NORTH-HOLLAND

Perturbation Bounds of the Krylov Bases and Associated Hessenberg Forms*

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ABSTRACT

This paper is devoted to further development of the method studying the condition numbers for the computation of the Krylov orthonormal bases and subspaces $\mathscr{K}_j(A, f) = \text{span}[f, Af, \dots, A^{j-1}f]$, where A is a matrix and f is a vector. The condition numbers were obtained by means of a first-order analysis of the sensitivity of the Krylov subspaces and their orthonormal bases under small perturbations of the matrix. We give perturbation bounds of the Krylov orthonormal basis and associated Hessenberg form of a matrix with respect to matrix and starting-vector perturbations. The bounds obtained depend on the condition number of the Krylov orthonormal basis. © 1997 Elsevier Science Inc.

1. INTRODUCTION

This paper contains further development of the idea proposed in [3]. That paper provided a first-order analysis of the sensitivity of the Krylov subspaces and their orthonormal bases under small perturbations of the matrix. It should be noted that perturbations of the starting vectors were not considered. The technique was to explore the structure of the skew-symmetric

*This research was supported by the Russian Foundation for Fundamental Researches (93-011-1515).

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LINEAR ALGEBRA AND ITS APPLICATIONS 265:1–28 (1997)

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655 Avenue of the Americas, New York, NY 10010

0024-3795/97/$17.00
PII S0024-3795(96)00299-6

matrices that are first-order approximations of orthogonal matrices. It is shown that the elements of these particular skew-symmetric matrices satisfy a triangular system of equations. The 2-norm of the inverse of this triangular matrix then gives the condition number of the Krylov basis, while the 2-norm of a subset of the rows of the inverse gives the condition number of the Krylov subspace. As shown in [3], the condition numbers of the Krylov subspace and its basis do not depend on the basis in which the matrix and the starting vector are expressed. Observe that the condition numbers a matrix and its transpose may be very different. Therefore the condition numbers depend on the structure of the matrix.

In the present paper we give perturbation bounds of a natural orthonormal basis of the Krylov subspace and the Hessenberg form of a matrix to matrix and starting-vector perturbations.

We consider the Krylov subspaces $\mathscr{K}_j(A, f) = \operatorname{span}\{f, Af, \dots, A^{j-1}f\}$, where $A \in \mathbb{R}^{n\times n}$ is a matrix and $f \in \mathrm{R}^n$ is a vector. A natural orthonormal Krylov basis of $\mathscr{K}_k(A, f)$ is an orthonormal basis $F_k = \{f_1, f_2, \dots, f_k\}$ such that for $1 \leqslant j \leqslant k$, F_j is an orthonormal basis of $\mathscr{K}_j(A, f)$. It is obvious that a Krylov natural orthonormal basis can be constructed by the Arnoldi process for example (see [1]). Observe that if F_k is a natural orthonormal basis of $\mathscr{K}_k(A, f)$, then the matrix $H_k = F_k^* A F_k$ is a Hessenberg matrix.

The paper is organized as follows. We first obtain in Section 2 some useful estimates for the solution of Sylvester's equation, in which the unknown matrix is a skew-symmetric matrix. Note that this equation for Hessenberg matrices was considered in [3]. Based on these estimates, we also prove some bounds for the condition numbers of the Krylov subspace and its orthonormal basis. The main problem is to obtain a perturbation bound of a natural orthonormal basis of the Krylov subspace built from a Hessenberg matrix and starting vector e_1 to matrix perturbations. We study a system of relations including an initial-value problem for the orthonormal basis of the Krylov subspace built from a perturbed Hessenberg matrix and e_1. Estimates for the solution of this Cauchy problem, as well as some properties of real orthonormal matrices presented in Section 4, are essential to establish a measure of sensitivity of a natural orthonormal basis of the Krylov subspace to matrix and starting vector perturbations in the general case.

We state the principal result of this paper in the form of the following theorem.

THEOREM. *Let F_j be a natural orthonormal basis of the subspace $\mathscr{K}_j(A, f)$, and let $l = \dim \mathscr{K}_n(A, f)$. Assume that the matrix $A_1 \in \mathrm{R}^{n\times n}$ and vector $f_1 \in \mathrm{R}^n$ are such that*

$$\|A - A_1\|_F \leqslant \varepsilon \|A\|_F, \qquad \|f - f_1\|_2 \leqslant \varepsilon \|f\|_2.$$

Then for $j \leqslant l$ and for sufficiently small ε there exists a natural orthonormal basis $Q_j = \{q_1, q_2, \dots, q_j\} \in \mathrm{R}^{n \times j}$ $(q_1 = f_1/\|f_1\|_2)$ of the Krylov subspace $\mathscr{K}_j(A_1, f_1) = \operatorname{span}[f_1, A_1 f_1, \dots, A_1^{j-1} f_1]$, such that the following inequality holds:

$$\|Q_j - F_j\|_2 \leqslant 20 \mu_b\{\mathscr{K}_j(A_0, f)\}\varepsilon,$$

where $\mu_b\{\mathscr{K}_j(A, f)\}$ denotes the condition number of the natural orthonormal basis F_j.

As a corollary, we obtain a perturbation bound of the Hessenberg form of a matrix with respect to matrix and starting-vector perturbations. Observe that this corollary may be treated as the essential supplement to the implicit Q theorem (see [4]). These results are obtained in Section 3.

We follow the notational convention used in numerical analysis. In particular, $\|\cdot\|_F$ denotes the Frobenius norm in the space of matrices, and $\|\cdot\|_2$ denotes the 2-norm for both vectors and matrices. Furthermore, e_i denotes the ith coordinate vector in R^n.

2. DEFINITIONS AND PRELIMINARIES

It is convenient to recall the main definitions and the results from [3].

Let $W \in \mathrm{R}^{n \times n}$ be an orthogonal matrix. It is well known that the set of eigenvalues of W, denoted by $\mathrm{Sp}(W)$, lies on the unit circle. It is evident that if $\lambda_j(W) = e^{i\omega_j} \in \mathrm{Sp}(W)$, where $\omega_j(W)$ is real, then $\overline{\lambda}_j(W) = e^{-i\omega_j} \in \mathrm{Sp}(W)$. We may suppose that for any $\omega_j(W)$ the following condition holds: $-\pi \leqslant \omega_j \leqslant \pi$.

DEFINITION 2.1.

$$\rho(W) = \left\{ 2\sqrt{\sum_j \omega_j^2} : e^{i\omega_j} \in \mathrm{Sp}(W) \text{ and } 0 < \omega_j < \pi \right\}.$$

Let $\mathscr{F}$ and $\mathscr{G}$ be two subspaces of R^n of dimension k, and let F and G be two orthonormal bases of $\mathscr{F}$ and $\mathscr{G}$, respectively. Then, there exist some matrices $W \in \mathrm{R}^{n \times n}$ such that $W^*W = I$ and $G = WF$. Let $\mathscr{W}$ be the set of such matrices.

DEFINITION 2.2. The distance between F and G is given by $d(F,G) = \min_{W \in \mathscr{W}} \rho(W)$, where $\mathscr{W} = \{W \in R^{n \times n} : G = WF \text{ and } W^*W = I\}$.

The distance between $\mathscr{F}$ and $\mathscr{G}$ is given by $d(\mathscr{F}, \mathscr{G}) = \min_{F,G} d(F,G)$ where F and G are respectively orthonormal bases of $\mathscr{F}$ and $\mathscr{G}$.

We recall here the definitions for the condition numbers of the Krylov subspace $\mathscr{K}_k(A, f)$ and of its Krylov basis through a matrix perturbation Δ. These condition numbers are denoted by $\mu\{\mathscr{K}_k(A,f)\}$ and $\mu_b\{\mathscr{K}_k(A,f)\}$, respectively.

