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# GLOBAL PLATE TECTONICS AND THE SECULAR MOTION OF THE POLE

by

Tomás Soler

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## PREFACE

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## ABSTRACT

Astronomical data compiled during the last 70 years by the international organizations (ILS-IPMS, BIH) providing the coordinates of the instantaneous pole, clearly shows a persistent drift of the "mean pole" (≡ barycenter of the wobble).

This study was undertaken with a specific objective in mind: to investigate the possibility of a true secular motion of the barycenter; that is, an actual movement of the earth pole of figure resulting from differential mass displacement due to lithospheric plate rotations.

The method developed assumes the earth modeled as a mosaic of $1^\circ \times 1^\circ$ crustal blocks, each one moving independently in accordance with their corresponding absolute plate velocities.

The differential contributions to the earth's second-order tensor of inertia were obtained and applied, resulting in no significant displacement of the earth's principal axis.

In view of the above, the effect that theoretical geophysical models for absolute plate velocities may have on an apparent displacement of the "mean pole" as a consequence of station drifting was analyzed.

The investigation also reports new values for the crustal tensor of inertia (assuming an ellipsoidal earth) and the orientation of its axis of figure, reopening the old speculation of a possible sliding of the whole crust over the upper mantle, including the supporting geophysical and astronomic evidence.

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"Quand on veut appliquer les théories de la mécanique rationnelle à l'étude des phénomènes naturels, on se trouve toujours en présence de questions d'une complication extrême. Si l'on voulait traiter ces questions en toute rigueur, on ne pourrait jamais y parvenir, et cela pour des raisons diverses que je n'ai pas besoin d'énumerer. Nous sommes donc obligés de nous contenter de résoudre, non pas les questions elles-mêmes que nous avons en vue, mais d'autres questions qui s'en rapprochent plus ou moins, et qui présentent un degré de simplicité assez grand pour que nous puissions en aborder la solution plus ou moins rigoureuse. C'est ainsi que nous sommes conduits à substituer aux solides de la nature de corps solides de forme absolument invariable; c'est ainsi encore que nous attribuons habituellement aux liquides une propiété de fluidité absolue qui n'existe nullement dans la nature, etc. Mais nous ne devons pas perdre de vue que, en agissant ainsi, nous nous mettons à côté de la réalité, et nous devons toujours nous préocuper de l'influence que les circonstances dont nous avons fait abstraction peuvent avoir sur le résultat auquel nous sommes parvenus".

C. E. Delaunay

"Sur l'hypothèse de la fluidité interieure du globe terrestre"; C. R. Acad. Sci. Paris, 67, p. 68 (1868).

# I INTRODUCTION

## 1.1 General Background

The earth rotation vector is characterized by its magnitude (instantaneous spin rate) and its direction, which can be referred about any prescribed reference system.

Due to the nature of the rotational motions of the earth and other related geodynamic phenomena, two basic reference frames are required to fully specify the orientation of the instantaneous rotation vector: an external system with directions fixed in space (inertial system) and a system rigidly linked to the earth (terrestrial system).

The rigorous definition and realization of these systems is a matter concerning the International Astronomic Union (IAU), the International Union of Geodesy and Geophysics (IUGG) and the International Association of Geodesy (IAG). A recent colloquium in Toruń (Poland) was held to initiate and coordinate efforts by the international scientific community in the arduous task of defining future reliable reference frames (accuracies of $5 \times 10^{-9}$ radians and about 1 cm. in orientation and in position, respectively for both the inertial and terrestrial systems ). Diverse viewpoints were presented and debated, but only the basic requirements seem to be fairly well established [Kołaczek and Weiffenbach, 1975, p. 13].

Since the earth rotation vector continually changes in magnitude and direction, its study requires the subdivision into three major domains:

(i) Variations of instantaneous rate of spin

(ii) Inertial orientation

(iii) Terrestrial orientation

Table 1.1 given in [Rochester, 1973] and adapted from [Kaula, 1969] tabulates the various components of the rotational motion of the earth.

Table 1.1 Spectrum of Changes in the Earth's Rotation*

| A. Inertial Orientation of Spin Axis | B. Terrestrial Orientation of Spin Axis (Polar Motion) | C. Instantaneous Spin Rate $\omega$ about Axis |
|---|---|---|
| 1. Steady precession: amplitude 23°.5; period $\cong$ 25,700 years. | 1. Secular motion of pole: irregular, $\cong 0''.2$ in 70 years. | 1. Secular acceleration: $\dot{\omega}/\omega \cong -5 \times 10^{-10}$/yr. |
| 2. Principal nutation: amplitude $9''.20$ (obliquity); period 18.6 years. | 2. 'Markowitz' wobble: amplitude $\cong 0''.02$(?); period 24-40 years(?). | 2. Irregular changes: (a) over centuries, $\dot{\omega}/\omega \leq \pm 5 \times 10^{-10}$/yr; (b) over 1-10 years, $\dot{\omega}/\omega \leq \pm 80 \times 10^{-10}$/yr; (c) over a few weeks or months ('abrupt'), $\dot{\omega}/\omega \leq \pm 500 \times 10^{-10}$/yr. |
| 3. Other periodic contributions to nutation in obliquity and longitude: amplitudes $< 1''$; periods 9.3 years, annual, semiannual, and fortnightly. | 3. Chandler wobble: amplitude (variable) $\cong 0''.15$; period 425-440 days; damping time 10-70 years(?). | 3. Short-period variations: (a) biennial, amplitude $\cong$ 9 msec; (b) annual, amplitude $\cong$ 20-25 msec; (c) semiannual, amplitude $\cong$ 9 msec; (d) monthly and fortnightly, amplitudes $\cong$ 1 msec. |
| 4. Discrepancy in secular decrease in obliquity: $0''.1$/century (?). | 4. Seasonal wobbles: annual, amplitude $\cong 0''.09$; semiannual, amplitude $\cong 0''.01$. | |
| | 5. Monthly and fortnightly wobbles: (theoretical) amplitudes $\cong 0''.001$. | |
| | 6. Nearly diurnal free wobble: amplitude $\leq 0''.02$(?); period(s) within a few minutes of a sidereal day. | |
| | 7. Oppolzer terms: amplitudes $\cong 0''.02$; periods as for nutations. | |

*From [Rochester, 1973].

The scope of this investigation is restricted, in general, to the terrestrial orientation of the instantaneous spin axis (Polar Motion) and in particular to its apparent secular motion.

More than fifteen years have passed since the publication of the monograph "The Rotation of the Earth: a Geophysical Discussion" by Munk and MacDonald [1960]; nevertheless, its contents remain surprisingly up-to-date and many of its conclusions substantially valid. The work describes and studies diverse geophysical questions, several still unsolved or without completely satisfactory answers.

The most recent advances in this area were discussed at the second Geodesy/Solid-Earth and Ocean Physics (GEOP) research conference on the Rotation of the Earth and Polar Motion held at the Ohio State University [Mueller, 1975 or Rochester, 1973].

Secular motion of the pole is a broad term embracing different possible interpretations. It seems necessary at this point to introduce the explicit terminology used in this report to avoid any future misinterpretations.

By secular motion of the pole is commonly understood the displacement of the astronomically-reduced center of the wobble (mean pole or barycenter) referred to the Conventional International Origin (CIO) (see section 2.4). The interval of time covered by observations is less than 100 years.

Polar wandering implies a geologic time scale and therefore possible large variations in the pole; hence it will not be the primary concern of this work.

The astronomically-derived secular fluctuations of the pole, as is well recognized, may in fact be

(a) true secular motion

(b) apparent secular motion caused by systemmatic-drifting of the observing stations or variations in their deflections of vertical

(c) true and apparent effects combined

There are speculations about the possibility of a true secular displacement of the pole, at this time still a controversial question, with several well-known authors for and against it. Observation data clearly shows a persistent drift of the barycenter, but the reasons for this drift are not yet clear. Recent geophysical theories have been postulated to explain this phenomenon. In the following two sections the available astronomical evidence and different geophysical interpretations will be reviewed.

## 1.2 Astronomical Evidence

Although the possibility of displacement of the earth pole of figure due to geological causes (earth mass displacements) was recognized a long time ago by geophysicists [Darwin, 1877], geodesists [Helmert, 1880, p. 420] and astronomers [Tisserand, 1891, p. 485], the observational verification needed to wait until accurate instruments and methods were available.

Several erroneous rates for the secular polar motion were reported prematurely, probably caused by over-anxiety to second the theoreticians. The indisputable fact that the pole has a progressive motion was established after the founding of the International Latitude Service (ILS) in 1899. The ILS uses five stations located at nearly the same latitude (38° 8′N) allowing them to observe the same stars, thus eliminating the effects of certain systematic errors in the star catalogs [Mueller, 1969, p. 81].

Lambert [1922] was the first to report a definite polar drift derived from ILS data. This initial estimate marks the beginning of an international campaign aimed at detecting the reality of a secular displacement of the pole.

Table 1.2, adapted from [Poma and Proverbio, 1976], presents various rates and directions of the secular drift of the mean pole based on ILS observations.

Table 1.2

Values of the Secular Motion and Direction of the Mean Pole Determined by Different Authors Based on ILS Latitude Observations*

| Author | Year | Angular Velocity | Direction (Longitude W) | Period of Observations |
|---|---|---|---|---|
| Lambert, W.D. | 1922 | 0".0066/yr | 83° | 1900-1918 |
| Kimura, H. | 1924 | 58 | 67 | 1900-1924 |
| Wanach, B. | 1927 | 47 | 42 | 1900-1925 |
| Wanach, B. and Mahnkopf, N. | 1932 | 63 | 84 | 1900-1911 |
| Wanach, B. and Mahnkopf, N. | 1932 | 51 | 62 | 1900-1923 |
| Hattori, T. | 1947 | 45 | 73 | 1900-1939 |
| Hattori, T. | 1959 | 359 | 65.2 | 1900-1947 |
| Markowitz, W. | 1960 | 32 | 60 | 1900-1959 |
| Yumi, S. and Wako, Y. | 1966 | 269 | 72.9 | 1933-1966 |
| Stoyko, A. | 1967 | 32 | 69.8 | 1890-1966 |
| Proverbio, E. et al. | 1971 | 294 | 65.6 | 1900-1962 |
| Proverbio, E. et al. | 1972 | 277 | 60.7 | 1900-1962 |
| Proverbio, E. and Quesada, V. | 1973 | 307 | 69.6 | 1900-1969 |

*From [Poma and Proverbio, 1976]

After reviewing all the available polar motion data, Sekiguchi [1954] showed for the first time that besides a progressive drift of the pole, some random discontinuities were also present. In 1960 when the secular motion of the barycenter seemed to be well confirmed, an empirically derived librational component (about a 24-year period) was reported by Markovitz [1960].

The ILS was reorganized in 1962 into the International Polar Motion Service (IPMS). Since then it provides the fundamental pole path derived from the five initial stations, and also publishes a pole path based on variations of latitude received from about 40 other cooperating stations (see Table 4.10)that observe different stars [Yumi, 1975].

Since 1955 another service, the Bureau International de l'Heure (BIH) has also reduced latitude observations. Since 1967 measurements of time and latitude from about 50 observatories have been used simultaneously for computing the coordinates of the pole, while previously only latitude determinations were utilized. Unlike the IPMS, the BIH deals with problems concerning the preservation of its newly-defined system (1968 BIH system), therefore providing the possibility of adding or cancelling cooperating observatories [Guinot et al., 1970].

A recent study by [Poma and Proverbio, 1976] shows unquestionably that the secular drift of the barycenter, as deduced from BIH polar coordinates, corresponds completely to that obtained with IPMS data, corroborating the reality of a secular motion of the mean pole. However, as mentioned above, because of crustal motions, etc., the observed secular drift of the pole may not represent a true polar drift.

## 1.3 Geophysical Theories

The intuitive assertion that changes in the earth's mass distribution may produce displacements in the direction of the earth's principal axes was originally formulated analytically by Darwin [1877]. The failure to detect any sensible polar shift through astronomical observations diminished the interest in the subject until 1922, when the first estimates of secular drift of the mean pole were reported.

Later, around 1950, the question was very much reopened after developments in paleomagnetism suggested that large-scale displacements of the rotation pole had occurred. Extensive material has been published showing different pole positions according to the location of the magnetic rock specimens. A recent investigation suggests that the polar wander path as derived from paleomagnetic data reflects mainly continental motions and not true polar shift, which remains within $4^{\circ}$ in 55 m.y. [Jurdy and Van der Voo, 1974].

The first modern considerations to explain a possible drift of the pole are due to Gold [1955], who following Darwin's reasoning, postulates the great importance of mass redistributions in the changes of the earth pole of figure.

Burgers [1955] in his analytical treatment, confirmed Gold's conclusions, primarily that asymmetric contributions to the moments and products of inertia of the earth are the main motive for polar wandering.

Several papers including [Pekeris, 1935] and [Runcorn, 1962] have suggested mantle convection as a possible cause for polar wandering.

Munk and MacDonald [1960] extensively reviewed the pros and cons of polar wandering, using different theoretical models. After claiming that from examination of various possible excitations of polar wandering it appears that the distribution of continents and oceans exceeds other possible causes by a factor of 1,000, their final conclusions recognize the complexity of the problem (polar wandering), stating that it remains unsolved.

New support to the theory that inertia changes in the earth may produce large-scale polar wandering was introduced in the often-quoted paper by Goldreich and Toomre [1969].

Takeuchi and Sugi [1972], using a crude calculation based on Darwin's equations (see section 4.5), concluded that mantle convection may cause secular changes in the earth products of inertia and consequently polar displacement. The direction of wandering calculated is about 90° W but its rate is about three times as large as the astronomically observed value.

While this work was in progress [Liu et al., 1974], using Darwin's approach, obtained changes in the products of inertia of the earth due to new creation of lithospheric material and found a secular shift of the pole toward the direction 67° W although only 10% of the astronomically observed rate.

Finally [Pan, 1975] speculates that polar wandering is dynamically linked with the secular tectonic movements in the earth's upper layers.

Thus, as can be seen, the majority of authors tends to agree with one hypothesis: changes in the inertia tensor of the earth may produce secular motion of the pole on a short term basis and polar wandering, if large redistribution of masses occurs.

## 1.4 Scope of the Investigation

The main objective of the present study is to calculate the effect of crustal mass displacements on the secular motion of the earth's pole within the past 70 years.

The differential contributions to the earth tensor of inertia from differential tectonic plate rotations are evaluated to verify the geophysical theories.

Borrowing the following quotation from Jeffreys [1974],

"A theory consists of three parts:
(1) a hypothesis p is set up for consideration;

(2) the consequences of p are worked out, as for example, $q_1, q_2 \ldots$

(3) finally, $q_1, q_2 \ldots$ are compared directly with observations.

Part (2) is generally mathematical; part (3) may involve considerable statistics.
If all $q_1, q_2 \ldots$ are found to agree with observations, we may say that p is strongly confirmed. If some agree but existing observations do not test the others, there is some confirmation, but less strong. If even one is definitely contradicted by observation, p must be either rejected or modified".

This investigation is concerned primarily with point (2). The hypothesis considered is the influence that tectonic plate motions may have on the true or apparent secular motion of the pole.

At this point one should consider that the astronomical observations cover an interval of time of less than a century, while the absolute plate velocity models used in this investigation have implicit longer-term rates. Thus the observations essentially reflect possible current changes of the mean pole, while the geophysical plate models involve velocity rates given for one year but obtained from data covering a geological time span.

The following chapter studies some basic geodynamic problems after reviewing the rigid earth case. The necessary theory for evaluating the differential contributions to the earth tensor of inertia due to differential mass displacements is introduced. A complete formulation for transforming tensors of inertia is included in Appendix B, where a novel approach for matrix diagonalization is presented. Contrary to the usual procedures, the orthogonal matrix of the transformation is obtained first, from which the eigenvalues are easily computed. Other derivations relevant to this as well as the other chapters are given in the different appendices in a convenient unified matrix notation.

The third chapter describes the various models adopted in the investigation. These include a crustal model based on the theory of isostatic compensation, plus the assumed break-up of tectonic plates and their absolute velocities with respect to the underlying mantle.

Chapter 4 is the presentation of the numerical experiments and the results obtained. The effect on the pole displacement of changes in inertia or possible station drifting is analyzed.

The last chapter contains the summary of the investigation and the conclusions reached.

## 2 GEODYNAMICS

The study of the rotation of the earth about its center of mass has constituted a fundamental problem in astronomy for centuries. Recent improvements in the theories of earth mechanics and the development of instruments and new observing techniques have forced the inclusion of the topic in the domains of geophysics.

Unfortunately the subject matter is greatly complicated by the heterogeneity and deformability of our planet, plus the action of changing forces mainly due to the attraction of other bodies in the solar system.

The effect on the rotational dynamics of the earth of such varied phenomena as winds and moving air masses; atmospheric, oceanic and bodily tides; sea level changes and tectonic motions; rigidity, elasticity or plasticity of the earth mantle, the nature of its core and their interactions, are not yet fully understood. The fact is, earth dynamics remains a challenging area for investigation proved by the intensive world-wide theoretical and applied research efforts devoted to the subject.

This chapter will present alternatives for formulating the general Lagrange-Liouville equations after consideration of specific geodynamic hypotheses. Some known kinematical and dynamical postulates will be reviewed for use in subsequent chapters.

For its simplicity and great didactic importance, the ideal case of rigid earth will be treated first. Besides the necessary formulation, this should provide a conceptual guidance for understanding ulterior developments.

## 2.1 Preliminary Concepts and Notation

To facilitate writing mathematical expressions, a simplified matrix notation will be used throughout this work. In this way the explicit representation of all elements in the equations is avoided.

The benefits of matrix calculus over vector or tensor methods are obvious in a computationally-oriented scientific community. The resulting formulas can be programmed immediately, taking advantage of matrix algebra and the subroutines available in most computer centers.

Unless otherwise stated, the following notation will be used in the present and remaining chapters:

### 2.1.1 Coordinate Systems

Right-handed rectangular coordinate systems will be represented in general by the convention

$$(x_1, x_2, x_3) \equiv (x) \tag{2.1-1}$$

with the implied origin denoted by Ox. In dynamics, when the system has the center of mass (CM) of a body as its origin, it will be called a "central system" and the following notation applies

$$(x_{01}, x_{02}, x_{03}) \equiv (x_0)$$

### 2.1.2 Vectors

In the treatment of vectors, several approaches are possible. In this study vectors referred to an (x) coordinate system will be presented by column matrices denoted as

$$\{a\}_x = \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}_x \tag{2.1-2}$$

A vector of this type is called a column vector and defines a point in the $E^3$ Euclidean space which can be specified by three numbers $a_i$ $(i = 1, 2, 3)$ expressing the coordinates along the Cartesian frame (x).

To conform with matrix multiplication rules, sometimes the components of the same vector $\{a\}_x$ will be arranged in horizontal array

$$\{a\}_x^T = \{a_1 \quad a_2 \quad a_3\}_x \tag{2.1-3}$$

and termed row vector.

One-dimensional vectors are called scalars.

In cases where the coordinate system used is clear by the context, the subindex x in the vectors will be omitted.

### 2.1.3 Matrices

In general a $3 \times 3$ real matrix will be denoted symbolically by $[M]$. Nevertheless throughout the text some well-known types of matrices are written without the brackets. For example, this is the case of the rotation matrix R between two Cartesian systems of coordinates.

The following special types of matrix notations are introduced and subsequently used:

a) Skew-symmetric (anti-symmetric)

To every vector $\{a\}_x$ it is possible to associate a skew-symmetric matrix denoted by

$$[\underline{a}]_x = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}_x \tag{2.1-4}$$

where clearly:

$$[\underline{a}]_x^T = -[\underline{a}]_x \tag{2.1-5}$$

The matrix in (2.1-4) is also referred to in mathematical literature as "axial vector" or "pseudovector". Anti-symmetric transformations are reviewed in Appendix A.

b) Identity matrix

The $3 \times 3$ unit matrix will always be denoted by $[1]$.

c) Symmetric

Such matrices usually will be denoted by the upper triangular elements only, and an "s" in the lower left of the matrix.

d) Diagonal

The symbolic notation $\lceil M \rfloor$ will apply to diagonal matrices excluding the unit matrix.

### 2.1.4 The Tensor-Matrix Concept

In general, an operator $L^{(r)}$, such that

$$(x) \xrightarrow{L^{(r)}} (x') \tag{2.1-6}$$

is called linear if $\forall \{x\}$, $\{y\} \in (x)$ and any real number k, the following two properties are satisfied,

$$L^{(r)}\{kx\} = k\,L^{(r)}\{x\} \tag{2.1-7}$$

and

$$L^{(r)}\{x+y\} = L^{(r)}\{x\} + L^{(r)}\{y\} \tag{2.1-8}$$

A linear operator $L^{(r)}$ ($L^{(0)}$ = scalar) is called a tensor of order (rank) r if

$$\forall\ r \geqslant 1 \quad \text{and} \quad \forall\ \{x\}$$

the application $L^{(r)}\{x\}$ is a tensor of rank $r - 1$.

A consequence of the above definition is that

(i) Any vector is a tensor of order 1, since when applied to other vectors it gives a scalar, i.e.,

$\{x\}^T\{y\}$ = scalar (order = 0) $\Longrightarrow$ $\{x\}$ is a tensor of order 1 .

(ii) Matrices are tensors of order 2, that is,

$[M]\{x\}$ = vector (order = 1) $\Longrightarrow$ $[M]$ is a tensor of order 2.

Accordingly, $3 \times 3$ matrices obtained through the components of any vector $\{x\}$ such as $\{x\}\{x\}^T$ and $[\underline{x}][\underline{x}]^T$ are tensors of second order. The term "dyadic," infrequently used today, occasionally is mentioned synonimously with the $\{x\}\{x\}^T$ tensor.

In general the number of components in a tensor of order r and dimension n is given by $n^r$. This study will be limited to tensors of $r \leq 2$ and $n = 3$.

At this point one should notice the difference between rank of a tensor and rank of a matrix (Rank $[M]$ = the number of linearly independent columns of $[M]$. Obviously $1 \leq$ Rank $[M] \leq 3$).

As a general rule, the algebra that has been developed for matrices may be used for tensors as well. In particular, tensors of second order, like matrices, may be diagonal, symmetric, anti-symmetric, orthogonal, etc.

Finally, the process by which from a tensor of order $r \geq 2$, other tensors of order $r - 2$ are derived, is known as "contraction" of a tensor. The contraction of a tensor of order 2 is a scalar equivalent to the trace of its matrix.

## 2.2 Inertia Tensors

There is little agreement in physical and mathematical literature about what is referred to as the tensor of inertia. Therefore, in order to avoid confusion, the following definitions and notations will be used:

### 2.2.1 Associate Tensor of Inertia (Associate Inertia Matrix)

The associate tensor of inertia of a body with respect to a system $(x)$ with origin $O_x$ is defined by

$$[I] = \int_M [\underline{x}][\underline{x}]^T dm = \int_M \begin{bmatrix} x_2^2+x_3^2 & -x_1 x_2 & -x_1 x_3 \\ & x_1^2+x_3^2 & -x_2 x_3 \\ s & & x_1^2+x_2^2 \end{bmatrix} dm = \begin{bmatrix} A & -F & -E \\ & B & -D \\ s & & C \end{bmatrix} \tag{2.2-1}$$

where A, B and C are referred to as the "moments of inertia" of the body with respect to the $x_1$, $x_2$ and $x_3$ axes respectively, and D, E and F are called "products of inertia" with regard to the axes $x_2$ & $x_3$, $x_1$ & $x_3$ and $x_1$ & $x_2$ respectively. The set of moments and products of inertia are known as the six constants of the body with respect to the $(x)$ system.

Considering (2.1-5) one may write:

$$[\,[\underline{x}][\underline{x}]^T\,]^T = [\underline{x}][\underline{x}]^T = [\underline{x}]^T[\underline{x}] = -[\underline{x}][\underline{x}] = -[\underline{x}]^2 \tag{2.2-2}$$

Thus the associate inertia tensor is clearly symmetric.

By an orthogonal transformation (which preserves the rectangular Cartesian character of the coordinates), the matrix $[I]$ may be reduced to the diagonal form

$$\lceil \mathbb{I} \rfloor = \begin{bmatrix} A_p & 0 & 0 \\ & B_p & 0 \\ s & & C_p \end{bmatrix} \qquad (2.2\text{-}3)$$

Then the values $A_p$, $B_p$, $C_p$ are called the "principal moments of inertia" (with respect to the reference point $Ox \equiv Ox_p$) and the $(x_p)$ axes of the transformed coordinate system are called the "principal axes of inertia".

The tensor of inertia referred to a $(x_o)$ system with origin at the CM will be called "central tensor of inertia". If the central tensor is principal (i.e., diagonal) its associate system is called "central principal axes" or sometimes "axes of figure" of the body. They will be represented by the notation

$$(x_{op1}, x_{op2}, x_{op3}) = (x_{op})$$

Observe that according to the above definitions the principal axes of inertia of a body are not necessarily central.

### 2.2.2 The Tensor of Inertia (Inertia Matrix)

The tensor of inertia of a body with respect to a system $(x)$ with origin Ox is defined as

$$[\mathbb{I}] = \int_M \{x\}\{x\}^T dm = \int_M \begin{bmatrix} x_1^2 & x_1 x_2 & x_1 x_3 \\ & x_2^2 & x_2 x_3 \\ s & & x_3^2 \end{bmatrix} dm \qquad (2.2\text{-}4)$$

$$= \begin{bmatrix} \mathbb{I}_{11} & \mathbb{I}_{12} & \mathbb{I}_{13} \\ & \mathbb{I}_{22} & \mathbb{I}_{23} \\ s & & \mathbb{I}_{33} \end{bmatrix} = \begin{bmatrix} \mathbb{I}_{11} & F & E \\ & \mathbb{I}_{22} & D \\ s & & \mathbb{I}_{33} \end{bmatrix}$$

where $\mathbb{I}_{11}$, $\mathbb{I}_{22}$, $\mathbb{I}_{33}$ are referred to as the moments of inertia with regard to the planes $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$ respectively, and D, E and F the products of inertia with respect to planes $x_2 = 0$ & $x_3 = 0$, $x_1 = 0$ & $x_3 = 0$ and $x_1 = 0$ & $x_2 = 0$.

The symmetry of $[\mathbb{I}]$ follows immediately from the matrix equation

$$[\{x\}\{x\}^T]^T = \{x\}\{x\}^T$$

Finally the moment of inertia with respect to the origin of the coordinate system is given by the contraction of the tensor $[\mathbb{I}]$ or

$$\mathrm{tr}\,[\mathbb{I}] = \int_M \{x\}^T\{x\}\,dm = \int_M (x_1^2 + x_2^2 + x_3^2)\,dm \qquad (2.2\text{-}5)$$

In Appendix B the reader may consult the different transformations between the tensors $[I]$ and $[\mathbb{I}]$ as well as the translation and rotational effects on the tensors of inertia.

## 2.3 Rigid Earth Case

Assumption 1: The earth is rigid.

Assumption 2: The central principal axes of the rigid earth $(x_{0p})$ are taken as system of reference. This system is body-fixed but moves with the earth in space. Under the above two hypotheses, the Lagrange-Liouville equations given in Appendix D by the general expression (D.2-12) will reduce to the well-known Euler's dynamical equations.

The following simplifications are a consequence of the two assumptions established:

$$\text{A 1) Rigid earth} \Longrightarrow \begin{cases} [\dot{I}_0] = [0] \\ [\Delta I] = [0] \\ \{\delta\omega\} = \{0\} \\ \{h\} = \{0\} \Longrightarrow \{\dot{h}\} = \{0\} \\ \{L_s\} = \{0\} \end{cases}$$

A 2) Reference system: $(x_{0p}) \Longrightarrow D_0 = E_0 = F_0 = 0 \Longrightarrow [I_0] \longrightarrow \lceil I_0 \rfloor$

Hence, equation (D.2-12) reduces to

$$\{L\} = [I_o]\{\dot{\omega}\} + [\underline{\omega}][I_o]\{\omega\} \tag{2.3-1}$$

After the matrix multiplications are performed, the above equation may be explicitly written as

$$L_1 = A\dot{\omega}_1 - (B-C)\omega_2\omega_3 \tag{2.3-2a}$$

$$L_2 = B\dot{\omega}_2 - (C-A)\omega_3\omega_1 \tag{2.3-2b}$$

$$L_3 = C\dot{\omega}_3 - (A-B)\omega_1\omega_2 \tag{2.3-2c}$$

where all the vector components and the moments of inertia clearly refer to the selected reference system ($x_{op}$).

### 2.3.1 Point of View from the Earth Surface

A variant of equations (2.3-2), interesting from the point of view of an observer on the earth, is obtained under the following assumption:

Assumption 3: The body torques acting on the earth are zero.

The solution of the differential equations (2.3-2) when $\{L\} = \{0\}$ is well documented in the classic works on dynamics (French literature commonly uses the notation $p = \omega_1$, $q = \omega_2$ and $r = \omega_3$).

An analytic solution of (2.3-2) under the constraint imposed by the third assumption is possible. The obtained values of $\{\omega\}$ are expressions containing Jacobi's elliptic functions. For a detailed discussion and extensive bibliography on the subject, consult [Leimanis, 1965].

Nevertheless, similar final conclusions may be derived from the simplified case when two principal moments of inertia are equal. This is a rather good approximation since the earth is nearly axially symmetric. Thus,

Assumption 4: A = B.

The above assumption implies that all the lines in the principal equatorial plane (plane through the CM normal to the axis of figure, also referred to as "dynamical" equatorial plane) are principal. Therefore the first two axes of the reference system are not uniquely defined. In the analysis that follows the original Cartesian frame ($x_{0p}$) will be kept as reference system. That is, the axes of reference are the principal axes of the rigid earth central momental ellipsoid satisfying the condition $A < B < C$.

Thus finally, equations (2.3-2), after assumptions 3 and 4 are implemented, become

$$\dot{\omega}_1 + n\omega_3 = 0 \qquad (2.3\text{-}3a)$$

$$\dot{\omega}_2 - n\omega_1 = 0 \qquad (2.3\text{-}3b)$$

$$\dot{\omega}_3 = 0 \qquad (2.3\text{-}3c)$$

where the following substitution has been introduced

$$n = \frac{C-A}{A}\,\omega_3 \qquad (2.3\text{-}4)$$

Equation (2.3-3c) immediately gives

$$\omega_3(t) = \omega_3(0) = \omega_3 \equiv \text{constant} \qquad (2.3\text{-}5)$$

Changing to complex variable notation, and defining

$$\omega_c = \omega_1 + i\omega_2 \qquad (2.3\text{-}6)$$

the remaining two differential equations (2.3-3a & b) can be combined in a single differential expression in $\omega_c$

$$\dot{\omega}_c - in\omega_c = 0 \qquad (2.3\text{-}7)$$

Equation (2.3-7) has the exponential solution

$$\omega_c(t) = \alpha e^{i(nt+\beta)} \qquad (2.3\text{-}8)$$

where $\alpha$ and $\beta$ are two constants of integration.

Equation (2.3-8) yields the known periodic solutions:

$$\omega_1(t) = \alpha\cos(nt+\beta) \qquad (2.3\text{-}9a)$$

$$\omega_2(t) = \alpha\sin(nt+\beta) \qquad (2.3\text{-}9b)$$

The earth rate of spin is the magnitude of the angular velocity vector $\{\omega\}$.

$$|\omega| = \sqrt{\omega_1^2+\omega_2^2+\omega_3^2} = \sqrt{\alpha^2+\omega_3^2} \quad \text{constant} \qquad (2.3\text{-}10)$$

From (2.3-6) and (2.3-9) one may conclude that the vector $\omega_a$ describes a circle of radius $\alpha$ in the principal equatorial plane while $\omega_3$ remains constant. Hence $\{\omega\}$ precesses uniformly in the body frame about the symmetry axis with angular velocity n (see Figs. 2.1 and 2.2). Once the differential equations are solved, other important quantities related to the rotation vector, such as its direction cosines, can be obtained immediately (see Fig. 2.1).

$$m_i = \frac{\omega_i}{|\omega|} \qquad (i = 1,2,3) \qquad (2.3\text{-}11)$$

The coordinates of the instantaneous rotation axis $P_i$ in a plane parallel to the dynamical equator through the north "inertia pole" of the earth (inertia pole ≡ point of intersection of the earth with its axis of figure), and referred to the axes of reference $x_{Op1}$ and $x_{Op2}$ are

$$x_{Op1} = \frac{\omega_1}{\omega_3} r_p \qquad (2.3\text{-}12a)$$

$$x_{Op2} = \frac{\omega_2}{\omega_3} r_p \qquad (2.3\text{-}12b)$$

where

$r_p$ ≡ semiminor earth axis

The rate at which the angular velocity vector (instantaneous spin axis) turns about the $x_{Op3}$ axis was given by

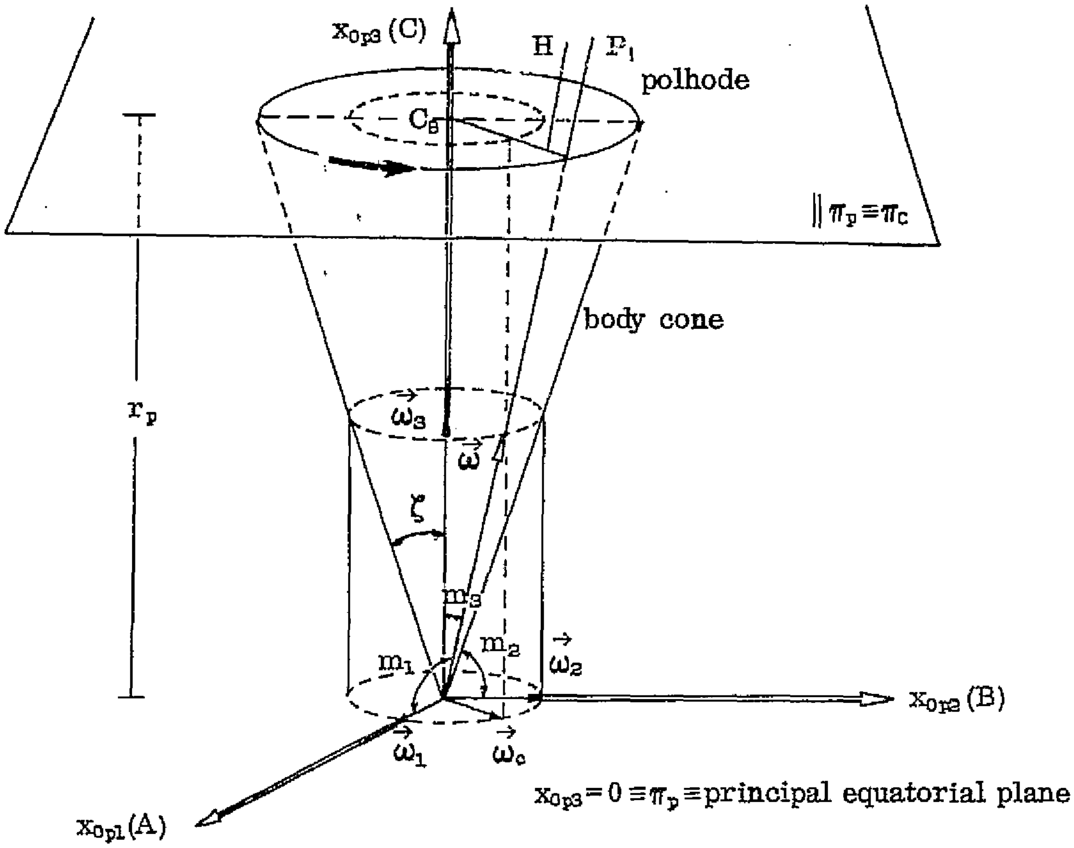



Fig. 2.1. Instantaneous Angular Velocity Vector $\{\omega\}$ in the Central Principal Frame ($x_{0p}$)

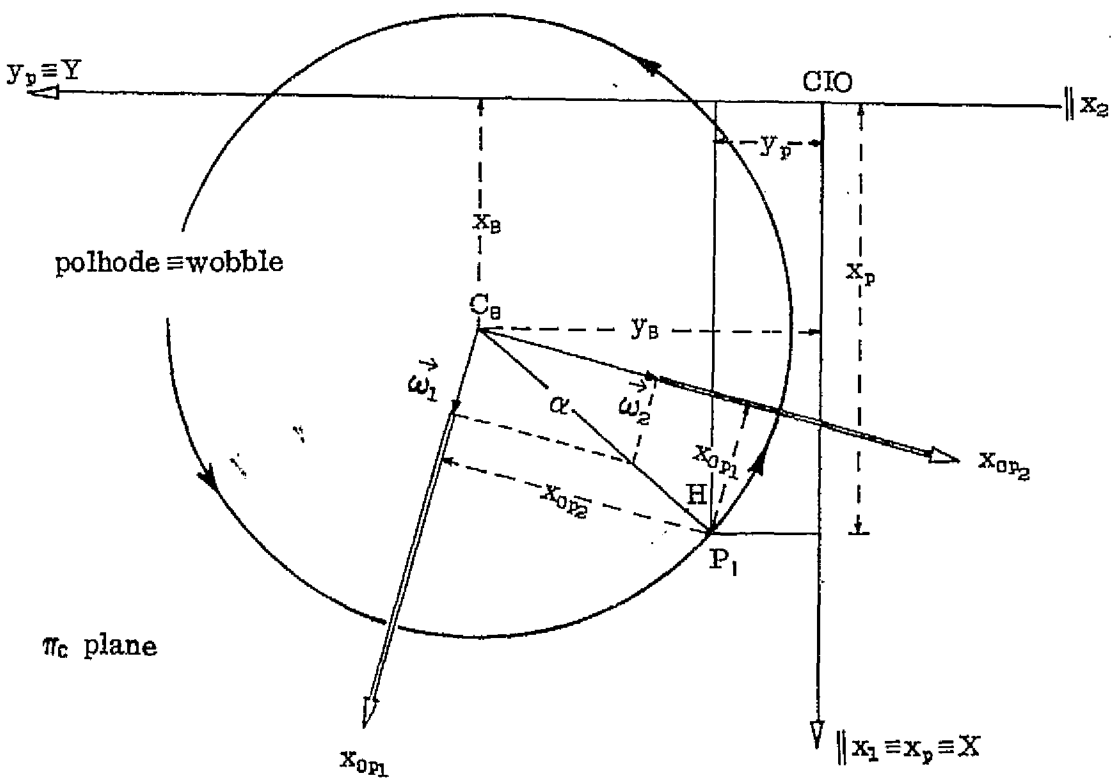



Fig. 2.2. Rigid Earth Wobble and Pole Coordinates

$$n = \frac{C-A}{A}\,\omega_3$$

The moments of inertia of the earth are approximately related by

$$\frac{C-A}{A} \approx 0.0033 \qquad (2.3\text{-}13)$$

The component $\omega_3$ of the angular velocity is for all practical purposes equal to $|\omega|$ (one rotation per day); thus finally,

$$n \approx 0.0033 \text{ rotations per day}$$

Therefore, the period of $\{\omega\}$ around the rigid earth axis of figure is

$$\tau = \frac{1}{n} \approx 305 \text{ sidereal days} \qquad (2.3\text{-}14)$$

This is generally termed the Eulerian period. In conclusion the following dynamical postulate may be stated when assumptions 1 to 4 are fulfilled:

Postulate 1: An observer at rest on the earth will see that the instantaneous rotation axis intersects its surface tracing out a circle (polhode) around the reference pole (mean pole = pole of figure) and moving in a retrograde direction with a period of about 305 sidereal days.

The polhode may also be referred to as wobble or free wobble.

As a corollary, one has: The mean position (center) of the wobble described by the spin axis coincides rigorously with the pole of figure.

Sometimes the name "free nutation" is applied to this motion on the earth of the instantaneous angular velocity vector. This denomination is nevertheless unfortunate, as rightly pointed out by Rochester et al., [1974] and should be applied only in connection with a space-fixed system, as is implied by the term "nutation".

The fact that the earth is not rigid introduces further difficulties discussed later in this chapter.

### 2.3.2 Point of View from Space

Up to now the described motion of the instantaneous spin axis of the rigid earth conforms with an observer at rest in the rotating body-fixed reference frame.

The situation changes if one considers an external observer related to an inertial (space-fixed) system.

As is known (see section D. 1) an immediate consequence of the assumption $\{L\} = \{0\}$ is that the total angular momentum vector $\{H\}$ is constant in the inertial frame. Hence the direction of $\{H\}$ will remain truly fixed in space and for this reason is also called the "invariable axis". The plane through the earth CM normal to $\{H\}$ is termed the "invariable plane".

The relation between $\{H\}$ and the earth angular rotation vector $\{\omega\}$ (the components of this vector are known for any particular instant at which the differential equations of motion are solved) is given by equation (D. 1-22), namely:

$$\{H\} = [I_o]\{\omega\} + \{h\} \tag{2.3-15}$$

Thus the direction cosines of the vector $\{H\}$ with respect to the earth-fixed system are immediately available:

$$\tilde{h}_i = \frac{H_i}{|H|} \qquad (i = 1,2,3) \tag{2.3-16}$$

After consideration of assumptions 1 and 2 equation (2.3-15) reduces to

$$\{H\} = [I_o]\{\omega\} \tag{2.3-17}$$

or

$$H_1 = A\omega_1 \tag{2.3-18a}$$

$$H_2 = B\omega_2 \tag{2.3-18b}$$

$$H_3 = C\omega_3 \tag{2.3-18c}$$

and

$$H^2 = A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 \tag{2.3-18d}$$

Denoting by $(0,0,1)$ the components of a unit vector along the earth axis of figure, the condition of coplanarity between this principal line and the vectors $\{\omega\}$ and $\{H\}$ is

$$\begin{vmatrix} 0 & 0 & 1 \\ \omega_1 & \omega_2 & \omega_3 \\ A\omega_1 & B\omega_2 & C\omega_3 \end{vmatrix} = 0 \tag{2.3-19}$$

or

$$B\omega_1\omega_2 - A\omega_1\omega_2 = 0 \tag{2.3-20}$$

The above equation implies that when the fourth assumption $(A = B)$ is satisfied, the three axes (rotation, figure and angular momentum) are in the same plane at any instant. Since $\{H\}$ is constant, the plane rotates around the invariable axis.

The axial vector representing the angular momentum forms an angle with the earth axis of figure defined by (see Figs. 2.3 and 2.4)

$$\tan\theta = \frac{\sqrt{H_1^2 + H_2^2}}{H_3} \tag{2.3-21}$$

where the components of $\{H\}$ in general may be computed from equation (2.3-15).

If rigidity and $A = B$ are assumed, one has

$$\tan\theta = \frac{\sqrt{A^2(\omega_1^2 + \omega_2^2)}}{C\omega_3} = \frac{A\alpha}{C\omega_3} = \text{constant} \tag{2.3-22}$$

Hence the following postulate can be stated:

Postulate 2: The axis of figure describes a cone in space whose axis is the invariable line ($\{H\}$) and whose semiangle is $\theta$.

The angle $\zeta$ between the instantaneous spin axis and the earth axis if figure is given by the relation

$$\tan \zeta = \frac{\sqrt{\omega_1^2 + \omega_2^2}}{\omega_3} \tag{2.3-23}$$

and under the four mentioned assumptions, it follows

$$\tan \zeta = \frac{\alpha}{\omega_3} \equiv \text{constant} \tag{2.3-24}$$

Thus the instantaneous rotation axis forms a cone (body cone) of constant angle $\zeta$ with the axis of figure. This was also implied previously by equations (2.3-5) and (2.3-9) and was summarized in Postulate 1.

As a consequence, one has the following corollary: The invariable line describes a cone in the body whose axis is the axis of figure and whose semiangle is $\theta$.

From (2.3-22) and (2.3-24) it is obvious that for a rigid earth and $A = B$,

$$\tan \zeta = \frac{C}{A} \tan \theta$$

The above implies that the displacement of the pole of figure from the pole of rotation is greater than its displacement from the pole of angular momentum by a factor $\frac{C}{A} \approx 1.00327$.

Finally, the angle between the angular momentum vector and the instantaneous rotation axis may be computed through the equation

$$\cos(\angle \mathrm{HOP}_1) = \frac{\{H\}^T \{\omega\}}{(\{H\}^T \{H\})^{1/2} (\{\omega\}^T \{\omega\})^{1/2}} \tag{2.3-25}$$

In the particular case of rigidity and $A = B$

$$\cos(\angle \mathrm{HOP}_1) = \frac{A\alpha^2 + C\omega_3^2}{\sqrt{A^2\alpha^2 + C^2\omega_3^2}\ \sqrt{\alpha^2 + \omega_3^2}} \equiv \text{constant} \tag{2.3-26}$$

Postulate 3: The instantaneous axis describes a cone in space (space cone) whose axis is the fixed angular momentum axis and whose semiangle is $\zeta - \theta$.

This motion of the instantaneous earth rotation axis around the invariable axis, even in the absence of torques, has been called "sway" by Munk and MacDonald [1960, p. 45], but as opportunely mentioned [Rochester et al., 1975] the name "free nutation" is more appropriate.

Clearly if A = B = C,

$$\cos(\angle HOP_1) = 1 \Longrightarrow \angle HOP_1 = 0$$

This implies the vectors $\{H\}$ and $\{\omega\}$ have the same direction.

Therefore, the instantaneous rotation axis and the total angular momentum axes are coincident only when A = B = C. (The momental ellipsoid is a sphere and every diameter is a principal axis).

Moreover, one has

$$A = B = C \Longrightarrow n = 0$$

Thus the spin axis is fixed in space and in the body.

Equations similar to (2.3-25) may be applied to obtain the cosine of the angles $\theta$ and $\zeta$

$$\cos\theta = \frac{C\omega_3}{\sqrt{A^2\alpha^2 + C^2\omega_3^2}} \tag{2.3-27a}$$

$$\cos\zeta = \frac{\omega_3}{\sqrt{\alpha^2 + \omega_3^2}} \tag{2.3-27b}$$

Therefore, recalling (2.3-22) and (2.3-24)

$$\sin\theta = \frac{A\alpha}{\sqrt{A^2\alpha^2 + C^2\omega_3^2}} \tag{2.3-28a}$$

$$\sin\zeta = \frac{\alpha}{\sqrt{\alpha^2 + \omega_3^2}} \tag{2.3-28b}$$

Thus, finally one can write (2.3-26) as

$$\cos(\angle \mathrm{HOP}_1) = \frac{\omega_3}{\sqrt{\alpha^2+\omega_3^2}} \frac{C\,\omega_3}{\sqrt{A^2\alpha^2+C^2\omega_3^2}} + \frac{\alpha}{\sqrt{\alpha^2+\omega_3^2}} \frac{A\,\alpha}{\sqrt{A^2\alpha^2+C^2\omega_3^2}}$$

$$= \cos\zeta\cos\theta + \sin\zeta\sin\theta = \cos(\zeta-\theta) \qquad (2.3\text{-}29)$$

and

$$\angle \mathrm{HOP}_1 = \zeta - \theta \qquad (2.3\text{-}30)$$

Similarly,

$$\sin(\angle \mathrm{HOP}_1) = \sin(\zeta-\theta) = \sin\zeta\cos\theta - \cos\zeta\sin\theta$$

$$= \frac{C\,\omega_3}{\sqrt{A^2\alpha^2+C^2\omega_3^2}} \frac{\alpha}{\sqrt{\alpha^2+\omega_3^2}} \frac{C-A}{C} = \cos\theta\sin\zeta\,\frac{C-A}{C} \qquad (2.3\text{-}31)$$

When the angles are small, as in this case,

$$\sin(\angle \mathrm{HOP}_1) \approx \angle \mathrm{HOP}_1 \quad , \quad \cos\theta \approx 1 \quad , \quad \sin\zeta \approx \zeta$$

From astronomic observations,

$$H = \frac{C-A}{C} = 0.003272 \equiv \text{dynamical ellipticity} \qquad (2.3\text{-}32)$$

and

$$\zeta < 0''.3 \approx 9.5 \text{ m} \qquad (2.3\text{-}33)$$

Thus

$$\angle \mathrm{HOP}_1 \approx 0''.00098 \approx 3.108 \text{ cm} \qquad (2.3\text{-}34)$$

Let an inertial system $(X)$ be introduced with origin $OX \equiv CM$ and the third axis in the direction of the fixed angular momentum vector $\{H\}$. This system can be related to the earth-fixed central principal axes through the Euler angles $\varphi$, $\theta$, $\psi$ (see Fig. 2.3) through the transformation

$$\{X\} = \mathbb{R}\,\{x_{0p}\} \qquad (2.3\text{-}35)$$

where the rotation matrix $\mathbb{R}$ is given by

$$\mathbb{R} = R_3(-\varphi)\,R_1(-\theta)\,R_3(-\psi) \qquad (2.3\text{-}36)$$

or

$$\{x_{O_p}\} = \mathbb{R}^T \{X\} \tag{2.3-37}$$

where

$$\mathbb{R}^T = R_3(\psi)\, R_1(\theta)\, R_3(\varphi)$$

In this case, the Euler kinematic equations are given by (see Appendix C)

$$\begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} = \begin{bmatrix} \sin\theta \sin\psi & \cos\psi & 0 \\ \sin\theta\cos\psi & -\sin\psi & 0 \\ \cos\theta & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{Bmatrix} \tag{2.3-38}$$

Clearly, the values of the angular velocities $\dot{\varphi}$, $\dot{\theta}$ and $\dot{\psi}$ will give the rate of precession, nutation and spin of the earth-fixed system ($x_{O_p}$) with respect to the inertial frame (X).

Applying the above to the ideal case of rigidity and A = B, it follows immediately from (2.3-22) that $\dot{\theta} = 0$.

Substituting the values of ($\omega_1, \omega_2, \omega_3$) from (2.3-5) and (2.3-9) in (2.3-38),

$$\dot{\varphi} \sin\theta \sin\psi = \alpha \sin(nt + \beta) \tag{2.3-39a}$$

$$\dot{\varphi} \sin\theta \cos\psi = \alpha \cos(nt + \beta) \tag{2.3-39b}$$

$$\dot{\psi} + \dot{\varphi} \cos\theta = \omega_3 \tag{2.3-39c}$$

From the first two equations it follows immediately that

$$\dot{\varphi}^2 \sin^2\theta = \alpha^2 \implies \dot{\varphi} \sin\theta = \pm\alpha \tag{2.3-40}$$

Substituting the above in (2.3-39a) one has

$$\psi = nt + \beta \implies \dot{\psi} = n \quad \text{(the rate of spin)} \tag{2.3-41}$$

where n is given by (2.3-4).

