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Representations of quivers with free module of covariants

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Abstract

Given a finite quiver Q of Dynkin type A_n, it is well known that the ring of semi-invariants $SI(Q, \mathbf{d})$ is a polynomial ring. We show that the ideal defined by semi-invariants of positive degree in $\mathrm{Rep}(Q, \mathbf{d})$ is a complete intersection. It follows that the action of $SL(Q, \mathbf{d})$ on $\mathrm{Rep}(Q, \mathbf{d})$ gives a cofree representation. In particular, we have that the modules of covariants are free $k[\mathrm{Rep}(Q, \mathbf{d})]^{SL(Q,\mathbf{d})}$-modules.

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1. Introduction

A quiver is an oriented graph $Q = (Q_0, Q_1)$ where Q_0 is the set of vertices and Q_1 is the set of arrows. For $\alpha \in Q_1$

$$\alpha : t\alpha \to h\alpha.$$

We say $x \in Q_0$ is *a sink* if for every $\alpha \in Q_1$, $x \neq t\alpha$. We say $x \in Q_0$ is *a source* if for every $\alpha \in Q_1$ $x \neq h\alpha$.

Let k be an algebraically closed field. A representation V of a quiver Q is a collection

$$V = \{(V_x, V(\alpha)) \mid x \in Q_0, \alpha \in Q_1\},$$

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where V_x is a finite dimensional vector space over k and $V(\alpha)$ is a linear map from $V_{t\alpha}$ to $V_{h\alpha}$. We usually denote the linear maps $V(\alpha)$ simply by α when the notation is unambiguous. The *dimension vector* of V is the function $\dim V : Q_0 \to \mathbb{N}$ given by

$$\mathbf{d} = (d(x))_{x\in Q_0},$$

where $d(x) = \dim V_x$. We denote the set of dimension vectors by $\mathbb{N}^{Q_0}$.

A *morphism* between two quiver representations, $\phi : V \to W$ consists of linear maps $\phi_x : V_x \to W_x$ for $x \in Q_0$ such that the following diagram commutes:

$$\begin{tikzcd} V_{t\alpha} \arrow[r, "V(\alpha)"] \arrow[d, "\phi_{t\alpha}"'] & V_{h\alpha} \arrow[d, "\phi_{h\alpha}"] \\ W_{t\alpha} \arrow[r, "W(\alpha)"] & W_{h\alpha} \end{tikzcd}$$

Two representations of Q, V and W, are isomorphic if and only if ϕ_x is invertible for all $x \in Q_0$.

Fix a quiver Q and a dimension vector $\mathbf{d} \in \mathbb{N}^{Q_0}$. A representation V of Q with $V_x = k^{d(x)}$ for all $x \in Q_0$ is determined by a point of the vector space

$$\mathrm{Rep}(Q,\mathbf{d}) = \bigoplus_{\alpha\in Q_1} \mathrm{Hom}_k(V_{t\alpha}, V_{h\alpha}).$$

There is an action of the group

$$GL(Q,\mathbf{d}) = \prod_{x\in Q_0} GL_{d(x)}(k)$$

as well as its subgroup

$$SL(Q,\mathbf{d}) = \prod_{x\in Q_0} SL_{d(x)}(k)$$

on $\mathrm{Rep}(Q,\mathbf{d})$ given as follows. For $V \in \mathrm{Rep}(Q,\mathbf{d})$ and $g \in GL(Q,\mathbf{d})$,

$$g \cdot V = (g_{t\alpha} \cdot V(\alpha) \cdot g_{h\alpha}^{-1})_{\alpha\in Q_1}.$$

Two representations V and W of Q are isomorphic if and only if they are in the same orbit under the action above. Thus, to study the representation theory of quivers, we only need to consider the orbits under the action of $GL(Q,\mathbf{d})$.

The ring of regular functions on $\mathrm{Rep}(Q,\mathbf{d})$ is

$$k[\mathrm{Rep}(Q,\mathbf{d})] = k[x_{ij}^{\alpha} \mid \alpha \in Q_1, 1 \leqslant i \leqslant d(t\alpha), 1 \leqslant j \leqslant d(h\alpha)].$$

There is a linear action of $GL(Q,\mathbf{d})$ on $k[\mathrm{Rep}(Q,\mathbf{d})]$ induced by the action on the vector space $\mathrm{Rep}(Q,\mathbf{d})$. Specifically, for $f \in k[\mathrm{Rep}(Q,\mathbf{d})]$,

$$g \cdot f(M) = f(g^{-1} \cdot M).$$

A polynomial $f \in k[\mathrm{Rep}(Q,\mathbf{d})]$ is called a semi-invariant of weight χ if there is a character χ of $GL(Q,\mathbf{d})$ such that $g \cdot f = \chi(g)f$ for all $g \in GL(Q,\mathbf{d})$. We denote by

$SI(Q,\mathbf{d})_\chi$ the space of semi-invariants of weight χ. The semi-invariants of various weights form a ring

$$SI(Q,\mathbf{d}) = \bigoplus_\chi SI(Q,\mathbf{d})_\chi.$$

This ring coincides with the ring of the invariants under the action of $SL(Q,\mathbf{d})$ on $\mathrm{Rep}(Q,\mathbf{d})$. Thus,

$$SI(Q,\mathbf{d}) = k[\mathrm{Rep}(Q,\mathbf{d})]^{SL(Q,\mathbf{d})}.$$

Henceforth, we use the term $GL(Q,\mathbf{d})$ semi-invariants and $SL(Q,\mathbf{d})$ invariants interchangeably.

Assume that characteristic of k equals zero. Let $\{V_\lambda\}$ be the set of irreducible representations of $G = SL(Q,\mathbf{d})$. For simplicity, let $S = k[\mathrm{Rep}(Q,\mathbf{d})]$. Then

$$S = \bigoplus_\lambda V_\lambda \otimes M_\lambda,$$

where M_λ is an S^G-module, known as the module of covariants. In this paper we prove that for $\mathrm{char}(k) = 0$ and for quivers Q of type A and arbitrary dimension vector $\mathbf{d}$ the module M_λ is a free S^G-module.

We recall first some generalities, working again with k of arbitrary characteristic. Let G be a reductive group acting linearly on a vector space $V = \mathrm{Rep}(Q,\mathbf{d})$. Let $S = k[V] = \mathrm{Sym}(V^*)$ and S^G be the invariants in S under the induced action of G on S. The representation is *cofree* if S is a free S^G-module. In general, it is difficult to determine the module structure of S. However, if S is cofree and if $\mathrm{char}(k) = 0$, then there is a G-invariant space H such that $S = S^G \otimes H$, and we can study the module structure of H.

The representation V of the reductive group G is *coregular* if S^G is a polynomial ring. Let $I = (S^G)^+S$ be the ideal generated by the positive degree invariants, and let $Z(I) \subseteq V$ be the zero set of the ideal I. V is cofree if and only if V is coregular and $\mathrm{codim}\, Z(I) = \dim S^G$, where the dimension on the right-hand side is the number of generating variables. In the case of $\mathrm{char}(k) = 0$ Schwarz [17,18] classified coregular and cofree representations of simple groups. Littelmann [12] classified coregular irreducible representations of semisimple groups.

We use the following result of Sato and Kimura [14] to first determine when a representation $\mathrm{Rep}(Q,\mathbf{d})$ of the group $GL(Q,\mathbf{d})$ is coregular.

Theorem 1.1 (Sato-Kimura [14]). *Let G be a connected linear algebraic group which acts on a vector space V. If the action of G has an open orbit, then $SI(G,V) = \oplus_\chi SI(G,V)_\chi$ is a polynomial ring*:

$$SI(G,V) = k[f_1,\ldots,f_s].$$

Moreover, if $f_i \in SI(G,V)_{\chi_i}$ then the χ_i are linearly independent in $\mathrm{char}\, G$.

If the underlying graph of a quiver Q is of Dynkin type A, D, or E, then $GL(Q,\mathbf{d})$ acts on $\mathrm{Rep}(Q,\mathbf{d})$ with a finite number orbits [8]. In fact for Dynkin quivers, it is shown in [3] that there is a bijection between the indecomposable representations and positive roots of the corresponding root system. Each orbit of $\mathrm{Rep}(Q, \mathbf{d})$ corresponds to a decomposition of $\mathbf{d}$ into a sum of positive roots. Since there is a finite number of orbits, we know that there is an open orbit. Thus, we know that all algebras of semi-invariants, $SI(Q, \mathbf{d})$, of Dynkin quivers Q are polynomial algebras.

Our main result shows that for any quiver of any orientation, whose underlying diagram is of Dynkin type A, the generating semi-invariants of $SI(Q, \mathbf{d})$ generate in the coordinate ring of $\mathrm{Rep}(Q, \mathbf{d})$ an ideal which is a complete intersection. More precisely,

Theorem 1.2. *Let Q be a quiver of Dynkin type A_n, and $SI(Q, \mathbf{d}) = k[f_1, \ldots, f_s]$. Then*

$$\operatorname{codim} Z(f_1, \ldots, f_s) = s.$$

Note that this is not the case for other Dynkin types, in particular for type D or E.

Example 1.3. Consider the representation of the quiver of type D_4 with the following dimension vector.

$$\begin{tikzcd} & 1 \arrow[d, "\gamma"] & \\ 1 \arrow[r, "\alpha"] & 2 & 1 \arrow[l, "\beta"'] \end{tikzcd}$$

The ring of semi-invariants is generated as a polynomial ring by

$$k[\det(\alpha, \beta), \det(\alpha, \gamma), \det(\beta, \gamma)].$$

Since these functions are 2×2 minors of a 2×3 matrix, we see that

$$\operatorname{codim} Z(\det(\alpha, \beta), \det(\alpha, \gamma), \det(\beta, \gamma)) = 2.$$

Corollary 1.4. *Assume that* $\mathrm{char}(k) = 0$. *Let Q be a Dynkin quiver of type A_n. Then the modules of covariants are always free S^G modules.*

This corollary follows easily from the following result of Kostant which can be found in [11] and in [18]. For all weights λ, let M_λ be the module of covariants, which is an S^G-module. We know that

$$\operatorname{depth} M_\lambda \geqslant \operatorname{depth} I.$$

If $\operatorname{depth} I = s$, $f_1, \ldots, f_s$ form a regular sequence on S.

Proposition 1.5 (Kostant [11]). *Assume that* $\mathrm{char}(k) = 0$. *Suppose S^G is a polynomial ring, i.e.,*

$$S^G = k[f_1, \ldots, f_s].$$

If $f_1, \ldots, f_s$ form a regular sequence on S then all the M_λ are free S^G modules.

Proof. See Section 1 of [11] and the discussion preceding Proposition 4.6. in [18]. □

In order to prove Theorem 1.2, we need to give a good description of the semi-invariants $f_1,\ldots,f_s$. Then to compute the codimension of $Z(f_1,\ldots,f_s)$, we only need to compute the codimensions of each of the orbits that occur in

$$Z(f_1,\ldots,f_s),$$

since if f_i vanishes on a point x it vanishes on the entire orbit of the point x under the action of our group. Thus, we need to identify on which orbits all our semi-invariants vanish. Then, to compute the codimension of an orbit we use the following result of Voigt which can be found in [13].

