ELSEVIER
Journal of Pure and Applied Algebra 179 (2003) 175–197
JOURNAL OF PURE AND APPLIED ALGEBRA
www.elsevier.com/locate/jpaa

On coconnected algebras ☆

J. Sichler[a], V. Trnková[b,*]

[a] Department of Mathematics, University of Manitoba, Winnipeg, Manitoba, Canada R3T 2N2
[b] Mathematical Institute of Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic

Received 17 May 2002; received in revised form 1 August 2002
Communicated by J. Adámek

Abstract

A universal algebra A is coconnected if every homomorphism from A^2 to A is essentially at most unary. The paper explores this notion and shows, inter alia, that for certain congruence distributive and unary varieties, the abstract endomorphism monoids of their members and their coconnectedness are fully independent.

MSC: 08A35; 08B25

For any two monoids M_0 and M_1 there exist a universal algebra A_1 and a subalgebra $A_0 \subseteq A_1$ such that the endomorphism monoid $\operatorname{End} A_i$ is isomorphic to M_i for $i=0,1$, and there is a similar representation by an algebra A_0 and its quotient algebra A_1. Such pairs of algebras exist in any *algebraically universal* variety $\mathbb{V}$, that is, in any variety for which there is a full embedding of the category $\mathbb{G}$ of all (directed) graphs into $\mathbb{V}$, see [9]. (Throughout the paper, the algebraic universality is abbreviated as universality.)

For any algebra A, let A^2 be its Cartesian square and let $\pi_0^{(2)} \in \hom(A^2, A)$ and $\pi_1^{(2)} \in \hom(A^2, A)$ denote the two Cartesian projections. Regardless of the variety the algebra A comes from, the monoids $\operatorname{End} A$ and $\operatorname{End} A^2$ certainly depend on each other: the monoid $\operatorname{End} A^2$ contains the Cartesian square of $\operatorname{End} A$, while for any $g \in \operatorname{End} A^2$ and the diagonal homomorphism $\delta_2 \in \hom(A, A^2)$ given by $\delta_2(x) = (x, x)$, the two composites $\pi_0^{(2)} \circ g \circ \delta_2$ and $\pi_1^{(2)} \circ g \circ \delta_2$ belong to $\operatorname{End} A$.

☆ Supported by NSERC of Canada and the Grant Agency of the Czech Republic under grants 201/02/0148 and 201/00/1466. Supported also by MSM 113200007.
* Corresponding author.
E-mail addresses: sichler@cc.umanitoba.ca (J. Sichler), trnkova@karlin.mff.cuni.cz (V. Trnková).

0022-4049/03/$ - see front matter
PII: S0022-4049(02)00245-1

In the present paper, we explore a situation in which the endomorphism monoids $\operatorname{End} A$ and $\operatorname{End} A^2$ are closely related. Specifically, we investigate algebras for which every $f \in \hom(A^2, A)$ is essentially at most unary. In abstract terms, this property can be formulated as follows.

Definition. Let A be an object in a category $\mathscr{A}$ with finite products and let $n \geqslant 2$ be finite. For the n-fold power A^n of A, let $\{\pi_i^{(n)} \mid i \in n\}$ be the collection of its product projections. We say that A is *n-coconnected* if every $\mathscr{A}$-morphism $f : A^n \to A$ has a decomposition

$$f = h \circ \pi_i^{(n)} \quad \text{for some } h \in \operatorname{End} A \text{ and } i \in n = \{0, \ldots, n-1\}.$$

Any 2-coconnected object A is also called *coconnected*. And when A is n-coconnected for every $n \geqslant 2$, we say that it is *fully coconnected*.

The notion of coconnectedness was considered for topological spaces in [19] as the abstract categorical dual to connectedness. A topological space X is connected if and only if every continuous map from X to the coproduct $X + X$ of two of its copies factors through one of the two coproduct injections. Subsequently, coconnected metric spaces with simultaneously prescribed monoids of nonconstant nonexpanding, uniformly continuous and continuous maps were constructed in [20].

We show that coconnectedness presents no obstacle to monoid representation by endomorphisms of algebras from certain universal varieties. Having first collected simple observations and various examples of coconnected algebras, in Section 2 we show that for every monoid M there exist arbitrarily large $(0,1)$-lattices A_0 and A_1 with $\operatorname{End} A_0 \cong \operatorname{End} A_1 \cong M$ such that A_0 is not coconnected while A_1 is fully coconnected, and that there exists an arithmetical variety having this property. In Section 3 it is shown that the variety $\mathrm{Alg}(1,1)$ of all algebras with two unary operations is also of this kind, and that coconnected monounary algebras are determined by their endomorphism monoids. In the concluding Section 4, we consider small n-coconnected algebras and coconnected algebras with zero, and give a partial answer to the question below.

Problem. *Is every* 3*-coconnected algebra A fully coconnected*?

If the answer is negative, examples cannot be found amongst algebras with zero, or in congruence distributive varieties. When of a countable similarity type, however, any such example can be transformed into an example in $\mathrm{Alg}(1,1)$.

1. Examples and observations

Let X be a set and let ω be the set of all finite cardinals $n = \{0, \ldots, n-1\}$. A system $\mathscr{S}$ of maps $X^n \to X^m$ with $n, m \in \omega$ is a *clone on the set* X if it contains every Cartesian projection $\pi_i^{(n)} : X^n \to X$ with $i \in n \in \omega$, is closed under the composition $\circ$ of its members and, for any choice $f_0, \ldots, f_{m-1} : X^n \to X$ in $\mathscr{S}$, the fibered product $f_0 \dot{\times} \cdots \dot{\times} f_{m-1} : X^n \to X^m$ sending each $s \in X^n$ to $(f_0(s), \ldots, f_{m-1}(s)) \in X^m$ also belongs to $\mathscr{S}$. Since $\pi_0^{(n)} \dot{\times} \cdots \dot{\times} \pi_{n-1}^{(n)}$ is the identity map on X^n, any clone $\mathscr{S}$ on the set X is a category fully determined by the collection of its hom-sets $\mathcal{S}(X^n, X)$ with $n \in \omega$. When viewed as an abstract category, any such $\mathcal{S}$ is an *abstract clone* in the sense of [13], and also an *algebraic theory* in the sense of [11,12].

In what follows, an algebra is a pair $A = (X, \Sigma)$ in which X is a set and Σ is a set of operations on X.

For any algebra $A = (X, \Sigma)$, the underlying maps of all homomorphisms between its finite powers $A^0, A, A^2, \ldots$ form a clone on the set X. We call this clone the *clone of* A, and denote it $\mathrm{Clo}\, A$. Thus, $\mathrm{Clo}\, A$ is what [13] calls the centralizer clone of A to distinguish it from the clone of all term functions of A. We note that the underlying set of the 0th power A^0 is the singleton consisting of the empty map $\emptyset \to X$. Therefore, A^0 is a singleton algebra in any variety containing the algebra A, and $\hom(A^0, A)$ consists of all maps sending the singleton algebra A^0 to a singleton subalgebra of A.

1.1. For any algebra A, and any finite $n \geqslant 2$ and $i \in n$, the following properties of an $f \in \hom(A^n, A)$ are equivalent:
(1) $f = h \circ \pi_i^{(n)}$ for some $h \in \mathrm{End}\, A$;
(2) $f = h \circ \delta_n \circ \pi_i^{(n)}$ for the diagonal $\delta_n \in \hom(A, A^n)$;
(3) $f(x_0, \ldots, x_{n-1}) = f(x_i, \ldots, x_i)$ for every $(x_0, \ldots, x_{n-1}) \in X^n$.

Indeed, since $f \circ \delta_n \in \mathrm{End}\, A$, the second claim implies the first. Conversely, if $f = h \circ \pi_i^{(n)}$ then $f \circ \delta_n = h \circ \pi_i^{(n)} \circ \delta_n = h$ because $\pi_i^{(n)} \circ \delta_n = 1_A$, and hence $f = f \circ \delta_n \circ \pi_i^{(n)}$. The third claim is just a reformulation of the second one.

In what follows, we shall use any of these properties without a specific reference.

If $f \in \hom(A^n, A)$ factors through two distinct projections $\pi_i^{(n)}$ and $\pi_j^{(n)}$, then its kernel $\mathrm{Ker}\, f$ contains the join $\mathrm{Ker}\, \pi_i^{(n)} \vee \mathrm{Ker}\, \pi_j^{(n)}$, which is the total congruence on A^n, and hence f is constant. Therefore

(4) if A is n-coconnected and if $f \in \hom(A^n, A)$ is not constant, then there is exactly one $i \in n$ such that f factors through $\pi_i^{(n)}$.

1.2. *Any coconnected algebra is directly indecomposable.*

Indeed, suppose that $A = B \times C$ is coconnected. Then the map $f: A^2 \to A$ defined by $f((b, c), (b', c')) = (b, c')$ is a homomorphism, and hence $f = f \circ \delta_2 \circ \pi_i^{(2)}$ for some $i \in \{0, 1\}$. For $i = 0$ we then have $(b, c') = (b, c)$, and hence $c' = c$ for any $c, c' \in C$. For $i = 1$ we get $(b', c') = (b, c')$, and hence $b' = b$ for any $b, b' \in B$. Thus C or B is a singleton, and hence A is directly indecomposable. $\square$

It is perhaps illustrative to have also a syntactic proof of this simple fact. To produce it, we recall that an algebra $A = (X, \Sigma)$ is directly indecomposable if and only if any $f \in \hom(A^2, A)$ satisfying $f(x, x) = x$ and $f(f(x, y), z) = f(x, f(y, z)) = f(x, z)$ for all $x, y, z \in X$ has the form $f = \pi_0^{(2)}$ or $f = \pi_1^{(2)}$, see [13], for instance. If A is coconnected, then any $f \in \hom(A^2, A)$ whatsoever has the form $f = f \circ \delta_2 \circ \pi_i^{(2)}$ for some $i = 0, 1$. Hence any f with $f \circ \delta_2 = 1_A$ equals $\pi_0^{(2)}$ or $\pi_1^{(2)}$, and hence A is indecomposable. $\square$

Indecomposable algebras need not be coconnected, however. For instance, for any Abelian group $G = (X, +)$, the mapping $f: X^2 \to X$ given by $f(x, y) = x + y$ is a homomorphism. From $f=f\circ\delta_2\circ\pi_i^{(2)}$ for some $i\in\{0,1\}$ it follows that $x+y=x+x$ for all $(x,y)\in X^2$, and this is true only in a trivial group. Hence, there are no nontrivial coconnected Abelian groups; yet any prime order cyclic group is directly indecomposable.

1.3. Other varieties lack nontrivial coconnected algebras as well. For instance, no coconnected commutative semigroup $S=(X,+)$ has more than one element. Indeed, if S is coconnected, then the identity $y+y=y+x=x+y=x+x$ holds in S. If $0\in S$ denotes the common value of all $z+z$ with $z\in S$, then $z+t=z+z=0$ for all $z,\ t\in S$, that is, S is a commutative zero semigroup. If $f:X^2\to X$ is any mapping, then for any $(x,y),(u,v)\in X^2$ we have $f((x,y)+(u,v))=f(x+u,y+v)=f(0,0)$ and $f(x,y)+f(u,v)=0$. Therefore $f\in\hom(S^2,S)$ exactly when $f(0,0)=0$. If card $X>1$ then no homomorphism with $f^{-1}\{0\}=\{(0,0)\}$ factors through either projection. Other varieties with only a few coconnected members will be discussed in Sections 2 and 3.

Are there any n-coconnected algebras?