Replacing (2.3-41) in (2.3-39c)

$$\dot{\varphi} \cos\theta = \omega_3 - n = \omega_3 \left(1 - \frac{C-A}{A}\right) = \frac{C}{A}\, \omega_3 \tag{2.3-42}$$

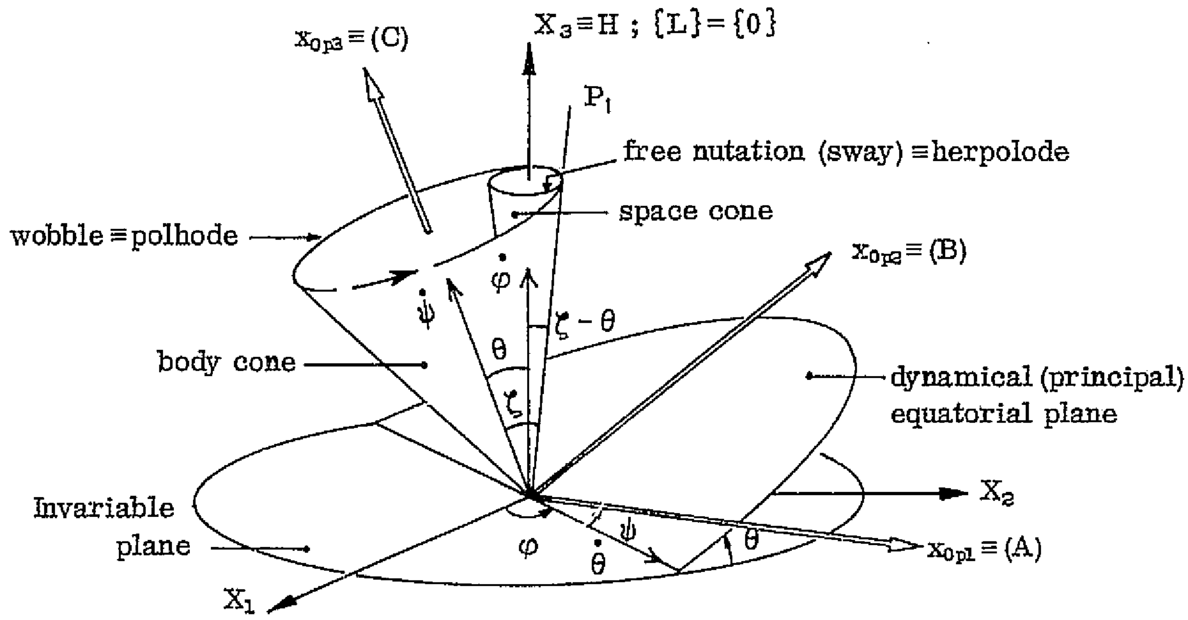



Fig. 2.3. Euler's Angles between the Inertial (X) and Central Principal ($x_{0p}$) Frames

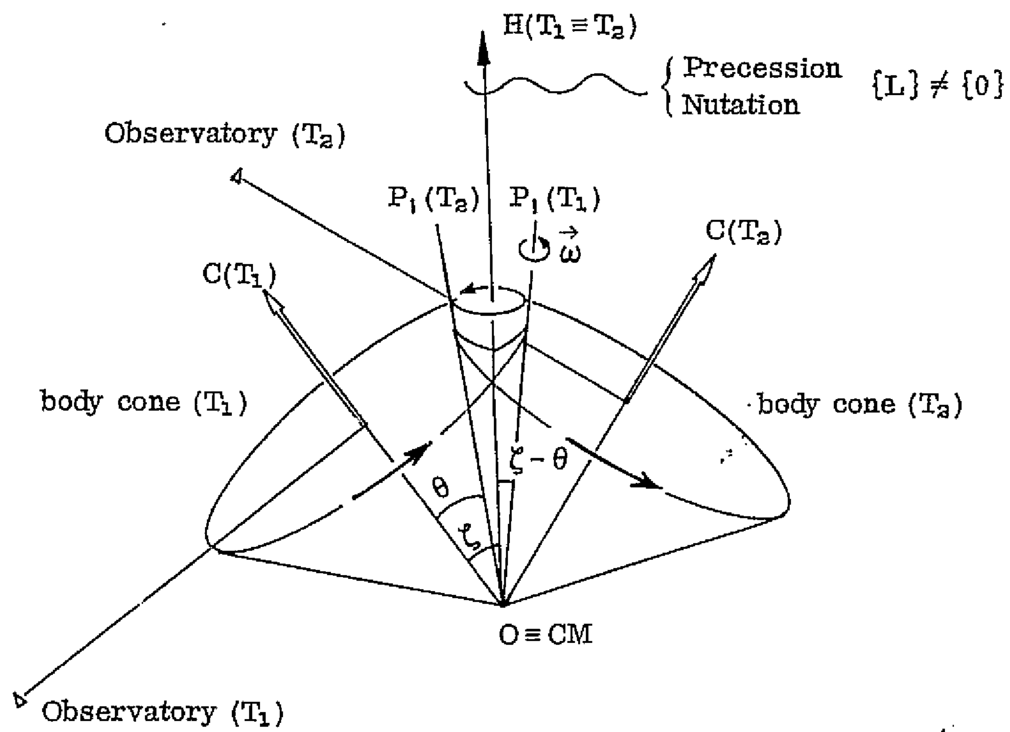



Fig. 2.4. Earth's Spatial Motion

and therefore, using (2.3-40) and (2.3-42)

$$\dot{\varphi}^2 = \alpha^2 + \left(\frac{C}{A}\right)^2 \omega_3^2 \tag{2.3-43}$$

and

$$\tan \theta = \frac{A}{C} \frac{\alpha}{\omega_3} \equiv \text{constant}$$

as was found before in (2.3-22).

The result of equation (2.3-41) may be interpreted kinematically as the rolling of the body cone on the fixed-space cone without slipping (see Fig. 2.3) at a rate of spin n precisely the same as found before for $\vec{\omega}$ when it was referred to an earth-fixed frame [Poinsot, 1851].

In order to visualize the complex behavior of the motion of the earth even under the simple assumption of rigidity, the description of Fig. 2.4 may be illustrative.

The position of the body cone has been drawn at two different epochs

$T_1$ (initial epoch)

$T_2 = T_1 + \frac{\tau}{2}$ ($\tau \sim 305$ days)

The invariable axis remains fixed in space at all times when $\{L\} = \{0\}$.

Attached to the body cone is represented an ideal observatory, the position of which with respect to the rigid earth will not change from epochs $T_1$ to $T_2$. Nevertheless, it will wobble in space as the earth does. (Notice that for the purpose of clarity the figure is not drawn to scale ).

The graph shows clearly the variation of observed instantaneous colatitude at epochs $T_1$ and $T_2$, while the reduced colatitude (referred to the axis of figure or mean pole) remains constant. Thus knowing the position of the instantaneous rotation axis with respect to the axis of figure (polar motion), one may obtain the observatory reduced colatitude. (In practice a conventional "geographic" reference system is used, although the same conclusions apply. In this case reduced colatitude will apply to the third axis selected frame).

Note that when two observatories are 180° apart, the observed colatitude variations should be opposite in phase, that is, at the same epoch $T_1$, maximum colatitude in one observatory will correspond to minimum colatitude in the other. As it turned out, this was the way polar motion was corroborated in 1891 after observations in Hawaii and Berlin were processed. For a complete account of the intricate history of the discovery of latitude variations, the review in [Lambert et al., 1931] is recommended. The methods currently used for obtaining latitude are properly explained in [Mueller, 1969] and [Melchior, 1957] where many other references on the subject are available. For specific data reduction procedures at the appropriate international organizations, consult the annual reports of the BIH and of the IPMS.

Up to now the condition $\{L\} = \{0\}$ was enforced. It is quite well known that this is not the case and that the action of external body torques (primarily due to the gravitational effects of the sun, moon and planets) cause motions of the invariable axis in the inertial system. Consequently the vector $\vec{H}$ will precess and nutate in space. Other periodic contributions to nutation (i.e., annual, semiannual) are possible because the external body forces acting on the earth depend on the ephemeris of the celestial bodies where clearly several periodicities are present.

No further considerations will be devoted to the case when the body torques are not zero. The interested reader may consult classical works like [Tisserand, 1870] or the modern treatment by Woolard [1953].

## 2.4 Non-Rigid Earth

Although in the previous section the hypothesis that the earth is a rigid body was considered, some given functional relations using $\{H\}$ or $\{\omega\}$ are general and may be applicable to any circumstance in which these vectors are known.

Under the assumption of absolute rigidity, the relative positions of all mass particles are constant, so the external form as well as the value of the central (principal) tensor of inertia is fixed and independent of time.

It has become increasingly clear that the assumption of rigidity for the earth is incorrect and a better modeling should be investigated. Because the mass distribution of our planet is subject to variations with time (tidal deformation, crustal motions, etc.) producing changes in its central inertia tensor, the rotational dynamics of the earth is better studied by the general Lagrange-Liouville equations as given by (D.2-9), (D.2-12) or (D.2-15), depending on the reference system adopted. In the case of a deformable body, the total action of the torques is expressed by the sum $\{L\} + \{L_s\}$ where $\{L\}$ are the body torques and $\{L_s\}$ the surface torques (see Appendix D).

Theoretically an infinite number of reference systems are possible, although it is convenient to select the one best suited to its specific purpose and which is practically feasible.

Previously a theoretical reference system was selected, the one defined by the central principal axes of the rigid earth. Considering that this frame is not available through observation, a crust-fixed or "geographic" system is generally adopted.

The present choice of this type of a terrestrial system conventionally used by geophysisists, astronomers and geodesists, is defined as follows (see Fig. 2.5):

Origin: Close as possible to the geocenter (center of mass of the earth including the atmosphere).

$x_3$ axis: Directed toward the CIO (Conventional International Origin) as defined by the IPMS (International Polar Motion Service) and the BIH (Bureau International de l'Heure). Some doubts about the definition of the CIO had been raised and other possible redefinitions mentioned [Melchior, 1975].

$x_1$ axis: Passes through the point of zero longitude as defined by the 1968 BIH system [Guinot et al., 1971].

$x_2$ axis: Forms a right-handed coordinate system with the $x_1$ and $x_3$ axes.

Clearly, for short periods of time (~ 1 year) the above conventional system may be considered earth-fixed. It is known that rigorously this is not the case, due primarily to crustal tectonic motions which produce systematic drifts of the observatories.

This system will be referred to as CIO terrestrial system, or simply "terrestrial system". In the geophysical literature the term "geographic system" is customarily used.

A second two-dimensional frame often mentioned and related to the CIO is the following: The axes are on a plane normal to the $x_3$ terrestrial axis at the point of intersection with the earth. They are parallel to the $x_1$ and $x_2$ axes, but the following sign convention applies:

$$X \equiv x_p = x_1$$

$$Y \equiv y_p = -x_2$$

This frame will be termed "CIO polar system."

Once the basic reference system has been agreed upon, the mathematical theory describing the orientation of the instantaneous rotation vector of the earth in the "crust-fixed" frame, assuming that body and surface torques are zero, is summarized by the following matrix differential equation (see equation D.3-13).

$$\{0\} = [I_o]\{\dot{\omega}\} + [\dot{I}_o]\{\omega\} + [\underline{\omega}][I_o]\{\omega\} + [\underline{\omega}]\{h\} + \{\dot{h}\} \qquad (2.4\text{-}1)$$

where

$[\underline{\omega}]$ ≡ skew-symmetric matrix of the earth rotation vector

$[I_o]$ ≡ central earth tensor of inertia of second order

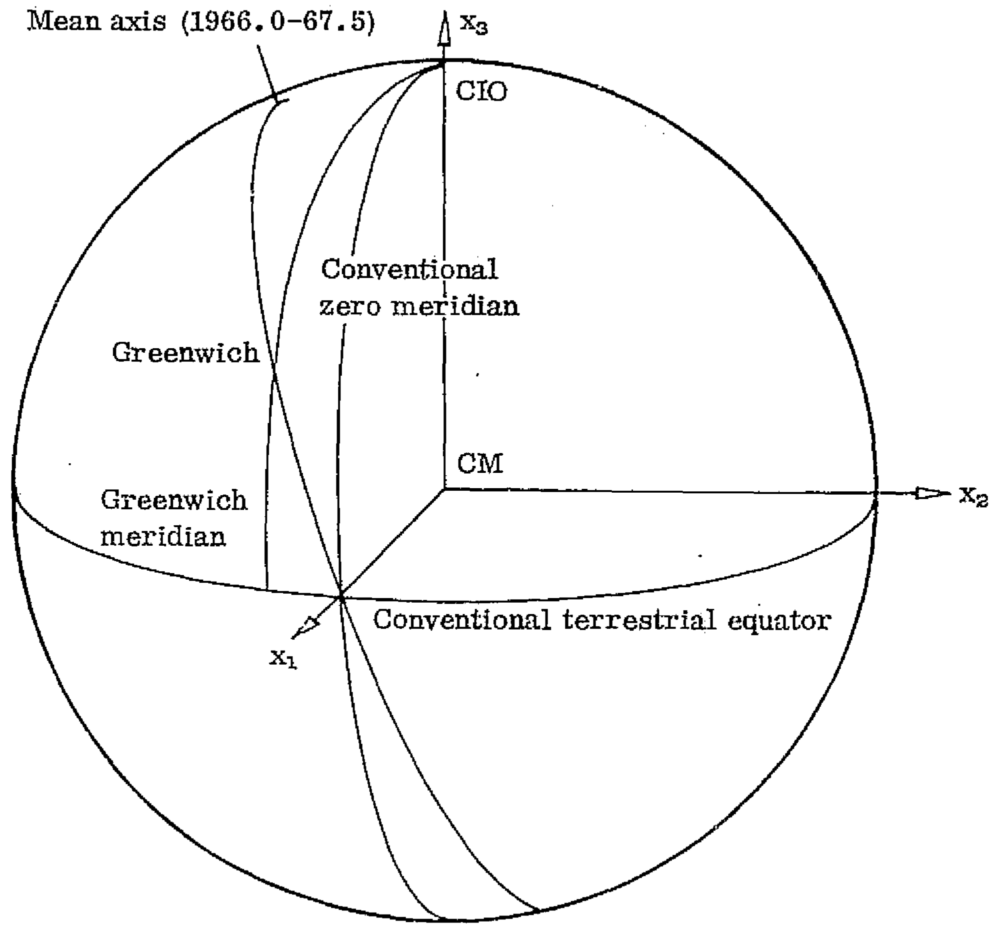



Fig. 2.5. Conventional Terrestrial (Geographic) System

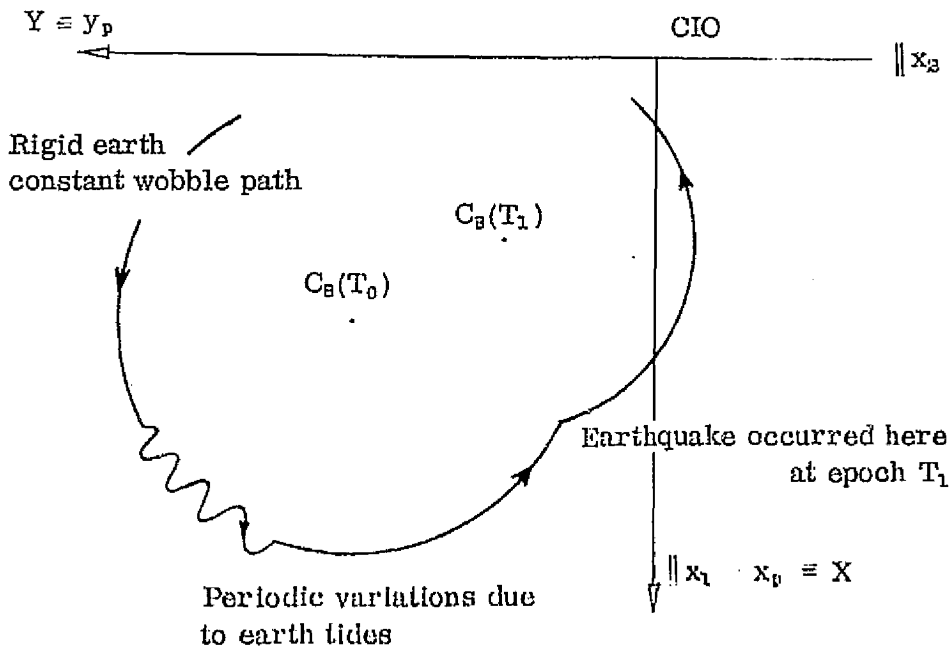



Fig. 2.6. CIO Polar System

$\{h\}$ $\equiv$ relative angular momentum vector

Equation (2.4-1) can be solved for the general case of a deformable earth by taking into consideration all the possible variations of $[I_o]$ and $\{h\}$.

For example, one may write

$$[I_o] = [I]_E + [\Delta I]_R + [\Delta I]_T + [\Delta I]_F + [\Delta I]_A + [\Delta I]_P + \text{other effects} \quad (2.4\text{-}2)$$

where

$[I]_E$ $\equiv$ initial value of the earth tensor of inertia (independent of time)

$[\Delta I]_R$ $\equiv$ contribution to $[I]_E$ due to rotational deformation

$[\Delta I]_T$ $\equiv$ contribution to $[I]_E$ due to tidal deformation

$[\Delta I]_F$ $\equiv$ contribution to $[I]_E$ due to major faulting (seismic effects)

$[\Delta I]_A$ $\equiv$ contribution to $[I]_E$ due to atmospheric air mass shifts

$[\Delta I]_P$ $\equiv$ contribution to $[I]_E$ due to crustal mass displacements or plate tectonics

By the same logic the different contributions to the initial relative angular momentum of the earth can be given as

$$\{h\} = \{h\}_E + \{h\}_R + \{h\}_T + \{h\}_F + \{h\}_P + \text{other effects} \qquad (2.4\text{-}3)$$

The values of $[I_o]$ and $\{h\}$ thus depend on the particular geophysical models adopted for describing the earth and its disturbing agents.

In effect, one may express

$$[\Delta I]_P = \sum_{i=1}^{n} [\Delta I]_{Pi} \qquad (2.4\text{-}4)$$

where n is the number of tectonic plates constituting the earth crust.

Likewise,

$$[\Delta I]_T = \sum_{j=1}^{k} [\Delta I]_{Tj} \qquad (2.4\text{-}5)$$

where $j = 1, 2 \ldots k$ and refers to the moon, sun and planets producing tidal deformation.

When all the contributions to the initial values $[I]_E$ and $\{h\}_E$ are known at some specified times, the differential equations (2.4-1) may be solved numerically. To this date an analytical solution of such a complex equation seems an insurmountable task, although it is conceivable that in the future it may be done by digital computers through symbolic math-logic programs. Nevertheless, numerical solutions of first-order differential equations are possible by using currently available tested subroutines, such as the so called DVDQ-available step, variable order Adams integrator [Krogh, 1969].

Once the equations of motion have been integrated and the vector $\{\omega\}$ obtained at different time intervals, the coordinates of the pole at these specific epochs may be computed by means of equations (2.3-12).

### 2.4.1 Inertia Tensor of a Deformable Earth

The contributions to the earth inertia tensor due to deformations (tidal, rotational, etc.) are obtained through relations involving the external potential of the deformed earth and the disturbing potential.

For example, explicit values for $[\Delta I]_T$ were derived in [Groves, 1971] following the above mentioned standard approach. The final equations involve the fluid Love number $k_f$ and the mass and positions (ephemeris) of the external bodies exerting torques.

By the same procedure Munk and MacDonald [1960, p. 25] give the corresponding expressions for $[\Delta I]_R$. Nevertheless, recently Rochester and Smylie [1974] have questioned these results for the first time, based on the fact that the trace of the inertia tensor is not invariant for deformations with spheroidal component of degree zero, as was assumed by Munk and MacDonald. (This is not the case for tidal deformations produced by an external body force derived from a potential that is a solid harmonic of degree 2).

According to [Rochester and Smylie, 1974] equation (5.2.4) in [Munk and MacDonald, 1960] should be rewritten as (with the notation adopted here)

$$[\Delta I]_R = \frac{k_2 a_e^5}{3G}\left[\{\omega\}\{\omega\}^T - \frac{1}{3}\left(tr(\{\omega\}\{\omega\}^T - [\Delta I_{RAD}])\right)[1]\right] \qquad (2.4\text{-}6)$$

where to the first order of $\{\delta u\}$

$$tr[\Delta I_{RAD}] = 4\int_V \rho_E(r)\{x\}^T\{\delta u\}\,dv \qquad (2.4\text{-}7)$$

and

$\rho_E(r)$ ≡ density of the undeformed earth (function of the distance to the CM)

$\{x\}$ ≡ position vector of any point P

$\{\delta u\}$ ≡ vector displacement at point P caused by the deformation

$k_2$ ≡ Love's number of degree 2 describing the elastic yielding of the earth to the second degree term in the centrifugal force potential

G ≡ gravitational constant

$a_e$ ≡ earth semi-major axis

Similar expressions for $[\Delta I]_T$ and $[\Delta I]_R$ may be consulted in [Sánchez, 1974] where a different approach through integration is also given.

Several models for the computation of the contribution to the moments and products of inertia due to earthquake faulting or slip displacements are available in literature [Mansinha and Smylie, 1967], [Dahlen, 1973], [Israel et al., 1973], but up to now only analytical solutions of the Lagrange-

Liouville equations using the approximations established in [Munk and MacDonald, 1960, p. 38] have been investigated.

Thus, the fact that the mass distribution of the earth is subject to variations which are functions of time, produces changes in the earth inertia tensor as expressed by equation (2.4-2). The nature of these changes will depend on the earth model adopted and its variations with time. Clearly periodic changes in $[I]_E$ may be caused by the effect of $[\Delta I]_T$ and $[\Delta I]_A$ ; sudden variations, if any, will be due to the effect of $[\Delta I]_F$ .

Results reported in [Sánchez, 1974] show that the effect of the lunisolar deformation on $\omega$ has a maximum periodic variation in the wobble of the order of 24 mm. The effects of the rotational deformations are important only in the lengthening of the Eulerian period.

A widely debated point today is the question of possible excitation of the earth rotation vector by earthquakes. This originated from the hypothesis advanced by Mansinha and Smylie [1967] about the cumulative effects of large earthquakes and their influence on the random excitation of the wobble. The resulting controversy has continued up to the present time with no definitive results established as yet [Mueller, 1975]. Large-scale crustal motion that occurs with major seismic activity may abruptly change the earth inertia tensor associated to the reference system, and therefore contribute to a sudden change in the earth rotation vector.

Schematically Fig. 2.6 shows the effect of lunisolar tides and sudden mass shifts (earthquakes) on the wobble and mean pole, compared to a regular curve for the case of rigid earth.

The earth atmosphere can produce periodic contributions $[\Delta I]_A$ to the initial tensor of inertia due to seasonal variations in the distribution of air masses. This can be estimated from observations of pressure on the ground. Munk and Hassan [1961] showed that the twelfth-month wobble is excited atmospherically, most of the contribution coming from Asia, due to the high pressure over Siberia in winter. A more detailed discussion in [Wilson and Haubrich, 1976] leads to essentially the same conclusion. These authors also indicate that contrary to previous beliefs, the motions of air and water play an important role in maintaining the Chandler wobble. Winds acting along the earth surface cause a change $\{h\}_A$ in the earth angular momentum. The semiannual and annual components in the spin rotation axis of the earth are caused by winds, according to Munk and MacDonald [1960, p. 131]. The important finding by Lambeck and Cazenave [1973] was to correlate the biennial component (in the rate of rotation) between atmospheric zonal winds and the observed astronomical values originally reported [Ijima and Okazaki, 1966]. However, today there is still no unanimous agreement among the authors as to whether the biennial zonal winds are an intermittent phenomenon or their period is 24 months, rather than 26 or 28 months. Hence a final conclusion cannot be drawn.

The wobble of the instantaneous rotation axis of a deformable earth, after being affected by all the random geophysical effects mentioned above, departs from a perfect conic and is irregular and complex. Other related phenomena, such as its excitation, damping or even its period, have not yet been fully studied and clarified.

For example, the rigid earth Eulerian period of about 305 sidereal days is extended to what is known, after its discoverer, as the Chandlerian period, which is of about 14 months [Chandler, 1892]. As early as 1892, Newcomb showed that the lengthening of the Eulerian period is mainly due to the yielding of the earth and oceans.

Chandler [1892] also discovered the twelve-month period, which is due primarily to metereological effects, as was mentioned. The composition of these two periods (Chandler and annual) produces a beat effect of about a six-year period.

Recently the question of the influence of the liquid core on a possible diurnal wobble has been reopened and theoretically analyzed. The reader is advised to consult the papers [McClure, 1973], [Rochester et al., 1974] or [Toomre, 1974] where a complete list of classical works on the subject is given.

Fig. 2.7 presents the polhode of the real earth as observed by the IPMS during the period 1968.0–1974.1. Also shown are the barycenters (computed centers of the wobble $\simeq$ poles of figure). The determination of the barycenters is somewhat arbitrary, due to the complexity of the observed wobble. These points will probably not coincide exactly with the axis of maximum moment of inertia of the earth at the particular epoch, although they should be very close to it. Note that a manifest secular motion of the barycenter is present in the figure.

The secular motion of the mean pole ( $\equiv$ barycenter), if any, will be caused by secular contributions to the tensor of inertia. As mentioned in the introduction of this work, one plausible cause, postulated by many authors, is the effect on $[I]_E$ of differential mass displacements due to the motion of different tectonic plates in which the earth crust is broken up. This being the subject of this investigation, it will be studied at length in the following sections and chapters. The next section presents the necessary formulation for computing the values of the contributions, as used in this study.

It should be emphasized here that the polhodes of the real earth as observed by the stations of the international organizations, IPMS (with the ILS stations) and BIH, are not coincident, and systematic differences as large as $0''.03$ are present.

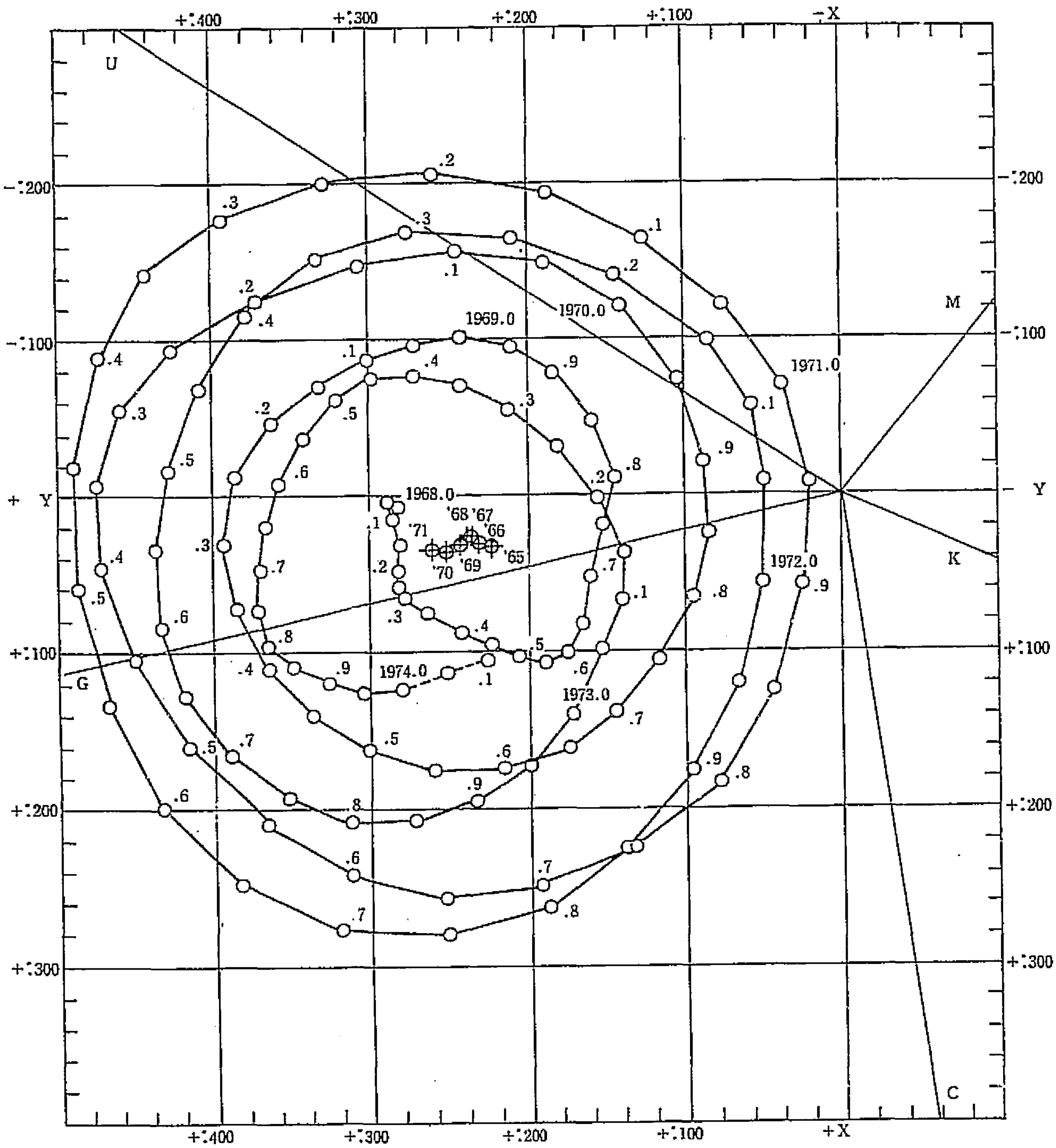



Fig. 2.7. Polar Orbit (from IPMS Latitude Observations) for 1962.0–1974.1. Near the Center of the Orbit the Barycenters at Epochs 1965.0...1971.0 Are Shown. From [Yumi, 1975].

## 2.5 Total Contribution $[\Delta I]$ to the Tensor of Inertia Due to Mass Displacements

### 2.5.1 Differential Changes in [ I ] Due to Infinitesimal Changes in the Cartesian Coordinates

By definition the value of the inertia tensor [ I ] may be written in one of the following forms

$$[I] = \int_M [\underline{x}][\underline{x}]^T dm = \int_V \rho_x [\underline{x}][\underline{x}]^T dv \tag{2.5-1}$$

where the element of volume is given by

$$dv = dx_1 \; dx_2 \; dx_3 \tag{2.5-2}$$

and the density $\rho_x$ is assumed constant for each mass element but not necessarily equal to the density in the neighboring elements.

Explicitly equation (2.5-1) may be written

$$[I] = \int_V \rho_x \begin{bmatrix} x_2^2 + x_3^2 & -x_1 x_2 & -x_1 x_3 \\ & x_1^2 + x_3^2 & -x_2 x_3 \\ s & & x_1^2 + x_2^2 \end{bmatrix} dv \tag{2.5-3}$$

The differential changes in [ I ] as a consequence of differential variations $\delta x_i$ $(i = 1,2,3)$ in the mass element Cartesian coordinates are (second and higher order partials are neglected)

$$[\Delta I] = \sum_{i=1}^{3} \frac{\partial [I]}{\partial x_i} \delta x_i = [\Delta I_1]\delta x_1 + [\Delta I_2]\delta x_2 + [\Delta I_3]\delta x_3 \tag{2.5-4}$$

where

$$\frac{\partial [I]}{\partial x_1} = [\Delta I_1] = \int_V \rho_x \begin{bmatrix} 0 & -x_3 & -x_3 \\ & 2x_1 & 0 \\ & & 2x_1 \end{bmatrix} dv$$

and similar equations hold for the other coefficients.

Thus, finally

$$[\Delta I] = \int_v \rho_x \left\{ \begin{bmatrix} 0 & -x_2 & -x_3 \\ & 2x_1 & 0 \\ s & & 2x_1 \end{bmatrix} \delta x_1 + \begin{bmatrix} 2x_2 & -x_1 & 0 \\ & 0 & -x_3 \\ s & & 2x_2 \end{bmatrix} \delta x_2 + \begin{bmatrix} 2x_3 & 0 & -x_1 \\ & 2x_3 & -x_2 \\ s & & 0 \end{bmatrix} \delta x_3 \right\} dv \tag{2.5-5}$$

### 2.5.2 Differential Changes in [ I ] Due to Infinitesimal Changes in the Spherical Coordinates

The earth being approximately a sphere, it will be more practical to obtain the differential changes in [ I ] as a function of variations in latitude $\delta\lambda$, colatitude $\delta\theta$ and radius $\delta r$, and at the same time to integrate in function of spherical coordinates.

The transformation between spherical and Cartesian coordinates is given by the well-known matrix equation

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} r \sin\theta \cos\lambda \\ r \sin\theta \sin\lambda \\ r \cos\theta \end{Bmatrix} \tag{2.5-6}$$

and the following linearized differential mapping applies between infinitesimal changes in spherical and Cartesian coordinates [Soler, 1976]

$$(\delta\lambda,\ \delta\theta,\ \delta r) \xrightarrow{J} (\delta x_1, \delta x_2, \delta x_3) \tag{2.5-7}$$

where the elements of the Jacobian matrix J are

$$\frac{\partial x_1}{\partial \lambda} = -r\sin\theta\sin\lambda = -x_2 \qquad \frac{\partial x_1}{\partial \theta} = r\cos\theta\cos\lambda = x_3\cos\lambda \qquad \frac{\partial x_1}{\partial r} = \sin\theta\cos\lambda = \frac{x_1}{r}$$

$$\frac{\partial x_2}{\partial \lambda} = r\sin\theta\cos\lambda = x_1 \qquad \frac{\partial x_2}{\partial \theta} = r\cos\theta\sin\lambda = x_3\sin\lambda \qquad \frac{\partial x_2}{\partial r} = \sin\theta\sin\lambda = \frac{x_2}{r}$$

$$\frac{\partial x_3}{\partial \lambda} = 0 \qquad \frac{\partial x_3}{\partial \theta} = -r\sin\theta = -\frac{x_1}{\cos\lambda} \qquad \frac{\partial x_3}{\partial \lambda} = \cos\theta = \frac{x_3}{r}$$

(2.5-8)

Thus

$$\delta x_1 = -x_2\,\delta\lambda + x_3\cos\lambda\,\delta\theta + \frac{x_3}{r}\,\delta r \tag{2.5-9a}$$

$$\delta x_2 = x_1\,\delta\lambda + x_3\sin\lambda\,\delta\theta + \frac{x_2}{r}\,\delta r \tag{2.5-9b}$$

$$\delta x_3 = -\frac{x_1}{\cos\lambda}\,\delta\theta + \frac{x_3}{r}\,\delta r \tag{2.5-9c}$$

and substituting the above equations in (2.5-5), follows

$$[\Delta I] = \int_V \rho_x \left\{ \begin{bmatrix} 2x_1x_2 & x_2^2 - x_1^2 & x_2x_3 \\ & -2x_1x_2 & -x_1x_3 \\ s & & 0 \end{bmatrix} \delta\lambda \right.$$

$$+ \begin{bmatrix} 2x_2x_3\sin\lambda - \frac{2x_1x_3}{\cos\lambda} & -x_2x_3\cos\lambda - x_1x_3\sin\lambda & -x_3^2\cos\lambda - \frac{x_1^2}{\cos\lambda} \\ & 2x_1x_3\cos\lambda - \frac{2x_1x_3}{\cos\lambda} & -x_3^2\sin\lambda - \frac{x_2x_3}{\cos\lambda} \\ s & & 2x_1x_3\cos\lambda + 2x_2x_3\sin\lambda \end{bmatrix} \delta\theta$$

$$\left. + \frac{2}{r} \begin{bmatrix} x_2^2 + x_3^2 & -x_1x_2 & -x_1x_3 \\ & x_1^2 + x_3^2 & -x_2x_3 \\ s & & x_1^2 + x_2^2 \end{bmatrix} \delta r \right\} dv \tag{2.5-10}$$

From the law of conservation of elements of mass, it follows

$$dm = \rho_x dv = \rho_{(\lambda,\theta,r)} |J| \, d\lambda \, d\theta \, dr \tag{2.5-11}$$

where $|J|$ is the Jacobian of the transformation between spherical and Cartesian coordinates. It is known [Soler, 1976] that

$$|J| = |H| = h_1 h_2 h_3 = r^2 \sin\theta$$

Thus, replacing the values of (2.5-6) in (2.5-10) with the above considerations, finally one has an expression of the form

$$[\Delta I] = [\Delta I_\lambda]\,\delta\lambda + [\Delta I_\theta]\,\delta\theta + [\Delta I_r]\,\delta r \tag{2.5-12}$$

where

$$[\Delta I_\lambda] = \int_v \rho_{(\lambda,\theta,r)} \begin{bmatrix} s\theta s2\lambda & -s\theta c2\lambda & c\theta s\lambda \\ & -s\theta s2\lambda & -c\theta c\lambda \\ s & & 0 \end{bmatrix} r^4 \sin^2\theta \, d\lambda \, d\theta \, dr \tag{2.5-13}$$

$$[\Delta I_\theta] = \int_v \rho_{(\lambda,\theta,r)} \begin{bmatrix} -s2\theta c^2\lambda & -s\theta c\theta s2\lambda & -c2\theta c\lambda \\ & -s2\theta s^2\lambda & -c2\theta s\lambda \\ s & & s2\theta \end{bmatrix} r^4 \sin\theta \, d\lambda \, d\theta \, dr \tag{2.5-14}$$

and

$$[\Delta I_r] = 2\int_v \rho_{(\lambda,\theta,r)} \begin{bmatrix} 1-s^2\theta c^2\lambda & -s^2\theta s\lambda c\lambda & -s\theta c\theta c\lambda \\ & 1-s^2\theta s^2\lambda & -s\theta c\theta s\lambda \\ s & & s^2\theta \end{bmatrix} r^3 \sin\theta \, d\lambda \, d\theta \, dr \tag{2.5-15}$$

In the above matrices, due to space limitations, the following notation equivalences hold:

$s \equiv \sin$

$c \equiv \cos$

Observe that as expected a rotation in $\lambda$ does not contribute any differential change to the original moment of inertia about the $x_3$ axis.

When the above integrals are evaluated over the volume of the earth, it is common to assume that the variation of density for neighboring "curvilinear mass elements" is only a function of the radius. Hence

$$\rho_{(\lambda,\theta,r)} = \rho(r)$$

### 2.5.3 Value of Trace $[\Delta I]$

Equation (2.5-12) may be written

$$[\Delta I] = [\Delta I_{ROT}] + [\Delta I_{RAD}] \tag{2.5-16}$$

where

$$[\Delta I_{ROT}] = [\Delta I_{\lambda}]\,\delta\lambda + [\Delta I_{\theta}]\,\delta\theta \tag{2.5-17}$$

is the contribution to the initial tensor of inertia $[I]_E$ due to differential rotations $\delta\lambda$ and $\delta\theta$ while

$$[\Delta I_{RAD}] = [\Delta I_r]\,\delta r \tag{2.5-18}$$

is the contribution due to expansions or contractions, that is, radial variations.

It is known that tr $[I]$ is invariant under rigid rotations of the reference system (see section B.3 in Appendix B).

Thus, if

$[I]_E \equiv$ initial earth inertia tensor

and

$[I] \equiv$ earth inertia tensor after rotations $\delta\lambda$ and $\delta\theta$

then

$$[I] = [I]_E + [\Delta I_{ROT}] \tag{2.5-19}$$

Therefore, the following implications must hold as a consequence of the tensorial properties of $[I]$

$$\mathrm{tr}[\,I\,] = \mathrm{tr}[\,I\,]_E \Longrightarrow \mathrm{tr}[\,\Delta I_{ROT}] = 0 \Longrightarrow \mathrm{tr}[\,\Delta I_{\lambda}] = \mathrm{tr}[\,\Delta I_{\theta}] = 0$$

The fulfillment of the above conditions is immediately seen from equations (2.5-13) and (2.5-14).

The situation is different with respect to the contribution due to radial deformations as expressed by equation (2.5-18). In this case after consideration of (2.5-15) and assuming that the density is only a function of r, one has

$$\mathrm{tr}[\,\Delta I_{RAD}] = \mathrm{tr}[\,\Delta I_r]\,\delta r = 4\int_V \rho(r)\, r\, \delta r\, dv \neq 0 \qquad (2.5\text{-}21)$$

which agrees with the previously given equation (2.4-7).

### 2.5.4 Inertia Tensor [ I ] in Spherical Coordinates

From equations (2.5-10) and (2.5-15) it immediately follows

$$[\,I\,] = \frac{r}{2}\,[\Delta I_r] \qquad (2.5\text{-}22)$$

Thus,

$$[\,I\,] = \int_V \rho_{(\lambda,\theta,r)} \begin{bmatrix} 1-s^2\theta\cos^2\lambda & -s^2\theta\, s\lambda\, c\lambda & -s^2\theta\, c\theta\, c\lambda \\ & 1-s^2\theta\, s^2\lambda & -s\theta\, c\theta\, s\lambda \\ s & & s^2\theta \end{bmatrix} r^4 \sin\theta\, d\lambda\, d\theta\, dr \qquad (2.5\text{-}23)$$

### 2.5.5 Value of Trace [ I ]

Using equations (2.5-21) and (2.5-22) it is possible to show that (only radial density variations are assumed)

$$\mathrm{tr}[\,I\,] = A + B + C = 2\int_V \rho(r)\, r^2\, dv \qquad (2.5\text{-}24)$$

But in the spherical case

$$v = \frac{4}{3}\pi r^3 \implies dv = 4\pi r^2 dr$$

Thus

$$\mathrm{tr}[I] = 8\pi \int_0^{r_e} \rho(r) r^4 dr = 8\pi r_e^5 \int_0^1 \rho(r) r^4 dr \tag{2.5-25}$$

2.5.6 Relative Angular Momentum $\{h\}$

By definition

$$\{h\} = \int_M [\underline{x}]\{\dot{x}\} dm = \int_V \rho_x [\underline{x}]\{\dot{x}\} dv = \int_V \rho_{(\lambda,\theta,r)} [\underline{x}]\{\dot{x}\} |J| \, d\lambda d\theta dr \tag{2.5-26}$$

but

$$[\underline{x}]\{\dot{x}\} = \begin{Bmatrix} -x_3\dot{x}_2 + x_2\dot{x}_3 \\ x_3\dot{x}_1 - x_1\dot{x}_3 \\ -x_2\dot{x}_1 + x_1\dot{x}_2 \end{Bmatrix} = \begin{Bmatrix} -r^2 \sin\lambda\dot{\theta} - r^2 \sin\theta\cos\theta\cos\lambda\dot{\lambda} \\ r^2\cos\lambda\dot{\theta} - r^2\sin\theta\cos\theta\sin\lambda\dot{\lambda} \\ r^2\sin^2\theta\dot{\lambda} \end{Bmatrix} \tag{2.5-27}$$

Thus, finally with

$$\dot{\theta} \equiv \dot{\delta}\theta$$

$$\dot{\lambda} \equiv \dot{\delta}\lambda$$

$$\{h\} = \dot{\delta}\theta \{\Delta h_\theta\} + \dot{\delta}\lambda \{\Delta h_\lambda\} \tag{2.5-28}$$

where

$$\{\Delta h_\theta\} = \int_V \rho_{(\lambda,\theta,r)} r^4 \sin\theta \begin{Bmatrix} -\sin\lambda \\ \cos\lambda \\ 0 \end{Bmatrix} d\theta d\lambda dr \tag{2.5-29}$$

and

$$\{\Delta h_{\lambda}\} = \int_{v} \rho_{(\lambda,\theta,r)} r^4 \sin\theta \begin{Bmatrix} -\sin\theta\cos\theta\cos\lambda \\ -\sin\theta\cos\theta\sin\lambda \\ \sin^2\theta \end{Bmatrix} d\theta\, d\lambda\, dr \qquad (2.5\text{-}30)$$

Notice that radial deformations (i.e. expansions and contractions) do not contribute to $\{h\}$; therefore the following implications are established:

$$\text{rigid body} \implies \{h\} = 0 \qquad \text{but} \qquad \{h\} = 0 \;\not\Longrightarrow\; \text{rigid body}$$

Finally, the equations given below hold:

$$\Delta h_{\lambda_1} = \Delta I_{\lambda_{23}} \quad ; \quad \Delta h_{\lambda_2} = -\Delta I_{\lambda_{13}} \quad ; \quad \Delta h_{\lambda_3} = I_{33} = C$$

## 2.6 Differential Contributions to the Earth Inertia Tensor Due to Plate Motions

As a real application to the earth of the equations given in section 2.5, one may compute the differential changes in the earth tensor of inertia due to differential tectonic plate motion. This may be expressed by equation (2.4-4), namely

$$[\Delta I]_P = \sum_{i=1}^{n} [\Delta I_{ROT}]_{Pi} \qquad (2.6\text{-}1)$$

where n is the number of tectonic plates constituting the earth crust. The elements in the summation of the right-hand side in the above equation will be given by

$$[\Delta I_{ROT}]_{Pi} = \sum_{k=1}^{m} (\delta\lambda_k [\Delta I_{\lambda}]_k + \delta\theta_k [\Delta I_{\theta}]_k) \qquad (2.6\text{-}2)$$

where

$m$ ≡ number of sample blocks on plate $P_i$

$[\Delta I_{\lambda}]_k$ ≡ differential changes in the tensor of inertia of block k due to an infinitesimal motion $\delta\lambda_k$

$[\Delta I_\theta]_k$ ≡ differential changes in the tensor of inertia of block k due to an infinitesimal motion $\delta\theta_k$

Observe that in equation (2.6-1) the assumption $\delta r = 0$ is implied; thus vertical crustal fluctuations are neglected.

Although conclusive evidence exists supporting the argument that in addition to horizontal tectonic displacements of the earth's crust (orogenic movements), vertical fluctuations (epirogenic movements) also take place (the Fennoscandian uplift being the best documented); the assumption $\delta r = 0$ will continue to be used until precise global data regarding vertical continental motion is available, and a unifying framework for vertical movements is known. The detection of such movements is hampered by the difficulty in separating rising of sea level (eustatic sea level change) with sinking of continents, or vice versa.

The computation of $[\Delta I_\lambda]_k$ and $[\Delta I_\theta]_k$ as expressed by equations (2.5-13) and (2.5-14) involves the integration over the mass of every sample crustal block k. Hence changes of density in the adopted earth model must be taken into consideration. Therefore in order to perform the integration in (2.5-13) and (2.5-14), a properly defined crustal model should be established.

On the other hand, the summation in equation (2.6-2) assumes knowledge of the plate boundaries and the differential motions for every block k (see Fig. 2.3). For this reason some kind of plate model and corresponding absolute plate velocities with respect to the underlying static mantle must be adopted.

In the following chapter, the particular models adopted in this investigation will be described.

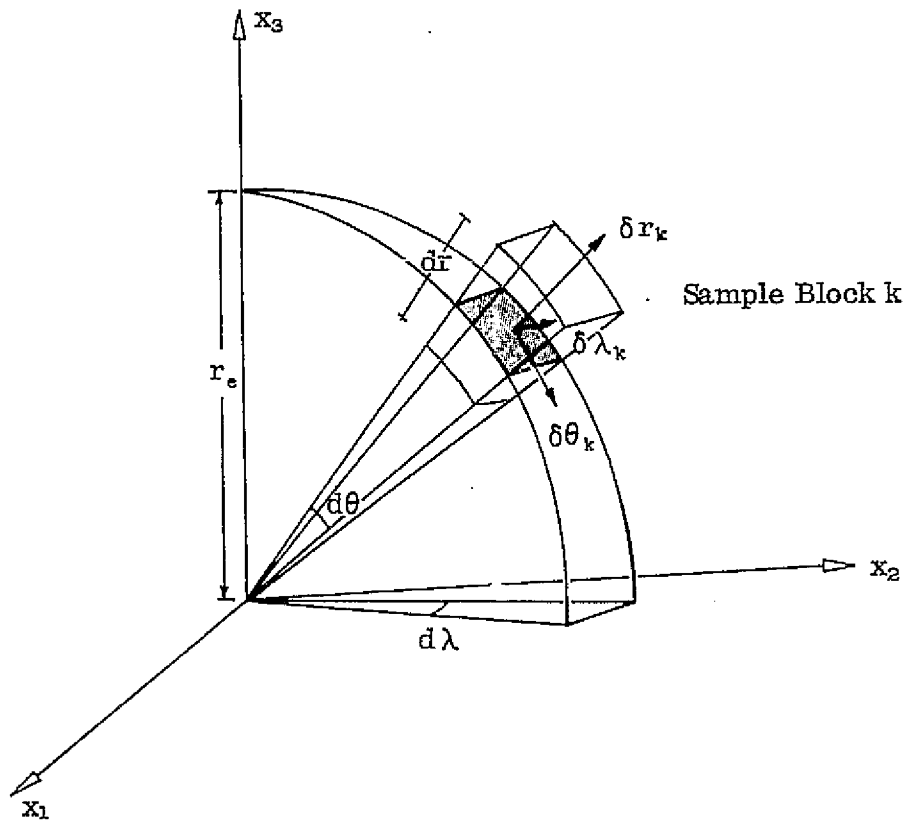



Fig. 2.8. Differential Motions of Sample Block k

# 3 TECTONIC MODELS

This chapter emphasizes the major assumptions inherent in the selected models used in the numerical integration and the values of the adopted constraints and parameters.

Two major models are required: a crustal model defining the composition of the thin upper layer of the earth, and a plate model describing the boundaries of the different tectonic blocks and absolute angular velocities with respect to the underlying mantle.

## 3.1 Crustal Model

Knowledge of the earth's crustal structure has advanced considerably in the last half century, primarily due to investigations in the fields of geodesy (gravimetry) and geophysics (seismology).

Seismic research shows a marked boundary between the crust and the upper mantle known as the Mohorovičić discontinuity (also referred to as Moho or M discontinuity), which separates two layers of very distinct density and seismic velocity.

It is now fairly well-established that the Airy-Heiskanen depth of isostatic compensation (partially formulated a long time before the seismological data was available) agrees well with the depth of the M discontinuity in the ocean basins as well as continental plateaus. This reinforces the hypothesis that on a continental scale the earth's crust is at least approximately in a state of isostatic equilibrium. Excess of mass above sea level

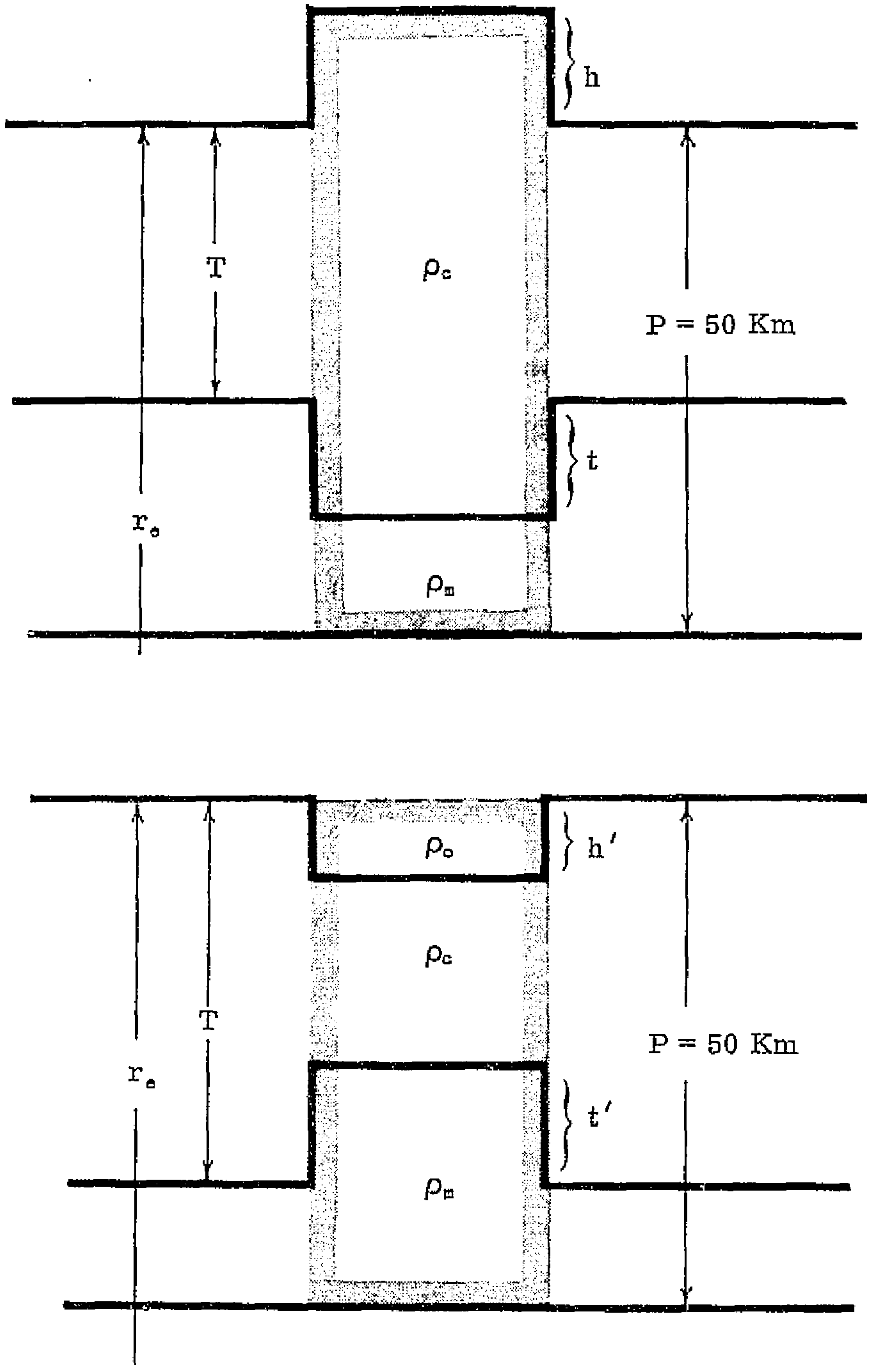



Fig. 3.1 Standard Continental and Oceanic Blocks

is balanced by a deficiency of mass inside the earth, while the lack of mass in the oceans is compensated by an excess of mass beneath.

Although there are regional disagreements which tend to conform with the Pratt-Hayford system of isostatic compensation, these are not globally significant and therefore will not be considered.

To model the upper shell of the earth as closely as possible to physical reality in a manner which is computationally feasible, the Airy-Heiskanen isostatic ideas and the formulation given by [Heiskanen, 1938] and [Heiskanen and Vening-Meinesz, 1958, p. 137] will be adopted.