Lemma 1.6. *Let V be a representation of Q with dimension vector $\mathbf{d}$. Let $\mathcal{O}(V)$ be the orbit of V under the action of $GL(Q,\mathbf{d})$. Then*

$$\operatorname{codim}\mathcal{O}(V)=\dim \operatorname{Ext}_Q(V,V).$$

This paper is organized as follows. In Section 2, we will review how to relate semi-invariants to indecomposable modules. This will also give us a recipe for determining when a semi-invariant vanishes. In Section 3 we outline the results of Abeasis and Del Fra on degenerations for type A quivers. In Section 4, we give some preliminary results required in the proof of the main theorem as well as the proof of our main theorem for the case when the quiver is equioriented. In Section 6 we describe the well-known Coxeter functors which we then use to reflect to simpler quivers.

The authors would like to thank Hanspeter Kraft for suggesting the ideas involved in Section 4 of lifting to the larger quiver to examine vanishing of certain semi-invariants.

2. Constructing semi-invariants

Due to recent progress by Derksen and Weyman [6], and also by Schofield and Van den Bergh [15,16] semi-invariants are easily computed using representations. A representation V defines a natural semi-invariant as follows.

Let V and W be two representations of a quiver Q such that $\dim V=\mathbf{d}$ and $\dim W=\mathbf{e}$. The Euler inner product is defined by

$$\langle V,W\rangle=\sum_{x\in Q_0} d(x)e(x)-\sum_{\alpha\in Q_1} d(t\alpha)e(h\alpha).$$

Ringel [13] shows that

$$\langle V,W\rangle=\dim \operatorname{Hom}_Q(V,W)-\dim \operatorname{Ext}_Q(V,W).$$

Furthermore, there is a natural sequence

$$0\to \operatorname{Hom}_Q(V,W)\to \bigoplus_{x\in Q_0}\operatorname{Hom}_k(V_x,W_x)\xrightarrow{d^V_W}\bigoplus_{\alpha\in Q_1}\operatorname{Hom}_k(V_{t\alpha},W_{h\alpha})$$
$$\to \operatorname{Ext}_Q(V,W)\to 0.$$

Schofield [15] showed that if $V \in \mathrm{Rep}(Q,\mathbf{d})$ and $W \in \mathrm{Rep}(Q,\mathbf{e})$ with $\langle \mathbf{d},\mathbf{e}\rangle = 0$ then d_V^W above is a square matrix and

$$c(V,W) = \det d_V^W$$

defines an $SL(Q,\mathbf{d}) \times SL(Q,\mathbf{e})$ invariant on $\mathrm{Rep}(Q,\mathbf{d}) \times \mathrm{Rep}(Q,\mathbf{e})$. Define c^V to be the restriction of $c(V,W)$ to $\{v\} \times \mathrm{Rep}(Q,\mathbf{e})$, and c_W the restriction of $c(V,W)$ to $\mathrm{Rep}(Q,\mathbf{d}) \times \{w\}$.

For a fixed V, we then have

$$c^V(W) = \det(d_W^V) \in SI(Q,\mathbf{e})_{\langle \mathbf{d},-\rangle}$$

and for a fixed W

$$c_W(V) = \det(d_W^V) \in SI(Q,\mathbf{d})_{-\langle -,\mathbf{e}\rangle},$$

where $\langle \mathbf{d},-\rangle$ and $\langle \mathbf{e},-\rangle$ are integer valued functions on the set of dimension vectors which can be realized as $GL(Q,\mathbf{e})$ or $GL(Q,\mathbf{d})$ weights. All semi-invariants are spanned by the semi-invariants of type c^V (resp. c_W).

Theorem 2.1 (Derksen and Weyman [6], Schofield and Van den Bergh [16]). *Let Q be a quiver without oriented cycles. The ring of semi-invariants $SI(Q,\mathbf{e})$ is spanned by the semi-invariants c^V with $\langle \mathbf{d}(V),\mathbf{e}\rangle = 0$ (c_W with $\langle \mathbf{e},\mathbf{d}(W)\rangle = 0$).*

When in doubt, denote by $c_{\mathbf{e}}^V$ the semi-invariant of weight V in $SI(Q,\mathbf{e})$, since c^V alone does not distinguish which dimension vector we are considering. For V an indecomposable representation of a quiver of Dynkin type, we may refer to V simply by its dimension vector.

Example 2.2. Let Q be the equioriented quiver of Dynkin type A_4, i.e.,

$$1 \xleftarrow{\alpha} 2 \xleftarrow{\beta} 3 \xleftarrow{\gamma} 4.$$

If $\mathbf{d} = (2,2,2,2)$, then the semi-invariants, $SI(Q,\mathbf{d})$, are generated by

$$c_{2222}^{0100} = \det\alpha, \quad c_{2222}^{0010} = \det\beta, \quad \text{and} \quad c_{2222}^{0001} = \det\gamma.$$

If $\mathbf{d} = (2,3,3,2)$, then the semi-invariants, $SI(Q,\mathbf{d})$, are generated by

$$c_{2322}^{0010} = \det\beta \quad \text{and} \quad c_{2322}^{0111} = \det\alpha\beta\gamma.$$

Example 2.3. Let Q be the following quiver of Dynkin type A_4:

$$1 \xrightarrow{\alpha} 2 \xleftarrow{\beta} 3 \xleftarrow{\gamma} 4.$$

If $\mathbf{d} = (1,3,3,2)$, then the semi-invariants, $SI(Q,\mathbf{d})$, are generated by

$$c_{1322}^{0010} = \det\beta \quad \text{and} \quad c_{1322}^{1111} = \det(\alpha,\beta\gamma).$$

Assume that $\langle V,W\rangle = 0$. A property of the c^V's is the following. Let V' and V'' be two representations with $\dim V' = \mathbf{d}'$ and $\dim V'' = \mathbf{d}''$. Consider the decomposition

$$V = V' \oplus V''.$$

If $\langle \mathbf{d}', \mathbf{e}\rangle = 0$ and $\langle \mathbf{d}'', \mathbf{e}\rangle = 0$ then $c^V(W) = c^{V'}(W)\cdot c^{V''}(W)$. If $\langle \mathbf{d}', \mathbf{e}\rangle \neq 0$ then $c^V(W) = 0$. Thus, to obtain generators of the semi-invariants, we simply look at the set of c^V such that V is indecomposable.

With the semi-invariants established, we also want to know when does $c^V(W)$ vanish identically. We can use King's theorem to identify when a semi-invariant vanishes.

Theorem 2.4 (King [10]). *Let W be a module with dimension vector* $\mathbf{e}$. *$f(W) = 0$ for all $f \in SI(Q, \mathbf{e})_{n\sigma}$ if and only if W has a submodule W' such that $\sigma(W') > 0$.*

Since for Dynkin quivers the weight spaces of rings of semi-invariants have dimensions $\leqslant 1$ by Theorem 1.1, in this case King's theorem says that $c^V(W)$ vanishes if and only if W has a submodule W' such that

$$\langle V, W'\rangle > 0. \tag{2.1}$$

Given an indecomposable V, this puts a necessary condition on what kind of submodules must occur in W in order to establish vanishing of c^V.

Example 2.5. Let Q be the equioriented quiver of Dynkin type A_4, i.e.,

$$1 \xleftarrow{\alpha} 2 \xleftarrow{\beta} 3 \xleftarrow{\gamma} 4,$$

as in our example above. Applying King's theorem, we have that for $\mathbf{d} = (1, 2, 2, 1)$,

- $c^{0010}_{1221} = \det \beta$ vanishes at W if and only if there is a submodule $W' \subseteq W$ such that $w'(3) - w'(2) > 0$.
- $c^{0111}_{1221} = \det \alpha\beta\gamma$ vanishes at W if and only if there is a submodule $W' \subseteq W$ such that $w'(4) - w'(1) > 0$.

Example 2.6. *Let Q be the following quiver of Dynkin type A_4*:

$$1 \xleftarrow{\alpha} 2 \xleftarrow{\beta} 3 \xleftarrow{\gamma} 4.$$

If $\mathbf{d} = (1, 3, 3, 2)$, the vanishing of the semi-invariants require the following submodules

- $c^{0010}_{1322} = \det \beta$ vanishes at W if and only if there is a submodule $W' \subseteq W$ such that $w'(3) - w'(2) > 0$.
- $c^{1111}_{1322} = \det(\alpha, \beta\gamma)$ vanishes at W if and only if there is a submodule $W' \subseteq W$ such that $w'(1) + w'(4) - w'(2) > 0$.

To establish vanishing of semi-invariants, we must study the degenerations of the open orbit.

3. Degenerations for type A quivers

We assume that the underlying diagram of our quiver is the Dynkin diagram of type A_n.

We want to characterize the degenerations of representations. For a representation $V \in \mathrm{Rep}(Q, \mathbf{d})$, we denote by either $\mathcal{O}_V$ or $\mathcal{O}(V)$ the orbit of V under the action of $GL(Q, \mathbf{d})$ on $\mathrm{Rep}(Q, \mathbf{d})$. We call W a degeneration of V, sometimes denoted $W \leqslant_g V$, if $\mathcal{O}_W \subseteq \overline{\mathcal{O}_V}$. For quivers of type A_n, Abeasis and Del Fra [1] give a simple combinatorial criterion for degenerations based on the decomposition of our modules into indecomposables. We sketch their results in this section.

For a type A_n quiver of any orientation, reading the diagram from left to right, the sources and sinks alternate. Let $1, \ldots, n$ be the n vertices of the Dynkin quiver of type A_n, and Q be a quiver of type A_n of any orientation.

$$\overset{1}{\circ} \leftarrow \overset{2}{\circ} \leftarrow \overset{3}{\circ} \rightarrow \overset{4}{\circ} \rightarrow \cdots \leftarrow \overset{n}{\circ}$$

Let $s_0 = 1 < s_1 < \cdots < s_v < s_{v+1} = n$ be alternating sequence of sources and sinks, see example below. A Dynkin quiver of type A_n is determined up to direction by the alternating sequence of sources and sinks. We refer to sinks and sources as critical points.

Example 3.1. $\overset{s_0}{\circ} \leftarrow \circ \leftarrow \overset{s_1}{\circ} \rightarrow \circ \rightarrow \circ \rightarrow \overset{s_2}{\circ} \leftarrow \circ \leftarrow \circ \leftarrow \overset{s_3}{\circ} \rightarrow \circ \rightarrow \overset{s_4}{\circ} \leftarrow \overset{s_5}{\circ}$

Recall that the indecomposable representations of a Dynkin quiver are in one to one correspondence with the positive roots of the Dynkin diagram, independent of the orientation. For type A_n, there is an indecomposable representation, denoted by E_{pq}, for each pair (p,q) with $1 \leqslant p \leqslant q \leqslant n$. In particular, this corresponds to a module with dimension vector $\mathbf{d} = (d_j) \in \mathbb{N}^{Q_0}$ with $d_j = 1$ for $p \leqslant j \leqslant q$ and $d_j = 0$ otherwise.

