SEQUENCE IN THE MATHEMATICS SYLLABUS

An investigation of the Senior Secondary Mathematics Syllabus (July 1984) of the Cape Education Department attempting to reconcile the demands of a strictly mathematical order and the developmental needs of pupils, modified by the mathematical potential of the electronic calculator. Some teaching strategies resulting from new influences in the syllabus.

by

Peter Roland Breetzke

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ABSTRACT

This study was motivated by the latest revision of the mathematics syllabuses of the Cape Education Department. The most important changes to content in the Senior Secondary Mathematics Syllabus (July 1984) are the introduction of calculus and linear programming, the substitution of a section on analytical geometry for vector algebra and the recall of the remainder and factor theorems.

The way in which these changes were introduced left the task of integrating them into the teaching process in the hands of individual teachers. This is a task of extreme importance. If one's classroom practice is to simply plough one's way through the syllabus, one loses many opportunities to make the study of mathematics meaningful and worthwhile. Accepting the view of the spiral nature of the curriculum where one returns to concepts and procedures at increasing levels of sophistication, one needs to identify the position of topics in this spiral and to trace their conceptual foundations.

Analytical geometry is in particular need of this treatment. Similarly there are many opportunities for preparing for the introduction of calculus. If the teaching of calculus is left until the last moments of the Standard 10 year without proper groundwork, the pupil will be left with little time to develop an understanding of the concepts involved.

It is the advent of calculators which presents the greatest challenge to mathematics education. We ignore this challenge to the detriment

of our teaching. Taken seriously calculators have the potential to exert a radical influence on the content of curricula and examinations. They bring into question the time we spend on teaching arithmetic algorithms and the priority given to algebraic manipulation. Numercial methods gain new prominence.

Calculators can even breathe new life into the existing curriculum. Their computing power can be harnessed not only to carry out specific calculations but also to introduce new topics and for concept reinforcement.

The purpose of this study has been to bring about a proper integration of the new sections into the existing syllabus and to give some instances of how the calculator can become an integral part of the teaching/learning process.

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CONTENTS

For everything there is a season, and a time for every
matter under heaven:

a time to break down, and a time to build up;

a time to cast away stones, and a time to gather stones together;

a time to keep silence, and a time to speak.

Ecclesiastes 3: 1; 3; 5 and 7

INTRODUCTION

Setting the scene

0.1 SOME PERSONAL BACKGROUND

I came into mathematics teaching via an initial degree in English and History with a slight smattering of Pure Mathematics. In many ways this has been an advantage. It has been the sort of background that has given me an appreciation for the achievements of the past and some knowledge of the background against which they were made. Not having mathematics as a major subject has also engendered a healthy sense of inferiority regarding topics of mathematical esotericism! I have been teaching mathematics for just over 20 years. During my initial years I was fortunate to serve an "apprenticeship" under a certain Scots gentleman, now retired, who was a mathematics teacher for whom I had the greatest respect. He had the most thorough knowledge of his subject and firm ideas on how it should be taught. His guidance I found invaluable.

For the past nine years I have been a Head of Department in charge of mathematics at a school under the Cape Education Department. I have been a sub-examiner for the Cape Senior Certificate examinations in both the standard and higher grade mathematics (second paper) under three different examiners. My experiences in the classroom, in talking to other teachers, in participating in and organizing MASA activities have given me insights into the state of mathematics teaching, particularly in the Cape. I am concerned that teachers are not making the most of available resources, specifically the electronic calculator and the appeal that is inherent in our subject when it is taught in a well-planned, well-ordered way by teachers who know where their pupils have come from, mathematically, and where they themselves are going.

0.2 REACTING TO CHANGE

Mathematics syllabuses have recently undergone a 10-yearly revision which is to be the future pattern, and will be examined at the Senior Certificate level in the Cape for the first time this year (1987). In addition to a certain amount of new material, two topics have reappeared in the syllabus while others have been dropped. Now is a good opportunity for assessing our classroom practice and its validity in terms of the latest developments.

Despite rumours to the contrary, teachers are human! As such they have a built-in resistance to change, no matter how well-motivated it is. Changes tend to be met on the basis of assimilation with the least disruption to the status quo. Mathematics teachers as a group undergo momentary pangs as vector algebra is excised from the syllabus but these are not too severe as the topic had not really been integrated into the body as a whole. In any event, analytical geometry neatly fills the slot and is welcomed back as a familiar friend by anyone who has been teaching for more than 10 years. As for calculus, we can always come round to that when we have finished everything else.

This is a rather cynical and emotional view of the situation and definitely an over-simplification. But because of the pressure of work on most teachers and the resulting lack of time to make a proper study of their subject, it is probably nearer the truth than most of us would care to admit. New topics tend to get slotted in in place of old and we get round to anything extra when we can find the time. The syllabus, in the minds of many, changes by a process of accretion and excision.

0.3 ON APPROACHING THE SYLLABUS

One reason for this attitude has already been cited - pressure of work. There are, of course, others. Part of the problem, I believe, lies in the way in which we approach the syllabus. When first confronted with the problem of making sense of a large body of knowledge, one's approach is usually to start at the beginning and master it piece by piece. Inexperienced teachers often tend to apply this order in their teaching. This approach places an unfair burden on the compilers of the mathematics curriculum as it assumes, albeit unwittingly, that they have arranged the syllabus in a logical order that should be followed exactly in the classroom situation.

Many teachers will have experienced the frustration caused by the inability of pupils to grasp immediately ideas developed in a neat and exact logical order. "It all makes such beautiful sense..." We forget that this is an order we have imposed on an unruly body of knowledge after many years of mankind's association with it. The original pangs mankind suffered in coming to terms with the content have been forgotten in our delight at imposing an aesthetically satisfying order on the chaos. This is an approach about which we should be extremely cautious. Recent history in the area of mathematics education contains the warning of the 'New Maths' movement. Sawyer (1943) makes the point that historically mathematics had a cautious beginning.

In taking geometry as an example, he says:

> "The Great Pyramid was built in 3900 B.C., by rules based on practical experience: Euclid's system did not appear until 3600 years later. It is quite unfair to expect children to start studying geometry in the form that Euclid gave it. One cannot leap 3600 years of human effort so lightly! The best way to learn geometry is to follow the road which the human race originally followed: Do things, make things, notice things, arrange things, and only then - reason about things."

This is also good advice to any teacher thinking about how to sequence his teaching. There may be a logical order in the content of the syllabus but logical order is often something which has been imposed on natural developments and does not necessarily reflect the initial groping which resulted in man first coming to terms with a topic. "While...logical order...makes good mathematical sense, it may not provide good learning experiences for students; for the logical order that we find so obvious today was not at all the way in which mathematical ideas have historically developed over the years" (Fremont, 1969).

0.4 THE OFFICIAL RECOMMENDATIONS

In the case of geometry, the syllabus at the Standard 5 and 6 level advocates a "practical and experimental" approach aimed at acquisition of knowledge "through calculations and intuitively by experimental methods." This appears to echo the sentiments expressed by both Sawyer and Fremont. In the preamble to both the Junior and Senior Secondary Courses, we are told that one of the aims to be borne in mind is that we should teach "to develop a love for, an interest in and a positive attitude towards mathematics, by presenting the subject meaningfully." It is also pointed out that "the arrangement of the content of the syllabus and of its subdivisions is not necessarily an indication of the sequence in which the work must be handled." In the case of the Senior Secondary Course, where for the first time there is a suggestion (in the Cape, at least) as to what work should be covered in Standard 9 and what in Standard 10, we are told that "the breakdown for standards 9 and 10 is not prescriptive."

The syllabus describes the end of year examinations in the Junior Secondary course as follows:

"6 THE EXAMINATION

The following guidelines for the examining are recommended:

6.1 Standard 5

6.1.1 TWO or THREE papers with an allocation of marks as indicated:

Algorithms (mechanical operations)	± 30 marks
Understanding of mathematical principles	± 50 marks
Differentiated problems (graded from simple to more advanced)	± 70 marks
TOTAL	150 marks

The above sections need not necessarily be examined in separate papers.

6.1.2 No paper should be more than one and a quarter hours in length.

6.1.3 The final examination should be set on the work of the whole year and should cover all sections of the syllabus.

6.2 Standards 6 and 7
6.2.1 The final paper(s) should be set on the work of the whole year and should cover all sections of the syllabus.
6.2.2 One or two papers may be set.
6.2.3 Cumulative year marks should contribute not more than one third to the final examination marks.
6.2.4 The total for the final examination is 300 marks.
6.2.5 A time-allocation of 1½ hours per 100 marks is suggested for examination papers."

With regard to the Senior Secondary Course the syllabus says:

"3 EXAMINATION
3.1 Standards 8 and 9 are examined internally.
3.2 Standard 10
3.2.1 The external Senior Certificate papers will be set on the syllabus for Standard 9 and Standard 10.
3.2.2 There will be two three-hour papers of equivalent value with the allocation of marks as follows:
3.2.2.1 First paper
Algebra : 75% (± 5%)
Differential Calculus : 25% (± 5%)
3.2.2.2 Second paper
Questions of a miscellaneous nature covering two or more sections of the whole syllabus may be asked : ± 5%
Euclidean Geometry : 30% (± 5%)
Analytical Geometry : 25% (± 5%)
Trigonometry : 40% (± 5%)"

0.5 A FLEXIBLE SYLLABUS?

It emerges from the above that within the framework of the syllabus for each standard (taking Standards 9 and 10 as a unit), teachers have the freedom to sequence the content in such a way as makes for meaningful presentation. As far as the internal examinations are concerned there also exists considerable room for the exercise of the teacher's discretion.

However, I do not believe that the majority of teachers make use of the freedom allowed. In their search for guidance and support in planning the sequencing of their instruction, many allow themselves to be bound by imaginary shackles to the order suggested by the syllabus.

The myth of a rigid sequence of content in the syllabuses is reinforced by most of the text books aimed at the South African market. In the past the Cape Education Department's syllabuses have suggested the topics to be covered in each standard, with Standards 9 and 10 being treated as a unit. The most recent Cape syllabuses, though emphasizing that the breakdown is not prescriptive, have suggested for the first time which sections should be covered in Standard 9 and which completed in Standard 10. (In some provinces this subdivision is prescribed). Where in the past publishers and authors produced a single book for the Standard 9/10 syllabus, the tendency now is to produce separate books for each year. I hesitate to suggest any advantages associated with this policy but in the Cape it tends to give force to what is, after all, only a suggested division of content. It also lends credence to the myth of a rigid and prescribed sequence of content. The index of the typical school text book often reads like the main headings of the syllabus and the inexperienced teacher seeking guidance in preparing his lessons is likely to be lead along these paths. Years of practice become a thick shell of habit from which he is unlikely to break free.

In our situation with its shortage of properly qualified mathematics teachers, text books have an extremely important role to play in the support and guidance of mathematics instruction. This places a great responsibility on the shoulders of authors to ensure that their books are more than copies of the syllabus with examples added and that they actually further the mathematical education of the learner, both pupil and teacher!

0.6 MATHEMATICS EDUCATION OR EDUCATION THROUGH MATHEMATICS?

I refer specifically to the promotion of mathematics education rather than the use of mathematics to further the education of pupils. This latter emphasis seems to me to be an approach very much open to abuse. Many teachers would be highly indignant if accused of having a cultural bias in their interpretation of the mathematics syllabus. However a not insiginificant number, on the other hand, would see it as their duty, to quote an editorial in one of the Murray Trust publications of the Faculty of Education of the University of Cape Town, to teach a subject that is "relevant, contextualised and conscientising." "Peoples' Education" and the cry of "relevant" mathematics are intruding more and more on our consciousness.

A policy statement contained in this same publication (Relevant Maths.) contains points with which one may sympathize but the worksheets which follow do little to put this policy into practice.

> "As teachers we aim to encourage an active multi-cultural teaching approach. Against what background do we do this? We live in a racist society. One result of colonialism and imperialism has been the suppression of the culture and science of the Third World peoples and the creation of the myth of 'European Science' as a seamless body of truth. In this way it promotes the inferiority of black people and encourages white chauvinism. Too much of the way we teach mathematics shares in this. We do little to show that mathematics is the product of the thinking and achievements of all the people in the world. We must not allow the universality of mathematics to be lost."
>
> "Anti-racist mathematics teaching...may involve a number of different methods - case study materials...or study of particular civilisations - or it may require the deliberate inclusion of more historical background material with existing activities. So far as possible, such work should be linked to other progressive cross-curricular material."

However, from the worksheets included in Relevant Maths there is no sign of historical material designed to illustrate the "universality of mathematics". On the contrary, the worsheets consist of pseudo-mathematical exercises based on largely out of date statistics and articles of emotional appeal. History in this context consists of examples from black/white confrontation. It would appear from this that 'Peoples' Education' is designed to exacerbate existing problems and has nothing of value to say on improvement of mathematics teaching. 'Relevant Maths' evidently is taken to mean the perversion of the subject to promote a political end above its application to solving everyday problems.

Conan Doyle has Sherlock Holmes remark to Dr Watson in The Sign of Four that "Detection is, or ought to be, an exact science and should be treated in the same cold and unemotional manner. You have attempted to tinge it with romanticism, which produces much the same effect as if you worked a love-story or an elopement into the fifth proposition of Euclid." I am not so sure about remaining unemotional in the face of the beauty and power of mathematics but I do think that relevant mathematics in the context of 'Peoples' Education' is an unedifying distortion of the raison d'être of the subject.

0.7 SYLLABUS CHANGES MAKE RE-ORDERING A PRIORITY

What actually is this subject, mathematics, that gives rise to such controversies? Sawyer (1955) describes it as "...the classification and study of all possible patterns. Pattern here is used...to cover almost any kind of regularity that can be recognized by the mind...Any theory of mathematics must account both for the power of mathematics, its numerous applications to natural science, and the beauty of mathematics, the fascination it has for the mind."

This is a definition which combines the "pure" and "applied" views of the subject. In presenting mathematics meaningfully, we aim at providing our pupils with the knowledge that will enable them to cope with the mathematical aspects of a scientific and technological age. Recent curriculum changes have the potential for making mathematics more relevant in the context of this aim.

Now is the time to push for a fresh evaluation of our ordering of the mathematics syllabus. Changes in the syllabus have been of an order that demand a major review of our approach to teaching mathematics. The most important influences necessitating this new look are the "legitimizing" of pocket calculators and the introduction of the basic ideas of the differential calculus, with analytical geometry as a necessary link in the mathematical chain.

0.8 THE POCKET CALCULATOR

The basic scientific calculator as prescribed for use in our schools has a vast and largely untapped potential as a means of introducing children to mathematical concepts, not least the ideas of limits and the calculus. Its appearance in American schools in the early 1970's resulted in an unprecendented flurry of research, investigation and indignation. Critics sought to have the use of calculators banned and predicted dire consequences for the mental arithmetic of the nation. The overwhelming evidence from research failed to confirm this view and indicated rather that the calculator could play a positive role in stimulating mathematical thought and improving attitudes towards the subject. Yet despite the potential suggested by many studies, it would appear that in the United States, as in South Africa, the calculator is in danger of being assigned the role of an electronic log book (Moursund, 1985; Hembree, 1986).

From what I have seen, it appears to me that many teachers do not appreciate the potential of the calculator and the vast range of mathematical ideas that flow from it. Because it deals with specific numbers rather than x's and y's does not mean that it cannot generate patterns which lend themselves to algebraic analysis or sequences which reinforce the concept of limit. Some careful thought and imagination will show that in the calculator we have an unequalled teaching aid through which we will be able to enrich our pupils' understanding of numbers and introduce them to many of the concepts in the school mathematics course. However, the realization of this potential requires a careful appraisal of what we are doing in the classroom. The calculator is useless in a passive role. It demands the active involvement of the pupil in a meaningful and directed plan of instruction. Teachers need to develop new lessons aimed specifically at integrating the calculator into their teaching and utilizing its capabilities. It should not be left on the periphery of our teaching, to be turned on when looking for the log of this value or the size of that value.

0.9 CALCULUS

The introduction of some of the basic ideas of the differential calculus underlines the need for us to take an urgent look at the mathematics curriculum and how we sequence the content of our syllabuses. We need to be aware of where we are coming from and where we are going - a teacher's vision of his subject should not stop at the boundaries of the standards he teaches. This is especially important with mathematics in view of the spiral nature of the curriculum with a topic being introduced low down in the spiral and being encountered again at various levels. It is just as important for teachers at the lower levels as it is higher up in the curriculum. The way in which a topic is first introduced to pupils can have serious implications for its study higher up the curriculum spiral. This is particularly true for the calculus where the school course serves as a brief introduction to concepts which will be developed further at the tertiary level.

The syllabus advocates a largely informal and intuitive approach to: the gradient of a curve between two points; average speed; the concept of a limit; differentiating a function of the type $f(x) = x^n$: and applications. The syllabus is worded in such a way that one can start laying the foundations for some of these topics as early as the first year of the junior secondary course (Standard 5). Teachers should guard against presenting the calculus as a watertight, neatly parcelled set of definitions and rules and pupils should appreciate that theirs is very much a first intuitive encounter with a most important section of mathematics. As is pointed out by Kriel et al

(1985), the aim of an open-ended intuitive approach will not be realised "...if calculus is simply taught as the application of rules of differentiation and the copying of standard applications. Also not if the pupil is confronted too early with formal definitions, theorems and subtle refinements of concepts. On the other hand we must also guard against an over-simplified "closed" intuitive base which inhibits the introduction, at tertiary level, of refined concepts...The pupil must not get the idea that his concept of limit, derivative, maximum and minimum etc. is complete and final."

0.10 MAKING THE SCHOOL MATHEMATICS COURSE MEANINGFUL

To teach mathematics meaningfully we need to study the syllabus in a variety of contexts. These include: the logical nature of the subject as well as the historical struggle that preceded the logical order; the idea of a spiralling curriculum where themes (such as the notion of function) can be traced through the syllabus and the foundations for the development of some concepts are laid, often intuitively, low down in the spiral and revisited at higher levels; the availability of devices such as the pocket calculator which help the learner to interact and experiment with mathematics with immediate feedback; the learner and his level of mathematical readiness.

Teaching involves the planning of meaningful learning experiences and as Ausubel (1968) points out "...new content becomes meaningful to the extent that it is substantively (nonarbitrarily) related to ideas already existing in the cognitive structure of the learner." If this is not the case, we would, in effect, be requiring our pupils to learn chunks of largely meaningless material by rote. That this happens on a not insignificant scale is obvious to anyone who has marked Senior Certificate papers. It is the type of teaching that results in minds closed against future development.

No one ever built a lasting structure by haphazardly throwing together a pile of stone. The syllabus presents us with the basic curriculum which needs to be interpreted to the learner and the teacher. Guidance may be forthcoming from colleagues and text books but the teacher must have developed a thorough familiarity with the curriculum and the strands that run through it, giving it structure. Only then will he have the self-confidence and knowledge to make all those adjustments for individual personalities that are necessary for successful teaching.

While making a careful study of the curriculum one must be aware of the fact that each level is only one stage in a progression that extends far beyond school mathematics. It requires initiative and sensitivity to cultivate a positive attitude towards mathematics in the minds of one's pupils. A passive regurgitation of the syllabus will not do it.

> "The teacher who says 'tell me what to teach and then I'll do it' is the one who can be safely and economically exchanged for minicomputers plus software" (Howson, 1983).

CHAPTER ONE

The logical sequencing of content and the reality of historical development.

1.1 A LOGICAL ORDER OF CONTENT IS NECESSARY FOR INTELLIGENT LEARNING

Teachers are familiar with the very real problems that arise if basic concepts which should have been acquired at an earlier stage in a course are missing from a pupil's experience. The result is that advanced content becomes a meaningless mass to be digested via the process of rote-learning. In this way negative attitudes towards mathematics develop and flourish.

Skemp (1971) expands on this point. He identifies two kinds of learning which he calls habit-learning (equated with rote-learning) and intelligent-learning. The latter is defined as "the formation of conceptual structures communicated and manipulated by means of symbols." Mathematics, he suggests, offers perhaps the clearest and most concentrated examples of such conceptual structures.

He identifies two principles which must be complied with before intelligent-learning can take place:
(1) Concepts of a higher order than those which a person already has cannot be communicated to him by a definition, but only by arranging for him to encounter a suitable collection of examples.
(2) Since in mathematics these examples are almost invariably other concepts, it must first be ensured that these are already formed in the mind of the learner.

The implications of these principles for the sequencing of instruction are very important. With regard to the first, it is sometimes very tempting to begin one's presentation of a new topic with a definition or two. "These are the laws of indices." "The trig. ratios are defined as follows." "This is how we prove Pythagoras' Theorem." "This is how we differentiate $f(x) = x^n$." "This is an electronic calculator." The list could be expanded considerably. This is all very neat and tidy and clear to the teacher who hopefully has the conceptual background on which these ideas are based. From the teacher's point of view it is perhaps easier than trying to find a collection of examples, each with the characteristic properties from which the concept to be learnt may be abstracted. For the pupils, on the other hand, it is again a case of being taught mathematical knowledge without consciously giving them the opportunity of developing mathematical understanding.

The second principle underlines the fact that if we wish our pupils to have a reasonable chance of mastering a particular concept we must ensure that it is taught as part of a logical sequence of related mathematical concepts, each of which contributes to the understanding of the next. These contributory concepts must be part of the learner's mathematical consciousness and available when required.

As an illustration of his point, Skemp (1971) refers to the dependence of the concepts of algebra on an understanding of the concepts of arithmetic.

> "To understand algebra without ever having really understood arithmetic is an impossibility, for much of the algebra we learn at school is generalized arithmetic. Since many pupils learn to do the manipulations of arithmetic with a very imperfect understanding of the underlying principles, it is small wonder that mathematics remains a closed book to them. Even those who get off to a good start may through absence, inattention, failure to keep up with the pace of the class, or other reasons, fail to form the concepts of some particular stage. In that case, all subsequent concepts dependent on these may never be understood and the pupil becomes steadily more out of his depth."

Such a situation may only be remedied by back-tracking to the point of conceptual breakdown or attempting to work up to it from the beginning. This provides yet another illustration of why it is so vital that a teacher knows his subject backwards!

1.2 BUILDING A CONCEPTUAL SCHEMA

The process of learning mathematics is one of building systems of concepts or schemas. To do this the learner has to process not raw data but the concepts of existing mathematics. This is an advantage in that it saves him from the mathematical equivalent of reinventing the wheel. However, it also means that the learner is that much further from the reality from which the concept was abstracted and that much deeper into the world of mathematical abstraction.

The function of a schema is to integrate existing knowledge and provide a mental tool for the acquisition of new knowledge. The advantages of schematic-learning (or intelligent-learning, as Skemp calls it) over rote-learning, when judged by the ability to recall concepts learnt, are well-known. The responsibility of the teacher is to see that the way in which concepts are learnt leaves the learner able to accommodate his conceptual structures to accept related new information. Learning mathematics is an evolutionary process and we must guard against schemas which block further development. Our own resistance to change should warn us about how difficult it can be to assimilate new ideas into an existing structure.

1.3 ON BRINGING ORDER TO THE CURRICULUM

Planning the sequencing of subject matter is a long term business - in addition to bearing in mind where the pupil has come from, mathematically, one must also prepare him for the future. Primarily in school mathematics one is attempting to lay a structured foundation of basic knowledge, a launching pad from which the learner can profitably move into the realms of higher mathematics or usefully employ what he has learnt in everyday life.

In planning for instruction, the teacher should be aware of difficulties inherent in the subject. Some obvious difficulties arise from the abstract nature of the concepts involved and also their complexity. Underestimating the complexity of those concepts which the teacher may consider "basic" creates many learning difficulties. We need to arrange content so as to prevent the occurrence of obstacles and emotional blocks to learning.

Macnab and Cummine (1986) identify the hierarchical nature of mathematics as another source of difficulty for many pupils. I have already referred to difficulties which arise when there are conceptual gaps in the pupil's mathematical background. As pointed out, these gaps occur for a number of reasons.

To this list Macnab and Cummine add that "difficulties can be caused for certain pupils because a necessary prerequisite skill for a new topic has not been encountered for some time. The benefits of a spiral curriculum may be greatly reduced if the spiral is not tight enough and there is not sufficiently frequent return to certain key topics."

These authors also suggest that if the hierarchical nature of the content is allowed to become too dominant and too much time is spent on developing a topic, boredom and apathy may result, no matter how well planned that unit of work may be. They suggest three strategies in an effort to lessen the consequences of too strict an adherence to the sequential nature of mathematics.

(1) Looking ahead. This involves motivating pupils by beginning the study of a section by discussion of the type of problem to be solved ultimately.

(2) Topic switching. The spiral curriculum notion implies that a given topic is not dealt with in a once-for-all manner but returned to again and again for further development. Such an approach should be built into the syllabus and one's classroom practice. It is also important that the teacher should feel free to switch topics when on the basis of his observations he sees that progress is slowing and pupils need a change of air. This should not be done too frequently lest an impression of fragmentation of the syllabus be imparted to the pupils.

(3) Starting afresh. Much has been said about the logical nature of the development of mathematical schemas. Sometimes the accommodation and assimilation of new concepts is aided if one goes further back in the sequence of supporting concepts than the one immediately preceding the concept to be learnt.

> "It is part of the professionalism of the good teacher of mathematics to have sufficient insight into both mathematics and the thought processes of children to know when the content hierarchy of the subject must be respected and when it may be ignored or side-stepped. As a general working rule, if the mathematics being taught does not require a particular sequencing of content, then the learning programme should be determined by consideration of pupil receptiveness" (Macnab and Cummine, 1986).

Macnab and Cummine are making a case for tempering the logical nature of mathematics with a sensitivity for the psychological development of the child.