Consequently, the following basic assumptions are made:

(a) isostatic compensation is complete

(b) the compensation is local (as opposed to regional)

(c) the density of the earth's crust is constant everywhere ($\rho_c = 2.67$ g/cm$^3$)

(d) the density of the underlayer upper mantle (tectosphere) is also constant everywhere ($\rho_m = 3.27$ g/cm$^3$)

(e) the normal depth of compensation is assumed to be T = 30 km

Fig. 3.1 shows schematically the cases of emerged and submerged blocks with the corresponding notation.

Written below are the two equations which give the values of the roots and antiroots for land and water compartments respectively.

$$t = \lambda_o h \left\{ 1 + \frac{2T + (\lambda_o + 1)h}{r_e} + \frac{(2T + \lambda_o h)[2T + (\lambda_o + 1)h]}{r_e^2} - \frac{T(T + \lambda_o h)}{r_e^2} - \frac{(\lambda_o^2 - 1)h^2}{3r_e^2} \right\} \tag{3.1-1}$$

$$t' = \mu h' \left\{ 1 + \frac{2T + (\mu + 1)h'}{r_e} + \frac{(2T + \mu h')[2T + (\mu + 1)h']}{r_e^2} - \frac{T(T + \mu h')}{r_e^2} - \frac{(\mu^2 - 1)h'^2}{3r_e^2} \right\} \tag{3.1-2}$$

where

$t \equiv$ length of mountain root for a land block

$t' \equiv$ length of ocean antiroot for a water block ($t' > 0$)

$h \equiv$ height above sea level

$h' \equiv$ depth below sea level ($h' < 0$)

$r_e \equiv$ earth radius

$\rho_o = 1.027\ g/cm^3 \equiv$ sea water density

and

$$\Delta\rho = \rho_m - \rho_c \tag{3.1-3}$$

$$\lambda_c = \frac{\rho_c}{\Delta\rho} \tag{3.1-4}$$

$$\mu = \frac{\rho_c - \rho_o}{\Delta\rho} \tag{3.1-5}$$

For the purpose of this investigation and for reasons explained in the following section, an ideal boundary 50 km deep (see Fig. 3.2) which contains all inequalities of the crust will be taken as depth limit of the upper shell of the solid earth model.

It should be understood that the final results are slightly dependent on the type of model selected. This work has used the best model obtainable at the present time. Naturally, in the future when a more thorough knowledge of the crustal structure becomes available, changes in the model may be expected and consequently the results may improve.

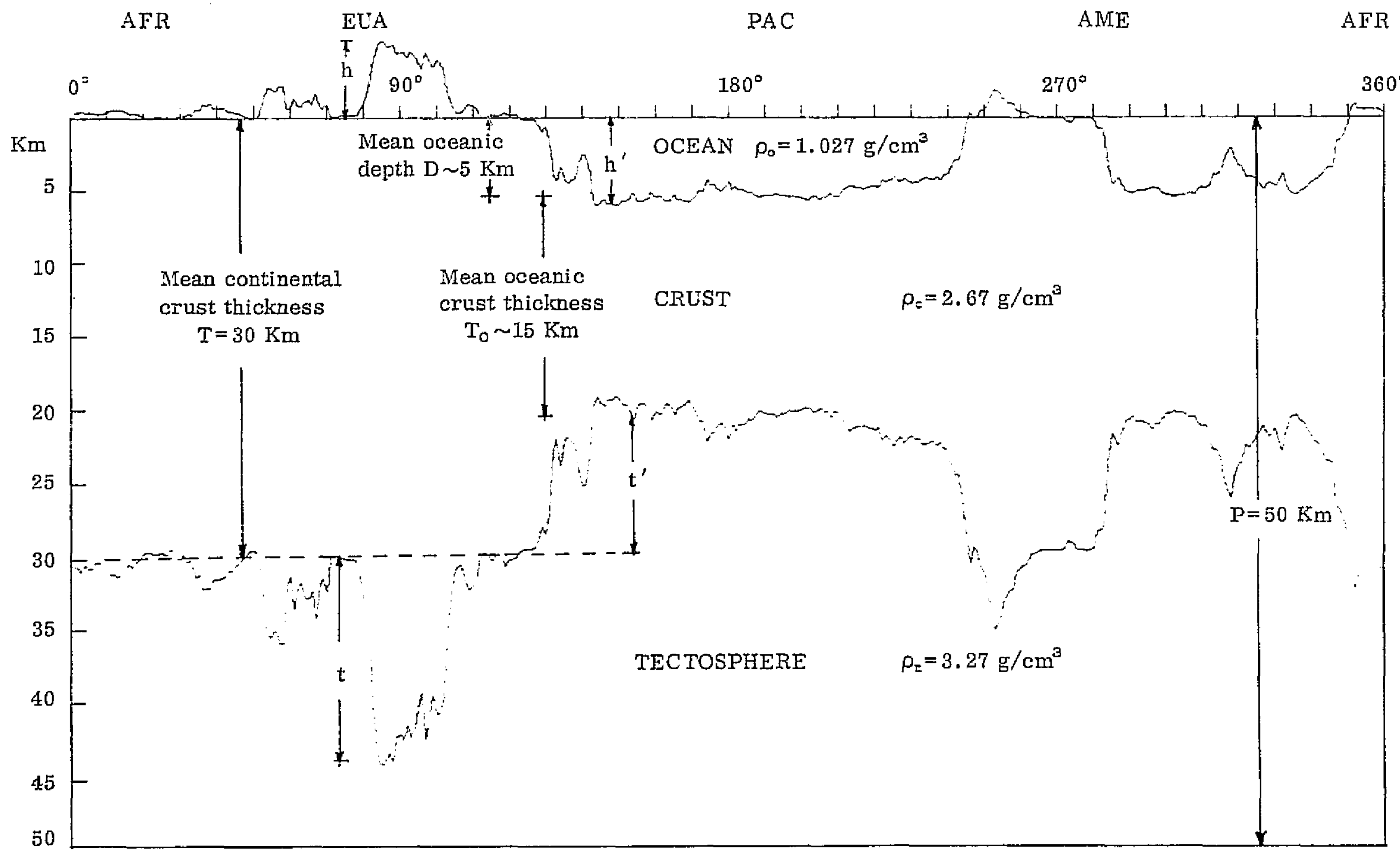
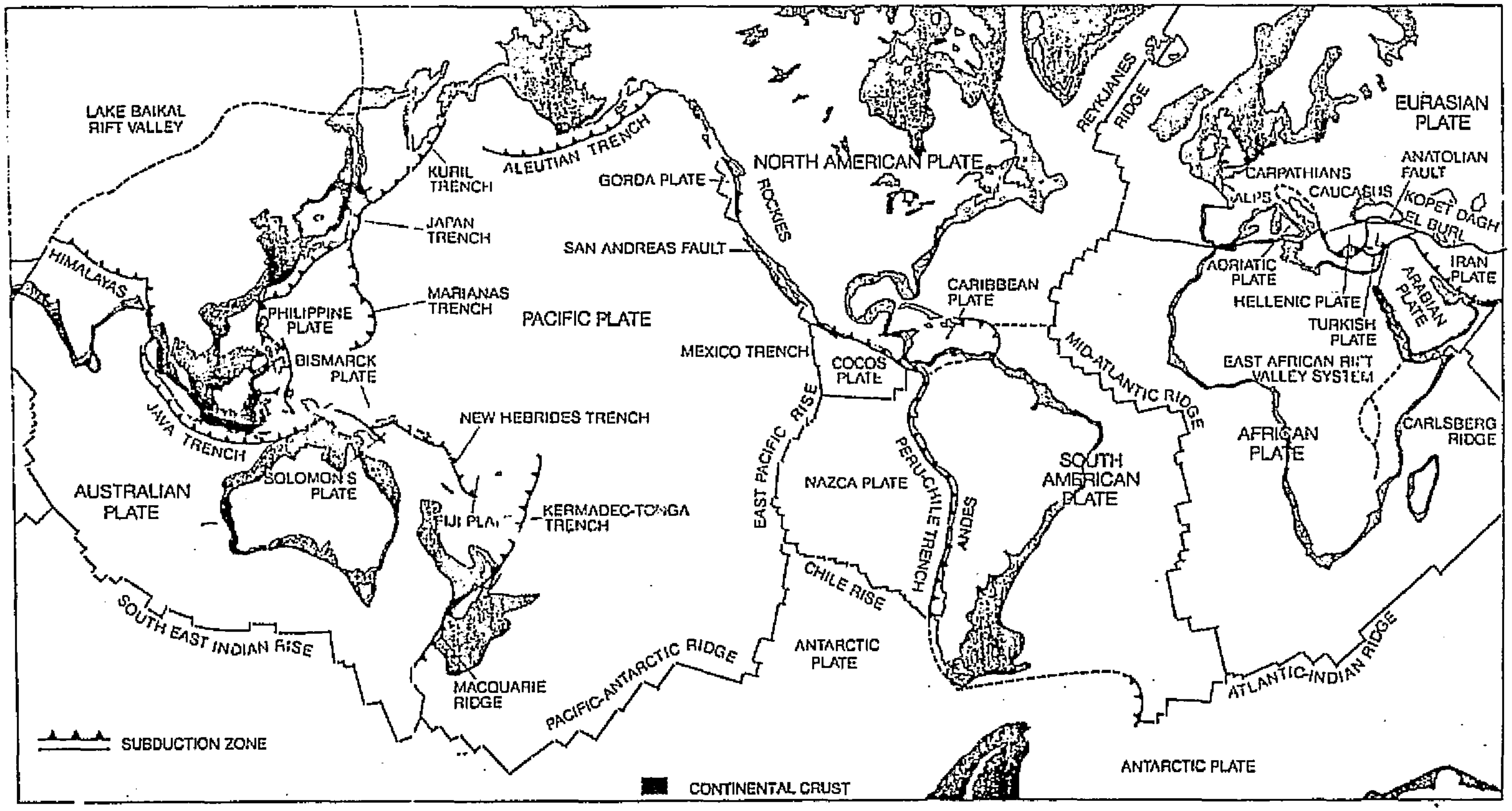



Fig. 3.2. Computer Generated Crustal Section Along $\theta = 60° \equiv \varphi = 30°$ Obtained Using Equations (3.1-1) and (3.1-2) (Vertical Scale Has Been Exaggerated by a Factor of About 360).




Mosaic of plates (adapted from "Plate Tectonics" by John Dewey, copyright ©by Scientific American, Inc. All rights reserved).

Fig. 3.3. Earth's Principal Tectonic Features.
(Fig. 2.1 in U. S. Program for the Geodynamics Project, National Academy of Sciences, 1973).

## 3.2 Absolute Plate Velocity Model

In order to model the earth's crustal motions, the most recent geophysical theories are taken into consideration.

The rigid outer shell of the planet (lithosphere) is divided into plates which are in continual motion. Depending on the authors, the number, boundaries and velocities of these plates vary, although there is consistent agreement with respect to the major important ones (macroplates).

### 3.2.1 Boundaries

The boundaries between the different plates have been defined classically by maps of intense seismic activity, even though recently a strong correlation between $1^\circ \times 1^\circ$ mean free air gravity anomalies and the plate perimeters was reported [Wilcox and Blouse, 1974].

Three types of tectonic boundaries are generally recognized [Cox, 1973]:

(i) Ridges: where two plates are diverging, permitting the upwelling of magma that creates new lithosphere.

(ii) Trenches: where two plates are converging, with one plate moving beneath the other, eventually to be absorbed into the mantle, or "destroyed".

The direction of relative motion of two plates need not be perpendicular to the ridge or trenches.

(iii) Transform faults: where two plates are moving tangential to each other. Lithosphere is neither created nor destroyed.

The direction of relative motion of the two plates is exactly parallel to the fault.

Fig. 3.3 shows the major tectonic features on the earth.

The plates are assumed to be internally rigid but uncoupled from each other. At their boundaries two plates may pull apart or one may slip beneath the other, but within the plates there is no deformation.

In this investigation the tectonic plate model described originally in [Solomon and Sleep, 1974] and shown in Fig. 3.4 was adopted. The surface of the earth is divided into eleven rigid blocks, avoiding the minor controversial plates (e.g., Gorda, Fiji, Iran, etc.) The division between North American and South American plates is not yet fully accepted, and therefore to combine them in a single plate (American) with a unique set of absolute velocities is reasonable. In the figure land masses are drawn up to the limit of the continental shelf ( $\sim$ 2000 m isobath) while subduction zones are shown as heavy dashed lines.

Among other plate models available in the geophysical literature and previously used by investigators are [Minster et al., 1974 and Kaula, 1975].

### 3.2.2 Absolute Angular Velocities

For the purpose of studying the movements of the earth's shell and to facilitate treatment of sections of oceanic and continental areas as units, it is assumed that the crust is joined rigidly to the upper part of the lithosphere (the tectosphere) and moves with it. This is in accordance with the new slab theory which supposes the crust to be a mere passenger driven by slabs and clearly contradicts the old concept that the continents just float

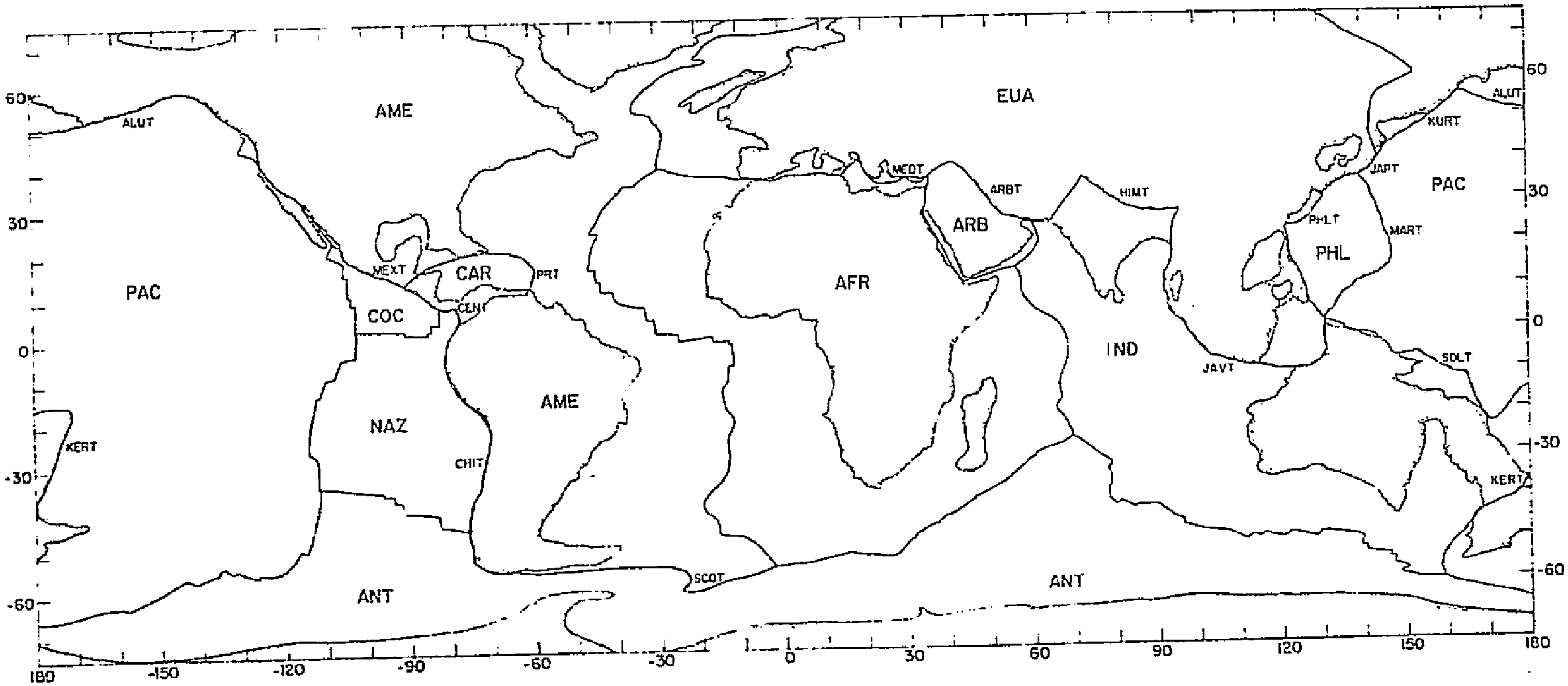



Fig. 3.4. Plate Model Adopted. From [Solomon and Sleep, 1974].

on a viscous mantle.

Consequently in this investigation crust and tectosphere move as a mechanical unit. An imaginary boundary at 50 km deep containing the whole crust is assumed to approximate as closely as possible the effect of the differential displacement of the irregularities of the crust due to plate motion. Below that boundary the earth is thought to be structured in some fashion not producing any variation in the crustal tensor of inertia.

Several theoretical models for absolute plate velocities have appeared recently in the geophysical literature. Small final differences depend mainly on the initial assumptions and the total number of plates considered.

One of the most complete sets of absolute plate velocities was presented in [Solomon et al., 1975] after postulating that no net torque is exerted on the lithosphere as a whole. Absolute angular velocities of the Pacific plate relative to the underlying mantle are given, after consideration of several driving mechanisms which can be modeled quantitatively and which are capable of affecting final absolute velocity.

The description of the different models and their corresponding absolute angular velocity for the Pacific plate is given in Table 3.1.

Also given in [Solomon et al., 1975] are the relative angular velocities of the other plates, assuming the Pacific plate to be stationary (see Table 3.2). Then the absolute angular velocity of any plate $P_i$ may be computed as follows

$$\{\omega_a\}_{P_i} = \{\omega_r\}_{P_i} + \{\omega_a\}_{RP} \qquad (3.2\text{-}1)$$

where

$\{\omega_r\}_{P_i}$ ≡ relative angular velocity vector of any plate $P_i$ with respect to an arbitrary reference plate (Pacific);

$\{\omega_a\}_{RP}$ ≡ absolute angular velocity vector of the reference plate (Pacific) with respect to the underlying mantle;

$\{\omega_a\}_{P_i}$ ≡ absolute angular velocity vector of any plate $P_i$ with respect to the underlying mantle.

Table 3.2 tabulates the assumed absolute angular velocity vectors $\{\omega_a\}_{P_i}$ for all the plates (other than the Pacific) computed from the values $\{\omega_r\}_{P_i}$ and $\{\omega_a\}_{RP}$ given in [Solomon et al., 1975].

## 3.3 Changes $(\delta\lambda_k, \delta\theta_k)$ in Each Block Due to Plate Motions

Once the absolute angular velocity vector $\{\omega_a\}_{P_i}$ for each plate is known, the changes $\delta\lambda_k$ and $\delta\theta_k$ in longitude and colatitude for a sample block k belonging to the particular plate may be computed.

Observe that the components of $\{\omega_a\}_{P_i}$ in Table 3.2 can be considered differentially small and that the given values conform with a right-hand rotation rule.

The required formulation for evaluating the differential changes in the curvilinear spherical coordinates at a point, after a differential rotation is performed, may be obtained following the general theory presented in [Soler, 1976].

The final equations are in the form

$$\begin{Bmatrix} \delta\theta \\ \delta\lambda \\ \delta r \end{Bmatrix}_k = H^{1}R[\,\underline{\delta\omega}]_{P_i} \begin{Bmatrix} r_e \sin\theta\cos\lambda \\ r_e \sin\theta\sin\lambda \\ r_e\cos\theta \end{Bmatrix}_k \qquad (3.3\text{-}1)$$

Table 3.1

Absolute Velocity of the Pacific Plate*

| Model | | $\omega_1$ | $\omega_2$ | $\omega_3$ | $\lvert\omega\rvert$ | Orientation of $\{\omega\}$ | |
|---|---|---|---|---|---|---|---|
| Name | Description | $10^{-7}$ deg/yr | | | | Latitude | Longitude |
| A3 | Uniform drag coefficient beneath all plates | -1.490 | 2.410 | -6.494 | 7.08 | $-66^{\circ}.4$ | $121^{\circ}.7$ |
| B3 | Drag beneath continents only | -1.800 | 3.412 | -8.040 | 8.92 | -64.4 | 117.8 |
| B4 | Continents have 3 times more drag than oceans | -1.605 | 2.876 | -7.142 | 7.86 | -65.2 | 119.2 |
| C3 | Drag opposing horizontal translations of slabs, oceanic subduction zone only | -1.333 | 2.211 | -6.147 | 6.67 | -67.2 | 121.1 |
| C4 | Same but including ARBT and HIMT (see Fig. 3.4) | -1.353 | 2.427 | -6.395 | 6.97 | -66.5 | 119.1 |
| D1 | Maximum pull by slabs plus plate drag | -1.335 | 3.558 | -6.570 | 7.59 | -60.0 | 110.6 |
| E2 | Drag beneath 8 mid-plate hot spots of [Morgan, 1971] | -0.755 | 2.460 | -6.972 | 7.43 | -69.7 | 107.1 |
| E3 | Drag beneath 19 hot spots of [Morgan, 1971] | -1.290 | 3.091 | -7.538 | 8.25 | -66.0 | 112.7 |

The components of $\{\omega\}$ are referred to the geographic system (x). A right hand rule applies to the vector $\{\omega\}$.

* From [Solomon et al., 1975].

where

$H$ $\equiv$ "metric matrix" of the differential transformation between curvilinear and Cartesian coordinates;

$R$ $\equiv$ rotation matrix of the transformation between the geocentric and "local moving" frames;

$[\underline{\delta\omega}]_{P_i}$ $\equiv$ skew-symmetric matrix of the absolute angular velocity vector for the particular plate $P_i$ containing the block k.

Observe that here $\{\delta\omega\}_{P_i}$ $\{\omega_a\}_{P_i}$

For the specific curvilinear coordinates and sign convention used in this work, one has

$$H = \begin{bmatrix} r_e & 0 & 0 \\ & r_e \sin\theta & 0 \\ s & & 1 \end{bmatrix}_k \tag{3.3-2}$$

$$R = R_2(\theta)\, R_3(\lambda) = \begin{bmatrix} \cos\theta\cos\lambda & \cos\theta\sin\lambda & -\sin\theta \\ -\sin\lambda & \cos\lambda & 0 \\ \sin\theta\cos\lambda & \sin\theta\sin\lambda & \cos\theta \end{bmatrix}_k \tag{3.3-3}$$

and for a given tectonic plate

$$[\underline{\delta\omega}]_{P_i} = \begin{bmatrix} 0 & -\delta\omega_3 & \delta\omega_2 \\ \delta\omega_3 & 0 & -\delta\omega_1 \\ -\delta\omega_2 & \delta\omega_1 & 0 \end{bmatrix}_{P_i}$$

Table 3.2

Relative and Absolute Angular Velocity of Plates Other Than Pacific

| Model | Veloc. | Plate Name | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $10^{-7}$ deg/yr | AFRICAN | | | AMERICAN | | | ANTARTIC | | | ARABIAN | | | CARIBBEAN | | |
| | $\{\omega_r\}_{P_1}$ * | 2.406 | -4.112 | 8.973 | 1.364 | -3.611 | 5.687 | 0.675 | -3.680 | 9.596 | 4.709 | -3.784 | 10.343 | 1.051 | -2.794 | 6.332 |
| A3 | | 0.916 | -1.702 | 2.479 | -0.126 | -1.201 | -0.807 | -0.815 | -1.270 | 3.102 | 3.219 | -1.374 | 3.349 | -0.439 | -0.384 | -0.162 |
| B3 | | 0.606 | -0.700 | 0.933 | -0.436 | -0.199 | -2.353 | -1.125 | -0.268 | 1.556 | 2.909 | -0.372 | 2.303 | -0.749 | 0.618 | -1.708 |
| B4 | | 0.801 | -1.236 | 1.831 | -0.241 | -0.736 | -1.455 | -0.930 | -0.804 | 2.454 | 3.104 | -0.908 | 3.201 | -0.554 | 0.082 | -0.810 |
| C3 | | 1.073 | -1.901 | 2.826 | 0.031 | -1.400 | -0.460 | -0.658 | -1.469 | 3.449 | 3.376 | -1.573 | 4.196 | -0.282 | -0.583 | 0.185 |
| C4 | $\{\omega_a\}_{P_1}$ | 1.053 | -1.686 | 2.578 | 0.011 | -1.184 | -0.708 | -0.678 | -1.253 | 3.201 | 3.356 | -1.357 | 3.948 | -0.302 | -0.367 | -0.063 |
| D1 | | 1.071 | -0.554 | 2.403 | 0.029 | -0.053 | -0.883 | -0.660 | -0.122 | 3.026 | 3.374 | -0.226 | 3.773 | -0.284 | 0.764 | -0.238 |
| E2 | | 1.651 | -1.652 | 2.001 | 0.609 | -1.151 | -1.285 | -0.080 | -1.220 | 2.624 | 3.354 | -1.324 | 3.371 | 0.296 | -0.334 | -0.640 |
| E3 | | 1.116 | -1.021 | 1.435 | 0.074 | -0.520 | -1.851 | -0.615 | -0.589 | 2.058 | 3.419 | -0.693 | 2.805 | -0.239 | 0.297 | -1.206 |

*From [Solomon et al., 1975].

Table 3.2 (Cont'd)

Relative and Absolute Angular Velocity of Plates Other Than Pacific

| Model | Veloc. $10^{-7}$ deg/yr | Plate Name | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | COCOS | | | EURASIAN | | | INDIAN | | | NAZCA | | | PHILIPPINES | | |
| | $\{\omega_r\}_{P_i}$ * | -4.715 | -14.425 | 13.332 | 1.313 | -3.569 | 8.267 | 6.334 | -0.221 | 10.890 | 0.693 | -9.001 | 13.692 | 5.836 | -1.362 | 0.806 |
| A3 | $\{\omega_a\}_{P_i}$ | -6.205 | -12.015 | 6.838 | -0.177 | -1.159 | 1.773 | 4.844 | 2.189 | 4.396 | -0.797 | -6.591 | 7.198 | 4.346 | 1.048 | -5.688 |
| B3 | | -6.515 | -11.013 | 5.292 | -0.487 | -0.157 | 0.227 | 4.534 | 3.191 | 2.850 | -1.107 | -5.589 | 5.652 | 4.036 | 2.050 | -7.234 |
| B4 | | -6.320 | -11.549 | 6.190 | -0.292 | -0.693 | 1.125 | 4.729 | 2.655 | 3.748 | -0.912 | -6.125 | 6.550 | 4.231 | 1.514 | -6.336 |
| C3 | | -6.048 | -12.214 | 7.185 | -0.020 | -1.358 | 2.120 | 5.001 | 1.990 | 4.743 | -0.640 | -6.790 | 7.545 | 4.503 | 0.849 | -5.341 |
| C4 | | -6.068 | -11.998 | 6.937 | -0.040 | -1.142 | 1.872 | 4.981 | 2.206 | 4.495 | -0.660 | -6.574 | 7.297 | 4.483 | 1.065 | -5.589 |
| D1 | | -6.050 | -10.867 | 6.762 | -0.022 | -0.011 | 1.697 | 4.999 | 3.337 | 4.320 | -0.642 | -5.443 | 7.122 | 4.501 | 2.196 | -5.764 |
| E2 | | -5.470 | -11.965 | 6.360 | 0.558 | -1.109 | 1.295 | 5.579 | 2.239 | 3.313 | -0.062 | -6.541 | 6.720 | 5.081 | 1.098 | -6.166 |
| E3 | | -6.005 | -11.334 | 5.794 | 0.023 | -0.478 | 0.729 | 5.044 | 2.870 | 3.352 | -0.597 | -5.910 | 6.154 | 4.546 | 1.729 | -6.732 |

*From [Solomon et al., 1975].

After these matrices are substituted in equations (3.3-1), it easily follows that

$$\delta\lambda_k = \delta\omega_3 - \delta\omega_1 \cos\lambda_k \cot\theta_k - \delta\omega_2 \sin\lambda_k \cot\theta_k \qquad (3.3\text{-}4a)$$

$$\delta\theta_k = -\delta\omega_1 \sin\lambda_k + \delta\omega_2 \cos\lambda_k \qquad (3.3\text{-}4b)$$

When the above values are expressed in radians, one may change to linear units as follows:

$$\overline{\delta\lambda}_k = r_e \sin\theta_k \,\delta\lambda_k \qquad (3.3\text{-}5a)$$

$$\overline{\delta\theta}_k = r_e \,\delta\theta_k \qquad (3.3\text{-}5b)$$

If the plate rotations were not differentially small, the matrix $[\,\underline{\delta\omega}\,]_{P_1}$ in (3.3-1) should be replaced by

$$R_\alpha(\omega) - [\,1\,] \qquad (3.3\text{-}6)$$

where the rotation matrix $R_\alpha(\omega)$ is given by (see Appendix A)

$$R_\alpha(\omega) = \cos\omega\,[\,1\,] - (1-\cos\omega)[\,P\,] + \sin\omega\,[\,\underline{\alpha}\,] \qquad (3.3\text{-}7)$$

and

$\omega \equiv$ rotation angle along the line of direction cosines $\{\alpha\}$

$$[\,P\,] = \{\alpha\}\{\alpha\}^T \equiv \text{projection matrix} \qquad (3.3\text{-}8)$$

Notice that for small angular rotations,

$$\cos\omega \approx 1$$

$$\sin\omega \approx \delta\omega$$

and (3.3-7) reduces to

$$R_\alpha(\delta\omega) = [\,1\,] + \delta\omega[\,\underline{\alpha}\,] \tag{3.3-9}$$

In this case,

$$\alpha_i = \frac{\delta\omega_i}{\delta\omega} \qquad \forall \quad i = 1, 2, 3 \tag{3.3-10}$$

where $\delta\omega_i$ ($i = 1.2.3$) are the components of the angular velocity along the three Cartesian axes.

Therefore equation (3.3-9) has the form

$$R_\alpha(\delta\omega) = [\,1\,] + [\,\underline{\delta\omega}\,] \tag{3.3-11}$$

and finally, expression (3.3-6), as expected, reduces to $[\,\underline{\delta\omega}\,]$.

# 4. NUMERICAL EXPERIMENTS AND RESULTS

Based on the theory discussed in sections 2.5 and 2.6, the differential contributions to the earth inertia tensor as a consequence of tectonic mass displacements can be evaluated.

The manner in which this is accomplished practically and the inherent restrictions will be discussed in the following section.

Afterwards, the computed values of the matrices involved will be given. As a preliminary calculation the earth tensor of inertia is obtained. Although there is a possibility of evaluating $[I]_E$ approximately for a spherical earth through integration, using equation (2.5-23) and a density model (e.g., [Bullen and Haddon, 1967]), in this study the latest satellite data is used. It should be mentioned here that in this investigation, as will be seen later, the matrix $[I]_E$ has only a relative role and therefore an approximate value is sufficient.

## 4.1 Integral Evaluation

The basic expressions to be evaluated, namely $[I]_E$, $[\Delta I_\lambda]$ and $[\Delta I_\theta]$ involve integrations over the total volume of the earth's upper layers, under the assumption that the density distribution is known. These triple integrals are evaluated along the domains of each plate, so that the individual plate contributions can be accounted for.

The integrations involve two parts:

(1) Integration with respect to r

(2) Integration with respect to $\lambda$ and $\theta$

### 4.1.1 Integration with Respect to r

As is known, from the characteristics of the crustal model discussed in section 3.1, all variables involved in the integration with respect to r are assumed to depend only on the height above or below sea level. The other parameters are adopted constants like T, P and the set of densities.

Thus a limitation on the integration is imposed by the requirement of a global set of earth elevations suitable for digital computation.

At the present a magnetic tape is available from the Defense Mapping Agency Aerospace Center (DMAAC), formerly Aeronautical Chart and Information Center (ACIC), Saint Louis, Mo., which contains $1° \times 1°$ mean elevations of the earth solid surface arranged in latitude belts [Czarnecki, 1970].

It is appropriate to mention now that in areas covered by ice, such as Antartica and Greenland, the value given in the tape was used as mean elevation of the terrain topography. No other alternative is left to the user because no ice elevation is specified. Therefore this work does not consider changes in density due to ice coverage. After this report was completed the author became aware of the availability of data sets containing Greenland and Antartic $1° \times 1°$ mean ice thicknesses and topographic surface elevations. This data was compiled by Anderson [1976] from maps supplied by the Australian National Antartic Research Expeditions (ANARE). Although no major changes are expected with respect to the basic conclusions published here, a better modeling may be accomplished after the changes in density due to ice coverage have been considered.

Consequently, in this investigation, the crustal layer is supposed to be divided into individual $1^\circ \times 1^\circ$ sample blocks which are not homogeneous, having variable heights and a constant depth of 50 km.

For every one of these blocks an analytical integration with respect to the variable $r$ may be performed. The following two cases are possible (consult section and figure 3.1 for the proper interpretation of parameters):

(a) Emerged block ($h > 0$)

In this case one has

$$\int_{\text{EMERGED}} \rho(r)\, r^4\, dr = \rho_c \int_{r_e}^{r_e+h} r^4\, dr + \rho_c \int_{r_e-(T+t)}^{r_e} r^4\, dr + \rho_m \int_{r_e-P}^{r_e-(T+t)} r^4\, dr$$

$$= \rho_c \left[ \frac{(r_e+h)^5 - [r_e-(T+t)]^5}{5} \right] + \rho_m \left[ \frac{[r_e-(T+t)]^5 - (r_e-P)^5}{5} \right] \qquad (4.1\text{-}1)$$

(b) Submerged block ($h' < 0$)

In a way similar to case (a), it follows

$$\int_{\text{SUBMERGED}} \rho(r)\, r^4\, dr = \rho_o \int_{r_e+h'}^{r_e} r^4\, dr + \rho_c \int_{r_e-(T-t')}^{r_e+h'} r^4\, dr + \rho_m \int_{r_e-P}^{r_e-(T-t')} r^4\, dr$$

$$= \rho_o \frac{r_e^{\,5} - (r_e+h')^5}{5} + \rho_c \frac{(r_e+h')^5 - [r_e-(T-t')]^5}{5} + \rho_m \frac{[r_e-(T-t')]^5 - (r_e-P)^5}{5} \qquad (4.1\text{-}2)$$

These integrals are determined for each block $k$ in a systematic computational manner. With the location of the block a search in the earth's mean elevation tape will immediately identify it as emerged or submerged, depending on the sign of h. Entering it into the proper equation (4.1-1) or (4.1-2) the integration with respect to r is easily evaluated.

To account for the earth's ellipticity in a reasonable way, the value of $r_e$ appearing in the above formulas is the geocentric radius of the earth, namely

$$r_e = a_e \left[ 1 - (f + \frac{3}{2} f^2) \cos^2\theta + \frac{3}{2} f^2 \cos^2\theta + \ldots \right] \qquad (4.1\text{-}3)$$

where

$$a_e = 6378140 \text{ m} \qquad \text{earth's equatorial radius} \qquad (4.1\text{-}4)$$

$$f = 1/298.257 \qquad \text{earth's flattening} \qquad (4.1\text{-}5)$$

are the current representative estimates recommended by the International Association of Geodesy (IAG) in the first resolution of the XVI$^{th}$ General Assembly [IAG, 1975].

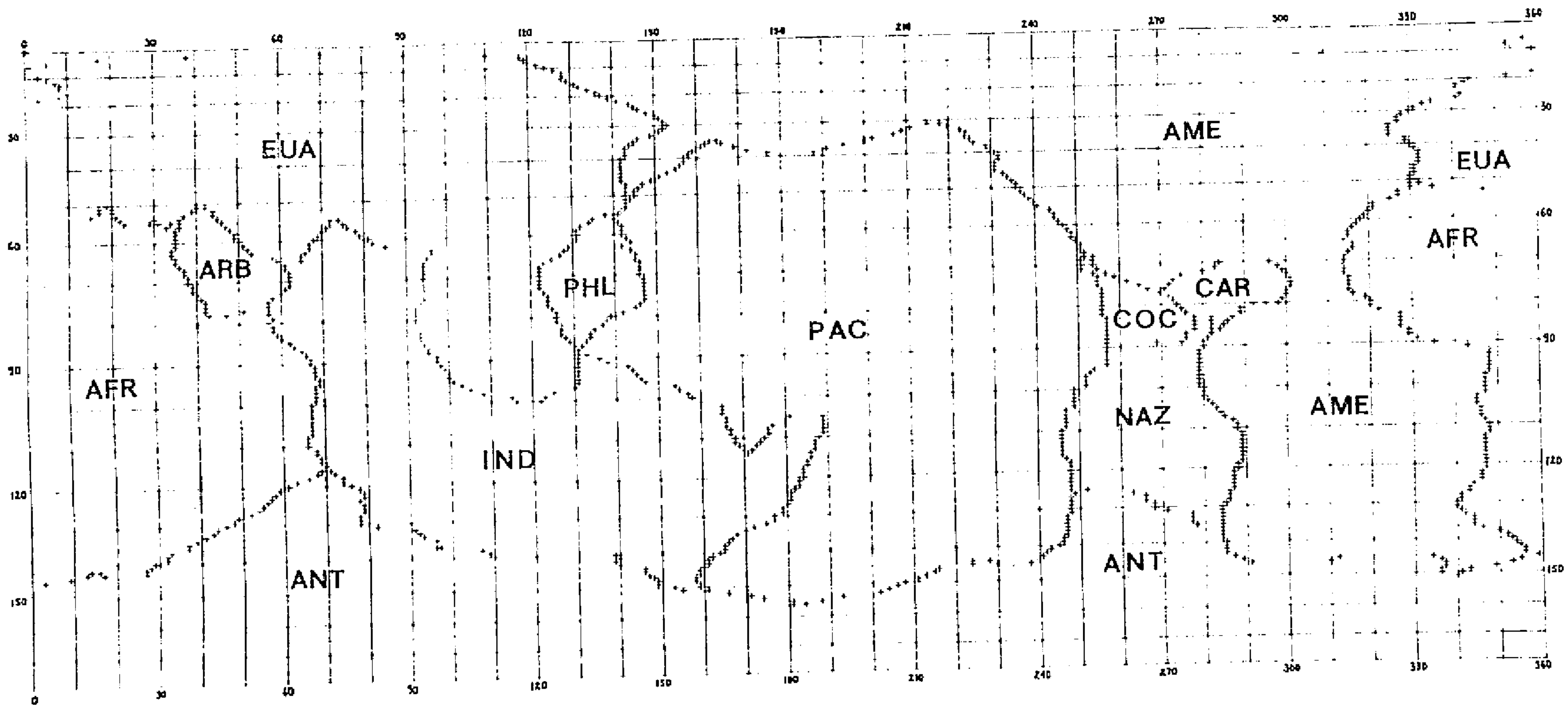
The use of the geocentric radius will introduce what may be considered a block deflection, i.e., each block k will be aligned with its central radius vector and therefore will not be normal to the earth ellipsoid. The error introduced is negligible considering that the maximum deflection at about 45° latitude for an ellipsoid the size of the earth is only 12′, clearly insignificant in the context of these computations.

### 4.1.2 Integration with respect to $\lambda$ and $\theta$

The integration with respect to $\lambda$ and $\theta$ requires the plate perimeter because evaluation must be done over individual tectonic plates.

It was mentioned in section 3.2 that the plate boundary selected was the one given in [Solomon and Sleep, 1974]. These investigators kindly

Fig. 4.1. Actual Plate Boundaries Used in Evaluating the Integrals



provided a string of data, punched on cards and containing latitude-longitude points of the boundary lines. This digitized data was not necessarily ordered in a continuous streamline; thus it was preprocessed to make it compatible with the information contained on the mean elevation tape.

Firstly, points at exact $1^\circ$ intervals (the nearest to the plate boundary lines) were selected. This set of points was still further reduced to evaluate the integration by colatitude belts from $\lambda = 0^\circ$ to $\lambda = 360^\circ$. Only a minimum number of points, all of them significant to the identification of each plate, were finally chosen. Fig. 4.1 presents the exact number of border points used in the integration process at their corresponding longitude-colatitude location.

Finally, the individual plate contribution to the initial tensor of inertia due to a differential motion of the crust was numerically implemented by using equation (2.6-1). The values $(\delta\lambda_k, \delta\theta_k)$ were computed for the center of each block from equations (3.3-1) which are dependent on the absolute angular plate velocity components.

Consequently each of the individual $1^\circ \times 1^\circ$ solid blocks forming the crustal model has independent differential motions $(\delta\lambda_k, \delta\theta_k)$ consistent with their plate absolute velocity.

## 4.2 Earth Tensor of Inertia $[I]_E$

It is well known that relationships exist between the earth moments and products of inertia and the satellite-derived potential coefficients. These may be derived by equating the expressions of the earth gravitational potential, as given by the spherical harmonic expansion, with the development of the same potential, as expressed by formulas of the MacCullagh type.

Thus one obtains

$$C'_{2,0} = \frac{1}{2}(A + B) - C \tag{4.2-1a}$$

$$C'_{2,2} = \frac{1}{2}(B - A) \tag{4.2-1b}$$

$$C'_{2,1} = E \tag{4.2-1c}$$

$$S'_{2,1} = D \tag{4.2-1d}$$

$$2\,S'_{2,2} = F \tag{4.2-1e}$$

Note that the potential coefficients as given above have the dimensions of the moments and products of inertia, that is, $ML^2$. Nevertheless, in practice they are usually in unitless dimensions and generally in a fully normalized form.

Hence the following known general transformations apply [Heiskanen and Moritz, 1967, p. 60].

$$\forall\, m \neq 0 : \begin{Bmatrix} C'_{nm} \\ \\ S'_{nm} \end{Bmatrix} = M_e a_e^{\,n} \begin{Bmatrix} C_{nm} \\ \\ S_{nm} \end{Bmatrix} = M_e a_e^{\,n} \sqrt{\frac{2(2n+1)(n-m)!}{(n+m)!}} \begin{Bmatrix} \bar{C}_{nm} \\ \\ \bar{S}_{nm} \end{Bmatrix} \tag{4.2-2a}$$

$$\forall\, m = 0 : \quad C'_{no} = M_e a_e^{\,n} \sqrt{2n+1} \quad \bar{C}_{no} \tag{4.2-2b}$$

where

$C'_{nm}$, $S'_{nm}$ $\equiv$ conventional spherical harmonic coefficients of degree $n$ and order $m$ (unit $L^n M$)

$C_{nm}$, $S_{nm}$ $\equiv$ unnormalized coefficients (unitless)

$\overline{C}_{nm}, \overline{S}_{nm}$ $\equiv$ fully normalized coefficients (unitless)

$M_e, a_e$ $\equiv$ mass and mean equatorial radius of the earth

From (4.2-1a) and (4.2-1b) one has

$$A = C'_{2,0} - 2C'_{2,2} + C \qquad (4.2\text{-}3a)$$

$$B = C'_{2,0} + 2C'_{2,2} + C \qquad (4.2\text{-}3b)$$

Thus finally

$$[\,I\,]_E = C[\,1\,] + \begin{bmatrix} C'_{2,0} - 2C'_{2,2} & -2S'_{2,2} & -C'_{2,1} \\ & C'_{2,0} + 2C'_{2,2} & -S'_{2,1} \\ s & & 0 \end{bmatrix} \qquad (4.2\text{-}4)$$

The value of the maximum moment of inertia may be determined under the assumption that the earth has rotational symmetry.

In this case

$$C'_{2,0} = A - C \qquad (4.2\text{-}5)$$

Knowing the dynamical ellipticity of the earth H, as expressed by equation (2.3-32) it follows that

$$C = -\frac{C'_{2,0}}{H} \qquad (4.2\text{-}6)$$

The value $a_e$ needed in equations (4.2-2) was given previously in (4.1-4).

The mass of the earth $M_e$ is obtained from the current representative estimates of $GM_e$ and $G$ as recommended by the International Association of Geodesy [IAG, 1975]:

$$GM_e = 3.986005 \times 10^{14}\ m^3 s^{-2}$$

$$G = 6.672 \times 10^{-11}\, m^3 s^{-2} kg^{-1}$$

resulting in

$$M_e = 5974.228118 \times 10^{30} g$$

The set of fully normalized potential coefficients used were those of the Smithsonian Astrophysical Observatory (SAO), Standard Earth III (denoted SE3) as given in [Gaposchkin, 1974], namely,

$$\bar{C}_{2,0} = -4.84170 \times 10^{-4}$$

$$\bar{C}_{2,2} = 2.3799 \times 10^{-6}$$

$$\bar{S}_{2,2} = -1.3656 \times 10^{-6}$$

The selection of the SE3 coefficients is such that

(i) $\bar{C}_{1,0} = \bar{C}_{1,1} = \bar{S}_{1,1} = 0$

(ii) $\bar{C}_{2,1} = \bar{S}_{2,1} = 0 \implies D = E = 0$

Thus the tensor of inertia $[\,I\,]_E$ will refer to a central system, the third axis of which is principal.

After the selected values are substituted in the general equations (4.2-2a and b), one has

$$C_{2,0} = -1082.637 \times 10^{-6}$$

$$C_{2,2} = 1.5362188 \times 10^{-6}$$

$$S_{2,2} = -0.88149101 \times 10^{-6}$$

$$C'_{2,0} = -0.26311809 \times 10^{43} g\, cm^2$$

$$C'_{2,2} = 0.37335410 \times 10^{40} g\, cm^2$$

$$S'_{2,2} = -0.21423268 \times 10^{40} g\, cm^2$$

and

$$C = 8.041506418 \times 10^{44}\, g\, cm^2$$

Thus, finally, recalling equation (4.2-4) the current estimate of the earth tensor of inertia $[\,I\,]_E$, as derived from satellite values is given by

$$[\,I\,]_E = \begin{bmatrix} 8.015119938 \times 10^{44} & 0.428465360 \times 10^{40} & 0 \\ & 8.015269280 \times 10^{44} & 0 \\ s & & 8.041506418 \times 10^{44} \end{bmatrix} \tag{4.2-7}$$

where all the units are expressed in $g\,cm^2$.

As a consequence of the corollary of postulate 1 discussed in section 2.3.1, it will be proper to assume that the computed center of the earth wobble (barycenter) practically coincides with the earth axis of figure, to which $[\,I\,]_E$ refers.

Thus one may transform the second-rank tensor of inertia $[\,I\,]_E$ to another system with the third axis coinciding with the CIO terrestrial axis and the two other axes in the terrestrial equatorial plane.

Making use of the tensor transformation equation (B.3-7) one may write

$$[\,\bar{I}\,]_E = R[\,I\,]_E R^T \tag{4.2-8}$$

where now the rotation angles being small, the value of $R$ is given by

$$R = R_2(-x_B)\, R_1(-y_B) \approx \begin{bmatrix} 1 & 0 & x_B \\ 0 & 1 & -y_B \\ -x_B & y_B & 1 \end{bmatrix} \tag{4.2-9}$$

and $(x_B, y_B)$ are the coordinates of the barycenter in the CIO polar system. These values may be obtained graphically from Fig. 2.7.

For the year 1970 one has

$$x_B \approx 0''.028 \tag{4.2-10a}$$

$$y_B \approx 0''.246 \tag{4.2-10b}$$

After transforming them to radians and substituting in (4.2-9), equation (4.2-8) finally gives the tensor of inertia with respect to the CIO system as (units in $g\,cm^2$)

$$[\bar{I}]_E = \begin{bmatrix} 8.015119938 \times 10^{44} & 0.4284653600 \times 10^{40} & 0.3633007984 \times 10^{36} \\ & 8.01526928 \times 10^{44} & -0.3129732002 \times 10^{37} \\ s & & 8.041506418 \times 10^{44} \end{bmatrix} \tag{4.2-11}$$

If the matrix $[\bar{I}]_E$ is now diagonalized, the principal moments of inertia of the earth and the direction of the central principal axis with respect to the CIO terrestrial system may be determined.

After diagonalization, using the eigentheory formalism as explained in section B.4.2 of Appendix B, one obtains

$$A_{op} = 8.015108518 \times 10^{44}\,g\,cm^2 \tag{4.2-12a}$$

$$B_{op} = 8.015280700 \times 10^{44}\,g\,cm^2 \tag{4.2-12b}$$

$$C_{op} = 8.041506418 \times 10^{44}\,g\,cm^2 \tag{4.2-12c}$$

The spherical coordinates of the positive direction of the central principal axes are given by

$$x_{op1} \begin{cases} \lambda_{p1} = -14°55'25''.2955 & (4.2\text{-}13a) \\ \varphi_{p1} = -\,0°\ 0'\ 0''.0904 & (4.2\text{-}13b) \end{cases}$$

$$x_{op2} \begin{cases} \lambda_{p2} = 75°\ 4'34''.7044 & (4.2\text{-}13c) \\ \varphi_{p2} = 0°\ 0'\ 0''.2305 & (4.2\text{-}13d) \end{cases}$$

$$x_{op3} \begin{cases} \lambda_{p3} = -83°30'23''.3089 & (4.2\text{-}13e) \\ \varphi_{p3} = 89°59'59''.7524 & (4.2\text{-}13f) \end{cases}$$

The direction of $x_{op3}$ was expected, because it corresponds to the used values of (4.2-10).

A major result is the direction of the earth's first principal axis which, according to the above values, is practically on the conventional terrestrial equator, but about $15°$ west of the meridian of zero longitude.

Therefore the earth's central momental ellipsoid will have its greatest axis along $x_{op1}$ and the semimajor axis of a triaxial earth lies $14°.92$ W of the geographic zero meridian. This agrees very well with the finding by Burša [1970] that the longitude of the meridian in which $a_a$ is situated for a best-fitting triaxial ellipsoid is $14°.8$ W.

The transformation of coordinates between the central principal and the terrestrial systems may be implemented as follows:

$$\{x_{op}\} \xrightarrow{R} \{x\} \qquad \text{or} \qquad \{x\} = R\{x_{op}\} \tag{4.2-14}$$

where

$$R = R_3(-\lambda_{p1})\,R_2(-x_B)\,R_1(-y_B) \tag{4.2-15}$$

with the meaning of the symbols $\lambda_{p1}$, $x_B$ and $y_B$ as mentioned above.

## 4.3 Crustal Tensor of Inertia $[I]_c$

In this section the results concerning the evaluation of the crustal tensor of inertia according to the model described in section 3.1 are reported.

The numerical integration of the matrix elements in equation (2.5-23) was performed following the formalism of section 4.1.

The results are presented in table 4.1. Tabulated are the individual tensors of inertia for each plate, as well as the corresponding total tensor of inertia for the whole crust up to 50 km depth.