1.4. *For any finite $n\geqslant 2$, the 'complete' unary algebra $A=(n,n^n)$ is n-coconnected.*

This is easy to see. Let $f\in\hom(A^n,A)$ be arbitrary. Then $f(0,\dots,n-1)=i\in n$. Since for every n-tuple $(x_0,\dots,x_{n-1})\in A^n$ there exists an operation $\sigma\in n^n$ such that $x_j=\sigma(j)$ for $j=0,\dots,n-1$, it follows that:

$$f(x_0,\dots,x_{n-1})=f(\sigma(0),\dots,\sigma(n-1))=\sigma(f(0,\dots,n-1))=\sigma(i)=x_i.$$

Therefore $f=\pi_i^{(n)}$ and hence A is n-coconnected. $\square$

In Section 4 we show that for any $n\geqslant 3$, the complete unary algebra (n,n^n) is fully coconnected.

1.5. *Let $2\leqslant m\leqslant n$ be finite. Then any n-coconnected algebra A is m-coconnected. In particular, any n-coconnected algebra is coconnected.*

Indeed, since $m\leqslant n$, there are maps $\kappa\in n^m$ and $\lambda\in m^n$ such that $\lambda\circ\kappa=1_m$. Then $k\in\hom(A^n,A^m)$ and $l\in\hom(A^m,A^n)$, respectively, given by $k=\pi^{(n)}_{\kappa(0)}\dot\times\cdots\dot\times\pi^{(n)}_{\kappa(m-1)}$ and $l=\pi^{(m)}_{\lambda(0)}\dot\times\cdots\dot\times\pi^{(m)}_{\lambda(n-1)}$ satisfy $\pi_i^{(n)}\circ l=\pi^{(m)}_{\lambda(i)}$ for every $i\in n$, and hence also $k\circ l=1_{A^m}$. Suppose that $f\in\hom(A^m,A)$. Then $f\circ k\in\hom(A^n,A)$, and hence $f\circ k=f\circ k\circ\delta_n\circ\pi_i^{(n)}$ for some $i\in n$ because A is n-coconnected. But $k\circ\delta_n=\delta_m$ and hence $f=f\circ k\circ l=f\circ k\circ\delta_n\circ\pi_i^{(n)}\circ l=f\circ\delta_m\circ\pi^{(m)}_{\lambda(i)}$ as required. $\square$

In general, the reverse implication fails to hold. A counterexample exists already on a two-element set $2=\{0,1\}$—the complete unary algebra $A=(2,2^2)$ is 2-coconnected, by 1.4, but not 3-coconnected: to see that A is not 3-coconnected, it is enough to note that the 'majority' map $m:2^3\to 2$ defined by

$$m(i,i,1-i)=m(i,1-i,i)=m(1-i,i,i)=m(i,i,i)=i \quad \text{for } i\in\{0,1\}$$

belongs to $\hom(A^3,A)$, but does not factor through any projection.

On the other hand, in certain varieties all coconnected algebras must be fully coconnected. Our first example is this.

1.6. *Any idempotent coconnected algebra A is fully coconnected.*

We recall that an algebra $A=(X,\Sigma)$ is idempotent if every singleton $\{x\}\subseteq X$ carries a subalgebra of A. When this is the case, the constant map $k_x^{(1)}: X\to X$ with the value x belongs to $\operatorname{End} A$ for every $x\in X$. Therefore, the n-ary constant map $k_x^{(n)}=c_x\circ\pi_i^{(n)}$ belongs to $\hom(A^n,A)$ for every $n\geqslant 2$ and $x\in X$ and, consequently, every map $\phi_x^{(n)}$ given by $\phi_x^{(n)}(t_0,\dots,t_{n-1})=(t_0,\dots,t_{n-1},x)$ belongs to $\hom(A^n,A^{n+1})$. The inductive argument presented in Section 3 of [19] uses only the maps $\phi_x^{(n)}$ to prove the full coconnectedness of any coconnected topological space, and can therefore be repeated verbatim to show that any coconnected idempotent algebra is n-coconnected for every $n\geqslant 2$. □

Do there exist any idempotent coconnected algebras?

1.7. To formulate a positive answer to this question, we recall that an algebra A is *rigid* whenever $\operatorname{End} A=\{1_A\}$. An idempotent algebra A with more than one element cannot be rigid since each constant selfmap is one of its endomorphisms. If A is an algebra such that $\operatorname{End} A$ consists of the identity map 1_A and all constant selfmaps of its underlying set, we say that A is *c-rigid*. Any c-rigid algebra must be idempotent, of course.

According to a result by Herrlich [4] strengthened by Taylor in [18], the clone on any set X with at least three elements whose unary members are just 1_X and all constant selfmaps of X is uniquely determined to the extent that its members $X^n\to X$ are exactly all n-ary projections and all n-ary constants. When applied to idempotent algebras, this result implies

Theorem (Herrlich [4] and Taylor [18]). *Any c-rigid idempotent algebra with at least three elements is fully coconnected.*

Commutative idempotent groupoids satisfying the identity $(xy)x=y$, the so-called Steiner quasigroups, form a variety in which arbitrarily large c-rigid objects do exist [15]. There also are arbitrarily large c-rigid modular lattices [10].

Corollary (Pigozzi and Sichler [15], Koubek and Sichler [10]). *There exist arbitrarily large fully coconnected Steiner quasigroups and arbitrarily large fully coconnected modular lattices.*

Thus, neither congruence permutability nor congruence distributivity of a variety prevent the existence of its fully coconnected members.

In Section 2 we investigate coconnected lattices and $(0,1)$-lattices a little further.

1.8. The full strength of c-rigidity is not needed to obtain large idempotent fully coconnected algebras. We illustrate this on the example of dual discriminator algebras.

Recall that the ternary dual discriminator $t(u,v,w)$ is defined as

$$t(u,v,w)=\begin{cases} u & \text{for } u=v,\\ w & \text{for } u\neq v.\end{cases}$$

Let $A=(X,t)$ be a dual discriminator algebra on a set X. It is easy to see that A is idempotent and simple, and that $\operatorname{End} A$ consists of all constants and all injective maps. Write $A^n=(X^n,t_n)$. For $\phi,\psi\in X^n$, denote $E(\phi,\psi)=\{i\in n \mid \phi(i)=\psi(i)\}$. Then

$$t_n(\phi,\psi,\theta)(i)=\begin{cases} \phi(i) & \text{for } i\in E(\phi,\psi),\\ \theta(i) & \text{for } i\in n\setminus E(\phi,\psi).\end{cases}$$

1.8.1. *Let $n\geqslant 2$, and let $f:X^n\to X$ be any mapping with* $\operatorname{Ker} f=\operatorname{Ker}\pi_i^{(n)}$ *for some $i\in n$. Then $f\in\hom(A^n,A)$.*

Proof. If $\operatorname{Ker} f=\operatorname{Ker}\pi_i^{(n)}$, then $f=h\circ\pi_i^{(n)}$ for some injective $h:X\to X$. But then $h\in\operatorname{End}A$ and hence f is a homomorphism. □

1.8.2. *If $n\geqslant 2$ and $f\in\hom(A^n,A)$ is nonconstant, then for some $i\in n$ there exist $\phi,\psi\in X^n$ with $E(\phi,\psi)=n\setminus\{i\}$ and $f(\phi)\neq f(\psi)$.*

Proof. Suppose that for every $i\in n$ and all $\phi,\phi'\in X^n$ with $E(\phi,\phi')=n\setminus\{i\}$ it follows that $f(\phi)=f(\phi')$. If $\phi,\psi\in X^n$ are selected arbitrarily and $n\setminus E(\phi,\psi)=\{i_j \mid j\in k\}$, then there is a finite sequence $\{\phi=\phi_0,\phi_1,\ldots,\phi_k=\psi\}$ with $E(\phi_j,\phi_{j+1})=n\setminus\{i_j\}$ for every $j\in k$—to obtain ϕ_{j+1} from ϕ_j simply replace $\phi_j(i_j)=\phi(i_j)$ by $\psi(i_j)$. But then $f(\phi)=f(\psi)$ as claimed. □

1.8.3. *If $n\geqslant 2$ and $f\in hom(A^n,A)$ is nonconstant, then* $\operatorname{Ker} f=\operatorname{Ker}\pi_i^{(n)}$ *for some $i\in n$.*

Proof. By 1.8.2, there exist $i\in n$ and $\phi,\psi\in X^n$ with $E(\phi,\psi)=n\setminus\{i\}$ and $f(\phi)\neq f(\psi)$. Select $\theta\in X^n$ arbitrarily and denote $\rho_\theta=t_n(\phi,\psi,\theta)$. Then $\rho_\theta(i)=\theta(i)$, and $\rho_\theta(j)=\phi(j)=\psi(j)$ for each $j\in n\setminus\{i\}$. Since $f\in\hom(A^n,A)$ and $f(\phi)\neq f(\psi)$, we have $f(\rho_\theta)=t(f(\phi),f(\psi),f(\theta))=f(\theta)$.

If $\theta'(i)=\theta(i)$, then $\rho_{\theta'}=\rho_\theta$ and hence $f(\theta')=f(\theta)$, and this shows that $\operatorname{Ker}\pi_i^{(n)}\subseteq\operatorname{Ker} f$. Since $\pi_i^{(n)}\in\hom(A^n,A)$ is surjective, there is a (unique) $h\in\operatorname{End}A$ with $h\circ\pi_i^{(n)}=f$. Since h is nonconstant, it must be injective, and hence $\operatorname{Ker} f=\operatorname{Ker}\pi_i^{(n)}$. □

The claim below now follows immediately from 1.8.1–1.8.3.

Proposition. *Let $A=(X,t)$ be a dual discriminator algebra and let $n\geqslant 1$. Then $f\in$* $\hom(A^n,A)$ *if and only if f is either one of the constants or else $f=h\circ\pi_i^{(n)}$ for a unique injective mapping $h:X\to X$ and a unique $i\in n$. Thus A is fully coconnected.*

Exchanging the roles of homomorphisms and operations, we obtain this

Observation. *No nontrivial unary algebra $B=(X,\Sigma)$ whose every operation σ is injective or constant is* 3*-coconnected.*

Proof. Clearly $t\in\hom(B^3,B)$, and t does not factor through any projection because $t(x,x,y)=t(x,y,x)=t(y,x,x)=x$ whenever $x,y\in X$ are distinct. □

2. Coconnected lattices

In this section, we show that for any monoid M there exist arbitrarily large $(0,1)$-lattices A_0 and A_1 such that A_0 is a nontrivial direct product, A_1 is fully coconnected and $\mathrm{End}\,A_0\cong\mathrm{End}\,A_1\cong M$. Abstract endomorphism monoids thus generally have no effect on coconnectedness or direct decomposability and do not depend on either property. It will be seen that this is also true in certain varieties of algebras whose reducts are distributive $(0,1)$-lattices.

2.1. Here we make two simple observations about algebras from congruence distributive varieties.

Recall that an algebra A is simple if it has more than one element and only two congruences: the diagonal congruence Δ and the total congruence τ.

Let $n\geqslant 2$ be finite. For any given set $\{\theta_i\mid i\in n\}$ of congruences on an algebra A, let $\theta_0\times\cdots\times\theta_{n-1}$ denote the 'product congruence' Θ defined on A^n by

$$(x_0,\ldots,x_{n-1})\Theta(y_0,\ldots,y_{n-1})\quad\text{iff } x_i\theta_i y_i \text{ for every } i\in n.$$

It is well-known that within any congruence distributive variety, any congruence Θ on A^n has the form $\Theta=\theta_0\times\cdots\times\theta_{n-1}$ for some set $\{\theta_i\mid i\in n\}$ of congruences on A.

Some initial examples of congruence distributive coconnected algebras can be obtained as follows.