Skemp (1971) refers to the same problem:

> "Some reformers try to present mathematics as a logical development. This approach is laudable in that it aims to show that mathematics is sensible and not arbitrary, but it is mistaken in two ways. First, it confuses the logical and the psychological approaches. The main purpose of a logical presentation is to convince doubters; that of a psychological one is to bring about understanding. Second, it gives only the end-product of mathematical discovery...and fails to bring about in the learner those processes by which mathematical discoveries are made. It teaches mathematical thought, not mathematical thinking."

This point is also made by Dienes (in Wain, 1978). He implies that one should be teaching through the medium of mathematics, rather than teaching the knowledge of mathematics per se. "The main aim of mathematics education...should be the development of certain patterns of thinking, certain types of strategies, that people might develop in the face of new situations which they have never encountered before."

A valuable argument in favour of learning mathematics, claims Dienes, would be if its study helped us acquire the skill of stripping situations of their cloak of irrelevancies, the skill of abstraction, of going from the general to the particular and from the particular to the general. This is the type of training which causes us to ask questions such as "What would happen if...?" and "Supposing that...?"

1.4 THE HISTORICAL PERSPECTIVE

The logical and psychological approaches are not the only views that need to be considered when contemplating the ordering of content.

> "Although the concepts and assumptions which form the basis of every mathematical system are inventions of the human mind, they are not the thoughtless gibberish which might be assembled by a robot. There is a rationale behind each such product of mathematics. Sometimes that rationale takes the form of a motivating force, a need in the physical world, or a desire for a simpler way of handling a chore. Sometimes the rationale has its roots in history, in common usage, or in the etymology of a word. At other times, the rationale for teaching a particular concept at this time and in this way may be explained in terms of a sensible match to the previous learnings of your students" (Farrell and Farmer, 1980).

Mathematics is a human activity and as such its development is inextricably linked with history. A British Ministry of Education Pamphlet (No 36 of 1958) points out that "the historical development often proves to be the most natural: the gropings of the pioneers are often closely allied to those of the schoolboy."

The mathematics syllabus consists largely of concepts which have been neatly refined in the processes of history. The question is, to what extent should we allow "the gropings of the pioneers" to influence the order and methods of our teaching?

There are some people who see the progress of mathematical knowledge in terms of gifted mathematicians revealing to lesser mortals new aspects of Platonic ideals. In other words, mathematics is a foreign land waiting to be discovered. This is an attitude encouraged by over formal teaching strategies. Mathematics is a man-made structure, a tool fine-honed through constant use. However, its human origins are not obvious when glancing at our syllabuses. Close investigation will reveal the names of three men, Euclid, Pythagoras and with dubious accuracy, John Venn. The syllabus is on the whole clear, clinical, and dehumanized.

There is a dangerous falsehood implicit in this. It is that the history of mathematics is irrelevant to the learning of the subject. While it may be true that the history of the development of mathematics is not necessary for one's <u>knowledge</u> of the subject, the struggle to develop a useful bit of mathematics, including the wrong directions taken and the different approaches, may well contribute to one's <u>understanding</u> of the topic.

Humphreys (Zweng, 1983) points out: "Perhaps there is no subject that suffers more than mathematics by being disassociated from its history: for an appreciation of the very nature of mathematics is impossible without some acquaintance with its history. Because mathematics is the study of ideas, understanding can be greatly facilitated by an analysis of the origin of these ideas."

A teaching style too closely centred on pure syllabus content and presented via an unimaginative text tends to encourage the view among pupils that mathematics is an esoteric, cut-and-dried system of knowledge with little to say about the real world. The various branches of mathematics encountered at school level come across as bearing little relation to each other and nothing is made of the connections between mathematics and other subjects. These are the dangers we lay ourselves open to if we ignore the human side of mathematics.

To quote Humphreys again:

> "Mathematics as viewed through its history is human. It grows and evolves as one idea leads to another...sometimes in linear progression, sometimes in complete independence

> and sometimes out of great bursts of vision. when students and teachers recognize that the excitement of mathematics originates in the mind, the teaching and learning of contemporary mathematics is lifted to a far more dynamic plane. Familiarity with the history of mathematics fosters an attitude which encourages students to use their intuition and imagination, to ask the question, "What would happen if ....?" As the dynamic and exciting aspects of mathematics become evident to students, they recognize that they too have the opportunity to experience the thrill of discovery and invention."

Humphreys' paper ends with a powerful plea for teachers to make use of the opportunities offered by the history of mathematics.

1.5 THE ROLE OF HISTORY IN MATHEMATICS EDUCATION

The important questions are: Just how large do we allow history to loom in our mathematics instruction? To what extent does it intrude directly into the classroom and to what extent indirectly through its influence on methodology? Is history to be seen as offering advice to the teacher or as something more for the pupil to learn?

The problem of finding a balance is dealt with by Grattan-Guinness (1978). He identifies a spectrum of presentations of a piece of mathematics:

← Historical presentations
← 'History-satire'-orientated presentations
← Orthodox heuristic presentations
← Some form of 'new' mathematics presentation
← Formal presentations

('History-satire' is an approach where rather than relying on an exact historical presentation, one uses the historical record as a bank from which to draw a sequence of developments as a means of teaching a particular piece of mathematics.)

At opposite ends of the above spectrum we have formal and historical presentations. The point is that historical development is often the exact reverse of the formal presentation. The latter turns out to be 'archaeological' mathematics where we spend our time digging down to the foundations, rather than building up from them. Most educators seem to be of the opinion that heuristic methods are better and the implication is that our methods should tend towards the middle of the spectrum. There are some topics which will lend themselves more to one type of presentation than another. Some of us have had the misfortune of having to teach groups and fields to reluctant school pupils where a formal presentation was a way of getting the job over quickly. In fact, if one were solely knowledge-orientated, then teaching through formal presentations is the way to go. If, on the other hand, one is more understanding-orientated in one's teaching then one would lean to the historical end of the spectrum.

Grattan-Guinness (1978) makes the point that a pure historical presentation is also an extreme approach and by no means straightforward. "For example, the history of mathematics is full of blind alleys, laborious manoeuvres and unnecessary delays, and there is little point in living through all these agonies again. Furthermore, it is not necessary or even desirable for pupils and students to become bogged down by the complex historiographical issues that inevitably attend a serious historical study." Hence his promotion of 'history-satire'.

When deciding on the best method of presentation of a piece of mathematics, it will soon become obvious that the more abstract that topic is, the more likely a teacher will be to opt for a formal presentation. Here abstract does not necessarily refer to the inherent properties of the mathematics itself but rather to the teacher's perception of it. Ill-qualified and inexperienced teachers are more likely to be the ones with this sort of perspective of the subject. This underlies the vital necessity for teachers to be more forthcoming in sharing ideas with their colleagues.

Many people nowadays take a perverse pride in their mathematical ignorance. This is largely a reaction to over-formal teaching methods. An opposite emphasis on the historical end of the spectrum would be likely to produce a similar revulsion through placing an unnecessary and undesirable premium on erudition.

1.6 LESSONS FROM HISTORY

It would be tedious to identify the historical pedigree of every topic in the mathematics syllabus, but there are some which immediately cry out for some historical perspectives appropriate to the level at which they are being treated. One thinks of the development of our number system; of Newton and Leibniz and the origins of the calculus. Grattan-Guinness (1978) identifies geometry as "by far the most fertile branch" of mathematics to use a middle 'history-satire'/heuristic approach and deplores the tendency to eliminate geometry in some forms of new mathematics education.

> "Geometry offers everything that a teacher could ask for. It interprets at once for the young in terms of shapes and things around them, and reinforces their learning of spatial concepts. For older pupils it provides an excellent vehicle for presenting other branches of mathematics, especially trigonometry, linear algebra and the calculus..."

(In establishing the claims of geometry and geometrical thinking to a central position in school mathematics education, Grattan-Guinness (1986) points out that a major part of the mathematical experience is that humans live their lives as specks on the surface of a rotating sphere. This has important syllabus implications. Should not provision be made for parts of astronomy and cartography - which have inspired so much mathematical discovery - to be treated at school?)

The most important lesson that teachers of mathematics can learn from the history of their subject is of a more general nature and it concerns methodology. Mathematics has developed as a response to man's interaction with his environment. The Egyptians and Babylonians only needed mathematics for practical purposes. In the Golden Age of

Greece, men had the opportunity to study mathematics from a more philosophical aesthetic point of view and Euclid used his genius for systematization in producing The Elements. The need to keep track of flourishing finance in late medieval Europe extended and refined concepts of number and the recording of numbers. Calculus was created to solve the scientific problems relating to motion, maxima and minima and astronomy. The obsession with axiomatisation, which began in the nineteenth century, started with Cauchy's efforts to put the calculus on a more sound theoretical basis. Mathematics has grown in response to needs and this is what we must establish in the minds of our pupils.

> "We should face our pupils with situations where they need to find or develop mathematical ideas. Teach the subject in such a way that they feel a need for proof. To introduce new ideas (like negative numbers), first create a situation where the pupils feel a need for the new ideas" (de Jager, 1977).

A legacy from Greek mathematics is its insistence on the establishment of mathematical facts through deductive reasoning. We must not lose sight of the fact that in the natural order of development, discovery precedes rigorous proof.

> "Mathematics must be developed, not deductively, but constructively. We must build up the concepts, techniques and theorems from the simplest cases to the slightly more complex and then to the still more complex. And only after they thoroughly understand what we have attained should we help our pupils deal with the work deductively. In fact, the very idea of a deductive structure must be learnt, and this is only learnt gradually. It is a lesson of history. If it took the human race thousands of years to understand the meaning and need of a proof, it will take your pupils a little while too. In no case should we start with the deductive approach, even after pupils know what it means. The deductive organisation is the last step. The

> constructive approach includes letting pupils do the building, the guessing, and the making up of proofs. This approach ensures understanding, not only of the specific things you are teaching at the moment, but of the mathematical process. It teaches independent, productive thinking. Sometimes" (de Jager, 1977).

Given a syllabus, what then does the history of mathematics tell us about the way we should interpret it? Firstly, history underlies the fact that mathematics is a human activity and we dare not present it as a pre-packaged perfect system. It developed in response to needs and establishing again the needs provides a way of introducing a topic. Secondly, problems are solved at a level appropriate to the situation in which they arise. With a more sophisticated situation, more sophisticated ways of dealing with the problem will be necessary. There are differing levels of rigour and abstraction. Thirdly, the need to establish axiomatic foundations of a piece of mathematics follows the development of the mathematics - it is not the basis from which the discoveries were made in the first place and therefore should not be used as a means of introducing a topic.

CHAPTER TWO

The development of the child and the ordering of mathematical structures.

2.1 TOWARDS A PHILOSOPHY OF TEACHING AND LEARNING

Matching the development of their pupils with the development of their subject matter would be the ideal of most teachers. Unfortunately they are subject to so many other pressures on their time that they find themselves basing their teaching practice mainly on what they have come to understand about the teaching-learning process. In many cases this leads to quite adequate performance and some teachers with a rare combination of talents may approach the ideal. In other cases the situation may verge on the disastrous as far as teaching for healthy intellectual development is concerned. Few would deny that successful classroom experience comes when the needs of the pupils and the requirements of the mathematics to be learned complement each other. To arrange such experiences requires knowledge of the learner and knowledge of the subject matter.

Hopefully teachers reject the view of the teaching-learning process as one where knowledge is poured into empty vessels. Such an approach requires no agonizing over the ordering of one's subject matter. The problem comes in deciding which model of intellectual development to adopt. Because there is so much variation in the stands taken by the learning psychologists on this issue, many teachers decide to do without their advice altogether.

2.2 THE VIEWS OF THE LEARNING THEORISTS

Hilgard (quoted in Fremont, 1969) attempted to find common ground in the work of the learning theorists and listed the following points (though there is even some disagreement over these):

(1) The capacity of the learner varies with his age.
(2) Motivation to learn makes acquisition of knowlege easier.
(3) Intense motivations (fear, anxiety) detract from the task.
(4) Success and reward yield more favourable outcomes than failure and punishment.
(5) Learning under intrinsic motivation is preferable to learning under extrinsic motivation.
(6) Tolerance for failure is built by success experiences.

(7) Practice in setting one's own goals leads to realistic goal-setting.
(8) Personal history may influence reaction to the teacher.
(9) Active participation is preferable to passive reception in learning.
(10) Meaningful tasks and responsibilities are learned more readily than nonsense.
(11) To elicit automatic responses, there is no substitute for practice.
(12) Learning is aided by knowledge of mistakes and successes.
(13) Transfer is aided by discovery and by experiencing ideas.
(14) Spaced recalls help fix the materials to be learned.

This list contains a number of important guidelines for general teaching method and classroom practice. However, for help in our problem of ordering content so that our teaching is more appropriate to the child's stage of development, we will need to examine the work of the learning specialists in more detail.

2.2.1 JEROME BRUNER

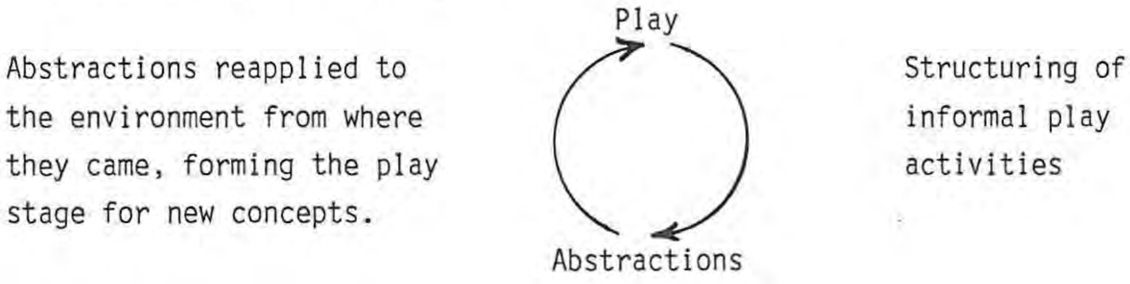
Bruner's studies into the improvement of learning contain a number of points of general value. He emphasizes the importance of structure in the material to be taught. Knowledge must be presented in such a form that the learner can understand it. His rather startling hypothesis that any child at any stage of development can be taught any subject in some honest form does not help us determine the stage of development of the child, or the appropriate form for the material. However, his hypothesis that an individual can experience an idea or concept at three different levels - the enactive, the iconic and the symbolic - does parallel mathematical structure. Enactive learning involves direct, hands-on encounters with the subject matter. Iconic learning takes place when the child is introduced to the subject matter via some visual medium such as pictures and films. Symbolic learning takes place, when one uses abstract symbols to represent reality. This complements the view that the development of a mathematical idea follows a pattern leading from action with concrete objects to logical thinking and intellectualisation away from the concrete situation.

From Bruner's standpoint the key to successful mathematical education is a teacher who is alive to the possibilities of his subject, who is sensitive to the state of readiness of his pupils and creates a meaningful environment which introduces them to the excitement inherent in successful learning. The idea of intrinsic motivation is very strong. Bruner's ideas pass implicit judgement on most school mathematics textbooks. They are inconsistent with the psychological makeup of children. As Post (in Lindquist, 1980) points out, "Bruner's three modes of representational thought are basically analogous to the proposition that "children learn by moving from the concrete to the abstract. A textbook can never provide enactive experiences. By its very nature it is exclusively iconic and symbolic."

2.2.2 Z.P. DIENES

Dienes is exclusively concerned with mathematics education. His view of learning is founded on the following basic principles:

(a) The Dynamic Principle: Dienes suggests that true understanding of a new concept is part of an ongoing evolutionary process consisting of three stages. The first is a preliminary or play stage where the learner should encounter the concept in an unstructured though not random manner. In the second stage structured activities are presented which broaden the learner's acquaintance with the concept to be learnt. The third stage is one of abstraction where the concept becomes part of the learner's intellectual makeup and is available for reapplication to the real world.



Dienes' Learning Cycle (from Post, op. cit.)

(b) The Perceptual Variability Principle: Dienes believes that the generalisation of a concept is facilitated when the learner sees the concept in various situations where the variables irrelevant to the concept are changed while the relevant variables are kept constant.

(c) The Constructivity Principle: Dienes claims that there are two types of thinking - analytical and constructive. Analytical thought consists of logical analysis and a systematic progression from problem to solution. Constructive thought proceeds in spurts and starts, intuitively, without always being aware of the whys and wherefores. Developmentally, the constructive stage occurs before the analytical stage. His claim is that the adult preoccupation with logical analysis has poisoned mathematics for many children. Children should be allowed to develop their concepts intuitively prompted by their experiences of reality. Analysis of their conceptual constructs is a task for the future. According to Dienes, the learning of mathematics requires the active physical and mental involvement of the learner.

2.2.3 JEAN PIAGET

Different views of the nature of learning all point towards some developmental pattern into which most learners fall. The best work on identifying the stages in this pattern was done by the French psychologist, Jean Piaget, and his collaborators.

Piaget viewed intelligence as effective adaptation to one's environment and knowledge as the transformation of experience. By means of carefully constructed experiments, he set about trying to find evidence of that transformation in the behaviour of children as they interacted with their environment. He attempted to discover how a child gets to know a concept - how he organizes a host of experiences of the concept into an ordered form. This analysis resulted in the identification of four major stages: the sensori-motor, the pre-operational, the concrete operational and the formal operational. He also predicted the approximate ages at which children pass through each stage. These age ranges need to be treated with extreme caution as they were postulated on the basis of experiments with children in Europe. Similar experiments have indicated considerable variation in different cultures. His age ranges for the stages which most concern the secondary teacher - the concrete operational and the formal operational - are from 7 to 12 years and from 12 to 16 years respectively. The first age indicates the stage at which characteristics begin to be exhibited by at least 75 percent of the children tested and the second is the age by which all the characteristics of that stage are stable intellectual characteristics of 75 percent of those tested.

Piaget believed that intelligence is developed through the individual's interaction with his environment. To know an idea or object means to get to grips with it, mentally or physically and thereby transform it. This transformation process takes place differently at the different stages. The way in which new ideas become part of the learner's cognitive structure Piaget called adaptation - new ideas are assimilated during interaction with the environment and prior cognitive structures are accommodated to the new input.

When it comes to planning potential learning situations it is obvious that they have to be carefully graded. "If the gap between the present level of understanding and that demanded by the new conceptualization is too great, then (the learner) is likely to assimilate the new idea with distortion, or turn away in apathy" (Wain, 1978). To gain some idea of how to grade material we consider the characteristics of the concrete and formal operational stages. Piaget saw the first indicator that a child was entering the concrete operational stage as the emergence of the ability to conserve number. Conservation of length, mass, volume and area are abilities which develop during this stage. Underlying all these are the abilities to classify and to work with relationships where order makes a difference. Piaget's data shows that "these abilities are developed gradually...as a result of interaction with sufficient experiences that require multiple classification. Ordering relationships is an inseparable part of learning to classify" (Farrell and Farmer, 1980).

An important point is that all these abilities are developed by interaction with concrete content. This does not necessarily imply the manipulation of physical models. What is necessary is that these experiences are real to the learner and that they reflect the concept being learned in as concrete a way possible. This reasoning obviously highlights the importance of the laboratory teaching mode.

What contrasts the formal operational stage with the concrete operational stage is that the form of the situation to be dealt with need no longer reflect concrete aspects of the problem. There should not be an attempt to draw a strict dividing line between concepts to be learned by children at these different stages.

> "During the period of concrete operations the pupil can and should be introduced to a number of other conceptualizations but they will only be understood intuitively, that is, they will always relate to a concrete referent... There will, of course, at this stage, be no formal analytic understanding and no precise relations between concepts elaborated" (Wain, 1978).

Round about ages 11 to 14 years (in average English pupils!) three new thinking skills merge which indicate the development of formal operational thought. These are an improvement in the pupil's ability to handle implications, the ability to move from being able to handle a system of two variables to being able to handle a three or more variable system, and, finally, an increasing ability to abstract.

Piaget identified four characteristics of formal operational thought:

(1) The first is the ability to treat the real as a subset of the possible. The concrete operational pupil transforms concrete experience while the formal operational child is able to transform non-observed and non-experienced phenomena. He is thus freed from the limitations of his senses.

(2) A combinational ability which implies an attempt to consider all possible combinations of sets of elements before experimentation occurs - it is also usual to see some systematic recording of the

effects of the experiments.

(3) Hypothetico-deductive reasoning. The concrete operational learner reasons 'This is true, therefore...,' whereas the formal operational learner reasons, 'If this were true, then...' The quality of necessity is a distinguishing feature. Pupils at this stage sought for necessary causes and were not satisfied with sufficient ones.

(4) Propositional thinking. The elements manipulated by the formal thinker are propositions. As a result of his interaction with new material, the formal thinker classifies it into classes and forms propositions using these results. He adapts these propositions by conjunction, disjunction, implication, negation and equivalence. This is what Piaget calls second degree thinking - operations which result in statements about statements.

These increased powers of abstraction are vital for many high school mathematical concepts and outline the manner in which the intellectually mature adolescent thinks.

> "Presented with a new situation, that adolescent begins by classifying and ordering the concrete elements of the situation. The results of these concrete operations are divested of their intimate ties with reality and become simply propositions that the adolescent may combine in various ways. Using combinational analysis, the student regards the totality of combinations as hypotheses that need to be verified and rejected or accepted" (Farrell and Farmer, 1980).

Note the implication that in the case of completely new material, the formal learner first passes through the concrete operational stage.

2.3 THE IMPLICATIONS OF A PIAGETIAN VIEWPOINT

Acceptance of a Piagetian viewpoint will have a profound effect on a teacher's view of his students. It becomes vital to guage their place in the scheme and errors become of diagnostic value, rather than indications of failure. Here follow some specific implications of Piagetian theory for instruction:

If adaptation (i.e. assimilation and accommodation) is to occur, the gap between the new experience and the past knowledge must not be too large. One must also ensure that all the necessary prior experiences have been provided. This underlines the importance of diagnosis and evaluation.

The way in which pupils transform experience is a very individual matter. This appears to be a case for individualised learning experiences. One should work towards this ideal in carefully monitored steps, getting feedback on different needs before treating each case differently. Individualised learning does not imply working alone. As interaction is so important in Piagetian theory, such a learning mode might not be the most profitable.

Some teachers use the sophisticated cognitive structure of the formal operational learner as justification for a highly verbal, highly symbolic treatment of content. This is a misreading of the situation. The fact that a learner has passed from one stage to another does not mean that he no longer makes use of the abilities characteristic of the previous stage(s). In fact sufficient concrete experience is necessary on which to base formal thought.

> "While it is true that when children reach adolescence their need for concrete experience is somewhat reduced because of the evolution of new and more sophisticated intellectual schemas, it is not true that this dependence is eliminated" (Post in Lindquist, 1980).

On the other hand, there is also a problem with the assumption that all pupils at the formal operational stage will approach all learning in a sophisticated way. They do not if there is no need to. Teachers might also unintentionally discourage formal operational thought if their lessons do not challenge pupils to think at this level.

> "Teachers who intentionally attack this problem use some or all of the following: 'What if' questions; tree diagrams to encourage a systematic approach to many combinations; a 'Predict, explain your prediction, now verify, modify predictions' sequence; or classwork on a problem for which the teacher has no answer" (Farrell and Farmer, 1980).

2.4 THE NEED TO EVALUATE TEACHING STRATEGIES

Having adopted the Piagetian standpoint the teacher needs to evaluate his teaching strategies. For example, should 'traditional skills' be taught alongside tasks designed to develop the child's understanding. In attempting an answer to this question Lavell considers the concept of number as a case in point. He comes to the conclusion that "it is certain that the teaching of traditional skills, like counting, and the reading and writing of numerals, serve as building blocks for the understanding as in conservation: and vice versa such understanding supports the traditional skills" (Wain, 1978). He suspects that this is the case across the curriculum.

When it comes to stragegies, such as the axiomatic method, one needs to consider at what stage they should be introduced. A report by the Mathematical Association on the use of the axiomatic method in secondary teaching points out that as children learn best from their own experience of concrete situations, no attempt should be made to develop the school course axiomatically from the beginning. However, as axiomatic formulation is the logical basis of mathematics, it is desirable that children should at some stage become aware of this aspect of the subject.

Geometry is the obvious example of an axiomatic structure. Pyshkalo (Servais and Varga, 1971) was responsible for distinguishing five levels in the development of geometrical ideas. These were:

(1) the child views a geometrical figure as a whole, without distinguishing its parts and without realising relationships between them or between different figures;
(2) the beginnings of analysis;

(3) the beginnings of deductive thinking;
(4) the view of geometry as a deductive system; and
(5) the appreciation of geometry as a system of axioms. His research indicated that at about age 11 or 12 years, 90 percent of children had not yet moved beyond stage one. (His findings were the result of surveys made in five Russian schools.) These results would appear to put the axiomatic method clearly as a field to be introduced in the formal operations stage.

This point raises a question as far as our practice is concerned. The syllabus requires that the axiomatic treatment of Euclidian geometry is begun in Standard 7 where the average age of pupils is 14 to 15 range. How many children at this level are ready to cope with an axiomatic exposition which corresponds to stages 3 and 4 in Pyshkalo's scheme?

2.5 THE ROLE OF SOCIOLOGICAL FACTORS

We have considered mainly the problem of matching intellectual development with mathematical structure. It is possible that social variations due to age, sex, socio-economic status and social class may require consideration over and above that called for by their intellectual development. R.P. Williams suggests that teachers be wary in seeking a sociologist's advice. "One should be cautious and not over optimistic in seeing sociological thinking as an additional tool that can easily be accommodated to mathematics education" (Wain, 1978). The type of questions teachers want to ask are not necessarily the ones sociologists want to ask. Maybe not all sociologists are as radical as Williams appears to suggest. Teachers should, however, be aware of the potential for problems with a sociological cause, including cultural differences between the teacher himself and his pupils.

> "The life style of children affects the ease with which they can use their intellectual skills in formal school problems. When the life style is such that pupils' thinking tends to be centred on the immediate, the concrete, the perceptible,

> the tangible, they may well be better at assessing perceptual cues as indicators of the properties of situations, but they are less good at using their logical structures in situations unfamiliar to them - although they readily do so in situations about which they know" (Wain, 1978).

