \operatorname{diag}(F'(x^*)) = \mathscr{D}_{F,i}, \quad i=1,2,\ldots,\alpha,$$

and

$$\langle F'(x^*)\rangle = |\mathscr{D}_{F,i}| - |\mathscr{L}_{F,i}| - |\mathscr{V}_{F,i}| \equiv |D| - |B|, \quad i=1,2,\ldots,\alpha,$$

where the matrices $\mathscr{D}_{F,i}$, $\mathscr{L}_{F,i}$ and $\mathscr{V}_{F,i}$, $i=1,2,\ldots,\alpha$, are defined in (3.1), *and $B=D-F'(x^*)$. If*

$$d_m = x_m^* \frac{\partial f_m(x^*)}{\partial x_m} + f_m(x^*) \neq 0, \quad m=1,2,\ldots,n,$$

then $x^* \in R^n$ is an attraction point of Method I *provided the relaxation parameters r and ω satisfy*

$$0 \leqslant r \leqslant \omega, \quad 0 < \omega < \frac{2}{1+\rho(|D|^{-1}|B|)}.$$

Proof. Because $x^* \in R^n$ is a solution of the NCP(F), from [1] we know that $(x^*;x^*)$ satisfies the NCP($F^{(i)}$) $(i=1,2,\ldots,\alpha)$. That is to say, it holds that

$$x^* \geqslant 0, \quad F^{(i)}(x^*;x^*) \geqslant 0, \quad (x^*)^{\mathrm{T}} F^{(i)}(x^*;x^*) = 0, \quad i=1,2,\ldots,\alpha.$$

Moreover, since it has been proved in [1] that for any fixed point $x \in R^n$, the nonlinear complementarity problems

$$y \geqslant 0, \quad F^{(i)}(x;y) \geqslant 0, \quad y^{\mathrm{T}} F^{(i)}(x;y) = 0, \quad i=1,2,\ldots,\alpha$$

are, respectively, equivalent to the nonlinear systems of equations

$$G^{(i)}(x; y)=0, \quad i=1,2,\ldots,\alpha,$$

we see that any solution $x^* \in R^n$ of the NCP(F) is evidently also a solution of the system of nonlinear equations $G(x)=0$, where $G^{(i)}: R^n \times R^n \to R^n$ $(i=1,2,\ldots,\alpha)$ are defined by (3.2), and $G: R^n \to R^n$ is defined by (3.3).

On the other hand, in light of the continuous differentiability of the nonlinear mappings $F^{(i)}: R^n \times R^n \to R^n$ $(i=1,2,\ldots,\alpha)$ and the definitions of the nonlinear mappings $G^{(i)}: R^n \times R^n \to R^n$ $(i=1,2,\ldots,\alpha)$, we see that $G^{(i)}: R^n \times R^n \to R^n$ $(i=1,2,\ldots,\alpha)$ are also continuously differentiable in a neighborhood of $(x^*; x^*)$.

Now, observing that the nonlinear equation (2.3) is equivalent to the nonlinear equation

$$g_m^{(i)}(x^{s^{(i)}(p)}; \hat{u}_m^{p,i})=0, \quad m \in J_i(p),$$

in accordance with [2] we see that Method I is substantially the asynchronous parallel nonlinear multisplitting AOR method for the system of nonlinear equations $G(x)=0$ with respect to the nonlinear multisplitting $(G^{(i)}, E_i)$ $(i=1,2,\ldots,\alpha)$ of the nonlinear mapping $G: R^n \to R^n$.

Furthermore, under the assumptions of this theorem, we can demonstrate the validity of the following four facts in an analogous way to [1]:

(1) the matrices $\mathscr{D}_{G,i}$, $i=1,2,\ldots,\alpha$, are nonsingular;

(2) $\langle G'(x^*)\rangle = |\mathscr{D}_{G,i}| - |\mathscr{L}_{G,i}| - |\mathscr{V}_{G,i}| \equiv |D_G| - |B_G|$, $i=1,2,\ldots,\alpha$;

(3) $\rho(|D_G|^{-1}|B_G|) \leqslant \rho(|D|^{-1}|B|) < 1$; and

(4) $G'(x^*) \in L(R^n)$ is an H-matrix.

Therefore, all assumptions of Theorem 1 in [2] hold for the nonlinear mapping $G: R^n \to R^n$ and its nonlinear multisplitting $(G^{(i)}, E_i)$ $(i=1,2,\ldots,\alpha)$. According to Theorem 1 in [2], we easily know that the sequence $\{x^p\}_{p \in N_0}$ generated by Method I starting from any initial vector $x^0 \in R^n$ within a neighborhood of x^*, the solution of the system of nonlinear equations $G(x)=0$, converges to this x^*. Hence, $x^* \in R^n$ is an attraction point of Method I. Here, we remark that a solution of the system of nonlinear equations $G(x)=0$ is also a solution of the NCP(F).

Theorem 4.2. *Under the conditions of Theorem* 4.1, *we additionally assume that for any* $i \in \{1,2,\ldots,\alpha\}$ *and any* $m \in \{1,2,\ldots,n\}$, $H_{mm}^{(i)}(x; y)$ *is continuously differentiable in a neighborhood of* $(x^*; x^*)$, *and*

$$\lim_{(x;y)\to(x^*;x^*)} H_{mm}^{(i)}(x; y) = \partial_2 f_{mm}^{(i)}(x^*; x^*)$$

holds. Then, $x^* \in R^n$ *is an attraction point of Method* II *provided the relaxation parameters* r *and* ω *satisfy*

$$0 \leqslant r \leqslant \omega, \quad 0 < \omega < \frac{2}{1+\rho(|D|^{-1}|B|)}.$$

Proof. By making use of Theorem 2 in [2], we can immediately fulfill the proof of this theorem. □

At the end of this section, we particularly point out that the local convergence theorems about the asynchronous parallel nonlinear multisplitting AOR-Newton method, the asynchronous parallel

nonlinear multisplitting AOR-chord method and the asynchronous parallel nonlinear multisplitting AOR-Steffensen method stated in section two can be set up in a quite similar way to Theorem 4.2.

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