Table 4.1

Tensor of Inertia for Each Individual Plate and the Total Crust (50 Km Deep)

| PLATE NAME | NO | TENSOR OF INERTIA [ I ]$_{P i}$ | | |
|---|---|---|---|---|
| AFRICAN | 1 | 0.1549375040D+43 | -.5746112666D+42 | 0.1434525736D+42 |
| | | | 0.3659923927D+43 | 0.2612359781D+42 |
| | | | | 0.3867953054D+43 |
| AMERICAN | 2 | 0.4566997943D+43 | 0.9976786627D+42 | 0.4988294053D+42 |
| | | | 0.3779194901D+43 | 0.4785378215D+42 |
| | | | | 0.3395438508D+43 |
| ANTARTIC | 3 | 0.3210393813D+43 | -.1098232871D+42 | 0.1450220562D+42 |
| | | | 0.2846084920D+43 | 0.1623310808D+42 |
| | | | | 0.8904670153D+42 |
| ARABIAN | 4 | 0.1779691548D+42 | -.1181730137D+42 | -.7282012894D+41 |
| | | | 0.1646764506D+42 | -.7472999848D+41 |
| | | | | 0.2432130936D+42 |
| CARIBBEAN | 5 | 0.1953842879D+42 | 0.5040995506D+41 | -.1326980322D+41 |
| | | | 0.2981484370D+41 | 0.4572475656D+41 |
| | | | | 0.2004365514D+42 |
| COCOS | 6 | 0.1941247325D+42 | -.1803535162D+41 | 0.3701302487D+40 |
| | | | 0.9776718418D+40 | 0.2837705655D+41 |
| | | | | 0.1924269429D+42 |
| EURASIAN | 7 | 0.3306476155D+43 | 0.2195409745D+42 | -.3817562569D+42 |
| | | | 0.2544419075D+43 | -.9673486707D+42 |
| | | | | 0.2067458743D+43 |
| INDIAN | 8 | 0.2735879769D+43 | 0.3761912903D+42 | -.5571617214D+42 |
| | | | 0.1626619564D+43 | 0.4907379365D+42 |
| | | | | 0.2958911977D+43 |
| NAZCA | 9 | 0.8188225755D+42 | -.4107306785D+41 | -.1381307608D+41 |
| | | | 0.1262283106D+42 | -.2244825863D+42 |
| | | | | 0.7465976881D+42 |
| PACIFIC | 10 | 0.2816670441D+43 | -.9297334383D+42 | 0.1782165703D+42 |
| | | | 0.4787631572D+43 | -.1343965475D+42 |
| | | | | 0.4950205370D+43 |
| PHILIPPINE | 11 | 0.1930333366D+42 | 0.1459722513D+42 | 0.6953150710D+41 |
| | | | 0.1835853592D+42 | -.7057441923D+41 |
| | | | | 0.2985204116D+42 |
| CRUST ≡ [I]$_c$ | | 0.1976512725D+44 | -.1656291368D+40 | -.6757155298D+38 |
| | | | 0.1975795564D+44 | -.4587592137D+40 |
| | | | | 0.1981162935D+44 |

NOTE : ALL TENSORS ARE SYMMETRIC AND REFER TO THE TERRESTRIAL SYSTEM. THE UNIT OF EACH ELEMENT IS g cm$^2$

Table 4.2

Principal Moments and Principal Axis Directions for Each Individual Plate and the Total Crust (50 Km Deep)

| PLATE NAME | NO | PRINCIPAL MOMENTS [I] | LONGITUDE | | | LATITUDE | | |
|---|---|---|---|---|---|---|---|---|
| AFRICAN | 1 | 0.4066027917D+43 | 99° | 4' | 32".791 | 49° | 54' | 49".976 |
| | | 0.3624965086D+43 | 108 | 38 | 36.258 | - 39 | 41 | 28.337 |
| | | 0.1386259018D+43 | 14 | 42 | 37.909 | - 4 | 43 | 27.103 |
| AMERICAN | 2 | 0.5470242709D+43 | 35 | 8 | 38.160 | 18 | 12 | 47.466 |
| | | 0.3250711257D+43 | 148 | 15 | 58.813 | 50 | 0 | 46.551 |
| | | 0.3020677385D+43 | 112 | 14 | 7.494 | - 34 | 8 | 52.129 |
| ANTARTIC | 3 | 0.3245025694D+43 | - 14 | 26 | 13.690 | 2 | 25 | 52.393 |
| | | 0.2834918709D+43 | 75 | 48 | 15.293 | 5 | 39 | 59.893 |
| | | 0.8670013455D+42 | -127 | 33 | 56.323 | 83 | 49 | 50.492 |
| ARABIAN | 4 | 0.2909396154D+42 | - 24 | 10 | 6.398 | - 36 | 54 | 19.474 |
| | | 0.2880189237D+42 | - 68 | 25 | 53.872 | 43 | 38 | 24.422 |
| | | 0.6900159962D+40 | 46 | 28 | 5.315 | 23 | 49 | 18.628 |
| CARIBBEAN | 5 | 0.2120511778D+42 | 140 | 33 | 40.178 | 73 | 32 | 1.283 |
| | | 0.2094564937D+42 | 17 | 54 | 51.556 | 9 | 3 | 34.405 |
| | | 0.4128011549D+40 | 105 | 41 | 53.187 | - 13 | 37 | 56.729 |
| COCOS | 6 | 0.1973532359D+42 | 7 | 45 | 28.528 | 56 | 41 | 40.086 |
| | | 0.1952809234D+42 | - 10 | 59 | 19.090 | - 31 | 53 | 17.655 |
| | | 0.3694234556D+40 | 84 | 25 | 18.234 | - 8 | 37 | 3.378 |
| EURASIAN | 7 | 0.3718348517D+43 | 36 | 7 | 56.223 | - 28 | 1 | 30.893 |
| | | 0.2904520938D+43 | 138 | 54 | 56.428 | - 22 | 34 | 21.285 |
| | | 0.1295484519D+43 | 81 | 54 | 56.202 | 52 | 38 | 49.527 |
| INDIAN | 8 | 0.3428986238D+43 | 170 | 57 | 2.616 | 53 | 9 | 30.414 |
| | | 0.2631121982D+43 | 34 | 50 | 4.264 | 28 | 22 | 9.125 |
| | | 0.1261303090D+43 | 112 | 35 | 0.746 | - 21 | 27 | 8.431 |
| NAZCA | 9 | 0.8213811173D+42 | 1 | 3 | 0.632 | - 13 | 26 | 44.779 |
| | | 0.8191929039D+42 | - 54 | 0 | 4.822 | 67 | 17 | 38.838 |
| | | 0.5107455311D+41 | 86 | 36 | 17.422 | 17 | 55 | 9.484 |
| PACIFIC | 10 | 0.5271775552D+43 | 113 | 4 | 57.369 | - 31 | 2 | 16.230 |
| | | 0.4840809704D+43 | 107 | 4 | 19.431 | 58 | 49 | 20.799 |
| | | 0.2441922627D+43 | 21 | 28 | 45.288 | - 2 | 39 | 43.845 |
| PHILIPPINE | 11 | 0.3346500451D+42 | 37 | 32 | 53.499 | 18 | 32 | 26.590 |
| | | 0.3320613880D+42 | - 93 | 42 | 4.642 | 63 | 2 | 20.187 |
| | | 0.8427674186D+40 | 134 | 7 | 30.534 | 18 | 51 | 19.971 |
| CRUST | | 0.1981201875D+44 | - 88 | 56 | 55.582 | 85 | 8 | 46.274 |
| | | 0.1976547396D+44 | - 11 | 46 | 47.845 | - 1 | 4 | 49.400 |
| | | 0.1975721953D+44 | 78 | 7 | 50.198 | 4 | 43 | 53.340 |

NOTE : THE UNIT OF THE PRINCIPAL MOMENTS IS g cm²

EAST LONGITUDES ARE CONSIDERED POSITIVE

One should notice that as a consequence of the way the integration with respect to $\lambda$ and $\theta$ was determined, the values of these tensors refer to the CIO terrestrial system.

Following the eigentheory treatment of section B. 4. 2 in Appendix B, the symmetric matrices in table 4.1 can be diagonalized. The resulting eigenvalues (principal moments of inertia) and the polar coordinates $(\lambda, \omega)$ of the corresponding eigenvectors (i.e., directions of the principal axes) with respect to the terrestrial system are shown in table 4.2.

Comparing the results for the crust at the end of table 4.2 with the values for the earth given in (4.2-12) and (4.2-13), the following conclusions can be drawn:

1) The principal moments of inertia of the earth crustal layer (50 km depth) represent only 2% of the principal moments of the earth;

2) The axis of maximum moment of inertia of the modeled crust is situated about $5°$ from the CIO pole in the direction of $\lambda \approx -89°$. Recall that the earth axis of figure was situated in the direction $\lambda \approx -83°.5$ according to the plotted barycenter (epoch 1970.0) as determined from astronomical observations processed at the IPMS.

3) The crust has a principal axis of inertia near the equatorial plane but at $\lambda \approx -11°$. As known from satellite data, the earth has a principal axis in the same plane and at $\lambda \approx -15°$.

A major disagreement with previously reported values appears to be in the above results. Milankovich in 1941 [see Scheidegger, 1963, p.179] and Munk [1958] as well as other investigators, have found the pole of the earth's "continental-ocean" distribution near Hawaii.

Before interpreting the results of the pole's location, published to date, the reader should be aware of the major differences between the assumptions inherent in the various solutions. These are as follows:

(i) Milankovich used a $20^\circ \times 20^\circ$ grid, while Munk employed a $10^\circ \times 10^\circ$ grid in the integration between continental boundaries.

(ii) Their model was much more simplified. Milankovich's work was restricted to standard continental coastlines. He weighted the areal mass density for continents differently as compared with oceans, without rigorous evaluation of the integrals. Munk included in his calculations the continental structure up to 1000 fathoms depth. He used a mean elevation of 0.9 km for land masses after postulating global isostatic compensation.

(iii) Both investigators assumed a spherical earth.

After the discrepancies between the different results were noticed, an attempt was undertaken to compare them and new evaluations of the integrals were made under various modeling hypotheses:

(a) Crust-M: only the standard crust is considered. (This includes the crustal masses contained above the Moho discontinuity and therefore excludes the layer of tectosphere down to 50 km depth).

(b) Crust-M-S: as above for a spherical earth.

(c) Crust-S: this case uses the crustal model selected originally in this investigation, but assumes the earth to be spherical.

The results concerning case (a) are presented in table 4.3 and clearly show agreement with previous findings by Milankovich and Munk [1958].

The maximum moment of inertia of the standard crust is about an



C'2

Table 4.3

Principal Moments and Principal Axis Directions for Each Individual Plate and the Total Crust (Including Only Down to the Moho Discontinuity)

| PLATE NAME | NO | PRINCIPAL MOMENTS [I] | DIRECTION OF PRINCIPAL AXES LONGITUDE ° | ′ | ″ | LATITUDE ° | ′ | ″ |
|---|---|---|---|---|---|---|---|---|
| AFRICAN | 1 | 0.1884897529D+43 | 104° | 0′ | 57″.391 | 45° | 59′ | 48″.644 |
| | | 0.1709100992D+43 | 108 | 20 | 24.359 | - 43 | 55 | 17.557 |
| | | 0.5838137625D+42 | 16 | 15 | 22.871 | - 2 | 9 | 43.920 |
| AMERICAN | 2 | 0.2698575420D+43 | 31 | 36 | 10.913 | 18 | 16 | 59.420 |
| | | 0.1645243004D+43 | 134 | 41 | 42.964 | 34 | 26 | 5.250 |
| | | 0.1444870314D+43 | - 81 | 20 | 2.375 | 49 | 42 | 34.877 |
| ANTARTIC | 3 | 0.1445811721D+43 | - 13 | 9 | 38.739 | 1 | 50 | 41.660 |
| | | 0.1287230115D+43 | 77 | 1 | 13.611 | 5 | 36 | 27.777 |
| | | 0.3468353969D+42 | -121 | 18 | 19.767 | 84 | 5 | 41.094 |
| ARABIAN | 4 | 0.1644522216D+42 | - 25 | 42 | 23.995 | - 35 | 2 | 18.218 |
| | | 0.1628719672D+42 | - 70 | 41 | 16.314 | 45 | 14 | 57.970 |
| | | 0.3767488542D+40 | 46 | 3 | 40.570 | 24 | 2 | 44.226 |
| CARIBBEAN | 5 | 0.1035760963D+42 | 137 | 26 | 18.805 | 74 | 23 | 46.013 |
| | | 0.1023242711D+42 | 17 | 10 | 42.964 | 8 | 0 | 39.266 |
| | | 0.2064343915D+40 | 105 | 16 | 19.031 | - 13 | 18 | 3.898 |
| COCOS | 6 | 0.8006675185D+41 | 7 | 44 | 46.306 | 56 | 49 | 34.903 |
| | | 0.7919275975D+41 | - 10 | 44 | 57.628 | - 31 | 47 | 49.946 |
| | | 0.1531613895D+40 | 84 | 33 | 23.924 | - 8 | 29 | 1.797 |
| EURASIAN | 7 | 0.2102051603D+43 | 36 | 5 | 31.056 | - 28 | 37 | 6.971 |
| | | 0.1695553623D+43 | - 40 | 2 | 49.734 | 23 | 42 | 14.783 |
| | | 0.6818762794D+42 | 83 | 10 | 35.473 | 51 | 17 | 37.752 |
| INDIAN | 8 | 0.1461500196D+43 | 174 | 29 | 12.996 | 53 | 44 | 45.420 |
| | | 0.1115313725D+43 | 38 | 20 | 50.218 | 27 | 52 | 5.160 |
| | | 0.5436504889D+42 | 116 | 29 | 20.108 | - 21 | 14 | 15.776 |
| NAZCA | 9 | 0.3116064944D+42 | 3 | 3 | 4.790 | - 19 | 43 | 6.794 |
| | | 0.3101563837D+42 | - 42 | 25 | 23.558 | 62 | 55 | 37.535 |
| | | 0.1982022846D+41 | 86 | 26 | 57.385 | 17 | 47 | 6.588 |
| PACIFIC | 10 | 0.1809966659D+43 | 113 | 1 | 27.209 | - 30 | 43 | 32.974 |
| | | 0.1687582251D+43 | 111 | 35 | 2.580 | 59 | 15 | 58.402 |
| | | 0.8669457759D+42 | 22 | 38 | 53.497 | - 0 | 37 | 57.464 |
| PHILIPPINE | 11 | 0.1162143837D+42 | 37 | 45 | 58.055 | 18 | 28 | 31.679 |
| | | 0.1152613601D+42 | - 92 | 47 | 9.307 | 62 | 48 | 1.642 |
| | | 0.3036507662D+40 | 134 | 27 | 36.319 | 19 | 13 | 57.492 |
| CRUST-M | | 0.9166113773D+43 | 9 | 20 | 11.869 | - 14 | 45 | 42.450 |
| | | 0.8860700313D+43 | 105 | 38 | 22.271 | - 22 | 37 | 7.216 |
| | | 0.8559947643D+43 | 68 | 53 | 22.897 | 62 | 31 | 34.217 |

NOTE : THE UNIT OF THE PRINCIPAL MOMENTS IS g cm$^2$

EAST LONGITUDES ARE CONSIDERED POSITIVE

axis with polar coordinates,

$$\begin{cases} \lambda = \phantom{-1}9^\circ 20' 11''.869 \\ \varphi = -14^\circ 45' 42''.450 \end{cases} \quad \text{or} \quad \begin{cases} \lambda = -170^\circ 39' 48''.131 \\ \varphi = \phantom{-}14^\circ 45' 42''.450 \end{cases}$$

This corresponds to an area west of the Hawaiian archipelago, close to Wake Island. The minimum moment of inertia is about an axis passing through the northern part of Russia near the city of Gorki.

The magnitude of the principal moments in this case represents only 1% of the total earth.

Case (b) gives basically the same results, as can be seen by inspecting table 4.4. Consequently when only the standard earth crust is integrated, the effect of the earth's ellipticity may be considered negligible. This implies that the asymmetric continental masses (dominant in this case) are distributed over the earth in such a way that their principal axes are not affected even when the effect of the equatorial bulge is considered. In general terms, this means that land masses are situated primarily north and south of the present equator, in a sense contradicting the "Polfluchtkraft" theory [see Scheidegger, 1963, p. 166] which postulates that continents as floating masses on the mantle, will tend to drift to the equator. The above results enforce the theory that continents are not just floating masses on the mantle, but inhomogeneities forming part of lithospheric plates.

The last part of this analysis concerns case (c) which introduces drastic changes with respect to the original results of table 4.2 where the ellipticity of the earth is taken into consideration.

Using the crust-tectosphere model as defined in section 3.1 but under the assumption of spherical earth, the maximum principal moment of inertia axis is no longer close to the CIO pole, but at the point with coordinates $\lambda = -18^\circ$ and $\varphi = 10^\circ$ as presented in table 4.5.

Consequently the bulge of the earth becomes an important factor when

Table 4.4

Principal Moments and Principal Axis Directions for Each Individua'l Plate and the Total Crust (Including only Down to the Moho Discontinuity) under the Assumption of Sphericity

| PLATE NAME | NO | PRINCIPAL MOMENTS [I] | LONGITUDE ° | ′ | ″ | LATITUDE ° | ′ | ″ |
|---|---|---|---|---|---|---|---|---|
| AFRICAN | 1 | 0.1879774250D+43 | 104 | 0 | 0.078 | 45 | 53 | 7.328 |
| | | 0.1704436039D+43 | 108 | 21 | 37.317 | - 44 | 1 | 53.890 |
| | | 0.5827028318D+42 | 16 | 15 | 2.708 | - 2 | 10 | 50.195 |
| AMERICAN | 2 | 0.2702458323D+43 | 31 | 36 | 16.348 | 18 | 14 | 40.570 |
| | | 0.1649227441D+43 | 134 | 13 | 26.209 | 33 | 32 | 4.287 |
| | | 0.1445090564D+43 | - 82 | 2 | 21.307 | 50 | 34 | 52.306 |
| ANTARTIC | 3 | 0.1454337697D+43 | - 13 | 8 | 48.634 | 1 | 50 | 27.806 |
| | | 0.1295372029D+43 | 77 | 1 | 59.835 | 5 | 35 | 10.133 |
| | | 0.3482772750D+42 | -121 | 19 | 19.188 | 84 | 6 | 59.230 |
| ARABIAN | 4 | 0.1640976624D+42 | - 25 | 42 | 44.710 | - 35 | 1 | 31.810 |
| | | 0.1625194285D+42 | - 70 | 42 | 27.783 | 45 | 15 | 21.969 |
| | | 0.3760111855D+40 | 46 | 3 | 29.033 | 24 | 3 | 10.598 |
| CARIBBEAN | 5 | 0.1031897742D+42 | 137 | 26 | 19.809 | 74 | 23 | 40.352 |
| | | 0.1019425571D+42 | 17 | 10 | 46.538 | 8 | 0 | 41.673 |
| | | 0.2056696970D+40 | 105 | 16 | 21.271 | - 13 | 18 | 8.965 |
| COCOS | 6 | 0.7973855473D+41 | 7 | 44 | 35.641 | 56 | 49 | 2.908 |
| | | 0.7886779705D+41 | - 10 | 45 | 13.421 | - 31 | 48 | 20.462 |
| | | 0.1525655971D+40 | 84 | 33 | 18.766 | - 8 | 29 | 8.561 |
| EURASIAN | 7 | 0.2106100667D+43 | 36 | 7 | 45.372 | - 28 | 35 | 28.608 |
| | | 0.1699685487D+43 | - 40 | 5 | 45.455 | 23 | 35 | 57.552 |
| | | 0.6824210458D+42 | 83 | 4 | 42.967 | 51 | 23 | 46.703 |
| INDIAN | 8 | 0.1458605405D+43 | 174 | 29 | 20.074 | 53 | 42 | 37.601 |
| | | 0.1113232652D+43 | 38 | 25 | 3.663 | 27 | 52 | 17.946 |
| | | 0.5425831677D+42 | 116 | 31 | 44.531 | - 21 | 17 | 0.550 |
| NAZCA | 9 | 0.3106990178D+42 | 3 | 3 | 1.574 | - 19 | 42 | 4.544 |
| | | 0.3092558780D+42 | - 42 | 28 | 6.508 | 62 | 55 | 51.125 |
| | | 0.1976431076D+41 | 86 | 26 | 57.369 | 17 | 47 | 55.591 |
| PACIFIC | 10 | 0.1807041555D+43 | 113 | 0 | 34.471 | - 30 | 21 | 9.043 |
| | | 0.1683952130D+43 | 111 | 35 | 54.136 | 59 | 38 | 23.671 |
| | | 0.8664118653D+42 | 22 | 38 | 56.977 | - 0 | 36 | 55.630 |
| PHILIPPINE | 11 | 0.1158815520D+42 | 37 | 46 | 10.956 | 18 | 27 | 43.117 |
| | | 0.1149305885D+42 | - 92 | 45 | 8.279 | 62 | 48 | 15.946 |
| | | 0.3028105098D+40 | 134 | 27 | 40.898 | 19 | 14 | 25.706 |
| CRUST-M-S | | 0.9174892481D+43 | 9 | 5 | 55.735 | - 14 | 5 | 30.808 |
| | | 0.8867746236D+43 | 104 | 41 | 38.107 | - 21 | 13 | 32.028 |
| | | 0.8550329394D+43 | 67 | 55 | 9.757 | 64 | 7 | 55.817 |

NOTE : THE UNIT OF THE PRINCIPAL MOMENTS IS g cm$^2$

EAST LONGITUDES ARE CONSIDERED POSITIVE

Table 4.5

Principal Moments and Principal Axis Directions for Each Individual Plate and the Total Crust (50 Km Deep) under the Assumption of Sphericity

| PLATE NAME | NO | PRINCIPAL MOMENTS [I] | LONGITUDE (°) | (′) | (″) | LATITUDE (°) | (′) | (″) |
|---|---|---|---|---|---|---|---|---|
| AFRICAN | 1 | 0.4055392814D+43 | 99° | 3′ | 42″.091 | 49° | 48′ | 30″.093 |
| | | 0.3615634658D+43 | 108 | 40 | 32.084 | − 39 | 47 | 33.449 |
| | | 0.1383570597D+43 | 14 | 42 | 23.688 | − 4 | 45 | 0.949 |
| AMERICAN | 2 | 0.5477107520D+43 | 35 | 8 | 42.866 | 18 | 11 | 25.594 |
| | | 0.3253321597D+43 | 146 | 51 | 2.959 | 48 | 22 | 44.691 |
| | | 0.3025351968D+43 | 111 | 23 | 18.819 | − 35 | 53 | 32.618 |
| ANTARTIC | 3 | 0.3263064242D+43 | − 14 | 25 | 38.867 | 2 | 25 | 35.103 |
| | | 0.2852009162D+43 | 75 | 48 | 44.493 | 5 | 38 | 28.667 |
| | | 0.8704827588D+42 | −127 | 36 | 30.814 | 83 | 51 | 21.235 |
| ARABIAN | 4 | 0.2903000348D+42 | − 24 | 10 | 31.322 | − 36 | 53 | 27.587 |
| | | 0.2873832037D+42 | − 68 | 27 | 7.456 | 43 | 38 | 54.704 |
| | | 0.6886435458D+40 | 46 | 27 | 53.174 | 23 | 49 | 45.119 |
| CARIBBEAN | 5 | 0.2112656225D+42 | 140 | 33 | 36.446 | 73 | 31 | 56.705 |
| | | 0.2086804883D+42 | 17 | 54 | 55.175 | 9 | 3 | 35.239 |
| | | 0.4112725850D+40 | 105 | 41 | 55.776 | − 13 | 38 | 1.595 |
| COCOS | 6 | 0.1965443738D+42 | 7 | 45 | 17.981 | 56 | 41 | 8.625 |
| | | 0.1944797445D+42 | − 10 | 59 | 34.703 | − 31 | 53 | 47.635 |
| | | 0.3679819167D+40 | 84 | 25 | 13.232 | − 8 | 37 | 10.070 |
| EURASIAN | 7 | 0.3725586922D+43 | 36 | 10 | 13.375 | − 27 | 59 | 50.431 |
| | | 0.2912031685D+43 | 138 | 51 | 39.554 | − 22 | 27 | 4.305 |
| | | 0.1296281111D+43 | 81 | 47 | 25.756 | 52 | 45 | 36.585 |
| INDIAN | 8 | 0.3422314728D+43 | 170 | 57 | 16.565 | 53 | 7 | 18.844 |
| | | 0.2626231903D+43 | 34 | 54 | 46.947 | 28 | 22 | 15.535 |
| | | 0.1258933699D+43 | 112 | 37 | 46.832 | − 21 | 30 | 5.907 |
| NAZCA | 9 | 0.8189878708D+42 | 1 | 2 | 24.065 | − 13 | 26 | 17.998 |
| | | 0.8168119743D+42 | − 54 | 6 | 27.613 | 67 | 18 | 33.010 |
| | | 0.5093077916D+41 | 86 | 36 | 18.585 | 17 | 55 | 57.085 |
| PACIFIC | 10 | 0.5263182494D+43 | 113 | 3 | 34.421 | − 30 | 44 | 57.404 |
| | | 0.4830337591D+43 | 107 | 1 | 57.006 | 59 | 6 | 39.560 |
| | | 0.2440444286D+43 | 21 | 28 | 42.941 | − 2 | 39 | 18.730 |
| PHILIPPINE | 11 | 0.3336701172D+42 | 37 | 33 | 4.049 | 18 | 31 | 43.843 |
| | | 0.3310871738D+42 | − 93 | 40 | 15.049 | 63 | 2 | 31.840 |
| | | 0.8404042012D+40 | 134 | 7 | 34.805 | 18 | 51 | 46.694 |
| CRUST−S | | 0.1978337131D+44 | − 18 | 22 | 30.683 | 10 | 39 | 37.666 |
| | | 0.1978001689D+44 | −119 | 17 | 35.909 | 45 | 10 | 38.267 |
| | | 0.1977111594D+44 | 81 | 41 | 4.670 | 42 | 51 | 36.037 |

NOTE : THE UNIT OF THE PRINCIPAL MOMENTS IS $g\ cm^2$

EAST LONGITUDES ARE CONSIDERED POSITIVE

the masses below the oceanic standard crust are considered.

Comparing the results pertaining to the crust in tables 4.2 and 4.5, one actually may conclude that at the present the more dense inhomogeneities of the earth's crustal layer are situated around the equatorial belt. Therefore, the role of the continents as sole agents in the phenomenon of balancing the earth's axis of figure lacks consistency. A better intuitive picture is attained when the whole plate tectonic structure is considered. The layer of higher density below the oceanic plates makes all the difference. These masses were systemmatically neglected in previous investigations.

Hence the present ellipsoidal earth seems to be quasi-dynamically balanced, as the distribution of plate masses proves, with the large dense ones (primarily oceanic) occupying the areas around the equatorial bulge. Thus the assumption of polar wandering on a large scale in recent geological history seems unlikely, primarily because the mechanism to explain it is not yet available. Nevertheless, a possibility of random secular polar motion due, for example, to small lithospheric motions and other redistribution of matter cannot be ruled out completely.

Further examination of tables 4.2 to 4.5 also shows that the location of the principal moments of inertia axes of each individual plate remains practically unchanged, corroborating the fact that ellipticity is not as important as mass distribution.

Small plates hardly change the direction of the principal axes in the case of spherical or ellipsoidal earth. The macroplate change (a few degrees only) mostly reflects their assymmetry with respect to the equator.

A final remark should be made here: If the plates as units are assumed to rotate without any friction or external forces, the dynamical possibility of rotation about their axes of figure always exists. Then the individual pole of rotation will correspond to the individual maximum principal moment of inertia axes.

Although the location of the pole of rotation for the absolute angular velocity of the Pacific plate according to table 4.2 agrees very well in longitude with the values found by Solomon et al. [1975], for the different models (see table 3.1), there is a significant unexplained disagreement in latitude (about 30°).

## 4.4 Contribution of Differential Plate Rotations $[\Delta I]_P$ to the Earth Tensor of Inertia

The contribution $[\Delta I]_P$ to the earth tensor of inertia $[\bar{I}]_E$ as a consequence of differential plate rotations was computed according to the theory presented in Chapter 2 and the numerical approach described in section 4.1.

In synthesis it will involve the double summation

$$[\Delta I]_P = \sum_{i=1}^{n} \sum_{k=1}^{m} (\delta\lambda_k [\Delta I_\lambda]_k + \delta\theta_k [\Delta I_\theta]_k) \quad (4.4\text{-}1)$$

where

$n$ ≡ number of plates (eleven)

$m$ ≡ number of $1° \times 1°$ blocks in any particular plate $P_i$

$[\Delta I_\lambda]_k$ : given by equation (2.5-13)

$[\Delta I_\theta]_k$ : given by equation (2.5-14)

$(\delta\lambda, \delta\theta)_k$ : given by equation (3.3-1). It depends on the absolute plate velocity model.

Since the expression in (4.4-1) is dependent on the plate's absolute velocity model, several choices are available from the set of eight models characterized in table 3.1 and given for other than the Pacific plate in table 3.2.

Table 4.6

Contributions to the Differential Rotation Tensor $[\Delta I_{ROT}]$ as a Consequence of Absolute Plate Rotations (Model B4) for Each Plate and the Total Crust

| PLATE | | MODEL B4 | | |
|---|---|---|---|---|
| NAME | NO | $\Sigma\delta\lambda[\Delta I_\lambda]$ | and | $\Sigma\delta\theta[\Delta I_\theta]$ |
| AFRICAN | 1 | -.8483951917D+30 | -.2962661296D+29<br>-.1034584122D+28 | 0.3089788182D+30<br>0.1078977810D+29<br>0.8494297758D+30 |
| | | 0.4783257395D+29 | -.6840376780D+30<br>-.4783257395D+29 | -.3417681716D+29<br>0.9786959952D+30<br>0.0 |
| AMERICAN | 2 | -.3683810047D+30 | 0.1340797206D+30<br>-.4880102730D+29 | 0.2263345136D+30<br>-.8237902592D+29<br>0.4171820320D+30 |
| | | 0.1413681311D+30 | 0.1684759781D+30<br>-.1413681311D+30 | -.1302855743D+30<br>-.3579566736D+30<br>0.0 |
| ANTARTIC | 3 | -.6258442499D+27 | 0.1092415200D+26<br>-.1906817819D+24 | 0.1792456445D+29<br>-.3128744359D+27<br>0.6260349317D+27 |
| | | 0.1162857966D+26 | 0.3329989810D+27<br>-.1162857966D+26 | -.3331511818D+27<br>-.1908622006D+29<br>0.0 |
| ARABIAN | 4 | 0.6917794706D+30 | 0.7163586124D+30<br>0.7418110588D+30 | -.2236728436D+31<br>-.2316200100D+31<br>-.1433590529D+31 |
| | | 0.3391309664D+31 | 0.1184271424D+30<br>-.3391309664D+31 | 0.5188486544D+30<br>-.5010463215D+30<br>0.0 |
| CARIBBEAN | 5 | 0.4663653460D+28 | -.2020050245D+29<br>0.8749798900D+29 | -.1475148393D+30<br>0.6389569674D+30<br>-.9216164246D+29 |
| | | 0.4408932047D+30 | -.9039650313D+30<br>-.4408932047D+30 | 0.6852658749D+29<br>0.1582060930D+29<br>0.0 |
| COCOS | 6 | 0.1519907921D+28 | -.2173569587D+29<br>0.3108349315D+30 | -.6239488001D+30<br>0.8922883574D+31<br>-.3123548394D+30 |
| | | -.1050204383D+31 | 0.7472592489D+31<br>0.1050204383D+31 | -.1313955878D+30<br>-.9188074535D+28<br>0.0 |

Table 4.6 (Cont'd)

Contributions to the Differential Rotation Tensor $[\Delta I_{ROT}]$ as a Consequence of Absolute Plate Rotations (Model B4) for Each Plate and the Total Crust

| PLATE NAME | NO | MODEL B4 $\Sigma\delta\lambda[\Delta I_\lambda]$ and $\Sigma\delta\theta[\Delta I_\theta]$ | | |
|---|---|---|---|---|
| EURASIAN | 7 | 0.6828396997D+29 | -.1136436102D+30<br>0.1891347288D+30 | -.3642617405D+30<br>0.6062333410D+30<br>-.2574186987D+30 |
| | | -.1111062252D+31 | 0.5907622788D+30<br>0.1111062252D+31 | -.1901900674D+30<br>-.1142777216D+30<br>0.0 |
| INDIAN | 8 | -.1860433120D+31 | 0.6771422781D+30<br>-.2464596335D+30 | -.1237136072D+31<br>0.4502807057D+30<br>0.2106892753D+31 |
| | | 0.2401100458D+30 | 0.2861520098D+30<br>-.2401100458D+30 | 0.2304849371D+30<br>0.6332521603D+30<br>0.0 |
| NAZCA | 9 | -.1877725314D+30 | 0.6548406379D+30<br>-.2283700698D+31 | -.2752349257D+30<br>0.9598582551D+30<br>0.2471473230D+31 |
| | | -.3918174142D+31 | 0.6270389380D+31<br>0.3918174142D+31 | 0.4794055988D+31<br>0.1374673432D+31<br>0.0 |
| PACIFIC | 10 | -.1109252706D+31 | -.5813347106D+29<br>-.3046646124D+28 | -.1000146288D+31<br>-.5241544600D+29<br>0.1112299352D+31 |
| | | -.3397361201D+29 | 0.3232373258D+30<br>0.3397361201D+29 | -.3820535006D+29<br>0.7290015051D+30<br>0.0 |
| PHILIPPINE | 11 | 0.2367765599D+30 | -.2723802740D+30<br>0.3133376619D+30 | 0.3433804559D+31<br>-.3950140280D+31<br>-.5501142217D+30 |
| | | 0.7759657071D+31 | -.1090548679D+31<br>-.7759657071D+31 | -.3099312539D+30<br>-.2694191288D+30<br>0.0 |
| TOTAL | | -.3371836835D+31 | 0.1666712007D+31<br>-.9404264102D+30 | -.1897928647D+31<br>0.5187554845D+31<br>0.4312263246D+31 |
| | | 0.5907767929D+31 | 0.1255181821D+32<br>-.5907767929D+31 | 0.4777398365D+31<br>0.2460469562D+31<br>0.0 |

NOTE : ALL TENSORS ARE SYMMETRIC AND REFER TO THE TERRESTRIAL SYSTEM.
THE UNIT OF EACH ELEMENT IS g cm²

Recently Kaula [1975] used an independent method to compute absolute plate velocities by minimizing the translational motion of tectonic plate boundaries. Solomon et al. [1975] model B4 (continents have three times more drag than oceans) appears to agree most closely with Kaula's published values. According to Kaula [1975] this is the most plausible model, being well supported by seismologic evidence.

For the reasons stated above the B4 model was used initially in this investigation. It will be seen later that all the models lead to the same basic conclusions.

Table 4.6 contains the values $\sum_k \delta\lambda [\Delta I_\lambda]$ and $\sum_k \delta\theta [\Delta I_\theta]$ for each plate and the total crust respectively. It can be noticed that, as was explained in section 2.5:

(a) there is no contribution to $[\bar{I}]_E$ due to a $\lambda$ rotation.

(b) the traces of the differential contributions (as a consequence of rotations) to the earth tensor of inertia are zero.

Table 4.7 tabulates the combined rotation effect. Point (b) mentioned above still holds if one considers the basic matrix property

$$tr([M] + [N]) = tr[M] + tr[N]$$

Because the final effect of each plate depends on diverse factors, mainly

(i) Type of plate (continental or oceanic)

(ii) Dimensions

(iii) Earth location

(iv) Absolute velocity

it is extremely difficult to make an "a priori" forecast of results.

Observe, for example, that the Nazca plate, although small, gives a significant contribution to the initial tensor of inertia $[\bar{I}]_E$, while the Antartic plate, primarily due to location, has a minimum effect.

Table 4.7

Differential Rotation Tensor $[\Delta I_{ROT}] \equiv [\Delta I]_P$ for Each Plate and the Total Crust Due to Absolute Plate Rotations (Model B4)

| PLATE | | MODEL B4 | | |
|---|---|---|---|---|
| NAME | NO | $[\Delta I]_{P1}$ | | |
| AFRICAN | 1 | -.8005626177D+30 | -.7136642909D+30<br>-.4886715807D+29 | 0.2748020010D+30<br>0.9894057733D+30<br>0.8494297758D+30 |
| AMERICAN | 2 | -.2270128736D+30 | 0.3025556987D+30<br>-.1901691584D+30 | 0.9604893922D+29<br>-.4403356995D+30<br>0.4171820320D+30 |
| ANTARTIC | 3 | -.6142156703D+27 | 0.3439231330D+27<br>-.1181926144D+26 | 0.1759141327D+29<br>-.1939909449D+29<br>0.6260349317D+27 |
| ARABIAN | 4 | 0.4083089134D+31 | 0.8347857549D+30<br>-.2649498605D+31 | -.1717879782D+31<br>-.2817246421D+31<br>-.1433590529D+31 |
| CARIBBEAN | 5 | 0.4455568582D+30 | -.9241655337D+30<br>-.3533952157D+30 | -.7898825179D+29<br>0.6547775767D+30<br>-.9216164246D+29 |
| COCOS | 6 | -.1048684475D+31 | 0.7450856793D+31<br>0.1361039315D+31 | -.7553443879D+30<br>0.8913695450D+31<br>-.3123548394D+30 |
| EURASIAN | 7 | -.1042778282D+31 | 0.4771186686D+30<br>0.1300196981D+31 | -.5544518079D+30<br>0.4919556194D+30<br>-.2574186987D+30 |
| INDIAN | 8 | -.1620323074D+31 | 0.9632942880D+30<br>-.4865696793D+30 | -.1006651135D+31<br>0.1083532866D+31<br>0.2106892753D+31 |
| NAZCA | 9 | -.4105946674D+31 | 0.6925230018D+31<br>0.1634473444D+31 | 0.4518821062D+31<br>0.2334531687D+31<br>0.2471473230D+31 |
| PACIFIC | 10 | -.1143226318D+31 | 0.2651038547D+30<br>0.3092696589D+29 | -.1038351638D+31<br>0.6765860591D+30<br>0.1112299352D+31 |
| PHILIPPINE | 11 | 0.7996433631D+31 | -.1362928953D+31<br>-.7446319409D+31 | 0.3123873305D+31<br>-.4219559409D+31<br>-.5501142217D+30 |
| TOTAL | | 0.2535931094D+31 | 0.1421853027D+32<br>-.6848194339D+31 | 0.2879469719D+31<br>0.7648024407D+31<br>0.4312263246D+31 |

NOTE : ALL TENSORS ARE SYMMETRIC AND REFER TO THE TERRESTRIAL SYSTEM.
THE UNIT OF EACH ELEMENT IS $g\,cm^2$

Table 4.8

Differential Rotation Tensor $[\Delta I_{ROT}] \equiv [\Delta I]_P$ Caused by Absolute Plate Rotations for All Models

| MODEL NAME | NO | TOTAL CONTRIBUTION TO EARTH TENSOR OF INERTIA $[\Delta I]_P$ | | |
|---|---|---|---|---|
| A3 | 1 | 0.9502470052D+30 | 0.1649147382D+32 | 0.33488[illegible]9128D+31 |
| | | | -.5199709473D+31 | 0.7874848246D+31 |
| | | | | 0.4249462468D+31 |
| B3 | 2 | 0.4726181552D+31 | 0.1111439970D+32 | 0.2202304196D+31 |
| | | | -.9161589827D+31 | 0.7574987070D+31 |
| | | | | 0.4435408275D+31 |
| B4 | 3 | 0.2535931094D+31 | 0.1421853022D+32 | 0.2879469719D+31 |
| | | | -.6848194339D+31 | 0.7648024407D+31 |
| | | | | 0.4312263246D+31 |
| C3 | 4 | 0.1044350038D+30 | 0.1768498427D+32 | 0.3707348543D+31 |
| | | | -.4239510340D+31 | 0.7642283584D+31 |
| | | | | 0.4135075337D+31 |
| C5 | 5 | 0.7137693517D+30 | 0.1679902451D+32 | 0.3550972278D+31 |
| | | | -.4850922470D+31 | 0.7436460337D+31 |
| | | | | 0.4137153118D+31 |
| D1 | 6 | 0.1207872429D+31 | 0.1577566927D+32 | 0.3335394159D+31 |
| | | | -.5256298225D+31 | 0.6018229126D+31 |
| | | | | 0.4048425797D+31 |
| E2 | 7 | 0.2100683629D+31 | 0.1490717325D+32 | 0.4029539687D+31 |
| | | | -.5750310776D+31 | 0.5539290979D+31 |
| | | | | 0.3649627147D+31 |
| E3 | 8 | 0.3500387515D+31 | 0.1284520232D+32 | 0.3065164654D+31 |
| | | | -.7542893181D+31 | 0.6408530605D+31 |
| | | | | 0.4042505666D+31 |

NOTE : ALL TENSORS ARE SYMMETRIC AND REFER TO THE TERRESTRIAL SYSTEM.
THE UNIT OF EACH ELEMENT IS g cm$^2$

Table 4.8 presents the differential tensor $[\Delta I_{ROT}]$ for all available models. It is evident from the tabulated values that all contributions from the different models are very similar. By the common sign of all the products of inertia in the given matrices, one can predict that the displacement of the principal axis of the original tensor of inertia will be in the same quadrant for all the models.

It should be mentioned here that the values in table 4.8 correspond to a one-year span. Thus they will be the contributions to the earth inertia tensor $[\overline{I}]_E$ as a consequence of the rearrangement of masses in the earth due to the motion of the plates for a period of one year.

To this date only a linear value for the plate absolute velocity is known, and this fact should always be kept in mind. Fortunately, the rates of rotation are so small (see table 3.2) that for a period of time $\leq 100$ years, extrapolation of velocities under the assumption of linearity may be considered satisfactory.

## 4.5 Final Mean Pole Displacements

The position of the mean pole after the lithospheric plate motions are considered may be computed by diagonalizing the matrix $[I]$ given by

$$[I] = [\overline{I}]_E + [\Delta I]_P \tag{4.5-1}$$

As implied by the evaluation of $[\overline{I}]_E$ and $[\Delta I]_P$, the tensor $[I]$ refers to the CIO terrestrial system. In the following analysis, this ideal reference frame is assumed to be mantle-fixed, and consequently is not going to be affected by the transfer of mass involved in the tectonic fluctuations. As mentioned before, this may be considered a reasonable hypothesis for short time intervals.

The differential contribution $[\Delta I]_P$ to the original earth tensor $[\overline{I}]_E$ alters the initial position of the earth principal axes, displacing them after the crustal masses have moved.

Hence a preliminary problem involves the diagonalization of $[\,I\,]$. After this, the new position of the disturbed earth axis of figure will be known, and a comparison with the undisturbed one possible (principal axis before any mass displacement is considered).

The diagonalization process may be expressed by

$$R\left[[\,\bar{I}\,]_E + [\,\Delta I]_P\right]R^T \;=\; \lceil I' \rfloor_E \qquad (4.5\text{-}2)$$

Thus the numerical computation of $\lceil I' \rfloor_E$ is reduced to the determination of the eigenvalues and eigenvectors of the matrix $[\,I\,]$ given in (4.5-1).

Before presenting the results, the relationship between the general expression (4.5-2) and previously given formulae [Darwin, 1877], [Helmert, 1880, p. 420], [Tisserand, 1891, p. 485] for obtaining the variation in the axis of figure as a result of a given earth mass displacement will be established.

Selecting the principal axis of the earth as reference system, equation (4.5-2) reduces to

$$R\left[\lceil I \rfloor_E + [\,\Delta I]_P\right]R^T \;=\; \lceil I' \rfloor_E \qquad (4.5\text{-}3)$$

and now

$$[\,I\,] \;=\; \lceil I \rfloor_E + [\,\Delta I]_P \qquad (4.5\text{-}4)$$

Assuming small counterclockwise rotation angles between the two principal frames, before and after the crustal motions, one has

$$R \approx \begin{bmatrix} 1 & \delta\alpha_3 & -\delta\alpha_2 \\ -\delta\alpha_3 & 1 & \delta\alpha_1 \\ \delta\alpha_2 & -\delta\alpha_1 & 1 \end{bmatrix} \;=\; [\,\delta\underline{\alpha}]^T + [\,1\,] \qquad (4.5\text{-}5)$$

Thus

$$[I']_E = \Big[[\delta\underline{\alpha}]^T + [1]\Big][I]\Big[[\delta\underline{\alpha}] + [1]\Big] \tag{4.5-6}$$

Neglecting second-order terms,

$$[I']_E \approx [\delta\underline{\alpha}]^T[I] + [I][\delta\underline{\alpha}] + [I] \tag{4.5-7}$$

and the following conditions are established

$$(I_{33}-I_{22})\delta\alpha_1 + I_{21}\delta\alpha_2 - I_{13}\delta\alpha_3 + I_{23} = 0 \tag{4.5-8a}$$

$$-I_{12}\delta\alpha_1 + (I_{11}-I_{33})\delta\alpha_2 + I_{23}\delta\alpha_3 + I_{13} = 0 \tag{4.5-8b}$$

$$I_{13}\delta\alpha_1 - I_{23}\delta\alpha_2 + (I_{22}-I_{11})\delta\alpha_3 + I_{12} = 0 \tag{4.5-8c}$$

From the above system of three equations with three unknowns, a unique set of values $\delta\alpha_i$ $(i = 1,2,3)$ can easily be obtained. For example, using Cramer's rule, the rotation angle about the first principal axis is given by

$$\delta\alpha_1 = \frac{\begin{vmatrix} -I_{23} & I_{12} & -I_{13} \\ -I_{13} & I_{11}-I_{33} & I_{23} \\ -I_{12} & -I_{23} & I_{22}-I_{11} \end{vmatrix}}{\begin{vmatrix} I_{33}-I_{22} & I_{12} & -I_{13} \\ -I_{12} & I_{11}-I_{33} & I_{23} \\ I_{13} & -I_{23} & I_{22}-I_{11} \end{vmatrix}} \tag{4.5-9}$$

with similar equations for $\delta\alpha_2$ and $\delta\alpha_3$.

A further simplification of the equations (4.5-8) is possible if one supposes that the products $\delta\alpha_i I_{ij}$ $(\forall i, j = 1,2,3$ and $i \neq j)$ are small, and all of the same order of magnitude. Then their differences as shown in equation (4.5-8) may be neglected.

Thus (4.5-8) simplifies to

$$\delta\alpha_1 (I_{33} - I_{22}) + I_{23} \;=\; 0 \qquad (4.5\text{-}10a)$$

$$\delta\alpha_2 (I_{11} - I_{33}) + I_{13} \;=\; 0 \qquad (4.5\text{-}10b)$$

$$\delta\alpha_3 (I_{22} - I_{11}) + I_{12} \;=\; 0 \qquad (4.5\text{-}10c)$$

which gives

$$\delta\alpha_1 \;=\; \frac{-I_{23}}{I_{33} - I_{22}} \qquad (4.5\text{-}11a)$$

$$\delta\alpha_2 \;=\; \frac{-I_{13}}{I_{11} - I_{33}} \qquad (4.5\text{-}11b)$$

$$\delta\alpha_3 \;=\; \frac{-I_{12}}{I_{22} - I_{11}} \qquad (4.5\text{-}11c)$$

if one writes

$$[\,I\,] \;=\; [\,I\,]_E + [\,\Delta I\,]_P \;=\; \begin{bmatrix} A & 0 & 0 \\ & B & 0 \\ s & & C \end{bmatrix} + \begin{bmatrix} a & -F & -E \\ & b & -D \\ s & & c \end{bmatrix} \qquad (4.5\text{-}12)$$

Then substituting the proper values of (4.5-12) in equations (4.5-11) one finally has

$$\delta\alpha_1 \;=\; \frac{D}{(C+c) - (B+b)} \qquad (4.5\text{-}13a)$$

$$\delta\alpha_2 \;=\; \frac{E}{(A+a) - (C+c)} \qquad (4.5\text{-}13b)$$

$$\delta\alpha_3 \;=\; \frac{F}{(B+b) - (A+a)} \qquad (4.5\text{-}13c)$$

The above equations are in the form given by Darwin, and have the disadvantage that in the case of rotational symmetry, $\delta\alpha_3$ is undefined.

In this investigation the rigorous approach using eigentheory was the selected operational process. Nevertheless, as a consequence of the smallness of the rotations, the same results are obtainable from equations (4.5-13), if the elements of the matrix $[\,I\,]_E$ given in (4.2-12) and $[\,\Delta I]_P$ from table 4.8 are used in (4.5-12).

In the present study the effect of tectonic plate motions on the displacement of the mean pole of the whole earth as well as on the crustal pole were determined.

The results are shown in table 4.9. Several conclusions may be drawn from the final tabulated results:

(a) The first and most important one is that lithospheric motions as described by recent geophysical theoretical models of absolute velocities [Solomon et al., 1975] do not produce any significant changes in the principal pole of inertia over a time period of approximately 70 years.

This contrasts emphatically with previous published results [Liu et al., 1974] claiming that tectonic plate movement may provide a secular drift of the mean pole (10% of the observed value).

In this respect two points should be clarified:

(i) The above authors divided the short paper in several parts. The "Analysis" follows literally the work of Darwin [1877]. Thus primarily equations 4.5-13 are applied.

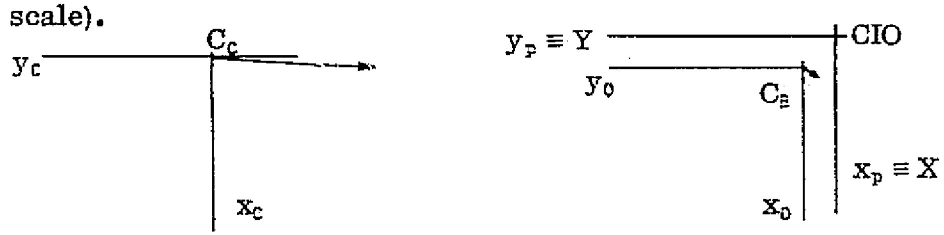
(ii) The part of the paper termed "Results of Computations" is deficient in the presentation of what may be considered to be basic intermediary results, such as the integrated values of the product of inertia used in Darwin's equations, or fundamental assumptions like the plate boundary model used. It is coincidental that the reported pole direction of secular motion agrees within 2°

Table 4.9

Displacement of the Polar Principal Axis of the Earth and Crust Due to Mass Displacements as a Consequence of Tectonic Plate Motions

| Model | Earth's Inertia Pole Displacement | | | | Crustal Inertia Pole Displacement | | | |
|---|---|---|---|---|---|---|---|---|
| | 1 Year | | 70 Years | | 1 Year | | 70 Years | |
| | $x_0$ (mm) | $y_0$ (mm) | $x_0$ (mm) | $y_0$ (mm) | $x_c$ (mm) | $y_c$ (mm) | $x_c$ (mm) | $y_c$ (mm) |
| A3 | 0.008 | -0.019 | 0.56 | -1.33 | 0.23 | -1.00 | 15.96 | -69.93 |
| B3 | 0.005 | -0.018 | 0.35 | -1.26 | 0.14 | -1.01 | 9.45 | -70.93 |
| B4 | 0.007 | -0.018 | 0.49 | -1.26 | 0.19 | -0.99 | 13.37 | -69.30 |
| C3 | 0.009 | -0.018 | 0.63 | -1.26 | 0.26 | -0.96 | 18.48 | -67.20 |
| C4 | 0.008 | -0.018 | 0.56 | -1.26 | 0.25 | -0.94 | 17.78 | -66.31 |
| D1 | 0.008 | -0.014 | 0.56 | -0.98 | 0.24 | -0.78 | 16.94 | -54.74 |
| E2 | 0.009 | -0.013 | 0.63 | -0.91 | 0.35 | -0.73 | 24.43 | -50.75 |
| E3 | 0.007 | -0.015 | 0.49 | -1.05 | 0.24 | -0.85 | 16.66 | -59.43 |

Note. The values of the above coordinates are referred to a coordinate system parallel to the CIO polar frame centered at the mean pole $x_0, y_0$ and crustal pole $x_c, y_c$ respectively. (See below; sketch not drawn to scale).



with the empirical value published by Markovitz [1960] and based on ILS data.

In the author's opinion the above-mentioned paper has a discontinuity between Darwin's theory and the final statement that plate tectonics may account for 10% of the secular drift of the pole, supposedly in the right direction. Practically everything in between is left to the reader's imagination.

If other changes of mass distribution in the earth (eustatic sea level changes, melting of ice in Greenland, epirogenic motions, etc.) are postulated as possible causes of secular drift of the pole, their effect should be simulated in the most rigorous way before any conclusive answer is established. One feels that the differential contribution to the earth inertia tensor as a result of plate tectonic motions is greater in importance, mainly because of the magnitude of the masses involved and their asymmetric distribution on the earth. Internal shift of masses (i.e., convection currents) is always a possibility, although there is no viable way today of knowing if they really exist. More precise satellite dynamical solutions in the future may clarify some questions in this area after reliable data for a number of consecutive decades is analyzed.

(b) While the earth axis of figure remains practically unaffected by plate motions, even during periods of a century or more, the pole of the crust moves only about one centimeter per decade in the general direction of the earth maximum moment of inertia axis. The vector displacement for the motion of the crustal pole is about 50 times larger than the one for the mean pole.