Independence is a key factor in intellectual and mathematical development. The pupil needs to be free from pressures which may retard his development. He needs freedom to indulge in wide-ranging, exploratory and adventurous thinking, freedom to reflect on actions and their possible consequences, freedom to find solutions to problems rather than having them provided by teachers. Although no person ever frees himself entirely from content and context, one needs the freedom to be able to reason independently of concrete examples. This freedom is essential for the development of a soundly structured knowledge of mathematics. It demands a carefully ordered mathematics curriculum that will not produce any checks to future mathematical development.

CHAPTER THREE

New influences in the mathematics curriculum

Major changes in the C.E.D. mathematics syllabus have been the substitution of analytical geometry for vector algebra, the recall of the remainder and factor theorems and the introduction of some basic differential calculus. These changes affect both the higher grade and standard grade while linear programming has found its way into the higher grade syllabus and compound increase/decrease into the standard grade at the expense of log theory. In addition to this the use of electronic calculators has been officially approved. Each of these revisions requires a greater or lesser degree of accommodation on the part of the teacher who has to assimilate them into his perception of the mathematics syllabus.

3.1 THE ROLE OF THE CALCULATOR

The factor in the above list which has sparked off the greatest amount of controversy among people interested in mathematics education is undoubtedly the electronic calculator. It has also been responsible for probably the largest body of research on any single topic in the history of mathematics education. Let us examine the reasons for this initial enthusiasm.

Bertrand Russell is alleged to have held the view that education has yet to catch up with the invention of printing. This may be a view tinged somewhat with cynicism but teachers as a body are renowned for their conservatism. Educational innovations are often greeted with uncompromising disdain or with uncritical enthusiasm which soon wanes when faced with the research and preparation necessary for their successful and meaningful implementation. Many of the innovations spawned by technological advances in recent years have failed to make the impact they deserved because of these attitudes, often assisted by factors such as cost. The availability of cheap electronic calculators has, however, presented us with a problem that is ignored at the risk of losing a valuable teaching aid. People in the real world are using calculators to perform tasks that we take years to teach.

> "Calculators are direct alternatives to the arithmetic and calculating methods that up to now have been the principal component of eight or more years of schooling in mathematics. I believe it would be difficult to overemphasize the challenge of that to mathematics education as we now know it" (Bell, 1978).

Calculators mushroomed into prominence in the early 1970's as a result of electronic advances due to space technology and as long ago as 1974, the N.C.T.M. of the United States made its standpoint with regard to calculators clear:

> "With the decrease in cost of the minicalculator, its accessibility to students at all levels is increasing rapidly. Mathematics teachers should recognize the potential contribution of the calculator as a valuable instructional aid. In the classroom, the minicalculator should be used in imaginative ways to reinforce learning and to motivate the learner as he becomes proficient in mathematics" (NCTM Newsletter, December 1974).

Proponents of calculators in the classroom claimed that at last we had a teaching aid which stimulates exploration, which, in freeing the learner from the threat of involved computation, enables him to concentrate on the thinking aspects of problem-solving, and in addition to all this, it is also psychologically motivating. Opponents claimed that there would be a decline in basic mathematical skills. These were the arguments that gave rise to the initial flood of research.

For as long as he has been faced with the problem of performing arithmetic calculations, man has attempted to devise means of facilitating computation. These attempts range from the abacus, through logarithms, the slide rule and various calculating "engines" (the development of which was often frustrated by inadequate technology) to electronic hand-held calculators.

We tend to think of numbers as the vehicles by which calculations can be done. Our concept of number is historically the result of many turns of the wheel of observation and abstraction. It is not the way our ancestors thought. As Hogben (1936) points out, this conception of figures was completely foreign to the most advanced mathematicians of ancient Greece - the ancient number scripts were merely labels to record the results of work on the abacus instead of for doing work with a pen or pencil.

3.2 THE GREAT CALCULATOR DEBATE

This view of number-work finds an echo in the minds of many who see calculators and computers as having been invented to replace paper and pencil. Typically, the enthusiasts see reactionary teachers clinging stubbornly to the old familiar methods, as the main stumbling blocks in the path of the calculator revolution. They argue that teaching pupils to go through the motions of the paper-and-pencil algorithms has not contributed much to the development of mathematical insight and intuition nor has it led to an adequate understanding of fundamental concepts.

As I have indicated, the most common criticisms of calculators stem from fears that they will lead to a decline in the standards of mathematics education. It is claimed that children will be prevented from acquiring basic skills and that if they are introduced to calculators at too early a stage, what they will be acquiring is mere practice in recognizing numerals rather than the opportunity to develop a concept of number. The case of computation offered by calculators will short-circuit the processes by which children master arithmetic algorithms. They will become dependent on calculators for all computation and this will discourage mathematical thinking and destroy all motivation for learning basic facts. The use of calculators will give rise to the notion that mathematics is nothing more than the pushing of buttons on a magic box, the machinations of which are clothed in mystery and the answers supplied quite infallible. This focussing on answers will lead to a neglect of the technical skills of neatness, accuracy and clear lay-out which remain important in areas of mathematics into which the calculator cannot intrude. Furthermore, it is claimed that if calculators can be of any assistance, it is only in the field of arithmetic. It is also proposed that calculators are inappropriate for slow learners.

One of the first things that strikes one on comparing the arguments presented for and against calculators is that many of them directly contradict each other.

Some arguments advocating the broad acceptance of calculators into the school curriculum start from the undeniable fact of their existence and that they can be ignored only to the detriment of mathematical education. The computational power of the calculator makes it a resource tool with far-reaching implications. Children will be enabled to tackle more realistic problems. In cutting down time spent in computation, more time will be available for practice. Children will be freed from pressures that contribute to anxiety and will be able to focus their minds and energy on the logical aspects of and the development of strategies for problem-solving. In providing immediate neutral feedback and reinforcement, as well as great computational power, calculators are seen as a means of motivating and encouraging curiosity, independence and creativity. They can be used to facilitate learning and understanding of basic number facts and algorithmic processes as well as aiding in concept development. Their use can encourage through practice an awareness of the need for the skills of estimation, approximation and verification. Because calculators are practical, convenient and efficient, they are of special help to low achievers as they decrease their frustration with the intricacies of calculation. They greatly improve the pupils accuracy and self-confidence, so leading to a better self-concept. Far from freeing children from the need to think, calculators force them to think logically - they have to translate their intentions into a clear, unambiguous sequence of steps if they are to operate the calculator efficiently.

3.3 RESEARCH ON CALCULATOR USAGE

Despite the publicity they have received and the controversy they have generated, calculators have failed to make a significant impact on mathematics curricula. The Cockcroft Report (1982) brought attention to the need for more work on the calculator as an aid to learning, especially in the primary school. An Open University Study (Hawkridge, 1983) has shown that teachers are still not aware of the potential of the calculator. Many see the next few years as a crucial period in mathematics education with the outcome dependent on how we react to the calculator revolution.

Maier (1983) expresses the views of the revolutionaries:

> "We must allow calculators and computers to become the primary computational tools in our schools...We can resist the revolution and try to keep it from affecting our classrooms, and watch the microcosm of school mathematics drift further and further from the rest of the world, until we are dismissed for our lack of perception about both mathematics and the reality around us. Or we can allow the revolution to reach our classrooms - put up with its discomforts and take advantage of its ability to vitalize school mathematics - and enjoy the excitement."

(Such views notwithstanding it would appear that total reliance on the calculator places the user in a very insecure position and that educational and social pressures are likely to require that pupils will have a certain facility in both written and mental arithmetic for quite a while to come.)

The findings of research on the question of whether or not the use of a calculator adversely affects the acquisition of basic skills appear quite clear. Provided that those basic ideas have first been developed with manipulative materials and paper and pencil, the answer is consistently no. This is obvious from the work of Roberts (1980), Suydam (1978, 1979, 1981, 1982) and Hembree and Dessart (1986). Nevertheless, calculators still attract the suspicion of teachers.

Hembree (1986) quotes a claim by Suydam that "fewer than 20 percent of elementary school teachers and fewer than 36 percent of secondary school teachers have employed the calculator during instruction." This situation is not confined to the United States. Wynands (in Zweng, 1983) in a paper delivered at ICME IV claimed that 95 percent of teachers in West Germany believed that the hand-held calculator would have a negative impact on mental and written calculation.

Negative teacher attitudes appear to be a major check to the successful integration of the calculator into existing mathematics curricula. Suydam (in Zweng, 1983) in a paper also delivered at ICME IV, titled An International Review of Calculator Uses in Schools, reported on the positive findings of research when investigating student achievement, motivation and attitude when using calculators. However, she noted that there was little cohesive planning of in-service activities for teachers to help them place the calculator into perspective and develop strategies for using it effectively. The improvement of teacher attitudes was the subject of a course of in-service education arranged by Bitter (1980). He pointed out that teacher reluctance may negatively effect the educational benefits desired from the use of a calculator. He found that pocket calculator in-service education, designed specifically for classroom teachers, was an excellent means to get them teaching their students about calculators and positively motivating the teachers to augment their mathematics instruction with the use of calculators - Cape Education Department refresher courses have included the calculator in their programmes.

We are faced with the fact of electronic calculators and a syllabus which is not designed around their availability. This is not a unique position. Suydam (in Zweng, 1983) reports that in Sweden there is a co-ordinated program of research and development, studying the effect of calculators as an aid in changing both the methods and the content of the present curriculum. This notwithstanding, Moursund (1985) is not aware of a single school system that has a mathematics curriculum designed around the ready availability of calculators (and computers).

3.4 THE CHALLENGE OF THE CALCULATOR

The calculator has the ability to motivate children by its sheer newness.

> "Time and again it has been demonstrated that even reluctant learners are anxious to use the calculator. Most teachers who have used the calculator in the classroom report that motivation continues as long as students are given interesting things to do...Success in using a calculator as a teaching tool is predicated on a sound educational program. Soon the novelty of the calculator will wear off, and if the student has learned its value as a mathematical tool, new horizons can be opened up to him or her" (Bitter and Mikesell, 1980).

Can we afford to let such opportunities slip?

The point we have reached is this - the calculator is now a standard resource in mathematics teaching. If it is integrated into our instruction and used in an effective and organized way, research has shown that it can contribute substantially to mathematical understanding and the reduction of learning difficulties. If, on the other hand, it is used in a casual, unthinking manner, it will only add to the difficulties of our pupils by adding yet another mysterious box of mathematical tricks to contribute to their confusion.

3.5 ANALYTICAL GEOMETRY

The second major new influence in the syllabus that I wish to discuss is the reappearance of analytical geometry. In his essay, La Géométrie, the third of three appended to his Discourse on Method, René Descartes set about showing how his method of problem-solving made short work of problems considered difficult by ancient mathematicians as well as his contemporaries. This method consisted of a reinterpretation of a geometrical problem in algebraic form and then the application of algebraic methods to find a solution. This work established Descartes as the father of analytical geometry.

Analytical geometry has replaced vector algebra in the Senior Secondary Syllabus. This appears to be a wise move. The basic concepts of analytical geometry are found in algebra and Euclidean geometry and its traditional role as a unifying theme bringing together elements of algebra, geometry and trigonometry is of great value in the school mathematics course. It also makes better sense to approach vector algebra with a knowledge of analytical geometry. In addition, the introduction of calculus is dependent on an understanding of some of the basic concepts of analytical geometry.

In view of these points, it is counter-productive to leave the classroom treatment of analytical geometry until the final stages of the senior secondary course as would arise if teachers were blindly to adhere to the order of the syllabus. Analytical geometry should be seen as an attitude or an approach based in algebra which one brings to problems in space defined in terms of co-ordinates. Consequently this approach needs to be developed as the pupil encounters the relevant concepts throughout the school mathematics course. This occurs as early as the first year of the junior secondary course when pupils encounter the idea of co-ordinates.

3.5.1 THE SYLLABUS REQUIREMENTS REGARDING ANALYTICAL GEOMETRY

The syllabus for analytical geometry prescribes the treatment of the following topics:

6.5 Analytical Geometry in a Plane
6.5.1 The distance between two points
6.5.2 The mid-point of a line segment (The division of a line segment in the ratio k:l is specifically excluded per Circular No. 16/1987)
6.5.3 Gradient of a line
6.5.4 Equation of a line and its sketch
6.5.5 Perpendicular and parallel lines (no proofs). (According to Circular No. 16/1987 these proofs may well be examined in other sections!)

6.5.6 Collinear points and intersecting lines
6.5.7 Intercepts made by a line on the axes
6.5.8 Equations of circles with any given centre and given radius (For S.G. the centre of the circle must be (0;0).)
6.5.9 Points of intersection of lines and circles
6.5.10 Equation of the tangent to a circle at a given point on the circle. (Tying up with 6.2.5.1, this is applicable to H.G. only.)
6.5.11 Other loci with respect to straight lines and circles. (This is qualified in the S.G. syllabus by saying "simple loci.")

3.5.2 INTEGRATING ANALYTICAL GEOMETRY INTO THE CURRICULUM

An analysis of the syllabus shows that all the concepts of analytical geometry will be encountered before the standard 10 year. The following analysis is an effort to integrate the analytical geometry into the curriculum for Standards 7, 8 and 9.

In the Standard 5 syllabus, pupils are introduced to the graphical representation of the relationship between two variables on two rectangular axes (3.8.3). In the Geometry section, a practical and experimental approach to the following is recommended: the concepts: Plane, point, line, line segment, ray, horizontal lines, vertical lines, perpendicular lines and parallel lines (3.9.1); drawing of circles with given radii (3.9.6); and (3.9.7) practice in the skilful use of compasses, set squares, protractor and ruler to establish amongst others concepts from 3.9.1, 3.9.2, 3.9.3, 3.9.4 and 3.9.6. So we have the chance of establishing a basic vocabulary of terms which will be used in analytical geometry and an opportunity for investigating their properties through an experimental approach - in other words it is an informal introduction to the geometric aspects of the topic.

In the Standard 6 year this acquaintance is extended and we start discussing some of the algebraic notions which will be applied to the geometrical problems. The geometrical aspects include construction of: line segments (4.9.5.1); bisectors of line segments and angles

(4.9.5.2), and perpendicular and parallel lines (4.9.5.5). The theorem of Pythagoras is also encountered for the first time.

During the Standard 7 year, the pupils' algebraic ability is further extended. They are expected to be familiar with the concepts: axes, origin, co-ordinates of a point (5.3.4). Also prescribed is a treatment of the function defined by $y = mx + c$, its graphical representation in the Cartesian plane, and its properties (5.3.5). Furthermore pupils are introduced to systems of linear equations and their solution (5.4) as well as the concept of proportion (5.5.1) and the graphical representation of direct and inverse proportion (5.5.3).

During the Standard 7 year pupils should be ready for their first exposure to the methods of analytical geometry. As they begin a formal study of Euclidean geometry, it is useful to be reminded of the essential unity of mathematics. Analytical geometry, by applying algebraic methods to problems in geometric space, helps achieve this.

The question is, how far should we go in introducing analytical geometry at this level. In order not to overwhelm pupils, I do not think it should be extended to topics not specifically implied by the syllabus. In drawing the graph of $y = mx + c$ we are already meeting part of the implications of 6.5.4. In considering the properties of the line $y = mx + c$, it is natural to talk about its gradient (6.5.3) and pupils will notice that parallel lines have the same gradient (6.5.5). As a technique for drawing the graph of $y = mx + c$, one could use the method of establishing where the graph cuts the axes thus dealing with 6.5.7. Finally the solving of systems of linear equations can easily be related to intersecting lines, thus making a first acquaintance with 6.5.9.

In the course of the Standard 8 year, pupils treat both straight lines and systems of linear equations at a more sophisticated level. They also encounter graphs of the type $y = r^2 - x^2$, $y = - r^2 - x^2$, $xy = k$ and $y = ax^2 + c$. In geometry they again meet Pythagoras and this time they also encounter its converse. In addition they deal with parallelograms and the mid-point and intercept theorems. All this gives us an opportunity to revise the work done in Standard 7 and to formalize the treatment of straight lines by introducing the gradient/intercept, gradient/point and two point formulae (6.5.4). By using the gradient formula we can also deal with the problem of collinear points (6.5.6). We can also develop methods for finding the distance between two points (6.5.1) and the mid-point of a line segment (6.5.2). Circle graphs could be introduced via the method for calculating the distance between two points (6.5.8).

In Standard 9, straight lines are again encountered in various forms. This leads on to linear programming. The solution of any quadratic equation is dealt with as is the solution of systems of equations, one of which is linear and the other quadratic. In geometry we deal with tangents to circles and in trigonometry we learn enough to prove that the product of the gradients of perpendicular lines is -1. In promoting the integrating function of analytical geometry, we can apply the definitions of trigonometric ratios to finding the co-ordinates of points and so derive alternative proofs for the area formula and, in conjunction with the distance formula, for the cosine rule. We are now in a position to calculate the intersection of a straight line and a circle as well as the equation of a tangent to a circle (6.5.9 and 6.5.10).

The remaining section of the analytical geometry syllabus concerns problems involving loci with respect to straight lines and circles. These problems presume a vocabulary of words such as equidistant, parallel, perpendicular and the distance formula is an essential key in their solution. They also require a familiarity with Euclidean geometry, especially the basic facts in connection with intersecting and parallel lines, as well as Pythagoras' Theorem. These give rise to a number of obvious locus problems. Some other areas for consideration are:

the locus of all points P (x;y)

(a) such that PA = k.PB where A, B and k are given;

(b) equidistant from a given line or point, a pair of given parallel lines or a pair of points;

(c) equidistant from a fixed given point and a fixed horizontal/vertical line (Kriel, 1985).

This type of problem could be dealt with in Standard 9 on completion of the analytical geometry. However, it might be profitable to discuss example (c) when introducing parabolas in Standard 8. The main point to bear in mind is that analytical geometry has a unifying function and one should be ready to introduce it in such a role as the opportunities arise.

3.6 CALCULUS

The major change in the new syllabus is the introduction of a section on calculus. This has come at a time when there is real concern in many countries for the age at which pupils are being introduced to the topic (see Orton, 1986) and the high failure rate of students taking introductory calculus courses at university (see Spectrum, October 1987).

Prior to the introduction of the current core syllabus for mathematics, there appeared to be great resistance to the teaching of calculus in schools, an exception being the advanced mathematics syllabus of the Joint Matriculation Board. Whatever the source of this resistance, whether it stemmed from a lack of confidence in the ability of teachers to handle the subject or from concern for the conceptual difficulties of the topic, every pupil who now takes higher or standard grade mathematics as a senior certificate subject enounters some elementary differential calculus.

3.6.1 EXPERIENCES WITH CALCULUS TEACHING IN OTHER COUNTRIES

Taking into account the inevitable lack of local experience in the topic, it is interesting to consider the position in other countries, such as the United Kingdom. The situation is complicated by differences in syllabus organisation but there are, nevertheless, numerous interesting parallels and much that can be learnt in the way of pitfalls to be avoided.

Writing 50 years ago, Nunn (1927) registered his indignation at the exclusion of the ordinary pupil from an acquaintance with calculus:

> "When we consider the position of the differential and integral calculus we have to protest against a tradition which forbids all but the exceptional pupils to become acquainted with the most powerful and attractive of mathematical methods."

This point of view was echoed in the Spens Report of 1938:

> "We hold that the ideas of calculus, both differential and integral, should be reached through the graph and through the course in algebraical methods before the majority of pupils leave school."

The actual introduction of calculus into examinations around the age of 16 was a direct result of the Jeffrey Report of 1944. Calculus has consequently been taught to pupils in this age group for about 40 years. During this time mathematics teachers have been able to identify areas of potential conceptual difficulty.

3.6.2 TEACHING CALCULUS FOR UNDERSTANDING RATHER THAN FOR MANIPULATIVE SKILL

According to a handbook of the Incorporated Association of Assistant Masters (1957):

> "The pitfalls are obvious, for there are few branches of the subject which can more easily afford the opportunity for blind manipulation of a notation or the mechanical application of rules. But these pitfalls can be avoided and the intrinsic importance of the subject is very great. The transition from the static mathematics of the formula, which enables one quantity to be calculated when another is known, to the dynamic mathematics of the function, which considers how one thing changes with another, is one of the chief ways

> in which mathematics has adapted itself to the consideration of practical problems. The ideas underlying differentiation and integration are not difficult to grasp if presented in appropriately simple circumstances. They grow naturally and easily out of the consideration of graphs which rightly now occupy so important a place in the elementary teaching of the subject."

The handbook goes on to warn that of all branches of mathematics, calculus requires very careful introduction and development. It points out that some areas of the greatest conceptual difficulty are encountered and that it is impossible and not desirable to deal with them rigorously at this level.

> "For example, although some idea of the meaning of 'limit' is essential, we cannot attempt any study of the theory of limits. On the other hand we must, at all costs, avoid letting pupils perform operations of differentiation and integration mechanically, without any understanding of what they are doing. It should be our aim to develop the subject so as to give them, at any rate, an intelligent comprehension of the processes they are using."

Orton (1986) points out that "the accumulation of ideas on teaching calculus in Britain has produced a very strong note of caution that the approach should be intuitive rather than rigorous." He goes on to suggest that the early steps should not be rushed but that the ideas should be allowed to develop naturally from, for example, graphical work. The importance of an exploratory and investigatory approach in the early stages is also advocated. Experience has shown, however, that if a method does not lend itself to examining purposes, it will tend to be neglected. This has unfortunate consequences for the teaching of calculus because many of the best ways for introducing children to its basic concepts are not examinable in terms of present syllabus requirements. However, pure manipulation of symbols is. So, when engaged in the socially acceptable pastime of teaching for results, it is easier to say to a pupil, this is how you differentiate rather than this is why you differentiate.

Skemp (1976) touches on this problem when he distinguishes between "relational" as opposed to "instrumental" understanding. Relational understanding implies a thorough understanding of the conceptual structure of the topic whereas instrumental understanding implies an ability to apply mechanical methods. Instrumental understanding in the early stages of a calculus course is relatively easy to acquire and very simple to examine. A number of writers refer to a tendency to teach with this end in mind to the detriment of the development of a sound understanding of the underlying concepts.

3.6.3 OTHER PROBLEM AREAS IN TEACHING CALCULUS

Difficulties connected with the teaching of calculus have an international flavour to them. Fremont (1969) writing of the situation in the United States, says that "a serious hindrance has been the lack of an experienced teacher who is qualified to do justice to the course. Perhaps one way to resolve this problem is to be clear about the specific aims for such a course. Woodly notes that while many high schools are teaching good quality calculus courses, he detects a trend towards the offering of calculus courses that are exercises in mechanical computation...This is an unfortunate point of emphasis for high school teachers to take. If we are to think in terms of offering a respectable course in the calculus, built upon a full semester course in analytic geometry..., it would seem that our concern should be the reverse of mechanical manipulation and center upon developing insight into the important ideas."

Fremont concludes the introduction to his section on teaching differentiation with a plea to avoid getting too immersed in the mechanics of the process but rather to evolve situations in which pupils can learn and apply the basic ideas of the calculus.

> "These attempts at rough calculations using fundamental ideas may well provide a basis in understanding that will make present work meaningful and at the same time provide a

> sound basis for continued study. In fact, the teacher himself may gain a great deal from such an introduction as he adds to his own background as well as to that of his students."

Orton's study of students' understanding of differentiation (Orton, 1983) led him to extract a number of implications for the school curriculum and teaching methods. He also rejects the procedure of introducing pupils to differentiation as a set of rules and advocates an investigatory approach, based on a thorough graphical foundation. Pupils must understand the tangent as the limit of secants and so avoid the confusion caused by the idea of a disappearing chord. The relationship between the gradient of the tangent and the rate of change of the curve must be explored. An electronic calculator, he points out, will be an invaluable aid.

Orton belives that much of the difficulty pupils experience with the concept rate of change, stems from a failure to get to grips with the ideas of ratio and proportion. He also finds that a lack of algebraic skills obscured the ideas of calculus for many children.

He makes the point that calculus is a topic that requires the most carefully planned introduction. The foundations of calculus need to be returned to and developed anew at various times throughout the students' mathematical education.

(In view of its omission from our syllabus, it is interesting to note that Orton does not advocate the teaching of the second derivative to investigate the nature of stationary points when dealing with the graphs of cubic functions. This is because the method does not always work and in further study different procedures need to be developed. He suggests that behaviour of the curve on either side of the stationary point be checked through considering the gradient of the tangent. Perhaps the purpose of the ban on points of inflection as per Circular No. 16/1987 was not designed to discourage reference to the second derivative. See page 82 .)

Orton concludes another paper on introducing pupils to calculus (Orton, 1986) with the following summary. It appears to be a healthy starting point for anyone setting out to teach calculus:

> "There are few really new ideas around in teaching, though there are many which lie forgotten or neglected by large numbers of teachers. In terms of the approach to calculus it is by no means new to advocate a practical/concrete approach, it is not new to recommend caution in the introduction of symbolism, nor is it new to suggest intuitive approaches to differentiation...The sad fact is the reluctance to act on such ideas. The computer and calculator present us with a wonderful opportunity to put them into practice with renewed determination to provide for the needs of the learners. Many teaching ideas have accumulated which we have yet to see used by the majority of teachers in the best interests of the students."

CHAPTER FOUR

Teaching with the calculator and teaching calculus

(The booklet and worksheets referred to are
included as appendices C, D and E)

4.1 CALCULATORS AND THE OFFICIAL POLICY

To be able to use their calculators in an intelligent and creative fashion, it is of the utmost importance that pupils become completely familiar with these instruments. One soon becomes rusty on the functioning of the more obscure keys if one does not use them regularly. A test of whether a pupil can use a calculator well is whether he can predict what the display will show when any particular keystroke is carried out.