2.1.1. Lemma. *If A is an algebra such that the congruence lattice of A^2 is the four-element Boolean lattice, then A is either coconnected or it contains an isomorphic copy of its square A^2. In particular, for a congruence distributive variety $\mathbb{V}$, a simple algebra $A\in\mathbb{V}$ is coconnected whenever it is finite or rigid.*

Proof. If $f\in\hom(A^2,A)$ does not factorize through either projection, then $\mathrm{Ker}\,f$ is the diagonal congruence, and hence f embeds A^2 into A. If $\mathbb{V}$ is congruence distributive and $A\in\mathbb{V}$ is simple, then $\mathrm{Ker}\,f$ has the form $\theta_0\times\theta_1$ with $\theta_0,\theta_1\in\{\Delta,\tau\}$. By the first claim, A is coconnected whenever it is finite. if A is rigid, then $f\circ\delta_2=1_A$ and hence f is also surjective. But then f is an isomorphism and hence A is directly decomposable, and thus not simple. □

2.1.2. Lemma. *Any coconnected algebra A from a congruence distributive variety is fully coconnected.*

Proof. Suppose that A is a coconnected algebra from a congruence distributive variety, and let $f\in\hom(A^n,A)$ for some $n\geqslant 3$.

For any given $j \in n$, let $e_j \in \hom(A^2, A^n)$ be defined by

$$\pi_i^{(n)} \circ e_j = \begin{cases} \pi_0^{(2)} & \text{for } i \in n \setminus \{j\}, \\ \pi_1^{(2)} & \text{for } i = j. \end{cases}$$

Thus, for each $j \in n$, for every $(x, y) \in A^2$ we have $e_j(x, y) = (x, \dots, x, y, x, \dots, x)$—where y occurs only once, in the jth coordinate.

Since A is coconnected, each composite $f \circ e_j \in \hom(A^2, A)$ factors through $\pi_0^{(2)}$ or through $\pi_1^{(2)}$; in other words, for each $j \in n$ we have $f \circ e_j(x, y) = f \circ e_j(x, x) = f(x, \dots, x)$ or $f \circ e_j(x, y) = f \circ e_j(y, y) = f(y, \dots, y)$. If $\Theta = \operatorname{Ker} f$, then $\Theta = \theta_0 \times \cdots \times \theta_{n-1}$ for some congruences θ_i of A.

There are two cases to consider.

Case 1: $f \circ e_j = f \circ \delta_n \circ \pi_1^{(2)}$ for some $j \in n$, that is, $f(x, \dots, x, y, x, \dots, x) = f \circ e_j(x, y) = f(y, \dots, y)$ for some $j \in n$. With no loss of generality, let $j = 0$. Then $(y, x, \dots, x) \Theta (y, \dots, y)$ for all $x, y \in A$, and hence θ_i is the total congruence τ for every $i \in n \setminus \{0\}$. Therefore $\Theta = \theta_0 \times \tau \times \cdots \times \tau \supseteq \Delta \times \tau \times \cdots \times \tau = \operatorname{Ker} \pi_0^{(n)}$ and, consequently, f factors through $\pi_0^{(n)}$.

Case 2: $f \circ e_j = f \circ \delta_n \circ \pi_0^{(2)}$ for every $j \in n$, that is, $f(x, \dots, x, y, x, \dots, x) = f \circ e_j(x, y) = f(x, \dots, x)$ for every $j \in n$. Therefore for any $x, y \in A$ and each $j \in n$ we have $(x, y) \in \theta_j$ and hence $\theta_j = \tau$. But then $\Theta = \theta_0 \times \cdots \times \theta_{n-1} = \tau \times \cdots \times \tau$, and hence f is constant; as such, it factors through all projections $\pi_j^{(n)}$ with $j \in n$. $\square$

2.2. Now we turn to the variety $\mathbb{L}_{01}$ of all $(0,1)$-lattices and to its subvarieties. The homomorphisms between members of $\mathbb{L}_{01}$ are all lattice homomorphisms that preserve 0 and 1, called $(0,1)$-homomorphisms.

Recall that an element e of a $(0,1)$-lattice $A = (X, \vee, \wedge, 0, 1) \in \mathbb{L}_{01}$ is *neutral* if

$$(e \wedge x) \vee (x \wedge y) \vee (y \wedge e) = (e \vee x) \wedge (x \vee y) \wedge (y \vee e)$$

for all $x, y \in A$. The elements $0, 1$ of A are neutral and form a complemented pair in A. If $A = B \times C$, then $\{(0,1), (1,0)\} \subseteq A$ is a complemented pair of elements neutral in A. Conversely, any complemented pair $\{e, e'\} \subseteq A$ of elements neutral in A gives rise to a direct decomposition $A \cong (e] \times (e']$ of A, see Chapter 4 in [13], for instance. Therefore, $A \in \mathbb{L}_{01}$ is directly indecomposable exactly when $\{0,1\} \subseteq A$ is the only complemented pair formed by neutral elements of A.

Let $\mathbb{D}_{01}$ denote the variety of all distributive $(0,1)$-lattices. Since all elements of any distributive lattice are neutral, a lattice $A \in \mathbb{D}_{01}$ is directly indecomposable if and only if $\{0,1\}$ is the only complemented pair of A.

While we do not have a structural characterization of coconnected members of $\mathbb{L}_{01}$, the condition (a) in the claim below will suit our purpose.

2.2.1 Lemma. *Let* $A \in \mathbb{L}_{01}$ *and* $f \in \hom(A^2, A)$. *Then* f *factors through a projection if and only if* $f(0,1) \in \{0,1\}$. *Consequently*,

(a) *if* $\{0,1\}$ *is the only complemented pair of* $A \in \mathbb{L}_{01}$, *then* A *is coconnected*;

(b) *if* $A \in \mathbb{D}_{01}$, *then* A *is coconnected exactly when it is directly indecomposable*.

Proof. Let $A \in \mathbb{L}_{01}$ and $f \in \hom(A^2, A)$. If f factors through a projection, then $f(x,y) = f(x,x)$ for all $x, y \in A$ or $f(x,y) = f(y,y)$ for all $x, y \in A$. But then $f(0,1) = f(0,0) = 0$ in the first case, and $f(0,1) = f(1,1) = 1$ in the second. For the converse, suppose that $f(0,1) \in \{0,1\}$. Since any $(x,y) \in A^2$ has the form $(x,y) = ((1,0) \wedge (x,x)) \vee ((0,1) \wedge (y,y))$, we have

$$f(x,y) = (f(1,0) \wedge f(x,x)) \vee (f(0,1) \wedge f(y,y)) \quad \text{for all } (x,y) \in A^2.$$

If $f(0,1) = 0$, then $f(1,0) = f(1,0) \vee f(0,1) = f(1,1) = 1$, and hence $f(x,y) = (1 \wedge f(x,x)) \vee (0 \wedge f(y,y)) = f(x,x)$. If $f(0,1) = 1$, then $f(1,0) = f(1,0) \wedge f(0,1) = f(0,0) = 0$, and hence $f(x,y) = (0 \wedge f(x,x)) \vee (1 \wedge f(y,y)) = f(y,y)$. Thus f factorizes through a projection, and the first claim is proved.

Let $\{0,1\}$ be the only complemented pair of A. Since $(0,1) \in A^2$ has a complement, for any $f \in \hom(A^2, A)$ we must have $f(0,1) \in \{0,1\}$, and (a) follows. If $A \in \mathbb{D}_{01}$ is indecomposable, then it has no nontrivial complemented pairs and hence it is coconnected, by (a). The converse in (b) is provided by 1.2. □

2.2.2 Lemma. *If* $B, C \in \mathbb{L}_{01}$ *have no nontrivial complemented pairs, and if* $\hom(B,C) = \emptyset$ *and* $\hom(C,B) = \emptyset$, *then* $\mathrm{End}(B \times C) = \mathrm{End}\, B \times \mathrm{End}\, C$.

Proof. For any $f \in \mathrm{End}\, B$ and $g \in \mathrm{End}\, C$, the mapping $f \times g : B \times C \to B \times C$ given by $(f \times g)(x,y) = (f(x), g(y))$ clearly belongs to $\mathrm{End}(B \times C)$.

To show that $B \times C$ has no other endomorphisms, suppose that $h \in \mathrm{End}(B \times C)$. Then h has the form $h(x,y) = (h_0(x,y), h_1(x,y))$ in which $h_0 \in \hom(B \times C, B)$ and $h_1 \in \hom(B \times C, C)$. Since $(x,y) = (x,0) \vee (0,y)$ for all $(x,y) \in B \times C$, we have $h_i(x,y) = h_i(x,0) \vee h_i(0,y)$ for $i = 0,1$ and all $(x,y) \in B \times C$. Since h is a $(0,1)$-homomorphism and because $(1,0) \wedge (0,1) = (0,0)$ in $B \times C$, it follows that $\{h_i(1,0), h_i(0,1)\}$ is a complemented pair for $i = 0,1$. The lattices B and C have only the trivial complemented pair, and thus $\{h_i(1,0), h_i(0,1)\} = \{0,1\}$ for $i = 0,1$. If $h_0(0,1) = 1$, then the mapping $\psi : C \to B$ defined by $\psi(y) = h_0(0,y)$ is a $(0,1)$-homomorphism, contrary to the hypothesis. Therefore $h_0(0,1) = 0$ and hence $h_0(0,y) = 0$ for all $y \in C$. Similarly we find that $h_1(x,0) = 0$ for all $x \in B$. But then $h_0(x,y) = h_0(x,0)$ and $h_1(x,y) = h_1(0,y)$. The mappings $f : B \to B$ and $g : C \to C$, respectively, defined as $f(x) = h_0(x,0)$ and $g(y) = h_1(0,y)$ are $(0,1)$-homomorphisms and $h(x,y) = (h_0(x,y), h_1(x,y)) = (f(x), g(y))$. Thus $h \in \mathrm{End}\, B \times \mathrm{End}\, C$ as claimed. □

2.3. Now we are prepared to formulate results on independence of coconnectedness or decomposability on the endomorphism monoid in certain universal varieties.

Definition. Let $\mathbb{V}$ be a variety of algebras. Then $\mathbb{V}$ is *expansive* if for every monoid M there exist $A_0, A_1 \in \mathbb{V}$ such that $\mathrm{End}\, A_0 \cong \mathrm{End}\, A_1 \cong M$ and A_1 is fully coconnected while A_0 is directly decomposable. If $\mathbb{V}$ is expansive and such that the algebras $A_0, A_1 \in \mathbb{V}$ can be chosen to be finite for any finite monoid M, we say that $\mathbb{V}$ is *f-expansive*. And when for any $A_0, A_1 \in \mathbb{V}$ we have $\mathrm{End}\, A_0 \cong \mathrm{End}\, A_1$ only when both algebras A_0 and A_1 are coconnected or neither one is, we say that $\mathbb{V}$ is *fractured.*

We recall that a universal variety $\mathbb{V}$ is *finite-to-finite universal* if the category $\mathbb{G}$ of graphs has a full embedding $\Phi_{\mathbb{V}}$ into $\mathbb{V}$ such that $\Phi_{\mathbb{V}}(G) \in \mathbb{V}$ is a finite algebra for every finite graph G. Since any (finite) monoid M is isomorphic to the endomorphism monoid of some (finite) graph [16], any (finite-to-finite) universal variety contains a (finite) algebra whose endomorphism monoid is isomorphic to M. Thus every (finite-to-finite) universal variety is also (*finite-to-finite*) *monoid universal*.

Below is the main result of [2].

Proposition (Goralčík et al. [2]). *The following properties of a variety* $\mathbb{V} \subseteq \mathbb{L}_{01}$ *are equivalent*:
(1) $\mathbb{V}$ *is* (*finite-to-finite*) *universal*;
(2) $\mathbb{V}$ *is* (*finite-to-finite*) *monoid universal*;
(3) $\mathbb{V}$ *contains a* (*finite*) $(0,1)$-*lattice with no prime ideal*;
(4) $\mathbb{V}$ *contains a* (*finite*) *simple non-distributive* $(0,1)$-*lattice*.

This result can be extended as follows.

Theorem. *Let* $\mathbb{V}$ *be a subvariety of* $\mathbb{L}_{01}$. *Then*
(1) $\mathbb{V}$ *is expansive exactly when it is universal*;
(2) $\mathbb{V}$ *is* f-*expansive exactly when it is finite-to-finite universal*.

Proof. If $\mathbb{V}$ is expansive (or f-expansive), then $\mathbb{V}$ is universal (or finite-to-finite universal) by the above Proposition.