Example 3.2. If the underlying diagram of our quiver is A_8, the indecomposable E_{35} is given by the diagram

$$0 \rightarrow 0 \leftarrow 1 \leftarrow 1 \rightarrow 1 \rightarrow 0 \rightarrow 0 \leftarrow 0$$

If we consider E_{pq} as an indecomposable representation of Q, then the pair (p,q) uniquely determines the pair of integers (a, b) such that

$$s_{a-1} < p \leqslant s_a, \quad s_b \leqslant q < s_{b_1}.$$

Moreover, it uniquely determines the nearest sources or sinks encompassing p and q. Thus (p,q) determines the subsequence $\{s_a, \ldots, s_b\}$, which is possibly empty. E_{pq} is called "even type" if the determined subsequence has an even number of critical points. Otherwise, E_{pq} is called "odd type".

Let V be a representation in $\mathrm{Rep}(Q, \mathbf{d})$. The isomorphism class $\mathcal{O}_V$ is determined by its decomposition into indecomposables. In particular, we can describe V by the set of non-negative integers m_{pq} such that

$$V = \bigoplus_{1 \leqslant p \leqslant q \leqslant m} m_{pq}^V E_{pq}.$$

3.1. Elementary degenerations

We recall some operations introduced by Abeasis and Del Fra on indecomposables E_{pq} called "elementary degenerations". This will ultimately give a partial ordering on the orbits giving the degenerations. However, we note that this does not necessarily determine minimal degenerations.

(e) For each pair of indecomposables $E_{hk} \oplus E_{rt}$ such that $h < r \leqslant t < k$ and E_{rt} is of even type, consider the operation

$$D^{e}_{hrtk} : E_{hk} \oplus E_{rt} \mapsto E_{ht} \oplus E_{rk}.$$

Example 3.3.

$$\begin{array}{lcl} \rightarrow\rightarrow\rightarrow \bullet \leftarrow\leftarrow \blacktriangle \rightarrow\rightarrow\rightarrow & \qquad & \rightarrow\rightarrow\rightarrow \bullet \leftarrow\leftarrow \blacktriangle \rightarrow \\ \quad\;\; \rightarrow \bullet \leftarrow\leftarrow \blacktriangle \rightarrow & & \quad\;\; \rightarrow \bullet \leftarrow\leftarrow \blacktriangle \rightarrow\rightarrow\rightarrow \end{array}$$

(e′) For each indecomposable E_{hk} and each integer t such that $h \leqslant t < k$ we consider the operation

$$D^{e'}_{hk} \mapsto E_{ht} \oplus E_{t+1,k}.$$

Example 3.4.

$$\begin{array}{lcl} \rightarrow \bullet \leftarrow\leftarrow\leftarrow \leftarrow \blacktriangle \rightarrow\rightarrow & \qquad & \rightarrow \bullet \leftarrow\leftarrow \\ & & \qquad\qquad \leftarrow \blacktriangle \rightarrow\rightarrow \end{array}$$

(o) For each pair of indecomposables E_{ht}, E_{rk} with $h < r \leqslant t < k$ and E_{rt} of odd type, we consider the operation

$$D^{o}_{hrtk} : E_{ht} \oplus E_{rk} \mapsto E_{hk} \oplus E_{rt}.$$

Example 3.5.

$$\begin{array}{lcl} \rightarrow\rightarrow \bullet \leftarrow \blacktriangle \rightarrow\rightarrow \bullet \leftarrow & \qquad & \rightarrow\rightarrow \bullet \leftarrow \blacktriangle \rightarrow\rightarrow \bullet \leftarrow\leftarrow\leftarrow \\ \quad \rightarrow \bullet \leftarrow \blacktriangle \rightarrow\rightarrow \bullet \leftarrow\leftarrow\leftarrow & & \quad \rightarrow \bullet \leftarrow \blacktriangle \rightarrow\rightarrow \bullet \leftarrow \end{array}$$

Note that the elementary degeneration of type (e′) is a special case of (e) where one is switching with the empty indecomposable direct summands.

Definition 3.6. Given $V, W \in \mathrm{Rep}(Q, \mathbf{d})$, we say that $\mathcal{O}_W \leqslant_c \mathcal{O}_V$ if and only if the set of indecomposable factors of W is obtained from the one of V with a finite number of elementary operations of types (e), (e′) or (o).

Proposition 3.7 (Abeasis and Del Fra [1]). *For $V, W \in \mathrm{Rep}(Q, \mathbf{d})$, the two orderings, $\mathcal{O}_W \leqslant_c \mathcal{O}_V$ and $\mathcal{O}_W \leqslant_g \mathcal{O}_V$, coincide.*

Henceforth, we refer to degenerations of representations as cuts or switches or anti-cuts or anti-switches depending on whether the indecomposables involved are of even or odd type. We denote switches of even type (e) or (e′) simply by D^e without the subscript.

The following theorem is necessary in computing the codimensions of degenerations of orbits.

Theorem 3.8 (Bongartz). *Let C be the category of representations of a Dynkin quiver. Consider two objects M and N in C. Then N is a minimal degeneration of M if and only if there is an exact sequence $E : 0 \to U \to M' \to V \to 0$ with the following properties*:

(a) *U and V are indecomposables with $M = M' \oplus U^{p-1} \oplus V^{q-1} \oplus X$ and $N = U^p \oplus V^q \oplus X$. Here $U \oplus V$ and $M' \oplus X$ are disjoint.*
(b) *$U \oplus V$ is a minimal degeneration of M'.*
(c) *Any common indecomposable direct summand $W \not\simeq V$ of M and N satisfies $[W,N] = [W,M]$.*
(d) *Dually, any common indecomposable direct summand $W \not\simeq U$ of M and N satisfies $[N,W] = [M,W]$.*

Here, U, V, M', p and q are uniquely determined by M and N. Furthermore, we have

$$\operatorname{codim}_{\overline{\mathcal{O}(M)}} \mathcal{O}(N) = \operatorname{codim}_{\overline{\mathcal{O}(M')}} \mathcal{O}(U \oplus V) + \varepsilon(p + q - 2),$$

where ε is 1 for $V \not\simeq U$ and 2 for $V \simeq U$.

Proof. The proof follows from Theorem 4 in [4] and Corollary 4.2 in [5]. □

4. Preliminary results and equioriented case

Let Q be a Dynkin quiver of Dynkin type A, D or E. Recall, there is a one to one correspondence between indecomposable modules and positive roots of the corresponding root system. In particular, the dimension vectors of indecomposables correspond to positive roots.

Let $\mathbf{d}$ be the dimension vector associated to a quiver Q. Then $\mathbf{d}$ has a canonical decomposition into a sum of positive roots,

$$\mathbf{d} = \alpha_1 + \cdots + \alpha_n,$$

which gives the open orbit in $\operatorname{Rep}(Q, \mathbf{d})$. This tells how a generic module in the open orbit in $\operatorname{Rep}(Q, \mathbf{d})$ decomposes into indecomposables. Note that other decompositions of $\mathbf{d}$ into a sum of positive roots will give other orbits of $GL(Q, \mathbf{d})$ in $\operatorname{Rep}(Q, \mathbf{d})$. By abuse of notation, let V_{generic} denote both the open orbit and a generic module in the open orbit of $\operatorname{Rep}(Q, \mathbf{d})$.

To determine if a decomposition of $\mathbf{d}$ gives the open orbit, we use the following criterion.

Proposition 4.1. *If* $\mathbf{d} = \alpha_1 + \cdots + \alpha_s$ *and* V_{α_i} *is the corresponding indecomposable such that* $\dim V_{\alpha_i} = \alpha_i$, *then the decomposition gives the open orbit if and only if* $\mathrm{Ext}_Q(V_{\alpha_i}, V_{\alpha_j}) = 0$ *for all* α_i *and* α_j *which occur in the sum.*

Proof. The codimension of the open orbit is 0. Thus, by Lemma 1.6 we have that $\operatorname{codim} \mathcal{O}(V_{\mathbf{d}}) = \dim \mathrm{Ext}_Q(V_{\mathbf{d}}, V_{\mathbf{d}}) = 0$. $\square$

Definition 4.2. Let $Z \subseteq \mathrm{Rep}(Q, \mathbf{d})$ be a Zariski closed $GL(Q, \mathbf{d})$-stable subset. We say that a representation $M \in \mathrm{Rep}(Q, \mathbf{d})$ is a component of Z if the closure of the orbit $\mathcal{O}_M$ of M is an irreducible component of Z.

This definition will be used for the sets Z which are zero sets of some semi-invariants. Sometimes by abuse of notation we will talk about a representation M being a subset of Z. In such cases we always mean orbit of M. Also, each of irreducible components of a set Z is a closure of an orbit of a representation M. We will often identify this component with the corresponding representation.

Lemma 4.3. *Let* Q *be the quiver of type* A_n *with an equioriented path from* z *to* x,

$$\cdots \longleftarrow \overset{x}{\circ} \xleftarrow{\ \alpha\ } \overset{y}{\circ} \xleftarrow{\ \beta\ } \overset{z}{\circ} \longleftarrow \cdots$$

and the representation of Q *with the dimension* $\mathbf{d}$ *such that* $\mathbf{d}(x)=n$, $\mathbf{d}(y)=p$, $\mathbf{d}(z)=m$ *with* $p > \max(m,n)$. *Consider the quiver* $\tilde{Q}$:

$$\cdots \overset{x}{\circ} \xleftarrow{\alpha\beta} \overset{z}{\circ} \cdots$$

and the corresponding map of representations,

$$\rho : \mathrm{Rep}(Q, \mathbf{d}) \to \mathrm{Rep}(\tilde{Q}, \tilde{\mathbf{d}})$$

given by

$$(\ldots, V(\alpha), V(\beta), \ldots) \mapsto (\ldots, V(\alpha\beta), \ldots)$$

where $\tilde{\mathbf{d}}(i)=\mathbf{d}(i)$ *for* $i \neq y$. *Then there is an isomorphism of the rings of semi-invariants, i.e.* $k[\mathrm{Rep}(Q, \mathbf{d})] \cong k[\mathrm{Rep}(\tilde{Q}, \tilde{\mathbf{d}})]$, *and the codimension of the nullcones are preserved.*

Proof. By Cauchy's formula, (see for example [7, Section A.1]) one can show that there cannot be a weight of a semi-invariant σ with $\sigma_y > 0$, hence there is a correspondence of semi-invariants. Let $SI(Q, \mathbf{d}) = K[c^{V_1}, \ldots, c^{V_s}]$ and $SI(\tilde{Q}, \tilde{\mathbf{d}}) = K[c^{\tilde{V}_1}, \ldots, c^{\tilde{V}_s}]$, where $\tilde{V}_i = \rho(V_i)$. Next, consider the Koszul complex $K(c^{V_1}, \ldots, c^{V_s})$. Then for $G := GL(p)$, $K(c^{\tilde{V}_1}, \ldots, c^{\tilde{V}_s}) = K(c^{V_1}, \ldots, c^{V_s})^G$ which implies

$$H_i(K(c^{\tilde{V}_1}, \ldots, c^{\tilde{V}_s})) = H_i(K(c^{V_1}, \ldots, c^{V_s}))^G.$$

Therefore $H_i(K(c^{V_1},\dots,c^{V_s})=0$ implies $H_i(K(c^{\tilde{V}_1},\dots,c^{\tilde{V}_s}))=0$. Hence

$$\operatorname{codim} Z(c^{V_1},\dots,c^{V_s})) \leqslant \operatorname{codim} Z(c^{\tilde{V}_1},\dots,c^{\tilde{V}_s}).$$

We claim that $\operatorname{codim} Z(c^{V_1},\dots,c^{V_s})) \geqslant \operatorname{codim} Z(c^{\tilde{V}_1},\dots,c^{\tilde{V}_s})$. First, observe that for $V,W\in \operatorname{Rep}(Q)$,

$$\langle V,W\rangle = \langle \rho(V),\rho(W)\rangle + (\dim V_z - \dim V_y)(\dim W_x - \dim W_y).$$

Hence the Euler form is preserved under ρ for all $V,W\in\operatorname{Rep}(Q)$ unless $\dim V_y \neq \dim V_z$ and $\dim W_x \neq \dim W_y$.