For $1 \leqslant k \leqslant l$, let:

(1) $\mathscr{K} = \mathscr{K}_k(A,f)$, and F be its Krylov basis. Since $k \leqslant l$, F is of dimension k.

(2) $\tilde{\mathscr{K}} = \mathscr{K}_k(A + \Delta, f)$, and $\tilde{F}$ be its Krylov basis.

We assume that $\|\Delta\|_F$ is small enough to ensure that $\tilde{F}$ is also of dimension k. We apply the usual definition of condition number [9] where the metric in the set of subspaces is defined by Definition 2.2 and we choose the Frobenius norm on the space of matrices.

DEFINITION 2.3. For $1 \leqslant k \leqslant l$,

$$\mu\{\mathscr{K}_k(A,f)\} = \inf_{\epsilon > 0} \left\{ \sup_{\|\Delta\|_F \leqslant \epsilon} \left(\frac{d(\mathscr{K}, \tilde{\mathscr{K}})}{\|\Delta\|_F} \|A\|_F \right) \right\}$$

and

$$\mu_b\{\mathscr{K}_k(A,f)\} = \inf_{\epsilon > 0} \left\{ \sup_{\|\Delta\|_F \leqslant \epsilon} \left(\frac{d(F, \tilde{F})}{\|\Delta\|_F} \|A\|_F \right) \right\}$$

For $l < k \leqslant n$,

$$\mu\{\mathscr{K}_k(A,f)\} = \mu_b\{\mathscr{K}_k(A,f)\} = \infty.$$

Let $A \in \mathrm{R}^{n \times n}$ be a Hessenberg matrix. In this case a practical method for estimating the condition numbers of the Krylov subspace $\mathscr{K}_j(A, e_1)$ is based on the solution of a large system of linear equations with a triangular matrix (see [3]). This system was obtained from Sylvester's equation for finding a first-order approximation of a small rotation that transforms the natural orthonormal basis of subspace $\mathscr{K}_j(A, e_1)$ to the natural orthonormal basis of subspace $\mathscr{K}_j(A + \Delta, e_1)$.

Let

$$J_i^j = \left(0_{j-i} \middle| \mathrm{I}_i\right) \in R^{i \times j},$$

and define the matrix S_k by

$$S_k = \begin{pmatrix} J_{n-k}^{n-2} & & & 0 \\ & J_{n-k}^{n-3} & & \\ 0 & & \ddots & \\ & & & J_{n-k}^{n-k} \end{pmatrix}.$$

The matrix $B^{(A,k)}$ is the following triangular matrix built with the elements of A:

$$\begin{pmatrix} a_{2,1} I_{n-2} & & & & \\ a_{2,2} J_{n-3}^{n-2} - \mathrm{A}^{(4)} & a_{3,2} I_{n-3} & & & \\ a_{2,3} J_{n-4}^{n-2} & \ddots & \ddots & 0 & \\ \vdots & \ddots & \ddots & \ddots & \\ a_{2,k-1} J_{n-k}^{n-2} & \cdots & a_{k-2,k-1} J_{n-k}^{n-k+2} & a_{k-1,k-1} J_{n-k}^{n-k+1} - \mathrm{A}^{(k+1)} & a_{k,k-1} I_{n-k} \end{pmatrix}. \tag{2.1}$$

where

$$\mathrm{A}^{(i)} = \begin{pmatrix} a_{i,i-1} & \cdots & \cdots & a_{i,n} \\ & \ddots & & \vdots \\ 0 & & a_{n,n-1} & a_{n,n} \end{pmatrix} \in \mathrm{R}^{(n-i+1) \times (n-i+2)}.$$

In [3] we proved the following theorem for the condition numbers of $\mathscr{K}_j(A, e_1)$ in the case of $A \in \mathrm{R}^{n \times n}$ is a Hessenberg matrix:

THEOREM 2.1. *Let l be the dimension of $\mathscr{K}_n(A, e_1)$. Then for $k \in [2, \min(l, n-1)]$, the matrix $C^{(A,k)} = (B^{(A,k)})^{-1}$ exists, and then*

(1) *The condition number of a natural orthonormal basis of $\mathscr{K}_k(A, e_1)$ is*

$$\mu_b\{\mathscr{K}_k(A, e_1)\} = \|C^{(A,k)}\|_2 \|A\|_F. \tag{2.2}$$

(2) *The condition number of $\mathscr{K}_k(A, e_1)$ is*

$$\mu\{\mathscr{K}_k(A, e_1)\} = \|\hat{C}^{(A,k)}\|_2 \|A\|_F, \tag{2.3}$$

where $\hat{C}^{(A,k)}$ is the matrix composed by a few rows of $C^{(A,k)}$ such that

$$\hat{C}^{(A,k)} = S_k C^{(A,k)}. \tag{2.4}$$

The proof of the last theorem given in [3] does not permit us to establish perturbation bounds for the matrix $B^{(A,k)}$ that are necessary for the following. Therefore we have to give another proof of this theorem and we obtain the explicit form for $B^{(A,k)}$ through the Kronecker product of matrices.

Let us consider Sylvester's equation

$$XA - AX = C, \tag{2.5}$$

where A, C, and $X \in \mathrm{R}^{n \times n}$ are arbitrary matrices.

DEFINITION 2.4. Let $M = \{m_{ij}\}_{1 \leqslant i,\, j \leqslant n} \in \mathrm{R}^{n \times n}$. Then $\operatorname{vec} M$ is the vector of dimension n^2 defined by

$$\operatorname{vec} M = \begin{pmatrix} M_{*1} \\ M_{*2} \\ \vdots \\ M_{*n} \end{pmatrix}$$

$$= \left(m_{1,1}, m_{2,1}, \dots, m_{n,1} \middle| m_{1,2}, \dots, m_{n,2}, \middle| \dots \middle| \dots, m_{n,n}\right)^T.$$

It is well known (see [7]) that the matrix X in (2.5) is the solution of the following system:

$$\left(A^T \otimes I_n - I_n \otimes A\right) \operatorname{vec} X = \operatorname{vec} C, \tag{2.6}$$

where $\otimes$ denotes the Kronecker product of matrices. It is obvious that the system of linear equations (2.6) has dimension n^2.

DEFINITION 2.5. Let $M \in \mathrm{R}^{n \times n}$. Then $\mathscr{L}_k\{M\}$ denotes the first $k-1$ columns below the subdiagonal of M, i.e.

$$\mathscr{L}_k\{M\} = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & & & & & \vdots \\ m_{3,1} & \ddots & & 0 & & \vdots \\ \vdots & \ddots & \ddots & & & \vdots \\ \vdots & & m_{k+1,k-1} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ m_{n,1} & \cdots & m_{n,k-1} & 0 & \cdots & 0 \end{pmatrix}.$$

REMARK 1. $\mathscr{L}_k$ is linear, and its kernel is the subspace of Hessenberg matrices for the first $k-1$ columns.