The key fact for the proof of the converse is that for any (finite-to-finite) universal variety $\mathbb{V} \subseteq \mathbb{L}_{01}$ the (finite-to-finite) full embedding $\Phi_{\mathbb{V}} : \mathbb{G} \to \mathbb{V}$ constructed in [2] has the property that for every graph G, the $(0,1)$-lattice $\Phi_{\mathbb{V}}(G)$ has only the trivial complemented pair. For a given (finite) monoid M, there exist (finite) nonsingleton graphs G_0 and G_1 such that G_0 is rigid, $\mathrm{End}\, G_1 \cong M$, and $\hom(G_0, G_1) = \hom(G_1, G_0) = \emptyset$, see [16]. Denote $A_0 = \Phi_{\mathbb{V}}(G_0) \times \Phi_{\mathbb{V}}(G_1)$ and $A_1 = \Phi_{\mathbb{V}}(G_1)$. Then A_0 is a nontrivial direct product, while A_1 is coconnected, by 2.2.1(a), and hence fully coconnected, by 2.1.2. Since the lattices $\Phi_{\mathbb{V}}(G_0)$ and $\Phi_{\mathbb{V}}(G_1)$ have no nontrivial complemented pairs and because there are no $\mathbb{L}_{01}$-morphism between them, from 2.2.2 it follows that $\mathrm{End}\, A_0 = \{1_{\Phi_{\mathbb{V}}(G_0)}\} \times \mathrm{End}\, A_1 \cong \mathrm{End}\, A_1 \cong M$. $\square$

Remark. There exist arbitrarily large graphs with a given endomorphism monoid M, see [16]. Thus, each expansive subvariety of $\mathbb{L}_{01}$ contains arbitrarily large coconnected $(0,1)$-lattices and also arbitrarily large directly decomposable $(0,1)$-lattices that represent M.

2.4. We turn to the variety $\mathbb{D}_{01}$ of distributive $(0,1)$-lattices and to varieties that have reducts in $\mathbb{D}_{01}$. As noted earlier, a lattice $A \in \mathbb{D}_{01}$ is coconnected exactly when it is directly indecomposable, and this occurs if and only if $\{0,1\}$ is the only complemented pair in A. It immediately follows that a Boolean algebra B is coconnected exactly when card $B \leqslant 2$. Since a Boolean algebra B is rigid exactly when it has at most two elements, its abstract endomorphism monoid determines whether B is coconnected or not. More generally, for any $A, A' \in \mathbb{D}_{01}$ we have $\operatorname{End} A \cong \operatorname{End} A'$ if and only if A' is isomorphic to A or to its order dual A^{d}, see [14]. Since $A \in \mathbb{D}_{01}$ has only the trivial complemented pair exactly when its order dual does, the endomorphism monoid $\operatorname{End} A$ determines whether A is coconnected or not. Therefore the variety $\mathbb{D}_{01}$ is fractured.

On the other hand, certain varieties whose algebras have reducts in $\mathbb{D}_{01}$ are expansive. This is true for double Heyting algebras and certain varieties of distributive double p-algebras, for reasons we now outline.

Recall that an algebra $(X, \vee, \wedge, \rightarrow, 0, 1)$ of the similarity type $(2,2,2,0,0)$ is a Heyting algebra if $(X, \vee, \wedge, 0, 1) \in \mathbb{D}_{01}$ and the remaining binary operation $\rightarrow$ is defined by the requirement that $t \leqslant x \rightarrow y$ exactly when $t \wedge x \leqslant y$. A double Heyting algebra $(X, \vee, \wedge, \rightarrow, \leftarrow, 0, 1)$ has an additional operation $\leftarrow$ given by the dual requirement that $u \geqslant x \leftarrow y$ exactly when $u \vee x \geqslant y$. A distributive double p-algebra $(X, \vee, \wedge, *, +, 0, 1)$ has the type $(2,2,1,1,0,0)$ in which $(X, \vee, \wedge, 0, 1) \in \mathbb{D}_{01}$, the unary operation $*$ is defined by the requirement that $t \leqslant x^*$ exactly when $t \wedge x = 0$, while $+$ is given by the dual requirement that $t \geqslant x^+$ exactly when $t \vee x = 1$.

We claim that any of these algebras is coconnected exactly when $\{0, 1\}$ is its only complemented pair. Indeed, suppose that $A = (X, \vee, \wedge, \rightarrow, \leftarrow, 0, 1)$ has only the trivial complemented pair. Then $f(0,1) \in \{0,1\}$ for any $f \in \hom(A^2, A)$ and, since f is also an $\mathbb{L}_{01}$-morphism, from 2.2.1 it follows that f factorizes through a projection. Thus A is coconnected. Conversely, if A has a complemented pair other than $\{0, 1\}$, then its $\mathbb{D}_{01}$-reduct $A' = (X, \vee, \wedge, 0, 1)$ is a nontrivial direct product $A' = B \times C$. To show that A has the same decomposition, let $b_0, b_1 \in B$. Select $z_0, z_1 \in C$ arbitrarily and let $(p, q) = (b_0, z_0) \rightarrow (b_1, z_1)$ in A. Since the projection onto B is a $\mathbb{D}_{01}$-morphism, we have $p \wedge b_0 \leqslant b_1$ in B. If $x \wedge b_0 \leqslant b_1$ in B, then $(x, 0) \wedge (b_0, z_0) = (x \wedge b_0, 0) \leqslant (b_1, z_1)$ in A, and hence $(x, 0) \leqslant (p, q)$. But then $x \leqslant p$ in B. This shows that setting $p = b_0 \rightarrow b_1$ defines the unique Heyting operation for which the projection $\pi_B : A' \rightarrow B$ satisfies

$$\pi_B(a_0 \rightarrow a_1) = \pi_B(a_0) \rightarrow \pi_B(a_1) \quad \text{for all } a_0, a_1 \in A.$$

The other operations are handled similarly.

According to [7], the category $\mathbb{G}$ of graphs has a full embedding $\Phi_{2\mathbb{H}}$ into the variety $2\mathbb{H}$ of double Heyting algebras such that $\Phi_{2\mathbb{H}}(G)$ has only the trivial complemented pair (and $\Phi_{2\mathbb{H}}(G)$ is infinite) for every $G \in \mathbb{G}$, see [7]. As in 2.3, we obtain

Proposition. *The variety $2\mathbb{H}$ of double Heyting algebras is expansive.*

It appears that $2\mathbb{H}$ is the only known arithmetical (that is, congruence distributive and congruence permutable) variety that is expansive.

We also note that no finitely generated subvariety of $2\mathbb{H}$ is monoid universal.

Now let $\mathbb{V}$ be a finitely generated variety of distributive double p-algebras. The variety $\mathbb{V}$ is (finite-to-finite) universal if and only if it is (finite-to-finite) monoid universal, see [5,8]. For any such $\mathbb{V}$, the (finite-to-finite) full embedding $\Phi_{\mathbb{V}} : \mathbb{G} \rightarrow \mathbb{V}$ constructed in these papers has the property that $\{0, 1\}$ is the only complemented pair in every $\Phi_{\mathbb{V}}(G)$. As in 2.3, we obtain the following result.

Theorem. *For any finitely generated variety $\mathbb{V}$ of distributive double p-algebras, these properties are equivalent*:
(1) $\mathbb{V}$ *is* (*finite-to-finite*) *universal*;
(2) $\mathbb{V}$ *is* (*finite-to-finite*) *monoid universal*;
(3) $\mathbb{V}$ *is* (*f*-)*expansive*.

2.5. We conclude this section with some comments on coconnected lattices.

2.5.1. First we note that any rigid $A \in \mathbb{L}_{01}$ is either coconnected or directly decomposable. Indeed, if $f(0,1) \in \{0,1\}$ for every $f \in \hom(A^2, A)$, then A is coconnected, by 2.2.1. Else $f(0,1), f(1,0) \in A \setminus \{0,1\}$ are neutral in A since $(0,1)$ and $(1,0)$ are neutral in A^2 and f is surjective (because A is rigid). Hence A has a nontrivial product decomposition, see 2.2. Rigids of either kind do exist, by the Theorem in 2.3.

2.5.2. There are directly indecomposable $(0,1)$-lattices which are not coconnected. For any $A \in \mathbb{L}_{01}$ with a prime ideal, let $p \in \hom(A, 2)$ denote the corresponding $(0,1)$-homomorphism onto the 2-element lattice $2 = \{0,1\}$. For any $e \in A$ having a complement, let $h_e \in \hom(2^2, A)$ be the homomorphism with $h_e(0,1) = e$. The composite $f = h_e \circ (p \times p)$ then satisfies $f(0,1) = h_e(0,1) = e$. If A is coconnected then $e = f(0,1) \in \{0,1\}$, by 2.2.1. In other words, no $A \in \mathbb{L}_{01}$ with a prime ideal and a nontrivial complemented pair can be coconnected. Directly indecomposable lattices of this kind certainly exist in any nondistributive subvariety of $\mathbb{L}_{01}$.

2.5.3. Let $\mathbb{M} \subseteq \mathbb{L}_{01}$ denote the variety generated by the simple modular lattice $M_{3,3} = \{0, a, b, c, d, e, f, 1\}$ with the order given by $0 < a, b, c < d < 1$ and $c < e, f < 1$. By [10], there is a finite-to-finite full embedding $\Phi_{\mathbb{M}} : \mathbb{G} \to \mathbb{M}$ such that any *lattice* homomorphism $\Phi_{\mathbb{M}}(G) \to \Phi_{\mathbb{M}}(G')$ is either a $(0,1)$-homomorphism or a constant map. Thus, in particular, the endomorphism monoid of any $\Phi_{\mathbb{M}}(G) = A = (X, \vee, \wedge, 0, 1)$ with $0 \neq 1$ is the disjoint union

$$\text{(c)} \quad \operatorname{End} A = \{k_x \mid x \in X\} \cup \operatorname{End}_{01} A,$$

where $k_x \in X^X$ is the constant map with the value $x \in X$ and $\operatorname{End}_{01} A$ consists of all $(0,1)$-endomorphisms of A. We now claim that any such lattice is coconnected. To prove this, let $f : A^2 \to A$ be a nonconstant lattice homomorphism (not necessarily $(0,1)$-preserving). Then $h_0(x) = f(x,0)$ and $h_1(y) = f(0,y)$ are endomorphisms of A and, since $(x,y) = (x,0) \vee (0,y)$ for every $(x,y) \in A^2$, it follows that $f(x,y) = h_0(x) \vee h_1(y)$. By (c), if h_0 is not constant then $h_0(1) = 1$ and hence $f(1,0) = 1 \vee h_1(0) = 1$. But then $h_1(0) = f(0,0) = f(1,0) \wedge f(0,1) = 1 \wedge h_1(1)$. Therefore h_1 is constant, and hence $f(x,y) = h_0(x) \vee h_1(0)$. Similarly, if h_1 is not constant then h_0 must be constant and hence $f(x,y) = h_0(0) \vee h_1(y)$. And if both h_0 and h_1 are constant then f is constant. Thus A is coconnected. Therefore, lattices satisfying (c) are examples of idempotent fully coconnected algebras with given monoids of nonconstant maps.

Any lattice A of the form $\Phi_{\mathbb{M}}(G)$ can also be viewed as a 0-lattice, that is, a member of the variety $\mathbb{L}_0$ in which homomorphisms are all lattice homomorphisms that preserve the least element 0. Then $\{0\}$ is the only singleton subalgebra of A, and hence (c) is replaced by $\mathrm{End}_0 A = \{k_0\} \cup \mathrm{End}_{01} A$. The argument from the above paragraph then shows that any lattice $A = \Phi_{\mathbb{M}}(G)$ with $G \in \mathbb{G}$ must be fully coconnected also in $\mathbb{L}_0$.