Recall by the Auslander–Reiten duality [2] that for the indecomposable representations V,W of a Dynkin quiver $\operatorname{Hom}(V,W)\neq 0$ implies $\operatorname{Ext}(V,W)=0$ and $\operatorname{Ext}(V,W)\neq 0$ implies $\operatorname{Hom}(V,W)=0$. Thus, we note, that for all V and W indecomposable representations of Q such that $\langle V,W\rangle = \langle \rho(V),\rho(W)\rangle$ we have to have $\dim\operatorname{Hom}(V,W) = \dim\operatorname{Hom}(\rho(V),\rho(W))$ and $\dim\operatorname{Ext}(V,W) = \dim\operatorname{Ext}(\rho(V),\rho(W))$. In fact, for $V = E_{a,y}$ and $W = E_{y,b}$ the discrepancy in the Euler form lies in the difference between $\operatorname{Hom}(V,W)$ and $\operatorname{Hom}(\rho(V),\rho(W))$, i.e. $\dim\operatorname{Ext}(V,W) = \dim\operatorname{Ext}(\rho(V),\rho(W))$. For $V=E_{z,b}$ and $W=E_{a,x}$ the discrepancy in the Euler form lies in that $\dim\operatorname{Ext}(V,W)\neq \dim\operatorname{Ext}(\rho(V),\rho(W))$.

Consider $c^{V_i}\in SI(Q,\mathbf{d})$, which has weight $\sigma=\langle V_i,-\rangle$. Then, $\sigma_y=\dim(V_i)_y-\dim(V_i)_z=0$. Hence, if c^{V_i} vanishes at W, then by King Theorem we must have a submodule $W_i\subseteq W$ *where* $\langle V_i,W_i\rangle>0$. Since $\dim(V_i)_y=\dim(V_i)_z$, the Euler form is preserved under ρ and we have $\langle V_i,W_i\rangle=\langle \tilde{V}_i,\rho(W_i)$. Thus $c^{\tilde{V}_i}$ vanishes at $\rho(W_i)$.

Let M be a generic representation of $\operatorname{Rep}(Q,\mathbf{d})$ and let M_{u_1}, for $1\leqslant u_1\leqslant n$, be a generic representation in each of the n components of $Z(c^{V_1})$. Further, let M_{u_1,u_2}, for $1\leqslant u_2\leqslant n(u_1)$, be a generic representation in each of $n(u_1)$ components of $Z(c^{V_2})\cap \overline{\mathcal{O}(M_{u_1})}$, and $M_{u_1,\dots,u_t}$ for $1\leqslant u_t\leqslant n(u_1,\dots,u_{t-1})$, be a generic representation in each of $n(u_1,\dots,u_{t-1})$ components of $Z(c^{V_t})$ on $\overline{\mathcal{O}(M_{u_1,\dots,u_{t-1}})}$. In particular, $M_{u_1,\dots,u_s}$ is a generic representation in a component of $Z(c^{V_1},\dots,c^{V_s})$.

We next claim that the indecomposables of the kind $V=E_{z,b}$ and $W=E_{a,x}$ do not occur as summands in any component of $Z(c^{V_1},\dots,c^{V_s})$. If we assume our claim, then for any component $M_{u_1,\dots,u_s}$ of $Z(c^{V_1},\dots,c^{V_s})$, $\rho(M_{u_1,\dots,u_s})\subseteq Z(c^{\tilde{V}_1},\dots,c^{\tilde{V}_s})$ and

$$\dim\operatorname{Ext}(M_{u_1,\dots,u_s},M_{u_1,\dots,u_s}) = \dim\operatorname{Ext}(\rho(M_{u_1,\dots,u_s}),\rho(M_{u_1,\dots,u_s})).$$

Since $\rho(M_{u_1,\dots,u_s})$ cannot be obtained as a sequence of s minimal degenerations,

$$\operatorname{codim} Z(c^{V_1},\dots,c^{V_s}) \geqslant \operatorname{codim} Z(c^{\tilde{V}_1},\dots,c^{\tilde{V}_s}).$$

To prove our claim that indecomposables of the kind $V=E_{z,b}$ and $W=E_{a,x}$ do not occur as summands in any component of $Z(c^{V_1},\dots,c^{V_s})$, we proceed by induction and show that these indecomposables do not occur in any component $M_{u_1,\dots,u_t}$ of $Z(c^{V_1},\dots,c^{V_t})$ for any $t\leqslant s$.

Consider the generic decomposition in $\operatorname{Rep}(Q,\mathbf{d})$, which can be computed by Proposition 4.1,

$$M=\bigoplus_{i=1}^{r(0)} E_{a_i(0),b_i(0)} \oplus \bigoplus_{i=1}^{s(0)} E_{c_i(0),d_i(0)} \oplus kE_{y,y}$$

where $y \in [a_i(0), b_i(0)]$, $y \notin [c_i(0), d_i(0)]$, and $k = \min(p-m, p-n)$. We see that $V = E_{z,b}$ and $W = E_{a,x}$ cannot be summands in M since $p > \max(m,n)$. Suppose

$$M_{u_1,\dots,u_t} = \bigoplus_{i=1}^{r(u_t)} E_{a_i(u_t),b_i(u_t)} \oplus \bigoplus_{i=1}^{s(u_t)} E_{c_i(u_t),d_i(u_t)} \oplus mE_{y,y}$$

and $V = E_{z,b}$ and $W = E_{a,x}$ are not summands in $M_{u_1,\dots,u_t}$. Then, if $c^{V_{t+1}} \neq c^{E_{y,z}}$,

$$M_{u_1,\dots,u_{t+1}} = \bigoplus_{i=1}^{r(u_{t+1})} E_{a_i(u_t t+1),b_i(u_{t+1})} \oplus \bigoplus_{i=1}^{s(u_{t+1})} E_{c_i(u_{t+1}),d_i(u_{t+1})} \oplus mE_{y,y}.$$

Since we did not make a cut between vertex x and y nor between vertex y and z, $V = E_{z,b}$ and $W = E_{a,x}$ cannot be summands in $M_{u_1,\dots,u_{t+1}}$. If $c^{V_{t+1}} = c^{E_{y,z}}$ then to force $c^{V_{t+1}}$ to vanish, a switch must be made between one of the $E_{a_i(u_t),b_i(u_t)}$ and $E_{y,y}$. Thus

$$M_{u_1,\dots,u_{t+1}} = \bigoplus_{i=1}^{r(u_{t+1})} E_{a_i(u_t t+1),b_i(u_{t+1})} \oplus \bigoplus_{i=1}^{s(u_{t+1})} E_{c_i(u_{t+1}),d_i(u_{t+1})} \oplus (m-1)E_{y,y}$$

and the indecomposables $V = E_{z,b}$ and $W = E_{a,x}$ still do not appear as summands in $M_{u_1,\dots,u_{t+1}}$. □

5. The equioriented case

In this section, let Q be the following equioriented quiver of type A_n:

$$\circ \xleftarrow{\alpha_1} \circ \xleftarrow{\alpha_2} \cdots \xleftarrow{\alpha_{n-1}} \circ .$$

Before we introduce our next result, we introduce a partial order $\leqslant_s$ on the indecomposables $V_1, \dots, V_s$ as follows.

Definition 5.1. Let V_i and V_j be two indecomposable representations of Q with dimension vectors $\mathbf{e}_i$ and $\mathbf{e}_j$, respectively. Then, $V_i \leqslant_s V_j$ if and only if $\mathbf{e}_i(x) \leqslant \mathbf{e}_j(x)$ for all $x \in Q_0$. Otherwise, we say that V_i and V_j are incomparable.

Consider the representation $M \in \mathrm{Rep}(Q, \mathbf{d})$. We define

$$\mathrm{rank}_M(a, b) = \mathrm{rank}(\alpha_a \circ \alpha_{a+1} \circ \dots \circ \alpha_{b-1}), \tag{5.1}$$

where α_i is a simplified notation for the linear map $M(\alpha_i)$.

Example 5.2. Let Q be the equioriented quiver of Dynkin type A_9, i.e.,

$$1 \leftarrow 2 \leftarrow 3 \leftarrow 4 \leftarrow 5 \leftarrow 6 \leftarrow 7 \leftarrow 8 \leftarrow 9.$$

Let $\mathbf{d} = (1,2,3,3,2,1,2,2,1)$. Then the semi-invariants, $SI(Q,\mathbf{d})$, are generated by $c^{V_1}, c^{V_2}, c^{V_3}, c^{V_4}$, and c^{V_5}, where

$$\dim V_1 = (0,0,0,0,0,0,0,1,0),$$

$$\dim V_2 = (0,0,0,0,0,0,1,1,1),$$

$$\dim V_3 = (0,0,0,1,0,0,0,0,0),$$

$$\dim V_4 = (0,0,1,1,1,0,0,0,0),$$

$$\dim V_5 = (0,1,1,1,1,1,0,0,0),$$

with $V_1 \leqslant_s V_2$ and $V_3 \leqslant_s V_4 \leqslant_s V_5$. Furthermore, for the general representation V with dimension vector $\mathbf{d}$, $\operatorname{rank}_V(1,6)=1$, $\operatorname{rank}_V(2,5)=2$, $\operatorname{rank}_V(3,4)=3$, $\operatorname{rank}_V(6,9)=1$, $\operatorname{rank}_V(7,8)=2$.

Throughout, let M, N denote representations in $\operatorname{Rep}(Q,\mathbf{d})$. Let $\mathcal{O}_M$, $\mathcal{O}_N$ be the orbits of M and N under the action of $GL(Q,\mathbf{d})$, and let $\overline{\mathcal{O}_M}$ be the orbit closure of M.