Let us choose a natural number $k \leqslant n$. In the system (2.2) we consider only equations with the entries $c_{i,j}$ for $i > j+1$, $j < k$ in the left-hand side, i.e., we will consider the equation

$$\mathscr{L}_k\{XA - AX\} = \mathscr{L}_k\{C\}. \tag{2.7}$$

As in [3], we want to find a skew-symmetric matrix X ($X^* = -X$) solving (2.7) and having the following structure:

$$X = \begin{pmatrix} 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & -x_{3,2} & \cdots & \cdots & \cdots & -x_{n,2} \\ \vdots & x_{3,2} & \ddots & \ddots & & & \vdots \\ \vdots & \vdots & \ddots & 0 & -x_{k+1,k} & \cdots & -x_{n,k} \\ \vdots & \vdots & & x_{k+1,k} & & & \\ \vdots & \vdots & & \vdots & & 0 & \\ 0 & x_{n,2} & \cdots & x_{n,k} & & & \end{pmatrix}. \tag{2.8}$$

We remark that the number of unknown entries of X in (2.7) denoted by m is equal to the number of equations where

$$m = (k-1)n + 1 - \frac{k(k+1)}{2}.$$

Let us define the matrices $L_k \in \mathrm{R}^{m \times n^2}$ and $M_k \in \mathrm{R}^{m \times n^2}$ as follows:

$$L_k = \left(\begin{array}{cccc|c} J^n_{n-2} & & 0 & & \\ & J^n_{n-3} & & & \\ 0 & & \ddots & & 0_{n^2-n(k-1)} \\ & & & J^n_{n-k} & \end{array}\right),$$

$$M_k = \left(\begin{array}{c|cccc|c} & J^n_{n-2} & & 0 & & \\ & & J^n_{n-3} & & & \\ 0_n & 0 & & \ddots & & 0_{n^2-nk} \\ & & & & J^n_{n-k} & \end{array}\right).$$

REMARK 2. It is easy to show that finding the solution $X \in \mathrm{R}^{n \times n}$ of Equation (2.7) is equivalent to determining the solution of the following system of linear equations:

$$L_k\left(A^T \otimes I_n - I_n \otimes A\right) \operatorname{vec} X = L_k \operatorname{vec} C.$$

Consider now that X in (2.7) has the structure (2.8). Then finding the solution of Equation (2.7) is equivalent to determining the solution of the system of linear equations

$$\left[L_k\left(A^T \otimes I_n - I_n \otimes A\right) M_k^*\right] M_k \operatorname{vec} X = L_k \operatorname{vec} C.$$

DEFINITION 2.6. Let $A \in \mathrm{R}^{n \times n}$. Then for $2 \leqslant k \leqslant n-1$, $B^{(A,k)} \in \mathrm{R}^{m \times m}$ is the matrix defined by

$$B^{(A,k)} = L_k\left(A^T \otimes I_n - I_n \otimes A\right) M_k^*.$$

REMARK 3. It is easy to show that

$$B^{(A,k)} = B^{(A-\lambda I_n,k)},$$

where λ is an arbitrary chosen real scalar.

Remark 2 gives us the following lemma:

LEMMA 2.1. *Let A, C, and $X \in \mathrm{R}^{n\times n}$. Then finding a skew-symmetric matrix X, with the structure (2.8), that is a solution of the system (2.7) is equivalent to determining the solution of the following system of linear equations:*

$$B^{(A,k)}x^{(k)} = c^{(k)},$$

where

$$x^{(k)} = M_k \operatorname{vec} X = \left(x_{3,2},\dots,x_{n,2}\middle| x_{4,3},\dots\middle|\dots\middle| x_{k+1,k},\dots,x_{n,k}\right)^T \in \mathrm{R}^m,$$

$$c^{(k)} = L_k \operatorname{vec} C = \left(c_{3,1},\dots,c_{n,1}\middle|\dots\middle| c_{k+1,k-1},\dots,c_{n,k-1}\right)^T \in \mathrm{R}^m.$$

Now we prove some useful estimates for the 2-norm of the matrix $B^{(A,k)}$. In particular, it is easy to verify the validity of the following statement.

LEMMA 2.2. *Let $A \in \mathrm{R}^{n\times n}$ and $2 \leqslant k \leqslant n-1$. Then the following estimate holds:*

$$\|B^{(A,k)}\|_2 \leqslant 2\|A\|_2 \leqslant 2\|A\|_F.$$

LEMMA 2.3. *Let A and $A_1 \in \mathrm{R}^{n\times n}$ be such that*

$$\|A - A_1\|_F \leqslant \varepsilon\|A\|_F.$$

Let us assume that the matrix $B^{(A,k)}$ is nonsingular. Then, if

$$\varepsilon \leqslant \frac{1}{4\left\|\left(B^{(A,k)}\right)^{-1}\right\|_2\|A\|_F},$$

the matrix $B^{(A_1,k)}$ is nonsingular and the following estimate holds:

$$\left\|\left(B^{(A_1,k)}\right)^{-1}\right\|_2 \leqslant 2\left\|\left(B^{(A,k)}\right)^{-1}\right\|_2.$$

Proof. Since

$$B^{(A-A_1,k)} = B^{(A,k)} - B^{(A_1,k)},$$

we have

$$\|B^{(A,k)} - B^{(A_1,k)}\|_2 \leqslant 2\varepsilon\|A\|_F.$$

Applying the inequalities for singular values from [5, p. 40, Theorem 4.4] to the last estimate, we ensure the fulfilment of the following inequality:

$$\left|\sigma_j\left(B^{(A,k)}\right) - \sigma_j\left(B^{(A_1,k)}\right)\right| \leqslant 2\varepsilon\|A\|_F,$$

where $\sigma_j(A)$ $(j = 1, 2, \ldots, n)$ are the singular values of $A \in \mathrm{R}^{n\times n}$.

Let $A \in \mathrm{R}^{n\times n}$. We know that the following equalities hold:

$$\|A\|_2 = \max_j \sigma_j(A), \qquad \|A^{-1}\|_2 = \frac{1}{\min_j \sigma_j(A)};$$

then the required inequality follows from the last estimates. ∎

It is not difficult to check that if $A \in \mathrm{R}^{n\times n}$ is a Hessenberg matrix for the first $k - 1$ columns, i.e.

$$\mathscr{L}_k A = 0,$$

then $B^{(A,k)}$ is the triangular matrix having the form (2.1).

REMARK 4. Based on Theorem 2.1 and Lemma 2.2, it is easy to establish the following estimate:

$$\mu_b\{\mathscr{K}_k(A, f)\} \geqslant \tfrac{1}{2}$$

for $k \geqslant 2$, $A \in \mathrm{R}^{n\times n}$, $f \in \mathrm{R}^n$.

Applying Theorem 2.1, we find the following lemma.

LEMMA 2.4. *Let* $A \in R^{n \times n}$ *be a Hessenberg matrix for the first* l *columns and* $a_{j+1,j} \neq 0$ *for* $j = 1, 2, \dots, l$. *Then for* $k < l - 1$ *the following estimates hold*:

$$\mu_b\{\mathscr{K}_k(A, e_1)\} \geqslant \max_{j=1,l} \frac{\|A\|_F}{|a_{j+1,j}|}, \qquad \mu\{\mathscr{K}_k(A, e_1)\} \geqslant \max_{j=1,l} \frac{\|A\|_F}{|a_{j+1,j}|}.$$

LEMMA 2.5. *Let* $l = \dim \mathscr{K}_n(A, f)$, *where* $A \in R^{n \times n}$, $f \in R^n$. *Then for* $k \in [2, \min(l, n-1)]$ *we have*

$$\mu\{\mathscr{K}_k(A, f)\} \geqslant \mu_b\{\mathscr{K}_k(A, f)\} - \mu_b\{\mathscr{K}_{k-1}(A, f)\}.$$

Proof. We have already mentioned that the condition numbers of the Krylov subspace and its basis do not depend on the orthonormal basis in which A and f are expressed. Therefore we can suppose that A is a Hessenberg matrix and that $f = e_1$. In this case the condition numbers $\mu\{\mathscr{K}_k(A, f)\}$ and $\mu_b\{\mathscr{K}_k(A, f)\}$ are given by (2.2)–(2.4).