3. Unary algebras

3.1. Let $\Delta = \{\lambda_i \mid i \in \kappa\}$ be a similarity type, that is, a sequence of cardinals λ_i indexed by a cardinal κ, and let $\mathrm{Alg}(\Delta)$ denote the category of all algebras of type Δ and all their homomorphisms.

Let A and B be algebras, not necessarily of the same similarity type. A functor

$$H:\ \mathrm{Clo}\, A \to \mathrm{Clo}\, B$$

is a *clone homomorphism* if $H(A^n) = B^n$ for every $n \in \omega$ and $H(\pi_i^{(n)}) = \pi_i^{(n)}$ for every projection $\pi_i^{(n)} : A^n \to A$ with $i \in n \in \omega$. Any clone homomorphism H which is one-to-one and such that every $g \in \hom(B^n, B)$ has the form $g = H(f)$ for some $f \in \hom(A^n, A)$ is called a *clone isomorphism.*

For any algebra $A \in \mathrm{Alg}(\Delta)$ of a *countable* similarity type Δ, that is, a type with $\kappa \leqslant \omega$ and $\lambda_i \leqslant \omega$ for every $i \in \kappa$, there is an algebra $B \in \mathrm{Alg}(1,1)$ such that $\mathrm{Clo}\, B$ is isomorphic to $\mathrm{Clo}\, A$, for reasons outlined in the paragraph below. Thus, B is n-coconnected (or directly decomposable, etc.) if and only if A is. Consequently, any existential result about the clone of an algebra of a countable similarity type Δ has its counterpart in the variety $\mathrm{Alg}(1,1)$.

In [17], for a countable type Δ and the countable type Δ_1 in which $\lambda_i = 1$ for all $i \in \omega$, the existence of the clone isomorphism was established via three consecutive product preserving full embeddings

$$\mathrm{Alg}(\Delta) \xrightarrow{\Psi_1} \mathrm{Alg}(\Delta_1) \xrightarrow{\Psi_2} \mathrm{Alg}(1,1,1) \xrightarrow{\Psi_3} \mathrm{Alg}(1,1),$$

each carried by the set functor $Q_\omega = \hom(\omega, -):\ \mathrm{Set} \to \mathrm{Set}$. Since this functor unnecessarily raises countable cardinalities for any finite finitary type Δ', for any such type we now describe a product preserving full embedding

$$\Psi' :\ \mathrm{Alg}(\Delta') \to \mathrm{Alg}(1,1)$$

that also preserves finiteness.

In this section, we also show that any coconnected algebra in $\mathrm{Alg}(1)$ has at most two elements. Therefore the similarity type cannot be further reduced.

3.2. Proposition. *For any finite finitary type Δ' there exists a product preserving full embedding $\Psi' :\ \mathrm{Alg}(\Delta') \to \mathrm{Alg}(1,1)$ carried by a functor $Q_p :\ \mathrm{Set} \to \mathrm{Set}$ with $p \in \omega$.*

Proof. We shall use the following observation about any functor $Q_m = \hom(m, -)$ with $m \neq \emptyset$.

For any set Z and each $j \in m$, define $p_j : Z^m \to Z^m$ by

$$p_j(\varphi)(l) = \varphi(j) \quad \text{for all } l \in m.$$

Let Δ_1 be the type of m unary operations. It is well-known and easy to verify that the functor Ψ_0: Set $\to$ Alg(Δ_1) given by $\Psi_0(Z)=(Q_m(Z),\{p_j \mid j\in m\})$ and $\Psi_0(f)=Q_m(f)$ for any $f:Z\to Z'$ is a full embedding, see [16], for instance.

Since Ψ_0 is carried by the set functor hom$(m,-)$, it preserves products.

Now we are ready to define the functor Ψ'. First, having replaced each nullary operation by a constant unary operation with the same value, we may assume that the type Δ' has the form $\Delta'=\{n_i \mid i\in k\}$ in which k is finite and $n_i\neq 0$ is finite for every $i\in k$. We set $n=\max\{k,n_0,\dots,n_{k-1}\}$ and, for every $i\in k$, denote $e_i:n_i\to n+1$ the inclusion map of the finite ordinal n_i into $n+1$. To a given algebra $A=(Z,\{\alpha_i \mid i\in k\})\in \mathrm{Alg}(\Delta')$, in which $\alpha_i:Z^{n_i}\to Z$ is the n_i-ary operation for each $i\in k$, we assign the algebra $\Psi'(A)=(Z^{n+1},\{\sigma,\gamma\})\in \mathrm{Alg}(1,1)$ whose two unary operations σ and γ are given for any $\varphi\in Z^{n+1}$ and $j\in n+1$ as follows.

$$\sigma(\varphi)(j)=\begin{cases}\varphi(j+1) & \text{for } j\in n,\\ \varphi(n) & \text{for } j=n\end{cases} \quad\text{and}\quad \gamma(\varphi)(j)=\begin{cases}\alpha_j(\varphi\circ e_j) & \text{for } j\in k,\\ \varphi(0) & \text{for } j=k,\dots,n.\end{cases}$$

For every $\varphi=(z_0,\dots,z_{n-1},z_n)\in Z^{n+1}$ we have thus defined

$$\sigma(\varphi)=(z_1,\dots,z_n,z_n),$$

$$\gamma(\varphi)=(\alpha_0(z_0,\dots,z_{n_0-1}),\dots,\alpha_{k-1}(z_0,\dots,z_{n_{k-1}-1}),z_0,\dots,z_0).$$

To any homomorphism f from $A=(Z,\{\alpha_i \mid i\in k\})$ to $A'=(Z',\{\alpha'_i \mid i\in k\})$ we assign the map $Q_{n+1}(f):Z^{n+1}\to (Z')^{n+1}$. Once we show that $g\in \hom(\Psi'(A),\Psi'(A'))$ if and only if $g=Q_{n+1}(f)$ for some $f\in\hom(A,A')$, setting $\Psi'(f)=Q_{n+1}(f)$ will give the desired full embedding.

For any maps $\beta:Q_{n+1}(Z)\to Q_{n+1}(Z)$ and $\beta':Q_{n+1}(Z')\to Q_{n+1}(Z')$, and for any $f:Z\to Z'$, for every $\varphi\in Q_{n+1}(Z)$ we have $(Q_{n+1}(f)\circ\beta)(\varphi)=Q_{n+1}(f)(\beta(\varphi))=f\circ\beta(\varphi)$ and $(\beta'\circ Q_{n+1}(f))(\varphi)=\beta'(Q_{n+1}(f)(\varphi))=\beta'(f\circ\varphi)$. Therefore $Q_{n+1}(f)\circ\beta=\beta'\circ Q_{n+1}(f)$ exactly when $(f\circ\beta(\varphi))(j)=\beta'(f\circ\varphi)(j)$ for every $\varphi\in Z^{n+1}$ and every $j\in n+1$.

We begin the actual argument with $\beta=\sigma$. For any $\varphi\in Z^{n+1}$ and each $j\in n$ we have $(f\circ\sigma(\varphi))(j)=f(\sigma(\varphi)(j))=f(\varphi(j+1))$ and $\sigma'(f\circ\varphi)(j)=(f\circ\varphi)(j+1)$. For the remaining $j=n$ we get $(f\circ\sigma(\varphi))(n)=f(\sigma(\varphi)(n))=f(\varphi(n))$ and $\sigma'(f\circ\varphi)(n)=(f\circ\varphi)(n)$. Therefore $Q_{n+1}(f)\circ\sigma=\sigma'\circ Q_{n+1}(f)$ for any map $f:Z\to Z'$ whatsoever.

Now we turn to $\beta=\gamma$. For any $\varphi\in Z^{n+1}$ and each $j=k,\dots,n$, we get $(f\circ\gamma(\varphi))(j)=f(\gamma(\varphi)(j))=f(\varphi(0))$ and $\gamma'(f\circ\varphi)(j)=(f\circ\varphi)(0)$, so that $(f\circ\gamma(\varphi))(j)=\gamma'(f\circ\varphi)(j)$ for $j=k,\dots,n$. Now, for $j\in k$ we have $(f\circ\gamma(\varphi))(j)=f(\gamma(\varphi)(j))=f(\alpha_j(\varphi\circ e_j))$ and $\gamma'(f\circ\varphi)(j)=\alpha'_j(f\circ\varphi\circ e_j)$. But $e_j(i)=i$ for all $i\in n_j$ and hence $(f\circ\gamma(\varphi))(j)=f(\alpha_j(\varphi(0),\dots,\varphi(n_j-1)))$ and $\gamma'(f\circ\varphi)(j)=\alpha'_j(f\varphi(0),\dots,f\varphi(n_j-1))$. For $j\in k$ it thus follows that $(f\circ\gamma(\varphi))(j)=\gamma'(f\circ\varphi)(j)$ holds for every φ exactly when f preserves the operation α_j. Therefore $Q_{n+1}(f):\Psi'(A)\to\Psi'(A')$ satisfies $Q_{n+1}(f)\circ\gamma=\gamma'\circ Q_{n+1}(f)$ if and only if $f\in\hom(A,A')$. Altogether, this shows that $f\in\hom(A,A')$ exactly when $Q_{n+1}(f)\in\hom(\Psi'(A),\Psi'(A'))$, and therefore implies that Ψ': Alg$(\Delta')\to$ Alg$(1,1)$ is a well-defined functor.

To prove that Ψ' is full, we need only show that every $g\in\hom(\Psi'(A),\Psi'(A'))$ has the form $g=Q_{n+1}(f)$ for some map $f:Z\to Z'$, for we already know that f is a homomorphism whenever $Q_{n+1}(f)$ is. In view of the initial observation, we need only show that the 'projection' operation $p_j : Z^{n+1} \to Z^{n+1}$ defined there is amongst the term operations of our algebra $\Psi'(A)$ for every $j = 0, \dots, n$. This is exactly what we set out to do now.

For any $\varphi \in Z^{n+1}$ we have $\sigma^n(\varphi)(j) = \varphi(n)$ for all $j \in n+1$, and therefore $\sigma^n(\varphi) = p_n(\varphi)$, that is, $\sigma^n = p_n$ is the desired 'last projection'. Using this fact, for every $\varphi \in Z^{n+1}$ we get $\sigma^n(\gamma(\varphi)) = p_n(\gamma(\varphi)) = \varphi(0) = p_0(\varphi)$, and conclude that $\sigma^n \circ \gamma = p_0$. To get the remaining 'projections', we first note that $p_0(\sigma(\varphi)) = \sigma(\varphi)(0) = \varphi(1) = p_1(\varphi)$ for every $\varphi \in Z^{n+1}$, and hence $p_0 \circ \sigma = p_1$. Then, continuing inductively, we find that $p_0 \circ \sigma^k = p_{k+1}$ for every $k = 1, \dots, n-2$ as well. We conclude that $p_0, p_1, \dots, p_n$ are term operations of every $\Psi'(A)$, as desired. We set $p = n+1$ to formally conclude the proof. □

3.3. Let $\Psi' : \mathrm{Alg}(\Delta') \to \mathrm{Alg}(1,1)$ be the functor from 3.2 carried by the set functor Q_p. Let A be an algebra in $\mathrm{Alg}(\Delta')$ and let Z be its underlying set. Then, for each $n \in \omega$, the mapping $e_Z^{(n)} : (Z^p)^n \to (Z^n)^p$ given, for any $\varphi_0, \dots, \varphi_{n-1} \in Z^p$ and all $i \in p$ by

$$[e_Z^{(n)}(\varphi_0, \dots, \varphi_{n-1})](i) = (\varphi_0(i), \dots, \varphi_{n-1}(i))$$

is the carrier of an isomorphism $\varepsilon_A^{(n)} : \Psi'(A)^n \to \Psi'(A^n)$ with the inverse

$$\eta_A^{(n)} = \Psi'(\pi_0^{(n)}) \dot{\times} \cdots \dot{\times} \Psi'(\pi_{n-1}^{(n)}).$$

Noting that $\eta_A^{(1)} = \Psi'(1_A)$ and $\pi_i^{(n)} = \Psi'(\pi_i^{(n)}) \circ \varepsilon_A^{(n)}$, we easily see that setting $H_A(A^n) = \Psi'(A)^n$ and $H_A(h) = \eta_A^{(m)} \circ \Psi'(h) \circ \varepsilon_A^{(n)}$ for every $\mathrm{Alg}(\Delta')$-morphism $h : A^n \to A^m$ defines a clone homomorphism $H_A : \mathrm{Clo}\, A \to \mathrm{Clo}\, H_A(A)$. Since Ψ' is a full embedding, H_A is a clone isomorphism; furthermore, $\mathrm{Clo}\, H_A(A)$ is isomorphic to $\mathrm{Clo}\, H_{A'}(A')$ if and only if $\mathrm{Clo}\, A$ is isomorphic to $\mathrm{Clo}\, A'$.