Lemma 5.3. *Let $f=c^V$, for $V_i=E_{a,b}$, be a nonzero irreducible semi-invariant on $\overline{\mathcal{O}_M}$. Then*

(1) $\mathbf{d}(a-1)=\mathbf{d}(b)$ *and* $\mathbf{d}(j)>\mathbf{d}(b)$ *for* $a\leqslant j<b$,
(2) $\operatorname{rank}_M(a-1,b)=\mathbf{d}(b)$, *and*
(3)
$$M = E_{a_1,b_1}\oplus\ldots\oplus E_{a_l,b_l}\oplus E_{a,u_1}\oplus\ldots\oplus E_{a,u_m}\oplus E_{t_1,b_1}\oplus\ldots\oplus E_{t_u,b_1}\oplus X\oplus Y,$$
where $a_i\leqslant a-1\leqslant b\leqslant b_i$, $l=\mathbf{d}(b)$, $u_i\leqslant b-1$, $a\leqslant t_i$, any $E_{t,u}\subseteq X$ has the property $a\leqslant t\leqslant u\leqslant b-1$, and any $E_{r,s}\subseteq Y$ has the property that either $b<r$ or $s<a-1$.

Proof. It is clear that $c^{E_{a,b}}$ is a non-zero semi-invariant if and only if

$$c^{E_{a,b}} = \det(\alpha_{a-1}\alpha_{i_1}\cdots\alpha_{b-1}) \neq 0.$$

Thus we must have that

$$\mathbf{d}(a-1)=\mathbf{d}(b)$$

and

$$\mathbf{d}(j)\geqslant\mathbf{d}(b) \tag{5.2}$$

for $a\leqslant j<b$. Since f is irreducible, the inequality in (5.2) must be strict. Our last point follows directly from the first two observations. □

Corollary 5.4. *Let c^V, c^W be two nonzero irreducible semi-invariants in $SI(Q,\mathbf{d})$. Let $V=E_{a,b}$ and $W=E_{c,d}$.*

(1) *If V and W are incomparable with respect to $<_s$, then without loss of generality $b<c$.*
(2) *If $V>_s W$ then $a<c\leqslant d<b$.*

Proof. The proof follows from Lemma 5.3. □

Lemma 5.5. *Let $f = c^V$, for $V = E_{a,b}$ be a nonzero irreducible semi-invariant on $\overline{\mathcal{O}_M}$, for $M \in \mathrm{Rep}(Q, \mathbf{d})$. Decompose M as follows*:

$$M = M_1 \oplus M_2 \oplus M_3$$

where

$$M_1 = \bigoplus_{i=1}^{u} X_i \quad \text{such that } E_{a,b} <_s X_i,$$

$$M_2 = \bigoplus_{i=1}^{v} Y_i \quad \text{such that } Y_i <_s E_{a,b},$$

$$M_3 = \bigoplus_{i=1}^{w} Z_i \quad \text{such that } E_{a,b} \text{ and } Z_i \text{ are incomparable}.$$

Assume that $X_1, \ldots, X_p$ are minimal among X_i's with respect to $\leqslant_s$, and that $Y_1, \ldots, Y_q$ are maximal among Y_i's with respect to $\leqslant_s$. Then the following statements are the consequences.

(1) *The irreducible components of $Z(f) \cap \overline{\mathcal{O}_M}$ are the orbit closures of the representations $N_{i,j}$ $(1 \leqslant i \leqslant p, 1 \leqslant j \leqslant q)$ where*

$$N_{i,j} = D^e(X_i \oplus Y_j) \oplus \bigoplus_{r=1, r\neq i}^{u} X_r \oplus \bigoplus_{s=1, s\neq j}^{v} Y_s \oplus \bigoplus_{k=1}^{w} Z_k,$$

where $D^e(X_i \oplus Y_j)$ is a switch of even type from Section 3.

(2) $\mathrm{codim}_{\overline{\mathcal{O}_M}} \mathcal{O}_{N_{i,j}} = 1$.

(3) *For $X_i = E_{i_1,i_2}$ minimal and $Y_j = E_{j_1,j_2}$ maximal, we have that* $\mathrm{rank}_{N_{i,j}}(r,s) = \mathrm{rank}_M(r,s) - 1$ *if $i_1 \leqslant r < j_1 \leqslant j_2 < s \leqslant i_2$ and* $\mathrm{rank}_{N_{i,j}}(r,s) = \mathrm{rank}_M(r,s)$ *otherwise.*

Proof. First notice that by Lemma 5.3, $E_{a,b}$ cannot be a summand of M. Recall that $f = c^V$ vanishes at $W \in \mathrm{Rep}(Q, \mathbf{d})$ if and only if there exists a submodule W' of W such that

$$\langle V, W' \rangle > 0. \tag{5.3}$$

Since $f = c^V$, where $V = E_{a,b}$, does not vanish on $\overline{\mathcal{O}_M}$ this implies for all $W' \subseteq M$, $\langle V, W' \rangle \leqslant 0$, i.e., for all $W' \subseteq M$ with dimension vectors $w' = \dim W'$,

$$\langle E_{a,b}, W' \rangle = w'(b) - w'(a-1) \leqslant 0. \tag{5.4}$$

In particular, for $W' = X_i, Y_i$ or Z_i in the decomposition of M, Eq. (5.4) holds.

In order to produce a W' with property 5.3, a switch must be made between one of the X_i's and Y_j's. If $X_i = E_{i_1,i_2}$, $Y_j = E_{j_1,j_2}$, then we know that

$$i_1 \leqslant a \leqslant b \leqslant i_2 \quad \text{and} \quad a \leqslant j_1 \leqslant j_2 \leqslant b$$

since $E_{a,b} \leqslant_s E_{i_1,i_2}$ and $E_{j_1 j_2} \leqslant_s E_{a,b}$. Thus,

$$D^e(E_{i_1,i_2} \oplus E_{j_1,j_2}) = E_{i_1,j_2} \oplus E_{i_2,j_1}$$

and

$$\langle E_{a,b}, E_{j_1,i_2}\rangle > 0.$$

Why must X_i be taken minimal and Y_j-maximal with respect to $\leqslant_s$? If D^e was applied to an X_r which is not minimal then the resulting component would not be a minimal degeneration of M, and can be obtained as a degeneration of a minimal one. Specifically, let $X_u >_s X_i$ for some $1 \leqslant i \leqslant u$. Then we claim that

$$P = D^e(X_u \oplus Y_j) \oplus \bigoplus_{r=1}^{u-1} X_r \oplus \bigoplus_{s=1, s\neq j}^{v} Y_s \oplus \bigoplus_{k=1}^{w} Z_k$$

is a degeneration of

$$N_{i,j} = D^e(X_i \oplus Y_j) \oplus \bigoplus_{r=1, r\neq i}^{u} X_r \oplus \bigoplus_{s=1, s\neq j}^{v} Y_s \oplus \bigoplus_{k=1}^{w} Z_k.$$

To see the exact path it takes, we set

$$X = \bigoplus_{r=1, r\neq i}^{u-1} X_r \oplus \bigoplus_{s=1, s\neq j}^{v} Y_s \oplus \bigoplus_{k=1}^{w} Z_k.$$

Then

$$N_{i,j} = D^e(X_i \oplus Y_j) \oplus X_u \oplus X = E_{i_1,j_2} \oplus E_{i_2,j_1} \oplus E_{u_1,u_2} \oplus X$$

and

$$P = D^e(X_u \oplus Y_j) \oplus X_i \oplus X = E_{u_1,j_2} \oplus E_{j_1,u_2} \oplus E_{i_1,i_2} \oplus X.$$

We show that P is obtained as a degeneration of $N_{i,j}$ by two switches. First,

$$P' = D^e(E_{u_1,u_2} \oplus E_{i_1,j_2} \oplus E_{j_1,i_2} \oplus X = E_{u_1,j_2} \oplus E_{i_1,u_2} \oplus E_{j_1,i_2} \oplus X,$$

and then

$$P = D^e(E_{i_1,u_2} \oplus E_{j_1,i_2} \oplus E_{u_1,j_2} \oplus X) = E_{u_1,j_2} \oplus E_{j_1,u_2} \oplus E_{i_1,i_2} \oplus X.$$

A similar argument shows that Y_j has to be chosen maximally.

To show that the codimension is correct, we use Bongartz's Theorem on minimal degenerations. We have $M = M' \oplus X$ and $N_{i,j} = U \oplus V \oplus X$, where $M' = E_{i_1,i_2} \oplus E_{j_1,j_2}$, $U = E_{i_1,j_2}$, $V = E_{j_1,i_2}$, and

$$X = \bigoplus_{r=1, r\neq i}^{u} X_r \oplus \bigoplus_{s=1, s\neq j}^{v} Y_s \oplus \bigoplus_{k=1}^{w} Z_k.$$

Furthermore,

$$\operatorname{codim}_{\overline{\mathcal{O}_m}} \mathcal{O}_{N_{i,j}} = \operatorname{codim}_{\overline{\mathcal{O}}_{M'}} \mathcal{O}_{U\oplus V} + \varepsilon(p+q-2).$$

In our case $\varepsilon(p+q-2) = 0$, since $p = q = 1$. Thus,

$$\operatorname{codim}_{\overline{\mathcal{O}}_{M'}} \mathcal{O}_{U\oplus V} = \dim \operatorname{Ext}_Q(U \oplus V, U \oplus V) = 1$$

for any quiver of type A_n. Finally, since all summands in $N_{i,j}$ are the same as in M with the exception that the indecomposable X_i was switched with the indecomposable Y_j,

$$\mathrm{rank}_M(r,s) = \mathrm{rank}_{N_{i,j}}(r,s)$$

except when $i_1 \leqslant r < j_1 \leqslant j_2 < s \leqslant i_2$, when the rank comes down by one. $\square$

Proposition 5.6. *Let c^V and c^W, for $V = E_{a,b}$, $W = E_{c,d}$, be two distinct nonzero irreducible semi-invariants on $\overline{\mathcal{O}}_M$ for $M \in \mathrm{Rep}(Q, \mathbf{d})$. Suppose that*

$$\mathrm{rank}_M(c-1,d) = \mathbf{d}(d).$$

Let N be a generic representation in any component of $Z(c^V) \cap O(M)$. Then

$$\mathrm{rank}_N(c-1,d) = \mathrm{rank}_M(c-1,d).$$

In other words, c^W is nonzero on every component N of $Z_M(c^V)$.

Proof. Suppose $V >_s W$, i.e. $a < c \leqslant d < b$.

Since c^V and c^W are nonzero irreducible semi-invariants, from Lemma 5.3 we know that

$$M = E_{a_1,b_1} \oplus \cdots \oplus E_{a_l,b_l} \oplus E_{a,u_1} \oplus \cdots \oplus E_{a,u_m} \oplus E_{t_1,b-1} \oplus \cdots \oplus E_{t_n,b-1} \oplus X$$
$$\oplus E_{c,r_1} \oplus \cdots \oplus E_{c,r_p} \oplus E_{s_1,d-1} \oplus \cdots \oplus E_{s_q,d-1} \oplus Y \oplus Z$$

where

(1) $a_i \leqslant a-1 \leqslant b \leqslant b_i$, $l = \mathbf{d}(b)$,
(2) either $u_i < c-1$ or $d \leqslant u_i$,
(3) either $d < t_i$ or $t_i \leqslant c-1$,
(4) for $E_{t,u} \subseteq X$, $a \leqslant t \leqslant c-1 \leqslant d \leqslant u \leqslant b-1$,
(5) $r_i \leqslant d-1$,
(6) $c \leqslant s_i$,
(7) for $E_{r,s} \subseteq Y$, $c \leqslant r \leqslant s \leqslant d-1$, and
(8) for $E_{p,q} \subseteq Z$, $q < a-1$ or $b < p$ or $d < p \leqslant q < b$ or $a-1 < p \leqslant q < c-1$.