Recall that $\hat{C}^{(A,k)}$ is the matrix composed by a few rows of $C^{(A,k)}$ and defined by (2.4). It is easy to construct the matrix $\tilde{C}^{(A,k)}$ of the same dimension as $C^{(A,k)}$ such that

$$\|\tilde{C}^{(A,k)}\|_2 = \|\hat{C}^{(A,k)}\|_2. \tag{2.9}$$

(replace in $C^{(A,k)}$ by zero all rows which are absent in $\hat{C}^{(A,k)}$).

Let

$$D^{(A,k)} = C^{(A,k)} - \hat{C}^{(A,k)}. \tag{2.10}$$

It follows from (2.1) and Definition 2.6 the matrix $C^{(A,k)}$ has the structure

$$C^{(A,k)} = \left(\begin{array}{c|c} C^{(A,k-1)} & 0 \\ \hline \cdots & \cdots \end{array}\right)$$

for $k \in [2, \min(l, n-1)]$.

If a row in the matrix $D^{(A,k)}$ is not equal to zero, then that row is present in the matrix

$$\tilde{D}^{(A,k)} = \left(\begin{array}{c|c} C^{(A,k-1)} & 0 \\ \hline 0 & 0 \end{array}\right).$$

In view of the last equality, we have

$$\|D^{(A,k)}\|_2 \leqslant \|\tilde{D}^{(A,k)}\|_2 = \|C^{(A,k-1)}\|_2. \tag{2.11}$$

Using (2.9)–(2.11), we obtain that

$$\|C^{(A,k)}\|_2 \leqslant \|D^{(A,k)}\|_2 + \|\tilde{C}^{(A,k)}\|_2 \leqslant \|C^{(A,k-1)}\|_2 + \|\hat{C}^{(A,k)}\|_2.$$

Multiplying both sides of the last estimate by $\|A\|_F$, we arrive at the required inequality. ∎

3. THEOREMS OF CONTINUITY

Let $A \in \mathrm{R}^{n\times n}$ be a Hessenberg matrix such that $a_{j+1,j} \neq 0$ for $j = 1, 2, \dots, l$. Under the above assumption, the following condition holds: $\dim \mathscr{K}_j(A, e_1) = j$ for $j \leqslant l$, where $e_1 = (1, 0, 0, \dots, 0)^T$. Observe that $G_j = \{e_1, e_2, \dots, e_j\}$ is the natural orthogonal basis of the subspace $\mathscr{K}_j(A, e_1)$.

Consider a continuously differentiable matrix function $A(t) \in \mathrm{R}^{n\times n}$ that satisfies the conditions

$$A(0) = A, \qquad \left\| \frac{dA(t)}{dt} \right\|_F \leqslant \nu \|A(t)\|_F,$$

where ν is a sufficiently small parameter. For $2 \leqslant j \leqslant l$, let $V_j(t) \in \mathrm{R}^{n\times n}$ be an orthogonal matrix defined as the solution of the Cauchy problem

$$\frac{dV_j(t)}{dt} = X_j(t)V_j(t), \qquad V_j(0) = I_n, \tag{3.1}$$

where the skew-symmetric matrix $X_j(t) \in \mathrm{R}^{n\times n}$ has the structure

$$X_j(t) = \begin{pmatrix} 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & -x_{3,2} & \cdots & \cdots & \cdots & -x_{n,2}(t) \\ \vdots & x_{3,2}(t) & \ddots & \ddots & & & \vdots \\ \vdots & \vdots & \ddots & 0 & -x_{j+1,j}(t) & \cdots & -x_{n,j}(t) \\ \vdots & \vdots & & x_{j+1,j}(t) & & & \\ \vdots & \vdots & & \vdots & & \mathbf{0} & \\ 0 & x_{n,2} & \cdots & x_{n,j}(t) & & & \end{pmatrix}. \tag{3.2}$$

For $j \leqslant l$ we consider the following system of equations:

$$\mathscr{L}_j\left(X_j(t)Z_j(t) - Z_j(t)X_j(t) + V_j(t)\frac{dA(t)}{dt}V_j^*(t)\right) = 0,$$
$$Z_j(t) = V_j(t)A(t)V_j^*(t), \tag{3.3}$$
$$Z_j(0) = A,$$

where $X_j(t) \in \mathrm{R}^{n\times n}$ and $V_j(t) \in \mathrm{R}^{n\times n}$ satisfy (3.1)–(3.2).

Recall that the condition numbers were obtained by means of a first-order analysis of the sensitivity of the Krylov subspaces and their orthonormal bases under small perturbations of the matrix. It follows from Lemma 2.3 and the above that there exists a small neighborhood of the matrix A such that the solution of the first equation in (3.3) for the matrix $X_j(t)$ will be a continuous function in a neighborhood of $t = 0$.

By virtue of (3.1), we obtain

$$\mathscr{L}_j Z_j(t) = 0.$$

Hence matrix $Z_j(t)$ is a Hessenberg matrix for the first $j-1$ columns.

REMARK 5. Let $Q_j(t) \in \mathrm{R}^{n\times j}$ be defined by

$$Q_j(t) = V_j^*(t)G_j, \tag{3.4}$$

where $G_j = \{e_1, e_2, \dots, e_j\} \in \mathrm{R}^{n\times j}$. Taking account of (3.1)–(3.3), we can easily prove that the columns of $Q_j(t)$ form a natural orthogonal basis of the subspace $\mathscr{K}_j(A(t), e_1) = \operatorname{span}[e_1, A(t)e_1, \dots, A^{n-1}(t)e_1]$.

Now we prove that for sufficiently small t there exists a natural orthogonal basis of the subspace $\mathscr{K}_j(A(t), e_1)$ close to the natural orthogonal basis G_j of the subspace $\mathscr{K}_j(A, e_1)$. Observe that this required basis can be found from the formula (3.4).

First we obtain the estimate of closeness of natural orthogonal bases of subspaces $\mathscr{K}_j(A(t), e_1)$ and $\mathscr{K}_j(A, e_1)$ for sufficiently small t.

THEOREM 3.1. *Let $V_j(t) \in \mathrm{R}^{n\times n}$ be the orthogonal matrix defined as the solution of the Cauchy problems* (3.1)–(3.3). *If*

$$0 \leqslant t \leqslant \frac{1}{16\nu\mu_b\{\mathscr{K}_j(A, e_1)\}\left[\mu_b\{\mathscr{K}_j(A, e_1)\} + 1\right]} \tag{3.5}$$

then

$$\left\| V_j^*(t)G_j - G_j \right\|_2 \leqslant 2\mu_b\{\mathscr{K}_j(A, e_1)\}\nu t. \tag{3.6}$$

Proof. Let us consider the first equation in (3.3), and let $x^{(j)}(t)$ and $\theta^{(j)}(t)$ be two vectors of dimension

$$m_j = (j-1)n + 1 - \frac{j(j+1)}{2}$$

defined by

$$x^{(j)}(t) = \left(x_{3,2}(t), \dots, x_{n,2}(t) \middle| x_{4,3}(t), \dots \middle| \dots \middle| x_{j+1,j}(t), \dots, x_{n,j}(t)\right)^T$$

$$\theta^{(j)}(t) = \left(\theta_{3,1}(t), \dots, \theta_{n,1}(t) \middle| \dots \middle| \theta_{j+1,j-1}(t), \dots, \theta_{n,j-1}(t)\right)^T,$$

where $x_{i,j}(t)$ are the entries of the matrix $X_j(t)$, and $\theta_{i,j}(t)$ are the entries of the matrix

$$\Theta_j = -V_j(t)\frac{dA(t)}{dt}V_j^*(t).$$

It follows from Section 2 that if $X_j(t)$ is the solution of the first equation of (3.3), then the vector $x^{(j)}(t)$ built with the elements of $X_j(t)$ is the solution of the following system of linear equations:

$$B^{(Z_j(t), j)}x^{(j)}(t) = \theta^{(j)}(t).$$

We will now prove that $B^{(Z_j(t), j)}$ is nonsingular for sufficiently small t, and then we obtain an estimate of the 2-norm of $(B^{(Z_j(t), j)})^{-1}$.