The result below now follows immediately from 2.3 and 3.2.

Corollary. *The variety* $\mathrm{Alg}(1,1)$ *is f-expansive.*

The situation changes when the similarity type is further reduced.

3.4. Proposition. *The variety* $\mathrm{Alg}(1)$ *is fractured.*

Proof. Any singleton algebra A_1 is clearly coconnected. We show that the two-element algebra $A_2 = (\{0,1\}, \alpha) \in \mathrm{Alg}(1)$ in which $\alpha(0) = 1$ and $\alpha(1) = 0$ is coconnected as well. Select any $f \in \hom(A_2^2, A_2)$, and denote $i = f(0,1)$. Then $f(1,0) = f(\alpha(0), \alpha(1)) = \alpha f(0,1) = 1 - i$. If $f(0,0) = 0$ then $f(1,1) = 1$, and it follows that $f = \pi_i^{(2)}$. If $f(0,0) = 1$ then $f(1,1) = 0$, and hence $f = \alpha \circ \pi_{1-i}^{(2)}$. Thus A_2 is coconnected.

It is clear that $\mathrm{End}\, A_2 = \{1_X, \alpha\}$, so that $\mathrm{End}\, A_2$ is isomorphic to the cyclic group C_2 of order two. On the other hand, if $A = (X, \alpha) \in \mathrm{Alg}(1)$ and $\mathrm{End}\, A \cong C_2$, then $\mathrm{card}\, X \geqslant 2$, and $\alpha^2 = 1_X$ because $\alpha \in \mathrm{End}\, A$. If $\alpha(x) = x$ for some $x \in X$ or if $\mathrm{card}\, X > 2$, then $\mathrm{End}\,(A)$ contains an idempotent other than 1_X. Therefore $\mathrm{End}\, A \cong C_2$ if and only if $A \cong A_2$. Since End A is trivial if and only if $A \cong A_1$, it remains to show that any coconnected $A \in \mathrm{Alg}(1)$ is isomorphic to A_1 or A_2.

Let $A=(X,\alpha) \in \mathrm{Alg}(1)$ have more than one element. Denote $\Delta = \{(x,x) \mid x \in A\}$. We will repeatedly use the following trivial observation:

(1) if $f \in \hom(A^2, A)$ satisfies $f(x,x)=x$ for all $x \in X$, and if there are $(a,b),(c,d) \in X^2\Delta$ such that $f(a,b)=a$ and $f(c,d)=d$, then f does not factor through either projection.

Recall that A is *connected* if the graph of its operation α is connected, that is, if for any $x, y \in X$ there exist integers $k, l \geqslant 0$ such that $\alpha^k(x) = \alpha^l(y)$. Any maximal connected subalgebra of A is called a *component* of A.

Case 1: The algebra A is disconnected. If so, then it has disjoint components D_0 and D_1. Then $D_0 \times D_1 \subseteq A^2 \setminus \Delta$ is a nonvoid subalgebra disjoint with $D_1 \times D_0$. Define

$$f(x,y) = \begin{cases} x & \text{for } (x,y) \in D_0 \times D_1, \\ y & \text{otherwise.} \end{cases}$$

Since $D_0 \times D_1$ and Δ are disjoint, we have $f(x,x)=x$ for all $x \in A$. For any choice of $d_0 \in D_0$ and $d_1 \in D_1$ we have $f(d_0,d_1)=d_0$ and $f(d_1,d_0)=d_0$. Thus f does not factor through a projection, by (1).

Case 2: The algebra A is connected, but A^2 is not.

Suppose first that $\alpha(a)=\alpha(b)$ for some $a \neq b$. Then (a,b) belongs to the component D of A^2 containing the diagonal Δ, and we set

$$f(x,y) = \begin{cases} x & \text{for } (x,y) \in D, \\ y & \text{otherwise.} \end{cases}$$

We have $f(a,b)=a$ and $f(c,d)=d$ for every $(c,d) \notin D$; clearly $f(x,x)=x$ for all $x \in X$. Thus f does not factorize through a projection, see (1).

Secondly, suppose that the operation α is one-to-one, and define

$$f(x,y) = \begin{cases} x & \text{if } y \neq x, \\ \alpha(x) & \text{if } y = x. \end{cases}$$

Then $f \in \hom(A^2, A)$. Indeed, $\alpha f(x,x) = \alpha^2(x) = f(\alpha(x),\alpha(x))$, and $\alpha f(x,y) = \alpha(x) = f(\alpha(x),\alpha(y))$ for $x \neq y$ because α is one-to-one. Suppose that A is coconnected. Then $f(x,y)=f(x,x)$ for all $x,y \in X$ or $f(x,y)=f(y,y)$ for all $x,y \in X$. In the first case, for each $x \in X$ select some $y_x \neq x$ to find that $x = f(x,y_x) = f(x,x) = \alpha(x)$. Thus $\alpha = 1_X$. But then any mapping $g: X^2 \to X$ belongs to $\hom(A^2,A)$. Since card $X > 1$, the algebra A is not coconnected, a contradiction. In the second case, for any two distinct $x,y \in X$ we obtain $x = f(x,y) = f(y,y) = \alpha(y)$. But then X is a two-element set, say $X = \{0,1\}$, and $\alpha(i) = 1-i$ for $i = 0,1$. Hence $A \cong A_2$.

Case 3: The algebra A^2 is connected. Pick $x \in A$ arbitrarily. Then there are $k,l \geqslant 0$ such that $(\alpha \times \alpha)^k(x,x) = (\alpha \times \alpha)^l(x,\alpha(x))$, that is, $\alpha^k(x) = \alpha^l(x)$ and $\alpha^k(x) = \alpha^{l+1}(x)$.

Therefore $\alpha^{l+1}(x) = \alpha^l(x) = a$, and $\{(a,a)\}$ is a singleton subalgebra of A^2. Since A^2 is connected, for any $(x,y) \in A^2$ there is some $k \geqslant 0$ such that $(\alpha \times \alpha)^k(x,y) = (a,a)$. Denote $E = (\alpha \times \alpha)^{-1}\{(a,a)\} \setminus \{(a,a)\}$. Since $\operatorname{card} A > 1$, there is some $b \in A \setminus \{a\}$ such that $(b,a), (a,b) \in E$. Let C be the component of the *partial* algebra $A^2 \setminus \{(a,a)\}$ with $(a,b) \in C$. Then $(b,a) \notin C$, so that the map

$$f(x,y) = \begin{cases} x & \text{for } (x,y) \in C, \\ y & \text{otherwise}, \end{cases}$$

satisfies $f(x,x) = x$ for all $x \in A$, $f(a,b) = a$ and $f(b,a) = b$. The map f thus does not factor through a projection, by (1); it is easy to see that f is a homomorphism.

Since there are no other cases to consider, the proof is complete. □

3.5. We close this section with a comparison of monounary and Boolean algebras.

The reason why only a few coconnected Boolean algebras exist is that there are only a few directly indecomposable ones. In contrast to this, there exist arbitrarily large directly indecomposable monounary algebras. To see this, on a disjoint union $X \cup \{0,1\}$ we define an algebra $A_X \in \mathrm{Alg}(1)$ by setting $\gamma(0) = 0$ and $\gamma(x) = \gamma(1) = 1$ for every $x \in X$. Let θ_0 and θ_1 be any congruences on A_X such that $\theta_0 \cap \theta_1 = \varDelta$ and $\theta_0 \circ \theta_1 = \theta_1 \circ \theta_0$ is the total congruence τ. If $i \in \{0,1\}$ and $(x,0) \in \theta_i$ for some $x \in X$, then $(1,0) = (\gamma(x), \gamma(0)) \in \theta_i$ and from $\theta_0 \cap \theta_1 = \varDelta$ it follows that the singleton $\{0\}$ is the class of θ_0 or θ_1. But then A_X must be indecomposable.

Recall that *monadic algebras* (a special case of cylindric algebras introduced by Henkin et al. in [3]) are Boolean algebras augmented by a unary 'closure' operation γ subject to the identities $\gamma(0) = 0$, $\gamma(x \vee y) = \gamma(x) \vee \gamma(y)$, $x \wedge \gamma(x) = x$ and $\gamma(x \wedge \gamma(y)) = \gamma(\gamma(x) \wedge \gamma(y))$. A monadic algebra $M = (B, \gamma)$ with the underlying Boolean algebra B is directly indecomposable exactly when its closure operation γ satisfies $\gamma(x) = 1$ for every $x \in B \setminus \{0\}$, see [13] for instance. Therefore, the unary reduct of any directly indecomposable M is the algebra $A_{B \setminus \{0,1\}}$ from the previous paragraph defined on the underlying set of B, and we may say that the monounary reduct $A_{B \setminus \{0,1\}}$ causes the indecomposability of M. Every directly indecomposable monadic algebra M is coconnected: for every $f \in \hom(M^2, M)$ we have $f(0,1) = f(\gamma(0), \gamma(1)) = \gamma(f(0,1))$ and, since $\gamma(m) = m$ in M only for $m \in \{0,1\}$, it follows that $f(0,1) \in \{0,1\}$.

4. On *n*-coconnectedness

While congruence distributivity implies that any 2-coconnected algebra is fully coconnected, there exist 2-coconnected unary algebras which are not 3-coconnected. For $n \geqslant 3$, we do not know whether or not every n-coconnected algebra A must be $(n+1)$-coconnected, but we show that this is the case when A has at most n elements. In the conclusion we discuss full coconnectedness of algebras with zero.

4.1. Let $A = (X, \Sigma)$ be an algebra and $n \geqslant 1$. For any $\kappa \in n^{n+1}$, let $e_\kappa : A^n \to A^{n+1}$ denote the morphism $e_\kappa = \pi^{(n)}_{\kappa(0)} \dot{\times} \cdots \dot{\times} \pi^{(n)}_{\kappa(n)}$. Thus e_κ is the morphism determined by the requirement that $\pi_i^{(n+1)} \circ e_\kappa = \pi_{\kappa(i)}^{(n)}$ for every $i \in n+1$. It is clear that $\delta_{n+1} = e_\kappa \circ \delta_n$ for every $\kappa \in n^{n+1}$.

On the underlying set X of A, the mapping $e_\kappa : X^n \to X^{n+1}$ is thus given by $e_\kappa(\psi) = \psi \circ \kappa$ for every $\psi \in X^n$. Since $\mathrm{Ker}(\psi \circ \kappa) \supseteq \mathrm{Ker}\,\kappa$, and because any $\varphi \in X^{n+1}$ which is not injective has the form $\varphi = \psi \circ \kappa$ for some $\psi \in X^n$ and some $\kappa \in n^{n+1}$, we have

$$\bigcup\{\mathrm{Im}\, e_\kappa \mid \kappa \in n^{n+1}\} = \{\varphi \in X^{n+1} \mid \varphi \text{ is not injective}\}.$$

For any $a \in X$ and $n \geqslant 1$, let $k_a^{(n)} : X^n \to X$ denote the constant map with $\mathrm{Im}\, k_a^{(n)} = \{a\}$.