(c) The general direction of the motion of the pole of the crust coincides with the prediction of Milankovich and Jardetzky [1962] and is in accordance with "Milankovich's theorem" [Scheidegger,

Table 4.10

Coordinates Used for the ILS-IPMS Observatories

| STATION NO | STATION LOCATION | PLATE CODE[1] | LATITUDE | | | LONGITUDE | | | INSTR. TYPE |
|---|---|---|---|---|---|---|---|---|---|
| 1 | MIZUSAWA | 7 | 39 | 08 | 03.299 | 141 | 07 | 51. | PZT |
| 2 | BLAGOVESHCHENSK | 7 | 50 | 19 | 9.300 | 127 | 28 | 45. | ZTL-180 |
| 3 | CHUREL TOGOT | 7 | 47 | 51 | 56.300 | 107 | 03 | 00. | VZT |
| 4 | IRKUTSK | 7 | 52 | 16 | 43.809 | 104 | 20 | 45. | ZTL-180 |
| 5 | KITAB | 7 | 39 | 08 | 01.850 | 66 | 52 | 51. | ZTL-180 |
| 6 | ENGELHARDT | 7 | 55 | 50 | 19.966 | 48 | 49 | 00. | ZTL-180 |
| 7 | GORKY | 7 | 56 | 15 | 31.000 | 43 | 59 | 00. | VZT |
| 8 | POLTAVA | 7 | 49 | 36 | 13.123 | 34 | 33 | 00. | VZT |
| 9 | PULKOVO | 7 | 59 | 46 | 15.534 | 30 | 19 | 45. | ZTF-135 |
| 10 | WARSAW | 7 | 52 | 05 | 56.021 | 21 | 02 | 15. | VZT |
| 11 | BELGRADE | 7 | 44 | 48 | 10.277 | 20 | 30 | 45. | VZT |
| 12 | BOROWIEC | 7 | 52 | 16 | 38.110 | 17 | 04 | 30. | VZT |
| 13 | PECNY | 7 | 49 | 54 | 56.000 | 14 | 47 | 15. | VZT |
| 14 | DRESDEN | 7 | 51 | 03 | 16.013 | 13 | 43 | 60. | TI |
| 15 | LA PLATA | 2 | -34 | 54 | 31.732 | 302 | 04 | 00. | VZT |
| 16 | MILANO | 7 | 45 | 27 | 59.267 | 9 | 11 | 30. | TI |
| 17 | TURKU | 7 | 60 | 24 | 57.510 | 22 | 26 | 45. | VZT |
| 18 | MT. STROMLO | 8 | -35 | 19 | 17.500 | 149 | 00 | 15. | PZT |
| (1) | MIZUSAWA | 7 | 39 | 08 | 03.216 | 141 | 07 | 51. | PZT |
| 19 | TOKYO | 7 | 35 | 40 | 20.626 | 139 | 32 | 30. | PZT |
| 20 | MOSCOW | 7 | 55 | 41 | 56.000 | 37 | 32 | 30. | PZT |
| 21 | HAMBURG | 7 | 53 | 35 | 50.100 | 10 | 01 | 00. | PZT |
| 22 | NEUCHATEL | 7 | 46 | 59 | 51.620 | 6 | 57 | 30. | PZT |
| 23 | GREENWICH | 7 | 50 | 52 | 17.812 | 0 | 20 | 15. | PZT |
| 24 | PUNTA INDIO | 2 | -35 | 20 | 40.500 | 302 | 42 | 45. | PZT |
| 25 | CAGIGAL | 2 | 10 | 30 | 23.000 | 293 | 04 | 15. | PZT |
| 26 | OTTAWA | 2 | 45 | 23 | 59.850 | 284 | 04 | 45. | PZT |
| 27 | WASHINGTON | 2 | 38 | 55 | 16.991 | 282 | 56 | 00. | PZT |
| 28 | RICHMOND | 2 | 25 | 36 | 47.020 | 279 | 37 | 00. | PZT |
| 29 | CALGARY | 7 | 50 | 52 | 22.000 | 245 | 42 | 30. | PZT |
| (1) | MIZUSAWA | 7 | 39 | 08 | 03.560 | 141 | 08 | 00. | ASTROLABE |
| 30 | ZI-KA-WEI | 7 | 31 | 11 | 25.000 | 121 | 25 | 45. | ASTROLABE |
| 31 | NOVOSIBIRSK | 7 | 55 | 01 | 46.000 | 82 | 54 | 30. | ASTROLABE |
| (8) | POLTAVA I | 7 | 49 | 36 | 13.123 | 34 | 33 | 00. | ASTROLABE |
| 32 | POTSDAM | 7 | 52 | 22 | 54.624 | 13 | 04 | 00. | ASTROLABE |
| 33 | GRASSE | 7 | 43 | 44 | 56.000 | 6 | 55 | 30. | ASTROLABE |
| 34 | BESANCON | 7 | 47 | 14 | 56.580 | 5 | 59 | 15. | ASTROLABE |
| (35) | UCCLE I | 7 | 50 | 47 | 51.890 | 4 | 21 | 30. | ASTROLABE |
| 35 | UCCLE II | 7 | 50 | 47 | 51.890 | 4 | 21 | 30. | ASTROLABE |
| 36 | ALGER | 1 | 36 | 48 | 06.677 | 3 | 02 | 00. | ASTROLABE |
| 37 | PARIS | 7 | 48 | 50 | 09.146 | 2 | 20 | 15. | ASTROLABE |
| 38 | SAN FERNANDO | 7 | 36 | 27 | 44.000 | 353 | 47 | 45. | ASTROLABE |
| 39 | SAO PAULO | 2 | -23 | 39 | 10.000 | 313 | 22 | 45. | ASTROLABE |
| 40 | SAN JUAN | 2 | -31 | 30 | 39.000 | 291 | 22 | 45. | ASTROLABE |
| 41 | SANTIAGO | 2 | -33 | 23 | 57.000 | 289 | 27 | 00. | ASTROLABE |
| (27) | WASHINGTON | 2 | 38 | 55 | 16.991 | 282 | 56 | 00. | ASTROLABE |
| 42 | QUITO | 2 | -00 | 12 | 57.000 | 281 | 15 | 00. | ASTROLABE |
| (28) | RICHMOND I | 2 | 25 | 36 | 47.500 | 279 | 37 | 00. | ASTROLABE |
| (13) | PECNY I | 7 | 49 | 54 | 56.000 | 14 | 47 | 15. | CZ |
| * 1 | MIZUSAWA | 7 | 39 | 08 | 03.602 | 141 | 07 | 51. | VZT |
| * 5 | KITAB | 7 | 39 | 08 | 01.850 | 66 | 52 | 51. | VZT |
| *57 | CARLOFORTE | 7 | 39 | 08 | 08.941 | 8 | 18 | 44. | VZT |
| *58 | GAITHERSBURG | 2 | 39 | 08 | 13.202 | 282 | 48 | 03. | VZT |
| *59 | UKIAH | 2 | 39 | 08 | 12.096 | 236 | 47 | 25. | VZT |
| *99 | CINCINNATI | 2 | 39 | 08 | 19.8 | 275 | 33 | 42. | |

* ILS OBSERVATORIES

1 REFERS TO TABLE 4.1

1963, p. 177]. This implies that the inertia of continental plates is less than that of oceanic plates. This is a fact established before, and does not contradict any isostatic model. Once again global tectonics fits the geophysical scenario perfectly, avoiding previously reported contradictions in the direction of polar wandering [see Munk and MacDonald, 1960, p. 277].

## 4.6 Apparent Motion of the Mean Pole Due to Station Shifts

It became evident by the results of the previous section that the earth axis of figure (i.e., the polar principal axis) does not suffer any significant displacement as a consequence of tectonic plate motions during a 70-year span. Thus, there is no true secular motion of the pole in this period of time, at least under the hypothesis and models used in this investigation.

This section studies the possibility of an apparent secular motion in the computed barycenter of the observed wobble, produced mainly by the drifts of the observatories monitoring polar motion.

Previous investigations in this area [Arur and Mueller, 1971] remained inconclusive, primarily because only the relative angular velocity for the plates available at the time and too noisy observations were used.

Depending on the selection of the set of observatories, two different cases will be analyzed:

(i) Apparent motion of the mean pole induced by the ILS stations;

(ii) Apparent barycenter displacements in the IPMS.

Table 4.10 compiled from [Yumi, 1975] shows the principal characteristics of the participating stations.

Systematic "errors" at the different observatories caused by plate rotations were determined according to the following procedure:

Table 4.11

Number of Years of Common Activity for the Different ILS Observatories*

| Stations Co-observing | | | | | | Time Span |
|---|---|---|---|---|---|---|
| Mizusawa | Tchardjui** | Carloforte | Gaithersburg | Ukiah | Cincinnati | 16 years |
| Mizusawa | Tchardjui | Carloforte | | Ukiah | Cincinnati | 4 years |
| Mizusawa | Tchardjui | Carloforte | | Ukiah | | 4 years |
| Mizusawa | | Carloforte | | Ukiah | | 11 years |
| Mizusawa | Kitab | Carloforte | | Ukiah | | 2 years |
| Mizusawa | Kitab | Carloforte | Gaithersburg | Ukiah | | 11 years |
| Mizusawa | Kitab | | Gaithersburg | Ukiah | | 3 years |
| Mizusawa | Kitab | Carloforte | Gaithersburg | Ukiah | | 19 years |

* Data source: [Dejaiffe, 1972].

** The coordinates of Tchardjui and Kitab are assumed to be equal.

First, differential changes in the Cartesian coordinates at each station j due to the differential rotations of the corresponding plate $P_i$ were obtained using the expression

$$\{dx\}_j = [\underline{\delta\omega}]_{P_i}\{x\}_j \tag{4.6-1}$$

where the vector $\{x\}_j$ was given as a function of spherical coordinates by equation (2.5-6); the skew-symmetric matrix $[\underline{\delta\omega}]_{P_i}$ was defined in section 3.3.

After the proper plate rotations are applied to each observatory, a new set of disturbed station coordinates $\{\tilde{x}\}_j$ is found

$$\{\tilde{x}\}_j = \{x\}_j + \{dx\}_j \tag{4.6-2}$$

These coordinates define a new Cartesian system (the final or rotated one.)

The problem is then reduced to determining the angles which will transform the original system (x), say at epoch $T_0$ to the new apparent system $(\tilde{x})$ at epoch $T_1$. This displacement, in a sense equivalent to the one between the barycenters of the wobble, at epochs $T_0$ and $T_1$, is a consequence of the station motions. Hence the position of the $(\tilde{x})$ system may be considered the apparent position of the (x) frame induced by the motion of the observatories themselves in the interval $T_1 - T_0$.

The coordinates of the two systems, assuming differential displacements, are related by the known transformation

$$\{\tilde{x}\}_j = R_\delta\{x\}_j = [[\underline{\delta\alpha}]^T + [1]]\{x\}_j \tag{4.6-3}$$

where in the usual notations:

$$\delta\alpha_1 = \epsilon$$
$$\delta\alpha_2 = \psi$$
$$\delta\alpha_3 = \omega$$

are the counterclockwise rotations about the initial (CIO) reference frame.

By a least square solution, the values of $\delta\alpha_i$ $(i = 1, 2, 3)$ for each particular absolute velocity plate model using ILS and IPMS stations were obtained.

The results are plotted in Figs. 4.2 and 4.3 for a 70-year span $(T_1 - T_0 = 70\ y)$. The figures show the apparent displacement of the mean pole caused by the motion of the ILS and IPMS observatories for the eight absolute velocity plate models described in Table 3.1.

It is known that the activity of the ILS observatories was sporadic and that all the stations were not observing on a continuous basis.

The discontinuity in the mean pole path in Fig. 4.2 reflects the changes in the number of active cooperating stations throughout the 70-year period covered by the observations. The values used are shown in Table 4.11 where the different groups of co-observing stations and the approximate number of years of simultaneous operation assumed, are given. Figures 4.2 and 4.3 thus reflect the apparent systematic error in the mean pole caused by individual station drifts and accummulated during a 70-year span. The figures also show the apparent displacement of the equatorial $x_1$ axis (point of zero longitude). All the models except B3 (drag beneath continents only) and E3 (drag beneath 19 hot spots) produce an apparent westward displacement at the equator.

The following conclusions may be drawn:

(a) An apparent displacement of the mean pole is evidently produced as a consequence of tectonic plate motions and should be implicit in the ILS data as well as the IPMS latitude observations.

(b) The rate and direction of the barycenter drift depend on the total number of observing stations and on the absolute velocity plate model. The magnitude of the pole displacement represents only 10% to 20% of the astronomically observed value. Model C3 (drag opposing horizontal translations of slabs, oceanic subduction zone only) produces the maximum shift while model D1 (maximum pull by slabs plus plate drag) gives the minimum displacement.

There is not a complete agreement in direction between the observed mean pole displacement and the one derived using absolute plate velocity models. In general models B3 (drag beneath continents only) and B4 (continents have three times more drag than oceans) give the best correlation with the observed pole displacement.

(c) It is clear comparing the ILS and IPMS results that if the number of stations is increased, the amount of displacement is reduced and the direction of the drift changes slightly toward the 90°W longitude. It also appears that greater numbers of observing stations located on different plates tend to average better and produce less apparent motion of the pole.

(d) While in the case of possible true displacement of the earth axis of figure studied previously, all the geophysical models provide practically the same answers; when the apparent position of the mean pole is obtained after consideration of station drifts, the dependence on the model increases. For example Model E2 (drag beneath eight mid-plate hot spots) is in complete disagreement with the others, producing an apparent displacement of the mean pole in a different quadrant ($\sim 10°$ E longitude). Consequently, caution should be exercised in the use of absolute velocity plate models, because they are theoretical and do not necessarily reflect the actual displacements of the plates. Hence

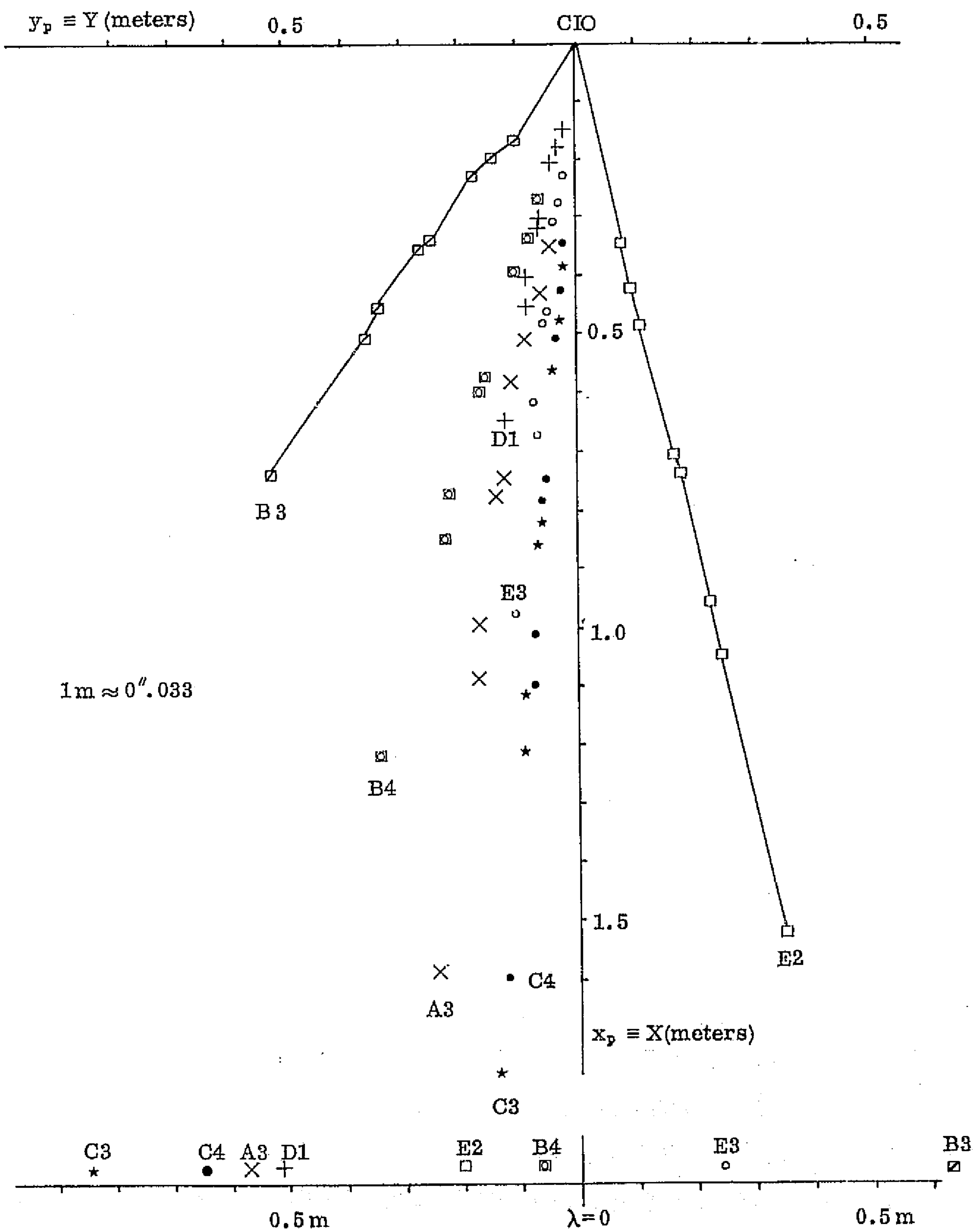



Fig. 4.2. Apparent Pole and Reference Meridian Displacement from ILS Stations for Different Absolute Plate Velocity Models

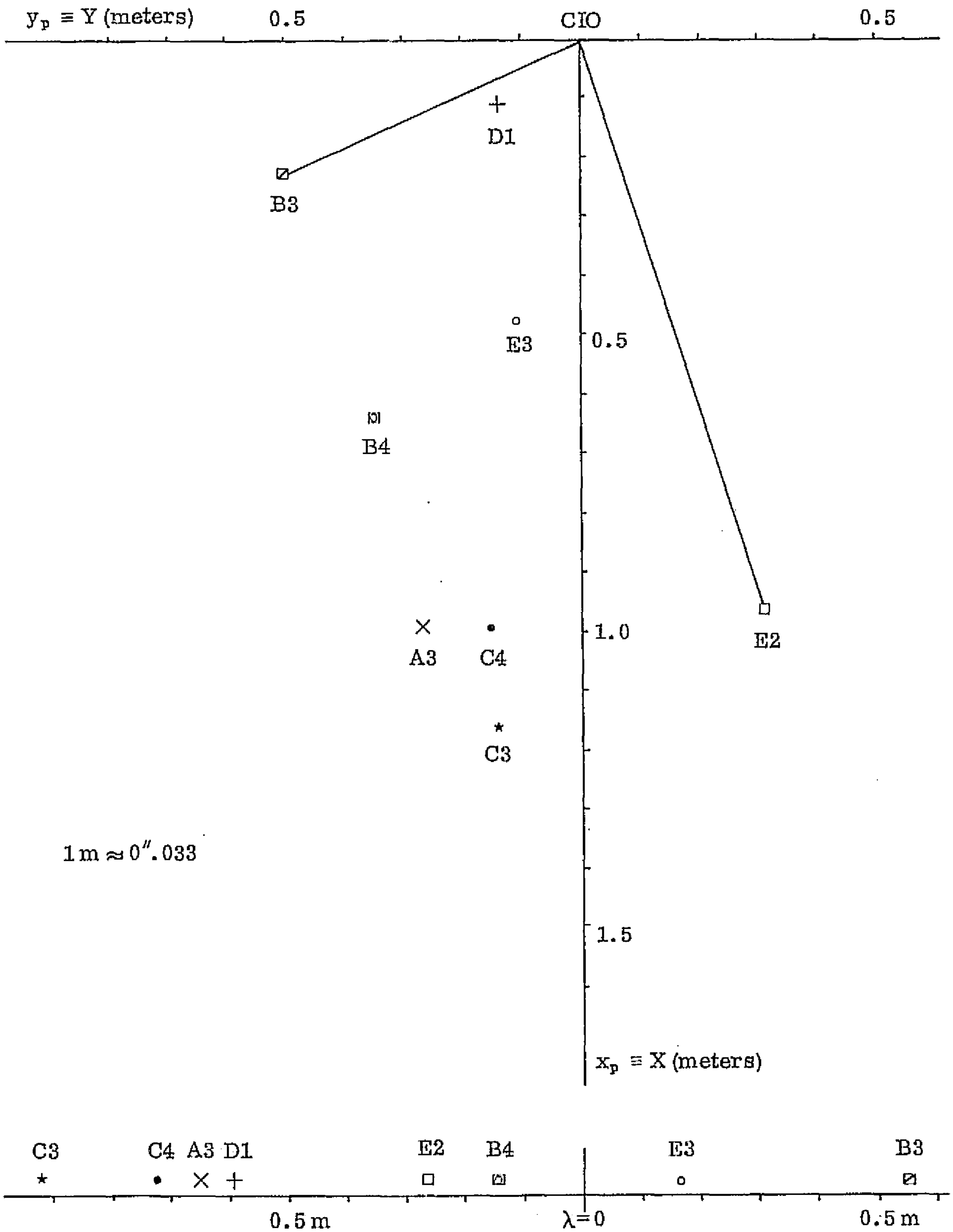



Fig. 4.3. Apparent Pole and Reference Meridian Displacement from IPMS Stations for Different Absolute Plate Velocity Models

these models should not be assumed valid and practically implemented for the correction of presently observing stations.

Current observational technology (VLBI, Lunar-Laser Range) provides the possibility of establishing a geodetic network capable of detecting relative tectonic motions of large plates on the earth with better accuracy than the optical astronomical observations. These undoubtedly may contribute to the fundamental understanding of global tectonics, providing ground truth for comparison with theoretical geophysical models. An initial study was conducted [Mueller and Schwartz, 1972] in order to decide if existing stations and baselines relative to surface plates could provide the precise measurements of intercontinental distances needed to recover the plate rates of displacement.

To this day no major geodetic results in this area have been reported, primarily because of the limited number of facilities available and the poor network geometry.

# 5 SUMMARY AND CONCLUSIONS

There is conclusive proof now (see Table 1.2) that the mean pole of the earth has a secular motion of the order of $0''.003$/yr toward the direction of 70° W latitude.

The fundamental objective of this investigation was directed to answer the controversial question:

Is there a <u>true</u> secular motion of the pole?

An examination of the geophysical hypotheses available to explain the astronomically observed drift of the mean pole suggest changes in the earth's inertia tensor as a plausible cause.

Rapid accumulation of geophysical and geologic evidence during the past decade strongly indicates that the earth surface is mobile. Large plates constituting the outer 50 to 100 km of the earth's crust appear to be moving with respect to one another at average rates from 1 cm/yr to as much as 15 cm/yr. This general structural scheme is termed "global plate tectonics".

On this basis a differential method was applied to determine the contributions to the earth's inertia tensor caused by infinitesimal plate motions. The approach is consistent for short intervals of time (<100 years), simplifying the earth model assumptions to present day configuration.

Nevertheless, the final results are subject to the following modelling constraints:

(1) Crustal model

(2) Plate boundaries

(3) Absolute velocity model

The most recent geophysical ideas have been implemented and used in the simulation. Among them, the eight theoretical models for absolute plate velocities established by Solomon et al. [1975] provide the framework of this investigation and should always count in any final interpretation.

The major conclusion of this quantitative analysis may be summarized as follows:

No evidence of a true secular motion of the earth axis of figure can be found from the earth's changes of inertia due to tectonic plate movements.

The indisputable astronomic evidence reflects the apparent motion of the wobble's barycenter due to drift of the observing stations as a consequence of lithospheric motions.

The above hypothesis is not fully supported by the results of section 4.6 where it was shown that the apparent fluctuations of the pole caused by the motion of the observatories due to tectonic plate rotations may account for only from 10% to 20% of the total observed displacement, depending on the absolute velocity model used and the number of stations involved.

Thus, neglecting other conjectural influences on the earth tensor of inertia due to mass redistributions (sea level changes and ice melting, convection currents, etc.), primarily because a unifying global framework describing the phenomenon is not yet available, the following old question should be reopened:

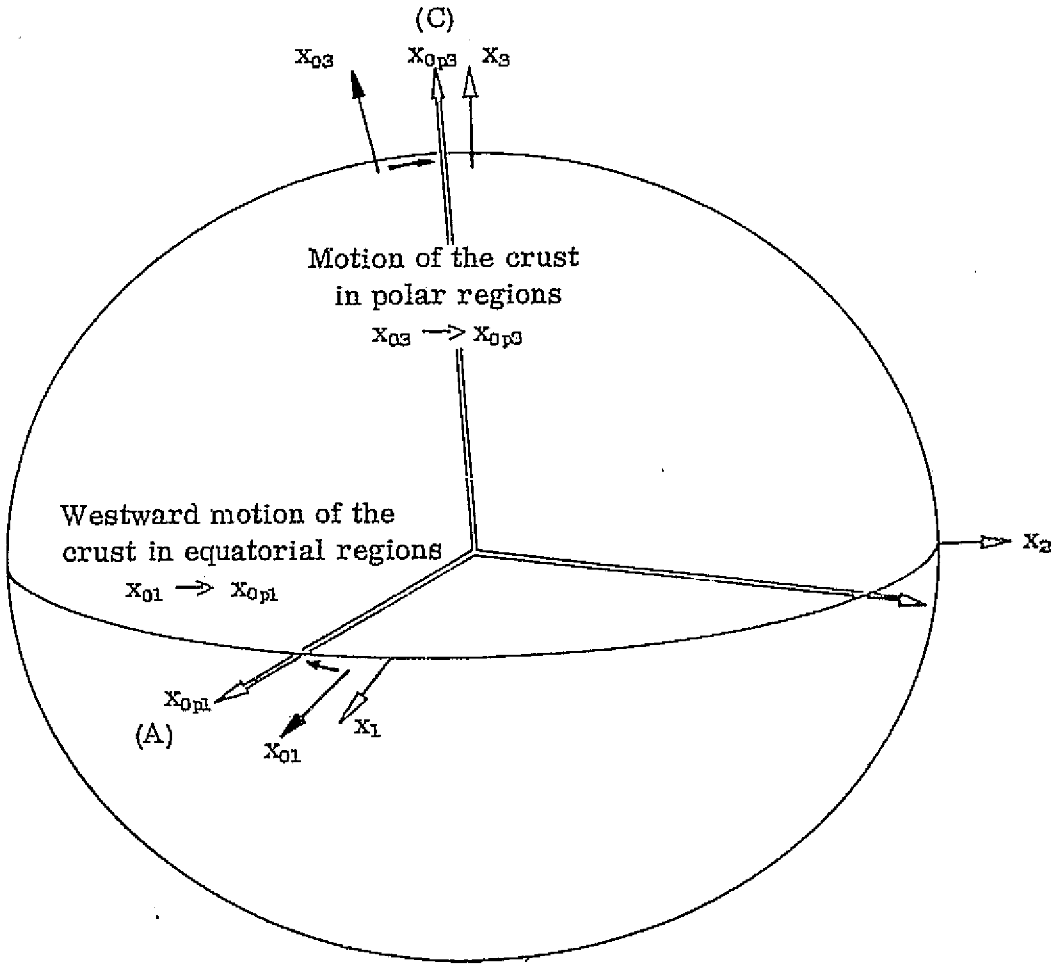
Is there a sliding of the whole lithospheric crust?

The idea preceded global plate tectonics by many years [Wegener, 1924, p. 152]. Several models have been proposed as possible driving mechanisms, although none of them is fully accepted. Among others, convection currents in the mantle, mantle plumes or hot spots and, recently, tidel drag are the most commonly referenced [Bostrom, 1971 and Bostrom et al., 1974]. Within a short time opposition arose to the theory of tidal drag as a driving mechanism producing westward motion of the crust. Jordan [1974] in a brief note showed that tidal torques are far too small to drag the lithosphere at any appreciable velocity.

A novel, conceptually simple driving mechanism appears implicit in the results of this investigation. Having a dynamical base, it fits the geophysical as well as the astronomical evidence.

Assume that the whole crustal upper layer, as modeled in this study, can glide over the mantle. According to basic gyrodynamic theory, the tendency of the thin shell is to attain dynamical stability, that is, the principal axes of the crust will constantly tend to pursue the principal axes of the whole earth until they interlock [Munk, 1958], [Ingles, 1957]. Thus the mechanism postulated in accordance with the results of section 4.3 will produce a westward drift of the lithosphere at the equator and an eastward displacement at the poles, as is schematically shown in Fig. 5.1.

Crustal gliding over the mantle, as theorized above, is not opposed by geophysical formalism. Munk and MacDonald [1960, p. 282] point out that a thin outer shell sliding over the interior could result in a displacement of a few degrees at most, and that the stress generated is too small to lead to failure. The maximum displacement considered here is about 5°, consistent with Munk and MacDonald [1960]. The same authors [p. 285] have also stated that a crustal gliding over the mantle does not appear to constitute a reasonable model for polar wandering; however, as mentioned in the introduction to this work, a clear difference should be established between secular motion of the mean pole and polar wandering. The results of this investigation leave the enigma of polar wandering as well as that of large-scale continental drift basically unanswered. All the conclusions of this study are in agreement with the hypothesis of present global tectonics, not implying the support of theories of polar wandering or major continental drift on a geologic time scale.




$(x_1, x_2, x_3)$ ≡ conventional terrestrial (geographic) system

$(x_{0p1}, x_{0p2}, x_{0p3})$ ≡ central principal axes

$(x_{01}, x_{02}, x_{03})$ ≡ crust's principal axes

Fig. 5.1. Crustal Sliding along the Equatorial and Polar Regions

Jardetzky [1962] considered the possibility of a gliding of the crustal shell over the mantle, analyzing the available paleographic evidence which supports the hypothesis: transgressions and regressions, the formation of guyots, vertical displacements of individual blocks forming a continent, and some features in connection with shear pattern distributions, were reviewed as favorable evidence in defense of a gliding of the crust.

Of all the above phenomena resulting from an infinitesimal sliding of the crustal shell, the formation of shear patterns may provide the needed correlation between real tectonic features on the earth and analytically-obtained shear patterns.

The first theoretical treatment for calculating such patterns under the assumption that the earth crust is ideally elastic, is due to Vening Meinesz [1947]. It is known that the final computed positions of the shear patterns with regard to the earth depend on the direction of polar path assumed [Scheidegger, 1973, p. 284]. Recently [Liu, 1974] essentially following Vening Meinesz' theory, but assuming the modern values of a shift of the poles over 80° N along the meridian of 75° W obtained a N-S fracture pattern which agrees roughly with the existing boundary system of the major plates. If an equatorial westward rotation of the crust about 14° W is assumed, it may provide the other E-W boundaries, such as the Pacific-Antartic ridge or South East Indian Rise.

A westward rotation of the crust is also supported by astronomic observations. Recently Proverbio and Poma [1976] considered a group of 16 BIH observatories located on six different plates. After analyzing time-scale data from 1962 to 1967, they obtained a final westward drift of the whole crust which accounts for the observed deceleration in the earth rotation.

Astronomically observed latitude also supports the proposed hypothesis of the sliding of the crust at the poles in the opposite direction of the

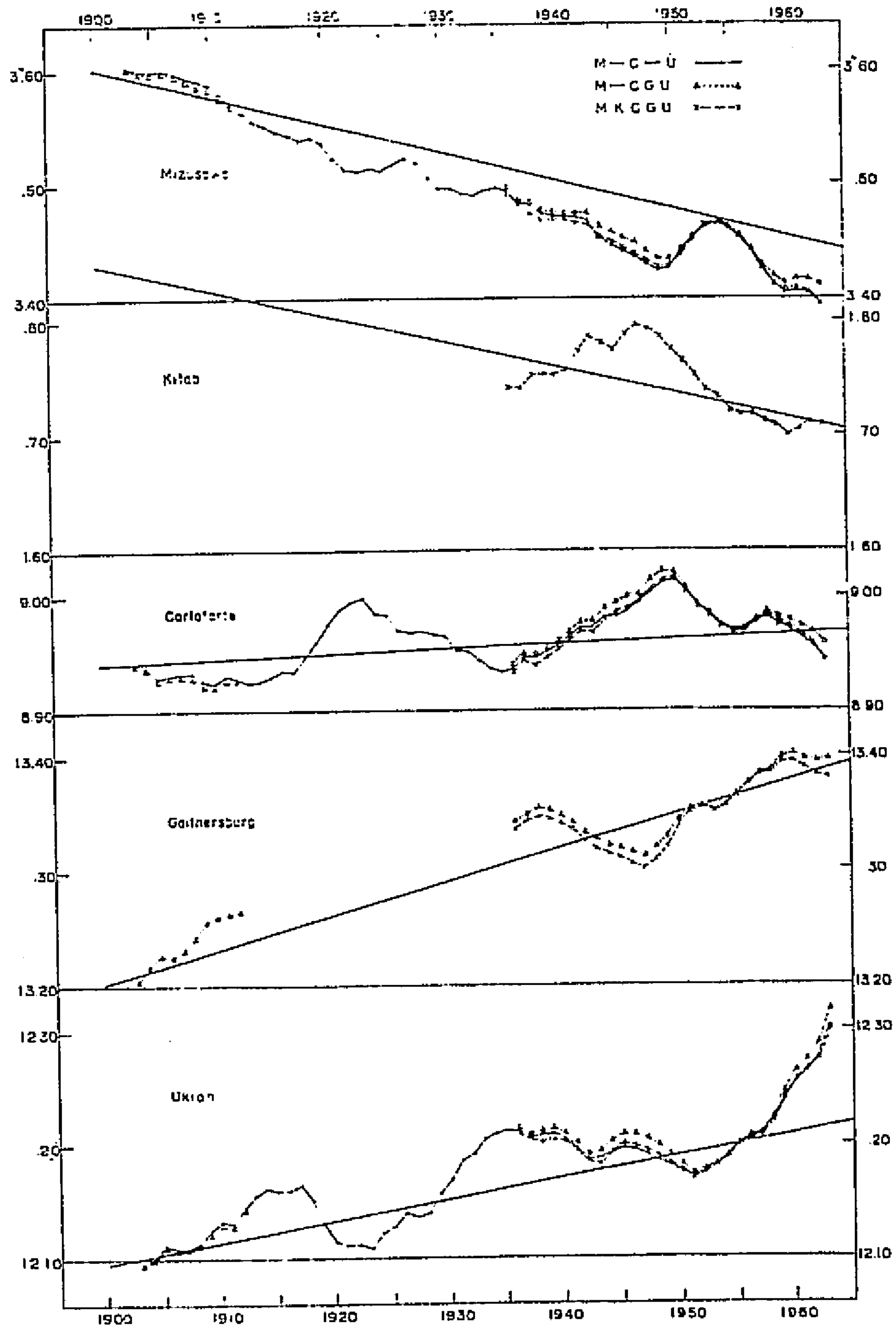



Fig. 5.2. Variation of Latitude at the Five ILS Observatories.
From [Yumi and Wako, 1970].

reduced secular motion of the mean pole. It should be mentioned here that Cecchini in an infrequently referenced paper [Cecchini, 1950] after analyzing astronomical data, shows skepticism about a true secular motion of the barycenter, supporting the almost absolute fixity of the mean pole. Fig. 5.2 from [Yumi and Wako, 1970] shows the variation of latitude at the five ILS stations for an interval of more than 60 years. The straight lines in the figure represent the variation of mean latitude of each observatory if the secular variation of the mean pole as obtained from the five ILS stations is considered. Note that for a sliding crust rotating as mentioned above, the variation of latitude will have the trend shown in the figure, with decrease or increase in station latitude depending on whether they are in the eastern or western hemisphere. It should be noted that the latitude of Carloforte, being close to the Greenwich meridian, does not vary greatly.

It would therefore appear that a crustal sliding over the mantle is a possibility that should not be overlooked, and further research should be devoted to this topic.

Fig. 5.3 represents schematically the actual situation at the north pole, if an assumed easterly slippage of the crust occurs. Two hypothetical epochs $T_0 \sim 1900.0$ and $T_1 \sim 1970.0$ are shown. The axis of figure of the whole earth will hardly change during that period of time. (Although changes in the inertia of the earth due to crustal sliding may move the polar principal axis a few millimeters, there is no way to detect this motion with the present instrumentation). The initial CIO point assumed to be a bench mark on the crust will move with it, thus causing an apparent motion of the mean pole or barycenter, precisely in the direction opposite to the crustal displacement.

In reality the actual apparent displacement of the crust will consist in the composition of two movements:

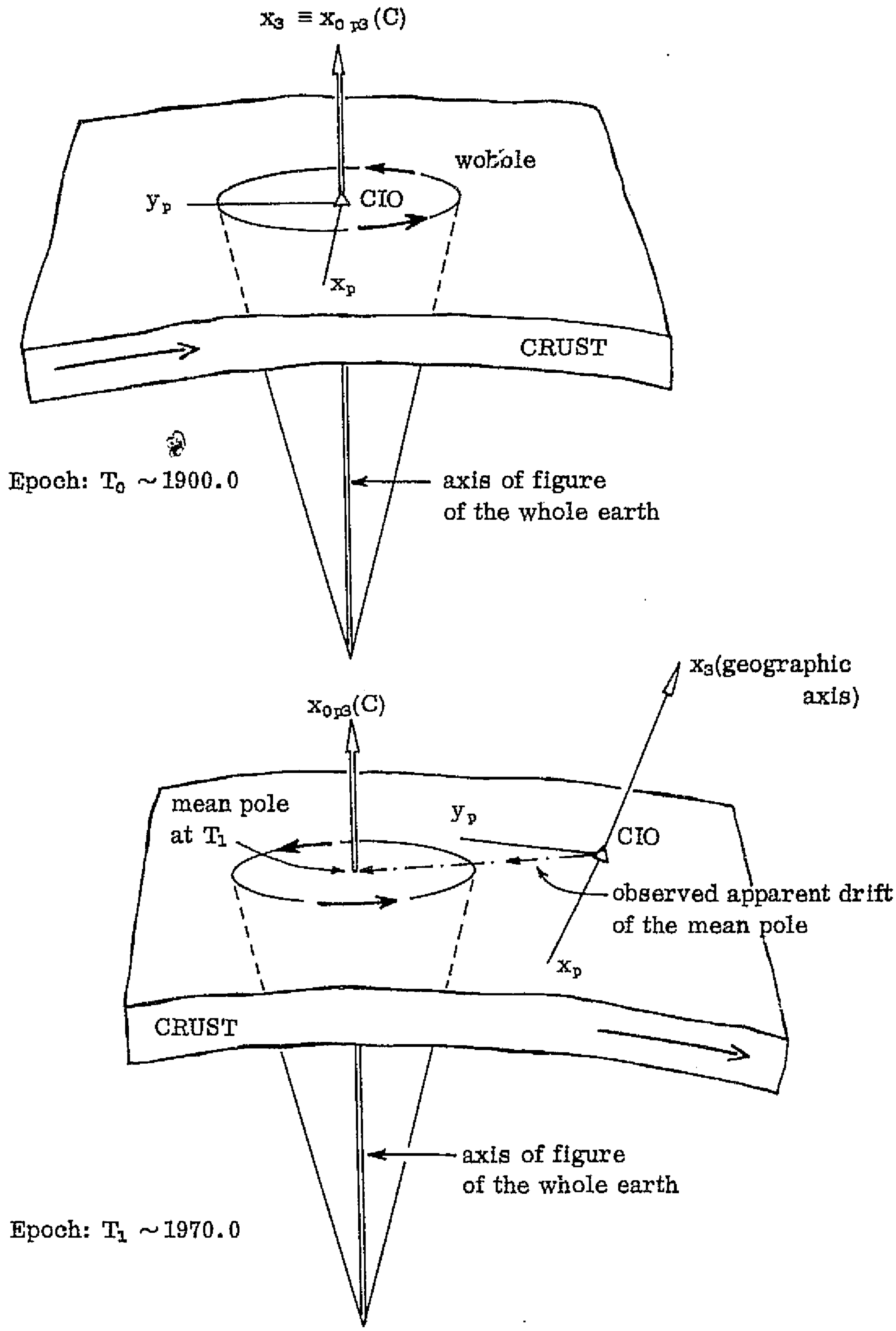



Fig. 5.3. Sliding of the Crust and Apparent Mean Pole Positions

(1) The apparent motion of the mean pole originated by the crustal slippage.

(2) The apparent motion of the barycenter of the wobble due to the drifting of the stations, as a consequence of plate tectonics.

In Fig. 5.4 these two apparent motions are illustrated for a 70-year span. One arrow represents the apparent direction of the principal pole of inertia due to the eastward motion of the crust (its axis of figure pursues the earth principal axis in the opposite direction). The other arrow is the apparent motion of the IPMS barycenter caused by station drifts (model B4 was used here). Observe that the resulting vector of these two motions closely agrees with the actual observed position of the barycenter.

All of the above evidence indicates the necessity of taking a new look at the secular motion of the pole.

For a short time span (< 100 years) the principal polar axis of the earth may be assumed to be fixed, that is, mantle-fixed, since it will pierce the crust at different points, given the apparent impression that it is moving.

Thus it is recommendable that all observations be referred to this quasi mantle fixed axis.

It is general practice today to apply what is known as polar motion correction to reduce the observations to a common CIO system. Unfortunately, what may occur is that, depending on the epoch of the reduction, different ficticious reference axes are obtained. Hence every polar motion reduction today may create a new reference system as a consequence of the possible movement of the crust and its attached CIO point. This is the primary disadvantage of a crust-fixed reference system: astronomically reduced latitudes (or other coordinates) at different epochs do not refer to a common system.

The principal polar axis of inertia of the earth (axis of figure) has the advantage of being dynamical in nature and not strictly geometrical,

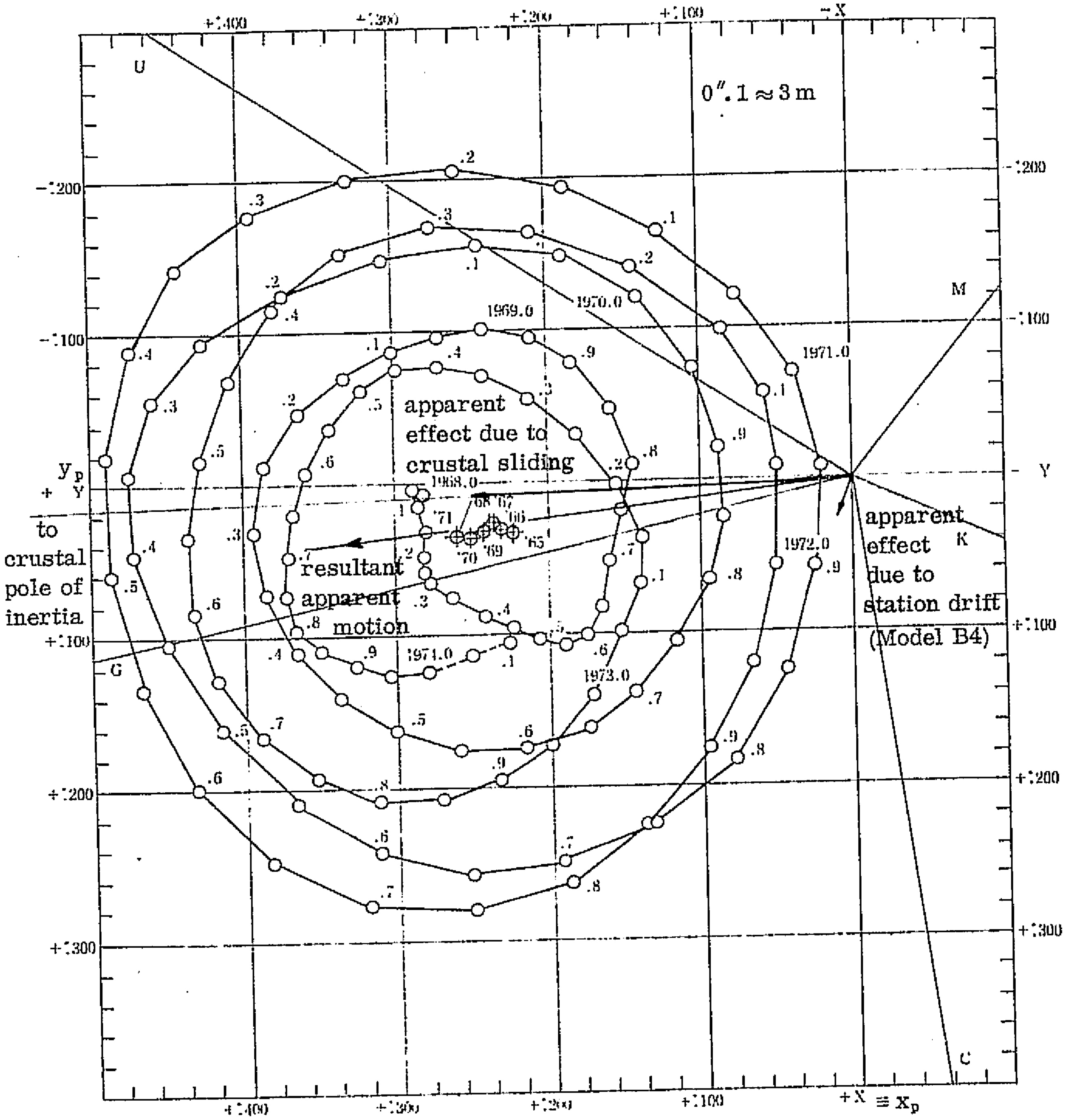



Fig. 5.4. Apparent Displacement of the Mean Pole Due to a Possible Combined Effect of Crust Sliding and IPMS Station Drift

as the CIO axis.

It will be preferible to refer all observations to a unique mantle-fixed system. To be able to do that, geodesists and astronomers need the pole wobble referred always to the same point (polar axis of figure), disregarding the apparent secular motion of the barycenter. In this way polar correction will refer the observatories to the unique principal axis of inertia of the earth. At the same time the international organizations in charge of latitude observation should give the secular displacement of the mean pole, since this would provide a way of knowing, for example, the rate and direction of crust slippage, an important factor in geophysical research.

After about 100 years a new mean pole should be defined to account for possible minor true displacement of the earth's axis of figure.

Proponents of a mantle-fixed reference system are not few in number; among them [Mueller, 1975], [Melchior, 1975] postulated its advantages. Previously Fedorov et al. [1972], after studying the effects of relative displacement of the zeniths at the observing stations, also recommended using the mean pole of the epoch of observation instead of the current CIO system.

Finally, it should be mentioned also that the possibility of a librational component is compatible with the above-described theory of crustal slippage.

If the earth axis of figure has a small periodic motion, as proposed by Busse [1970] due to the coupling between the inner core and the mantle of the earth, this may create an apparent libration, as seen in the astronomically-reduced barycenter.

In all, a careful study about these matters should be undertaken before new resolutions on the subject of reference systems are adopted by the IUGG or IAU.

# APPENDIX A

(Referenced in Sections 2.1.3 and 3.3)

## A.1 Special Types of Linear Transformations

In what follows several important particular cases of the general linear transformation L in the Euclidean space $E^3$ will be studied.

Consider in general the linear transformation given by

$$\{x\} \xrightarrow{[L]} \{x'\} \implies \{x'\} = [L]\{x\} \qquad \text{(A.1-1)}$$

where [L] is known as the matrix of the transformation and satisfies the properties specified in section 2.1.4.

Explicitly equation (A.1-1) may be written:

$$\begin{aligned} x_1' &= \ell_{11}x_1 + \ell_{12}x_2 + \ell_{13}x_3 \\ x_2' &= \ell_{21}x_1 + \ell_{22}x_2 + \ell_{23}x_3 \\ x_3' &= \ell_{31}x_1 + \ell_{32}x_2 + \ell_{33}x_3 \end{aligned} \qquad \text{(A.1-2)}$$

where

$$\ell_{ij} \in \mathbb{R} \quad (\mathbb{R} \equiv \text{set of real numbers})$$

The following specific examples of transformation matrices will be reviewed.

$[P] \equiv$ projective transformation matrix

$[A] \equiv$ antisymmetric transformation matrix

$$[N] \equiv \text{orthogonal transformation matrix} \begin{cases} R_\alpha \equiv \text{rotational} \\ S_\alpha \equiv \text{symmetric} \\ [1] \equiv \text{identity} \end{cases}$$

## A.2 Projective Transformations

When [L] is symmetric, and of the form

$$[P] = \{\alpha\}\{\alpha\}^T = \begin{bmatrix} \alpha_1^2 & \alpha_{12} & \alpha_1\alpha_3 \\ & \alpha_2^2 & \alpha_2\alpha_3 \\ s & & \alpha_3^2 \end{bmatrix} \qquad \text{(A.2-1)}$$

where $\alpha_i \; (i = 1,2,3) \in \mathbb{R}$

and the condition

$$\mathrm{tr}[\{\alpha\}\{\alpha\}^T] = \sum_{i=1}^{3} \alpha_i^2 = 1 \qquad \text{(A.2-2)}$$

is fulfilled, then

$$\{x\} \xrightarrow{[P]} \{x_p'\} \qquad \text{(A.2-3)}$$

is called a projective transformation and [P] its matrix.

The above implies

$$\{x_p'\} = [P]\{x\} = \begin{Bmatrix} \alpha_1(\Sigma\,\alpha_i x_i) \\ \alpha_2(\Sigma\,\alpha_i x_i) \\ \alpha_3(\Sigma\,\alpha_i x_i) \end{Bmatrix} = (\Sigma\alpha_i x_i)\{\alpha\} \qquad \text{(A.2-4)}$$

Geometrically this means that the point $\{x_p'\}$ is the projection of $\{x\}$ on the line with direction cosines $\{\alpha\}$. In this respect the following equivalences are implied

$$\alpha_i \equiv \cos\alpha_i \quad (i = 1,2,3) \qquad \text{(A.2-5)}$$

The above geometric interpretation may be shown very easily using classical vector representations.

If (see Fig. A.2)

$\vec{u} \equiv$ unit vector with direction cosines $\{\alpha\}$

then

$$OP = \{x_p'\} = (\vec{u}.\vec{OQ})\,.\,\vec{u} = (\Sigma\,\alpha_i x_i)\{\alpha\}$$

as was already given by equation (A.2-4).

Among the important properties of the projective transformation matrix are:

(i) $[P]$ is idempotent $\Longrightarrow [P]^2 = [P]$

This is immediately proved,

$$[P]^2 = [P][P] = (\Sigma\,\alpha_i^2)[P] = [P]$$

(ii) The matrix $[P]$ is singular

$[P]$ is singular $\Longrightarrow \det[P] = 0$

$$\det[P] = \begin{vmatrix} \alpha_1^2 & \alpha_1\alpha_2 & \alpha_1\alpha_3 \\ & \alpha_2^2 & \alpha_2\alpha_3 \\ s & & \alpha_3^2 \end{vmatrix} = \alpha_1\alpha_2\alpha_3 \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{vmatrix} = 0$$

and from the above it follows that

$$\text{Rank}\,[P] = 1 = \text{tr}\,[P]$$

Therefore the transformation in (A.2-3) is an application from $E^3$ to $E^1$ (i.e., vectors (points) in the three dimensional Euclidean space $E^3$ are projected onto the line with $\{\alpha\}$ direction cosines).

(iii) The following equalities are easily proved

$$\{x\}^T[P]\{x\} = (\Sigma\,\alpha_i x_i)^2$$

and

$$\{x\}^T\Big[[P] + [P']\Big]\{x\} = (\Sigma\,\alpha_i x_i)^2 + (\Sigma\,\alpha_i' x_i)^2$$

## A.3 Antisymmetric Transformations

The linear transformation [L] given by (A.1-1) is called antisymmetric and denoted by [A] if

$$\{x\} \xrightarrow{[A]} \{x_A'\} \implies \{x_A'\} \text{ is normal to } \{x\}$$

that is,

$$\{x_A'\}^T\{x\} = \{x\}^T\{x_A'\} = 0 \tag{A.3-1}$$

but

$$\{x_A'\} = [A]\{x\} \tag{A.3-2}$$

thus one has

$$\begin{aligned}\{x\}^T\{x_A'\} = \{x\}^T[A]\{x\} &= a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + (a_{23}+a_{32})x_2x_3 \\ &+ (a_{31}+a_{13})x_3x_1 + (a_{12}+a_{21})x_1x_2 = 0\end{aligned} \tag{A.3-3}$$

Hence, in order to fulfill (A.3-3) it is necessary and sufficient that

$$a_{11} = a_{22} = a_{33} = 0$$

$$a_{23} = -a_{32}$$

$$a_{31} = -a_{13}$$

$$a_{12} = -a_{21}$$

Using the notation,

$$a_{32} = a_1 \qquad a_{13} = a_2 \qquad a_{21} = a_3$$

finally one may write

$$[A] \equiv [\underline{a}] = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{A.3-4}$$

and clearly

$$[\underline{a}]^T = -[\underline{a}] \tag{A.3-5}$$

Note that

$$\{x_A'\} = [\underline{a}]\{x\} = [\underline{x}]^T\{a\} = -[\underline{x}]\{a\} = -[\underline{a}]^T\{x\} \tag{A.3-6}$$

and

$$\{x_A'\}^T = \{x\}^T[\underline{a}]^T = \{a\}^T[\underline{x}] = -\{a\}^T[\underline{x}]^T = -\{x\}^T[\underline{a}] \qquad \text{(A.3-7)}$$

which implies in the classical vector representation

$$\vec{x}_A' = \vec{a} \wedge \vec{x} \quad \text{where } \wedge \equiv \text{vector or cross product}$$

and

$$|\vec{x}_A'| = |\vec{a}|\,|\vec{x}| \sin\theta \qquad \text{(A.3-8)}$$

A precautionary note should be made here with respect to the sign convention implicit in the cross product. The rule for the sign of $\vec{x}_A'$ is that of a right-handed (or clockwise as viewed along $\vec{x}_A'$) rotation which carries $\vec{a}$ into $\vec{x}$ through an angle $< 180°$ and making $\sin\theta > 0$.