In next section (see Lemma 4.5) we establish that the matrix $V_j(t)$ defined as the solution of the Cauchy problem (3.1)–(3.2) satisfies the following inequality:

$$\left\| V_j(t) - I_n \right\|_2 \leqslant 2 \sin \frac{\int_0^t \left\| X_j(\xi) \right\|_2 d\xi}{2}.$$

Let us introduce the notation

$$\delta_j(t) = 2\sin\frac{\int_0^t \|X_j(\xi)\|_2\,d\xi}{2},$$
$$\rho_j(t) = \{2\delta_j(t) + t\nu[1 + \delta_j(t)]\}\|A\|_F. \tag{3.7}$$

Since

$$\|A(t) - A\|_F = \left\|\int_0^t \frac{A(\xi)}{d\xi}\,d\xi\right\|_F \leqslant t\nu\|A\|_F,$$

the following inequalities hold:

$$\begin{aligned}\|Z_j(t) - A\|_2 &= \|V_j(t)A(t)V_j^*(t) - A\|_2\\ &= \|V_j(t)A(t)V_j^*(t) - A(t)V_j^*(t) + A(t)V_j^*(t)\\ &\qquad -AV_j^*(t) + AV_j^*(t) - A\|_2\\ &\leqslant (1 + t\nu)\|V_j(t) - I_n\|_2\|A\|_F\\ &\quad + \|V_j(t) - I_n\|_2\|A\|_F + t\nu\|A\|_F\\ &\leqslant \{2\delta_j(t) + t\nu[1 + \delta_j(t)]\}\|A\|_F = \rho_j(t).\end{aligned}$$

In view of continuity of the function $\rho_j(t)$ and the condition $\rho_j(0) = 0$, we establish the estimate

$$2\rho_j(t)\left\|\left(B^{(A,j)}\right)^{-1}\right\|_2 \leqslant \tfrac{1}{2} \tag{3.8}$$

for sufficiently small t. It follows from Lemma 2.3 in Section 2 that the matrix $B^{(Z_j(t),j)}$ is nonsingular and the estimate

$$\left\|\left(B^{(Z_j(t),j)}\right)^{-1}\right\|_2 \leqslant 2\left\|\left(B^{(A,j)}\right)^{-1}\right\|_2$$

is valid for all t satisfying (3.8).

Applying the last inequality, it is easy to show that

$$\|X_j(t)\|_2 \leqslant \frac{\|X_j(t)\|_F}{\sqrt{2}} = \|x^{(j)}(t)\|_2 \leqslant \left\|\left(B^{(Z_j(t),j)}\right)^{-1}\right\|_2 \|\theta^{(j)}(t)\|_2$$

$$\leqslant 2\left\|\left(B^{(A,j)}\right)^{-1}\right\|_2 \left\|V_j(t)\frac{dA(t)}{dt}V_j^*(t)\right\|_F$$

$$\leqslant 2\left\|\left(B^{(A,j)}\right)^{-1}\right\|_2 \left\|\frac{dA(t)}{dt}\right\|_F$$

$$\leqslant 2\mu_b\{\mathscr{K}_j(A, e_1)\}\nu.$$

It is obvious that if t satisfies (3.8), then

$$0 \leqslant \frac{d\delta_j(t)}{dt} \leqslant \left(\cos\frac{\int_0^t\|X_j(s)\|_2\,ds}{2}\right)\|X_j(t)\|_2 \leqslant \|X_j(t)\|_2.$$

Thus we have

$$\frac{d\delta_j(t)}{dt} \leqslant 2\mu_b\{\mathscr{K}_j(A, e_1)\}\nu.$$

Integrating both sides of the last inequality, we easily obtain the following estimate:

$$\|V_j(t) - I_n\|_2 \leqslant \delta_j(t) \leqslant 2\mu_b\{\mathscr{K}_j(A, e_1)\}\nu t. \tag{3.9}$$

Observe that the estimate (3.6) follows from (3.9).

We now consider the restriction (3.8) and prove that this inequality holds for all t satisfying (3.5).

Note that

$$2\rho_j(t)\left\|\left(B^{(A,j)}\right)^{-1}\right\|_2 = 2\{2\delta_j(t) + t\nu[1 + \delta_j(t)]\}\mu_b\{\mathscr{K}_j(A, e_1)\}.$$

In view of (3.9), it is sufficient to prove the estimate

$$4t\nu\mu_b\{\mathscr{K}_j(A, e_1)\} + t\nu\left[1 + 2t\nu\mu_b\{\mathscr{K}_j(A, e_1)\}\right] \leqslant \frac{1}{4\mu_b\{\mathscr{K}_j(A, e_1)\}}.$$

It is easy to verify that the last inequality must hold for $z = t\nu$ satisfying the condition

$$0 \leqslant z = t\nu \leqslant \frac{1}{2\mu_b\{\mathscr{K}_j(A, e_1)\}}.$$

If $z = t\nu$ satisfies the last inequality, then

$$4t\nu\mu_b\{\mathscr{K}_j(A, e_1)\} + t\nu\left[1 + 2t\nu\mu_b\{\mathscr{K}_j(A, e_1)\}\right]$$
$$\leqslant 2t\nu\left[1 + 2\mu_b\{\mathscr{K}_j(A, e_1)\}\right].$$

Thus the estimate (3.8) is true for all t satisfying (3.5). ■

Let $\tilde{A} \in \mathrm{R}^{n \times n}$ be such that

$$\|A - \tilde{A}\|_F \leqslant \varepsilon \|A\|_F, \tag{3.10}$$

and let us define the continuously differentiable matrix function $A(t) \in \mathrm{R}^{n \times n}$ by

$$A(t) = A + \frac{A - \tilde{A}}{s} t, \tag{3.11}$$

where s is a number from the segment [0, 1]. In view of (3.10)–(3.11), we have

$$\left\| \frac{dA(t)}{dt} \right\|_F \leqslant \frac{\varepsilon}{s} \|A\|_F.$$

Applying Theorem 3.1 to the matrix $A(t)$ defined by (3.11) and with $\nu = \varepsilon / s$, we obtain the following theorem.

THEOREM 3.2. *Let $\tilde{A} \in \mathrm{R}^{n \times n}$ be such that*

$$\|A - \tilde{A}\|_F \leqslant \varepsilon \|A\|_F,$$

where

$$\varepsilon \leqslant \frac{1}{16\nu\mu_b\{\mathscr{K}_j(A, e_1)\}[\mu_b\{\mathscr{K}_j(A, e_1)\} + 1]}.$$

Then there exists a natural orthonormal basis $Q_j \in \mathrm{R}^{n\times j}$ *of Krylov subspace* $\mathscr{K}_j(\tilde{A}, e_1) = \mathrm{span}[e_1, \tilde{A}e_1, \dots, \tilde{A}^{n-1}e_1]$ *such that*

$$\|Q_j - G_j\|_2 \leqslant 2\mu_b\{\mathscr{K}_j(A, e_1)\}\varepsilon.$$

Let us consider the Krylov subspaces $\mathscr{K}_j(A_0, f)$ where $A_0 \in \mathrm{R}^{n\times n}$ and $f \in \mathrm{R}^n$. Let l be such that $\dim \mathscr{K}_j(A_0, f) = j$ for $j \leqslant l$ and let F_j be a natural orthonormal basis of subspace $\mathscr{K}_j(A_0, f)$ for $j \leqslant l$. Let $V = (F_l, F')$ $\in \mathrm{R}^{n\times n}$ be an orthogonal matrix such that F' is an orthonormal basis of $\mathscr{K}_l^{\perp}(A, f)$, and such that the matrix H_0 defined by the equality

$$H_0 = V^*A_0V$$

is a Hessenberg matrix.