This familiarity, even with keys which are not going to be used much, is one way of attempting to dispel any mystery about the calculator. We often tend to refer to the machine in anthropomorphic terms with expressions such as "the calculator will tell us." It is essential that pupils realize that the calculator is a fallible instrument, quite capable of misfunctioning or suffering from battery failure. It is unable to evaluate input critically and the results it displays need to be regarded with intelligent scepticism.

At present the Cape Education Department supplies a particular model (the Sharp EL-531) to pupils in their Standard 8 year at a tender price. Originally it was specified that this would be the only model permitted at Cape Senior Certificate examinations but that instruction has been modified to allow any calculator with similar functions, as long as it is not programmable.

4.2 THE CALCULATOR WORKSHEETS

I drew up the booklet, Understanding your Calculator, and the worksheets with two purposes in mind. Firstly, they are an attempt to introduce the pupil to the effective use of the calculator and, secondly, they try to show something of the vast amount of mathematics that can be initiated through imaginative use of the calculator. The aim has been to integrate the calculator into the mathematics curriculum so that its potential to motivate learning and encourage curiosity is harnessed to classroom mathematics.

Some of the ideas which prompted the worksheets have been gleaned from books and journal articles. Others (such as worksheet 6) have come via pupils. There are even some which are original. It was never the intention that pupils should be given a worksheet and left entirely to their own devices. There is much that pupils can work through on their own but the stand I would like to see adopted is that the worksheets are like maps - every now and then it is advantageous to pause and consider the progress made, collect stragglers and look at the way ahead.

As I pointed out, a great deal of mathematics can be initiated via calculator usage. These worksheets only scratch the surface. In compiling this batch, I was very conscious that I should have included a few on trigonometry and, even though it is not in the syllabus, on probability, a topic most pupils find fascinating when it is introduced via a slightly naughty topic like gambling.

Because the calculator deals only with real numbers, many fail to recognize the role it can play in the highly symbolic world of algebra. I have tried to show that much algebra can be initiated through calculator applications - after all algebra is generalized arithmetic and arithmetic is very much the domain of the calculator. The methods used in worksheet 21 (i.e. investigations leading to the establishment of laws and relationships) would offer a concrete introduction to many algebraic and trigonometric generalizations.

Here is a list of the worksheets included as Appendix D. It should be noted that they are devised specifically for the Sharp EL-531.

Worksheet	
1	Basic Operations I
2	Basic Operations II
3	Basic Operations III
4	The Whole Truth
5	Getting to the root of the problem
6	A guesstimation game
7	Estimation I

Worksheet

8	Estimation II
9	Estimation III - exercising your powers
10	An investigation into decimal fractions
11	Rational numbers - simplifying fractions
12	Verifying patterns
13	The number 37
14	Drawing graphs
15	Solving equations by trial and error
16	Solving equations - another look
17	Functions I
18	Functions II
19	What is a logarithm?
20	Changing the base
21	Introducing the laws of logarithms

4.2.1 COMMENTARY ON THE CALCULATOR WORKSHEETS

The first aim of both booklet and worksheets is to make the user familiar with the basic functions of the calculator. Consequently they concentrate on the operations addition and subtraction, multiplication and division, raising to powers and extracting roots. In worksheets 1 - 3, the numbers in the examples have been deliberately kept small. This is to enable the pupil to concentrate on how the calculator is working without being distracted by other arithmetic considerations. It is hoped that familiarity with the calculator's methods will dispel any idea that it has magic properties.

Worksheets 4 and 5 investigate the extent of the internal accuracy of the calculator and attempt to explain some of the apparent contradictions that arise if one is unaware of this property. A point that should be emphasized is that what appears in the display is not the whole story, especially when one is dealing with the decimal representation of irrational numbers.

Worksheet 6 describes a game which utilizes the automatic constant feature of the calculator. As pupils develop a strategy for playing this game, a record of results could be used to reinforce an intuitive approach to limits.

Worksheets 7 - 9 are an effort to develop estimation skills. These skills depend on a thorough knowledge of basic number facts. In the initial debate on calculator usage, many attempts were made to define what basic arithmetic a pupil should be able to perform without having to resort to electronic aids. The general expectation as far as multiplication was concerned was a knowledge of products up to 10 x 10. In practice, pupils must develop a first hand feeling for the tools of the mathematical trade. One must encourage a sense of proportion and responsibility when it comes to using a calculator.

Worksheets 10 and 11 are investigations into rational numbers. The former explores the recurring nature of the decimal representation of a rational number. The pupil is led to take a calculator-guided look at the long division algorithm, a process which has been said to mark the beginning of many a child's disaffection with mathematics. The long division algorithm is used to uncover the full sequence of recurring numbers in the decimal expansion of the rational number. There follows an exploration into this process and its results.

In the second worksheet, a method is considered for writing an improper fraction as a mixed number, using a calculator. There follows a discussion of the Euclidean algorithm for finding highest common factors and from this point one proceeds to lowest common denominators.

In worksheets 10 and 11 and some of those that follow, the topics themselves, though not unimportant, serve as a means to other less tangible ends. They aim to give the child the opportunity to actually do mathematics and perhaps discover something for himself and develop an appreciation for mathematics as an art. These are difficult aims to achieve when one is working strictly to a syllabus.

Following in this vein, worksheets 12 and 13 are also investigatory in nature. They also introduce the child to the use of algebra to prove suspected patterns and thus, hopefully, help to reinforce the notion of proof.

Worksheet 14 continues with a theme begun in the booklet and expands on the use of the calculator in finding the value of algebraic expressions. It uses the calculator's ability to generate pairs of numbers, (input; output), to provide a basic way of plotting points and drawing graphs. This theme is continued in worksheets 17 and 18 on functions.

Worksheets 15 and 16 deal with the solution of equations by trial and error and also by iterative methods. These processes are both useful in helping develop the concept of limits and this is developed further in the calculus worksheets (C_1 and C_4).

Worksheets 19 - 21 are an introduction to the theory of logarithms, using the calculator as a means of initiating the basic ideas.

4.2.2 THE NEED TO PLAN TEACHING AROUND THE CALCULATOR

Through these worksheets I have attempted to show that the calculator does not need to sit on the side-lines of classroom mathematics, to be reached for whenever it is necessary to carry out a calculation. It is possible to make it a part of our teaching. Such an approach is the only way in which we will be able to capitalize on the calculator's potential as a teaching aid. This is going to demand much rethinking on the part of the teacher. It is not something that will be achieved during a few brief Departmental in-service training courses. It will require a concerted effort on the part of organized mathematics teachers. Unless this is done, the calculator, as a teaching aid, will be yet another failure.

4.3 THE CALCULUS REQUIREMENTS OF THE CAPE EDUCATION DEPARTMENT'S MATHEMATICS SYLLABUSES

The syllabus requirements are: (These are the higher grade requirements - sections omitted at standard grade level are noted.)

6.2 DIFFERENTIAL CALCULUS

6.2.1 The average gradient of a curve between two points; average speed.

6.2.2 Limits

6.2.2.1 Intuitive approach to the concept of a limit

6.2.2.2 Determining $\lim_{h \to o} \frac{f(x+h)-f(x)}{h}$, for f(x) one of the following functions: k; $\frac{k}{x}$; ax; ax+b; ax^2; ax^2+bx+c and ax^3 (last two omitted in S.G.)

6.2.2.3 The derivative of a function; the notations: D_x; $\frac{d}{dx}$; and $f'(x)$.

6.2.2.4 The gradient of a curve at any point on the curve

6.2.3 $D_x \ x^n = nx^{n-1}$; n real (without proof)

6.2.4 Rules for differentiating (No proofs in S.G.)

6.2.4.1 $D_x \ f(x) \pm g(x) = D_x \ f(x) \pm D_x \ g(x)$

6.2.4.2 $D_x \ k.f(x) = k. \ D_x \ f(x)$

6.2.5 Applications

6.2.5.1 The equations of tangents to graphs (Not in S.G.)

6.2.5.2 Turning-points and sketches of polynomials of at most the third degree. (See note.)

6.2.5.3 Simple practical problems in connection with maxima and minima and (excluded from S.G.) rates of change.

NOTE. In a subsequent instruction (Circular No.16/1987), points of inflection were specifically excluded from the section on turning points and the sketching of polynomials of the third degree (6.2.5.2 H.G. and 6.2.5.1 S.G.). As a literal interpretation of this instruction would exclude the drawing of such simple curves as $y=x^3$ and the point of inflection is the point about which the graph of the third degree polynomial is symmetric, I would ignore this instruction.

A point emphasized by many authors and backed up by personal experience is that though the mechanics of the calculus may be grasped easily, understanding of what exactly one is doing often only develops after some considerable acquaintance with the topic. A first encounter with calculus should not overwhelm the student with a flood of new concepts camouflaged in a welter of unnecessary variations of notation.

> "In a school course on the calculus there can hardly be any place for the development of manipulative skill in connection with differentiation or integration, except in so far as it is necessary for simple practical applications. The rate of change of a function ought not to be a difficult concept to develop if formulae and graphs have been taught correctly. The development of the meaning of differentiation in terms of gradient of a curve can be well adapted to school circumstances. The pupil will especially appreciate the value of calculus for finding maxima and minima of functions. He will have at his disposal a more powerful and easier method for doing many of the things he now does by clumsier methods" (Van Zyl, 1942).

> "A first introduction aims only to give the pupil mastery of certain elementary techniques and their applications, based on a reasonable acquaintance with the essential ideas. A more detailed account of these ideas must await a later stage, and at that stage practical experience of the use of calculus puts the student in a better position to understand the more careful treatment" (SMG, 1969).

Having considered various approaches I decided that the sequence I would follow would be firstly to attempt the development of a feeling for limits and then, via graphs, approach the idea of rate of change. Through focussing on the tangent as the limiting case of a secant, the idea of the derivative would be introduced. Differentiation from first principles and the rules for differentiating would then be dealt with, proceding subsequently to finding the equations of tangents to curves, curve sketching and problems in maxima and minima.

In referring to these activity units as worksheets, it is not intended to imply that the presence and participation of the teacher is dispensed with. For example, C_1 was prefaced with a discussion which went something like this:

> A mathematical model is something we construct when we attempt to parallel a real-life situation with a mathematical description. A problem that arises is that we get into the habit of expecting exact, perfect answers to our problems and when something does not work out exactly, we start suspecting that we have made a mistake. This is an artificial situation. In real life we usually do not need 100% accuracy in problem-solving. Consider recipes, carpetting your house, painting a room... Take the last example. I suppose there is theoretically an exact quantity of paint that will be required. In making our estimate we can get as close to this quantity as we like by making our model as accurate as possible - provide sketch with windows, doors, areas around windows, etc. This is the first idea we have to appreciate in our study of calculus, the idea of a limit. In the decorating example the limit would be the exact amount of paint required for the job... A limit is the final goal towards which one is striving.

There are numerous opportunities in the syllabus for introducing pupils to the concept of a limit. Some of these occur as early as the Standard 5 level where the syllabus requires experiments in geometric drawing (3.9.6 and 3.9.7) and number patterns and number sequences (3.2.5). This provides an opportunity to launch into curve-stitching and curves of pursuit and an elementary discussion of converging series introduced perhaps by considering Zeno's paradoxes. Further syllabus opportunities to discuss limits occur when dealing with the graphs of hyperbolas (Standard 8, 4.6.3.4 H.G. and S.G.) and the sum to infinity of a geometric series (Standard 10, 6.1.2.4, H.G. only).

In addition, the electronic calculator is ideal for exploring converging number sequences. Examples used are the successive approximations to an irrational root by taking ever-thinner slices of that region on the number line where that root occurs. This gives a feeling for the limit being approached from both directions. One must, however, guard against the idea that the number in the display is the exact value of the root being approximated. Other explorations that could be attempted are an investigation of the Fibonacci sequence, the Newton method for finding square roots, and iterative methods of solving equations and finding various roots and inverses. This also provides some insight into the workings of the calculator and should help to counteract the "magic-box-syndrome."

4.4 THE CALCULUS WORKSHEETS

C_1 An encounter with limits
C_2 Stitching curves and curves of pursuit
C_3 Some more limits
C_4 Recurrence Relations - another way of generating sequences
C_5 Rates of change
C_6 The gradient of a tangent at any point on a curve
C_7 Derivatives from first principles
C_8 Some rules for differentiating
C_9 Tangents to curves
C_{10} Curve sketching

These worksheets were produced in order to meet the demands of teaching the calculus section of the Cape Senior Secondary mathematics syllabus. Circumstances forced the teaching of this section for the first time as a block with little if any prior conscious emphasis on the conceptual foundations of the topic. Consequently the purpose of the first few worksheets was to provide this emphasis, and much of the work done could and should have been covered at an earlier stage.

4.4.1 USING THE CALCULUS WORKSHEETS IN PRACTICE

The worksheets were tried out on two occasions. On the first they were formally tested in the classroom situation with a class of 25 Standard 10 girls. Originally the plan had been to work only with the higher grade group, but the teacher of the standard grade group asked if they might attend and as he was headmaster of the school...! The time-table provided for hour-long periods and these proved ideal for handling one of the worksheets with time for work in class and a bit left over for completion of homework.

Having higher and standard grade pupils together was not ideal. The latter required far more explanation and more revision of basic concepts. This was particularly noticeable when it came to carrying out computations with a calculator. When presented with the sequence of keystrokes for carrying out a particular calculation, it often appeared that pupils did not comprehend the connection between the keystrokes and the calculation they represented. This applied particularly to the more advanced calculations involved in the iterative processes. It appeared to be part of the familiar difficulty pupils have in establishing a correspondence between language and a highly symbolic notation. Nevertheless, the worksheet format gave quicker pupils some activities to get on with while the problems of others were being sorted out.

In teaching calculus I have been particularly conscious of the obligation to lay foundations for further study. Consequently, with higher grade pupils, the concern is to lay a thorough conceptual foundation. Most pupils appeared to find the mechanical processes relatively easy to grasp, but it is another matter with the concept of a limit and the appreciation of what exactly one is finding when one goes through the mechanics of finding the gradient of a tangent to a curve. In future I would be inclined to omit sheets C_2 and C_4 with standard grade classes.

The worksheets were also discussed in two 90 minute sessions with prospective high school mathematics teachers, at present Higher Diploma in Education students. My general impression was that these students wanted to work through the sheets in great detail. Unfortunately there was not time for this and I wanted comments from their point of view on how effective they found the content. In retrospect I appreciate that this was an unfair expectation in view of their relative lack of familiarity with mathematics at the school level - I do not intend this in any derogatory sense. However, I do believe that one can learn about teaching in an academic environment and this is very important but one starts to learn how to teach and carries on learning for the rest of one's career from the time that one enters the classroom situation. Nevertheless, some important points emerged in these sessions and they will be dealt with in considering the individual worksheets.

4.4.2 THE PREPARATION OF THE WORKSHEETS

In preparing these sheets a number of sources were consulted in addition to the syllabus. They included The Mathematics Curriculum: From Graphs to Calculus by Shuard and Neill (1977), Modern Mathematics for Schools Book 8 (Teacher's Edition) prepared by the Scottish Mathematics Group (SMG) (1969), Calculator Calculus by McCarthy (1982) and the Guide to the Senior Secondary Syllabus published by MASA (1985). From the treatment of the topic as expounded in these books I attempted to identify the prior knowledge on which the worksheets should be based and the key concepts that needed emphasis in a course on basic calculus.

4.4.3 COMMENTARY ON THE CALCULUS WORKSHEETS

The above ideas on an intuitive approach to limits formed the substance of the first four worksheets on the calculus. Certain assumptions were made. These were that the pupils would be familiar with the electronic calculator, functional notation, the basics of analytical geometry and the remainder theorem. The most daring of these assumptions was the first, emphasizing again the way in which a calculator is often thrust into pupils' hands and they are then left to get on with the job of becoming familiar with it.

I would like to expand on this point for a moment. As I have indicated, these worksheets were tried out in a formal classroom environment and with prospective mathematics teachers. Even in this latter group there was an evident lack of familiarity with calculators and one young lady confessed to being intimidated by the machine.

Worksheet C_5 is an attempt to introduce rates of change and link them to gradients of lines. Though the first example makes sense to anyone involved in long distance running, it proved confusing to the uninitiated. The confusion arises from the runners habit of thinking of speed in terms of so many minutes per kilometre. In view of the confusion, I do not think it is worth catering to the idiosyncrasies of runners and will rephrase this example, reverting to speed in terms of kilometres per hour.

Worksheet C_6 expands on the idea of rate of change at a point being given by the gradient of the tangent at that point. After an initial example in which a graph is plotted and actual tangents drawn whereupon their gradients are measured, the idea of the tangent as the limiting case of a secant is introduced. Here the pupils require some familiarity with the ideas of analytical geometry. It is also an approach where the pocket calculator is a very useful tool. At the end of this unit, the formal definition of the gradient of the tangent to $y = f(x)$ as $\lim_{h-o} \frac{f(x+h)-f(x)}{h}$ is introduced.

Worksheet C_7 links the limit and the derivative and practice is given in differentiating from first principles. There is also a first reference to the conclusions one can draw about the behaviour of the graph of $f(x)$ from the value of $f'(x)$. (In teaching this unit I had the feeling that this perhaps went too quickly.)

In worksheet C_8 pupils are introduced to the prescribed rules for differentiating by extrapolation from the results of differentiation from first principles. Worksheet C_9 again makes assumptions regarding a knowledge of analytical geometry and deals with finding the equations of tangents to curves. It concerns only pupils following the higher grade syllabus. And worksheet C_{10} deals with curve sketching.

Originally I had planned to conclude this series of worksheets with one on problems. In the end this idea was dropped mainly because most school texts included a variety of suitable problems and the aim of developing an introduction to calculus utilising the potential of the electronic calculator had been realized in this set of worksheets.

Classroom reaction to the worksheets varied from the extremely positive to that of the tail-end of the standard grade group whose knowledge of mathematics remained vague. Both teachers who attended the lessons were also very positive about the course. Neither had taught calculus before and in the case of one about thirty years had elapsed since the subject had last been studied. In the case of the second, the gap was about five years. Both found the course a useful refresher and their opinions were probably influenced by the extent to which it served to clear the dust from their previous experience. Hopefully it served to introduce their pupils to the calculus in a meaningful way.

CHAPTER FIVE

Recommendations and Conclusions

This work was prompted largely by two factors.

The first was the introduction of the revised syllabus with exciting "new" topics. The appearance was that these topics had been haphazardly tacked on to the existing framework. There was a need to have them properly integrated into the body of school mathematics.

The second factor was the official sanctioning of the use of electronic calculators. Calculators made their appearance amid a grand flurry of research and debate in world-wide mathematical circles. In the midst of great expectations, teachers were suddenly faced with the fact that the calculator is a very inanimate object unless it is actively worked into the school mathematics programme. I have attempted to provide some indications of how the calculator can provide a valuable stimulus to learning within the constraints of the present syllabus.

In the course of this investigation various problem areas identified themselves. They arose firstly in my own personal experience but I mention them for what they may be worth:

(a) The nature of our system and the way in which syllabuses are presented encourages a very compartmentalized approach to mathematics teaching.

It is interesting to compare, for example, a British mathematics text with a South African school text. The local product tends to stick closely to the syllabus breakdown. Its foreign counterpart usually pays far more attention to the notion of a spiral curriculum.

We do not make enough of tracing the development of a topic through the syllabus and teaching with the conscious awareness that the work done at various levels is forming the essential foundations of topics that will be introduced at a later stage. This type of approach leads to anomalies in the syllabus, the prime example of which is the section on analytical geometry, suggested for treatment in Standard 10 when most of its ideas have already been dealt with long before that stage.

There is a need for investigations to identify these themes in the syllabus, to identify the points where various content spirals touch each other and to propagate the findings amongst teachers.

(b) The electronic calculator presents a major challenge, but unless we react to the challenge by developing teaching programmes to capitalize on its potential, it will be nothing more than another wasted opportunity.

Though the obvious function of the calculator is to calculate, this does not mean that it cannot be used to motivate learning in more symbolic areas. After all, algebra is often defined as generalized arithmetic and the symbolism of algebra is there to parallel work with numbers. Consequently we should be able to introduce much of the theory via numerical methods.

There is a need for the development of teaching/learning programmes which harness the potential of the calculator.

(c) I have been concerned with the use of the calculator at the Senior Secondary level but it appears clear from research (see Hembree and Dessart, 1986, amongst others) that the calculator has much to offer in the earlier stages of the learning process. If programmes are not developed to utilize the calculator in primary and junior secondary mathematics we will be neglecting the area where it is most relevant and where children's attitudes towards mathematics are being moulded.

(d) The advent of the calculator also raises the question as to what model should be used in schools. At present only the use of a non-programmable basic scientific calculator is permitted officially.

Are there any advantages in the use of programmable calculators? Would not the very act of having to programme the machine provide the pupil with insights into the nature of the procedures he is carrying out?

In the case of programmable calculators the actual procedures do not involve learning a complicated programming language and consequently do not make too great inroads into teaching time. Some possible present applications are the solution of equations, the generation of sequences to illustrate limits or plot graphs and the solution of triangles.

An obvious extension of this line of thought is to investigate the position of pocket computers. As these become more common and more directly applicable to school mathematics, they too will stake a claim for inclusion in the curriculum.

(e) The sanctioning of calculators has important implications for the content of syllabuses. The present position does little to encourage the integration of the calculator into the teaching programme. It is there merely as a more convenient calculation aid than logarithmic tables.

If we are to show any growth in this direction, then we must examine the curriculum possibilities opened up by the calculator. Proper integration of the calculator into our curricula is going to give more opportunity for the exploration of topics with greater emphasis on numerical methods. We will need to decide on a new set of priorities in our teaching. The present emphasis on algebraic manipulation will be brought into question. What will be the mathematics curriculum of the future?

(f) An immediate concern is the effect that calculators will have on examinations. Those viewing the calculator as an electronic log book will not find much difficulty here. The official position is to limit the scoring of any calculation involving calculators to a maximum of three marks. This is, however, an approach which will limit the exploitation of the calculator's potential.

There is a need to investigate evaluation, just what it is we are testing, the best way in which to examine and the form of the examinations themselves.

This thesis set out to examine two basic problem areas, new topics in the syllabus and the use of the calculator. It poses far more questions that it answers. For the present, we can carry on doing our best to teach mathematics in a meaningful way, thankful that pupils manage to survive and even prosper, sometimes in spite of our teaching. However, it appears to me that we are approaching a stage when we will need to decide whether what we are doing is still relevant. In the words of the parable, "No one patches up an old coat with a piece of new cloth, for the new patch will shrink and make an even bigger hole in the coat."

APPENDIX A

The Mathematics Syllabuses of the Cape Education Department.

(a) The Junior Secondary Syllabus (July 1983)
(b) The Senior Secondary Syllabus (July 1984)

4. STANDARD 8

4.1 Products

The following types by inspection:

4.1.1 $(a \pm b)(c \pm d)$

4.1.2 $(ax \pm b)(cx \pm d)$ and $(ax \pm by)(cx \pm dy)$

4.1.3 $(ax \pm b)^2$ and $(ax \pm by)^2$

4.1.4 $(ax + b)(ax - b)$ and $(ax + by)(ax - by)$

4.1.5 $(ax + b)(a^2x^2 - abx + b^2)$ and $(ax + by)(a^2x^2 - abxy + b^2y^2)$ *

4.1.6 $(ax - b)(a^2x^2 + abx + b^2)$ and $(ax - by)(a^2x^2 + abxy + b^2y^2)$ *

4.2 Factors

Factorization of the following types:

4.2.1 Quadrinomials by grouping

4.2.2 Quadratic trinomials

4.2.3 Difference of squares

4.2.4 Sum and difference of cubes *

4.2.5 Preceding types including a common factor.

4.3 Algebraic fractions

4.3.1 Simplification

4.3.2 Main operations

4.4 Equations and inequalities in one unknown

4.4.1 Solution of linear equations with numerical and literal coefficients

4.4.2 Solution of linear inequalities with numerical coefficients

4.4.3 Solutions of problems with the aid of linear equations

4.4.4 Solutions of quadratic equations by means of factors in which only integers may occur as coefficien

4.5 Formulae

4.5.1 Construction of formulae for area and volume of right prisms and right cylinders

4.5.2 Substitution in formulae

4.5.3 Changing the subject of formulae

4.6 Functions

4.6.1 Concept of a function, functional notation and values of a function

4.6.2 Domain and range of a function

4.6.3 Graphical representation of the functions (and deduction of the characteristics of each from its equat graphical representation) defined by:

4.6.3.1 $ax + by + c = 0$

4.6.3.2 $y = \sqrt{r^2 - x^2}$ (r rational and $r \neq 0$)

4.6.3.3 $y = -\sqrt{r^2 - x^2}$ (r rational and $r \neq 0$)

4.6.3.4 $xy = k$ (k an integer and $k \neq 0$)

4.6.3.5 $y = ax^2 + c$ (a en c rational numbers)

4.7 Systems of linear equations in two unknowns

4.7.1 Solution of systems of linear equations — graphically and algebraically

4.7.2 Application of systems of linear equations in the solution of problems *

4.8 Exponents (Pocket calculators may not be used)

4.8.1 The meaning of a^{-n}, a^0 en $a^{\frac{m}{n}}$, $a \neq 0$ (m, n natural numbers), (for $a^{\frac{m}{n}}$ examples $a > 0$ only)

4.8.2 Intuitive extension of the laws of exponents to include integers and rational exponents

4.10 Trigonometry

4.10.1 Definitions of the six trigonometric functions for an angle θ and $\theta \in [0°; 90°]$

4.10.2 Applications for the six trigonometric functions in a right-angled triangle

4.10.3 Solution of right-angled triangles

4.10.4 The definitions of the six trigonometric functions for any angle in terms of co-ordinates with respect to perpendicular axes *

5. STANDARD 9

5.1 Algebra

5.1.1 *A brief intuitive review of the real numbers*

5.1.2 *Absolute value*

5.1.2.1 Definition *

5.1.2.2 Algebraic solution of $|x - a| \leqq b$ *

5.1.3 *Functions*

(No point-by-point plotting of graphs will be required for examination purposes.)