It is customary in geodesy to use the convention that a positive rotation is one which appears counterclockwise to an observer looking from the positive region of an axis toward the origin. Thus

$$-[\underline{a}] = [\underline{a}]^T \equiv \text{implies positive counterclockwise rotation}$$

Fig. A.1 illustrates the above sign convention with a simple example

Counterclockwise rotation : $[\underline{k}]^T\{j\} = \{i\} \equiv \vec{j} \wedge \vec{k}$

Clockwise rotation : $[\underline{k}]\{j\} = -\{i\} \equiv \vec{k} \wedge \vec{j}$

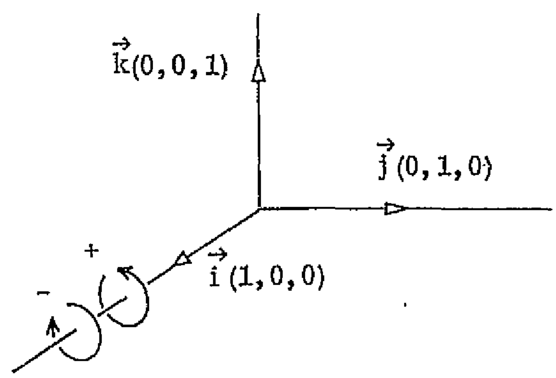



Fig. A.1. Sign Conventions for Clockwise and Counterclockwise Rotations

From (A.3-6) and (A.3-7) it is obvious that in the particular case $\{x\} = \{a\}$

$$[\underline{x}]\{x\} = [\underline{x}]^T\{x\} = -[\underline{x}]\{x\} = \{0\} \qquad \text{(A.3-9)}$$

and

$$\{x\}^T[\underline{x}]^T = \{x\}^T[\underline{x}] = -\{x\}^T[\underline{x}]^T = \{0\}^T \qquad \text{(A.3-10)}$$

It is immediate to prove that

$$\det[\underline{a}] = 0 \quad \text{and} \quad \text{Rank}[\underline{a}] = 2$$

Notice that if

$$|\vec{a}| = 1 \Longrightarrow \Sigma a_i^2 = 1 \Longrightarrow \vec{a} = \vec{u} \text{ (unit vector)}$$

then

$$[\underline{u}] = \begin{bmatrix} 0 & -\alpha_3 & \alpha_2 \\ \alpha_3 & 0 & -\alpha_1 \\ -\alpha_2 & \alpha_1 & 0 \end{bmatrix} = [\underline{\alpha}]$$

where $\{\alpha\}$ are the direction cosines of the unit vector $\vec{u}$.

## A.4 Orthogonal Transformations

It is very well known that in general a transformation like (A.1-1) is called orthogonal when the matrix of the transformation fulfills the property

$$[N]^T = [N]^{-1} \qquad \text{(A.4-1)}$$

If det $[N] = +1$ the transformation is called "proper", or "direct". Every orthogonal direct transformation conserves the length and the angles between vectors.

### A.4.1 Relations between Orthogonal and Antisymmetric Transformation Matrices

(a) Direct orthogonal matrix as a function of antisymmetric matrix

It may be proved that if $[\underline{a}]$ is the matrix of an antisymmetric transformation, then

$$[N] = \Big[[1] - [\underline{a}]\Big]\Big[[1] + [\underline{a}]\Big]^{-1} \qquad (A.4\text{-}2)$$

is the matrix of a direct orthogonal transformation.

Observe that $\det[\underline{a}] = 0$ but $\det\big[[1] + [\underline{a}]\big] = 1 \neq 0$

Given a square matrix $[\underline{a}]$ one has

$$\big[[\underline{a}] + [1]\big]\big[[\underline{a}] - [1]\big] = [\underline{a}]^2 - [1] = \big[[\underline{a}] - [1]\big]\big[[\underline{a}] + [1]\big]$$

In the same way it may be proved that

$$\big[[1] - [\underline{a}]\big]\big[[1] + [\underline{a}]\big]^{-1} = \big[[1] + [\underline{a}]\big]^{-1}\big[[1] - [\underline{a}]\big]$$

Then, if $[\underline{a}]$ is antisymmetric, and

$$[N] = \big[[1] - [\underline{a}]\big]\big[[1] + [\underline{a}]\big]^{-1} \qquad (A.4\text{-}3)$$

one has

$$[N]^T = \big[[1] + [\underline{a}]^T\big]^{-1}\big[[1] - [\underline{a}]^T\big] = \big[[1] - [\underline{a}]\big]^{-1}\big[[1] + [\underline{a}]\big]$$

but

$$[N]^{-1} = \Big[[1] + [\underline{a}]\Big]\Big[[1] - [\underline{a}]\Big]^{-1}$$

thus

$$[N]^T = [N]^{-1}$$

and the statement related to equation (A.4-2) is proved.

(b) Antisymmetric matrix as a function of a direct orthogonal matrix

Reciprocally, if [N] is direct orthogonal, then

$$[\underline{a}] = \Big[[N] + [1]\Big]^{-1}\Big[[1] - [N]\Big] \qquad (A.4\text{-}4)$$

and the following condition must be fulfilled

$$\det\big[[N] + [1]\big] \neq 0 \qquad (A.4\text{-}5)$$

Let equation (A.4-3) be written in the form

$$[N]\big[[1] + [\underline{a}]\big] = [1] - [\underline{a}] \qquad (A.4\text{-}6)$$

or

$$[N][1] + [N][\underline{a}] = [1] - [\underline{a}]$$

which gives

$$\left[[N] + [1]\right][\underline{a}] = [1] - [N]$$

Consequently, if condition (A.4-5) is fulfilled

$$[\underline{a}] = \left[[N] + [1]\right]^{-1}\left[[1] - [N]\right]$$

which proves equation (A.4-4).

Now it will be proved that the given expression for $[\underline{a}]$ in (A.4-4) is antisymmetric. After transposing (A.4-6)

$$\left[[1] + [\underline{a}]^T\right][N]^T = [1] - [\underline{a}]^T \Longrightarrow [1] + [\underline{a}]^T = \left[[1] - [\underline{a}]^T\right][N] \tag{A.4-7}$$

Multiplying (A.4-6) on the left by $[N]^{-1}$

$$[1] + [\underline{a}] = [N]^{-1}\left[[1] - [\underline{a}]\right] \tag{A.4-8}$$

Multiplying (A.4-7) on the right by $[1] + [\underline{a}]$ as given by (A.4-8), one has

$$\left[[1] + [\underline{a}]^T\right]\left[[1] + [\underline{a}]\right] = \left[[1] - [\underline{a}]^T\right]\left[[1] - [\underline{a}]\right]$$

thus finally,

$$\cancel{[1]} + [\underline{a}] + [\underline{a}]^T + \cancel{[\underline{a}]^T[\underline{a}]} = \cancel{[1]} - [\underline{a}] - [\underline{a}]^T + \cancel{[\underline{a}]^T[\underline{a}]}$$

$$2\left[[\underline{a}]^T + [\underline{a}]\right] = [0] \Longrightarrow [\underline{a}]^T = -[\underline{a}]$$

### A.4.2 Orthogonal Transformation Matrix for a Rotation around a Line of Direction Cosines $\{\alpha\}$

In the following treatment all counterclockwise rotations as defined in section A.3 are going to be considered positive.

Suppose one has a vector $\{x\} = \vec{OQ}$ (see Fig. A.2) referred to a Cartesian frame $(x)$. A line of direction cosines $\{\alpha\}$ is given and a rotation $\theta$ around this line performed.

After the rotation the vector $\{x\}$ will be transformed into the $\{x'_R\}$. The problem is to find the orthogonal matrix $R_\alpha(\theta)$ of the transformation,

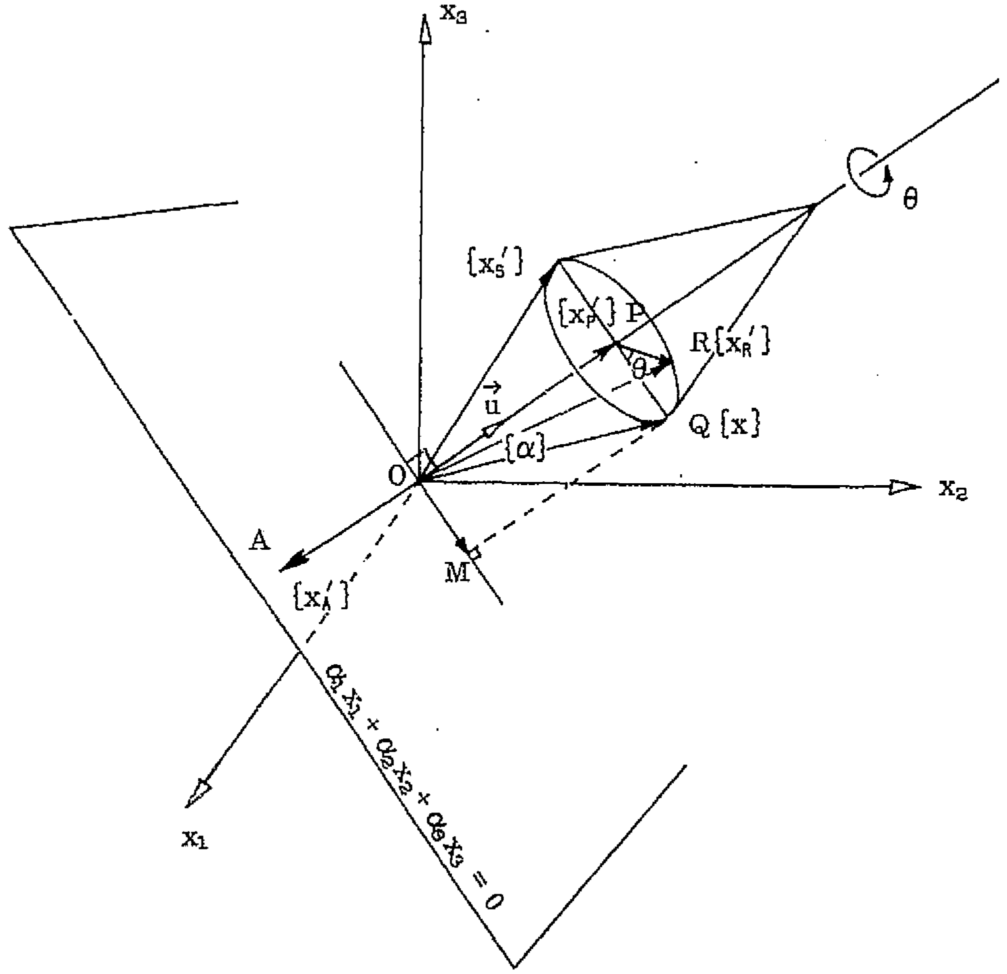



Fig. A.2. Projective, Antisymmetric and Orthogonal Transformations. Geometric Interpretations.

$$\{x\} \xrightarrow{R_\alpha(\theta)} \{x'_R\} \quad \text{or} \quad \{x'_R\} = R_\alpha(\theta)\{x\} \tag{A.4-9}$$

In Fig. A.2 the vectors $\vec{OP} \equiv \{x'_P\}$ and $\vec{OA} \equiv \{x'_A\}$ transformed from $\{x\}$ by the previously studied projective and antisymmetric transformations are shown.

That is,

$$\vec{OP} \equiv \{x'_P\} = [P]\{x\} \tag{A.4-10}$$

where, recalling equation (A.2-1)

$$[P] = \{\alpha\}\{\alpha\}^T$$

and

$$\vec{OA} \equiv \{x'_A\} = [\underline{\alpha}]^T\{x\} = [\underline{x}]\{\alpha\} \tag{A.4-11}$$

From the picture, clearly

$$\{x'_R\} \equiv \vec{OR} = \vec{OP} + \vec{PR} \tag{A.4-12}$$

but

$$\vec{PR} = \vec{OM}\cos\theta - \vec{OA}\sin\theta \tag{A.4-13}$$

and

$$\vec{OM} = \vec{OQ} - \vec{OP} \tag{A.4-14}$$

Therefore, replacing the value given by (A.4-14) in (A.4-13) and this result in (A.4-12), one has

$$\{x'_R\} = \vec{OP} + (\vec{OQ} - \vec{OP})\cos\theta - \vec{OA}\sin\theta \tag{A.4-15}$$

Substituting above the values from (A.4-10) and (A.4-11) one obtains

$$\{x'_R\} = [P]\{x\} + \left\{ \{x\} - [P]\{x\} \right\}\cos\theta + [\underline{\alpha}]\ \{x\}$$

or

$$\{x'_R\} = \Big[ [P] + \big[ [1] - [P] \big]\cos\theta + [\underline{\alpha}]\ \sin\theta \Big]\{x\}$$

Hence, finally recalling (A.4-9)

$$R_\alpha(\theta) = \cos\theta[1] + (1 - \cos\theta)[P] + \sin\theta[\underline{\alpha}] \tag{A.4-16}$$

which gives the value of the rotation matrix $R_\alpha(\theta)$ as a function of the known parameters: the rotation $\theta$ and the direction cosines $\{\alpha\}$.

Note that

$$R_\alpha^T(\theta) = R_\alpha(-\theta) = \cos\theta[1] + (1-\cos\theta)[P] - \sin\theta[\underline{\alpha}] \qquad (A.4\text{-}17)$$

Clearly the above equation will apply to the clockwise rotation case.

In order to prove that $R_\alpha^T(\theta) = R_\alpha^{-1}(\theta)$ (observe that this cannot be done by direct inversion of (A.4-16) because $[P]$ and $[\underline{\alpha}]$ are singular matrices) one will show that

$$R_\alpha^T(\theta)R_\alpha(\theta) = 1$$

After multiplying the matrices

$$R_\alpha^T(\theta) = [P] + [1]\cos\theta - [P]\cos\theta - [\underline{\alpha}]\sin\theta$$

$$R_\alpha(\theta) = [P] + [1]\cos\theta - [P]\cos\theta + [\underline{\alpha}]\sin\theta$$

and after simplification, considering that

(i) $[P]$ is idempotent $\Longrightarrow [P]^2 = [P]$

(ii) $[\underline{\alpha}][P] = [P][\underline{\alpha}] = [0]$

one has

$$R_\alpha^T(\theta)R_\alpha(\theta) = [P] + \cos^2\theta[1] - \cos^2\theta[P] - \sin^2\theta[\underline{\alpha}]^2$$

It may be proved that (see B.1-9 in Appendix B)

$$[\underline{\alpha}]^2 = [P] - (\mathrm{tr}[P])[1]$$

but $\mathrm{tr}[P] = 1$, thus

$$[\underline{\alpha}]^2 = [P] - [1] \qquad (A.4\text{-}18)$$

Then finally

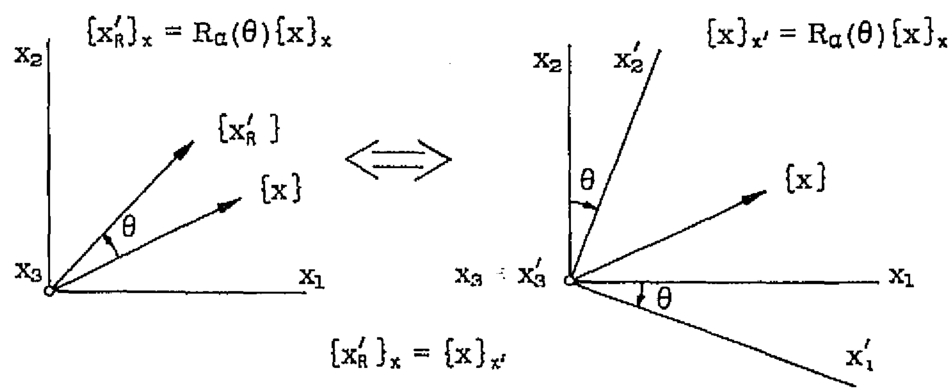
$$R_\alpha^T(\theta)R_\alpha(\theta) = [P] + \cos^2\theta[1] - \cos^2\theta[P] - \sin^2\theta[P] + \sin^2\theta[1] = [1]$$

and the orthogonality condition of $R_\alpha(\theta)$ is fulfilled.

Remark. The transformation matrix $R_\alpha(\theta)$ as derived above rotates (counterclockwise) vectors in $E^3$ while keeping the basic frame $(x)$ fixed.

That is, it transforms the vector components (w.r.t. the same fixed Cartesian system) before the rotation to the components after the rotation. Instead, when the reference frame is rotated (counterclockwise) while keeping the vectors fixed, their components will be transformed by the matrix $R^T_\alpha(\theta)$. The basic concept is self-explained below in Fig. A.3.

a) Rotation of Vectors (Frame-Fixed) b) Rotation of Frame (Vector-Fixed)




Frame-Fixed Vector-Fixed

$R_\alpha(\theta)$ : Counterclockwise rotation of vectors <=> Clockwise rotation of frame

$R^T_\alpha(\theta)$ : Clockwise " " " <=> Counterclockwise " "

Fig. A.3. Rotation of Vectors and Frames

In particular, the following notations apply when the counterclockwise rotations are taken around the three Cartesian axes: $R^T_{\alpha_i}(\theta) \equiv R_i(\theta) \;\; \forall i = 1,2,3$ where $\alpha_i$ $(i = 1,2,3)$ are the direction cosines of the $x_i$ Cartesian axes. This can be easily proved using (A.4-17).

### A.4.3 Knowing the Orthogonal Matrix $R_\alpha(\theta)$, to Obtain the Line about which The Rotation $\theta$ Is Performed

The problem will be divided into two parts:

(a) Computation of the direction cosines $\{\alpha\}$

As shown in the previous section, every orthogonal rotation matrix keeps the direction cosines $\{\alpha\}$ of the rotation axis invariant.

This is also true for the matrix $R^T_\alpha(\theta)$.

Thus, for any vector $\{x\}$ of $\{\alpha\}$ direction cosines, it follows that

$$R_\alpha(\theta)\ \{x\} = R_\alpha^T(\theta)\{x\} = \{x\}$$

whence

$$[R_\alpha(\theta) - R_\alpha^T(\theta)]\ \{x\} = [\underline{a}]\{x\} = \{0\}$$

which implies

$$\vec{a} \wedge \vec{x} = \vec{0} \implies \vec{a} \text{ and } \vec{x} \text{ are collinear}$$

Thus the components of $\vec{a}$ will define the axis of rotation, that is,

$$[\underline{a}] = R_\alpha(\theta) - R_\alpha^T(\theta) = \begin{bmatrix} 0 & r_{12}-r_{21} & r_{13}-r_{31} \\ r_{21}-r_{12} & 0 & r_{23}-r_{32} \\ r_{31}-r_{13} & r_{32}-r_{23} & 0 \end{bmatrix}$$

If the counterclockwise sign rule as defined in section A.2 is followed,

$$a_1 = r_{23}-r_{32} \tag{A.4-19a}$$

$$a_2 = r_{31}-r_{13} \tag{A.4-19b}$$

$$a_3 = r_{12}-r_{21} \tag{A.4-19c}$$

and the direction cosines sought will be given by

$$\alpha_i = \frac{a_i}{|a|} \qquad \forall\ i = 1, 2, 3 \tag{A.4-20}$$

(b) Computation of the rotation angle $\theta$

It is known that the trace of a second-order tensor is invariant under rotation (see section B.3.1). Therefore from equation (A.4-16) it follows

$$\text{tr}\,[R_\alpha(\theta)] = 3\cos\theta + (1-\cos\theta) = 1 + 2\cos\theta$$

and

$$\cos\theta = \frac{\text{tr}[R_\alpha(\theta)] - 1}{2} \tag{A.4-21}$$

As an illustrative example, assume that the following orthogonal matrix is given

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

(i) Direction cosines of the rotation axis

From (A.4-19) it can be found

$$a_1 = 2\sin\theta$$
$$a_2 = 0$$
$$a_3 = 0$$

Thus

$$\alpha_1 = \frac{2\sin\theta}{\sqrt{4\sin^2\theta}} = 1$$

$$\alpha_2 = \alpha_3 = 0$$

(ii) Angle rotated

Using (A.4-21)

$$\cos\theta = \frac{2\cos\theta + 1 - 1}{2} = \cos\theta$$

Therefore, the given matrix is, as expected, $R_1(\theta)$ : rotation of an angle $\theta$ around the first axis.

## A.4. Symmetric (Orthogonal) Transformations

In order that $R_\alpha(\theta)$ be symmetric, it is necessary and sufficient that

$$\sin\theta[\underline{\alpha}] = [0]$$

which implies

$$\sin\theta = 0 \Longrightarrow \theta = n\pi \qquad \forall n \in \mathbb{R}$$

Two particular cases may be studied:

(a) n is an even number. $n = 2k$ , $\forall k \in \mathbb{R}$

Then

$$\theta = 2k\pi \quad \text{and} \quad R_\alpha(2k\pi) = [1] \tag{A.4-22}$$

This was obviously expected, because when a rotation of a multiple of $2\pi$ around a line is given, the transformed vector is itself. Thus the unit matrix is a particular case of rotation for $\theta = 2k\pi$.

(b) n is an odd number $\Longrightarrow n = 2k+1 \quad , \forall k \in \mathbb{R}$

In this case,

$$R_\alpha(\pi) = 2[P] - [1]$$

and

$$\{x_s'\} = R_\alpha(\pi)\{x\} \Longrightarrow \{x_s'\} = 2[P]\{x\} - [1]\{x\}$$

thus

$$\{x_s'\} + \{x\} = 2[P]\{x\}$$

which implies that $\{x_s'\}$ is symmetric to $\{x\}$ with respect to the line of direction cosines $\{\alpha\}$.

Therefore, the matrix of a symmetric transformation [S] is given by

$$S_\alpha = R_\alpha(\pi) = 2[P] - [1] \tag{A.4-23}$$

Thus the matrices of the symmetric transformations are orthogonal and symmetric.

Clearly the symmetry with respect to the origin may be obtained from (A.4-23) when $\{\alpha\} = \{0\}$.

$$S_0 = -[1] = \begin{bmatrix} -1 & 0 & 0 \\ & -1 & 0 \\ s & & -1 \end{bmatrix}$$

In particular, symmetries with respect to the three axes are easily obtained. For example the symmetry respect to the first axis is given as

$$S_1 = \begin{bmatrix} 1 & 0 & 0 \\ & -1 & 0 \\ s & & -1 \end{bmatrix}$$

In general, $S_i$ denotes the symmetry matrix whose elements $s_{mn}$ are equal to zero except

$$s_{ii} = 1$$

$$s_{jj} = s_{kk} = -1$$

## APPENDIX B

(Referenced in Sections 2.2, 2.5.3, 4.2, 4.3, A.4, D.1, D.2 and C.2)

### B.1 General Comments

This appendix summarizes the transformations between the second-order inertia tensors [ I ] and [ II ] as defined in Section 2.2. When the context makes the difference between the symbols [ I ] and [ II ] clear, the word "associate" for the inertia tensor will be omitted.

The accuracy of present observations (VLBI, Lunar Laser Range) will require consideration of these tensor transformations between selected frames of reference. It is standard practice today to give the principal moments of inertia of the earth without clearly specifying the system to which they are practically referred. The same situation applies to the set of satellite-derived potential coefficients.

This work treats the tensors of inertia as individual entities. This generalization results, in particular cases relating moments and products of inertia, some of which may be found in classical textbooks on dynamics [Routh, 1905, I]. From the point of view of the reader, the anlysis is greatly simplified because only the basic concepts in matrix algebra are needed to establish the dynamical formalism.

Based on Binet's theorem, a new approach is presented for computing the moments of inertia and the directions of the principal axes at any point of a body. This is generalized to the diagonalization of matrices using particular rotation matrices.

#### B.1.1 Relations between the Tensors [ II ] and [ I ]

The following matrix equality is easy to prove,

$$\{x\}\{y\}^T = [\underline{y}][\underline{x}] + (\{x\}^T\{y\})[1] \qquad \text{(B.1-1)}$$

where the coefficient in parenthesis may be expressed in one of the following forms:

$$\{x\}^T\{y\} = \{y\}^T\{x\} = \mathrm{tr}[\{x\}\{y\}^T] = \mathrm{tr}[\{y\}\{x\}^T] \qquad \text{(B.1-2)}$$

Similarly,

$$\{y\}\{x\}^T = [\{x\}\{y\}^T]^T = [\underline{x}][\underline{y}] + (\{x\}^T\{y\})[1] \qquad \text{(B.1-3)}$$

and properties analogous to (B.1-2) hold.

From (B.1-1) and (B.1-3) one has

$$\{x\}\{y\}^T = [\underline{y}][\underline{x}] - [\underline{x}][\underline{y}] + \{y\}\{x\}^T \qquad \text{(B.1-4)}$$

or

$$\{y\}\{x\}^T = [\underline{x}][\underline{y}] - [\underline{y}][\underline{x}] + \{x\}\{y\}^T \qquad \text{(B.1-5)}$$

Equations (B.1-4) or (B.1-5) immediately yield

$$[\underline{x}][\underline{y}]^T = [\underline{y}]^T[\underline{x}] + \{x\}\{y\}^T - \{y\}\{x\}^T \qquad \text{(B.1-6)}$$

Consequently, in general

$$[\underline{x}][\underline{y}]^T \neq [\underline{y}]^T[\underline{x}]$$

In the particular case when $\{x\} = \{y\}$ the following matrix relations may be established:

(i) From (B.1-6)

$$[\underline{x}][\underline{x}]^T = [\underline{x}]^T[\underline{x}]$$

which is obvious from the basic definition of skew-symmetric matrices and was already given by equation (2.2-2).

(ii) From (B.1-1)

$$\{x\}\{x\}^T = [\underline{x}]^2 + (\{x\}^T\{x\})[1] \tag{B.1-7}$$

where clearly recalling (B.1-2)

$$\{x\}^T\{x\} = \mathrm{tr}[\{x\}\{x\}^T] \tag{B.1-8}$$

Thus finally, one concludes

$$[\underline{x}]^2 = \{x\}\{x\}^T - (\{x\}^T\{x\})[1] \tag{B.1-9}$$

The above equation gives another way of expressing the associate tensor of inertia,

$$[I] = -\int_M [\underline{x}]^2 dm = \int_M \Big[(\{x\}^T\{x\})[1] - \{x\}\{x\}^T\Big] dm \tag{B.1-10}$$

Taking into consideration (2.2-4) and (2.2-5) the relation between the matrix tensor $[\mathrm{II}]$ and $[I]$ follows immediately,

$$[I] = (\mathrm{tr}[\mathrm{II}])[1] - [\mathrm{II}] \tag{B.1-11}$$

Applying traces on both sides of the above matrix equation, one has

$$\mathrm{tr}[I] = 3(\mathrm{tr}[\mathrm{II}]) - \mathrm{tr}[\mathrm{II}]$$

which results in the relation between the traces of $[\mathrm{II}]$ and $[I]$

$$\mathrm{tr}[I] = 2\,\mathrm{tr}[\mathrm{II}] \tag{B.1-12}$$

Hence it is now possible to express $[\mathrm{II}]$ as a function of $[I]$. Introducing (B.1-12) in (B.1-11)

$$[\mathrm{II}] = \frac{1}{2}(\mathrm{tr}[I])[1] - [I] \tag{B.1-13}$$

From (B.1-13) or (B.1-11) it immediately follows,

$$A < B < C \Longrightarrow \mathrm{II}_{11} > \mathrm{II}_{22} > \mathrm{II}_{33}$$

B.1.2 Variations of $[\mathrm{II}]$ and $[\mathrm{I}]$ with respect to Time

The following notation will be employed

$$[\dot{\mathrm{I}}] = \frac{d}{dt}[\mathrm{I}]$$

where as usual the dot means differentiation with respect to time. Similar representation will apply for $[\mathrm{II}]$.

Differentiating (B.1-11) with respect to time

$$[\dot{\mathrm{I}}] = \frac{d}{dt}(\mathrm{tr}[\mathrm{II}])[1] - [\dot{\mathrm{II}}] \tag{B.1-14}$$

but

$$\frac{d}{dt}(\mathrm{tr}[\mathrm{II}]) = \frac{d}{dt}\left(\int_M \{x\}^T\{x\}\,dm\right) = \int_M \{\dot{x}\}^T\{x\}dm + \int_M \{x\}^T\{\dot{x}\}\,dm$$

and recalling (B.1-2) one may write

$$\int_M \{\dot{x}\}^T\{x\}dm = \int_M \{x\}^T\{\dot{x}\}dm = \int_M \mathrm{tr}[\{\dot{x}\}\{x\}^T]\,dm = \int_M \mathrm{tr}[\{x\}\{\dot{x}\}^T]\,dm \tag{B.1-15}$$

Thus

$$\frac{d}{dt}(\mathrm{tr}[\mathrm{II}]) = 2\int_M \{\dot{x}\}^T\{x\}\,dm \tag{B.1-16}$$

Similarly from (B.1-12)

$$\frac{d}{dt}(\mathrm{tr}[\mathrm{I}]) = 2\frac{d}{dt}(\mathrm{tr}[\mathrm{II}]) = 4\int_M \{\dot{x}\}^T\{x\}\,dm \tag{B.1-17}$$

Therefore (B.1-14) may be rewritten in one of the following forms

$$[\dot{\mathrm{I}}] = 2\left(\int_M \{\dot{x}\}^T\{x\}\,dm\right)[1] - [\dot{\mathrm{II}}] \tag{B.1-18}$$

$$[\dot{I}] = \frac{1}{2}\frac{d}{dt}(\mathrm{tr}[I])[1] - [\dot{\mathbb{I}}] \qquad (B.1\text{-}19)$$

or finally,

$$[\dot{I}] = \frac{1}{2}(\dot{A} + \dot{B} + \dot{C})[1] - [\dot{\mathbb{I}}] \qquad (B.1\text{-}20)$$

The values of $[\dot{\mathbb{I}}]$ are readily available from any one of the above equations.

## B.2 Effect of Coordinate System Translations on the Tensors of Inertia

As implied by their definition, the values of $[\mathbb{I}]$ and $[I]$ refer to a particular Cartesian coordinate system with origin at some specified point.

The effects of a translation of the coordinate system on the tensors $[\mathbb{I}]$ and $[I]$ and their traces will be investigated. In the first place, the formulation of the problem under the assumption that one of the coordinate systems is central is considered.

### B.2.1 General Theorem of Parallel Planes

This theorem relates the moments of inertia of a body about a plane through the center of mass to the moments of inertia about any other parallel plane, and viceversa. The proof given in this section is general and based on tensors of inertia.

Suppose that the following values are known,

$M$ ≡ total mass of the body in question

$[\mathbb{I}_o]$ ≡ value of the tensor of inertia referred to a system $(x_o)$ centered at the CM ("central tensor of inertia")

$\{\delta x_o\} \;\equiv\;$ coordinates of the origin of the (final) system $(x)$ after a parallel displacement with respect to the $(x_o)$ frame

From figure B.1 one has immediately

$$\{x\} = \{x_o\} - \{\delta x_o\} \tag{B.2-1}$$

Thus, after substituting (B.2-1) in the definition of $[\mathbb{I}]$ given by equation (2.2-4),

$$[\mathbb{I}] = \int_M \{x\}\{x\}^T dm = \int_M \{x_o\}\{x_o\}^T dm - \{\delta x_o\}\int_M \{x_o\}^T dm$$
$$- \{\delta x_o\}^T \int_M \{x_o\} dm + \{\delta x_o\}\{\delta x_o\}^T \int_M dm \tag{B.2-2}$$

It is well known that

$$\int_M dm = M \quad ; \quad \int_M \{x_o\}^T dm = \{0\}^T \quad ; \quad \int_M \{x_o\} dm = \{0\} \tag{B.2-3}$$

Therefore, finally

$$[\mathbb{I}] = [\mathbb{I}_o] + M\{\delta x_o\}\{\delta x_o\}^T \tag{B.2-4}$$

The second-rank tensor $\{\delta x_o\}\{\delta x_o\}^T$ will be called the "translation tensor" and represented by the (translation) matrix,

$$[\Delta_o] = \{\delta x_o\}\{\delta x_o\}^T \tag{B.2-5}$$

The following theorem holds: The tensor of inertia with respect to any orthogonal frame parallel to the central coordinate system, is equal to the central tensor of inertia plus the product of the mass of the body and the translation tensor between the two systems.

It should be noted that equation (B.2-4) is a generalization of the often-quoted theorem of parallel planes, according to which: The moment of inertia of a body with respect to any plane is equal to the moment of inertia with respect to the plane through the CM plus the product of the mass of

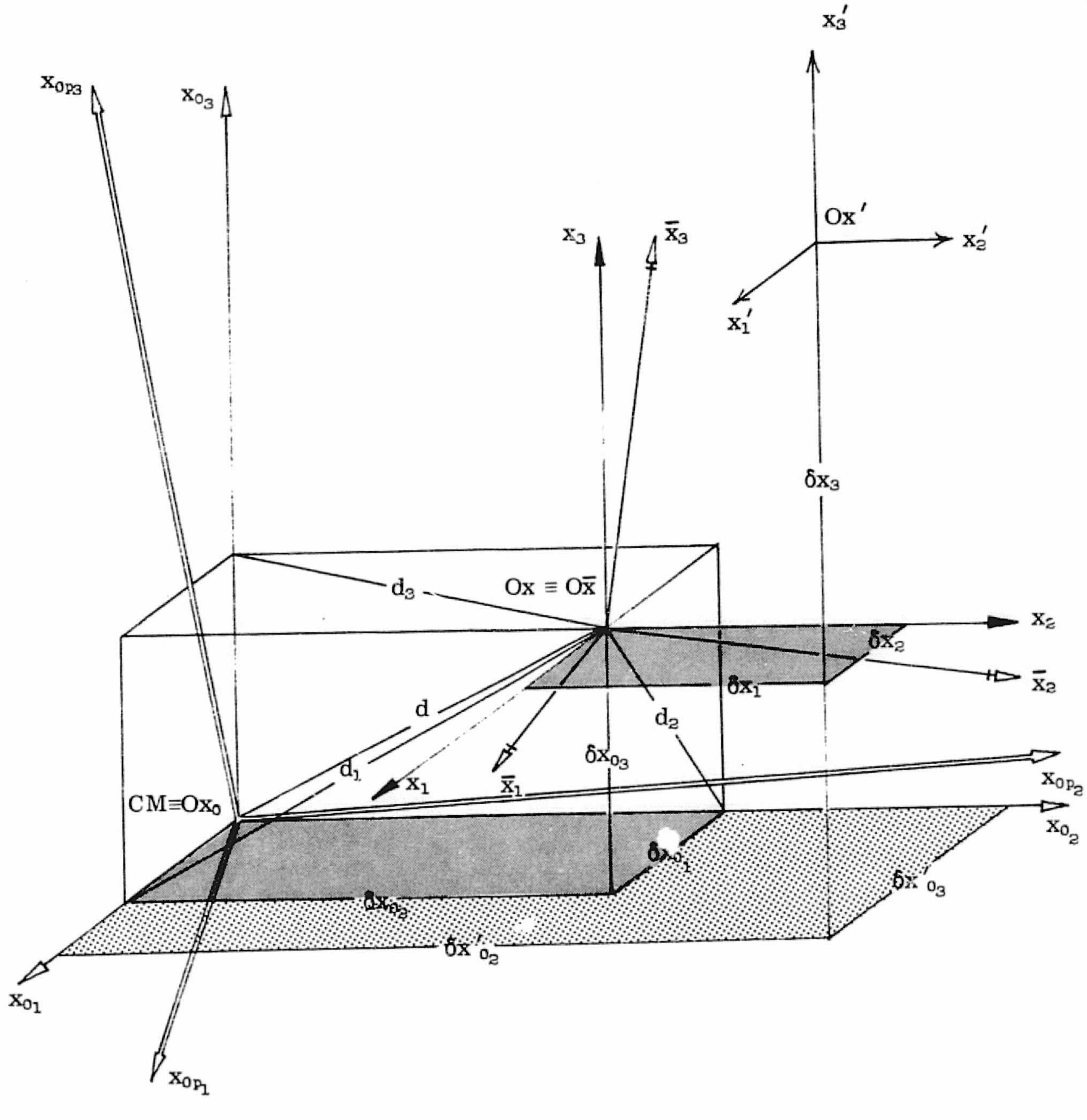



Fig. B.1. Transformation of Tensors of Inertia Associated to Particular Coordinate Systems the body and the square of the distance between the two planes.

$$\mathbb{I}_{PL} = \mathbb{I}_{O\,PL} + M d^2_{PL}$$

From (B.2-4) it is obvious that

$$[\mathbb{I}_O] = [\mathbb{I}] - M[\Delta_O] \qquad \text{(B.2-6)}$$

which gives the central tensor of inertia as a function of the parallel tensor of inertia about any point and its translation matrix.

Taking traces on both sides of the matrix equation (B.2-4)

$$\mathrm{tr}[\mathbb{I}] = \mathrm{tr}[\mathbb{I}_O] + M\,\mathrm{tr}[\Delta_O] \qquad \text{(B.2-7)}$$

but (see fig. B.2-1)

$$\mathrm{tr}[\Delta_O] = \delta x^2_{01} + \delta x^2_{02} + \delta x^2_{03} = d^2 \qquad \text{(B.2-8)}$$

where

$d \equiv$ distance between the origins of the $(x_O)$ and $(x)$ systems.

Thus (B.2-7) expresses what may be called the Theorem of Origin: The moment of inertia with respect to the origin of any system is equal to the moment of inertia with respect to the CM plus the product of the mass of the body and the square of the distance between the origin of the two systems. The above can be applied to any point in $E^3$ and may also be obtained independently after substitution of (B.2-1) in (2.2-5).

B.2.2 General Theorem of Parallel Axes

As in the preceding section, the well-known theorem of parallel axes will be proved through a more general theorem relating the associate tensors of inertia.

Substituting (B.1-12) into (B.2-7)

$$\frac{1}{2}\,\mathrm{tr}[\,I\,] = \frac{1}{2}\,\mathrm{tr}[\,I_o\,] + M\,\mathrm{tr}[\,\Delta_o\,]$$

or

$$\mathrm{tr}[\,I\,] = \mathrm{tr}[\,I_o\,] + 2M\,\mathrm{tr}[\,\Delta_o\,] \tag{B.2-9}$$

Recalling (B.1-13) and (B.2-9) one may write

$$[\,\mathrm{II}\,] = \frac{1}{2}(\mathrm{tr}[\,I_o\,])[\,1\,] + M(\mathrm{tr}[\,\Delta_o\,])[\,1\,] - [\,I\,] \tag{B.2-10}$$

From (B.1-13) it is known that

$$[\,\mathrm{II}_o\,] = \frac{1}{2}(\mathrm{tr}[\,I_o\,])[\,1\,] - [\,I_o\,] \tag{B.2-11}$$

Replacing the values of (B.2-10) and (B.2-11) in (B.2-4) and after simplification

$$[\,I\,] = [\,I_o\,] + M[\,\Delta_o\,] \tag{B.2-12}$$

where the "associate translation tensor" $[\,\Delta_o\,]$ is given by the matrix

$$[\,\Delta_o\,] = (\{\delta x_o\}^T\{\delta x_o\})[\,1\,] - \{\delta x_o\}\{\delta x_o\}^T = [\underline{\delta x_o}][\underline{\delta x_o}]^T = -[\underline{\delta x_o}]^2 \tag{B.2-13}$$

Hence the following theorem may be stated: The associate tensor of inertia with respect to any frame $(x)$ parallel to a central coordinate system $(x_o)$ is equal to the associate central tensor of inertia plus the product of the mass of the body and the associate translation tensor between the origins.

Observe that as an application of equation (B.2-12) the moments of a body with respect to a given line or axis can be obtained.

From (B.2-13) and fig. B.2.1, one may write,

$$[\,\Delta_o\,] = \begin{bmatrix} d^2_1 & -\delta x_{o1}\delta x_{o2} & -\delta x_{o1}\delta x_{o3} \\ & d^2_2 & -\delta x_{o2}\delta x_{o3} \\ s & & d^2_3 \end{bmatrix} \tag{B.2-14}$$

Thus from (B.2-12) the parallel axis rule follows immediately

$$I_{AX} = I_{O\,AX} + M d^2_{AX} \qquad \text{(B.2-15)}$$

The moment of inertia of a rigid body about a given axis through a point is equal to the moment of inertia about a parallel axis through the CM plus the product of the mass and the square of the distance between the two axes.

Finally one should notice that equations (B.2-4) and (B.2-12) give exactly the same values for the products of inertia. This was expected by simple consideration of the definition of the quantities D, E and F.

### B.2.3 Relation between Two Non-Central "Parallel" Tensors of Inertia

In this section the equation relating two non-central tensors of inertia $[\mathbb{I}]$ and $[\mathbb{I}']$ referred to parallel systems $(x)$ and $(x')$ respectively, will be given.

From (B.2-4) and the shift sign convention implied by fig. B.1,

$$[\mathbb{I}] = [\mathbb{I}_o] + M\{\delta x_o\}\{\delta x_o\}^T$$

and

$$[\mathbb{I}'] = [\mathbb{I}_o] + M\{\delta x'_o\}\{\delta x'_o\}^T$$

Thus

$$[\mathbb{I}'] = [\mathbb{I}] + M[\{\delta x'_o\}\{\delta x'_o\}^T - \{\delta x_o\}\{\delta x_o\}^T] \qquad \text{(B.2-16)}$$

or

$$[\mathbb{I}'] = [\mathbb{I}] + M \;\; [\Delta'_o] - [\Delta_o] \qquad \text{(B.2-17)}$$

If the shifts $\{\delta x\}$ between the $(x)$ and $(x')$ systems are known

$$\{\delta x\} = \{\delta x'_o\} - \{\delta x_o\} \implies \{\delta x'_o\} = \{\delta x\} + \{\delta x_o\}$$

which can be substituted in (B.2-16) and gives

$$[\mathbb{I}'] = [\mathbb{I}] + M[\{\delta x\}\{\delta x\}^T + \{\delta x\}\{\delta x_o\}^T + \{\delta x_o\}\{\delta x\}^T] \qquad (B.2\text{-}18)$$

A consequence of (B.2-18) is that to obtain the relation between two non-central "parallel" tensors of inertia $[\mathbb{I}]$ and $[\mathbb{I}']$, the position of at least one of them with respect to the central $(x_o)$ system is required, in addition to knowing the shifts between the $(x)$ and $(x')$ systems.

Similarly to (B.2-17) one has for the associate tensor of inertia,

$$[I'] = [I] + M \ [\Delta'_o] - [\Delta_o] \qquad (B.2\text{-}19)$$

and the same conclusion mentioned above also applies here.

## B.3 Effect of System Rotations on Tensors of Inertia

In this case one is seeking the tensor transformation

$$[\mathbb{I}] \xrightarrow{R} [\overline{\mathbb{I}}] \qquad (B.3\text{-}1)$$

where $R$ is the orthogonal matrix of the mapping between the coordinate systems $(x)$ and $(\bar{x})$, to which the two tensors are referred, that is

$$\{x\} \xrightarrow{R} \{\bar{x}\} \qquad \text{or} \qquad \{\bar{x}\} = R\{x\} \qquad (B.3\text{-}2)$$

Applying (B.3-2) to the definition of $[\overline{\mathbb{I}}]$

$$[\overline{\mathbb{I}}] = \int_M \{\bar{x}\}\{\bar{x}\}^T dm = \int_M R\{x\}\{x\}^T R^T dm = R\int_M \{x\}\{x\}^T dm\, R^T ,$$

which gives

$$[\overline{\mathbb{I}}] = R[\mathbb{I}]R^T \qquad (B.3\text{-}3)$$

This proves an important theorem in tensor analysis for transforming second-rank tensors under rotation. Observe the simplicity by which it was obtained here, using only the notions of matrix calculus.

### B.3.1 Invariants under Rotation

By definition,

$$\mathrm{tr}\,[\,\overline{\mathbb{I}}\,] = \int_M \{x\}^T \{x\}\, dm$$

Substituting (B.3-2) above gives

$$\mathrm{tr}\,[\,\overline{\mathbb{I}}\,] = \int_M \{x\}^T R^T R\, \{x\}\, dm = \int_M \{x\}^T \{x\}\, dm = \mathrm{tr}\,[\,\mathbb{I}\,]$$

Thus

$$\mathrm{tr}\,[\,\overline{\mathbb{I}}\,] = \mathrm{tr}\,[\,\mathbb{I}\,] \qquad \text{(B.3-4)}$$

and recalling (B.1-17)

$$\mathrm{tr}\,[\,\overline{I}\,] = \mathrm{tr}\,[\,I\,] \qquad \text{(B.3-5)}$$

Consequently the trace of the inertia matrix is invariant under rotation. This invariance of the traces of the inertia tensors makes an important check when equation (B.3-3) is numerically computed.

It also follows that the determinant of the inertia matrix is invariant under rotation. Taking determinants in (B.3-3) and after consideration of the orthogonality condition of $R$ ($R$ orthogonal $\Longrightarrow$ $\det R = \det R^T = +1$),

$$\det\,[\,\overline{\mathbb{I}}\,] = \det R \,\det\,[\,\mathbb{I}\,]\,\det R^T = \det\,[\,\mathbb{I}\,] \qquad \text{(B.3-6)}$$

The same properties and transformations apply to the associate tensor of inertia.

Equation (B.3-3) holds in general for any transformation under rotation of second-rank Cartesian tensors. Thus, for the associate tensor of inertia one can write,

$$[\,\overline{I}\,] = R\,[\,I\,]\,R^T \qquad \text{(B.3-7)}$$

The above relation may be proved independently after substituting (B.3-18) in (B.3-3) and considering (B.3-5).

Recalling that the matrix R is orthogonal, from (B.3-3) and (B.3-7) it immediately follows that

$$[\mathrm{II}] = R^T[\overline{\mathrm{II}}]R \qquad (B.3\text{-}8)$$

and

$$[\mathrm{I}] = R^T[\overline{\mathrm{I}}]R \qquad (B.3\text{-}9)$$

It is known that R is the matrix of direction cosines of the three Cartesian axes of the $(\bar{x})$ system with respect to the $(x)$ system.

### B.3.2 Applications

(1) Transformations of variance-covariance matrices

At this point one should notice that an immediate application of (B.3-3) to geodetic problems is the transformation of point variance-covariance matrices (VCM) between different Cartesian coordinate systems. In this case, if

$$\{Y\} = R\{X\}$$

then

$$[\Sigma_Y] = R[\Sigma_X]R^T \qquad (B.3\text{-}10)$$

where

$[\Sigma_X] \equiv$ VCM with respect to the X system

$[\Sigma_Y] \equiv$ VCM with respect to the Y system

When the transformation of variance-covariance matrices between Cartesian and curvilinear geodetic coordinates is intended, then the following linearized relation applies [Soler, 1976],

$$(dx_1, dx_2, dx_3) \xrightarrow{J} (d\lambda, d\varphi, dh)$$

or

$$\begin{Bmatrix} d\lambda \\ d\varphi \\ dh \end{Bmatrix} = J^{-1} \begin{Bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{Bmatrix} \qquad \text{(B.3-11)}$$

where $J$ is the Jacobian matrix of the transformation between Cartesian and curvilinear geodetic coordinates.

Thus if the transformation between local VCM of Cartesian and curvilinear geodetic coordinates is desired, one has

$$[\Sigma_{\lambda\varphi h}] = J^{-1}[\Sigma_{x_1x_2x_3}](J^{-1})^T \qquad \text{(B.3-12)}$$

where

$$J^{-1} = \begin{bmatrix} -\dfrac{\sin\lambda}{(N+h)\cos\varphi} & \dfrac{\cos\lambda}{(N+h)\cos\varphi} & 0 \\ -\dfrac{\sin\varphi\cos\lambda}{M+h} & -\dfrac{\sin\varphi\sin\lambda}{M+h} & \dfrac{\cos\varphi}{M+h} \\ \cos\varphi\cos\lambda & \cos\varphi\sin\lambda & \sin\varphi \end{bmatrix}$$

Note that in this case the second-rank tensor $[\Sigma_{\lambda\varphi h}]$ is not "Cartesian" and the transformation matrix is not orthogonal, but of the Jacobian type. As a consequence, the following conversion of units will hold:

$$\underset{(\text{all } m^2)}{[\Sigma_{x_1x_2x_3}]} \longrightarrow \begin{bmatrix} (\text{rad})^2 & (\text{rad})^2 & \text{rad} \times \text{m} \\ & (\text{rad})^2 & \text{rad} \times \text{m} \\ s & & m^2 \end{bmatrix}$$

Observe that the analytical form of equation (B.3-12) cannot be obtained directly through the classical propagation of errors approach, because the curvilinear geodetic coordinates are not expressable as simple functions of the Cartesian coordinates.

(2) Momental and gyration ellipsoids

A particular problem often mentioned in dynamics is the computation of the moment of inertia with respect to a line of known direction cosines $\{\alpha\}$ through the origin of coordinates. In this case the matrix R in (B.3-7) will be reduced to a single row matrix $\{\alpha\}^T$.

Thus, the moment of inertia with respect to the line (axis) when the direction cosines are known is given by

$$I_{AX} = \{\alpha\}^T [I] \{\alpha\} \tag{B.3-13}$$

Similarly, the moment of inertia with regard to a plane whose normal has direction cosines $\{\alpha\}$, is

$$\mathbb{I}_{PL} = \{\alpha\}^T [\mathbb{I}] \{\alpha\} \tag{B.3-14}$$

If one now considers the following relations

$$\{x\} = \tilde{\rho}\{\alpha\} \tag{B.3-15}$$

where $\tilde{\rho}$ is the radius vector of the $\{x\}$ point, that is,

$$\tilde{\rho}^2 = \{x\}^T\{x\} = x_1^2 + x_2^2 + x_3^2$$

then

$$\{\alpha\} = \frac{1}{\tilde{\rho}}\{x\} \qquad \text{and} \qquad \{\alpha\}^T = \frac{1}{\tilde{\rho}}\{x\}^T$$

After substituting the above equation in (B.3-13),

$$\tilde{\rho}^2 I_{AX} = \{x\}^T [I] \{x\}$$

and taking the value of the radius vector equal to

$$\tilde{\rho} = \sqrt{\frac{M\epsilon^4}{I_{AX}}}$$

one will have,

$$\{x\}^T [I] \{x\} = M\epsilon^4 = k = \text{constant} \tag{B.3-16}$$

which is the equation of a quadric associate to the tensor $[I]$ and the point Ox; $\epsilon$ is an arbitrary length introduced to keep the dimensions of equation (B.3-16) correct. This quadric (known as the inertia or momental ellipsoid) has the property that the moment of inertia about any radius vector is inversely proportional to the square of that radius vector.

Thus $I_{AX}$ will be maximum when $\tilde{\rho}$ is minimum, and viceversa. Furthermore, if $a_m$, $b_m$, $c_m$ are the semiaxes of the momental ellipsoid at any point P with respect to which $A_p$, $B_p$ and $C_p$ are the principal moments of inertia and if $A_p < B_p < C_p$, then $a_m > b_m > c_m$.

The axes of this ellipsoid are the principal axes of the body at the point. The relation between the principal moments of inertia and the semiaxes of the momental ellipsoid is immediate. Using equation (B. 3-16) one may write

$$\{x\}^T [I] \{x\} = k \qquad \text{(B.3-17)}$$

or

$$A_p x_1^2 + B_p x_2^2 + C_p x_3^2 = k \quad \Rightarrow \quad \frac{x_1^2}{\frac{k}{A_p}} + \frac{x_2^2}{\frac{k}{B_p}} + \frac{x_3^2}{\frac{k}{C_p}} = 1$$

and finally,

$$a_m^2 = k/A_p \ , \qquad b_m^2 = k/B_p \ , \qquad c_m^2 = k/C_p \qquad \text{(B. 3-18)}$$

which gives

$$A_p < B_p < C_p \implies a_m > b_m > c_m$$

If two principal moments at any point P are equal, the ellipsoid becomes a rotational ellipsoid; if the three principal moments are equal, the associate quadric to the tensor $[I]$ becomes a sphere, and every diameter is a principal axis.

It should be noticed that the longest and the shortest axes of the central momental ellipsoid (which is unique in a solid body) coincide in direction with the longest and the shortest axes respectively of the material body. Thus the central momental ellipsoid resembles the general shape of the body, is elongated when the body is elongated and flattened when the body is flattened.

The reciprocal surface of the momental ellipsoid is termed "ellipsoid of gyration". Its equation is given by

$$\frac{x^2_1}{A_p} + \frac{x^2_2}{B_p} + \frac{x^2_3}{C_p} = \frac{1}{M} \tag{B.3-19}$$

and thus

$$a^2_g = A_p/M \ , \quad b^2_g = B_p/M \ , \quad c^2_g = C_p/M \tag{B.3-20}$$

Therefore

$$A_p < B_p < C_p \implies a_g < b_g < c_g$$

Hence the (central) ellipsoid of gyration has the reciprocal shape of the body and its (central) momental ellipsoid.