As shown in [3], for $j \leqslant l$ the following equalities hold:

$$\mu\{\mathscr{K}_j(A_0, f)\} = \mu\{\mathscr{K}_j(H_0, e_1)\}, \qquad \mu_b\{\mathscr{K}_j(A_0, f)\} = \mu_b\{\mathscr{K}_j(H_0, e_1)\}. \tag{3.12}$$

Along with subspaces $\mathscr{K}_j(A_0, f)$ we shall consider subspaces $\mathscr{K}_j(A_1, f_1)$, where $A_1 \in \mathrm{R}^{n\times n}$ and $f_1 \in \mathrm{R}^n$ such that

$$\|A_0 - A_1\|_F \leqslant \varepsilon\|A_0\|_F, \qquad \|f - f_1\|_2 \leqslant \varepsilon\|f\|_2.$$

Under the above assumptions the following theorem holds.

THEOREM 3.3. *Let*

$$\varepsilon \leqslant \frac{1}{112\mu_b\{\mathscr{K}_j(A_0, f)\}[\mu_b\{\mathscr{K}_j(A_0, f)\} + 1]}. \tag{3.13}$$

Then for $j \leqslant l$ there exists a natural orthonormal basis $Q_j = \{q_1, q_2, \dots, q_j\} \in \mathrm{R}^{n \times j}$ $(q_1 = f_1 / \|f_1\|_2)$ *of the subspace* $\mathscr{K}_j(A_1, f_1) = \operatorname{span}[f_1, A_1 f_1, \dots, A_1^{j-1} f_1]$ *such that*

$$\|Q_j - F_j\|_2 \leqslant \big[3 + 14\mu_b\{\mathscr{K}_j(A_0, f)\}\big]\varepsilon \leqslant 20\mu_b\{\mathscr{K}_j(A_0, f)\}\varepsilon. \quad (3.14)$$

Proof. Observe that $V^* f = e_1 \|f\|_2$. Since

$$\left\| \frac{f}{\|f\|_2} - \frac{f_1}{\|f_1\|_2} \right\| \leqslant \frac{\|f - f_1\|_2}{\|f\|_2} + \frac{\big|\|f\|_2 - \|f_1\|_2\big|}{\|f\|_2} \leqslant 2\varepsilon,$$

it follows from Lemma 4.6 (see next section) that there exists a real orthogonal matrix $\tilde{V} \in \mathrm{R}^{n \times n}$ such that

$$\tilde{V}^* f_1 = e_1 \|f_1\|_2, \qquad \|V - \tilde{V}\|_F \leqslant 2\sqrt{2}\,\varepsilon. \quad (3.15)$$

By virtue of the last inequality, we obtain

$$\|\tilde{V}^* A_0 \tilde{V} - V^* A_0 V\|_F \leqslant \|\tilde{V}^* A_0 \tilde{V} - H_0\|_F \leqslant 4\sqrt{2}\,\varepsilon \|A\|_F.$$

Using the inequality

$$\|\tilde{V}^* A_0 \tilde{V} - \tilde{V}^* A_1 \tilde{V}\|_F \leqslant \varepsilon \|A\|_F,$$

it is easy to ensure the fulfilment of the following estimate:

$$\begin{aligned} \|\tilde{V}^* A_1 \tilde{V} - H_0\|_F &= \|\tilde{V}^* A_1 \tilde{V} - V^* A_0 V\|_F \\ &\leqslant (4\sqrt{2} + 1)\varepsilon \|A\|_F \leqslant 7\varepsilon \|A\|_F. \end{aligned} \quad (3.16)$$

Under the restriction (3.13) and $j \leqslant l$, it follows from Theorem 3.2 there exists a natural orthonormal basis $\tilde{Q}_j = \{\tilde{q}_1, \tilde{q}_2, \dots, \tilde{q}_j\} \in \mathrm{R}^{n \times j}$ of the subspace $\mathscr{K}_j(\tilde{V}^* A_1 \tilde{V}, e_1) = \operatorname{span}[e_1, \tilde{V}^* A_1 \tilde{V} e_1, \dots, \tilde{V}^* A_1^{j-1} \tilde{V} e_1]$ such that

$$\|\tilde{Q}_j - G_j\|_2 \leqslant 14\mu_b\{\mathscr{K}_j(H_0, e_1)\}\varepsilon, \quad (3.17)$$

where G_j is a natural orthonormal basis of the subspace $\mathscr{K}_j(H_0, e_1)$ defined by

$$G_j = \begin{pmatrix} I_j \\ 0 \end{pmatrix}.$$

Note that the columns of the matrix $\tilde{V}\tilde{Q}_j = Q_j \in \mathrm{R}^{n \times j}$ form a natural orthonormal basis of the subspace $\mathscr{K}_j(A_1, f_1)$ and the columns of $F_j = VG_j$ form a natural orthonormal basis of $\mathscr{K}_j(A_0, f)$.

Remark 4 and Equations (3.12), (3.15)–(3.17) imply the required inequality:

$$\begin{aligned} \|Q_j - F_j\|_2 &= \|\tilde{V}\tilde{Q}_j - VG_j\|_2 \leqslant \|\tilde{V}\tilde{Q}_j - V\tilde{Q}_j\|_2 + \|V\tilde{Q}_j - VG_j\|_2 \\ &\leqslant \|\tilde{V} - V\|_2\|\tilde{Q}_j\|_2 + \|\tilde{Q}_j - G_j\|_2 \leqslant \|\tilde{V} - V\|_F + \|\tilde{Q}_j - G_j\|_2 \\ &\leqslant \left[3 + 14\mu_b\{\mathscr{K}_j(A_0, f)\}\right]\varepsilon \leqslant 20\mu_b\{\mathscr{K}_j(A_0, f)\}\varepsilon. \end{aligned}$$

■

Let us prove two important corollaries of the last theorem.

The first corollary provides a measure of the sensitivity of a natural orthonormal basis of the Krylov subspace to starting vector perturbation only.

COROLLARY 1. *Let us assume that* $\dim \mathscr{K}_j(A_0, f) = j$, *and* F_j *is a natural orthonormal basis of the Krylov subspace* $\mathscr{K}_j(A_0, f)$. *Let* f_1 *be such that*

$$\|f - f_1\|_2 \leqslant \varepsilon\|f\|_2,$$

where

$$\varepsilon \leqslant \frac{1}{96\mu_b\{\mathscr{K}_j(A_0, f)\}\left[\mu_b\{\mathscr{K}_j(A_0, f)\} + 1\right]}.$$

Then for $j \leqslant l$ *there exists a natural orthonormal basis* $Q_j = \{q_1, q_2, \ldots, q_j\} \in \mathrm{R}^{n \times j}$ $(q_1 = f_1/\|f_1\|_2)$ *of the subspace* $\mathscr{K}_j(A_0, f_1) = \mathrm{span}[f_1, A_0 f_1, \ldots, A_0^{j-1} f_1]$ *such that*

$$\|Q_j - F_j\|_2 \leqslant 2\sqrt{2}\left[1 + 4\mu_b\{\mathscr{K}_j(A_0, f)\}\right]\varepsilon \leqslant 12\sqrt{2}\,\mu_b\{\mathscr{K}_j(A_0, f)\}\varepsilon.$$

It is easy to obtain a measure of the sensitivity of the Hessenberg form of a matrix to starting-vector and matrix perturbation.