By Lemma 5.5 any generic representation N in a component of $Z_M(c^V)$ has the property that it was obtained from M by a switch between one of the summands $X_i = E_{a_i,b_i}$ and $Y_j = E_{j_1,j_2}$ where $a \leqslant j_1 \leqslant c-1 \leqslant d \leqslant j_2 \leqslant b-1$. Furthermore we know that a switch of this type affects the ranks as follows.

$$\mathrm{rank}_N(r,s) = \mathrm{rank}_M(r,s) - 1 \quad \text{if } a_i \leqslant r < j_1 \leqslant j_2 < s \leqslant b_i,$$

and otherwise $\mathrm{rank}_N(r,s) = \mathrm{rank}_M(r,s)$. Hence,

$$\mathrm{rank}_N(c-1,d) = \mathrm{rank}_M(c-1,d).$$

Similarly, any generic representation N' in a component of $Z_M(c^W)$ has the property that it was obtained by a switch between one of the summands $X_i = E_{i_1,i_2}$ where $a \leqslant i_1 \leqslant c-1 \leqslant d \leqslant i_2 \leqslant b-1$ and $Y_j = E_{j_1,j_2}$ where $c \leqslant j_1 \leqslant j_2 \leqslant d-1$. Hence

$$\mathrm{rank}_{N'}(a-1,b) = \mathrm{rank}_M(a-1,b).$$

Suppose V and W are incomparable. Without loss of generality we may assume that $a < b < c < d$. By Lemma 5.3 we have that

$$M = E_{a_1,b_1} \oplus \ldots \oplus E_{a_h,b_h} \oplus E_{a,u_1} \oplus \ldots \oplus E_{a,u_l} \oplus E_{t_1,b-1} \oplus \ldots \oplus E_{t_m,b-1} \oplus X$$

$$\oplus E_{c_1,d_1} \oplus \ldots \oplus E_{c_n,d_n} \oplus \oplus E_{c,r_1} \oplus \ldots \oplus E_{c,r_p} \oplus E_{s_1,d-1} \oplus \ldots \oplus E_{s_q,d-1} \oplus Y \oplus Z,$$

where

(1) $a_i \leqslant a - 1 \leqslant b \leqslant b_i$,
(2) $u_i \leqslant b - 1$,
(3) $a \leqslant t_i$,
(4) for $E_{t,u} \subseteq X$ we have $a \leqslant t \leqslant u \leqslant b - 1$,
(5) $c_i \leqslant c - 1 \leqslant d \leqslant d_i$,
(6) $r_i \leqslant d - 1$,
(7) $c \leqslant s_i$,
(8) for $E_{r,s} \subseteq Y$ we have $c \leqslant r \leqslant s \leqslant d - 1$, and
(9) for $E_{p,q} \subseteq Z$ we have $b < p \leqslant q < c$ or $q < a - 1$ or $d < p$.

By Lemma 5.5 any component generic representation N in a component of $Z_M(c^V)$ has the property that it was obtained from M by a switch between one of the summands $X_i = E_{a_i,b_i}$ and $Y_j = E_{j_1,j_2}$ where $a \leqslant j_1 \leqslant j_2 \leqslant b-1$. Furthermore, we know that a switch of this type affects the ranks as follows.

$$\mathrm{rank}_N(r,s) = \mathrm{rank}_M(r,s) - 1 \quad \text{if } a_i \leqslant r < j_1 \leqslant j_2 < s \leqslant b_i,$$

and otherwise $\mathrm{rank}_N(r,s) = \mathrm{rank}_M(r,s)$.

Since $j_1 \leqslant j_2 \leqslant b - 1 < c$,

$$\mathrm{rank}_N(c-1,d) = \mathrm{rank}_M(c-1,d). \qquad \square$$

Theorem 5.7. *Let Q be an equioriented quiver of Dynkin type A_n, and $SI(Q,\mathbf{d}) = k[f_1,\ldots,f_s]$ where $f_1 = c^{V_1},\ldots,f_s = c^{V_s}$ are the irreducible semi-invariants. Then for any subset $\{i_1,\ldots,i_m\} \subseteq \{1,\ldots,s\}$,*

$$\operatorname{codim} Z(f_{i_1},\ldots,f_{i_m}) = m.$$

Proof. It is enough to prove the theorem for $\{i_1,\ldots,i_m\} = \{1,\ldots,s\}$. Indeed, if the set of some subset of semi-invariants $f_1,\ldots,f_s$ would have a component of codimension smaller than the cardinality of this subset, then the same would be true for the set $\{1,\ldots,s\}$. We claim that $f_t,\ldots,f_s$ are nonzero on every component $M_{u_1,\ldots,u_{t-1}}$ of $Z(f_1,\ldots,f_{t-1})$ for $1 \leqslant t \leqslant m-1$. We show it by induction on t. For $t=1$ the statement is obvious. For $t=2$, consider a generic representation $M \in \mathrm{Rep}(Q,\mathbf{d})$. Since $f_1,\ldots,f_s$ are distinct nonzero irreducible semi-invariants on $\overline{\mathcal{O}_M}$, by Proposition 5.6 we know that $f_2,\ldots,f_s$ are nonzero on every component M_{u_1} of $Z(f_1)$. Suppose $f_{t-1},\ldots,f_s$ are nonzero on every component

$$M_{u_1,\ldots,u_{t-2}} \subseteq Z(f_1,\ldots,f_{t-2}).$$

Again by Proposition 5.6, $f_t, \dots, f_s$ are nonzero on every component $M_{u_1,\dots,u_{t-1}}$ of $Z(f_{t-1}) \cap M_{u_1,\dots,u_{t-2}}$. Thus we have proved our claim and may conclude that

$$\operatorname{codim} Z(f_1, \dots, f_s) = s. \qquad \square$$

6. Reflection functors

In this section we recall the notion of reflection functors which we will use to obtain more complicated orientations from the equioriented quiver of type A. We show that some of the properties of representations are preserved under reflection functors. Let Q be a quiver. Let $x \in Q_0$ be a sink or a source.

Then $\sigma_x Q$ is the new quiver obtained by reversing the orientation of all the arrows going into or out of x. If x is a sink, at the representation level, we get a functor

$$C_x^+ : \mathrm{Rep}(Q) \to \mathrm{Rep}(\sigma_x Q)$$

$$V \mapsto W = C_x^+(V),$$

where $W_y = V_y$ for $y \neq x$ and $W_x = \ker(V_{t\alpha_1} \oplus \cdots \oplus V_{t\alpha_n} \to V_x)$. If x is a source, we obtain a similar functor,

$$C_x^- : \mathrm{Rep}(Q) \to \mathrm{Rep}(\sigma_x Q)$$

$$V \mapsto W = C_x^-(V),$$

where $W_y = V_y$ for $y \neq x$ and $W_x = \mathrm{coker}(V_x \to V_{h\alpha_1} \oplus \cdots \oplus V_{h\alpha_n})$. If $\mathbf{d} = \dim V$, we denote by $\sigma_x \mathbf{d}$ the resulting dimension vector for $C_x^+(V)$ or $C_x^-(V)$.

Suppose x is a sink. The following theorem has an analogue when x is a source, which we will omit.

Theorem 6.1 (Bernstein–Gelfand–Ponomarev [3]).

(1) *If* $V = S_x$ *then* $C_x^+ = 0$.
(2) *If* $V \neq S_x$ *is indecomposable,* $C_x^+(V)$ *is indecomposable,* $C_x^- C_x^+(V) = V$, *and*

$$\dim(C_x^+(V))_y = \begin{cases} \dim V_y & if \;\; y \neq x, \\ \sum \dim V_{t\alpha_i} - \dim V_x & if \;\; y = x. \end{cases}$$

Throughout this section, let Q be a quiver with a sink at vertex x. Further, let $x_1, \dots, x_n$ be the neighboring vertices of x and $\alpha_1, \dots, \alpha_n$ the corresponding maps between the vertices x_i and x. The next proposition follows easily from Theorem 6.1 and also holds when x is a source and when C_x^+ is replaced with C_x^-.

Corollary 6.2. *Let* V *and* W *be representations of* Q. *Suppose the simple module* S_x *is not a summand of* V *or* W. *Then*

$$\langle V, W \rangle = \langle C_x^+ V, C_x^+ W \rangle.$$

Proposition 6.3. *Suppose the simple module S_x is not a summand of V or W. Then*

$$\dim \mathrm{Hom}_Q(V,W) = \dim \mathrm{Hom}_Q(C_x^+ V, C_x^+ W)$$

and

$$\dim \mathrm{Ext}_Q(V,W) = \dim \mathrm{Ext}_Q(C_x^+ V, C_x^+ W).$$

Proof. Consider the map $C_x^+ : \mathrm{Hom}_Q(V,W) \to \mathrm{Hom}_Q(C_x^+ V, C_x^+ W)$. Suppose $\phi \in \mathrm{Ker}\, C_x^+$. Then $\phi_y = 0$ for all $y \neq x$. We claim that $\phi_x = 0$. If $\phi_x \neq 0$ and $\phi_y = 0$ for all $y \neq x$, then we define the submodule V' of V by setting $V'_y = V_y$ for $y \neq x$ and $V''_x = \mathrm{Ker}\, \phi_x$. We have an exact sequence

$$0 \to V' \to V \to \oplus S_x \to 0.$$

Since S_x is a projective representation, this sequence splits and S_x is a direct summand of V, which contradicts our assumption. Hence $\phi_x = 0$ and we see that the map C_x^+ is injective. Consider the map $C_x^- : \mathrm{Hom}_Q(C_x^+ V, C_x^+ W) \to \mathrm{Hom}_Q(C_x^- C_x^+ V, C_x^- C_x^+ W)$. Let us take $\psi \in \mathrm{Ker}\, C_x^-$. Then $\psi_y = 0$ for $y = 0$. We claim $\psi_x = 0$. If not, we define the quotient module $\tilde{V}$ on $C_x^+ V$ by setting $\tilde{V}_y = (C_x^+ V)_y$ for $y \neq x$ and $\tilde{V}_x = (C_x^+ V)_x / \mathrm{Ker}\, \psi_x$. We have an exact sequence

$$0 \to \oplus S_x \to C_x^+ V \to \tilde{V} \to 0.$$

Since S_x is injective in $\mathrm{Rep}(\sigma_x Q)$ the sequence splits, and S_x has to be a direct summand in $C_x^+ V$. This contradicts our assumption. Hence C_x^- is injective. But also we see that by our assumption the modules V and $C_x^- C_x^+ V$ are isomorphic, and similarly W and $C_x^- C_x^+ W$ are isomorphic. Thus both maps C_c^+, C_x^- have to be isomorphisms. This implies first statement of the Proposition. The second follows from the first and the corollary.