5.1.3.1 Graphical representation of the functions defined by:

(a) $y = ax^2 + bx + c$ $(a \neq 0)$

(b) $y = |x|$, $y = |x - a|$ and $y = |x| + a$, a is a rational number. *

5.1.3.2 The deduction of the characteristics of the functions in 5.1.3.1 from their equations and gra representation

5.1.3.3 Graphical representation of simultaneous equalities with respect to functions from 5.1.3.1, includin intersection with $ax + by + c = 0$

5.1.3.4 The inverses of the functions defined by $y = mx + c$, $y = ax^2$, $xy = k$, $y = |x|$ *

5.1.4 *Linear programming*

5.1.4.1 Graphical representation of $ax + by + c \leqq 0$ *

5.1.4.2 Problem solving by means of programming *

5.1.5 *Quadratic equations and inequalities*

5.1.5.1 The roots of $ax^2 + bx + c = 0$ where a, b and c are rational

(a) The solution of $ax^2 + bx + c = 0$

(b) Conditions for which the equation is solvable on the set of real numbers

(c) Equal and unequal roots; rational and irrational roots; real and non-real roots.

5.1.5.2 The solution of $ax^2 + bx + c \leqq 0$ *

5.1.5.3 Problems which lead to quadratic equations *

5.1.6 *The remainder and factor theorem*

Applications including solution of equations of the third degree

5.1.7 *Systems of equations*

5.1.7.1 Solving simultaneous equations in two unknowns of which one equation is of the first and the othe second degree

5.1.7.2 Solving of problems which lead to equations as shown in 5.1.7.1 *

5.1.8 *Exponents*

5.1.8.1 Solving of equations of the form

$ax^{\frac{m}{n}} - b = 0$ where m and n are integers, $n \neq 0$

5.1.8.2 Relationship between surds and exponents and the corresponding basic properties where a and positive and m and n are positive integers:

$$\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$$

$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$$

$$(\sqrt[n]{a})^m = \sqrt[n]{a^m}$$

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

5.1.8.3 Rationalization of surds (S.G. denoms limited to monomials)

5.1.8.4 Solving of simple exponential equations in one variable

5.2 Trigonometry

5.2.1 The definitions of the six trigonometric functions for any angle in terms of co-ordinates with respect to perpendicular axes (S.G. Limited to interval [0°; 360°])

5.2.2 Graphs of $y = \sin\theta$, $y = \cos\theta$ en $y = \tan\theta$

5.2.3 Function values for $(90° - \theta)$; $(180° \pm \theta)$ and $(360° \pm \theta)$, expressed in function values for θ, where $\theta \in [0°; 90°]$

5.2.4 Function values for 0°, 30°, 45° and multiples thereof without the use of calculators (S.G. [0°; 360°])

5.2.5 *Identities*

× 5.2.5.1 The mutual relationships between the trigonometric function values

5.2.5.2 (a) $\sin^2\theta + \cos^2\theta = 1$

(b) $\tan^2\theta + 1 = \sec^2\theta$

(c) $\cot^2\theta + 1 = \operatorname{cosec}^2\theta$

5.2.6 *Formulae*

5.2.6.1 The sine formula

5.2.6.2 The cosine formula

5.2.6.3 Area of a triangle $ABC = \frac{1}{2}\,ab\,\sin C$

5.2.6.4 Application of the above formulae in the solution of

(a) triangles;

(b) problems in two (and three dimensions.) *

6. STANDARD 10

6.1 Algebra

6.1.1 *Logarithms*

6.1.1.1 The exponential function $y = a^x$, $a > 0$; its graph and deductions from the graph *

6.1.1.2 The logarithmic function $y = \log_a x$, $a > 0$ and $a \neq 1$; its graph and deductions from the graph *

6.1.1.3 The basic properties of logarithms *

6.1.1.4 Change of base of a logarithm *

6.1.1.5 Simple logarithmic equations and inequalities *

6.1.2 *Sequences and series*

6.1.2.1 Characteristics and the general terms of arithmetic and geometric sequences

6.1.2.2 The Σ-notation (S.G. Writing a series in expanded form when given Σ-notation, but not the reverse.)

6.1.2.3 Calculations involving the sum to n terms of arithmetic and geometric series

6.1.2.4 Convergence of a geometric series and its sum to infinity *

6.1.2.5 Solving simple problems using the above

6.1.1. Compound Increase and Decrease
6.1.1.1 Calculation of initial and final sum
6.1.1.2 Calculation of rate
6.1.1.3 Calculation of intervals
} S.G. Only

6.3 Trigonometry

6.3.1 Function values for $-\theta$ and $(\theta + 360° n)$ where n is an integer, expressed in functionvalues for θ, and $\theta \in [0°; 9$ *

6.3.2 The sine, cosine and tangent functions

6.3.2.1 Description of domain and range [S.G. Domain limited to [0°; 360°])

6.3.2.2 Maximum and minimum function values and period

6.3.2.3 Sketches of curves of the following types:

(where a is an integer or a fraction of the form $\frac{1}{n}$; n an integer)

(a) $y = a\sin\theta$, $y = a\cos\theta$, $y = a\tan\theta$ (S.G. limited to [0°; 360°]

(b) $y = \sin a\theta$, $y = \cos a\theta$, ($y = \tan a\theta$) *

(c) $y = a + \sin\theta$, $y = a + \cos\theta$, $y = a + \tan\theta$ *

(d) $y = \sin(\alpha + \theta)$, $y = \cos(\alpha + \theta)$ *

× 6.3.3 $\cos(A - B) = \cos A\cos B + \sin A\sin B$, and identities for *

(a) $\cos(A + B)$

(b) $\sin(A \pm B)$

(c) $\tan(A \pm B)$

(d) $\sin 2\theta$

(e) $\cos 2\theta$

(f) $\tan 2\theta$ (SG. Types as in 6.3.2.3 (a) and (b))

× 6.3.4 General and specific solutions of elementary trigonometric equations (Equations of the type $a\sin x + b\cos x = c$ are included only if $c = 0$)

3. SYLLABUS FOR STANDARD 5

3.1 *Sets*

3.1.1 The concepts: equal sets, equivalent sets, subsets, universal sets, the complement of a set, intersection of sets and union of sets.
3.1.2 Application of the above concepts to known sets.
3.1.3 Venn diagrams to illustrate the above.

3.2 *Natural numbers and whole numbers*

3.2.1 Ready knowledge of addition and subtraction combinations and multiplication and di
3.2.2 Order of operations.
3.2.3 Rules of divisibility by 2, 3, 4, 5, 6, 7, 8, 9, 10 and 11.
3.2.4 Short methods of computing, estimation, approximation, testing of solutions and answers.
3.2.5 Number patterns and number sequences.

3.3 *Common fractions*

3.3.1 Forming equivalent fractions.
3.3.2 Determining common denominators.
3.3.3 Magnitude and order of proper and improper fractions as indicated by the symbols: =
3.3.4 Operations with proper and improper fractions.

3.3.4.1 The four basic operations (including "of") restricted to examples containing at most operations. The use of brackets.
3.3.4.2 Expressing one quantity as a fraction of another.

3.4 *Number sentences*

3.4.1 Writing open sentences, equations and inequalities in which letters are used as place numbers, using the symbols: =, >, <
3.4.2 Solution of open sentences by inspection.

3.5 *Decimal fractions*

3.5.1 Rounding off to nearest whole number and to at most three decimals; appropriateness off.
3.5.2 Conversion of decimal fractions to common fractions and vice versa, rounded off to a decimals.
3.5.3 Operations.
3.5.3.1 Multiplication.
3.5.3.2 Division.

3.6 *Percentages*

3.6.1 Concept and notation
3.6.2 Relation between common fractions, decimal fractions and percentages and the conv one to the other.
3.6.3 Calculation of a percentage of a given quantity.
3.6.4 Expression of one quantity as a percentage of another.
3.6.5 Use of percentages for the purpose of comparisons and simple applications.

3.7 *Physical quantities*

Calculations with respect to money, time, length (kilometre, metre, centimetre, millimetr (kilolitre, litre, millilitre), area, volume, mass (ton, kilogram, gram) and appropriate applied to practical situations.

3.8 *Graphical representations*

(Graphs should be looked upon as a unifying concept and hence graphical representatic used throughout the syllabus wherever applicable and suitable.)

3.8.1 Simple graphs of statistics.
3.8.2 Discussion and interpretation of and deductions from graphs already drawn.
3.8.3 Graphical representation of the relationship between two variables on two rectangular a

4. SYLLABUS FOR STANDARD 6

4.1 *Natural numbers and whole numbers*

4.1.1 Factors and multiples, prime numbers and composite numbers, prime factors.
4.1.2 Squares and cubes; square roots and cube roots by factorization.

4.2 *Integers*

4.2.1 Extension of the number concept to the set of integers.
4.2.2 Order.
4.2.3 The properties of zero.
4.2.4 Additive inverse.
4.2.5 The four main operations.

4.3 *Rational numbers*

4.3.1 Extension of the number concept to the set of rational numbers.
4.3.2 The four main operations with rational numbers.
4.3.3 Informal acquaintance with irrational numbers and their approximation by rational numbe appear in the syllabus (e.g. π and the theorem of Pythagoras).

4.7 *Ratio*

4.7.1 The concept of ratio.
4.7.2 Division of a quantity in a given ratio.
4.7.3 The concept of rate.
4.7.4 Percentage increase and decrease.
4.7.5 Profit percent and loss percent on cost price.

4.8 *Statistics*

4.8.1 Methods to represent statistical data, including different types of graphs.

4.8.2 Critical discussion of statistics appearing in newspapers and periodicals.
4.8.3 Calculation of the arithmetic mean, median and mode of ungrouped data.

5.5 *Proportion*

5.5.1 The concept.
5.5.2 Calculations incorporating simple direct and inverse proportion.
5.5.3 Graphical representation of direct and inverse proportion.
5.5.4 Practical application including scale drawing.

5.6 *Interest*

5.6.1 Calculation of simple interest and amount. (Period restricted to months and years.)
5.6.2 Calculation of compound interest (annually compounded) on a given sum for a full numbe (to a maximum of three years) at a given rate n%, where n is a natural number.

5.7 *Statistics*

5.7.1 Tabulated data: range, class intervals, class marks and class boundaries.
5.7.2 Frequency tables
5.7.2.1 Determination of modal class.
5.7.2.2 Calculating of arithmetic mean.
5.7.2.3 Drawing of histograms and frequency polygons.

JUNIOR SECONDARY COURSE: SYLLABUS FOR MATHEMATICS STANDARD 5 TO 7

1. AIMS

1.1 To develop a love for, an interest in and a positive attitude towards mathematics, by presenting the subject meaningfully.
1.2 To enable pupils to gain mathematical knowledge and proficiency.
1.3 To develop clarity of thought and the ability to make logical deductions.
1.4 To develop mathematical insight.
1.5 To develop accuracy in both calculation as well as mathematical expression.
1.6 To instil in pupils the habit of estimating answers where applicable and where possible of verifying their answers.
1.7 To develop the ability of the pupils in applying mathematical knowledge and methods in other subjects and in their daily life.

1.8 To provide basic training for future study and careers.

2. GENERAL REMARK

The arrangement of the content of the syllabus and of its subdivisions is not necessarily an indication of the sequence in which the work must be handled.

6. THE EXAMINATION

The following guidelines for the examining are recommended:

6.1 *Standard 5*

6.1.1 TWO or THREE papers with an allocation of marks as indicated:
Algorithms (mechanical operations) .... = 30 marks
Understanding of mathematical principles .... = 50 marks
Differentiated problems (graded from simple to more advanced) .... = 70 marks
TOTAL......... 150 marks
The above sections need not necessarily be examined in separate papers.
6.1.2 No paper should be more than 1¼ hours in length.
6.1.3 The final examination should be set on the work of the whole year and should cover all sections of the syllabus.

6.2 *Standards 6 and 7*

6.2.1 The final paper(s) should be set on the work of the whole year and should cover all sections of the syllabus.
6.2.2 One or two papers may be set.
6.2.3 Cumulative year marks should contribute not more than 33⅓% to the final examination marks.
6.2.4 The total for the final examination is 300 marks.
6.2.5 A time-allocation of 1½ hours per 100 marks is suggested for examination papers.

APPENDIX B

Official circulars in connection with the implementation of the new Syllabuses

KAAPLANDSE
ONDERWYSDEPARTEMENT

CAPE
EDUCATION DEPARTMENT

Circular No. 16/1987
File L.16/22/10/1
Telephone: 45-9309

P.O. Box 13
CAPE TOWN
8000

27 February 1937

TO CHIEF SUPERINTENDENTS AND
SUPERINTENDENTS OF EDUCATION
(EDUCATIONAL GUIDANCE) AND
PRINCIPALS OF HIGH SCHOOLS

SENIOR SECONDARY COURSE: EXTERNAL EXAMINATIONS: SYLLABUSES FOR MATHEMATICS HG AND SG: INTERPRETATION OF THE SYLLABUSES FOR PURPOSES OF EXAMINING

1. The following general principle is applicable to laws and formulae which are included in the syllabuses:

   Deductions and proofs may be examined, except where specifically indicated in the syllabus that they are excluded.

2. Further guidelines:

SYLLABUS REFERENCE		INTERPRETATION
HG	SG	
4.6.3.2 4.6.3.3 4.6.3.4	4.6.3.2 4.6.3.3 4.6.3.4	Graphical representations of two semicircles (circle) and the hyperbola are dealt with in Std 8. May they be examined in Std 10 and, if so, may they be included with graphs? Answer HG: Yes, but not explicitly. Only with other work and preferably in analytical geometry. SG: As for HG, but the hyperbola may not be examined for SG.
5.1.4		Is the following terminology acceptable in linear programming: Feasible region Objective function Profit line? Answer Yes

HG	SG	
5.1.4	-	Linear programming: Should it be accepted that graph paper will be supplied when problems must be solved graphically? Answer Yes, when the question is worded accordingly.
5.1.6	5.1.4	Are proofs required in the case of the remainder and factor theorem? Answer HG and SG: Yes
5.1.8.2	5.1.6.2	Are proofs needed for surds and exponents? Answer HG and SG: Proofs are indeed necessary for surds, but not for exponents.
5.1.8.4	5.1.6.4	Is an exponential equation such as $b.a^{2x} + c.a^{x} + d = 0$ permissible in this section? Answer HG: Yes SG: No
5.2.5 6.3.3	5.2.5.3	Are proofs required for identities? Answer HG and SG: Yes
5.2.6	5.2.6	Can proofs be examined in the case of trigonometrical formulae? Answer HG and SG: Yes
5.3(i) 6.4	5.3(i) 6.4	May axioms and deductions which do NOT appear on the list for the syllabus, be used as reasons for proofs (e.g. the product of the segments of chords)? Aswer HG and SG: Yes

HG	SG	
5.3 6.4	5.3 6.4	May coordinate geometry techniques be used in proofs of problems in Euclidean geometry and vice versa? <u>Answer</u> HG and SG: Yes
5.3 6.4	5.3 6.4	If candidates have gained knowledge of vector algebra as a result of enrichment, may this knowledge be used in examinations? <u>Answer</u> HG and SG: Yes
-	6.1.1	Should examples in the case of compound increase and decrease be limited only to financial applications? <u>Answer</u> No, other applications are possible. Candidates should also know the formula, but need not know how to deduce it.
6.1.1.1	-	Do problems and examples of exponential growth fall within the HG syllabus? <u>Answer</u> Yes
6.1.1.3 6.1.1.4	- -	Do the logarithmic laws and their proofs, as well as change of base, fall within the HG syllabus? <u>Answer</u> Yes
6.1.2.3 6.1.2.4	6.1.2.3	Are bookwork and the deduction of formulae required for arithmetic and geometric series? <u>Answer</u> HG and SG: Yes
6.2.4	6.2.4	Are proofs required for the rules for differentiation? <u>Answer</u> HG: Yes SG: No (See syllabus)

HG	SG	
6.2.5.2	6.2.5.1	Are points of inflection included in the section on turning-points and the sketch of polynomials of the third degree? Answer HG and SG: No
6.3	6.3	To what degree of accuracy should be measurement of angles be given in trigonometrical calculations done with the aid of the pocket calculator? Answer HG and SG: One or two decimals may be required.
6.3.2.3	6.3.1.2	Sketches as prescribed in the syllabus (e.g. $y = a \sin x$ and $y = \sin bx$) may be examined, but what about their combinations (for example $y = a \sin bx$)? Answer HG and SG: Combinations will not be examined.
6.5	6.5	Paragraph 6.5.5 specifies that "no proofs" are required for perpendicular and parallel lines. Does this imply that proofs may well be examined in the other sections? Answer HG and SG: Yes (see paragraph 1 above)
6.5.	6.5	May the division of a line segment in the ratio $k : \ell$ be examined? Answer HG and SG: No

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8000

27 February 1987

TO: CHIEF SUPERINTENDENT AND
SUPERINTENDENTS OF EDUCATION
(EDUCATIONAL GUIDANCE) AND
PRINCIPALS OF HIGH SCHOOLS

SENIOR CERTIFICATE EXAMINATION: MATHEMATICS HG, SG AND LG, FUNCTIONAL MATHEMATICS SG: IMPLICATIONS OF THE USE OF THE POCKET CALCULATOR

1. Your attention is drawn to Circular No. 25 of 1 April 1986.

2. The use of the pocket calculator has the following implications regarding examining:

2.1 Number of marks for calculations and information requiring the use of a pocket calculator

HG: Maximum of 20 marks per question paper
SG: Maximum of 20 marks per question paper
LG: First Paper: Maximum of 20 marks
LG: Second Paper: No limit
Functional Mathematics: No limit

2.2 Influence on the setting of questions

2.2.1 Drastic changes to the contents of question paper are not envisaged.

2.2.2 Questions where a calculator may not be used, may still be asked.

2.2.3 The instructions each question should be given very clearly.

2.2.4 The examiner should specify to how many figures/decimals the answer should be given.

2.3 Influence on mark allocation

2.3.1 No single calculation requiring the use of the calculator should count more than 3 marks.

2.3.2 Candidates are not expected to write down their interim steps (displayed on the calculator) or key stroke order.

2.3.3 The awarding of marks for partially correct work will be left to the discretion of the examiner.

APPENDIX C

Understanding your Calculator

INDEX

INTRODUCTION

Your calculator, treated with care and understanding, will prove a faithful and reliable friend for many years. It will always do exactly what you ask of it; it will be willing to work - or play - whenever you require it; it will never complain about being bored or shout at you when you get an answer wrong and tell you that you are stupid - unlike some teachers!

When calculators first became common, many parents and teachers were very worried because they thought that pupils would no longer think for themselves. This has been shown not to be the case. A calculator is no good to you UNLESS you think carefully and logically about how you are going to use it.

In getting to know your calculator, it is most important to realize that although it can compute with great speed and accuracy, it can only do what you tell it to do. So good communication between you and your calculator is essential. You must know exactly what it is going to do when you press a particular key.

You need to cultivate an attitude of intellectual integrity towards your calculator! A great advantage of the calculator is that it helps you tackle realistic problems. Do not insult your intelligence by using it for simple arithmetic. Critically consider your keystrokes and ask yourself if there is not perhaps a different, more economical way of solving the problem. Anybody can punch away at the keys and arrive at an answer but you must remember that what you see in the display is not automatically correct. The calculator is not a magic machine churning out infallible answers. There is one very fallible part of the whole process and that is the operator. In addition, batteries do go flat and machines do develop faults.

It is important to use your calculator critically and develop the habit of estimating answers. If what the calculator gives is very different from your estimate, work through the calculation again, checking on the logic and correctness of your keystrokes. There are three stages in using a calculator for problem solving. These are planning your procedure; carrying out the keystrokes; and evaluating the answer. The amount of time you spend on each stage depends on the complexity of the problem but each does need intelligent consideration.

Calculators bring immense computational power within reach of your finger-tips. They have the potential to lead you on exciting journeys of exploration in the world of mathematics. They free you from the hassle of time-consuming arithmetic. However, the power and the potential will remain untapped unless you learn to use your calculator intelligently. That is the object of this guide and worksheets. The numbers used in most examples have deliberately been kept small so that you can more readily discover and explain any errors you may have made.

BASIC KEYS OFF , ON/C and CE

These three keys are grouped in the upper right corner of the keyboard. When the calculator is first turned on, a "0." will appear and the word "DEG" will be shown at the top of the display, left of centre. Once the calculator is on, pressing ON/C will clear the machine entirely - except for the memory. Pressing CE will clear the last number entered, provided it was not followed by an operation or a function key.

2ndF

You will have noticed that most of the keys on your calculator can be used for more than one purpose. To select any of the functions printed in brown, you must first press 2ndF which you will find at the top left of the keyboard.

DATA ENTRY KEYS 0 .... 9 , • and +/-

These keys are used to enter numbers into the calculator. When the calculator is turned on the decimal point appears at the extreme right of the display. Once • is pressed, it will float to the left as the fractional part of the number is entered.

All numbers entered are assumed to be positive. If you wish to change the sign of a number you have just entered, press +/- .

You will notice that your calculator displays at the most

8 digits. The display is a window on the internal workings of the calculator - we cannot see everything that is going on inside. The calculator works internally to 11 digits - this is to improve its accuracy - and the display is this 11 digit number rounded off to 8 digits. We can directly enter numbers of up to 8 digits. By being a little crafty, it is possible to enter numbers with more than 8 digits. See Worksheet 4

THE BASIC OPERATIONS

$\boxed{+}$ and $\boxed{-}$, $\boxed{\times}$ and $\boxed{\div}$, $\boxed{y^x}$ and $\boxed{\sqrt[x]{y}}$, $\boxed{=}$

Your calculator is one of the latest developments in Man's efforts at inventing a device to help him perform calculations. Its ancestors go back thousands of years and include such objects as your own 10 fingers, the abacus, Napier's Bones, Charles Babbage's Difference Engine and many others now consigned to the dust of history. It is easy to take for granted the immense calculating power you have at your fingertips - power that you can get some inkling of when you try to do by hand some of the computations that your calculator scarcely blinks at.

In doing these calculations the $\boxed{=}$ key plays a most important role. It is the key which makes things happen. When it is pressed all pending operations in the machine are carried out and the calculator is ready to begin a new calculation without your having to clear the display. The number in the display is dropped if a new number is entered. It will form part of the next calculation if an operation key is pressed before the next number is entered.

The calculator is designed for ease of operation. It uses what is known as an Algebraic Operating System (AOS) where we enter the calculation from left to right, just as we would read it. The calculator performs all the necessary intermediate steps, giving each operation its due mathematical priority. To achieve this it has 4 levels of operational memory. For calculations without brackets, the calculator uses only 3 of these levels. Here are some examples of how the calculator handles computations:

(1.) $1 + 2$

	4.				
OPERATIONAL MEMORIES	3.				
	2.				
	1.		1+	1+	
DISPLAY		1	1	2	3
KEYSTROKES		1	$\boxed{+}$	2	$\boxed{=}$

(2.) $1 + 2 \times 3$

	4.						
	3.						
	2.				1+	1+	
	1.		1+	1+	2×	2×	
DISPLAY		1	1	2	2	3	7
KEYSTROKES		1	$\boxed{+}$	2	$\boxed{\times}$	3	$\boxed{=}$

(3.) $1 + 2 \times 3^4$

	4.								
	3.						1+	1+	
	2.				1+	1+	2×	2×	
	1.		1+	1+	2×	2×	$3y^x$	$3y^x$	
DISPLAY		1	1	2	2	3	3	4	163
KEYSTROKES		1	$\boxed{+}$	2	$\boxed{\times}$	3	$\boxed{y^x}$	4	$\boxed{=}$

(4) $1 + 2 \times 3^4 \div 5$

4.										
3.						1+	1+			
2.				1+	1+	2x	2x		1+	
1.		1+	1+	2x	2x	$3y^x$	$3y^x$	1+	162÷	
DISPLAY	1	1	2	2	3	3	4	162	5	33,4
KEYSTROKES	1	$\boxed{+}$	2	$\boxed{\times}$	3	$\boxed{y^x}$	4	$\boxed{\div}$	5	$\boxed{=}$

AOS - ALGEBRAIC OPERATING SYSTEM

As has been pointed out, your calculator uses an Algebraic Operating System. The advantage of this system is that calculations can generally be entered into the calculator as we would read them, without our first having to translate them into some machine language - as when we program a computer.

The other major calculator language is RPN (Reverse Polish Notation - named, with an obvious concession to pronounciation, after its Polish inventor, Jan Lucasiewicz). Most people who use RPN calculators come to them via a calculator using some form of algebraic logic.

An essential of arithmetic is that each calculation can only have one answer. This is ensured by a universally accepted set of rules for dealing with mixed calculations. The calculator is provided with a function that judges the priority level of individual operations according to these rules - a sort of electronic BODMAS! Operations are performed in this order :

(1.) Single variable functions such as log, ln, sin, cos, tan, x^2 and their inverses, as well as n!, 1/x and % and the keys for addressing the memory, x → M and RM

(2.) y^x and $\sqrt[x]{y}$

(3.) × and ÷

(4.) + and −

(5.) = and M+

BRACKETS (or PARENTHESES) $\boxed{(}$ and $\boxed{)}$

It happens that sometimes we need to give special priority to one part of a calculation. To do this we use brackets.

Consider the calculation $\frac{4}{2+3}$. If we entered it in the order $4 \boxed{\div} 2 \boxed{+} 3 \boxed{=}$, the answer given would be 5. To get the correct answer of 0,8 we must instruct the calculator to FIRST add 2 and 3 BEFORE dividing. We do this by inserting brackets: $4 \boxed{\div} \boxed{(} 2 \boxed{+} 3 \boxed{)} \boxed{=}$. There are, of course, other ways of doing this calculation which involve changing the order and using the memory or the $\boxed{1/x}$ key.