COROLLARY 2. *Let the Krylov subspaces* $\mathcal{K}_j(A_0, f)$ *and* $\mathcal{K}_j(A_1, f_1)$ *be such that*

$$\|A_0 - A_1\|_F \leqslant \varepsilon \|A_0\|_F, \qquad \|f - f_1\|_2 \leqslant \varepsilon \|f\|_2,$$

and let F_j *be a natural orthonormal basis of* $\mathcal{K}_j(A_0, f)$ *for* $j \leqslant l$, *where* $l = \dim \mathcal{K}_n(A_0, f)$.

If ε *satisfies* (3.13), *then for the Hessenberg matrix* H_j *defined by*

$$H_j = F_j^* A_0 F_j$$

there exists a Hessenberg matrix $\tilde{H}_j$ *such that the following conditions hold:*

$$\tilde{H}_j = Q_j^* A_1 Q_j,$$

$$\|H_j - \tilde{H}_j\|_2 \leqslant \varepsilon\left[7 + 28\mu_b\{\mathcal{K}_j(A_0, f)\}\right]\|A_0\|_F \tag{3.18}$$

$$\leqslant 42\varepsilon\mu_b\{\mathcal{K}_j(A_0, f)\}\|A_0\|_F.$$

where Q_j *is a natural orthonormal basis of* $\mathcal{K}_j(A_1, f_1)$.

Let us assume that there exists k such that the following condition holds:

$$\mu_b\{\mathcal{K}_k(A_0, f)\} = \mu\{\mathcal{K}_k(A_0, f)\} = \infty. \tag{3.19}$$

We shall prove that if the matrix $A_1 \in \mathrm{R}^{n\times n}$ and the vector $f_1 \in \mathrm{R}^n$ are close to the matrix A_0 and vector f respectively, then the condition numbers $\mu_b\{\mathcal{K}_k(A_1, f_1)\}$ and $\mu\{\mathcal{K}_k(A_1, f_1)\}$ will be sufficiently large.

LEMMA 3.1. *For* $l < k \leqslant n - 1$, *let the matrix* $A_0 \in \mathrm{R}^{n\times n}$ *and the vector* $f \in \mathrm{R}^n$ *satisfy* (3.19) *where* $l = \dim \mathcal{K}_n(A_0, f_1)$. *Let us assume that* $A_1 \in \mathrm{R}^{n\times n}$ *and* $f_1 \in \mathrm{R}^n$ *satisfy the following estimates*

$$\|A_0 - A_1\|_F \leqslant \varepsilon\|A_0\|_F, \qquad \|f - f_1\|_2 \leqslant \varepsilon\|f\|_2.$$

Then for small ε and $k > l$ the following estimates hold:

$$l_1 = \dim \mathscr{K}_n(A_1, f_1) \geqslant l,$$

$$\mu_b\{\mathscr{K}_k(A_1, f_1)\} \geqslant \frac{1}{\varepsilon[14 + 56\mu_b\{\mathscr{K}_l(A_0, f)\}]},$$

$$\mu\{\mathscr{K}_k(A_1, f_1)\} \geqslant \frac{1}{\varepsilon[14 + 56\mu_b\{\mathscr{K}_l(A_0, f)\}]}. \tag{3.20}$$

Proof. The first inequality in (3.20) immediately follows from Theorem 3.3. Let us prove the other two estimates.

Let $F_l = \{f_1, f_2, \dots, f_l\}$ be a natural orthonormal basis of the Krylov subspace $\mathscr{K}_l(A_0, f)$. It follows from Theorem 3.3 that for sufficiently small ε there exists a natural orthonormal basis $Q_l = \{q_1, q_2, \dots, q_l\}$ of the Krylov subspace $\mathscr{K}_l(A_1, f_1)$ such that

$$\|Q_l - F_l\|_2 \leqslant [3 + 14\mu_b\{\mathscr{K}_l(A_0, f)\}]\varepsilon$$

$$\|H_l - \tilde{H}_l\|_2 \leqslant \varepsilon[7 + 28\mu_b\{\mathscr{K}_l(A_0, f)\}]\|A_0\|_F,$$

where $H_l = F_l^* A_0 F_l$, $\tilde{H}_l = Q_l^* A_1 Q_l$.

Let $V = (F_l, \tilde{F}) \in \mathrm{R}^{n\times n}$ and $V_1 = (Q_l, \tilde{Q}) \in \mathrm{R}^{n \times n}$ be orthogonal matrices such that the matrices H and W defined by

$$H = VA_0V, \qquad W = V_1^* A_1 V_1,$$

are the Hessenberg matrices. Recall that the condition $\mu_b\{\mathscr{K}_k(A_0, f)\} = \infty$ for $k > l$ is equivalent to the equality $h_{l+1,l} = 0$. Prove that the absolute value of entry $w_{l+1,l}$ will be sufficiently small. Indeed, by virtue of the equalities

$$w_{l+1,l}q_{l+1}e_l^T = A_1Q_l - Q_l\tilde{H}_l, \qquad A_0F_l - F_lH_l = 0,$$

we have the estimates

$$|w_{l+1,l}| \leqslant \varepsilon[14 + 56\mu_b\{\mathscr{K}_l(A_0, f)\}]\|A_0\|_F.$$

Then the required estimates immediately follow from Lemma 2.4. ∎

4. PROPERTIES OF ORTHOGONAL MATRICES

Let $V \in \mathrm{R}^{n\times n}$ be an orthogonal matrix. It is well known that all eigenvalues of an orthogonal matrix lie on the unit circle. Therefore an eigenvalue $\lambda_j(V)$ of V may be represented in the form $\lambda_j(V) = e^{i\omega_j(V)}$ where $\omega_j(V)$ is real. Without loss of generality we suppose that $\omega_j(V)$ are ordered in the following manner:

$$-\pi \leqslant \omega_1(V) \leqslant \omega_2(V) \leqslant \cdots \leqslant \omega_n(V) \leqslant \pi.$$

LEMMA 4.1. *Let $V \in \mathrm{R}^{n\times n}$ be an orthogonal matrix. Then*

$$\|V - I_n\|_2 = 2\sin\frac{\omega_n(V)}{2}.$$

Proof. Let us consider the symmetric matrix S ($S = S^*$) defined by the formulae

$$S = (V^* - I_n)(V - I_n) = 2I_n - (V^* + V).$$

It is known that (see [5])

$$\|V - I_n\|_2 = \max_k \sqrt{\lambda_k(S)}, \tag{4.1}$$

where $\lambda_k(S)$ are eigenvalues of S.

Observe that $\lambda_k(S)$ can be calculated from the equality

$$\lambda_k(S) = 2 - \left[\lambda_k(V) + \bar{\lambda}_k(V)\right] = 2[1 - \cos\omega_k(V)] = 4\sin^2\frac{\omega_k(V)}{2}. \tag{4.2}$$

The equalities (4.1) and (4.2) immediately prove the statement. ∎

Next we prove the following statement:

LEMMA 4.2. *Let $V \in \mathrm{R}^{n\times n}$, $V_1 \in \mathrm{R}^{n\times n}$, $V_2 \in \mathrm{R}^{n\times n}$ be orthogonal matrices where $V = V_1V_2$ and $\omega_n(V_1) + \omega_n(V_2) \leqslant \pi$. Then*

$$\omega_n(V) \leqslant \omega_n(V_1) + \omega_n(V_2).$$

Proof. Let $x \in \mathrm{R}^n$ ($\|x\|_2 = 1$) be an arbitrary vector. Consider the vectors $y \in \mathrm{R}^n$ and $z \in \mathrm{R}^n$ defined by

$$y = V_2 x, \qquad z = V_1 y.$$

It is not difficult to check that (see [2])

$$\angle(x, z) \leqslant \angle(x, y) + \angle(y, z),$$

where $\angle(u, v)$ is the angle between the vectors u and v defined by

$$\angle(u, v) = \left| \arccos \frac{(u, v)}{\|u\|_2 \|v\|_2} \right|.$$

Since

$$\angle(x, V_2 x) = \angle(x, y) \leqslant \omega_n(V_2), \qquad \angle(y, V_1 y) = \angle(y, z) \leqslant \omega_n(V_1),$$

we have

$$\angle(x, Vx) \leqslant \omega_n(V_2) + \omega_n(V_1) \tag{4.3}$$

for any vector $x \in \mathrm{R}^n$.