Lemma. *Let $n \geqslant 3$. If $A = (X, \Sigma)$ has more than one element and is n-coconnected, then every $f \in \hom(A^{n+1}, A)$ has one of these two properties*:
(a) *$f \circ \delta_{n+1} = k_a^{(1)}$ is constant and then $f \circ e_\kappa = k_a^{(n)}$ for every $\kappa \in n^{n+1}$, or*
(b) *$f \circ \delta_{n+1}$ is not constant and then there exists a unique $i \in n+1$ such that $f \circ e_\kappa = (f \circ \delta_{n+1} \circ \pi_i^{(n+1)}) \circ e_\kappa$ for every $\kappa \in n^{n+1}$.*

Remark. In terms of the underlying maps, the conclusion says that either there exists a unique $a \in X$ such that $f(\varphi) = a$ for every non-injective $\varphi \in X^{n+1}$, or else there is a unique $i \in n+1$ such that $f(\varphi) = \varphi(i)$ for every non-injective $\varphi \in X^{n+1}$. In either case, the conclusion says nothing about the injective members of X^{n+1}.

Proof of the Lemma. Since $f \circ e_\kappa \in \hom(A^n, A)$ for every $\kappa \in n^{n+1}$ and because A is n-coconnected, we have $f \circ e_\kappa = (f \circ e_\kappa) \circ \delta_n \circ \pi_j^{(n)}$ for some $j \in n$. But $e_\kappa \circ \delta_n = \delta_{n+1}$ and hence

$$\text{(nc)} \quad (\forall \kappa \in n^{n+1})(\exists j \in n)(f \circ e_\kappa = f \circ \delta_{n+1} \circ \pi_j^{(n)}).$$

For (a), suppose that $f \circ \delta_{n+1} = k_a^{(1)}$ is constant. Then $f \circ e_k = k_a^{(1)} \circ \pi_j^{(n)} = k_a^{(n)}$ for every $\kappa \in n^{n+1}$, and this proves (a).

To prove (b), suppose that $f \circ \delta_{n+1}$ is not constant. Select any $\kappa \in n^{n+1}$. Then $f \circ e_\kappa = f \circ \delta_{n+1} \circ \pi_j^{(n)}$ for some $j \in n$, and we claim that $j \in \mathrm{Im}\,\kappa$. Indeed, if $j \notin \{\kappa(i) \mid i \in n+1\}$, then for any $x, y \in X$ we choose $\psi \in X^n$ so that $\psi(j) = x$ and $\psi(j') = y$ for all $j' \in n \setminus \{j\}$. Then $(f \circ e_\kappa)(\psi) = f(e_\kappa(\psi)) = f(\psi \circ \kappa) = f(\psi\kappa(0), \ldots, \psi\kappa(n)) = f(y, \ldots, y)$, while $f \circ \delta_{n+1} \circ \pi_j^{(n)}(\psi) = f(\psi(j), \ldots, \psi(j)) = f(x, \ldots, x)$. Therefore $f(x, \ldots, x) = f(y, \ldots, y)$ for any $x, y \in X$, and hence $f \circ \delta_{n+1}$ is constant, a contradiction. Therefore $j = \kappa(i)$ for some $i \in n+1$, and hence

$$(\forall \kappa \in n^{n+1})(\exists i \in n+1)(f \circ e_\kappa = f \circ \delta_{n+1} \circ \pi_{\kappa(i)}^{(n)})$$

Therefore the set $I_\kappa = \{i \in n+1 \mid f \circ e_\kappa = f \circ \delta_{n+1} \circ \pi_{\kappa(i)}^{(n)}\}$ is nonvoid for every $\kappa \in n^{n+1}$.

We now aim to show that the intersection of all the sets I_κ with $\kappa \in n^{n+1}$ is nonvoid as well.

The sets $I_\kappa \subseteq n+1$ have these two properties:
(1) if $i \in I_\kappa$ then $i' \in I_\kappa$ exactly when $\kappa(i) = \kappa(i')$;
(2) if $\kappa, \kappa' \in n^{n+1}$ and $\mathrm{Ker}\,\kappa \subseteq \mathrm{Ker}\,\kappa'$, then $I_\kappa \subseteq I_{\kappa'}$.

To prove (1), assume that $i \in I_\kappa$. If $\kappa(i') = \kappa(i)$, then obviously $i' \in I_\kappa$ as well. For the converse, suppose that $i \in I_\kappa$ and $\kappa(i') \neq \kappa(i)$. For any $x, y \in X$, choose $\psi \in X^n$ so that $\psi\kappa(i) = x$ and $\psi(j) = y$ for every $j \in n \setminus \{\kappa(i)\}$. If $i' \in I_\kappa$ then $f \circ \delta_{n+1} \circ \pi^{(n)}_{\kappa(i')} = f \circ e_\kappa = f \circ \delta_{n+1} \circ \pi^{(n)}_{\kappa(i)}$ and hence $f(y, \ldots, y) = f \circ \delta_{n+1} \circ \pi^{(n)}_{\kappa(i')}(\psi) = f \circ \delta_{n+1} \circ \pi^{(n)}_{\kappa(i)}(\psi) = f(x, \ldots, x)$. But then $f \circ \delta_{n+1}$ is constant, a contradiction.

For (2), suppose that $\operatorname{Ker} \kappa \subseteq \operatorname{Ker} \kappa'$. Then there is a $\mu \in n^n$ such that $\mu \circ \kappa = \kappa'$, and hence $e_\kappa \circ e_\mu = e_{\kappa'}$. If $i \in I_\kappa$ then $f \circ e_{\kappa'} = f \circ e_\kappa \circ e_\mu = f \circ \delta_{n+1} \pi^{(n)}_{\kappa(i)} \circ e_\mu = f \circ \delta_{n+1} \circ \pi^{(n)}_{\mu k(i)} = f \circ \delta_{n+1} \circ \pi^{(n)}_{\kappa'(i)}$ and hence $i \in I_{\kappa'}$.

To show that $I = \cap \{I_\kappa \mid \kappa \in n^{n+1}\} \neq \emptyset$, in view of (2) we need only show that the intersection of all sets I_κ with minimal $\operatorname{Ker} \kappa$ is nonvoid. The kernel $\operatorname{Ker} \kappa$ is minimal if and only if it has exactly one non-singleton class and this class has exactly two elements. If this is the case, we say that the mapping κ itself is *minimal*.

Next we show that

(3) there exists a minimal $\kappa_0 \in n^{n+1}$ such that I_{κ_0} is a singleton.

To prove (3), suppose that $\operatorname{card} I_\kappa > 1$ for every minimal κ. Since $n \geqslant 3$, there exist disjoint doubletons $\{p, q\}, \{r, s\} \subseteq n+1$ and minimal $\kappa, \kappa' \in n^{n+1}$ such that $\kappa(p) = \kappa(q)$ and $\kappa'(r) = \kappa'(s)$. By (1), we have $I_\kappa = \{p, q\}$ and $I_{\kappa'} = \{r, s\}$. For any mapping $\kappa'' \in n^{n+1}$ with $\kappa''(p) = \kappa''(q) \neq \kappa''(r) = \kappa''(s)$ we then have $\{q, r\} \subseteq \{p, q, r, s\} = I_\kappa \cup I_{\kappa'} \subseteq I_{\kappa''}$ by (2). But then $\kappa''(q) = \kappa''(r)$ by (1), a contradiction. Hence (3) holds.

With no loss of generality, assume that the minimal $\kappa_0 \in n^{n+1}$ from (3) is such that

(3′) $I_{\kappa_0} = \{0\}$.

The nontrivial class of $\operatorname{Ker} \kappa_0$ thus does not contain 0.

Next, let $\kappa \in n^{n+1}$ be any map such that $\kappa(0) \neq \kappa(r)$ for all $r \in (n+1) \setminus \{0\}$. Let $\kappa' \in n^{n+1}$ be a map with $\kappa'(0) \neq \kappa'(1) = \cdots = \kappa'(n)$. From (1) it follows that either $I_{\kappa'} = \{0\}$ or $I_{\kappa'} = \{1, \ldots, n\}$. Clearly $\operatorname{Ker} \kappa'$ contains $\operatorname{Ker} \kappa_0$ and $\operatorname{Ker} \kappa$, and hence $\{0\} \cup I_\kappa = I_{\kappa_0} \cup I_\kappa \subseteq I_{\kappa'}$ by (2). Thus $0 \in I_{\kappa'}$ and this excludes the case of $I_{\kappa'} = \{1, \ldots, n\}$. Therefore $I_{\kappa'} = \{0\}$ and consequently $I_\kappa = \{0\}$ as well. Whence

(4) if $\kappa^{-1}\{\kappa(0)\} = \{0\}$ then $I_\kappa = \{0\}$.

Finally, let $\kappa \in n^{n+1}$ be minimal with $\kappa^{-1}\{\kappa(0)\} \neq \{0\}$. Then there is a unique $r \in (n+1) \setminus \{0\}$ for which $\{0, r\}$ is the non-singleton class of $\operatorname{Ker} \kappa$. Since $n \geqslant 3$, there exists a minimal $\kappa' \in n^{n+1}$ such that the non-singleton class $\{p, q\}$ of its kernel is disjoint with $\{0, r\}$. From (4) it then follows that $I_{\kappa'} = \{0\}$. Let $\kappa'' \in n^{n+1}$ be such that $\{p, q\}$ and $\{0, r\}$ are the only two non-singleton classes of $\operatorname{Ker} \kappa''$. Then $\operatorname{Ker} \kappa' \subseteq \operatorname{Ker} \kappa''$, and hence $0 \in I_{\kappa''}$ by (2), and from (1) it follows that $I_{\kappa''} = \{0, r\}$. Since $\operatorname{Ker} \kappa \subseteq \operatorname{Ker} \kappa''$, we have $I_\kappa \subseteq I_{\kappa''}$ by (2), and hence $I_\kappa \subseteq \{0, r\}$. Since $\{0, r\}$ is a class of $\operatorname{Ker} \kappa$, from (1) we get $I_\kappa = \{0, r\}$. Therefore $0 \in I_\kappa$ again, and hence

(5) if $\kappa^{-1}\{\kappa(0)\} \neq \{0\}$ then $0 \in I_\kappa$.

Altogether $\bigcap \{I_\kappa \mid \kappa \in n^{n+1}\} = I_{\kappa_0} = \{0\}$, and from $\pi^{(n)}_{\kappa(0)} = \pi^{(n+1)}_0 \circ e_\kappa$ it follows that $f \circ e_\kappa = (f \circ \delta_{n+1} \circ \pi^{(n+1)}_0) \circ e_\kappa$ for every $\kappa \in n^{n+1}$. The projection $\pi^{(n+1)}_0$ in this formula is uniquely determined because $\kappa_0^{-1}\{\kappa(0)\} = I_{\kappa_0} = \{0\}$. $\square$

Corollary. *Let $n \geqslant 3$. Then any n-coconnected algebra A with $\operatorname{card} A \leqslant n$ is fully coconnected.*

Proof. Let $f \in \hom(A^{n+1}, A)$ be nonconstant. Since A is n-coconnected, there exists an $i \in n+1$ such that $f \circ e_\kappa = f \circ \delta_{n+1} \circ \pi_i^{(n+1)} \circ e_\kappa$ for every $\kappa \in n^{n+1}$. But $\operatorname{card} A \leqslant n$, and hence every $\varphi \in A^{n+1}$ has the form $\varphi = e_\kappa(\psi)$ for some $\psi \in A^n$ and some $\kappa \in n^{n+1}$. Thus

$$f(\varphi) = (f \circ e_\kappa)(\psi) = (f \circ \delta_{n+1} \circ \pi_i^{(n+1)})(\varphi)$$

for the same $i \in n+1$. But then A is $(n+1)$-coconnected. The conclusion follows by induction. □

Remark. None of the arguments in the two claims above used the fact that A is an algebra. These claims hold in any category with concrete products.

From the Corollary and 1.4 it now follows that any complete finite unary algebra (n, n^n) with $n \geqslant 3$ is fully coconnected.