If x is a source, we can make a similar conclusion whenever S_x is not a summand of W. Furthermore, if $\mathrm{Hom}_Q(V,W)$ is preserved, since the Euler product is preserved, $\mathrm{Ext}_Q(V,W)$ is also preserved under reflections. □

The next proposition follows easily from the previous two propositions.

Proposition 6.4. *Let $f = c^V \in k[\mathrm{Rep}(Q,\mathbf{d})]$ and $S_x \neq V$. Suppose S_x is not a summand of W. If $f = c^V$ vanishes at W, then $\hat{f} = c^{C_x^+ V}$ vanishes at $C_x^+ W$.*

Proof. Recall by King's criterion, $c^V(W)$ vanishes if and only if W has a submodule W' such that $\langle V, W' \rangle > 0$. Since $\langle V, W' \rangle = \langle C_x^+ V, C_x^+ W' \rangle$, $\hat{f} = c^{C_x^+ V}$ vanishes at $\sigma_x W$. □

Corollary 6.5. *Let $k[\mathrm{Rep}(Q,\mathbf{d})]^{SL(Q,\mathbf{d})} = k[c^{V_1}, \ldots, c^{V_s}]$ and $Z(c^{V_1}, \ldots, c^{V_s}) = X_1 \cup \cdots \cup X_n$. Suppose the simple module S_x is not a summand of X_i for any i. Then*

$$C_x^+ X_1 \cup \cdots \cup C_x^+ X_n \subseteq Z(c^{C_x^+ V_1}, \ldots, c^{C_x^+ V_s}).$$

Proof. The proof follows immediately from Proposition 6.4. □

Lemma 6.6. *Let Q be a quiver of type A. Consider the representation of Q with the following dimension* $\mathbf{d}$:

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & & & & & & d_m \arrow[dl] \arrow[dr] & & \\
 & & & & & & \cdots \arrow[dl] & & d_{m+1} \arrow[dr] & \\
 & & & d_{x-1} \arrow[dl] \arrow[dr] & & d_{x+1} \arrow[dl] & & & & \cdots \\
 & & \cdots \arrow[dl] & & d_x & & & & & \\
 & d_2 \arrow[dl] & & & & & & & & \\
d_1 & & & & & & & & &
\end{tikzcd}$$

where $d_1 < d_2 < \cdots < d_m$. *Let*

$$SI(Q, \mathbf{d}) = k[c^{V_1}, \ldots, c^{V_s}].$$

Then

(1) *There is no semi-invariant* c^V *with* $V = S_x$.
(2) $(\sigma_x \mathbf{d})_1 < (\sigma_x \mathbf{d})_2 < \cdots < (\sigma_x \mathbf{d})_m$.
(3) *There is a one to one correspondence between the components of* $Z(c^{V_1}, \ldots, c^{V_s})$ *and the components of* $Z(c^{C_x^+ V_1}, \ldots, c^{C_x^+ V_s})$ *given by applying the Coxeter functor* C_x^+ *to the representative of each component of* $Z(c^{V_1}, \ldots, c^{V_s})$.
(4) *If* X_i *is a component of* $Z(c^{V_1}, \ldots, c^{V_s})$, *then* $\operatorname{codim} X_i = \operatorname{codim} C_x^+ X_i$.

Note, the results is also true when we reverse all the arrows.

Proof. Let A be the Euler matrix for our quiver

$$A = \begin{pmatrix}
1 & 0 & \ldots & & & & & 0 \\
-1 & 1 & 0 & \ldots & & & & \\
0 & -1 & 1 & 0 & \ldots & & & \\
\ldots & \ldots & \ldots & & & & & \\
0 & \ldots & 0 & -1 & 1 & 0 & & \\
 & & & 0 & -1 & 1 & 0 & \\
 & & & & 0 & 1 & 0 & \\
 & & & & 0 & 1 & -1 & 0 \\
 & & & & \ldots & & &
\end{pmatrix}.$$

Examining the rows of A we see that the weight of S_x cannot be the weight of a semi-invariant.

To prove our second point, since $(\sigma_x \mathbf{d})_y = d_y$ for $y \neq x$ we must only compare the dimensions at vertex $x-1, x$ and $x+1$. At these vertices, $d_{x-1} < d_{x+1} + d_{x-1} - d_x < d_x$. Hence, we prove our second point.

To prove the third and fourth points, consider a generic representation M in $\mathrm{Rep}(Q, \mathbf{d})$:

$$M = E_{1,b_{11}} \oplus \cdots \oplus E_{1,b_{1d_1}} \oplus E_{2,b_{21}} \oplus \cdots \oplus E_{2,b_{2,d_2-d_1}} \oplus \cdots \oplus E_{m,b_{m1}} \oplus \cdots \oplus E_{m,b_{m,d_m-d_{m-1}}} \oplus E,$$

where $b_{ij} \geqslant m$ and the summands $E_{a,b} \subseteq E$ are such that $m < a$. Since $d_1 < d_2 < \cdots < d_m$, there is no semi-invariant c^V with $V = E_{i,i}$ for $i < m$. Hence, degenerations which force semi-invariants to vanish involve switches between two summands of M or a cut along a summand $E_{a,b}$ of M where $m < a$. Thus M_{u_1}, a component of $Z(c^{V_1})$, must be of the form

$$M_{u_1} = E_{1,c_{11}} \oplus \cdots \oplus E_{1,c_{1d_1}} \oplus E_{2,c_{21}} \oplus \cdots \oplus E_{2,c_{2,d_2-d_1}} \oplus \cdots \oplus E_{m,c_{m1}} \oplus \cdots \oplus E_{m,c_{m,d_m-d_{m-1}}} \oplus E_{u_1},$$

where $c_{ij} \geqslant m$ and the summands $E_{ab} \subseteq E_{\alpha_1}$ are such that $m < a$. Let $M_{u_1,\ldots,u_{t-1}}$ be a component of $Z(c^{V_1}, \ldots, c^{V_{t-1}})$. Suppose

$$M_{u_1,\ldots,u_{t-1}} = E_{1,f_{11}} \oplus \cdots \oplus E_{1,f_{1d_1}} \oplus E_{2,f_{21}} \oplus \cdots \oplus E_{2,f_{2,d_2-d_1}} \oplus \cdots \oplus E_{m,f_{m1}} \oplus \cdots \oplus E_{m,f_{m,d_m-d_{m-1}}} \oplus E_{u_1,\ldots,u_{t-1}},$$

where $f_{ij} \geqslant m$ and the summands $E_{a,b} \subseteq E_{\alpha_1,\ldots,\alpha_{t-1}}$ are such that $m < a$. Then, again since $d_1 < d_2 < \cdots < d_m$, degenerations which force semi-invariants to vanish involve switches between summands of $M_{u_1,\ldots,u_{t-1}}$ or cuts along $E_{a,b}$ where $m < a$.

Hence, for $M_{u_1,\ldots,u_{t-1}}$ a component of $Z(c^{V_t})$ in $\overline{\mathcal{O}}_{M_{u_1,\ldots,u_{t-1}}}$,

$$M_{u_1,\ldots,u_t} = E_{1,g_{11}} \oplus \cdots \oplus E_{1,g_{1d_1}} \oplus E_{2,g_{21}} \oplus \cdots \oplus E_{2,g_{2,d_2-d_1}} \oplus \cdots \oplus E_{m,g_{m1}} \oplus \cdots \oplus E_{m,g_{m,d_m-d_{m-1}}} \oplus E_{\alpha_1,\ldots,\alpha_t},$$

where $g_{ij} \geqslant m$ and the summands $E_{a,b} \subseteq E_{\alpha_1,\ldots,\alpha_t}$ are such that $m < a$.

Since in each of the degenerations, M, $M_{u_1}, \ldots, M_{u_1,\ldots,u_s}$, we never made a cut along a summand $E_{a,b}$ with $b < m$, we see that the simple module $S_x = E_{x,x}$ does not occur as a summand in $M_{u_1,\ldots,u_s}$.

Finally, since a component X_i of $Z(c^{V_1}, \ldots, c^{V_s})$ is just one of the components $M_{u_1,\ldots,u_s}$ by Corollary 6.5

$$C_x^+ X_1 \cup \cdots \cup C_x^+ X_n \subseteq Z(c^{C_x^+ V_1}, \ldots, c^{C_x^+ V_s}),$$

and by Proposition 6.3 $\dim \mathrm{Ext}(X_i, X_i) = \dim \mathrm{Ext}(C_x^+ X_i, C_x^+ X_i)$. Hence $\operatorname{codim} X_i = \operatorname{codim} C_x^+ X_i$.

Conversely, consider the quiver $\sigma_x Q$ with dimension $\sigma_x \mathbf{d}$:

$$d_1 \leftarrow d_2 \leftarrow \cdots \leftarrow d_{x-1} \rightarrow d_{x+1} + d_{x-1} - d_x \rightarrow d_{x+1} \leftarrow \cdots \leftarrow d_m \rightarrow d_{m+1} \cdots,$$

where $d_1 < d_2 < \cdots < d_m$. The general decomposition in this dimension $\sigma_x\mathbf{d}$ is of the form

$$\begin{aligned} M' = {} & E_{1,h_{11}} \oplus \cdots \oplus E_{1,h_{1d_1}} \oplus E_{2,h_{21}} \oplus \cdots \oplus E_{2,h_{2,d_2-d_1}} \\ & \oplus \cdots \oplus E_{x-1,h_{x-1,1}} \oplus \cdots \oplus E_{x-1,h_{x-1,d_{x-1}-d_{x-2}}} \oplus E_{x,h_{x1}} \oplus \cdots \oplus E_{x,h_{x,d_x-d_{x-1}}} \\ & \oplus E_{x+1,h_{x+1,1}} \oplus \cdots \oplus E_{x+1,h_{x+1,d_x-d_{x-1}}} \oplus \cdots \oplus E_{m,h_{m,1}} \oplus \cdots \oplus E_{m,h_{m,d_m-d_{m-1}}} \oplus E, \end{aligned}$$

where $h_{ij} \geqslant m$, and the summands $E_{a,b} \subseteq E$ are such that $m < a$.