## B.4 Final Transformation Equations

The final transformation equations for the inertia tensors $[\mathbb{I}]$ and $[I]$ after consideration of parallel shifts with respect to the CM and rotations of the Cartesian system, may be written by the following respective equations:

$$[\mathbb{I}] = R\Big[[\mathbb{I}_0] + M[\Delta'_0]\Big]R^T \tag{B.4-1}$$

and

$$[I] = R\Big[[I_0] + M[\Delta_0]\Big]R^T \tag{B.4-2}$$

As an example of the utility and application of the general equations, two particularly interesting problems may be reviewed:

(i) Moment of inertia of a body of mass M about any straight line having equation

$$\frac{x_1 - a}{\alpha_1} = \frac{x_2 - b}{\alpha_3} = \frac{x_3 - c}{\alpha_3} \tag{B.4-3}$$

with respect to a central system. The above implies that the line has direction cosines $\{\alpha\}$ and contains the point $(a, b, c)$; thus as a particular case of (B.4-2) one may write:

$$I_{AX} = \{\alpha\}^T \Big[ [I_o] + M[\Delta_o] \Big] \{\alpha\} \qquad \text{(B.4-4)}$$

or

$$I_{AX} = \{\alpha\}^T [I_o] \{\alpha\} + M \{\alpha\}^T [\Delta_o] \{\alpha\}$$

where in this case

$$[\Delta_o] = \begin{bmatrix} b^2 + c^2 & -ab & -ac \\ & a^2 + c^2 & -bc \\ s & & a^2 + b^2 \end{bmatrix} \qquad \text{(B.4-5)}$$

(ii) Similarly, as an application of (B.4-1) the moment of inertia of a body about a plane containing the point (a, b, c) and having normal vector of direction cosines $\{\alpha\}$ is:

$$\mathbb{I}_{PL} = \{\alpha\}^T \Big[ [\mathbb{I}_o] + M[\Delta\!\!\backslash_o] \Big] \{\alpha\} \qquad \text{(B.4-6)}$$

and

$$[\Delta\!\!\backslash_o] = \begin{bmatrix} a^2 & ab & ac \\ & b^2 & bc \\ s & & c^2 \end{bmatrix} \qquad \text{(B.4-7)}$$

The following subsections will try to clarify when a tensor of inertia is principal (i.e., diagonal), depending on the orientation and origin of the system to which this tensor is referred. In other words, the condition under which the matrix equation

$$R \Big[ [I_o] + M[\Delta_o] \Big] R^T = [\,I\,] \qquad \text{(B.4-8)}$$

is fulfilled will be investigated. Although not described here, the same process may be followed for the tensor $[\,\mathbb{I}\,]$. In conclusion it will be found that a body has infinite sets of principal axes, one at every point. However, a body has only one set of central principal axes. This intuitive assertion may be proved easily from (B.4-2) as follows:

$$[I] = [I_o] \text{ only if } \begin{cases} [\Delta_o] = [0] \implies \{\delta x_o\} = \{0\} \\ \qquad\qquad \text{and} \\ R = [1] \end{cases}$$

i.e., the systems $(x)$ and $(x_o)$ are coincident.

Clearly the fulfillment of the same conditions will apply to the particular case

$$\lceil I \rfloor = \lceil I_o \rfloor$$

implying

$$(x_p) \equiv (x_{op})$$

### B.4.1 Principal Axes Which Are Parallel to a Central System

This section investigates the conditions under which a non-central tensor of inertia referred to a frame $(x)$ parallel to a system centered at the CM is principal.

By the parallelism assumption, one has $R = [1]$. It follows from equation (B.4-2) that only three cases are possible:

(i) The central tensor of inertia is principal

This is equivalent to solving the following diagonalization problem,

$$\lceil I_o \rfloor + M[\Delta_o] = \lceil I \rfloor \qquad \text{(B.4-9)}$$

The above equation is true only when the matrix $[\Delta_o]$ is diagonal. That is, one of the three conditions given below must be fulfilled.

$$\delta x_{o1} = \delta x_{o2} = 0 \qquad \text{(B.4-10a)}$$

$$\delta x_{o1} = \delta x_{o3} = 0 \qquad \text{(B.4-10b)}$$

$$\delta x_{o2} = \delta x_{o3} = 0 \qquad \text{(B.4-10c)}$$

Consequently the following theorem may be stated: Every Cartesian system $(x)$ parallel to the central principal axes $(x_{op})$ and with origin in any of its points is also principal.

The following inequalities clearly hold from (B.4-9)

$$A_p > A_{op}\ , \quad B_p > B_{op}\ , \quad C_p > C_{op}$$

That is, the minimum principal moments of inertia are obtained when the reference system has the CM as origin.

Consequently, recalling (B.3-18) the momental ellipsoid of maximum volume is the central momental ellipsoid.

(ii) Two central products of inertia are zero

This condition implies that only one of the central axes is principal. Assuming, for example, that $D_o = E_o = 0$ (i.e. the $x_3$ axis is principal), one has

$$\begin{bmatrix} A_o & -F_o & 0 \\ & B_o & 0 \\ s & & C_o \end{bmatrix} + M[\Delta_o] = \lceil I \rfloor \qquad \text{(B.4-11)}$$

which immediately gives the following three conditions to be satisfied:

$$F_o + M\delta x_{o1}\delta x_{o2} = 0 \qquad \text{(B.4-12a)}$$

$$\delta x_{o1}\delta x_{o3} = 0 \qquad \text{(B.3-12b)}$$

$$\delta x_{o2}\delta x_{o3} = 0 \qquad \text{(B.3-12c)}$$

Therefore,

$$F_o \neq 0 \implies \delta x_{o1} \neq 0 \quad \text{and} \quad \delta x_{o2} \neq 0$$

The above implies that to fulfill the conditions b or c in equations (B.4-12), $\delta x_{o3} = 0$. This means that the origin of the new ($x_p$) system must be on the plane $x_{o1}x_{o2}$, that is, the plane through the CM and normal to the only central principal axis.

Equation (B.4-12a) gives

$$M\,\delta x_{o1}\delta x_{o2} = -F_o$$

or

$$\delta x_{01}\,\delta x_{02} = -\frac{F_o}{M} \qquad \text{(B.4-13)}$$

which is the equation of a rectangular hyperbola referred to its asymptotas as coordinate axes and parameter $K = \left|\frac{F_o}{M}\right|$.

Hence all the coordinate systems parallel to $(x_o)$ and with origin in the hyperbola given by (B.4-13), which lies in the $x_{01}x_{02}$ plane, are principal (see Fig. B.2 ). Similar solutions can be obtained with $D_o = F_o = 0$ or $E_o = F_o = 0$.

(iii) None of the central products of inertia are zero.

In this case the given central tensor of inertia is not principal. Consequently one should solve the general diagonalization problem,

$$[I_o] + M[\Delta_o] = [\ulcorner I \lrcorner] \qquad \text{(B.4-14)}$$

This implies the simultaneous fulfillment of the conditions

$$F_o + M\,\delta x_{01}\,\delta x_{02} = 0 \qquad \text{(B.4-15a)}$$

$$E_o + M\,\delta x_{01}\,\delta x_{03} = 0 \qquad \text{(B.4-15b)}$$

$$D_o + M\,\delta x_{02}\,\delta x_{03} = 0 \qquad \text{(B.4-15c)}$$

After solving for $\delta x_{oi}$ $(i = 1,2,3)$ above, one has

$$\delta x_{01} = \sqrt{\frac{E_o F_o}{M D_o}} \qquad \text{(B.4-16a)}$$

$$\delta x_{02} = \sqrt{\frac{D_o F_o}{M E_o}} \qquad \text{(B.4-16b)}$$

$$\delta x_{03} = \sqrt{\frac{E_o D_o}{M F_o}} \qquad \text{(B.4-16c)}$$

Therefore, in any body there are two specific points where one has a principal system $(x_p)$ which is parallel to any given central frame $(x_o)$. The

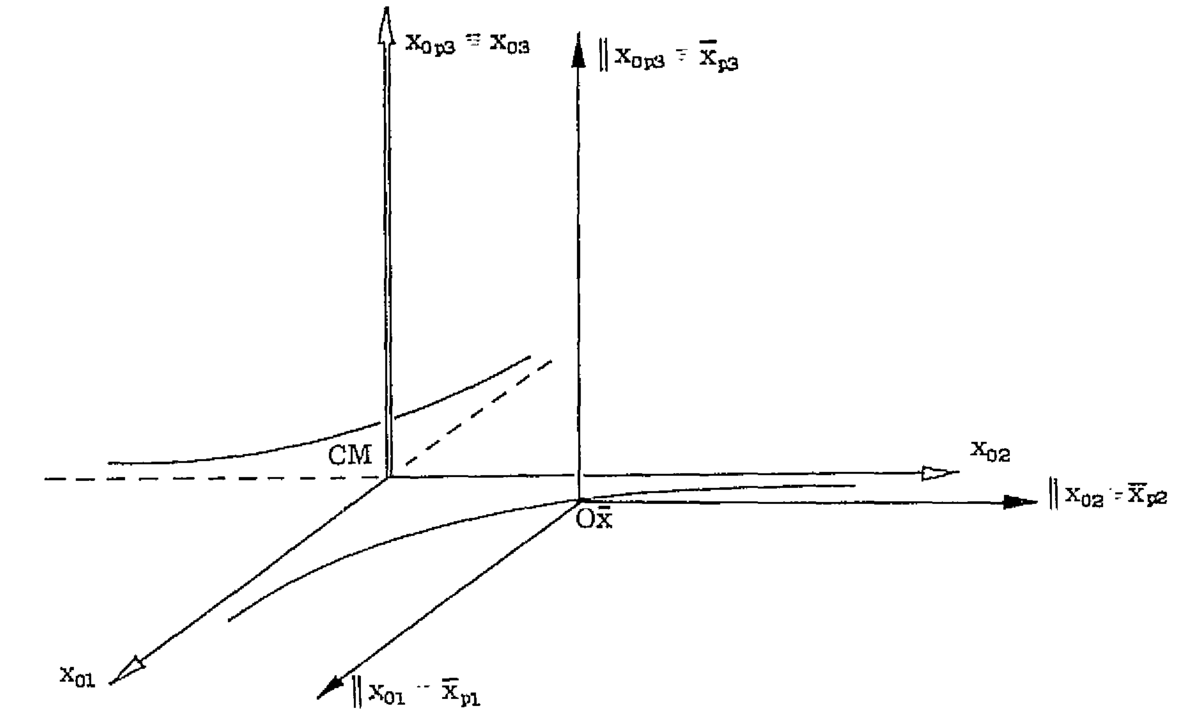



Fig. B.2. Principal Axes of Inertia Parallel to the Frame $(x_{01}, x_{02}, x_{0p3})$

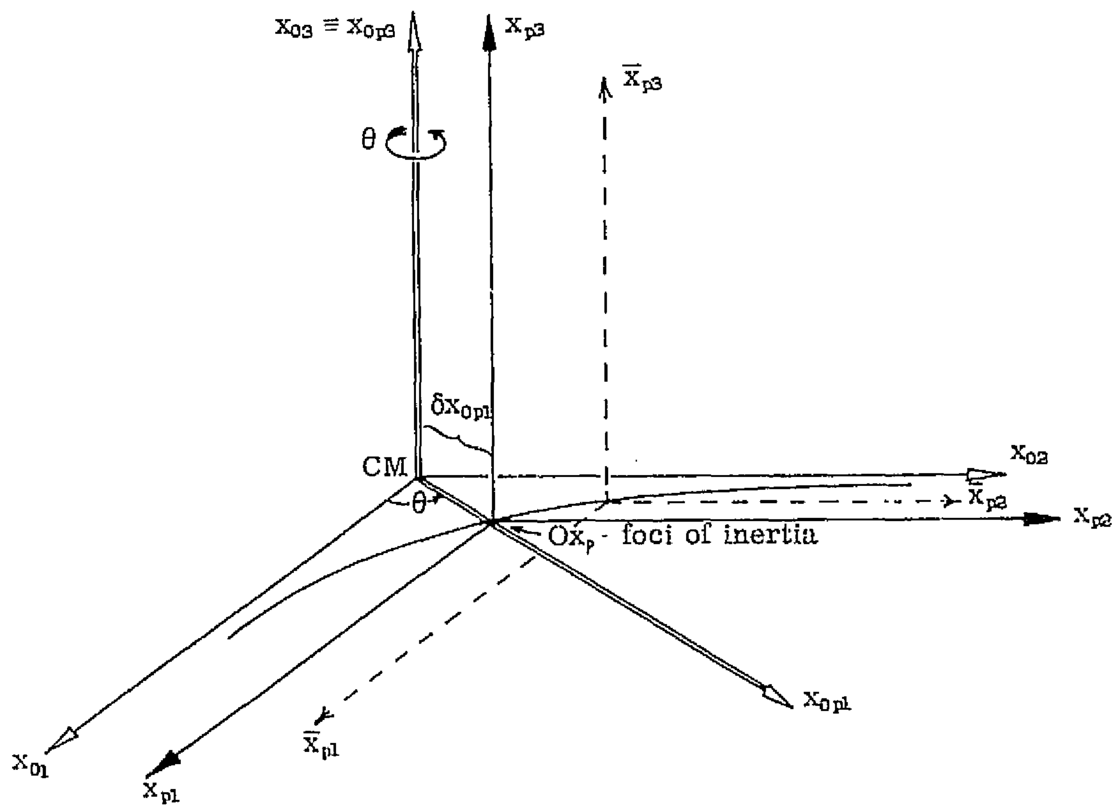



Fig. B.3. Foci of Inertia

two points are in opposite quadrants and equidistant from the center of mass, and their coordinates are given by equations (B.4-16).

### B.4.2 Eigentheory and the General Case with Rotations

The most general case will be the computation of the principal tensor of inertia at a given point of a body when the central tensor of inertia is known and no assumption of parallelism is made.

This may be represented in matrix notation by the equation

$$\lceil I \rfloor = R[\bar{I}]R^T \tag{B.4-17}$$

where

$$[\bar{I}] = [I_o] + M[\Delta_o] \tag{B.4-18}$$

Equation (B.4-17) expresses a well-known theorem in matrix diagonalization theory, implying that any symmetric matrix $[\bar{I}]$ may be diagonalized by premultiplying and postmultiplying it by some peculiar orthogonal matrix and its transpose.

It may be proved that

$R$ = matrix of orthonormal eigenvectors of $[\bar{I}]$

$\lceil I \rfloor$ = diagonal matrix of the eigenvalues of $[\bar{I}]$

Thus, by the use of eigentheory the computation of $\lceil I \rfloor$ from $[\bar{I}]$ may be achieved. The problem is then reduced to finding the vectors $\{x\}_i$ $(i = 1, 2, 3)$ such that

$$[\bar{I}]\{x\}_i = \lambda_i\{x\}_i \tag{B.4-19}$$

Observe that equation (B.4-19) implies that the vectors $\{x\}_i$ $(i = 1, 2, 3)$ after being multiplied by the symmetric tensor $[\bar{I}]$ do not change their direction but only their magnitude, i.e., any vector in a given direction, after being premultiplied by a symmetric tensor, is stretched into a vector in the same direction but of proportionate size.

This is not the case for tensors in general, which transform vectors into others of different magnitude and orientation. As is known, for example, the vectors transformed by orthogonal tensors keep their magnitude but change their orientation.

It is well known that by definition

$\lambda_i \quad (i = 1,2,3) \;=\;$ eigenvalues of $[\bar{I}]$

$\{x\}_i \quad (i = 1,2,3)$ eigenvectors of $[\bar{I}]$ corresponding to the $\lambda_i$ eigenvalues

In general, equation (B.4-19) is equivalent to

$$[\bar{I}(\lambda)]\{x\} = \{0\} \tag{B.4-20}$$

where the abbreviated notation

$$[\bar{I}(\lambda)] = [\bar{I}] - \lambda[1] \tag{B.4-21}$$

has been used.

The system of homogeneous linear equations given by (B.4-20) has a non-trivial solution $\{x\} \neq \{0\}$ if and only if

$$\det[\bar{I}(\lambda)] = 0 \tag{B.4-22}$$

which is known as the characteristic equation of $[\bar{I}(\lambda)]$

Equation (B.4-20) implies that the eigenvectors corresponding to the different eigenvalues $\lambda_i$ are orthogonal to the row vectors of the matrices $[\bar{I}(\lambda_i)]$. Thus the eigenvector direction cosines are proportional to the rows of $[\bar{I}(\lambda_i)]$, i.e., to the rows of det $[\bar{I}(\lambda_i)]$, and consequently they will also be proportional to the minors of the elements of its rows.

The left-hand side of (B.4-22) is called the characteristic polynomial and represented by $\varphi(\lambda)$.

Observe that

$$\varphi(\lambda) = \det[\bar{I}(\lambda)] = \lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0 \qquad \text{(B.4-23)}$$

where the coefficients and the roots of $\varphi(\lambda) = 0$ are related by the equations

$$a_1 = \mathrm{tr}[\bar{I}] = \mathrm{tr}[I] = \lambda_1 + \lambda_2 + \lambda_3 \qquad \text{(B.4-24a)}$$

$$a_2 = \frac{1}{2}\left( (\mathrm{tr}[\bar{I}])^2 - \mathrm{tr}[\bar{I}]^2 \right) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 \qquad \text{(B.4-24b)}$$

$$a_3 = \det[\bar{I}] = \det[I] = \lambda_1\lambda_2\lambda_3 \qquad \text{(B.4-24c)}$$

Notice that starting with equation (B.4-19) one is referring to the eigenvalues and eigenvectors of the matrix $[\bar{I}]$ and not of $R[\bar{I}]R^T$ as should be done according to equation (B.4-17), which expresses the initial diagonalization problem.

It will be shown now that the matrices $R[\bar{I}]R^T$ and $[\bar{I}]$ have identical characteristic equations and therefore their eigenvalues and eigenvectors are the same.

By simple consideration of the orthogonality properties of R, one may write

$$\begin{aligned} \det\left[ R[\bar{I}]R^T - \lambda[1] \right] &= \det\left[ R[\bar{I}]R^T - R\lambda[1]R^T \right] \\ &= \det\left[ R\left[[\bar{I}] - \lambda[1]\right]R^T \right] \\ &= \det R \det\left[ [\bar{I}] - \lambda[1] \right] \det R^T \\ &= \det\left[ [\bar{I}] - \lambda[1] \right] \end{aligned}$$

Thus, as a consequence, all coefficients in the characteristic equation are invariant under orthogonal transformation. They are called the principal invariants. Observe that the invariance under rotation of the coefficients $\mathrm{tr}[\bar{I}]$ and $\det[\bar{I}]$ was already proved independently in section B.3.1.

It is known that the values of $\lambda_i$ $(i = 1, 2, 3)$ for a real symmetric matrix are all real but not necessarily all distinct.

Assuming now that $\lambda_1$ and $\lambda_2$ are two distinct eigenvalues of $[\bar{I}]$ and $\{x\}$ and $\{y\}$ their two corresponding eigenvectors, and making use of (B.4-19), one may write,

$$[\bar{I}]\{x\} = \lambda_1\{x\} \tag{B.4-25a}$$

$$[\bar{I}]\{y\} = \lambda_2\{y\} \tag{B.4-25b}$$

Premultiplying (B.4-25a) by $\{y\}^T$ and (B.4-25b) by $\{x\}^T$ and subtracting,

$$\{y\}^T[\bar{I}]\{x\} - \{x\}^T[\bar{I}]\{y\} = \lambda_1\{y\}^T\{x\} - \lambda_2\{x\}^T\{y\}$$

but $[\bar{I}]$ being symmetric, the following equality holds

$$\{y\}^T[\bar{I}]\{x\} = \{x\}^T[\bar{I}]\{y\} \tag{B.4-26}$$

Therefore, finally

$$(\lambda_1 - \lambda_2)\{x\}^T\{y\} = 0$$

By assumption $\quad \lambda_1 \neq \lambda_2 \implies \lambda_1 - \lambda_2 \neq 0$

thus $\quad \{x\}^T\{y\} = 0 \implies \{x\}$ and $\{y\}$ are orthogonal. Hence the eigenvectors of a symmetric tensor corresponding to different eigenvalues are orthogonal (In general this is not valid for non-symmetric tensors).

If the three eigenvalues are distinct, the three corresponding eigenvectors form a unique orthogonal basis in $E^3$. The associated quadric to the $[\bar{I}]$ tensor will be an ellipsoid of three parameters. If two eigenvalues are equal, the associated quadric will be a rotational ellipsoid, implying that all the systems obtained by rotation of the axis with the only distinct eigenvalue are possible. If all the roots of the cubic equation (B.4-22) are equal, any system through the origin associated to the tensor are principal.

As a practical application of the above eigentheory to the analysis of inertia tensors, suppose the general case of finding the three principal axes and moments of inertia of a body of mass M at a point $(\delta x_{01}, \delta x_{02}, \delta x_{03})$ when the central tensor of inertia $[I_0]$ is known.

This is equivalent to determining the eigenvalues and eigenvectors of the matrix given by (B.4-18), namely

$$[\bar{I}] = [I_0] + M[\Delta_0]$$

Therefore, the principal moments of inertia at $(\delta x_{01}, \delta x_{02}, \delta x_{03})$ are the roots of the cubic equation

$$\begin{vmatrix} A_0 + M(\delta x_{02}^2 + \delta x_{03}^2) - \lambda & -F_0 - M\delta x_{01}\delta x_{02} & -E_0 - M\delta x_{01}\delta x_{03} \\ & B_0 + M(\delta x_{01}^2 + \delta x_{03}^2) - \lambda & -D_0 - M\delta x_{02}\delta x_{03} \\ s & & C_0 + M(\delta x_{01}^2 + \delta x_{03}^2) - \lambda \end{vmatrix} = 0 \tag{B.4-27}$$

Hence

$$[I] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ & \lambda_2 & 0 \\ s & & \lambda_3 \end{bmatrix} = \begin{bmatrix} A_p & 0 & 0 \\ & B_p & 0 \\ s & & C_p \end{bmatrix} \tag{B.4-28}$$

and the direction of the principal axes are the orthogonal eigenvectors corresponding to the values of $\lambda_i (i = 1, 2, 3)$. Notice that as mentioned above, not all the $\lambda_i$ need necessarily be distinct.

### B.4.3 Some Geometric Interpretations

(1) Foci of inertia

It was proved in section B.4.1 that all the Cartesian principal systems parallel to a given central system (with only one principal axis) have their origin in a rectangular hyperbola, situated in the plane through the CM and normal to the only central principal axis.

Clearly the system $(x_{o1}, x_{o2}, x_{o3} \equiv x_{op3})$ can be made coincident with the central principal frame $(x_{op})$ after a rotation $\theta$ around the principal axis $x_{op3}$ (see Fig. B.3).

The rotated coordinate axes $x_{o1}$ or $x_{o2}$ will intersect the rectangular hyperbola in two points symmetric with respect to the CM and with coordinates $(\delta x_{op1}, 0, 0)$ or $(0, \delta x_{op2}, 0)$ respectively.

One may write the equation of the hyperbola in the new central principal coordinate system $(x_{op})$ after considering the transformation of coordinates:

$$\{x_o\} \xrightarrow{R_3(\theta)} \{x_{op}\} \qquad \text{(B.4-29)}$$

Thus

$$\{x_o\} = R_3^T(\theta)\{x_{op}\}$$

which gives

$$x_{o1} = \cos\theta x_{op1} - \sin\theta x_{op2} \qquad \text{(B.4-30a)}$$

$$x_{o2} = \sin\theta x_{op1} + \cos\theta x_{op2} \qquad \text{(B.4-30b)}$$

$$x_{o3} = x_{op3} \qquad \text{(B.4-30c)}$$

and similar equations hold for $\delta x_{oi}$ $(i = 1, 2, 3)$. $\delta x_{o3} = \delta x_{op3} = 0$.

Substituting the above values in the equation of the hyperbola (B.4-13) and after considering that

$$F_o = \int_M x_{o1} x_{o2} dm$$

and

$$\tan 2\theta = \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}$$

one has, after algebraic simplification,

$$2M\,\delta x_{op1}\delta x_{op2} + M\tan 2\theta(\delta x^2_{op1} - \delta x^2_{op2}) + \tan 2\theta \int_M (x^2_{op1} - x^2_{op2})dm = 0 \qquad \text{(B.4-31)}$$

The above represents the equation of the rectangular hyperbola in the central principal coordinate system ($x_{op}$). Observe that the variables are the coordinates ($\delta x_{op1}$, $\delta x_{op2}$).

By simple inspection of equation (B.4-31), if

$$\int_M (x^2_{op1} - x^2_{op2})dm > 0 \implies \mathrm{I\!I}_{op11} > \mathrm{I\!I}_{op22} \implies B_{op} > A_{op} \tag{B.4-32}$$

then

$$\delta x_{op1} = 0$$

which implies that the central principal axis will always intersect the hyperbola along the axis of greater moment of inertia.

Assuming (see Fig. B.3)

$$A_{op} > B_{op} \implies \delta x_{op2} = 0 \tag{B.4-33}$$

then equation (B.4-31) gives

$$\delta x_{op1} = \sqrt{\frac{1}{M}\int_M (x^2_{op2} - x^2_{op1})dm} = \sqrt{\frac{\mathrm{I\!I}_{op22} - \mathrm{I\!I}_{op11}}{M}} = \sqrt{\frac{A_{op} - B_{op}}{M}} \tag{B.4-34}$$

Clearly all the lines through the points ($\delta x_{op1}$, 0, 0) and ($-\delta x_{op1}$, 0, 0) in the plane $x_{op1}\, x_{op2}$ are principal at this point and with moments of inertia equal to $A_{op}$. The momental ellipsoid at Ox is a rotational ellipsoid. Every body has six of these points known as foci of inertia, two in each principal plane.

(2) <u>Binet's theorem</u>

Assume the following diagonalization problem

$$R\left[\ulcorner I_o \lrcorner + M[\,\Delta_o\,]\right] R^{T} = \ulcorner I \lrcorner \tag{B.4-35}$$

where the diagonal elements of the central principal tensor of inertia are given by

$$A_{o_p} = B_{o_p} = Ma^2_{o_g}\ , \quad C_{o_p} = Mc^2_{o_g} \tag{B.4-36}$$

For simplicity the condition that the two equatorial moments are the same is enforced ($\Longrightarrow a_{o_g} = b_{o_g}$).

The problem is then equivalent to determining the principal axes and moments of an inertia tensor [ I ] given by

$$[I] = [I_o] + M[\Delta_o] \tag{B.4-37}$$

which is referred to a frame parallel to the central principal axes and with origin at the point P ($\delta x_{o1}$, $\delta x_{o2}$, $\delta x_{o3}$).

According to section B.4.2, this may be reduced to obtaining the eigenvalues and eigenvectors of the symmetric matrix [ I ]. Consequently the following equation in $\lambda$ must be solved:

$$\begin{vmatrix} M(a_{o_g}^2+\delta x_{o2}^2+\delta x_{o3}^2)-\lambda & -M\,\delta x_{o1}\delta x_{o2} & -M\,\delta x_{o1}\delta x_{o3} \\ & M(a_{o_g}^2+\delta x_{o3}^2+\delta x_{o1}^2)-\lambda & -M\,\delta x_{o2}\delta x_{o3} \\ s & & M(c_{o_g}^2+\delta x_{o1}^2+\delta x_{o2}^2)-\lambda \end{vmatrix} = 0 \tag{B.4-38}$$

Making the substitution,

$$M(\delta x_{o1}^2+\delta x_{o2}^2+\delta x_{o3}^2)-\lambda = M\tilde{\lambda} \tag{B.4-39}$$

equation (B.4-38) simplifies to

$$\begin{vmatrix} -\delta x_{o1}^2+(a_{o_g}^2+\tilde{\lambda}) & -\delta x_{o1}\,\delta x_{o2} & -\delta x_{o1}\,\delta x_{o3} \\ & -\delta x_{o2}^2+(a_{o_g}^2+\tilde{\lambda}) & -\delta x_{o2}\,\delta x_{o3} \\ s & & -\delta x_{o3}^2+(c_{o_g}^2+\tilde{\lambda}) \end{vmatrix} = 0 \tag{B.4-40}$$

and the corresponding cubic equation becomes

$$(a_{o_g}^2+\tilde{\lambda})(c_{o_g}^2+\tilde{\lambda})-\delta x_{o1}^2(c_{o_g}^2+\tilde{\lambda})-\delta x_{o2}^2(c_{o_g}^3+\tilde{\lambda})-\delta x_{o3}^2(a_{o_g}^2+\tilde{\lambda}) = 0 \tag{B.4-41}$$

Dividing through the above equation by $(a_{O_g}^2+\tilde{\lambda})(c_{O_g}^2+\tilde{\lambda})$ finally the following quadratic equation in $\tilde{\lambda}$

$$\frac{\delta x_{O1}^2+\delta x_{O2}^2}{a_{O_g}^2+\tilde{\lambda}} + \frac{\delta x_{O3}^2}{c_{O_g}^2+\tilde{\lambda}} = 1 \qquad \text{(B.4-42)}$$

will have two roots $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$. An ellipsoid of revolution will correspond to one of the $\tilde{\lambda}$, and a hyperboloid of revolution to the other.

These surfaces will be confocal with

$$\frac{\delta x_{O1}^2+\delta x_{O2}^2}{a_{O_g}^2} + \frac{\delta x_{O3}^2}{c_{O_g}^2} = 1 \qquad \text{(B.4-43)}$$

which is the rotational ellipsoid of gyration centered at the CM. As is known, the two confocal surfaces will intersect orthogonally.

The orientation of the principal axes at the given point $(\delta x_{o1}, \delta x_{o2}, \delta x_{o3})$ will be the same as that of the eigenvectors corresponding to the computed eigenvalues. It was shown previously in relation to equation (B.4-20) that for any $\lambda_1$ the direction cosines of the eigenvectors are proportional to the minors of the rows of the determinant given by (B.4-38), that is to say, to the determinant in (B.4-40).

They are thus proportional to the quantities

$(a_{O_g}^2+\tilde{\lambda})(c_{O_g}^2+\tilde{\lambda})-\delta x_{O3}^2(a_{O_g}^2+\tilde{\lambda})-\delta x_{O2}^2(c_{O_g}^2+\tilde{\lambda})$ ; $\delta x_{o1}\,\delta x_{o2}(c_{O_g}^2+\tilde{\lambda})$ ;

$\delta x_{o1}\,\delta x_{o3}(a_{O_g}^2+\tilde{\lambda})$; dividing by $(a_{O_g}^2+\tilde{\lambda})(c_{O_g}^2+\tilde{\lambda})$ the above three expressions

reduce to

$$1-\frac{\delta x_{o3}^2}{c_{O_g}^2+\tilde{\lambda}}-\frac{\delta x_{o2}^2}{a_{O_g}^2+\tilde{\lambda}}\ ,\quad \frac{\delta x_{o1}\delta x_{o2}}{a_{O_g}^2+\tilde{\lambda}}\ ,\quad \frac{\delta x_{o1}\delta x_{o3}}{c_{O_g}^2+\tilde{\lambda}} \qquad \text{(B.4-44)}$$

but recalling (B.4-42), the above simplifies to

$$\frac{\delta x_{O1}^2}{a_{O_g}^2+\tilde{\lambda}}\ ,\quad \frac{\delta x_{o1}\delta x_{o2}}{a_{O_g}^2+\tilde{\lambda}}\ ,\quad \frac{\delta x_{o1}\delta x_{o3}}{c_{O_g}^2+\tilde{\lambda}}$$

or finally, the eigenvector direction cosines are proportional to the quantities

$$\frac{\delta x_{01}}{a_{0_\xi}^2+\tilde{\lambda}} \quad , \quad \frac{\delta x_{02}}{a_{0_\xi}^2+\tilde{\lambda}} \quad , \quad \frac{\delta x_{03}}{c_{0_\xi}^2+\tilde{\lambda}} \tag{B.4-45}$$

which are the direction cosines of the normal to the confocal surfaces passing through the point $(\delta x_{01}, \delta x_{02}, \delta x_{03})$.

Two of the principal axes at P are the normals to the respective members of the confocal surfaces which pass through the point. The third principal axis must therefore be the axis mutually orthogonal to them, thus it will be on a plane through P parallel to the $x_{01}x_{02}$ plane. Two of the principal moments of inertia are given by the eigenvalues $\lambda_1$ and $\lambda_2$ that may be determined from equation (B.4-39) after substitution of $\tilde{\lambda}$. The third moment can be computed using the parallel axis theorem as follows,

$$M(\delta x_{0\,1}^2+\delta x_{0\,2}^2+\delta x_{0\,3}^2+a_{0_\xi}^2)$$

This remarkable geometric finding that the principal inertia axes at any point P are the normals to the three orthogonal families of surfaces passing through P confocal with the ellipsoid of giration, was originally given by Binet [1813].

In the next section some important consequences of this seldom referenced theorem in the diagonalization of symmetric tensors of second-order will be examined.

## B.5 Diagonalization by Rotation Matrices

After the geometric conclusions derived from Binet's theorem, which unusually, links dynamics and geometry, a novel approach (to the author's best knowledge) for determining the principal axes and moments of inertia at any point of a body is presented here.

In [Soler, 1976] the theory for obtaining the rotation matrix R between the central and local frames (the axes of which are normal to the confocal surfaces) was studied. The problem here is thus reduced to computing the orthogonal matrix of the transformation between central and local frames from the available data.

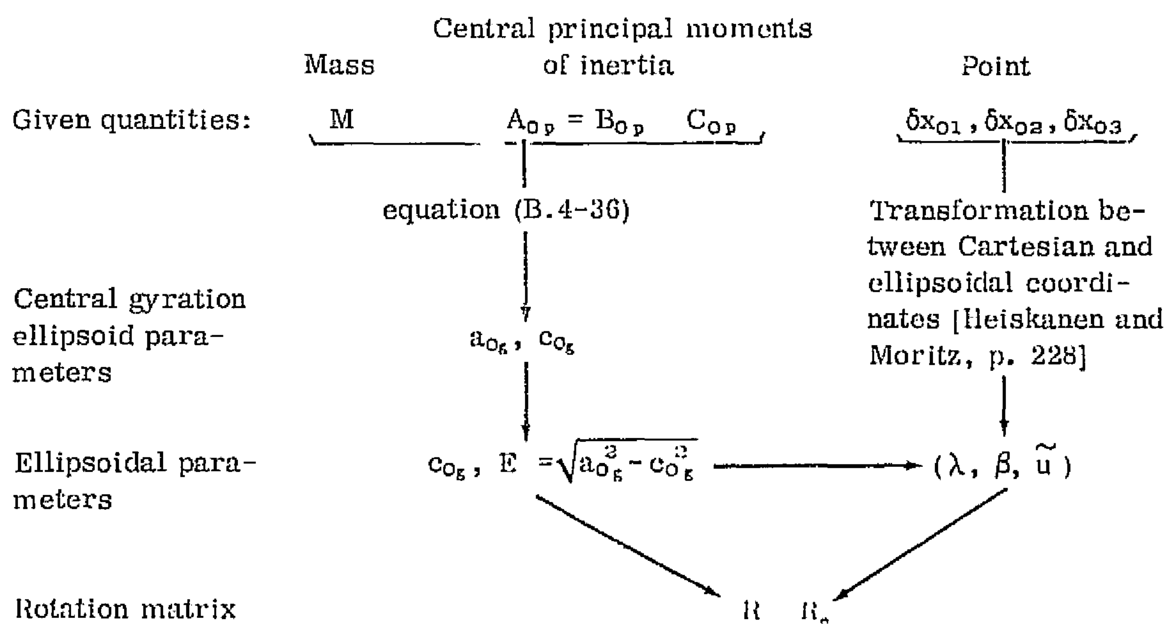
The initial values of M and $\ulcorner I_0 \lrcorner$ will provide the ellipsoid of gyration parameters from which R may be determined at the specified point $(\delta x_{01}, \delta x_{02}, \delta x_{03})$.

Once R is known the matrix [ I ] given by (B.4-37) can immediately be diagonalized

$$R[I]R^T = \ulcorner I \lrcorner$$

By this simple geometric interpretation implicit in Binet's theorem, a different method is found for obtaining the principal axes of inertia independently of the standard approach through eigenvalues and eigenvectors.

A simple practical algorithm is given below for the particular case of (rotational) momental ellipsoid treated in the previous section.



The value of $R_e$ in the case of rotational ellipsoidal coordinates was given in [Soler, 1976] as

$$R_e = \begin{bmatrix} -\sin\lambda & \cos\lambda & 0 \\ -\frac{1}{w_e}\sin\beta\cos\lambda & -\frac{1}{w_e}\sin\beta\sin\lambda & \frac{\tilde{u}\cos\beta}{w_e(\tilde{u}^2 \pm E^2)^{1/2}} \\ \frac{\tilde{u}}{w_e(\tilde{u}^2 \pm E^2)^{1/2}}\cos\beta\cos\lambda & \frac{\tilde{u}}{w_e(\tilde{u}^2 \pm E^2)^{1/2}}\cos\beta\sin\lambda & \frac{\sin\beta}{w_e} \end{bmatrix} \quad \text{(B.5-1)}$$

where

$$w_e = \left( \frac{\tilde{u}^2 \pm E^2 \sin^2\beta}{\tilde{u} \pm E^2} \right)^{1/2} \quad \text{(B.5-2)}$$

In the above equations the + and − signs refer to oblate and prolate (rotational) ellipsoids respectively. In the latter case, $E = \sqrt{c_{o_E}^2 - a_{o_E}^2}$. The rows of $R_e$ are the direction cosines of the three principal (local) axes at the point $P(\delta x_{01}, \delta x_{02}, \delta x_{03}) = P(\lambda, \beta, \tilde{u})$.

The principal moments at P are the elements of the diagonal matrix obtained as a function of the given moments of inertia through the transformation

$$R_e[I]R_e^T = [\ulcorner I \lrcorner] \quad \text{(B.5-3)}$$

where

$$[I] = [\ulcorner I_o \lrcorner] - M[\underline{\delta x}_o]^2 \quad \text{(B.5-4)}$$

The method can even be applied to the case of diagonalization of $3 \times 3$ symmetric positive definite non-singular matrices.

Every symmetric matrix [ M ] may be expressed as the addition of two other matrices, one diagonal $\lceil M' \rfloor$ and the other of the form $\{x\}\{x\}^T$. Thus

$$[M] = \lceil M' \rfloor + \{x\}\{x\}^T \qquad \text{(B.5-5)}$$

To $\lceil M' \rfloor$ one may associate a central quadric that in the most general case will have three orthogonal families of confocal surfaces (equivalent to the ellipsoid of gyration in dynamics).

At the point $\{x\}$ the local moving frame will have the direction of the eigenvectors of $[M]$. Thus through this point a confocal quadric to the one represented by $\lceil M' \rfloor$ will pass. (In dynamics this point will have principal axes parallel to the system associated to the tensor $[M]$).

The point $\{x\}$ then is found in form similar to the one explained in section B.4.1. It will satisfy the condition

$$x_1 x_2 = m_{12}$$

$$x_1 x_3 = m_{13}$$

$$x_2 x_3 = m_{23}$$

and

$$x_1 = \sqrt{\frac{m_{12} m_{13}}{m_{23}}} \qquad \text{(B.5-6a)}$$

$$x_2 = \sqrt{\frac{m_{12} m_{23}}{m_{13}}} \qquad \text{(B.5-6b)}$$

$$x_3 = \sqrt{\frac{m_{13} m_{23}}{m_{12}}} \qquad \text{(B.5-6c)}$$

Once the point $\{x\}$ is known, the rotational matrix R between central and local systems may be found as formerly described, using the parameters of the quadric associate to $\lceil M' \rfloor$. Finally, $[M]$ can be diagonalized using the standard approach

$$R[\,M\,]R^{T} = \lceil M \rfloor$$

The following subsection shows two simple examples where the rotation matrix R is immediately obtainable, although the method is general and may be applied to any $3 \times 3$ symmetric positive definite non-singular matrix.

### B.5.1 Examples

(1) <u>Spherical case</u>

Diagonalize the matrix $[\,M\,] = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

From equations (B.5-6) one has

$$x_1 = 1 \quad ; \quad x_2 = 1 \quad ; \quad x_3 = 1$$

Therefore the given matrix may be decomposed according to (B.5-5) as

$$[\,M\,] = \lceil M' \rfloor + \{x\}\{x\}^{T}$$

or

$$\begin{bmatrix} 2 & 1 & 1 \\ & 2 & 1 \\ s & & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ s & & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ s & & 1 \end{bmatrix}$$

In this case it is clear that a sphere of radius 1 is associated to the matrix $\lceil M' \rfloor$. Thus it will be easy to obtain the spherical coordinates of the point $\{x\}$. From Fig. B.4 it immediately follows

$$\sin\lambda = \frac{1}{\sqrt{2}} \qquad\qquad \sin\psi = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$$

$$\cos\lambda = \frac{1}{\sqrt{2}} \qquad\qquad \cos\psi = \frac{\sqrt{2}}{\sqrt{3}}$$

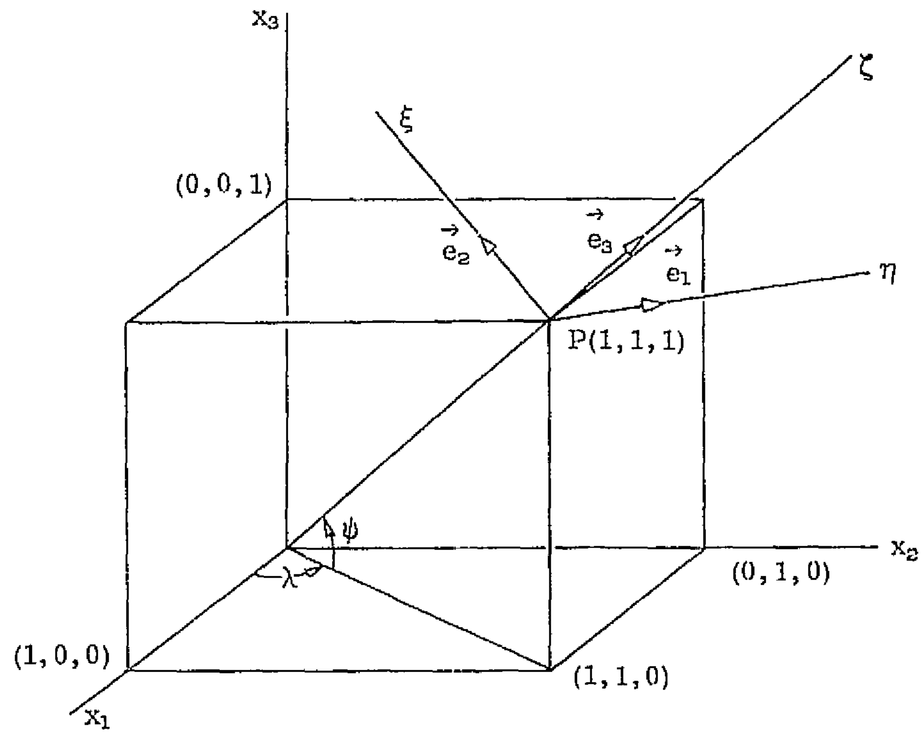



Fig. B.4. Local Frame $(\eta, \xi, \zeta)$ and Base $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$

Then the rotation matrix of the transformation between central and local systems is

$$R = R_1\left(\frac{\pi}{2} - \psi\right) R_3\left(\lambda + \frac{\pi}{2}\right) = \begin{bmatrix} -\sin\lambda & \cos\lambda & 0 \\ -\sin\psi\cos\lambda & -\sin\psi\sin\lambda & \cos\psi \\ \cos\psi\cos\lambda & \cos\psi\sin\lambda & \sin\psi \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{\sqrt{2}}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

Thus the eigenvectors have the direction of the local base $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$. Finally the given matrix [ M ] is diagonalized as follows

$$R[M]R^T = \begin{bmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ s & & 4 \end{bmatrix} \quad [M]$$

By the classical formalism of eigenvalues and eigenvectors, the same result is obtained. According to (B.4-22)

$$\det[M(\lambda)] = \begin{vmatrix} 2-\lambda & 1 & 1 \\ & 2-\lambda & 1 \\ s & & 2-\lambda \end{vmatrix} = 0 \implies (2-\lambda)^3 - 3(2-\lambda) + 2 = 0$$

Therefore the eigenvalues are $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 4$.

It may be shown that the unit vectors in the direction of the eigenvectors corresponding to the eigenvalues $\lambda_i$ $(i = 1,2,3)$ are the rows of the R matrix.

Finally, it should be observed that the properties of the principal invariants described in section B.4.2 hold:

(i) $\mathrm{tr}[M] = \mathrm{tr}\lceil M \rfloor = \lambda_1 + \lambda_2 + \lambda_3 = 6$

(ii) $\frac{1}{2}\left((\mathrm{tr}[M])^2 - \mathrm{tr}[M]^2\right) = \frac{1}{2}\left((\mathrm{tr}\lceil M \rfloor)^2 - \mathrm{tr}\lceil M \rfloor^2\right)$

$$= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = 9$$

(iii) $\det[M] = \det\lceil M \rfloor = \lambda_1\lambda_2\lambda_3 = 4$

(2) <u>Ellipsoidal (rotational) case</u>

Diagonalize the matrix $[M] = \begin{bmatrix} 2 & 1 & 0 \\ & 2 & 0 \\ s & & 2 \end{bmatrix}$

Following a procedure similar to example (1), one has

$x_1 = x_2 = 1 \quad ; \quad x_3 = 0$

Thus

$$[M] = \begin{bmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ s & & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 \\ & 1 & 0 \\ s & & 0 \end{bmatrix}$$

In this case the quadric associated to the matrix $[M']$ is

$$x_1^2 + x_2^2 + 2x_3^2 = 1 \Longrightarrow \frac{x_1^2 + x_2^2}{1} + \frac{x_3^2}{\frac{1}{2}} = 1 \Longrightarrow \text{flattened ellipsoid}$$

of revolution with parameters $a = b = 1$ and $c = \dfrac{1}{\sqrt{2}}$ .

Thus the linear excentricity E is given by

$$E = \sqrt{a^2 - c^2} = \frac{1}{\sqrt{2}}$$

The curvilinear ellipsoidal (rotational) coordinates at P(1,1,0) may be obtained using equations (6-8) in [Heiskanen and Moritz, 1967, p. 228]

$$\tilde{u}^2 = (x_1^2 + x_2^2 + x_3^2 - E^2)\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4E^2x_3^2}{(x_1^2 + x_2^2 + x_3^2 - E^2)^2}}\right) = \frac{3}{2}$$

$$\tan\beta = \frac{x_3\sqrt{\tilde{u}^2 + E^2}}{\tilde{u}\sqrt{x_1^2 + x_2^2}} = 0 \Longrightarrow \beta = 0$$

$$\tan\lambda = \frac{x_2}{x_1} = 1 \Longrightarrow \lambda = \frac{\pi}{4}$$

Thus through the point P(1,1,0) passes a confocal ellipsoid with parameters

$$\tilde{u} = \sqrt{\frac{3}{2}} \quad \text{and} \quad E = \frac{1}{\sqrt{2}}$$

and from (B.5-2) it immediately follows that $w_o = \dfrac{\sqrt{3}}{2}$ .

Thus the rotation matrix which makes the central (principal) system parallel to the local (moving) frame at $P(x_1 = 1, x_2 = 1, x_3 = 0)$ $P(\lambda = \pi/4,$ $\beta = 0,\ \tilde{u} = \sqrt{3}/\sqrt{2})$ can be obtained from equation (B.5-1).

$$R_o = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$$

and finally

$$R_o[\,M\,]R_o^T = \begin{bmatrix} 1 & 0 & 0 \\ & 2 & 0 \\ s & & 3 \end{bmatrix}$$

Observe that when the confocal ellipsoids are flattened, the resulting elements in the diagonal (eigenvalues) are obtained in ascending order

$$\lambda_1 < \lambda_2 < \lambda_3$$

and correspond t‹ the directions of the $\eta$, $\xi$ and $\zeta$ local axes respectively.

It may be shown as in example (1) that the principal invariant properties hold.

By the formal approach of eigenvalues and eigenvectors the same results are derived.

Eigenvalues:

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ & 2-\lambda & 0 \\ s & & 2-\lambda \end{vmatrix} = 0 \implies (2-\lambda)[(2-\lambda)^2 - 1] = 0 \implies \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 2 \\ \lambda_3 = 3 \end{cases}$$

Eigenvectors:

$$\lambda_1 = 1 \;:\; [\,M\,]\{x\} = \{x\} \implies \begin{Bmatrix} x_3 = 0 \\ x_2 = -x_1 \end{Bmatrix} \implies \text{unit vector} \quad \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$\lambda_2 = 2 \;:\; [\,M\,]\{x\} = 2\{x\} \implies \begin{Bmatrix} x_1 = x_2 = 0 \\ \text{any } x_3 \end{Bmatrix} \implies \text{unit vector} \quad (0, 0, 1)$$

$$\lambda_3 = 3 \;:\; [\,M\,]\{x\} = 3\{x\} \implies \begin{Bmatrix} x_1 = x_2 \\ x_3 = 0 \end{Bmatrix} \implies \text{unit vector} \quad \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

In the general case the quadric associated to the matrix $[M']$ will be an ellipsoid of three parameters (i.e. the three elements in the diagonal will be different). Although the same approach may be followed, the introduction of general orthogonal ellipsoidal coordinates is required and consequently the computation of R is more involved.

By the same procedure explained above the errors ellipsoids from point variance-covariance matrices may be obtained . Observe that the result has strictly a geometrical meaning of difficult statistical interpretation.

## APPENDIX C

(Referenced in Sections 2.3.2, D.1 and D.1.1)

### C.1 Velocity and Acceleration in the Inertial System

In this appendix the following notations will apply:

$(X_1, X_2, X_3)$ = inertial (space fixed) system

$(x_1, x_2, x_3)$ = body fixed rotating system

$\{\omega\}_X$ = angular velocity components along the (X) system

$\{\delta X\}$ = coordinates of the CM with respect to the (X) system

It is well known that the transformation of coordinates between the two systems is given by the matrix relation

$$\{X\} = R\{x\} + \{\delta X\} \tag{C.1-1}$$

where R is the rotation matrix between the (x) and (X) systems.

#### C.1.1 Velocity

The derivative of the vectors and matrices with respect to time will be denoted as usual by a dot. Let

$\{\dot{X}\}$ = velocity components of any point relative to the (X) system.