Here is an example of how your calculator copes with a calculation involving brackets: find the value of $1 + 2(3 - \frac{4}{5})$

4.									1+	1+		
3.							1+	1+	2×	2×		
2.				1+	1+	1+	2×	2×	3−	3−	1+	
1.		1+	1+	2×	2×	2×	3−	3−	4÷	4÷	2×	
DISPLAY	1	1	2	2	0	3	3	4	4	5	2.2	5.4
KEYSTROKES	1	$\boxed{+}$	2	$\boxed{\times}$	$\boxed{(}$	3	$\boxed{-}$	4	$\boxed{\div}$	5	$\boxed{)}$	$\boxed{=}$

This calculation raises a couple of points: firstly, pressing $\boxed{)}$ causes the calculator to carry out all the pending calculations going back to the previous (; and, secondly, the machine cannot handle implied multiplication - we had to insert a × between 2 and (.

With some more complicated division sums, it is necessary to bracket the whole numerator and the whole denominator:

$$\frac{5-3 \ + \ 4 \times -2}{(3+5\div4) \times 8}$$

If we do not insert these extra brackets the calculator will divide the product of 4 and -2 by the bracketed section in the denominator, multiply the result by 8 and then add this quantity to 2, which is 5 - 3. The answer to the sum is -0,1764705. How many different ways can you devise for doing it?

Your calculator can handle up to 15 levels of brackets, as long as you do not exceed the 4 levels of pending operations.

ADDRESSING THE MEMORY $\boxed{x\to M}$, $\boxed{RM}$ and $\boxed{M+}$

These keys, outlined in blue, will be found in the extreme right hand column of the keyboard. These keys are used for communicating with the calculator's addressable memory (as opposed to the operational memories). This memory is a useful store for a number which is going to be used again or for cumulative totals. We can communicate with this memory in three ways:

$\boxed{x\to M}$ This key transfers the number in the display into the memory, automatically replacing any number already there. When a number is being kept in the memory, an M appears at the left of the screen. If you wish to clear the memory, press $\boxed{ON/C}$ and $\boxed{x\to M}$, this displays a 0 and then transfers it to the memory.
$\boxed{RM}$ This key recalls to the display the contents of the memory. It does not change the contents of the memory.
$\boxed{M+}$ This key causes whatever number is in the display to be added to the contents of the memory.

The facility of a memory is useful in many situations. In algebra we often have to evaluate expressions for specific values of the unknown. For example, say we had to find the values of $f(x) = x^3 + 2x^2 - 4x - 3$ for integral values of x from -4 to $+4$. The contents of the memory are going to be the value of x so our first step is to put -4 into the memory. The keystrokes for finding $f(-4)$ are:

DISPLAY	4	−4	−4	−4	3	−64	2	2	−4	16	−32	4	4	−4	−16	3	−16
KEYSTROKES	4	$\boxed{+/-}$	$\boxed{x\to M}$	$\boxed{y^x}$	3	$\boxed{+}$	2	$\boxed{\times}$	$\boxed{RM}$	$\boxed{x^2}$	$\boxed{-}$	4	$\boxed{\times}$	$\boxed{RM}$	$\boxed{-}$	3	$\boxed{=}$

We would then repeat the procedure for $x = -3$ and so on. On completion, your table of answers should be:

x	−4	−3	−2	−1	0	1	2	3	4
$f(x)$	−19	0	5	2	−3	−4	5	30	77

You could use this table to sketch the graph of the expression or, with a bit more calculation, you could even find the roots of $x^3 + 2x^2 - 4x - 3 = 0$. This is followed up in Worksheets 14 – 16.

AUTOMATIC CONSTANTS

Your calculator has a built-in constant feature which is useful when you have to do a number of similar calculations. The object of the built-in constant is to save the user from having to enter the same number and operation into the calculator again and again. It automatically comes into play with all the operations combining two numbers, that is $\boxed{+}$ and $\boxed{-}$, $\boxed{\times}$ and $\boxed{\div}$, $\boxed{y^x}$ and $\boxed{\sqrt[x]{y}}$.

For example, working with the numbers {3; 4; 7; 9} :

(1.) If you need to increase each one by 5, do the first calculation, entering the number which is to be the constant second, 3 $\boxed{+}$ 5 $\boxed{=}$, and then simply enter the remaining numbers, one by one, pressing $\boxed{=}$ after each number. $\boxed{+}$ 5 will remain as a constant until some other operation is carried out.

If you had to subtract a constant from a set of numbers, you would follow the same procedure.

(2.) If each number had to be multiplied by a constant factor of, for example, 1,12 (as one would do when working out the price of an article with sales tax added), <u>first</u> enter the constant you wish to multiply by, 1,12 $\boxed{\times}$ 3 $\boxed{=}$, and then enter the remaining numbers, once again pressing $\boxed{=}$ after each.

(3.) For dividing, raising to a power or finding a particular root, follow the same procedure as for addition and subtraction.

For practice try (a) dividing each of the numbers given above by 4 ; (b) raising each to the power of 4; and

(c) finding the cube root of each.

THE INVERSE OR RECIPROCAL KEY $\boxed{1/x}$

Your calculator is equipped with a number of "short cut" keys for directly carrying out certain useful calculations. The $\boxed{1/x}$ key takes whatever number is in the display and divides it into one. So, for example, 2 $\boxed{1/x}$ gives $\frac{1}{2}$ - except that as this model cannot display fractions in this form and so it will display the answer as 0,5. This key has applications in arithmetic and trigonometry.

INSTANT SQUARES AND SQUARE ROOTS $\boxed{x^2}$ and $\boxed{\sqrt{\ }}$

Because squares and square roots occur so frequently in calculations, your calculator has keys which cause these operations to be carried out immediately without having to resort to a slightly longer process using $\boxed{y^x}$ and $\boxed{\sqrt[x]{y}}$.

When we enter a number and press the keys $\boxed{\sqrt{\ }}$ and $\boxed{x^2}$, we should arrive back at the original number. Try this sequence of keystrokes : 7 $\boxed{\sqrt{\ }}$ $\boxed{x \to M}$. The display should read 2,6457513. Now enter this number into the calculator and press $\boxed{x^2}$. The display is now 6,9999999. Yet on recalling the contents of the memory and pressing $\boxed{x^2}$, the display is 7. Can you explain this?

$\boxed{\pi}$

Another number frequently used in calculations is the quantity "π" (the Greek letter Pi). We encounter it in calculating facts in connection with shapes involving circles. Another area where you will encounter π is in trigonometry when working with angles measured in radians.

The number that appears in the display when $\boxed{\pi}$ is pressed is 3,1415927 , π correct to 8 figures. Can you find the actual number stored in the calculator?

THE PERCENTAGE KEY $\boxed{\%}$

This is another of the "short cut" keys which is of particular use in commercial calculations when working with discounts and mark-ups and interest rates. Percentage means "rate per cent", in other words, parts per 100. So 25% of R300 means 25 hundredths of R300 i.e. $\frac{25}{100}$ of 300. Percentages play a common role in everyday life. In addition to the uses already mentioned, they help us evaluate information and pass judgements on claims made by advertisers and others who are trying to influence our way of thinking.

The $\boxed{\%}$ key enables you to calculate percentages of a given quantity and to increase or decrease amounts by a certain percentage. These calculations can be done without the $\boxed{\%}$ key, as the following examples illustrate:

Examples

(1.) What is the sales tax (currently 12%) on a motorcycle costing R1250?

K/S 1250 $\boxed{\times}$ 12 $\boxed{\%}$ $\boxed{=}$

Without the $\boxed{\%}$ key

$\frac{12}{100}$ of 1250

12 $\boxed{\div}$ 100 $\boxed{\times}$ 1250 $\boxed{=}$

(2.) You score 47 out of 73 for a maths test. What percentage is this?

K/S 47 $\boxed{\div}$ 73 $\boxed{\%}$ $\boxed{=}$

$\frac{47}{73} \times 100$

47 $\boxed{\div}$ 73 $\boxed{\times}$ 100 $\boxed{=}$

(3.) What would be the final cost of a radio marked R98-75 once sales tax of 12% has been added on?

K/S 98.75 $\boxed{+}$ 12 $\boxed{\%}$ $\boxed{=}$

Final price of the radio is 112% of marked price. $\frac{112}{100} \times 98{,}75$

112 $\boxed{\div}$ 100 $\boxed{\times}$ 98,75 $\boxed{=}$

(4.) The price of petrol, the Government announces, is to be reduced by 4,5%. At present it costs 77,5c per litre. What will the new price be?

K/S 77,5 $\boxed{-}$ 4,5 $\boxed{\%}$ $\boxed{=}$

New price will be 95,5% of the old price.

$77{,}5 \times \frac{95{,}5}{100}$

77,5 $\boxed{\times}$ 95,5 $\boxed{\div}$ 100 $\boxed{=}$

THE FACTORIAL KEY $\boxed{n!}$

This key is most useful when it comes to calculations in the field of probability. Say, for example, you were trying to work out how many different 6 digit numbers you could make with the digits 1, 2, 3, 4, 5 and 6. For the first digit you have 6 possibilities and as you have used one digit, there are 5 possibilities for the second place, making a total of 30. For each of these 30, there are 4 choices for third digit, making a total of 120 – and so on. The total number of possibilities is therefore 6×5×4×3×2×1 which in mathematical Shorthand is 6! We read this as "6 factorial." What the factorial key does is, once you have entered a whole number, it tells the calculator to multiply 1×2×3×4×... up to the number entered.

These numbers get very big very rapidly. Work out how many ways you can arrange the batting order of a cricket team or the number of different ways a rugby team could run onto the field.

Although probability is not a topic in our mathematics syllabus, it is an interesting field.

SCIENTIFIC NOTATION EXP, F↔E and ◀▶

Scientific notation is invaluable for dealing with very large or very small numbers. For example, your calculator cannot display the number 6723500000 because it can only show 8 digits. A number in scientific notation is expressed as a base number (or mantissa) multiplied by 10 raised to a power (or exponent). Technically, the mantissa is a number between 1 and 10. So in the above example, the mantissa should be entered in the form 6,7235 and the exponent is 9. The keystrokes are 6,7235 EXP 9. The mantissa could have been entered in a non-standard form and on pressing = or one of the operation keys, the calculator would convert it to the standard form. Watch the display as you enter

67235 $\boxed{\text{EXP}}$ 5 $\boxed{=}$.

So, to summarize the procedure for entering a number in scientific notation:

(1.) Enter the mantissa, pressing $\boxed{+/-}$ if it is negative. The mantissa can be entered to 8 digits but in scientific notation the calculator will only display 5 digits.

(2.) Press $\boxed{\text{EXP}}$, standing for "exponent".

(3.) Enter the exponent, pressing $\boxed{+/-}$ if it is negative.

In scientific notation, the exponent tells us where the decimal point would lie if the number were written out in the normal floating decimal notation. A positive exponent tells us how many places the decimal should be shifted to the right and a negative exponent how many places it should be shifted to the left.

Example : Convert to scientific notation: 0,000 000 7324.
Predict what the calculator will display.

It is very difficult to comprehend the size of numbers in scientific notation. Numbers which are just within our comprehension seem innocuously small in scientific notation. For example: there are approximately $3{,}2 \times 10^9$ seconds in 100 years ; the moon is about $3{,}8 \times 10^8$ metres from the earth ; the nearest star is about $3{,}9732 \times 10^{13}$ km from us.

Pressing the key $\boxed{\text{F}\leftrightarrow\text{E}}$ converts any 8 digit number in the display to scientific notation, having first pressed $\boxed{=}$. Pressing the key again will return the number to the floating decimal notation.

When a number is in scientific notation, pressing $\boxed{\blacktriangleleft\blacktriangleright}$ displays the mantissa to 8 digits. Pressing $\boxed{\text{F}\leftrightarrow\text{E}}$ returns it to scientific notation.

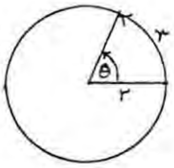
THE TRIGONOMETRIC FUNCTIONS

$\boxed{\sin}$ $\boxed{\cos}$ $\boxed{\tan}$ and their inverses $\boxed{\sin^{-1}}$ $\boxed{\cos^{-1}}$ $\boxed{\tan^{-1}}$

Trigonometry has been described as that branch of mathematics which uses the fact that numerous problems may be solved by the calculation of the unknown parts of a triangle when three parts are known.

We know from our study of geometry - in particular of congruency - that a triangle is determined by any one of four sets of minimum conditions. These are when we know the lengths of its sides <u>or</u> two sides and the included angle (if the angle were NOT included there would be the possibility of constructing two triangles with the information) <u>or</u> two angles and a side <u>or</u> the hypotenuse and a side of a right-angled triangle. Trigonometry provides us with formulae which we can use to solve all such triangles. However, our introduction to trigonometry is usually via right-angled triangles.

Before pursuing this topic, you will have noticed that the moment your calculator is turned on, the word DEG appears at the top of the display, left of centre. This indicates that the calculator is set to work with angles measured in degrees. There are two other systems for measuring angles, though we usually do not encounter them at school. These are radians and grads. A radian is the angle subtended at the centre of a circle by an arc on the circumference equal in length to the radius of the circle. The circumference of a circle with radius r is $2\pi r$. So one revolution (i.e. 360°) is equivalent to $\frac{2\pi r}{r} = 2\pi$ radians.



$\theta = 1$ radian.

Can you work out the equivalents of 180°, 90°, 60°, 45° and 30° in radians? Radian measure is very important in higher mathematics. A right angle is defined as 100 grads. These different systems may be selected by using the DRG key in the middle of the bottom row of the keyboard.

Many calculators display angles in the sexagesimal scale - that is, degrees - minutes - seconds. Our calculator does not. It is, however, an easy matter to convert an angle from decimal to sexagesimal form and vice versa.

Examples: (1.) Express 51,976° in sexagesimal form.

DISPLAY	0,976	0,976	60	58,56	58	0,56	0,56	60	33,6
KEYSTROKES	0,976	$\boxed{\times}$	60	$\boxed{-}$	58	$\boxed{=}$	$\boxed{\times}$	60	$\boxed{=}$

Answer:
51,976° = 51° 58′ 33,6″

↗
Could this have been omitted? If not, why not?

(2.) Express 37° 42′ 9,8″ in decimal form.

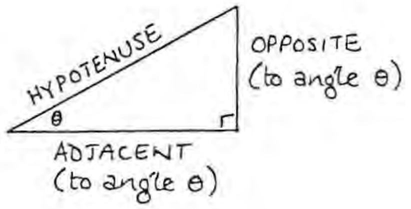
KEYSTROKES: 9,8 $\boxed{\div}$ 60 $\boxed{+}$ 42 $\boxed{=}$ $\boxed{\div}$ 60 $\boxed{+}$ 37 $\boxed{=}$

Answer: 37° 42′ 9,8″ = 37,702722

The words sin, cos and tan on your calculator are the accepted abbreviations for three of the six trigonometric functions, sine, cosine and tangent.

There are various ways of defining these ratios and one method is to name the sides of a right-angled triangle according to their positions relative to a particular angle as shown below:

$$\sin\theta = \frac{\text{OPPOSITE SIDE}}{\text{HYPOTENUSE}}$$



$$\cos\theta = \frac{\text{ADJACENT SIDE}}{\text{HYPOTENUSE}}$$

$$\tan\theta = \frac{\text{OPPOSITE SIDE}}{\text{ADJACENT SIDE}}$$

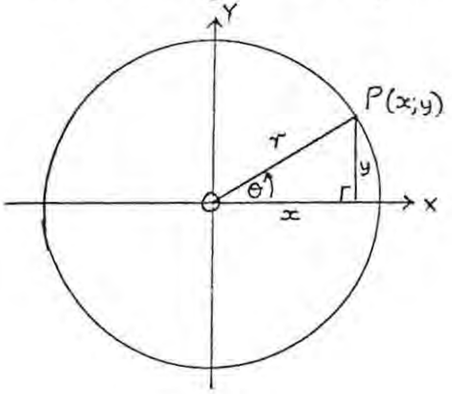
So, for example, sin 50° is the ratio of the side opposite 50° to the hypotenuse in a right-angled triangle with an angle of 50°. To find this value using the calculator, we first enter the angle (in decimal form) and then select the ratio we want. This immediately gives us the value we want:

DISPLAY	50	0,7660444
KEYSTROKES	50	$\boxed{\sin}$

If we know the ratio and wish to find the angle then we use the keys $\boxed{\sin^{-1}}$ $\boxed{\cos^{-1}}$ and $\boxed{\tan^{-1}}$. These functions are often called arcsin, arccos and arctan and mean "the angle whose sine (or cosine or tangent) is ....." So $\sin^{-1}(0{,}678)$ means the angle whose sine is 0,678 and the calculator tells us this is 42,687553°.

DISPLAY	0,678	42,687553
KEYSTROKES	0,678	$\boxed{\sin^{-1}}$

The problem with defining trig ratios as we have above is that it gives the impression that they only apply to angles between 0° and 90°. This is not the case and one way of breaking free from this impression is to look at the triangle in different surroundings. We place it in the co-ordinate plane:



P lies on the circumference of a circle with radius r, so the hypotenuse is r. The adjacent side is x and the opposite, y. The co-ordinates of point P are, therefore, $(x;y)$. The trig ratios can now be redefined: $\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$ and $\tan\theta = \frac{y}{x}$. These definitions should be easy to interpret no matter what the size of angle θ.

Investigation will show that there are an infinite number of angles which give the same value for a particular trig function. (Find the cosine of ± 40°; ± 320°; ± 400°. Can you find a pattern and predict some more angles with the same cosine? The relationships between such angles is an important study.) Your calculator, however, can only display one of all the possibilities. So, for $\sin^{-1}$ and $\tan^{-1}$, the angle displayed will be between −90° and 90° and for $\cos^{-1}$, the angle will be between 0° and 180°.

You will learn much about the trig functions if you draw their graphs. This is an easy matter using your calculator...

LOGARITHM KEYS $\boxed{\log}$ and $\boxed{10^x}$, $\boxed{\ln}$ and $\boxed{e^x}$

The calculator has actually done away with a lot of the applications of logarithms that used to be an important part of previous maths syllabi.

Logs, as they are usually called, were first used to simplify computations. It was noticed that calculations involving multiplication and division, raising to powers and extracting roots were much simplified if the numbers involved were expressed as powers of the same base. Here are some examples to illustrate the point:

$$64 \times 32 = 2^6 \times 2^5 = 2^{6+5} = 2^{11} = 2048$$

$$243 \div 27 = 3^5 \div 3^3 = 3^{5-3} = 3^2 = 9$$

$$8^3 = (2^3)^3 = 2^9 = 512$$

$$\sqrt[3]{4096} = \sqrt[3]{2^{12}} = 2^{\frac{12}{3}} = 2^4 = 16$$

The tricky stages in the above examples come when one has first to express the original numbers as powers of a particular base and then when one has to convert back to an ordinary number again. Tables were devised which enabled the user to express every number as a power of the base 10. So, when we enter 2 and press $\boxed{\log}$, the calculator displays 0,30103, telling us that $10^{0,30103} = 2$. You can verify this using the $\boxed{y^x}$ key.

If we know what the log is (say 0,1234) and we want to find the number, then we enter 0,1234 and press $\boxed{10^x}$. The display reads 1,3286176. The calculator is telling us that $10^{0,1234} = 1,3286176$.

The keys $\boxed{\ln x}$ (read "lin x") and $\boxed{e^x}$ perform similar functions except that the base in this system is no longer 10 but a very special number called "e". Logs with the base e are called natural logs while those with 10 as base are called common logs. (Can you devise a method to get your calculator to tell you the value of e ?)

Logarithms have proved useful in the description of many natural phenomena.

STATISTICAL KEYS

The most common statistical calculation that we do is the finding of the mean or average of a set of data. Here, for example, are the marks scored by a class in a maths test: 23, 18, 17, 27, 28, 23, 14, 20, 22, 25, 19 and 23.

The first step in entering this information into the calculator is to get it into the statistical mode. This is done by pressing $\boxed{\text{2ndF}}$ $\boxed{\text{ON/C}}$. The word STAT appears in the right of the display. (Note that once the calculator is in the statistical mode there are a number of functions that are no longer available - these are brackets and the memory keys.)

We then enter the scores one by one, pressing $\boxed{\text{DATA}}$ after each entry. If you enter the wrong score in error, you can cancel your mistake by re-entering the incorrect number and pressing $\boxed{\text{CD}}$. If the same score has to be entered more than once, it can be done as follows 23 $\boxed{\times}$ 3 $\boxed{\text{DATA}}$. The calculator is designed to give us a great deal of information.

The statistical keys are in the extreme right hand column with a fine black border:

$\boxed{\bar{x}}$ will give us the average (21,583333)
$\boxed{n}$ will display again the number of entries (12)
$\boxed{\Sigma x}$ and $\boxed{\Sigma x^2}$. In maths the Greek letter Σ (sigma) is used to indicate a sum, and in statistics x stands for the scores that make up the data. So Σx means "the sum of all the x's." For certain statistical calculations, one also needs to know the sum of the squares of each of the scores. This is what $\boxed{\Sigma x^2}$ tells us.
$\boxed{S}$ and $\boxed{\sigma}$. Another useful statistic is known as "standard deviation". This is a measure of spread. It tells us how much variation there is in the scores entered. Here is a simplified example to illustrate one way in which it could be used: Tests were run on two makes of torch battery. Samples of type A were found to last 20, 21, 22 and 25 hours respectively, giving an average of 22 hours battery life. The average for samples of type B was found to be 24,5 hours but the batteries themselves lasted 14, 15, 20 and 49 hours respectively. The standard deviation for type A ($\boxed{\sigma}$) is 1,87 and that for type B is 14,33 – as you can see, the bigger the standard deviation, the bigger the spread in the scores. So when a manufacturer claims that the average life of his product is that much greater than that of his competitors, remember that it is not the whole story – there may be a bigger chance of picking a dud.

There are two symbols for standard deviation. "s" stands for the standard deviation of the actual sample with which you are working and "σ" stands for the predicted standard deviation of the whole population from which that sample is taken. This illustrates an important point about statistics and that is that its aim is to enable us to generate information about the whole population based on results of measurements taken from small random groups from that population.

Your calculator is a means whereby you can explore the interesting world of statistics. For anyone interested in mathematics and people, it is a fascinating study.

E : ON ASKING THE IMPOSSIBLE

Occasionally your calculator might object to some calculation you ask it to perform. It will register its displeasure by showing an E at the bottom left of the display and will not accept any new calculation until you press $\boxed{\text{ON/C}}$. This E (which stands for "error") appears whenever the answer to a particular calculation is too big to be displayed or undefined or imaginary.

APPENDIX D

The Calculator Worksheets

WORKSHEET 1

BASIC OPERATIONS 1

1. Do the following calculations WITHOUT using your calculator. In the space provided predict the display on completion of the keystroke below. Check your answers by working through the calculation with your calculator.

(a)

DISPLAY						
KEYSTROKES	2	$\boxed{\times}$	3	$\boxed{\times}$	4	$\boxed{=}$

(b)

DISPLAY						
KEYSTROKES	4	$\boxed{\times}$	3	$\boxed{\div}$	2	$\boxed{=}$

(c)

DISPLAY						
KEYSTROKES	4	$\boxed{\times}$	3	$\boxed{+}$	2	$\boxed{=}$

(d)

DISPLAY						
KEYSTROKES	4	$\boxed{+}$	3	$\boxed{\times}$	2	$\boxed{=}$

(e)

DISPLAY						
KEYSTROKES	4	$\boxed{+}$	3	$\boxed{-}$	2	$\boxed{=}$

2. (a)

DISPLAY								
KEYSTROKES	2	$\boxed{+}$	3	$\boxed{\times}$	4	$\boxed{\times}$	5	$\boxed{=}$

(b)

DISPLAY									
KEYSTROKES	2	$\boxed{+}$	3	$\boxed{\times}$	4	$\boxed{=}$	$\boxed{\times}$	5	$\boxed{=}$

(c)

DISPLAY									
KEYSTROKES	2	$\boxed{+}$	3	$\boxed{=}$	$\boxed{\times}$	4	$\boxed{\times}$	5	$\boxed{=}$

What was the effect of inserting the extra $\boxed{=}$ in (b) and (c)?

3. Complete the following as with the previous examples but in the block above each operation indicate the order in which the calculator performs the operations:

(a)

ORDER								
DISPLAY								
KEYSTROKES	3	$\boxed{\times}$	2	$\boxed{+}$	5	$\boxed{\times}$	4	$\boxed{=}$

(b)

ORDER												
DISPLAY												
KEYSTROKES	2	$\boxed{+}$	3	$\boxed{\times}$	4	$\boxed{\div}$	6	$\boxed{-}$	3	$\boxed{\times}$	5	$\boxed{=}$

(c)

ORDER										
DISPLAY										
KEYSTROKES	2	$\boxed{+}$	3	$\boxed{\times}$	4	$\boxed{\times}$	2	$\boxed{-}$	8	$\boxed{=}$

(d)

ORDER										
DISPLAY										
KEYSTROKES	2	$\boxed{+}$	3	$\boxed{\times}$	4	$\boxed{y^x}$	2	$\boxed{-}$	8	$\boxed{=}$

(e)

ORDER											
DISPLAY											
KEYSTROKES	2	$\boxed{+}$	3	$\boxed{\times}$	4	$\boxed{=}$	$\boxed{y^x}$	2	$\boxed{-}$	8	$\boxed{=}$

Examine (c), (d) and (e) and comment on the operation $\boxed{y^x}$.