It is obvious that there exists a vector $x \in \mathrm{R}^n$ such that

$$\angle(x, Vx) = \omega_n(V). \tag{4.4}$$

If we apply the inequality (4.3) to a vector x satisfying (4.4), we obtain the required inequality. ■

Taking into account the last two lemmas it is easy to prove the validity of the following statement.

LEMMA 4.3. *Assume that* $V_1, V_2 \in \mathrm{R}^{n \times n}$ *are orthogonal matrices where* $\omega_n(V_1) + \omega_n(V_2) \leqslant \pi$. *Then*

$$\|V_1 V_2 - I_n\|_2 \leqslant 2 \sin \frac{\omega_n(V_1) + \omega_n(V_2)}{2}$$

$$= 2 \sin\left(\sum_{j=1}^{2} \arcsin \frac{\|V_j - I_n\|_2}{2} \right).$$

By induction on the statement of Lemma 4.3 we easily obtain the result of the following lemma:

LEMMA 4.4. *Let* $V_1, V_2, \dots, V_m \in \mathrm{R}^{n \times n}$ *be orthogonal matrices, so that* $\sum_{j=1}^{m} \omega_n(V_j) \leqslant \pi$. *Then*

$$\|V_1 V_2 \cdots V_m - I_n\|_2 \leqslant 2 \sin\left(\sum_{j=1}^{m} \arcsin \frac{\|V_j - I_n\|_2}{2} \right).$$

We now consider a matrix function $V(t) \in \mathrm{R}^{n \times n}$ defined as the solution of the Cauchy problem

$$\frac{dV(t)}{dt} = X(t)V(t), \qquad V(0) = I_n, \tag{4.5}$$

where $X(t)$ is a continuous skew-symmetric matrix function, i.e. $X^*(t) = -X(t)$. It is obvious that $V(t)$ is a orthogonal matrix, i.e. $V^*(t)V(t) = I_n$.

LEMMA 4.5. *Let* $X(t)$ *in* (4.5) *be such that*

$$\int_0^s \|X(\xi)\|_2 \, d\xi \leqslant \pi.$$

Then for $V(t)$ *a solution of* (4.5) *the following estimate holds*:

$$\|V(s) - I_n\|_2 \leqslant 2 \sin \frac{\int_0^s \| X(\xi) \|_2 \, d\xi}{2}.$$

Proof. Let us split the segment $[0, s]$ into equal parts by the points

$$t_0 = 0 < t_1 < t_2 < \cdots < t_M = s, \qquad t_j - t_{j-1} = \Delta = s/M.$$

The orthogonal matrices $V_1^{(M)}, V_2^{(M)}, \dots, V_m^{(M)} \in \mathrm{R}^{n \times n}$ are defined as the solutions of the Cauchy problems:

$$\frac{dV_j^{(M)}(t)}{dt} = X(t)V_j^{(M)}(t), V_j^{(M)}(t_{j-1}) = I_n.$$

It is evident that

$$V(s) = V(t_M) = V_M^{(M)}(t_M)V_{M-1}^{(M)}(t_{M-1}) \cdots V_1^{(M)}(t_1).$$

On the basis of Lemma 4.4, we establish the inequality

$$\|V(s) - I_n\|_2 \leqslant 2\sin\left(\sum_{j=1}^{m} \arcsin \frac{\|V_j^{(M)}(t_j) - I_n\|_2}{2}\right). \tag{4.6}$$

It is easy to check that the matrix function $\tilde{V}_j^{(M)}$ defined as the solution of the Cauchy problem

$$\frac{d\tilde{V}_j^{(M)}(t)}{dt} = X(t_{j-1})\tilde{V}_j^{(M)}(t), \qquad \tilde{V}_j^{(M)}(t_{j-1}) = I_n$$

satisfies

$$\left\|\tilde{V}_j^{(M)}(t) - I_n\right\|_2 = 2\sin\left(\frac{(t - t_{j-1})\| X(t_{j-1})\|_2}{2}\right),$$

Letting M tend to ∞ in (4.6), we obtain the required estimate. ■

LEMMA 4.6. *Let $f \in \mathrm{R}^n$ and $\tilde{f} \in \mathrm{R}^n$ be such that*

$$\|f - \tilde{f}\|_2 \leqslant \varepsilon, \qquad \|f\|_2 = \|\tilde{f}\|_2 = 1.$$

Assume that the orthogonal matrix $V \in \mathrm{R}^{n\times n}$ ($V^*V = I_n$) satisfies the following equality:

$$V^*f = e_1 = (1,0,0,\ldots,0)^T.$$

Then there exists a matrix $\tilde{V} \in \mathrm{R}^{n\times n}$ such that

$$\tilde{V}^*\tilde{V} = I_n, \qquad \tilde{V}^*\tilde{f} = e_1, \qquad \|V - \tilde{V}\|_F \leqslant \sqrt{2}\,\varepsilon.$$

Proof. Without loss of generality we may suppose that

$$f = e_1, \qquad V = I_n, \qquad \tilde{f} = (1 - \alpha_1, \alpha_2, \ldots, \alpha_n)^T,$$

where

$$\alpha_1 \geqslant 0, \qquad \sqrt{\sum_{j=1}^{n} \alpha_j^2} \leqslant \varepsilon. \tag{4.7}$$

Introduce the notation

$$\sigma = \sqrt{\sum_{j=2}^{n} \alpha_j^2}.$$

Let us consider the orthogonal basis $Q_n = \{q_1, q_2, \dots, q_n\}$ where

$$q_1 = f = e_1, q_2 = (0, \alpha_2/\sigma, \dots, \alpha_k/\sigma, \dots, \alpha_n/\sigma)^T.$$

Observe that $\tilde{f} = Q_n \beta$, where

$$\beta = (1 - \alpha_1, \sigma, 0, 0, \dots, 0)^T, \qquad \|\beta\|_2 = 1.$$

Let us define the matrix $\mathscr{P} \in \mathrm{R}^{n \times n}$ by the equality

$$\mathscr{P} = \begin{pmatrix} \cos\varphi & \sin\varphi & & & \\ -\sin\varphi & \cos\varphi & & \mathbf{0} & \\ & & 1 & & \\ & \mathbf{0} & & \ddots & \\ & & & & 1 \end{pmatrix},$$

where $\cos\varphi = 1 - \alpha_1$, $\sin\varphi = \sigma$. Since $\mathscr{P}\beta = e_1$, then $\tilde{V}^*\tilde{f} = e_1$, where

$$\tilde{V} = Q_n \mathscr{P} Q_n^* \in \mathrm{R}^{n \times n}.$$

Using (4.7), we easily obtain

$$\|\tilde{V} - I_n\|_2 \leqslant \|\tilde{V} - I_n\|_F = \|\mathscr{P}^* - I_n\|_F = \sqrt{2\left[(\cos\varphi - 1)^2 + \sin^2\varphi\right]}$$

$$= \sqrt{2\sum_{j=1}^{n} \alpha_j^2} \leqslant \sqrt{2}\,\varepsilon.$$

■

The author is indebted to S. K. Godunov, V. I. Kostin, and J.-F. Carpraux for their helpful remarks about this paper.

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Received 2 February 1995; final manuscript accepted 17 April 1996