From (C.1-1)

$$\{\dot{X}\} = \dot{R}\{x\} + R\{\dot{x}\} + \{\delta\dot{X}\} \tag{C.1-2}$$

but

$$\{x\} = R^T\{X\} - R^T\{\delta X\} \tag{C.1-3}$$

thus

$$\{\dot{X}\} = \dot{R}R^T\{X-\delta X\} + R\{\dot{x}\} + \{\delta\dot{X}\} \tag{C.1-4}$$

where the "angular velocity matrix" is given by

$$[\underline{\omega}]_x = \dot{R}R^T \tag{C.1-5}$$

Thus finally

$$\{\dot{X}\} = [\underline{\omega}]_x\{X-\delta X\} + R\{\dot{x}\} + \{\delta\dot{X}\} \tag{C.1-6}$$

### C.1.2 Acceleration

Differentiating (B.1-6) with respect to time, one has

$$\{\ddot{X}\} = [\dot{\underline{\omega}}]_x\{X-\delta X\} + [\underline{\omega}]_x\{\dot{X}-\delta\dot{X}\} + \dot{R}\{\dot{x}\} + R\{\ddot{x}\} + \{\delta\ddot{X}\} \tag{C.1-7}$$

or, after simplification,

$$\{\ddot{X}\} = [\dot{\underline{\omega}}]_x\{X-\delta X\} + [\underline{\omega}]_x^2\{X-\delta X\} + 2\dot{R}\{x\} + R\{\ddot{x}\} + \{\delta\ddot{X}\}$$

but from (C.1-5)

$$\dot{R} = [\underline{\omega}]_x R$$

Thus finally,

$$\{\ddot{X}\} = [\dot{\underline{\omega}}]_x\{X-\delta X\} + [\underline{\omega}]_x^2\{X-\delta X\} + 2[\underline{\omega}]_x R\{\dot{x}\} + R\{\ddot{x}\} + \{\delta\ddot{X}\} \tag{C.1-8}$$

where the different terms in (C.1-6) and (C.1-8) have the following meaning:

$$\left.\begin{matrix}[\underline{\omega}]_x\{X-\delta X\} \\ [\dot{\underline{\omega}}]_x\{X-\delta X\}\end{matrix}\right\} = \left\{\begin{matrix}\text{velocity} \\ \text{acceleration}\end{matrix}\right\} \text{ of a particle P of the body in the (X) system resulting from}$$

$$\text{the angular}\left\{\begin{matrix}\text{velocity} \\ \text{acceleration}\end{matrix}\right\}\text{ of the (x) system}$$

$[\underline{\omega}]_x^2\{X-\delta X\}$ = the centrifugal acceleration in the (X) system of a particle P stationary in the (x) system

$2[\underline{\omega}]_x R\{\dot{x}\} \equiv$ Coriolis acceleration of the moving particle

$$\left.\begin{array}{l} R\{\dot{x}\} \equiv \{\dot{X}'\} \\ \\ R\{\ddot{x}\} \equiv \{\ddot{X}'\} \end{array}\right\} \equiv \left\{\begin{array}{l} \text{velocity} \\ \\ \text{acceleration} \end{array}\right\} \begin{array}{l} \text{of a particle P in the (X) system con-} \\ \text{sidered relative to the (x) system} \\ \text{treated as stationary} \end{array}$$

$$\left.\begin{array}{l} \{\delta\dot{X}\} \\ \\ \{\delta\ddot{X}\} \end{array}\right\} \equiv \left\{\begin{array}{l} \text{velocity} \\ \\ \text{acceleration} \end{array}\right\} \text{ of the CM in the (X) system}$$

### C.1.3 Velocity and Acceleration under the Assumption $\{\delta X\} = 0$

If $OX \equiv Ox \Longrightarrow \{\delta X\} = \{0\}$ and $\{X\} = R\{x\}$, then equations (C.1-6) and (C.1-8) simplify to

$$\{\dot{X}\} = [\underline{\omega}]_x\{X\} + R\{\dot{x}\} = [\underline{\omega}]_x\{X\} + \{\dot{X}'\} \qquad \text{(C.1-9)}$$

$$\{\ddot{X}\} = [\dot{\underline{\omega}}]_x\{X\} + [\underline{\omega}]_x^2\{X\} + 2[\underline{\omega}]_x R\{\dot{x}\} + R\{\ddot{x}\}$$

or

$$\{\ddot{X}\} = [\dot{\underline{\omega}}]_x\{X\} + [\underline{\omega}]_x^2\{X\} + 2[\underline{\omega}]_x\{\dot{X}'\} + \{\ddot{X}'\} \qquad \text{(C.1-10)}$$

## C.2 Angular Velocity Matrices

The angular velocity matrix $[\underline{\omega}]_x$ respect to the (X) system was given in (C.1-5) by

$$[\underline{\omega}]_x = \dot{R}\,R^T \qquad \text{(C.2-1)}$$

It may be proved easily that $[\underline{\omega}]_x$ is a skew-symmetric matrix. Differentiating

$$R\,R^T = [1] \qquad \text{(C.2-2)}$$

one has

$$\dot{R}\,R^T + R\,\dot{R}^T = [0] \Longrightarrow \dot{R}\,R^T = -R\,\dot{R}^T = -(\dot{R}\,R^T)^T$$



C 3

Thus

$$[\underline{\omega}]_X = -[\underline{\omega}]_X^T \qquad (C.2\text{-}3)$$

The angular velocity matrix with respect to the body-fixed or rotating system may be obtained after transforming the tensor $[\underline{\omega}]_X$ under rotation (see section B.3).

If

$$\{x\} \xrightarrow{R} \{X\} \qquad \text{or} \qquad \{X\} = R\{x\} \qquad (C.2\text{-}4)$$

then

$$[\underline{\omega}]_X = R[\underline{\omega}]R^T \implies [\underline{\omega}] = R^T[\underline{\omega}]_X R \qquad (C.2\text{-}5)$$

and substituting equation (C.2-1) above

$$[\underline{\omega}] = R^T\dot{R} \qquad (C.2\text{-}6)$$

From (C.2-6) one immediately has

$$\dot{R} = R[\underline{\omega}] \qquad (C.2\text{-}7)$$

which are the Poisson equations, and with

$$R = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}$$

are generally given in the form

$$\frac{d\alpha_1}{dt} = \alpha_2\omega_3 - \alpha_3\omega_2 \; ; \; \frac{d\alpha_2}{dt} = \alpha_3\omega_1 - \alpha_1\omega_3 \; ; \; \frac{d\alpha_3}{dt} = \alpha_1\omega_2 - \alpha_2\omega_1$$

and with similar equations for $\frac{d\beta_i}{dt}$ ; $\frac{d\gamma_i}{dt}$ $\quad \forall\, i = 1,2,3$

In particular if $\{\alpha\}$ are the direction cosines of a line respect to a moving frame with angular velocity $\{\omega\}$ one has

$$\{\dot{\alpha}\}^T = \{\alpha\}^T[\underline{\omega}] \qquad (C.2\text{-}8)$$

## C.3 Euler's Kinematic (Geometric) Equations

The so-called Euler's kinematic equations give the components $(\omega_1, \omega_2, \omega_3)$ of the angular velocity as a function of Euler's angles $\varphi$, $\theta$ and $\psi$. The term "geometric equations" is applied by some authors.

Assume that the transformation between the inertial and body-fixed systems is given by (see Fig. 2.3)

$$\{X\} = \mathbb{R}\{x\} \tag{C.3-1}$$

with

$$\mathbb{R} = R_3(-\varphi)\, R_1(-\theta)\, R_3(-\psi) \tag{C.3-2}$$

thus

$$\mathbb{R}^T = R_3(\psi)\, R_1(\theta)\, R_3(\varphi) \tag{C.3-3}$$

The components of $\{\omega\}$ along the $(x)$ frame may be obtained from equation (C.2-6)

$$[\underline{\omega}] = \mathbb{R}^T \dot{\mathbb{R}} \tag{C.3-4}$$

where

$$\dot{\mathbb{R}} = \dot{R}_3(-\varphi)\, R_1(-\theta)\, R_3(-\psi) + R_3(-\varphi)\, \dot{R}_1(-\theta)\, R_3(-\psi) + R_3(-\varphi)\, R_1(-\theta)\, \dot{R}_3(-\psi) \tag{C.3-5}$$

Considering that in general, for any angle $\chi$ one has

$$\dot{R}_i(-\chi)\, R_i(\chi) = R_i(\chi)\, \dot{R}_i(-\chi) = \dot{\chi}\,[\underline{\alpha}]_i \quad \forall\, i = 1, 2, 3 \tag{C.3-6}$$

where

$$\dot{R}_i(-\chi) = \dot{R}_i^T(\chi) \tag{C.3-7}$$

and $[\underline{\alpha}]_i$ are the antisymmetric matrices corresponding to the unit vectors which have the direction cosines of the three Cartesian axes. That is:

$$i = 1 \Longrightarrow \{\alpha\}^T = \{1, 0, 0\} \Longrightarrow [\underline{\alpha}]_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{C.3-8a}$$

$$i = 2 \Longrightarrow \{\alpha\}^T = \{0,\ 1,\ 0\} \Longrightarrow [\underline{\alpha}]_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad \text{(C.3-8b)}$$

$$i = 3 \Longrightarrow \{\alpha\}^T = \{0,\ 0,\ 1\} \Longrightarrow [\underline{\alpha}]_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{(C.3-8c)}$$

From (C.3-6) it obviously follows

$$R_i^T(\varkappa)\,\dot{R}_i(\varkappa) = \dot{R}_i(\varkappa)\,R_i^T(\varkappa) = \dot{\varkappa}[\underline{\alpha}]_i^T \qquad \forall\, i = 1,2,3$$

Therefore from (C.3-6) one may write

$$\dot{R}_i(-\varkappa) = \dot{\varkappa}[\underline{\alpha}]_i R_i(-\varkappa) \qquad \forall\, i = 1,2,3 \qquad \text{(C.3-9)}$$

which can be substituted in (C.3-5) giving

$$\dot{\mathbb{R}} = \dot{\varphi}\,[\underline{\alpha}]_3\,\mathbb{R} + \dot{\theta}\,R_3(-\varphi)\,[\underline{\alpha}]_1\,R_1(-\theta)\,R_3(-\psi) + \dot{\psi}\,R_3(-\varphi)\,R_1(-\theta)\,[\underline{\alpha}]_3\,R_3(-\psi) \qquad \text{(C.3-10)}$$

and the angular velocity matrix takes the form

$$[\underline{\omega}] = \mathbb{R}^T\dot{\mathbb{R}} = R_3(\psi)\left[R_1(\theta)\left[\dot{\omega}R_3(\varphi)[\underline{\alpha}]_3R_3(-\varphi) + \dot{\theta}[\underline{\alpha}]_1\right]R_1(-\theta) + \dot{\psi}[\underline{\alpha}]_3\right]R_3(-\psi)$$

$$= R_3(\psi)\left[R_1(\theta)\begin{bmatrix} 0 & -\dot{\varphi} & 0 \\ \varphi & 0 & -\dot{\theta} \\ 0 & \dot{\theta} & 0 \end{bmatrix}R_1(-\theta) + \dot{\psi}[\underline{\alpha}]_3\right]R_3(-\psi)$$

$$= R_3(\psi)\begin{bmatrix} 0 & -\dot{\varphi}\cos\theta - \dot{\psi} & \dot{\varphi}\sin\theta \\ & 0 & -\dot{\theta} \\ \text{ANTISYMMETRIC} & & 0 \end{bmatrix}R_3(-\psi)$$

and finally

$$[\underline{\omega}] = \begin{bmatrix} 0 & -\dot{\varphi}\cos\theta - \dot{\psi} & \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ & 0 & -\dot{\varphi}\sin\theta\sin\psi - \dot{\theta}\cos\psi \\ \text{ANTISYMMETRIC} & & 0 \end{bmatrix} \qquad \text{(C.3-11)}$$

or

$$\omega_1 = \dot{\varphi} \sin\theta \sin\psi + \dot{\theta} \cos\psi \tag{C.3-12a}$$

$$\omega_2 = \dot{\varphi} \sin\theta \cos\psi - \dot{\theta} \sin\psi \tag{C.3-12b}$$

$$\omega_3 = \dot{\varphi} \cos\theta \qquad\qquad + \dot{\psi} \tag{C.3-12c}$$

and in matrix form:

$$\{\omega\} = \begin{bmatrix} \sin\theta \sin\psi & \cos\psi & 0 \\ \sin\theta \cos\psi & -\sin\psi & 0 \\ \cos\theta & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{Bmatrix} \tag{C.3-13}$$

By the same approach one may compute the components of $\{\omega\}$ along the inertial frame (X)

$$[\underline{\omega}]_X = \dot{\mathbb{R}} \, \mathbb{R}^T$$

After matrix operation and simplification, one arrives at

$$\{\omega\}_X = \begin{bmatrix} 0 & \cos\varphi & \sin\theta \sin\varphi \\ 0 & \sin\varphi & -\sin\theta \cos\varphi \\ 1 & 0 & \cos\theta \end{bmatrix} \begin{Bmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{Bmatrix} \tag{C.3-14}$$

Although it is possible to obtain equations (C.3-13) and (C.3-14) by geometric considerations only, the approach followed here is general, and may be applied to any case where the rotation matrix $\mathbb{R}$ between the inertial and body-fixed frames is known.

## APPENDIX D

(Referenced in Sections 2.3, 2.3.2 and 2.4)

Although recently many investigators have studied the mathematical theory of the rotational motion of deformable bodies about their center of mass, it is common to resort to the vector or tensor notation to formulate the principal equations.

In this appendix the general Lagrange-Liouville equations extended to deformable bodies are derived only by means of matrix algebra, which one feels may also be used to advantage in formulating the basic equations.

In another respect, the approach presented here is different from previously published ones. The point of departure is the fundamental axiom of mechanics called "Principle of balance and moment of momentum" that can be stated as: The time rate of change of moment of momentum about a fixed point is equal to the resultant moment about the point.

One arrives at the same final results as previously published papers using varied approaches; i.e. [Kopal, 1968] bases his study on the fundamental equations of hydrodynamics, while [Tokis, 1974] starts from Cauchy's first law of continuum mechanics; the analysis by Grafarend [1973 and 1977] originates in the so-called virial equations of second order.

### D.1 Basic Dynamical Equations of Motion

This section emphasizes primarily the general principles by which the differential equations describing the motion of any material system of particles can be set up. Obviously the specification of the positions of the particles defines the body and its configuration.

The equations of motion of a body about its center of mass with respect to a space-fixed (inertial) system denoted by

$$(X_1, X_2, X_3) \equiv (X)$$

is

$$\{L\}_X = \{\dot{H}\}_X \qquad (D.1-1)$$

where

$$\{\dot{H}\}_X = \frac{d}{dt}\{H\}_X$$

and

$$\{H\}_X \equiv \int_M [\underline{X}]\{\dot{X}\}\, dm \equiv \text{column vector of total angular momentum about OX} \qquad (D.1-2)$$

$$\{L\}_X \equiv \int_M [\underline{X}]\{f_b\} dm \equiv \text{column vector of total body torques (about OX) acting on the body} \qquad (D.1-3)$$

$\{f_b\} \equiv$ body force per unit mass exerted on each particle of the body (see section D.3-1)

If $\{f_b\} = \{0\}$ the system is said to be "isolated". In this case one has a torque-free body and the angular momentum vector is constant (i.e., $\{\dot{H}\}_X = 0 \implies \{H\}_X =$ constant), having a fixed direction in inertial space, hence forming an invariant axis.

Let a second rotating system $(x_{o1}, x_{o2}, x_{o3}) = (x_o)$ be introduced, centered at the CM and not necessarily fixed to the body, but related to the inertial frame at any instant through the orthogonal rotation matrix R, i.e.,

$$\{x_o\} \xrightarrow{R} \{X\} \qquad \text{or} \qquad \{X\} = R\{x_o\} \qquad (D.1-4)$$

(In what follows the vectors referred to the $(x_o)$ Cartesian system will be unsubscribed). Then,

$$\{H\}_X = R\{H\} \qquad (D.1-5)$$

and

$$\{\dot{H}\}_x = \dot{R}\{H\} + R\{\dot{H}\} \qquad (D.1-6)$$

Therefore, equation (D. 1-1) may be written

$$\{L\}_x = \dot{R}\{H\} + R\{\dot{H}\} \qquad (D.1-7)$$

but

$$\{L\}_x = R\{L\}$$

Thus

$$\{L\} = R^T\dot{R}\{H\} + R^T R\{\dot{H}\} \qquad (D.1-8)$$

The matrix

$$R^T\dot{R} = [\underline{\theta}] \qquad (D.1-9)$$

is the angular velocity matrix with which the system ($x_o$) is rotating. The components of $\{\theta\}$ are referred to the ($x_o$) frame.

Observe that applying the theorem of section B.3 for transforming second-rank tensors under rotation, one may write

$$[\underline{\theta}]_x = R[\underline{\theta}]R^T = \dot{R}R^T \qquad (D.1-10)$$

which gives the angular velocity matrix of the rotating system ($x_o$) with respect to the fixed frame (see also Appendix C).

Substituting (D. 1-9) in (D. 1-8) and recalling the orthogonality condition of R, it follows

$$\{L\} = [\underline{\theta}]\{H\} + \{\dot{H}\} \qquad (D.1-11)$$

which is the equation of motion of the body with respect to the moving (rotating) system when $OX \equiv Ox_o \equiv CM$.

In the particular case when the rotating frame ($x_o$) is body-fixed

$$\{\theta\} = \{\omega\} \equiv \text{angular velocity vector of the rotating body} \qquad (D.1-12)$$

and

$$\{L\} = [\underline{\omega}]\{H\} + \{\dot{H}\} \qquad (D.1-13)$$

this being the basic equation of motion in a rotating reference system tied to the body.

### D.1.1 Total Angular Momentum

The total angular momentum with respect to the fixed (X) system was defined in (D.1-2) by

$$\{H\}_X = \int_M [\underline{X}]\{\dot{X}\}\,dm$$

The absolute velocity $\{\dot{X}\}$ may be expressed as (see Appendix C)

$$\{\dot{X}\} = [\underline{\omega}]_X\{X - \delta X\} + R\{\dot{x}\} + \{\delta\dot{X}\} \qquad \text{(D.1-14)}$$

Let us obtain the value of $\{H\}_X$ along the rotating frame $(x_o)$. Using the law for transforming second-rank tensors under rotation discussed above, one may write

$$[\underline{X}] = R[\underline{x}_o]R^T$$

Thus after multiplying the above matrix by equation (D.1-14) and referring the resulting free vector components to the $(x_o)$ system, one has

$$[\underline{X}]\{\dot{X}\} = R[\underline{x}_o][\underline{\omega}]\{x_o\} + R[\underline{x}_o][\underline{\omega}]\{\delta x_o\} + R[\underline{x}_o]\{\dot{x}_{o\,r}\} - R[\underline{x}_o]\{\delta\dot{x}\} \qquad \text{(D.1-15)}$$

Recalling equations (D.1-2) and (D.1-5) the value of the total angular momentum referred to the $(x_o)$ system is

$$\{H\} = R^T\{H\}_X = \int_M R^T[\underline{X}]\{\dot{X}\}\,dm \qquad \text{(D.1-16)}$$

where $[\underline{X}]\{\dot{X}\}$ is given by (D.1-15).

Therefore, after (D.1-15) is substituted in (D.1-16) the total angular momentum $\{H\}$ may be given in the form

$$\{H\} = \{H\}_{TR} + \{h\} + \{H\}_{ROT} \qquad \text{(D.1-17)}$$

where

(a) $\{H\}_{TR}$ is the contribution to $\{H\}$ due to the translation of the origin $Ox_o$ and is expressed by

$$\{H\}_{TR} = \int_M [\underline{x}_o][\delta\underline{x}_o]^T dm\{\omega\} - \int_M [\underline{x}_o]\{\delta\dot{x}_o\}dm \qquad (D.1-18)$$

Clearly if $\{\delta x_o\} = \{0\} \Longrightarrow Ox_o \equiv CM$ and

$$\{H\}_{TR} = \{0\} \qquad (D.1-19)$$

(b) $\{h\}$ is the relative angular momentum given by

$$\{h\} = \int_M [\underline{x}_o]\{\dot{x}_{o\,r}\}dm \qquad (D.1-20)$$

where $\{\dot{x}_{or}\}$ = relative velocity of the body particles with respect to the $(x_o)$ rotating frame

This contribution to $\{H\}$ is due to the proper motion of the particles themselves besides the motion of the body as a whole due to rotation.

(c) Finally $\{H\}_{ROT}$ is the angular momentum caused by the rotation and has the form

$$\{H\}_{ROT} = \int_M [\underline{x}_o][\underline{x}_o]^T dm\{\omega\} = [I_o]\{\omega\} \qquad (D.1-21)$$

where by definition

$$[I_o] = \int_M [\underline{x}_o][\underline{x}_o]^T dm$$

is the central tensor of inertia associated to the system $(x_o)$.

Hence, the total angular momentum respect to the rotating $(x_o)$ system when $Ox_o \equiv CM$ is given by

$$\{H\} = [I_o]\{\omega\} + \{h\} \qquad (D.1-22)$$

D.2 Classical Lagrange-Liouville Equations

The equations of motion of a changing body may be established by substituting equation (D.1-22) in (D.1-11). The final differential matrix equation

$$\{L\} = [I_o]\{\dot{\omega}\} + [\dot{I}_o]\{\omega\} + [\underline{\theta}][I_o]\{\omega\} + [\underline{\theta}]\{h\} + \{\dot{h}\} \qquad \text{(D.2-1)}$$

is not restricted to the rigid body case, as clearly can be seen by the presence of the time derivative of the inertia tensor and the relative angular momentum vector $\{h\}$. Thus, equation (D.2-1) may be applied to more general situations, such as quasi-rigid bodies having particles moving among themselves, gyrostatic motions, expansion or contraction of systems of particles, etc.

All the vector components in equation (D.2-1) and the associate tensor of inertia refer to a central reference system of coordinates $(x_o)$ not necessarily attached to the body and rotating with angular velocity $\{\theta\}$. The vector $\{\omega\}$ represents the angular velocity of the body about the axes of reference. Obviously when the axes of reference are body-fixed, $\{\theta\} = \{\omega\}$.

Although equation (D.2-1) is usually credited to Liouville [1858] because it appears that he formulated it explicitly for the first time, Tisserant [Mecanique Celeste, Vol. II, p. 500] remarks that the so-called Liouville equations are implicit in the general equations of motion of a dynamical system given already by Lagrange [1815].

In effect, the Lagrange equations may be expressed by

$$L_1 = \frac{d}{dt}\left(\frac{\partial T}{\partial \omega_1}\right) + \theta_2\left(\frac{\partial T}{\partial \omega_3}\right) - \theta_3 \frac{\partial T}{\partial \omega_2}$$

$$L_2 = \frac{d}{dt}\left(\frac{\partial T}{\partial \omega_2}\right) + \theta_3\left(\frac{\partial T}{\partial \omega_1}\right) - \theta_1 \frac{\partial T}{\partial \omega_3}$$

$$L_3 = \frac{d}{dt}\left(\frac{\partial T}{\partial \omega_3}\right) + \theta_1\left(\frac{\partial T}{\partial \omega_2}\right) - \theta_2 \frac{\partial T}{\partial \omega_1}$$

or in the abbreviated matrix notation used in this work,

$$\{L\} = \left\{\frac{\dot{\partial T}}{\partial \omega}\right\} + [\underline{\theta}]\left\{\frac{\partial T}{\partial \omega}\right\} \tag{D. 2-2}$$

where

$$T = \frac{1}{2}\int_M \{\dot{x}\}^T\{\dot{x}\}\, dm = \text{total kinetic energy of the system} \tag{D. 2-3}$$

and

$\{\dot{x}\}$ = absolute velocity of the particles.

Assuming the origin of the (x) system at the CM (i.e. $(x) = (x_o)$ )

$$\{\dot{x}\} \equiv \{\dot{x}_o\} = [\underline{\omega}]\{x_o\} + \{\dot{x}_{o\,r}\} = [\underline{x}_o]^T\{\omega\} + \{\dot{x}_{or}\}$$

where

$\{\dot{x}_{or}\}$ = components of relative velocity w. r. t. the $(x_o)$ frame, and

$$\{\dot{x}_o\}^T\{\dot{x}_o\} = \{\omega\}^T[\underline{x}_o][\underline{x}_o]^T\{\omega\} + 2(\{\omega\}^T[\underline{x}_o]\{\dot{x}_{or}\}) + \{\dot{x}_{or}\}^T\{\dot{x}_{or}\}$$

Thus integrating the above equation over the mass of the body and making use of (D. 1-19) and (D. 1-22),

$$T = \frac{1}{2}\{\omega\}^T[I_o]\{\omega\} + \{\omega\}^T\{h\} + \frac{1}{2}\int_M \{\dot{x}_{or}\}^T\{x_{or}\}\, dm \tag{D. 2-4}$$

where the value

$$\frac{1}{2}\int_M \{\dot{x}_{or}\}^T\{x_{or}\}\, dm = \text{kinetic energy of the relative motion}$$

Hence, taking the partials of equation (D. 2-4) with respect to $\{\omega\}$

$$\left\{\frac{\partial T}{\partial \omega}\right\} = [I_o]\{\omega\} + \{h\} \tag{D. 2-5}$$

and finally computing the derivative respect to time of the above equation gives

$$\left\{\frac{\partial \dot{T}}{\partial \omega}\right\} = [\dot{I}_o]\{\omega\} + [I_o]\{\dot{\omega}\} + \{\dot{h}\} \tag{D.2-6}$$

After replacement of (D.2-5) and (D.2-6) in (D.2-2) one arrives at equation (D.2-1), which from now on will be referred to as the Lagrange-Liouville equation. The recent popularity of the Lagrange-Liouville type of equation in geophysical literature should be credited to [Munk and MacDonald, 1960, p. 10] who applied them to the solution of some geodynamic problems.

As mentioned previously, the Lagrange-Liouville equations (D.2-1) refer to a Cartesian system with origin at the CM. A more general expression with respect to a parallel non-central system may be obtained by making use of equation (B.2-12), namely

$$[I] = [I_o] + M[\Delta_o]$$

Consideration of the above will introduce the following contribution to equation (D.2-1)

$$M[\underline{\theta}][\Delta_o]\{\omega\} + M[\dot{\Delta}_o]\{\omega\} + M[\Delta_o]\{\dot{\omega}\} \tag{D.2-7}$$

where clearly the velocities of the origin of the new frame are involved through the matrix $[\dot{\Delta}_o]$.

A final remark should be made here about the Lagrange-Liouville equations as given by (D.2-1). They are functions of the associated central tensor of inertia $[I_o]$. Recalling the relationship between the tensors $[I]$ and $[\mathrm{II}]$ and their time derivatives as expressed by equations (B.1-11) and (B.1-14), a new form of the Lagrange-Liouville equations in function of the tensor $[\mathrm{II}_o]$ may be obtained easily,

$$\begin{aligned}\{L\} = {} & (\mathrm{II}_{o11} + \mathrm{II}_{o22} + \mathrm{II}_{o33})\{\dot{w}\} - [\mathrm{II}_o]\{\dot{\omega}\} + (\dot{\mathrm{II}}_{o11} + \dot{\mathrm{II}}_{o22} + \dot{\mathrm{II}}_{o33})\{\omega\} \\ & - [\dot{\mathrm{II}}_o]\{\omega\} - [\underline{\theta}][\mathrm{II}_o]\{\omega\} + [\underline{\theta}]\{h\} + \{\dot{h}\}\end{aligned} \tag{D.2-8}$$

D.2.1 Selection of Reference Coordinate Systems

In the application of the Lagrange-Liouville differential expressions, various approaches are possible in order to select the reference coordinate system. It is a common practice to choose the reference system in such a way as to facilitate the maximum simplification of the equations. Several options follow.

(i) Absolute body-fixed reference system

For convenience it is usual to select a reference system fixed to the body in some way. It is known that in this particular case $\{\theta\} = \{\omega\}$; thus from (D.2-1)

$$\{L\} = [I_o]\{\dot{\omega}\} + [\dot{I}_o]\{\omega\} + [\underline{\omega}][I_o]\{\omega\} + [\underline{\omega}]\{h\} + \{\dot{h}\} \tag{D.2-9}$$

Observe that in this instance the tensor of inertia associated with the reference coordinate system will vary if the mass configuration of the body changes. Thus the value of $[I_o]$ should be a known function of time. Hence the components of the tensor $[I_o]$ may be considered to be composed of a constant part $[I]_E$ and a contribution $[\Delta I]$ (time dependent) which will account for the differential variations in the tensor of inertia due to the changes involved in the body. Therefore,

$$[I_o] = [I]_E + [\Delta I] \tag{D.2-10}$$

Similar reasoning may be applied to the relative angular momentum vector that should be expressed as

$$\{h\} = \{h\}_E + \{\Delta h\}$$

A note of precaution. Although in a non-rigid body one may find such types of absolute body-fixed axes, the situation is complicated greatly in the case of the real earth, primarily because the crust moves and any bench marks fixed to it defining the coordinate axes will inevitably change with time.

(ii) Central principal axes as reference system

If the instantaneous positions of the central principal axes are taken as the axes of reference, then $D_o = E_o = F_o = 0$ and the tensor of inertia will be diagonal. This simplifies considerably the explicit form of equations (D.2-1). Nevertheless, observe that the axis to which the diagonal tensor is referred will move if transport of mass is assumed. This is a major disadvantage of this option.

Thus only at the initial instant

$$\{\theta\} = \{\omega\} \tag{D.2-11}$$

but for any subsequent time $\{\theta\} = \{\omega + \delta\omega\}$ where the vector $\{\delta\omega\}$ represents the variation in the components of $\{\omega\}$ as a consequence of the motion of the principal system itself; that is, the angular velocities with which the central principal axes (axes of figure) are separating from the initial reference system.

Hence equation (D.2-1) will reduce to:

$$\{L\} = [\ulcorner I_o \lrcorner]\{\dot{\omega}\} + [\ulcorner \dot{I}_o \lrcorner]\{\omega\} + [\underline{\omega + \delta\omega}][\ulcorner I_o \lrcorner]\{\omega\} + [\underline{\omega + \delta\omega}]\{h\} + \{\dot{h}\} \tag{D.2-12}$$

All the values in the above equation are referred to the instantaneous central principal axes.

In this instance, as in (i) the principal moments of inertia must be known as a function of time, thus

$$[\ulcorner I_o \lrcorner] = [\ulcorner I_o \lrcorner]_E + [\ulcorner \Delta I \lrcorner] \tag{D.2-13}$$

In the particular case that the central principal axes move only differentially in the body, the following simplification applies

$$[\Delta I] \approx [0] \qquad \text{(D.2-14a)}$$

and

$$\{\delta\omega\} \approx \{0\} \qquad \text{(D.2-14b)}$$

(iii) Mean axes as reference system

A third alternative often mentioned, advocates the selection of the reference system in such a way that at the instant of selection

$$\{h\} = \{0\} \implies \{\dot{h}\} = \{0\}$$

simplifying equation (D.2-1) accordingly to

$$\{L\} = [I_0]\{\dot{\omega}\} + [\dot{I}_0]\{\omega\} + [\underline{\theta}][I_0]\{\omega\} \qquad \text{(D.2-15)}$$

Although the imposition of the above condition will "freeze" rotations (currents) of the moving particles, the body still can have radial deformation (see 2.5.6). This set of axes has been called "mean axes" [Tisserand, 1891, Ch. 30] but they have the disadvantage that they are not uniquely determined [Routh, II, 1905, p. 21].

## D.3 Lagrange-Liouville Equations Extended to Deformable Bodies

In section D.1 the total torque acting on the body was defined by equation (D.1-3)

$$\{L\} = \int_M [\underline{x}]\{f_b\}\,dm$$

where $\{f_b\}$ represents the body forces per unit mass exerted on each particle of the system.

These are not the only forces that may act on the deformable body. It is also possible to consider surface forces per unit area $\{f_s\}$ which will create surface torques of the type

$$\{L_s\} = \int_s [\underline{x}]\{f_s\}\, ds \qquad (D.3\text{-}1)$$

Body and surface forces are the resultant effect of several components originated in different ways (i.e., directly applied forces, reaction forces due to constraints on the system, etc.) and which finally may produce deformations on the body.

The individual description of these forces and the mathematical formulation to compute them follow.

### D.3.1 Body Forces

The body forces acting upon a body may be divided into two large groups:

(i) External forces resulting from the interactions with objects outside the system (i.e., gravitational attractions, electrostatic forces, etc.)

(ii) Internal forces corresponding to different interaction of particles located in the interior of the body (i.e., Newtonian gravitational forces, centrifugal forces associated with rotation, etc.)

Body forces are expressed in general by the vector equation

$$\{f_b\} = \left\{\frac{\partial}{\partial x}\right\} W \qquad (D.3\text{-}2)$$

where the symbol $\left\{\frac{\partial}{\partial x}\right\}$ is a vector-operator and the scalar $W$ represents the total potential acting on the body.

The value of $W$ is given by the sum of all the contributing potentials in the particular problem at hand.

For example,

$$\mathbb{W} = V_E + V_M + V_{\odot} + V_{☽} + V_{\beta} + \text{other potentials}$$

where

$V_E$ ≡ potential of the stationary (equilibrium) state of undistorted configuration

$V_M$ ≡ the total potential arising from the mass

$V_{\odot}$, $V_{☽}$, $V_{\beta}$ ≡ gravitational potentials due to the attraction of the sun, moon and planets.

## D.3.2 Surface Forces

Surface (contact) forces, also referred to as surface stresses, derive from the action of one portion of the material on another through the bounding surface (i.e., forces from hydrostatic pressure, wind drag, friction, viscous force, etc.)

These forces are given by

$$\{f_s\} = [\sigma]\{n\} \tag{D.3-3}$$

where

$$[\sigma] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ & \sigma_{22} & \sigma_{23} \\ s & & \sigma_{33} \end{bmatrix} = \text{stress tensor} \tag{D.3-4}$$

$\{n\}$ = vector normal to the element of surface $ds$ and with magnitude equal to the area

The corresponding torque, after consideration of the Gauss integral equation, may be written

$$\{L_s\} = \int_s [\underline{x}][\sigma]\{n\}\,ds = \int_v [\underline{x}]\,\mathrm{div}[\sigma]\,dv = \int_v [\underline{x}][\sigma]\left\{\frac{\partial}{\partial x}\right\} dv \tag{D.3-5}$$

The stress tensor of an isotropic elastic medium (one whose elastic properties at any point are independent of direction) is given by

$$[\sigma] = \lambda(\mathrm{tr}[e])[1] + 2\mu[D] \tag{D.3-6}$$

where the following notations are introduced

$$[e] = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ & e_{22} & e_{23} \\ s & & e_{33} \end{bmatrix} = \left\{\frac{\partial}{\partial x}\right\}\{\dot{x}\}^T = \begin{bmatrix} \frac{\partial \dot{x}_1}{\partial x_1} & \frac{\partial \dot{x}_2}{\partial x_1} & \frac{\partial \dot{x}_3}{\partial x_1} \\ & \frac{\partial \dot{x}_2}{\partial x_2} & \frac{\partial \dot{x}_3}{\partial x_2} \\ s & & \frac{\partial \dot{x}_3}{\partial x_3} \end{bmatrix} \equiv \text{strain tensor} \tag{D.3-7}$$

$\mathrm{tr}[e] = e_0 =$ dilatation (increase in volume per unit of volume of the medium

$$[D] = \frac{1}{2}\left(\left\{\frac{\partial}{\partial x}\right\}\{\dot{x}\}^T + \{\dot{x}\}\left\{\frac{\partial}{\partial x}\right\}^T\right) = \frac{1}{2}\left([e] + [e]^T\right) \equiv \text{deformation rate tensor} \tag{D.3-8}$$

$\lambda, \mu \equiv$ Lame's elastic constants. These quantities are the most convenient parameters to use in theoretical work. In applications to the earth they are methodically substituted by Love's numbers. Nevertheless, Lame's constants have the advantage of being directly related to other common elastic parameters, such as Young's modulus and Poisson's ratio.

In the case of a linear viscous fluid, one has

$$[\sigma] = -(p - \lambda\,\mathrm{tr}[e])[1] + 2\mu[D] \tag{D.3-9}$$

where now

$p \equiv$ hydrostatic pressure

$\mu \equiv$ coefficient of viscosity

It can be proved that in this instance

$$\lambda = -\frac{2}{3}\mu \tag{D.3-10}$$

and equation (D.3-9) may be written in the form

$$[\sigma] = -(p + \frac{2}{3}\mu \, \mathrm{tr}[e])[1] + 2\mu[D] \tag{D.3-11}$$

In the particular case of perfect fluids $\mu = 0$, thus

$$[\sigma] = -p[1] \tag{D.3-12}$$

When applying the above equations, it is typical to begin with the assumption that the body initially is in hydrostatic equilibrium

$$[\sigma] = -p_0[1]$$

and then examine small deviations from these equilibrium conditions.

### D.3.3 Final Equations

The final Lagrange-Liouville equations applicable to a deformable body, when elasticity, viscosity, etc. are taken into consideration, is given by

$$\{L\} + \{L_s\} = [I_0]\{\dot{\omega}\} + [\dot{I}_0]\{\omega\} + [\underline{\theta}][I_0]\{\omega\} + [\underline{\theta}]\{h\} + \{\dot{h}\} \tag{D.3-13}$$

The above matrix differential equation is in a form sufficiently general to impose few restrictions on deformation when the contributions to $[I_0]$ and $\{h\}$ are properly computed.

The torques $\{L\}$ and $\{L_s\}$ must be obtained for the particular assumptions involved, and in general are expressed by

$$\{L\} = \int_M [\underline{x}]\left\{\frac{\partial}{\partial x}\right\} \mathbb{V}\, dm \tag{D.3-14}$$

where $\mathbb{V}$ represents the total potential created by the body forces acting on the system, and

$$\{L_s\} = \int_V [\underline{x}][\sigma]\left\{\frac{\partial}{\partial x}\right\} dv \tag{D.3-15}$$

which gives the equations of the torques governing the appropriate stress-strain relations.

All the vectors and second-order tensors in equation (D.3-13) refer to a pre-selected reference system with the simplifications described in section D.2.1 possible.

## REFERENCES


Anderson, E.G., 1976. "The Effect of Topography on Solutions of Stokes' Problem," Unisurv Report No. S14, School of Surv., University of New South Wales, Sydney, Australia.

Arur, M.G. and I.I. Mueller, 1971. "Latitude Observations and the Detection of Continental Drift," J. Geophys. Res., 76, 2071-2076.

Binet, J., 1813. "Sur la Théorie des Axes Conjugués et des Moments d'Inertie des Corps," Journal de l'Ecole Polytechnique (1), Cah. 16, 41-67.

Bostrom, R.C., 1971. "Westward Displacement of the Lithosphere," Nature, 234, 536-538.

Bostrom, R.C., M. A. Sherif and R. H. Stockman, 1974. "Deformation of Earth's Lithosphere with Reference to Tidal Couples," in Plate Tectonics--Assessments and Reassessments. Kahle, C. (Editor), American Association of Petroleum Geologists, Memoir 23, Tulsa, Oklahoma.

Bullen, K.E. and R. A. W. Haddon, 1967. "Derivation of an Earth Model from Free Oscilation Data," Proc. U. S. Nat. Acad. Sci., 58, 846.

Burgers, J., 1955. "Rotational Motion of a Sphere Subject to Visco-Elastic Deformation, I, II, III," Nederland. Akad. Wetensch. Proc., Ser. B, 58, 219-237.

Bursa, M., 1970. "Best-fitting Tri-axial Earth Ellipsoid Parameters Derived from Satellite Observations," Studia Geoph. et Geod., 14, 1-9.

Busse, F.H., 1970. "The Dynamical Coupling Between Inner Core and Mantle of the Earth and the 24-Year Libration of the Pole," in Earthquake Displacement Fields and the Rotation of the Earth, Mansinha, L., D. E. Smylie and A. E. Beck (Editors), Springer-Verlag, New York.

Cecchini, G., 1950. "Le Variationi di Latitudine e il Movimento del Polo di Rotazione Terrestre," Bulletin Géodesique, 17, 325-361.

Chandler, S. C., 1892. "On the Variation of Latitude," Astronomical Journal, 11, 59-61, 65-70, 75-79, 83-86, 107-109, 134-135.

Cox, A., (Editor), 1973. Plate Tectonics and Geomagnetic Reversals. Freeman and Co., San Francisco.

Czarnecki, W., 1970. "The Geodetic Importance of Mean Elevation Data," Technical Papers from the 30th Annual Meeting, American Congress on Surveying and Mapping, Washington, D. C.

Dahlen, F. A., 1973. "A Correction to the Excitation of the Chandler Wobble by Earthquakes," Geophys. J. R. astr. Soc., 32, 203-217.

Darwin, G. H., 1877. "On the Influence of Geological Changes on the Earth's Axis of Rotation," Philosophical Transactions of The Royal Society, A, 167, 271-312. Also in Scientific Papers, 1910, III, 1-46, Cambridge University Press.

Dejaiffe, R., 1972. "La Détermination Astronomique des Mouvements du Pôle par les Procédés en Usage dans les Stations Internationales de Latitude," Série Géophysique No. 105, Observatoire Royal de Belgique.

Fedorov, E. P., A. A. Korsun and N. T. Mironov, 1972. "Non-Periodic Latitude Variations and the Secular Motions of the Earth Pole" in Rotation of the Earth, Melchior, P. and S. Yumi (Editors), Reidel Publishing Co., Dordrecht, Holland.

Gaposchkin, E. M., 1974. "Earth's Gravity Field to the Eighteenth Degree and Geocentric Coordinates for 104 Stations from Satellite Terrestrial Data," J. Geophys. Res., 79, 5377-5411.

Gold, T., 1955. "Instability of the Earth's Axis of Rotation," Nature, 175, 526-529.

Goldreich, P. and A. Toomre, 1969. "Some Remarks on Polar Wandering," J. Geophys. Res., 74, 2555-2567.

Grafarend, E. W., 1973. "A l. o. d. and Wobble Analysis of Generalized Euler-Liouville Type Based on Virial Planetary Equations," in Proceedings of Symposium on the Earth's Gravitational Field and Secular Variations in Position. R. S. Mather and P. V. Angus-Leppan (Editors), The School of Surveying, The University of New South Wales, Sydney, Australia.

Grafarend, E. W., 1977. "Space-Time Differential Geodesy" in The Changing World of Geodetic Science, U. A. Uotila (Editor), Dept. of Geod. Sci., Report No. 250, The Ohio State University, Columbus, Ohio.

Groves, G. W., 1971. "Dynamics of the Earth-Moon System," in Physics and Astronomy of the Moon. Z. Kopal (Editor), Academic Press, New York.

Guinot, B., M. Feissel and M. Granveaud, 1971. Annual Report for 1970, Bureau International de l'Heure, Paris.

Heiskanen, W. A., 1938. "New Isostatic Tables for the Reduction of Gravity Values Calculated on the Basis of Airy's Hypothesis," Publication of The Isos. Inst. IAG, 2, Helsinki.

Heiskanen, W. A. and H. Moritz, 1967. Physical Geodesy. Freeman and Co., San Francisco.

Heiskanen, W. A. and F. A. Vening-Meinesz, 1958. The Earth and Its Gravity Field. McGraw Hill, New York.

Helmert, F. R., 1880. Theorien der Höheren Geodäsie, II. B. G. Teubner, Leipzig.

IAG, 1975. International Association of Geodesy, First Resolution, XVI General Assembly, Bulletin Géodésique, 118, 364-365.

Ijima, S. and S. Okazaki, 1966. "On the Biennial Component in the Rate of Rotation of the Earth," J. Geod. Soc. Japan, 12, 91.

Inglis, D. R., 1957. "Shifting of the Earth's Axis of Rotation," Rev. of Modern Physics, 28, 9-19.

Israel, M., A. Ben-Menahem and S. Singh, 1973. "Residual Deformation of Real Earth Models with Application to the Chandler Wobble, Geophys. J. R. astr. Soc., 32, 219-247.

Jardetzky, W. S., 1962. "Aperiodic Pole Shift and Deformation of the Earth's Crust," J. Geophys. Res., 67, 4461-4472.

Jeffreys, H., 1974. "Theoretical Aspects of Continental Drift" in Plate Tectonics--Assessments and Reassessments. Kahle, C. (Editor), American Association of Petroleum Geologists, Memoir 23, Tulsa, Oklahoma.

Jordan, T. H., 1974. "Some Comments on Tidal Drag as a Mechanism for Driving Plate Motions," J. Geophys. Res., 79, 2141-2142.

Jurdy, D. M. and R. Van der Voo, 1974. "A Method for the Separation of True Polar Wander and Continental Drift, Including Results for the Last 55 m.y.," J. Geophys. Res., 79, 2945-2951.

Kaula, W. M. (Chairman), 1969. "The Terrestrial Environment, Solid-Earth and Ocean Physics, Application of Space and Astronomic Techniques," Report to a Study at Williamstown, Mass., to the National Aeronautics and Space Administration, NASA CR-1579.

Kaula, W. M., 1975. "Absolute Plate Motions by Boundary Velocity Minimizations," J. Geophys. Res., 80, 244-248.

Kołaczek, B. and G. Weiffenbach (Editors), 1975. On Reference Coordinate Systems for Earth Dynamics, International Astronomical Union, Colloquium No. 26, Torun, Poland. Institute of Higher Geodesy and Geodetic Astronomy, Warsaw Technical University, Warsaw, and The Smithsonian Astrophysical Observatory, Cambridge, Massachusetts.

Krogh, F. T., 1969. "VODQ/SVDQ/DVDQ - Variable Order Integrators for the Numerical Solution of Ordinary Differential Equations," Section 314, Subroutine Write-up, Jet Propulsion Laboratory, Pasadena, California.

Lagrange, J. L., 1788. Mécanique Analytique. Veuve Desaint, Paris or Oeuvres, 1889, 11 and 12. Gauthier-Villars, Paris.

Lambeck, K. and A. Cazenave, 1973. "The Earth's Rotation and Atmospheric Circulation--I. Seasonal Variations," J. R. astr. Soc., 32, 79-93.

Lambert, W., 1922. "The Interpretation of Apparent Changes in Mean Latitude," The Astronomical Journal, 34, 103-110.

Lambert, W., F. Schlesinger and E. Brown, 1931. "The Variation of Latitude," Bulletin 78 of the National Research Council, 16, 245.

Leimanis, E., 1965. The General Problem of the Motion of Coupled Rigid Bodies about a Fixed Point. Springer-Verlag, New York.

Liouville, J., 1858. "Développements sur un Chapitre de la Mécanique de Poisson," Journal de Mathématiques pures et Appliquées, (2), 3, 1-25.

Liu, H., 1974. "On the Break-up of Tectonic Plates," J. Geophys. Res., 79, 2568-2573.

Liu, H., I. Carpenter and R. W. Agreen, 1974. "Plate Motion and Secular Shift of the Poles, J. Geophys. Res., 79, 4379-4384.

Mansinha, L. and D. E. Smylie, 1967. "Effect of Earthquakes on the Chandler Wobble and the Secular Polar Shift," J. Geophys. Res., 72, 4732-4745.

Markowitz, W., 1960. "Latitude and Longitude and the Secular Motion of the Pole," in Methods and Techniques in Geophysics, S. K. Runcorn (Editor), Interscience, New York.

McClure, P., 1973. "Diurnal Polar Motion," Goddard Space Flight Center Doc. No. X-592-73-259, Greenbelt, Maryland.

Melchior, P., 1957. "Latitude Variation" in Progress in Physics and Chemistry of the Earth, Vol. 2, L. H. Ahrens, K. Rankama and S. K. Runcorn (Editors), Pergamon Press, New York.

Melchior, P., 1975. "On Some Problems of BIH and IPMS Services" in On Reference Coordinate Systems for Earth Dynamics. B. Kołaczek and G. Wieffenbach (Editors), International Astronomical Union, Colloquium No. 26, Torun, Poland. Institute of Higher Geodesy and Geodetic Astronomy, Warsaw Technical University, Warsaw, and The Smithsonian Astrophysical Observatory, Cambridge, Massachusetts.

Minster, J. B., T. H. Jordan, P. Molnar and E. Haines, 1974. "Numerical Modelling of Instantaneous Plate Tectonics," Geophys. J. R. astr. Soc., 36, 541-576.

Morgan, W. J., 1971. "Convection Plumes in the Lower Mantle," Nature, 230, 42-43.

Mueller, I. I., 1969. Spherical and Practical Astronomy as Applied to Geodesy, Frederick Ungar Publishing Co., New York.

Mueller, I. I., 1975. "Tracking Station Positioning from Artificial Satellite Observations," Geophysical Surveys, 2, 243-276.

Mueller, I. I. and C. R. Schwarz, 1972. "Separating the Secular Motion of the Pole from Continental Drift--Where and What to Observe?" in Rotation of the Earth, P. Melchior and S. Yumi (Editors), Reidel Publishing Co., Dordrecht, Holland.

Mueller, I. I. (Editor), 1975. "Proceedings of the Geodesy/Solid Earth and Ocean Physics (GEOP) Research Conferences," Dept. of Geodetic Sci. Report No. 231, The Ohio State University, Columbus, Ohio.

Munk, W., 1958. "Remarks Concerning the Present Position of the Pole," Geophysica, 6, 335-355.

Munk, W. H. and G. J. F. MacDonald, 1960. The Rotation of the Earth, a Geophysical Discussion, Cambridge University Press, London.

Munk, W. and E. Hassan, 1961. "Atmospheric Excitation of the Earth's Wobble," Geophys. J. R. astr. Soc., 4, 339-358.

Pan, C., 1975. "Polar Motion of a Triaxial Earth and Dynamic Plate Tectonics," Tectonophysics, 25, 1-40.

Pekeris, C. L., 1935. "Thermal Convection in the Interior of the Earth," Roy. Astron. Soc., Month. Not., Geophys. Suppl., 3, 346-367.

Poinsot, L., 1851. "Théorie nouvelle de la rotation des corps," J. Math. Pures Appl. (1), 16, 9-129, 289-336.

Poma, A. and E. Proverbio, 1976. "The Secular Motion of the Pole from BIH Results," Astron. & Astrophys., 47, 105-111.

Proverbio, E. and A. Poma, 1976. "Astronomical Evidence of Change in the Rate of the Earth's Rotation and Continental Motion" in Growth Rhythmus and History of the Earth's Rotation. G. D. Rosemberg and S. K. Runcorn (Editors), John Wiley, Chichester, England.

Rochester, M. G., 1973. "The Earth's Rotation," EOS, Transactions American Geophysical Union, Vol. 54, 8, 769-781.

Rochester, M. G. and D. E. Smylie, 1974. "On Changes in the Trace of the Earth's Inertia Tensor," J. Geophys. Res., 79, 4948-4951.

Rochester, M. G., O. G. Jensen and D. E. Smylie, 1974. "A Search for the Earth's 'Nearly Diurnal Free Wobble,'" Geophys. J. R. astr. Soc., 38, 349-363.

Routh, E. J., 1905. Dynamics of a System of Rigid Bodies, I & II. Macmillan and Co., Ltd. Reprinted by Dover Publications, Inc., New York, 1960.

Runcorn, S. K., 1962. "Towards a Theory of Continental Drift," Nature, London, 193, 311-314.

Sánchez, B. V., 1974. "Rotational Dynamics of Mathematical Models of the Nonrigid Earth," (Dissertation), Applied Mechanics Research Laboratory, The University of Texas at Austin.

Scheidegger, A. E., 1963. Principles of Geodynamics. Academic Press, New York.

Sekiguchi, N., 1954. "On a Character about the Secular Motion of the Earth," Publ. Astr. Soc. Japan, 5, 109.

Soler, T., 1976. "On Differential Transformations between Cartesian and Curvilinear (Geodetic) Coordinates," Dept. of Geod. Sci., Report No. 236, The Ohio State University, Columbus, Ohio.

Solomon, S. C. and N. H. Sleep, 1974. "Some Simple Physical Models for Absolute Plate Motions," J. Geophys. Res., 79, 2557-2567.

Solomon, S. C., N. H. Sleep and R. M. Richardson, 1975. "Forces Driving Plate Tectonics: Inferences from Absolute Plate Velocities and Interplate Stress," Geophys. J. R. astr. Soc., 42, 769-801.

Takeuchi, H. and N. Sugi, 1972. "Polar Wandering and Mantle Convection" in Rotation of the Earth, P. Melchior and S. Yumi (Editors), Reidel Publishing Co., Dordrecht, Holland.

Tisserand, F., 1891. Traité de Mécanique Céleste, II. Gauthier-Villars, Paris.

Tokis, J. N., 1974. "Rotational Dynamics of Deformable Celestial Bodies," Astrophysics and Space Science, 26, 447-476.

Toomre, A., 1974. "On the 'Nearly Diurnal Wobble' of the Earth," Geophys. J. R. astr. Soc., 38, 335-348.

Vening Meinesz, F. A., 1947. "Shear Patterns of the Earth's Crust," EOS Trans Amer. Geophys. Un. 28, 1-61.

Wegener, A., 1929. Die Entstehung der Kontinente und Ozeane, Friedr. Vieweg & Sohn, Braunschweig or The Origin of Continents and Oceans, (English translation), Dover, New York, 1966.

Wilcox, L. E. and R. S. Blouse, 1974. "Mapping Plate Boundaries with Reference to Mean Gravity Anomalies," Proceedings of the International Symposium on Applications of Marine Geodesy, Battelle Memorial Institute, Columbus, Ohio, Marine Technology Society, Washington, D. C.

Wilson, C. R. and R. A. Haubrich, 1976. "Meterological Excitation of the Earth's Wobble," Geophys. J. R. astr. Soc., 46, 707-743.

Woolard, E. W., 1953. "Theory of the Rotation of the Earth around Its Center of Mass," Astronomical Papers Prepared for the Use of the American Ephemeris and Nautical Almanac, Vol. XV, Part 1, U.S. Government Printing Office, Washington, D. C.

Yumi, S., 1975. "Annual Report of the International Polar Motion Service for the Year 1973. Central Bureau of the International Polar Motion Service, Mizusawa.

Yumi, S. and Y. Wako, 1970. "Secular Motion of the Pole" in Earthquake Displacement Fields and the Rotation of the Earth, L. Mansinha, D. E. Smylie and A. E. Beck (Editors), Springer-Verlag, New York.