WORKSHEET 2

BASIC OPERATIONS II

1. WITHOUT using a calculator, complete the following indicating the order of operations and predicting the display. Use your calculator to check your predictions :

(a)

ORDER								
DISPLAY								
K/S	2	+	3	×	4	+	5	=

(b)

ORDER										
DISPLAY										
K/S	2	+	(	3	×	4	+	5	)	=

(c)

ORDER										
DISPLAY										
K/S	2	+	3	×	(	4	+	5	)	=

(d)

ORDER												
DISPLAY												
K/S	(	5	+	3	)	÷	(	4	−	2	)	=

(e)

DISPLAY				
K/S	2	+	3	=

(f)

DISPLAY				
K/S	2	+	3	)

Discuss the role of brackets in performing calculations .

2. Complete the following as with the previous examples. In addition, write each calculation in the normal arithmetic form. For example, the keystrokes in 1(d) above represent the expression $\frac{5+3}{4-2}$.

(a)

DISPLAY															
K/S	$\boxed{(}$	5	$\boxed{\times}$	3	$\boxed{-}$	2	$\boxed{)}$	$\boxed{\div}$	$\boxed{(}$	5	$\boxed{x^2}$	$\boxed{+}$	3	$\boxed{)}$	$\boxed{=}$

The expression is

(b)

D.																		
K/S	$\boxed{(}$	$\boxed{(}$	5	$\boxed{\times}$	7	$\boxed{-}$	11	$\boxed{\times}$	3	$\boxed{)}$	$\boxed{y^x}$	5	$\boxed{-}$	5	$\boxed{)}$	$\boxed{\sqrt[x]{y}}$	3	$\boxed{=}$

The expression is

(c)

D.																
K/S	$\boxed{(}$	$\boxed{(}$	2	$\boxed{-}$	3	$\boxed{)}$	$\boxed{\times}$	2	$\boxed{+}$	4	$\boxed{)}$	$\boxed{\div}$	2	$\boxed{-}$	7	$\boxed{=}$

The expression is

3. Draw up Display/Keystroke tables for each of the following. See how many different ways you can find for doing each calculation. Do you think any particular method is better than the others? If yes, why?

(a) $\frac{5\times4-2}{3^2}$

(b) $\frac{3^2}{5\times4-2}$

(c) $\frac{9+2}{3-4}$

(d) $\frac{6}{2\times3}$

(e) $\frac{6}{2\div3}$

(f) $27^{\frac{2}{3}}$

WORKSHEET 3

BASIC OPERATIONS III

WITHOUT using your calculator, draw up Display/Keystroke tables for each of the following calculations:

1. (a) $\sqrt{16}$ (b) $25 - \sqrt{16}$ (c) $\sqrt{25-16}$

2. (a) 4^2 (b) $9 + 4^2$ (c) $(9+4)^2$

3. (a) $2+3\times\sqrt{4}\times 5$ (b) $2+\sqrt{3\times 12}\times 5$ (c) $2+3\times\sqrt{4+5}$

4. (a) $5+3\times 2^2$ (b) $5+(3\times 2)^2$

For each of the following predict the display, indicate the order in which your calculator will perform each operation and write an arithmetic expression for each set of keystrokes. Use your calculator to check each table.

5.

ORDER															
DISPLAY															
K/S	(	5	+	4	×	(	3	+	6	)	√	)	÷	2	=

6.

O.																	
D.																	
K/S	(	5	+	(	4	×	(	3	+	6	)	)	√	)	÷	2	=

7.

ORDER															
DISPLAY															
K/S	(	13	+	4	×	(	3	+	6	)	)	√	÷	2	=

Page 2

Draw up Display/Keystroke tables and indicate the order of operations for each of the following expressions. Once you have done this use your calculator to check your answers.

8. $$\frac{6^2 - 2^3}{7 \times \sqrt{5^2 - 3^2}}$$

9. $$\frac{\sqrt[3]{(3+2)^2 + 2} + 2^3 \times 3^2}{4^2 + 3^2}$$

10. Use your calculator to find the value of:

$$\frac{\frac{4}{5} + \frac{33}{52} \div 1\frac{9}{13}}{2\frac{1}{5} + \frac{2}{3} \times \left(\frac{5}{8} - \frac{2}{5}\right)}$$

If you enter this problem in the normal way, using brackets, you are likely to get a most frustrating error reading just as you are about to get an answer. Can you explain this and find an alternative method to get round the problem?

WORKSHEET 4.

THE WHOLE TRUTH!

(1.) We have pointed out that the display is a window on the internal workings of the calculator. For example, if you are busy with a calculation involving $\sqrt{8}$, the number your calculator works with is $\boxed{2.8284271}247$.

The section of the number outlined is what appears in the display. The number in the display is correct to 8 digits. This means, for example, that if you ask it for $\sqrt{5}$, the machine will display 2.236068 while the actual number stored in the calculator is 2.2360679775. Though the display is correct to 8 digits, your calculator is internally accurate to 11 digits.

If we want to see these digits that are off the display, we have, in effect, to persuade the display to move three places to the left. Before it will do that, however, we have to get rid of the three digits on the left of the display.

Referring back to $\sqrt{8}$, the calculator displays $\boxed{2.8284271}247$ and we want it to display $282\boxed{84271247}$ Can you devise a method to achieve this?

Use your method to find each of the following correct to 11 digits. Remember that the answer will not always have 11 digits – this is merely the limit to which the calculator can work. Be sure you have uncovered all the digits.

$\sqrt{11}$ $\sqrt{45}$ $\sqrt[3]{458}$ $\sin 40°$ $\cos 89°$ 5^{12}

(2.) Though it is seldom if ever necessary to do so in practise, it is possible to enter an 11-digit number into your calculator. Remembering that the calculator will not accept such a number directly and yet it holds the results of calculations to 11-digits, devise a method to enter 123456.89098.

Use the method you found in (1.) to check whether you have managed to persuade your calculator to accept this number.

WORKSHEET 5

GETTING TO THE ROOT OF THE PROBLEM!

(1.) Use your calculator to find $\sqrt{5}$. The display should read 2.236068. Press $\boxed{x \to M}$. Now enter 2.236068 and press $\boxed{x^2}$. The display reads 5.0000001. If we press $\boxed{RM}$ the display again reads 2.236068 and yet if we next press $\boxed{x^2}$ it will read 5. What is happening? You should be able to explain this if you have understood the ground covered in Worksheet 4.

(2) The fact that the display is accurate to 8 digits while the calculator functions internally to 11-digits gives rise to some apparent contradictions. Explain the following:

(a) 4 $\boxed{\div}$ 9 − 0.4444444 $\boxed{=}$
5 $\boxed{\div}$ 9 − 0.5555555 $\boxed{=}$

(b) 0.3333333 $\boxed{\times}$ 3 $\boxed{=}$
1 $\boxed{\div}$ 3 $\boxed{\times}$ 3 $\boxed{=}$

(c) 11111111 $\boxed{+}$ 0.2 $\boxed{=}$ $\boxed{-}$ 11111111 $\boxed{=}$

(d) 11111111 $\boxed{+}$ 0.2 $\boxed{=}$ $\boxed{\times}$ 2 $\boxed{=}$
11111111 $\boxed{+}$ 0.2 $\boxed{=}$ $\boxed{\times}$ 3 $\boxed{=}$

(e) 89 $\boxed{+}$ 0.9999999 $\boxed{=}$ $\boxed{\tan}$
90 $\boxed{\tan}$

Make sure your calculator is set to cope with angles measured in DEGrees.

WORKSHEET 6

A GUESSTIMATION GAME

Can you predict what number your calculator will display if you enter the number 5 and then press the keys $\boxed{\div}$ $\boxed{=}$? Carry out this key sequence to check your guess.

Due to the calculator's automatic constant facility, if you now enter any number followed by $\boxed{=}$, it will automatically be divided by 5.

There is a useful game you can play based on this procedure. Get a friend to enter any number into the calculator - to start with, make it a number between 1 and 10 with 3 decimal places - and then press $\boxed{\div}$ $\boxed{=}$. The display will read an anonymous 1.

Your task is to deduce the original number by making an estimate, entering it, and pressing $\boxed{=}$. If your estimate is too big the number in the display will be less than 1 ; if it is too small, the number displayed will be greater than 1 ; if you are correct, the display will be 1. You could make a competition of it by seeing who can arrive at the answer after the fewest attempts.

It is actually possible to discover the hidden number in just a few keystrokes (though this would defeat the object of the game). Can you work out what these keystrokes are ?

WORKSHEET 7

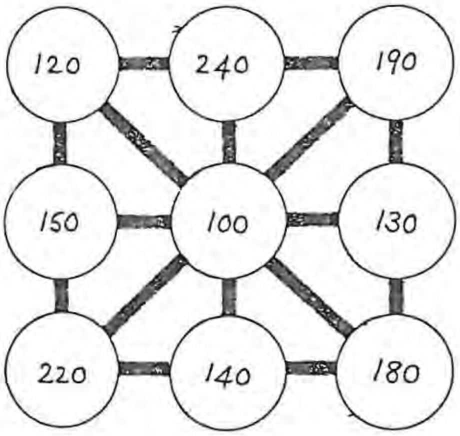
ESTIMATION I

Noughts-and-crosses must be a game with which most people are familiar. The object is, of course, to make your mark three times in a row on a 3x3 grid. Here is a mathematical version of noughts-and-crosses, the object of which is to exercise your ability in estimation.

This is a game for two players and the aim is to mark three numbers in a row. To mark a number you have to choose two numbers from the list below the game pattern and get closer to the number you want than any other number on the grid by performing the prescribed operation. Check your estimate with your calculator once you have specified the numbers you have chosen. If the number nearest that displayed on your calculator is already marked, mark the next free number closest in value.

The first player to get three in a row wins. The loser starts the next game. If the game is a draw, try a second time before moving on to the next pattern.

Game 1

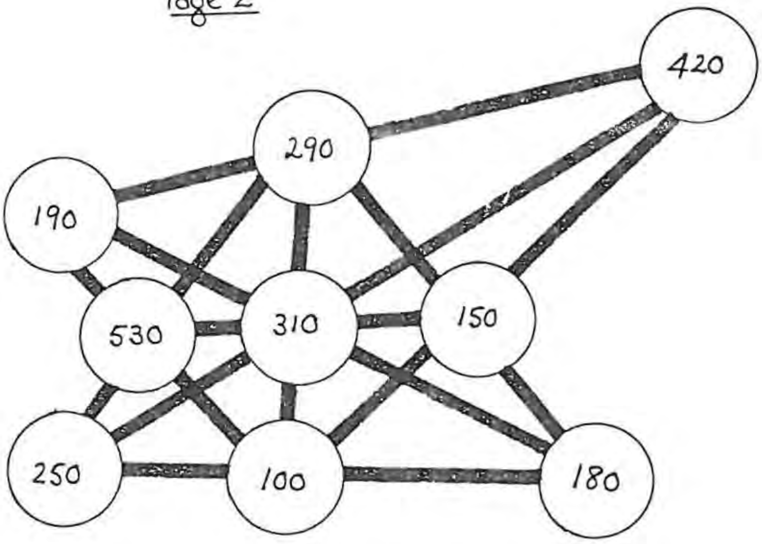


PRODUCTS

9; 11; 13; 14; 17

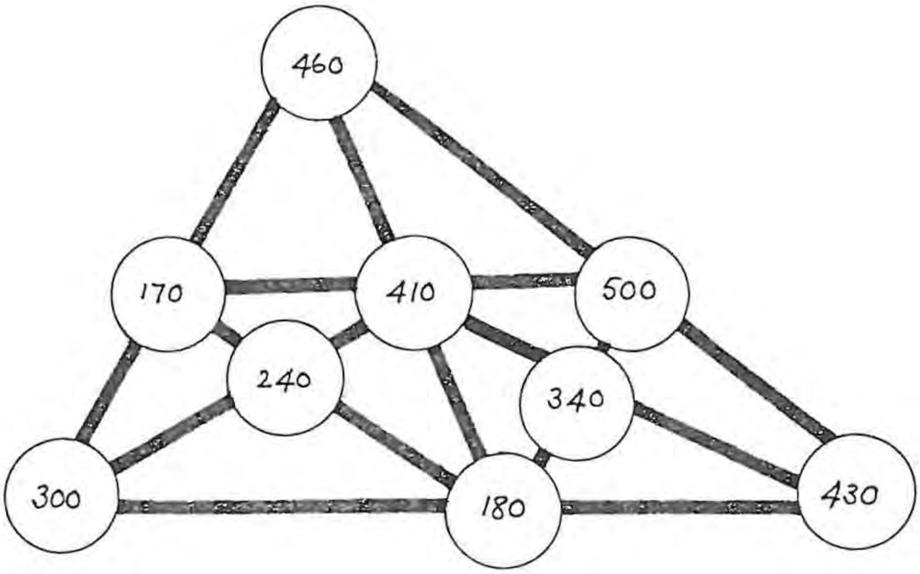
Page 2

Game 2



8; 13; 19; 22; 24

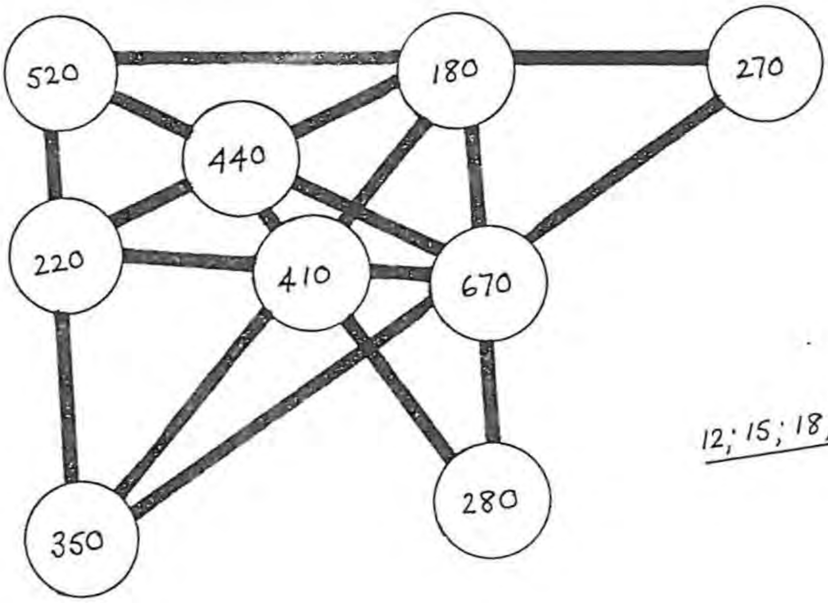
Game 3



11; 15; 16; 27; 31

Page 3

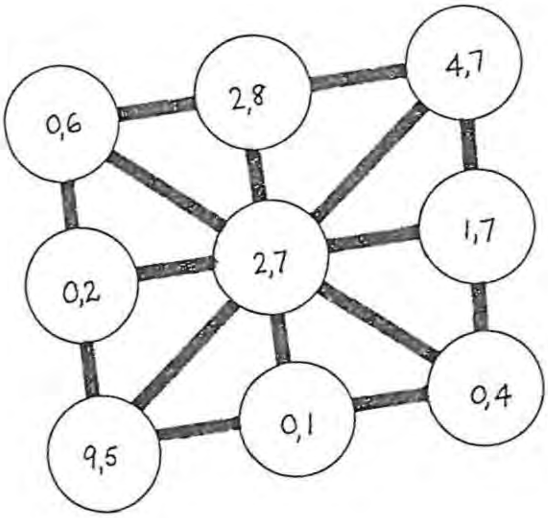
Game 4



12; 15; 18; 23; 29

QUOTIENTS

Game 5



7; 12; 19; 33

WORKSHEET 8

ESTIMATION II

(1.) Here are some examples to exercise your knowledge of how numbers combine with each other under the operations × and ÷. Each box represents a missing digit, which you have to find.
Consider the example 93 × 8□ = 8□□1 . The question to ask is "What number multiplied by 3 will give an answer ending in 1 ?" There is only one possibility and that is 7. So we use our calculators to find 93 × 87. The answer is 8091. The first and last digits agree with what is given so the answer is

$$93 \times 8\boxed{7} = 8\boxed{0}\boxed{9}1$$

Try the following:

(a) 53□ × □9 = 3□□□6
(b) 8□ × □4 = 4□9□
(c) 4□2 × 3□ = 1□8□2
(d) □5□ × 41 = □□□3
(e) 3□□2 ÷ □6 = 47
(f) □2□6 ÷ 2□ = □2
(g) 38 × □3 × □4 = 38456
(h) □7 × 31 × 6□ = 1□4□□5
(i) 37 × (8□ − □6) = 2□□9
(j) (129 + 26□) × □4 = 1□3□2

(2.) In each of the following examples the box represents a missing operation, +, −, × or ÷ .

(a) 83 □ 52 □ 63 = 72
(b) 42 □ 23 □ 66 = 900
(c) (842 □ 553) □ 31 = 45
(d) (136 □ 47) □ 17 = 1513
(e) 24 □ (52 □ 27) = 1896
(f) $2^5 \square (5^3 \square 3^4) = 1408$
(g) (84 □ 61) − (92 □ 43) = 1168
(h) (3055 □ 47) □ (18 □ 26) = 533

ESTIMATION III : EXERCISING YOUR POWERS

The route to the solution of a problem in mathematics often begins with the recognition of a pattern in the information given or a good guess. An educated guess – which is the only guess worth venturing – is based on experience of how numbers behave in different circumstances. Exponential growth (the way in which a^x increases as we increase the value of x) is a concept which is tricky to grasp, partly because it is so rapid.

Consider the problem: What number should be placed in the box to complete the following statement?

$$\square^7 = * * * * * * 2$$

(Each * stands for one digit.)

To be in a position to make an educated guess, we need to do some preliminary investigation. Firstly, which is bigger, 5×6 or 6^5? The answer is the second number. The point is that 5×6 is shorthand for 6+6+6+6+6, which is 30, while 6^5 is short for 6×6×6×6×6, which is 7776.

It becomes harder to decide on the relative sizes of numbers when both involve bases to powers. One way of making a guess would be to "decompose" the numbers concerned. For example, if we were trying to decide which of 12^3 and 7^4 is the greater, we could regard 12^3 as $(6 \times 2)^3 = 6^3 \times 2^3$ which is $6 \times 6 \times 6 \times 8$ which must be smaller than $7 \times 7 \times 7 \times 7$. Here are some examples to try. First guess which is the bigger and then check:

(1) 18^3 or 6^5 (3) 5^4 or 4^5

(2) 27^3 or 12^4 (4) 32^2 or 2^{10}

Looking at a different type of problem, what number has been omitted from the box? First guess and then check.

(1) $\square^5 = 243$ (3) $\square^8 = 1679616$

(2) $\square^7 = 78125$ (4) $\square^5 = 16807$

In developing a strategy for guessing, the easiest thing to decide would be whether the number is bigger or less than 10. How would you do this?

Secondly we could decide whether it is even or odd. How? As you can see, the final digit of the expansion is an important clue. Complete this table:

FINAL DIGIT OF POWER EXPANSION

FINAL DIGIT \ INDEX	1	2	3	4	5	6	7	8	9	10
1										
2										
3										
4										
5										
6										
7										
8										
9										
0										

Using the table as a guide to guessing, we can draw several conclusions:

(a) powers of numbers ending in 1, 5, 6 and 0 always end in 1, 5, 6 and 0 respectively;

(b) powers of numbers ending in 2 and 8 end in 2, 4, 6 or 8;

(c) powers of numbers ending in 4 end in 4 or 6;

(d) powers of numbers ending in 3 or 7 end in 1, 3, 7 or 9;

(e) powers of numbers ending in 9 end in 1 or 9.

The patterns revealed in the table could be analysed in more detail and one could use it to answer problems of the following sort:

Give the final digits of the expansions of:

(a) 6^{11} 4^{8} 3^{16} 5^{231} 9^{82}

(b) 11^{37} 28^{7} 117^{4} 12^{23} 7^{22}

Now if we are confronted with guessing an answer to a problem such as $\square^5 = 14348907$, our guessing procedure would be as follows: First, the number must lie between 10 and 100 because $10^5 < 14348907 < 100^5$. If we were familiar with powers of numbers less than 10, we could narrow it down still further:

$$20^5\,(3200000) < 14348907 < 30^5\,(24300000)$$

A glance at our table will show that any number whose 5th power ends in 7 must itself end in 7. So the answer is 27

Page 4

Here again is the problem we started with plus a few others to practise on :

What number must be placed in the box to complete the statement ? First guess and then test your answer. Each * stands for one digit .

$$\square^{7} = * * * * * * 2$$
$$\square^{4} = * * * * 1$$
$$\square^{5} = * * * 6$$
$$\square^{9} = * * * * 3$$
$$\square^{4} = * * * * * * * 6$$

WORKSHEET 10

AN INVESTIGATION INTO DECIMAL FRACTIONS

The decimal representations of rational numbers fall into two groups : terminating and recurring decimals.

(1.) Our first question is, how can we predict whether a rational number in decimal form will recur or terminate?

(a) Give the decimal form of the following rational numbers :

$\frac{3}{7}$ =	$\frac{7}{15}$ =
$\frac{3}{4}$ =	$\frac{17}{20}$ =
$\frac{5}{6}$ =	$\frac{37}{50}$ =
$\frac{7}{22}$ =	$\frac{17}{125}$ =

(b) The answer to our question must lie in the denominators. Listed below are the denominators of the fractions in (a). Write each one as a product of its prime factors.

Recurring decimals	Terminating decimals
7 =	4 =
6 =	20 =
22 =	50 =
15 =	125 =

(c) What conclusions can we draw about the denominators of rational numbers that lead to terminating decimal fractions?

Some recurring decimals have short repeating cycles (for $\frac{5}{6}$ and $\frac{7}{15}$ it is one digit and for $\frac{7}{22}$ it is two digits) whereas others are longer (for $\frac{3}{7}$ the period is 6 digits).

Page 2

And for many fractions the period is going to be more than the 8-digit display of your calculator.

(2.) How to uncover digits outside the limits of the calculator display.

One method is to consider all the decimal representations (to 8 digits) of fractions with a particular denominator and then extrapolate (what a beautiful word!) from that information:
Give the decimal forms (to 8 digits) of the following fractions:

$\frac{1}{17} =$
$\frac{2}{17} =$
$\frac{3}{17} =$
$\frac{4}{17} =$
$\frac{5}{17} =$
$\frac{6}{17} =$
$\frac{7}{17} =$
$\frac{8}{17} =$
$\frac{9}{17} =$
$\frac{10}{17} =$
$\frac{11}{17} =$
$\frac{12}{17} =$
$\frac{13}{17} =$
$\frac{14}{17} =$
$\frac{15}{17} =$
$\frac{16}{17} =$

Examining these decimals, it is possible to list all the numbers of the recurring cycle. Take the first fraction, $\frac{1}{17} = 0{,}0588235....$ and continue the decimal until it starts repeating. How many digits are there in the cycle?
Another way of unmasking these digits is to use the dreaded long division, aided and abetted by your calculator of course!
Let's convert $\frac{1}{19}$ to decimal form, stopping when the

fraction starts to recur.

$$
\begin{array}{rl}
 & \overbrace{0.052631}^{a}\overbrace{578947}\overbrace{368421}\overbrace{05263} \\
19 & \overline{)1.000000} \\
 & \underline{0.999989} \\
 & 11000000 \\
 & \underline{10999993} \\
 & 700000 \\
 & \underline{699998} \\
 & 2
\end{array}
$$

The recurring cycle, consisting of 18 digits is 052631578947368421. The steps in the method were:

① divide 1 by 19. The display reads 0.0526315.

* Discarding the final digit, this number, a, was entered into the space for the quotient.

② Multiply a by 19, getting 0.999989 and subtract in the usual way, leaving 11.

③ The process is then repeated as often as is necessary.

(* The reason why we do not make use of the full number as given in the display is that when we multiply by 19, we might exceed the display and this would give an inaccurate remainder on subtraction, so throwing the rest of the answer out of sequence. This would have happened if we had used 0.5789473 instead of 0.578947 as the second batch of digits in the quotient.)

(3.) Can you work out what is the maximum number of digits in the recurring decimal cycle of a rational number with a prime denominator, n? Consider the examples already done. The reason for the answer has something to do with the remainders.

You will notice that some prime denominators give rise to more than one recurring cycle. For example, $\frac{1}{3} = 0.333....$ and $\frac{2}{3} = 0.666...$ Another example is the denominator 11 which gives the cycles 09, 18, 27, 36 and 45

Do the necessary calculations and complete the following table :

LENGTHS OF PERIODS AND THE NUMBER OF SEQUENCES OF FRACTIONS WITH PRIME DENOMINATORS

DENOMINATOR (D)	PERIOD (P)	No. OF SEQUENCES (N)
3	1	2
7	6	1
11	2	5
13		
17	16	1
19	18	1
23		
29		
31		
37		

Can you find the relationship between D, P and N using this table?

There are a number of directions in which these investigations could be pursued:

(a) Explore the situation regarding rational numbers with composite denominators, first establishing just how many fractions there are with that particular denominator. For example, there are only 2 fractions smaller than one with a denominator of 6. They are $\frac{1}{6}$ and $\frac{5}{6}$ as both $\frac{2}{6}$ and $\frac{4}{6}$ both belong to those rationals with a denominator of 3 and $\frac{3}{6}$ is equivalent to $\frac{1}{2}$. Tabulate your findings under the headings: Denominator; No. of Fractions; Period; No. of sequences.

(b) Can you predict the lengths of the periods of recurring decimals? If the decimal has a non-recurring and a recurring part, can you predict the length of each?

(c) Why is the sum of the digits of each recurring cycle of a rational number with a prime denominator (excluding 3) divisible by 9?