$$Z(c^{C_x^+V_1}, \ldots, c^{C_x^+V_s}) = Y_1 \cup \cdots \cup Y_t.$$

Again we see that the simple module S_x does not occur in any summands of the nullcone. Hence,

$$C_x^- Y_1 \cup \cdots \cup C_x^- Y_t \subseteq Z(c^{C_x^- C_x^+ V_1}, \ldots, c^{C_x^- C_x^+ V_s}),$$

and we see that $n = t$ and $Y_i = C_x^+ X_i$. $\square$

Corollary 6.7. *Let Q be quiver of type A. Consider the representation of Q with one of the following dimension vectors* $\mathbf{d}$

$$\begin{tikzcd}[column sep=small] & & & n_m \arrow[dl] \arrow[dr] & \\ & & \cdots \arrow[dl] & & \cdots \\ & n_2 \arrow[dl] & & & \\ n_1 & & & & \end{tikzcd} \quad or \quad \begin{tikzcd}[column sep=small] n_1 \arrow[dr] & & & & \\ & n_2 \arrow[dr] & & & \\ & & \cdots \arrow[dr] & & \cdots \arrow[dl] \\ & & & n_m & \end{tikzcd}$$

where $n_1 < n_2 < \cdots < n_m$. Let $SI(Q, \mathbf{d}) = k[c^{V_1}, \ldots, c^{V_s}]$, and

$$Z(c^{V_1}, \ldots, c^{V_s}) = X_1 \cup \cdots \cup X_n.$$

Using the reflection functors

$$C^+ = C_1^+ C_2^+ C_1^+ \cdots C_{m-2}^+ \cdots C_1^+ C_{m-1}^+ \cdots C_1^+, \ \textit{or respectively},$$

$$C^- = C_1^- C_2^- C_1^- \cdots C_{m-2}^- \cdots C_1^- C_{m-1}^- \cdots C_1^-,$$

we obtain the quiver $\tilde{Q}$ with dimension $\mathbf{e}$

$$(n_m - n_{m-1}) \to (n_m - n_{m-2}) \to \cdots \to (n_m - n_1) \to n_{m+1} \cdots,$$

or respectively

$$(n_m - n_{m-1}) \leftarrow (n_m - n_{m-2}) \leftarrow \cdots \leftarrow (n_m - n_1) \leftarrow n_{m+1} \cdots,$$

where

(1) $SI(\tilde{Q}, \mathbf{e}) = k[c^{C^+V_1}, \ldots, c^{C^+V_s}]$,
(2) $Z(c^{C^+V_1}, \ldots, c^{C^+V_s}) = C^+(X_1) \cup \cdots \cup C^+(X_n)$, *and*
(3) $\operatorname{codim} X_i = \operatorname{codim} C^+(X_i)$.

Proof. The first point follows from [9].

Notice that our quiver representation with dimension $\mathbf{d}$ satisfies the conditions of Lemma 6.6. Hence, we can apply Lemma 6.6 as long as the conditions are satisfied. □

7. Main theorem

Lemma 7.1. *Let Q be a quiver of type A. Consider the representation of Q with one of the following dimension vectors* $\mathbf{d}$

$$n_1 \leftarrow n_2 \leftarrow \cdots \leftarrow n_m \rightarrow \cdots,$$

where $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_m$. Let $n_i = n_{i+1}$ for $i < m$. Hence $c^{E_{i+1,i+1}}$ is a semi-invariant in $k[\mathrm{Rep}(Q, \mathbf{d})]$. Assume that c^V, for $V \neq E_{i+1,i+1}$, is nonzero on $\overline{\mathcal{O}_n}$, for $N \in \mathrm{Rep}(Q, \mathbf{d})$, and let P be any component of $Z_N(c^V)$. Then

$$\mathrm{rank}_P(i, i+1) = \mathrm{rank}_N(i, i+1).$$

In words, the rank associated to $c^{E_{i+1,i+1}}$ does not change when we force c^V to vanish.

Proof. Every component of $Z_n(c^V)$ is obtained by making a switch or a cut between the summands of N. Since $V \neq E_{i+1,i+1}$, we do not need to make a cut between vertices i and $i+1$ to obtain any component of $Z_n(c^V)$. Hence, we see that the rank from vertex $i+1$ to i does not change. □

Theorem 7.2. *Let Q be a quiver of Dynkin type A_n, and $SI(Q, \mathbf{d}) = k[c^{V_1}, \ldots, c^{V_s}]$. Then*

$$\mathrm{codim}\, Z(c^{V_1}, \ldots, c^{V_s}) = s.$$

Proof. First, let $N(Q, \mathbf{d}) = Z(c^{V_1}, \ldots, c^{V_s})$.

For a quiver of type A_n, let $1 = s_0, \ldots, s_{v+1} = n$ be the alternating sequence of sources and sinks. If $v + 1 = 1$ then we have the equioriented quiver.

To prove our main theorem, we proceed by induction on the number of vertices in the quiver, n, the number of changes in orientation, $v+1$, the sum $S = \sum_{x \in Q_0} \mathbf{d}(x)$, and the length until the first change in direction, s_1. When $n = 1$ we are done since there are no semi-invariants. In fact, when $n = 2$, we are done since this implies when $v + 1 = 1$, we are done by Theorem 5.7 since this is the equioriented case.

Let Q be the quiver of type A_n and consider the representation with dimension vector

$$n_1 \leftarrow \cdots \leftarrow n_{i-1} \leftarrow n_i \leftarrow n_{i+1} \leftarrow \cdots \leftarrow n_{s_1} \rightarrow n_{s_1+1} \rightarrow \cdots \rightarrow n_{s_2} \leftarrow \cdots.$$

If $n_1 > n_2$ then there is no semi-invariant with weight σ where $\sigma(1) \neq 0$. Hence, we may assume that $n_1 \leqslant n_2$. If $n_{i-1} \leqslant n_i$ and $n_i > n_{i+1}$ then we may reduce using Lemma 4.3 to the quiver

$$n_1 \leftarrow \cdots \leftarrow n_{i-1} \leftarrow n_{i+1} \leftarrow \cdots \leftarrow n_{s_1} \rightarrow n_{s_1+1} \rightarrow \cdots \rightarrow n_{s_2} \leftarrow \cdots.$$

Hence we can assume that $n_i \leqslant n_{i+1}$ for all $1 \leqslant i \leqslant s_1$.

We induct on the length of s_1. If $s_1 = 2$, then Q is a quiver with the following dimension vector

$$n_1 \overset{\phi}{\leftarrow} n_2 \to n_3 \to \cdots n_{s_2} \leftarrow \cdots.$$

If $n_1 < n_2$ then we can apply Corollary 6.7 and reduce to the quiver

$$n_2 - n_1 \overset{\tilde{\phi}}{\to} n_2 \to n_3 \to \cdots n_{s_2} \leftarrow \cdots$$

and we are done by induction on the number of changes in orientation. If $n_1 = n_2$ then all semi-invariants other than $\det\phi$ do not depend on the entries of ϕ, hence we can reduce to looking at the following quiver

$$n_2 \to n_3 \to \cdots \to n_{s_2} \to \cdots.$$

Again, by induction on number of changes in orientation we are done.

Now, for $s_1 > 2$, consider the quiver with dimension vector

$$n_1 \leftarrow \cdots \leftarrow n_{i-1} \leftarrow n_i \leftarrow n_{i+1} \leftarrow \cdots \leftarrow n_{s_1} \to n_{s_1+1} \to \cdots \to n_{s_2} \leftarrow \cdots.$$

If $n_1 < n_2 < \cdots < n_{s_1}$ then we can apply Lemma 6.7 to reduce to the quiver

$$(n_{s_1} - n_{s_1-1}) \to (n_{s_1} - n_{s_1-2}) \to \cdots \to (n_{s_1} - n_1) \to n_{s_1+1} \cdots,$$

and we are done by induction of number of changes in orientation. If $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_{s_1}$ then let i be the smallest index such that $n_i = n_{i+1}$. Hence our quiver dimension vector is as follows:

$$n_1 \leftarrow \cdots n_{i-1} \leftarrow n_i \overset{\phi}{\leftarrow} n_i \leftarrow \cdots \leftarrow n_{s_1} \to \cdots.$$

Let $c^{V_1} = \det\phi$. First, to find $Z(c^{V_1})$ we look at the following larger quiver with dimension $\mathbf{d}^1$

$$Q^1 : n_1 \leftarrow \cdots n_{i-1} \leftarrow n_i \xleftarrow{\phi_1} n_i - 1 \xleftarrow{\phi_2} n_i \leftarrow n_{i+2} \leftarrow \cdots \leftarrow n_{s_1} \to \cdots$$

and consider the map

$$\pi_1 : \mathrm{Rep}(Q^1, \mathbf{d}^1) \to \mathrm{Rep}(Q, \mathbf{d}),$$

where $\pi_1(\phi_1\phi_2)=\phi$. The image of the general representation of $\mathrm{Rep}(Q^1, \mathbf{d}^1)$ in $\mathrm{Rep}(Q, \mathbf{d})$ will be equal to $Z(c^{V_1})$. By the First Fundamental Theorem of Invariant Theory,

$$\operatorname{im}\pi_1 = \mathrm{Rep}(Q^1, \mathbf{d}^1)/GL(n_i - 1).$$

Clearly no other semi-invariants $c^{V_2}, \ldots, c^{V_s}$ vanish on $\operatorname{im}\pi_1$. Let $M = Z(c^{V_1})$. Then

$$\begin{array}{ccccc} Z_M(c^{V_2}, \ldots, c^{V_s}) & \subseteq & M & \subseteq & \mathrm{Rep}(Q, \mathbf{d}) \\ \uparrow \bar{\pi}_1 & & \uparrow \pi_1 & & \\ Z(\pi_1^*(c^{V_2}), \ldots, \pi_1^*(c^{V_s})) & \subseteq & \mathrm{Rep}(Q^1, \mathbf{d}^1) & & \end{array}$$

By Lemma 7.1, we see that the dimension of the generic fibre of π_1 and $\overline{\pi_1}$ are the same. Denote this dimension by p. Hence since $\dim \mathrm{Rep}(Q^1, \mathbf{d}^1) = \dim M + p$ and $\dim Z(\pi_1^*(c^{V_2}), \ldots, \pi_1^*(c^{V_s})) = \dim Z_M(c^{V_2}, \ldots, c^{V_s}) + p$, we have that

$$\operatorname{codim} Z_M(c^{V_2}, \ldots, c^{V_s}) = \operatorname{codim} Z(\pi_1^*(c^{V_2}), \ldots, \pi_1^*(c^{V_s})).$$

Now to find the vanishing of the rest of the semi-invariants, we may project to the quiver Q^2 with the dimension vector $\mathbf{d}^2$:

$$n_1 \leftarrow \cdots n_{i-1} \leftarrow n_i \leftarrow n_{i+2} \leftarrow \cdots \leftarrow n_{s_1} \rightarrow \cdots .$$

Now consider $SI(Q^2, \mathbf{d}^2) = k[c^{W_1}, \ldots, c^{W_t}]$. We may have introduced a few more semi-invariants. However, the image of $c^{V_2}, \ldots, c^{V_s}$ in $SI(Q^2, \mathbf{d}^2)$ are a subset of the semi-invariants $C^{W_1}, \ldots, c^{W_t}$. By induction on the sum of dimensions, S, $\operatorname{codim} Z(c^{W_1}, \ldots, c^{W_t}) = t$. Moreover $c^{W_1}, \ldots, c^{W_t}$ form a regular sequence. Since these are homogeneous elements, any subset of these form a regular sequence. Hence we are done. □

When this article was in press, the authors learned that C. Riedtmann and G. Zwara obtained similar results for arbitrary Dynkin quiver.

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