4.2. The Lemma in 4.1 cannot be extended down to $n = 2$, for reasons other than ‘small’ counterexamples. Indeed, consider any unary algebra $A = (X, \Sigma)$ with $\operatorname{card} X \geqslant 4$ for which Σ consists of sufficiently many injective operations $\sigma : X \to X$ to make the monoid $I(\Sigma)$ they generate 3-transitive (in the sense that for any two triples (x_0, x_1, x_2) and (y_0, y_1, y_2) with pairwise distinct entries there exists some $\sigma' \in I(\Sigma)$ such that $\sigma'(x_i) = y_i$ for $i = 0, 1, 2$). To see that A is 2-coconnected, let $f \in \hom(A^2, A)$. Should there be distinct $a, b, c \in X$ such that $c = f(a, b)$, for any $\sigma' \in I(\Sigma)$ with $\sigma'(a) = a$ and $\sigma'(b) = b$ we would have $\sigma'(c) = \sigma' f(a, b) = f(\sigma'(a), \sigma'(b)) = f(a, b) = c$, contrary to the hypothesis of 3-transitivity. If $f(a, b) = a$ in one instance, then $f(x, y) = x$ for all $(x, y) \in X^2$, and if $f(a, b) = b$ then $f(x, y) = y$ for all $(x, y) \in X^2$. Thus A is 2-coconnected, but it cannot be 3-coconnected because of the observation made at the end of 1.8.

4.3. We recall that an algebra $A = (X, \Sigma)$ is an *algebra with zero* if it has a singleton subalgebra $\{0\}$ and a binary operation $+$ such that $x + 0 = 0 + x = x$ for every $x \in X$. Examples are lattices with zero, join semilattices with zero, but also groups and monoids—for which the unit element 1 plays the role of the zero.

Proposition. *Any* 2-*coconnected algebra* A *with zero is fully coconnected.*

Proof. Since $\{0\} \subseteq A$ is a subalgebra, the n-ary constant $k_0^{(n)} : A^n \to A$ with the value 0 is a homomorphism for any $n \geqslant 1$. For any $j \in n$, let $z_j^{(n)} : A^n \to A^n$ be the map given by

$$z_j^{(n)}(x_0, \ldots, x_{n-1}) = (0, \ldots, x_j, \ldots, 0).$$

Thus $z_j^{(n)}$ replaces all its variables except x_j by 0, and hence $z_j^{(n)} \in \hom(A^n, A^n)$. We claim that A is n-coconnected exactly when

$$\text{(nc)} \quad (\forall f \in \hom(A^n, A))(\exists j \in n)(f = f \circ z_j^{(n)}).$$

Indeed, let A be n-coconnected and let $f \in \mathrm{hom}(A^n, A)$. Then $f = f \circ \delta_n \circ \pi_j^{(n)}$ for some $j \in n$, and $f \circ z_j^{(n)} = f \circ \delta_n \circ \pi_j^{(n)} \circ z_j^{(n)} = f \circ \delta_n \circ \pi_j^{(n)} = f$ because $\pi_j^{(n)} \circ z_j^{(n)} = \pi_j^{(n)}$. For the converse, assume that $j \in n$ is such that $f = f \circ z_j^{(n)}$. Since

$$z_j^{(n)} \circ \delta_n \circ \pi_j^{(n)}(x_0, \ldots, x_{n-1}) = z_j^{(n)}(x_j, \ldots, x_j) = (0, \ldots, x_j, \ldots, 0) = z_j^{(n)}(x_0, \ldots, x_{n-1}),$$

we have $z_j^{(n)} \circ \delta_n \circ \pi_j^{(n)} = z_j^{(n)}$, and hence $f \circ \delta_n \circ \pi_j^{(n)} = f \circ z_j^{(n)} \circ \delta_n \circ \pi_j^{(n)} = f \circ z_j^{(n)} = f$ as claimed.

Assume now that A is n-coconnected for some $n \geqslant 2$, and let $f \in \mathrm{hom}(A^{n+1}, A)$. Since $+$ is defined componentwise on any power of A and because $x + 0 = 0 + x = x$ for every $x \in A$, any $(x_0, \ldots, x_n) \in A^{n+1}$ can be written in the form

$$(x_0, \ldots, x_n) = (x_0, \ldots, x_{n-1}, 0) + (0, \ldots, 0, x_n).$$

Then $f(x_0, \ldots, x_n) = f(x_0, \ldots, x_{n-1}, 0) + f(0, \ldots, 0, x_n)$ because f preserves $+$. Since A is n-coconnected, the homomorphism $h(x_0, \ldots, x_{n-1}) = f(x_0, \ldots, x_{n-1}, 0)$ satisfies $h = h \circ z_j^{(n)}$ for some $j \in n$, and this means that $f(x_0, \ldots, x_{n-1}, 0) = f(0, \ldots, 0, x_j, 0, \ldots, 0)$. Therefore $f(x_0, \ldots, x_n) = f(0, \ldots, 0, x_j, 0, \ldots, 0, x_n)$ for some $j \neq n$. Since A is 2-coconnected, the homomorphism $g : A^2 \to A$ given by $g(x_j, x_n) = f(x_0, \ldots, x_n)$ satisfies $g(x_j, x_n) = g(x_j, 0)$ or $g(x_j, x_n) = (0, x_n)$, and therefore $f(x_0, \ldots, x_n) = f(0, \ldots, x_j, \ldots, 0)$ for some $j \in n$ or else $f(x_0, \ldots, x_n) = f(0, \ldots, 0, x_n)$. But then A is $(n+1)$-coconnected by the condition established earlier in the proof. □

Any finite simple non-Abelian group A is coconnected because its square A^2 has only the four obvious normal subgroups, see [1] for instance. The result below will supply coconnected monoids with widely varying endomorphism monoids. In what follows, we switch to the usual multiplicative notation.

Theorem (Koubek and Sichler [6]). *For any monoid variety $\mathbb{V}$ containing all commutative monoids, and such that the identity $(xy)^n = x^n y^n$ fails to hold in $\mathbb{V}$ for each $n \geqslant 2$, there exist a noncommutative $N \in \mathbb{V}$ and an embedding $\Phi_{\mathbb{V}} : \mathbb{G} \to \mathbb{V}$ of the category $\mathbb{G}$ of all graphs such that for any graphs $G, G' \in \mathbb{G}$ and any $\mathbb{G}$-morphism $g : G \to G'$*

(a) $\Phi_{\mathbb{V}}(g)^{-1}\{1\} = \{1\}$;

(b) *if $f \in \mathrm{hom}(\Phi_{\mathbb{V}}(G), \Phi_{\mathbb{V}}(G'))$ then either $f = \Phi_{\mathbb{V}}(g)$ for some $g : G \to G'$ or else f is the constant onto $\{1\}$;*

(c) *$N \subseteq \Phi_{\mathbb{V}}(G)$ and the restriction of $\Phi_{\mathbb{V}}(g)$ to N is the identity map.*

We claim that every monoid $A = \Phi_{\mathbb{V}}(G)$ is fully coconnected. By the above Proposition, it is enough to show that A is 2-coconnected. Since $(x, y) = (x, 1) \cdot (1, y) = (1, y) \cdot (x, 1)$ in A^2, any $f \in \mathrm{hom}(A^2, A)$ has the form $f(x, y) = f(x, 1) \cdot f(1, y) = f(1, y) \cdot f(x, 1)$. If the endomorphism $h_0(x) = f(x, 1)$ of A is not of the form $\Phi_{\mathbb{V}}(g)$, then $h_0(x) = 1$ for all $x \in A$ by (b), so that $f(x, y) = f(1, y)$ and hence $f(x, y) = f(y, y)$. If $h_1(y) = f(1, y)$ does not have the form $\Phi_{\mathbb{V}}(g)$ for any $g \in \mathrm{End}\, G$, then similarly $f(x, y) = f(x, x)$. In the remaining case of $h_0 = \Phi_{\mathbb{V}}(g_0)$ and $h_1 = \Phi_{\mathbb{V}}(g_1)$, we have $h_0(n) = n$ and $h_1(n) = n$ for every $n \in N \subseteq \Phi_{\mathbb{V}}(G)$, by (c). Then for any $n, n' \in N$ we obtain $n' \cdot n = h_1(n') \cdot h_0(n) = f(n, n') = h_0(n) \cdot h_1(n') = n \cdot n'$, in contradiction to the fact that N is not commutative.

Thus any $A = \Phi_{\mathbb{V}}(G)$ is fully coconnected.

Using (a), we then conclude that

for any monoid M there exists a coconnected monoid $A \in \mathbb{V}$ such that $\operatorname{End} A \cong M \cup \{0\}$, where $0 \notin M$ is the two-sided zero of $\operatorname{End} A$ representing the constant endomorphism of A onto $\{1\}$.

In view of 1.3, the noncommutativity of the monoid A is essential.

Acknowledgements

We would like to thank Václav Koubek and the referee for valuable comments and suggestions.

References

[1] S. Burris, H.P. Sankappanavar, A Course in Universal Algebra, Springer, New York, Heidelberg, Berlin, 1981.
[2] P. Goralčík, V. Koubek, J. Sichler, Universal varieties of (0,1)-lattices, Canad. J. Math. 42 (1990) 470–490.
[3] L. Henkin, D. Monk, A. Tarski, Cylindric Algebras, Part I, North-Holland, Amsterdam, 1971.
[4] H. Herrlich, On the concept of reflections in general topology, Proceedings of the Symposium on Extension Theory of Topological Structures, Berlin 1967.
[5] V. Koubek, Finite-to-finite universal varieties of distributive double p-algebras, Comment. Math. Univ. Carolin. 31 (1990) 67–83.
[6] V. Koubek, J. Sichler, Almost universal varieties of monoids, Algebra Universalis 19 (1984) 330–334.
[7] V. Koubek, J. Sichler, Categorical universality of regular double p-algebras, Glasgow Math. J. 32 (1990) 329–340.
[8] V. Koubek, J. Sichler, Finitely generated universal varieties of distributive double p-algebras, Cahiers Topologie Géom. Différentielle Catégoriques 35 (1994) 139–164.
[9] V. Koubek, J. Sichler, On full and faithful Kan extensions, Appl. Categorical Structures 6 (1998) 291–332.
[10] V. Koubek, J. Sichler, On almost universal varieties of modular lattices, Algebra Universalis 45 (2001) 191–210.
[11] F.W. Lawvere, Functorial semantics of algebraic theories, Proc. Nat. Acad. Sci. U.S.A. 50 (1963) 869–872.
[12] F.W. Lawvere, Some algebraic problems in the context of functorial semantics of algebraic theories, Lecture Notes in Mathematics, Vol. 61, Springer, Berlin, Heidelberg, New York, 1968, pp. 41–46.
[13] R. McKenzie, G. McNulty, W. Taylor, Algebras, Lattices, Varieties, Vol. 1, Brooks/Cole, Monterey, CA, 1987.
[14] R. McKenzie, C. Tsinakis, On recovering a bounded distributive lattice from its endomorphism monoid, Houston J. Math. 7 (1981) 525–529.
[15] D. Pigozzi, J. Sichler, Homomorphisms of partial and of complete Steiner triple systems and quasigroups, Proceedings of the UA& LT Conference, Springer Lecture Notes, Vol. 1149, Charleston, 1984, pp. 224–237.
[16] A. Pultr, V. Trnková, Combinatorial, Algebraic and Topological Representations of Groups, Semigroups and Categories, North-Holland, Amsterdam, 1980.

[17] J. Sichler, V. Trnková, Clones in topology and algebra, Acta Math. Univ. Comenianae 66 (1997) 243–260.
[18] W. Taylor, The clone of a topological space, in: Research and Exposition in Mathematics, Vol. 13, Helderman Verlag, Berlin, 1986.
[19] V. Trnková, Co-connected spaces, Serdica Math. J. 24 (1998) 25–36.
[20] V. Trnková, Clones, coclones and coconnected spaces, Acta Math. Univ. Comenianae 69 (2000) 241–259.