WORKSHEET II

RATIONAL NUMBERS - SIMPLIFYING FRACTIONS

A few calculators are designed to express rational numbers in fractional form. A calculator with a $\boxed{a\,b/c}$ key will allow us to enter $\frac{14}{5}$, displaying it in the form 14⌋5. Press $\boxed{=}$ and the calculator will convert it to a mixed number displaying 2⌋4⌋5 for $2\frac{4}{5}$. Pressing $\boxed{a\,b/c}$ again will convert the display to decimal form, 2.8 .

How could we use our calculators which do not have this capability to convert a rational number to a mixed number? Let us look at $\frac{14}{5}$ again. It is easy to express this in decimal form:

14 $\boxed{\div}$ 5 $\boxed{=}$ gives us 2.8

So the integer part of the mixed number is 2. The decimal part, 0.8, represents the fractional part, so many fifths. The question is, how many. Putting this in the form of an equation

$$\frac{x}{5} = 0.8$$

Multiplying both sides by 5, we get

$$x = 0.8 \times 5 = 4$$

So the mixed number form of $\frac{14}{5}$ is $2\frac{4}{5}$.

If we had used our calculators for this entire procedure:

KEYSTROKES	DISPLAY	REMARKS
14 $\boxed{\div}$ 5 $\boxed{=}$	2.8	The integer is 2
$\boxed{-}$ 2 $\boxed{=}$	0.8	0.8 represents $\frac{x}{5}$
$\boxed{\times}$ 5 $\boxed{=}$	4	so x is 4

It is worth examing this procedure more closely.

The calculator tells us that 14 divided by 5 is 2.8
Writing this in the normal arithmetic notation :

$\frac{14}{5} = 2.8$

$\therefore \frac{14}{5} = 2 + 0.8$

$\therefore 14 = 2 \times 5 + 4$

ALSO

$$\begin{array}{r} 2 \\ 5\overline{)14} \\ \underline{10} \\ 4 \end{array} = \text{Remainder}$$

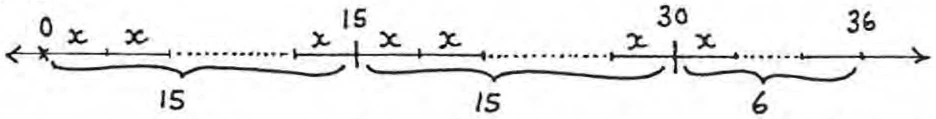
This is important as it serves as a starting point for the study of the Remainder Theorem in Std 9 algebra. What it is saying is that when we divide an integer m by an integer n, there exists a quotient q and a remainder r so that $m = nq + r$.
So for the integers 9 and 4, 9 can be expressed as a quotient, 2, multiplied by 4, plus a remainder of 1 : $9 = 2 \times 4 + 1$

Complete the following table :

$\frac{m}{n}$ RATIONAL No.	$m = nq + r$	MIXED No.
$\frac{9}{4}$	$9 = 4 \times 2 + 1$	$2\frac{1}{4}$
$\frac{14}{3}$		
	$23 = 4 \times 5 + 3$	
		$5\frac{5}{6}$

Simplifying fractions involves the cancelling of common factors. For example, $\frac{15}{12} = \frac{5 \times 3}{4 \times 3} = \frac{5}{4}$.
It is possible to find the HCF (highest common factor) of two numbers using a calculator by adapting a procedure known as the Euclidean algorithm. The Euclidean algorithm is based on the fact that there

exists an HCF for any two integers, even if it is only 1. This means that each of the numbers can be expressed as a multiple of that common factor, and if we divide the smaller into the larger, the remainder will also be a multiple of that common factor. Consider the numbers 36 and 15. You can probably see the HCF but let us assume for the moment that it is an unknown number x and so both 36 and 15 are multiples of x. Now 15 goes into 36 twice with a remainder of 6, which must also be a multiple of x. We can illustrate this on a number line:



Now take 15 and 6, both of which are multiples of the mystery (!) number x. Repeating the same procedure, 6 goes into 15 twice, leaving a remainder of 3 which must also be a multiple of x.

This leaves us with 6 and 3, both of which must be multiples of x. 3 goes into 6 exactly twice - there is no remainder. This means that 6 can be divided into precisely two equal parts of length 3. This means that x, the number which we said goes exactly into 36 and 15, must be 3.

Mechanically, the procedure for finding the HCF of two numbers a and b amounts to this:

(1) Divide the smaller number into the larger and find the remainder R_1

(2) Divide R_1 into the smaller of a and b and

find the remainder R_2

(3) Divide R_2 into R_1 and find the remainder R_3.

(4) Repeat this procedure until the remainder is 0. When you reach this stage the divisor is the HCF you are looking for.

Here is an example giving the keystrokes for finding the HCF of 2921 and 2159.

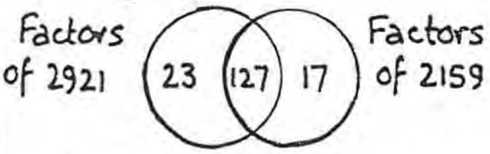
KEYSTROKES	DISPLAY
2921 $\boxed{\div}$ 2159 $\boxed{=}$	1.3529412
$\boxed{-}$ 1 $\boxed{=}$	0.3529412
$\boxed{\times}$ 2159 $\boxed{=}$	762 (Remainder)
2159 $\boxed{\div}$ 762 $\boxed{=}$	2.8333333
$\boxed{-}$ 2 $\boxed{=}$	0.8333333
$\boxed{\times}$ 762 $\boxed{=}$	635 (Remainder)
762 $\boxed{\div}$ 635 $\boxed{=}$	1.2
$\boxed{-}$ 1 $\boxed{=}$	0.2
$\boxed{\times}$ 635 $\boxed{=}$	127 (Remainder)
635 $\boxed{\div}$ 127 $\boxed{=}$	5 with NO remainder

∴ the HCF of 2921 and 2159 is 127.

We could use this to simplify the fraction $\frac{2921}{2159}$:

$$\frac{2921}{2159} = \frac{23 \times 127}{17 \times 127} = \frac{23}{17}$$

Setting out this information in a diagram :



From this we can deduce that the LCM of 2921 and 2159 is 23 × 127 × 17 = 49657.

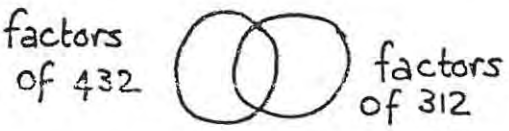
Page 5

Here is an example for you to try. (To help you with the lay-out, the basic framework has been provided.) Find the HCF of 432 and 312.

KEYSTROKES	DISPLAY
432 $\boxed{\div}$ 312 $\boxed{=}$	.
$\boxed{-}$ $\boxed{=}$	0.
$\boxed{\times}$ $\boxed{=}$	(Remainder)
$\boxed{\div}$ $\boxed{=}$	.
$\boxed{-}$ $\boxed{=}$	0.
$\boxed{\times}$ $\boxed{=}$	(Remainder)
$\boxed{\div}$ $\boxed{=}$	.
$\boxed{-}$ $\boxed{=}$	0.
$\boxed{\times}$ $\boxed{=}$	(Remainder)
$\boxed{\div}$ $\boxed{=}$	.
$\boxed{-}$ $\boxed{=}$	0.
$\boxed{\times}$ $\boxed{=}$	(Remainder)
$\boxed{\div}$ $\boxed{=}$	with no remainder

∴ the HCF of 432 and 312 is ——————→

Having got this far, we might as well find the LCM of 432 and 312. Fill in the factors in the diagram below and then calculate the LCM.



The LCM = ... × .... × ....

=

——————→

Page 6

Now use your calculator to find the HCF and LCM of each of the following pairs of numbers:

(1) 63 and 81
(2) 195 and 104
(3) 210 and 135 .

WORKSHEET 12

THE SEARCH FOR PATTERN

The calculator makes it very easy to check number patterns.

Example : Consider the following statements. Are they true ?

$$5^2 - 5 = 4^2 + 4$$
$$6^2 - 6 = 5^2 + 5$$

Write down the next three examples and check whether they are true or not.
Give another example of this pattern involving numbers of 4 digits and check its veracity.

It would be rather tempting at this stage to think that we had shown that this pattern was always true. However, to produce a proof that will convince other mathematicians, we have to show that the pattern is generally true. This means that we have to construct a model of the pattern using algebraic variables and show that this is true. In this example you will have noticed that the numbers on the left are 1 more than the numbers on the right. So if we use n to represent the numbers on the right, then those on the left can be represented by $n+1$ and we have to prove that

$$(n+1)^2 - (n+1) = n^2 - n$$

Show that this is true by simplifying the LHS.

Example 2: Here is a more tricky pattern. Study the two examples:

$$\begin{cases} 2^2 + 3^2 = 13 \quad \text{and} \quad 1^2 + 2^2 = 5 \\ 13 \times 5 = 65 = 49 \underset{*}{+} 16 = 7^2 + 4^2 \end{cases}$$

$$\begin{cases} 3^2 + 4^2 = 25 \quad \text{and} \quad 2^2 + 5^2 = 29 \\ 25 \times 29 = 725 = 625 \underset{*}{+} 100 = 25^2 + 10^2 \end{cases}$$

This pattern is concerned with numbers which are the sum of two squares. It claims that the product of two such numbers can also be expressed as the sum of two squares. It is quite hard to generate some more examples of this pattern unless we use small numbers. Find three more examples, two using numbers from the set {0; 1; 2; 3; 4} and a harder one using numbers from {3; 5; 6; 8}.

Producing a general proof of this pattern involves using four unknowns. Let these be a, b, c and d. Construct models of the first line of the pattern. Then use these numbers to begin the second line. When you get to the stage equivalent to *, you are going to have to be quite crafty. Think of perfect square trinomials. Once you have proved the pattern in general it will be a simple matter to find more complicated examples.

Example 3:

Show that $12 \times 42 = 21 \times 24$

$12 \times 63 = 21 \times 36$

$24 \times 63 = 42 \times 36$

There are some more pairs of two digit numbers whose products remain unchanged when the digits of each number are reversed. Once we have worked out the way in which the pairs are constructed, finding further examples is easy. To find them, first construct an algebraic model of the pattern: let a be the ten's digit and b the unit's digit of the first number and c the ten's digit and d the unit's digit of the second. Write expressions for the L.H.S. and the R.H.S. of the pattern, multiply them out and then ask yourself what relationship must exist between a, b, c and d that will ensure that the L.H.S. will be equal to the RHS.

Excluding such obvious examples as 19×91, how many more numbers can you find that fit this pattern?

WORKSHEET 13

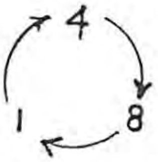
THE NUMBER 37

Calculate the following products:

3×37 6×37 9×37
12×37 15×37 18×37
21×37 24×37 27×37

These results may appear quite remarkable at first but consider the value of 3×37 and then consider the other multiples of 37 in order.

Multiples of 37 do have some interesting properties. Take any 3-digit multiple of 37, for example, 481, which is 37×13. Arrange the digits in a circle as follows:



Reading clockwise from the figure, we get the numbers 814 and 148 which are called the cyclic permutations of 481. Show that each is a multiple of 37.

Can you prove that every cyclic permutation of a 3-digit multiple of 37 is also a multiple of 37? To do this we must first construct a general model of the situation. We are dealing with a 3-digit number which is a multiple of 37,

i.e. ⓐⓑⓒ $= 37x$

The hundred's digit is a, the ten's digit is b and the unit's digit is c, so we can write:

$$100a + 10b + c = 37x$$

The first cyclic permutation of ⓐⓑⓒ is ⓑⓒⓐ, so what we have to prove is that

$100b + 10c + a$ is also a multiple of 37.

In mathematical proofs we often start from what is given and work towards the desired end. So the question is, how can we get from

$$100a + 10b + c = 37x$$

to $100b + 10c + a$? The answer is by being a little crafty and multiplying $100a + 10b + c = 37x$ by 10.

This gives $1000a + 100b + 10c = 370x$. Remembering that we want $100b + 10c + a$ on the L.H.S., take all the unwanted a's across to the R.H.S. and show that the new R.H.S. is a multiple of 37.

To practise this proof, consider the 5-digit multiples of 41.

For further examples of numbers whose cyclic permutations all have the same factor, investigate some of the sequences generated by recurring decimals. (See worksheet 10)

WORKSHEET 14

DRAWING GRAPHS

The calculator enables us to tackle the graphs of many functions which it would otherwise have been best to avoid.

Given a particular equation, $y = f(x)$, an obvious way of drawing its graph would be to choose a value for x, substitute it into $f(x)$ to find the corresponding value of y, and then plot the point $(x;y)$ on the XOY plane. Without a calculator this would be a time-consuming and tedious process.

Let us consider, for example, the graph of $y = x^3 + x^2 - 7x + 2$. A method has already been suggested on page 9 of the booklet introducing the calculator:

1. First decide on the values we are going to substitute for x. Here let's use integral values of x from -5 to 5.
2. Find the values of y as each of these is substituted into the equation, using your calculator where necessary.
3. Plot the resulting points.

X	−5	−4	−3	−2	−1	0	1	2	3	4	5
Y	−63	−18	5	12	8	2	−3	0	17	54	117

In calculating the values of the function, we start by putting -5 into memory and then substituting directly into the equation, reading from left to right.

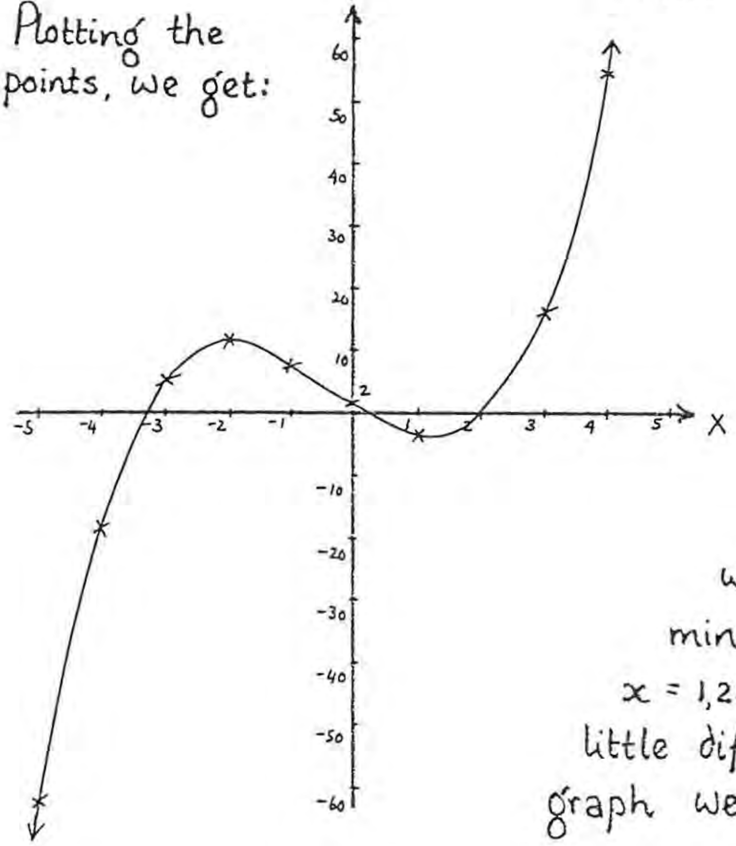
The suggested calculator keystrokes are:

Page 2

$5\ \boxed{+/-}\ \boxed{x{\leftrightarrow}M}\ \overset{*}{\boxed{y^x}}\ 3\ \boxed{+}\ \boxed{Rm}\ \boxed{x^2}\ -7\ \boxed{\times}\ \boxed{Rm}\ \boxed{+}\ 2\ \boxed{=}\ 1\ \boxed{M+}\ \boxed{Rm}$ return to *

record results at this stage

Plotting the points, we get:



The accuracy of this graph can be improved by plotting more points in between those we already have. For example, the graph actually reaches a local maximum of 12,05 when $x = -1,9$ and a local minimum of $-3,24$ when $x = 1,23$. However, this makes little difference to the basic graph we have drawn.

Draw the graphs of the following, taking values of x from -5 to $+5$:

(1.) $y = x^2 - 3x - 2$ (You will see from your sketch that this graph has a single turning point. Can you find its co-ordinates?)

(2.) $y = \frac{3}{x+2}$ (Are there any values x cannot have?)

(3.) $y = \frac{x+1}{x^2+x-2}$ (Which values can x not have?)

(4.) $y = x^3 + 2x^2 - 6x - 9$

(5.) $y = -1 \pm \sqrt{x+4}$ (Which values can x not have?)

WORKSHEET 15

SOLVING EQUATIONS BY TRIAL AND ERROR

In Worksheet 14 we sketched the graph of $y=x^3+x^2-7x+2$ and found that it cut the x-axis in 3 places. The x-axis is the line with the equation $y=0$, so the values of x for which the graph cuts the x-axis are the values which make x^3+x^2-7x+2 equal zero, the roots of the equation. Of the three, we only know the exact value of one. This root is $x=2$. We are not completely in the dark about the other two, however. The graph shows us that one of these roots lies between -4 and -3 and the other between 0 and 1. We can use our calculators to magnify these sections of the graphs and find these roots with as great a degree of accuracy as we like.

Here are the calculations for tracking down the root between -4 and -3:

x	-4	……	-3.5	-3.4	-3.3
y	-18	……	-4.5	-1.944	0.053

So the root lies between -3.4 and -3.3 and would appear to be closer to -3.3 Now we look for the second decimal place:

x	-3.32	-3.31	-3.30
y	$-0.3319\ldots$	$-0.138\ldots$	0.053

The gap has been narrowed to -3.31 to -3.30

Now for the third decimal:

x	-3.31	-3.305	-3.304	-3.303	-3.302
y	$-0.138\ldots$	$-0.042\ldots$	$-0.023\ldots$	-0.004	0.01

and the best value appears to be -3.303. We could proceed in this way getting an ever more accurate approximation of the root. But let's stop at three decimal places. Now you find the root of $x^3+x^2-7x+2=0$ which lies between 0 and 1 to the same degree of accuracy.

Here are some equations for you to solve by trial and error:

(1.) $x^x = 50$

(2.) $4^x = \frac{2}{x}$

(3.) $x^2-3x-2=0$ (for graph see Worksheet 14)

(4.) $x^3+2x^2-7x-12=0$ (a graph or at least a table of values would be a useful starting point, giving you an idea of where the roots lie.)

WORKSHEET 16

SOLVING EQUATIONS : ANOTHER LOOK

The trial-and-error method of solving equations can be more or less cumbersome depending on how intelligently we make our estimates.

Here is another approach to solving equations. Let us reconsider an equation we have already looked at: $x^2 - 3x - 2 = 0$.
Some people try to solve the equation as follows:

$$x^2 - 3x = 2$$
$$\therefore x(x-3) = 2$$

So far, so good - but without imaginative use of a calculator this is a dead-end. Using a calculator, though, we can actually solve the equation using what is known as the method of successive substitution or, to give it its proper name, iteration. First it would be helpful if we made the LHS of our equation equal to x. There are two possibilities:

$$x = \frac{2}{x-3} \quad \text{or} \quad x - 3 = \frac{2}{x}$$
$$\text{which gives} \quad x = \frac{2}{x} + 3$$

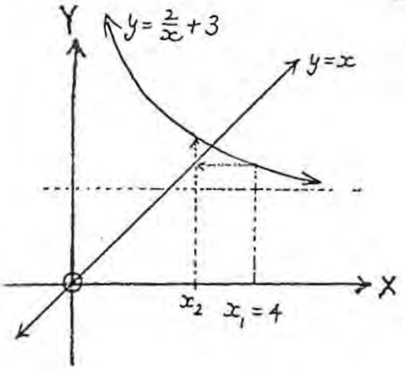
Now look what happens when we take a value for x, substitute it into the RHS and then repeat the process using the result as the next value of x to be substituted, again into the RHS.

Page 2

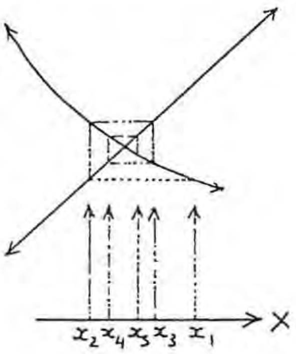
	$x = \frac{2}{x-3}$	$x = \frac{2}{x} + 3$
x_1	-1	4
x_2	-0.5	3.5
x_3	-0.5714285	3.5714286
x_4	-0.56	3.56
x_5	-0.5617977	3.5617978
x_6	-0.5615142	3.5615142
x_7	-0.5615589	3.5615589
x_8	-0.5615518	3.5615519
x_9	-0.5615529	3.561553
x_{10}	-0.5615527	3.5615528
x_{11}	-0.5615528	3.5615528
x_{12}	-0.5615528	

The process is begun with an estimate of the answer and with each successive substitution the result gets closer to an actual root of the equation – you can check this by substituting either of the above results into the original equation. It is important to remember that the roots are only as accurate as the display of your calculator can accommodate – they are NOT the exact roots of the equation.

One way of illustrating what is happening in the tables above is by looking at the graphs of the two functions concerned. If we took $x = \frac{2}{x} + 3$, the functions concerned would be $f(x) = x$ and $f(x) = \frac{2}{x} + 3$.



We chose a value for x_1, our first guess, 4, and substituted into $\frac{2}{x}+3$. This gives a new value for x, labelled x_2 in the diagram, which in turn is substituted into $\frac{2}{x}+3$. And so the process continues....



It is quite interesting to draw a rough enlargement of the area under consideration – it shows how this method homes in on the actual value of x of the point where the graphs intersect. If you try a few such rough sketches yourself, you may be able to see why this method does not always work!

There are more efficient ways of solving quadratic equations, but if we are presented with a cubic or higher equation which does not have rational roots, this method is invaluable.

Here are two equations for you to practise on:

(1.) $3x^2+4x-2=0$ Roots $\begin{cases}0,3874258\\-1,7207592\end{cases}$

(2.) $x^3-5x^2+6x-1=0$ Roots $\begin{cases}0,1980622\\1,5549581\\3,2469796\end{cases}$

(Persevere with the cubic equation. There are many ways in which you can set up the basic equations. Some of them will give the same answer. Do not be put off by this - try another arrangement.)

(3.) $3^x + x = 0$ Root $-0,5478086$

Be careful with this example. Entering -3 and pressing $\boxed{y^x}$ followed by any number which is not a positive integer, will give an error reading. (I'm assuming that you will write the equation in the form $x = -3^x$.) Remember that you are dealing with -3^x and NOT $(-3)^x$. Change the sign once you have worked out 3^x.

WORKSHEET 17

FUNCTIONS I

Before the advent of calculators, the treatment of functions at a school level seldom progressed beyond a knowledge of notation and definitions - a function is a set of pairs of elements; each pair represents the association of a second element with a first according to some or other rule. We are usually left to absorb a lot intuitively.

Your calculator is a very versatile function machine. This business of associating a second element with a first is what it does every time you enter a number and press a key. For example, if I carry out the following sequence of keystrokes $1 \boxed{+} 1 \boxed{=}$, the calculator will give the result 2. If I now press $2 \boxed{=}$, the calculator will display 3. Pressing $3 \boxed{=}$ will cause the display to read 4, and so on. What it is doing is carrying out the "+1" function, at the same time generating a set of pairs {(1;2); (2;3); (3;4) .....}. Translating the calculator language into algebraic notation, we entered a number, 'x', added 1, and the machine displayed a result 'y' : $y = x + 1$.

The calculator is capable of carrying out a great many functions. Using algebraic notation, can you identify and name the following (Decimal displays have been truncated to 3-digits.)

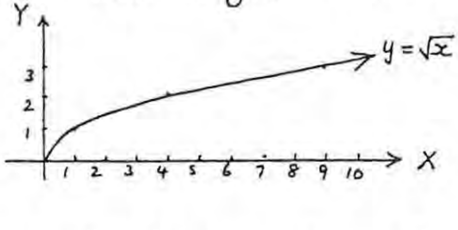
1. (a) { (1; 0,5); (2;1); (3;1,5) .... }
   (b) { (1;1); (2;4); (3;9) ..... }
   (c) { (1;1); (2; 1.414); (3; 1.732) ..... }
   (d) { (1; 0.017); (2; 0.035); (3; 0.052) .... }

(e) $\{(1; 1.557); (2; 74.686); (3; -0.143) \ldots\}$
(f) $\{(1; 0.01); (2; 0.02); (3; 0.03) \ldots\}$
(g) $\{(1; 0); (2; 0.301); (3; 0.477) \ldots\}$
(h) $\{(1; 1); (2; 0.5); (3; 0.333) \ldots\}$
(i) $\{(1; 2.718); (2; 7.389); (3; 20.086) \ldots\}$
(j) $\{(1; 1); (2; 2); (3; 6); (4; 24) \ldots\}$
(k) $\{(1; 1); (2; 8); (3; 27); (4; 64) \ldots\}$
(l) $\{(1; 1); (2; 1.260); (3; 1.442); (4; 1.587) \ldots\}$

(Answers are given at the end of the worksheet.)

The ideas of domain and range are defined respectively as the sets of all the possible first and second elements in each pair. In terms of the calculator, the domain is the set of acceptable input (or entries) and the range is the set of possible output (or displays). The calculator reacts to unacceptable input with an error display.

For example, when using the calculator function $\boxed{\sqrt{\ }}$, you will find that any negative entry produces an error display. This is because the square root of a negative number is not real and the calculator only deals in real numbers. A graph provides useful insight into the behaviour of a function and helps with identifying the domain and range. Here is the graph of $y = \sqrt{x}$.



It shows plainly that the domain of $y = \sqrt{x}$ is $\mathbb{R}_0^+$, that is, the set of all positive real numbers including 0 and the range is also $\mathbb{R}_0^+$.

2. Using your calculator to generate the function values, draw rough sketches of the graphs of each of the following and use them to deduce the domain and range.

(a) $y = x^2$

(b) $y = \frac{1}{x}$

(c) $y = 10^x$

(d) $y = \log x$

(e) $y = \sin x$ (measuring x in degrees)

(f) $y = \sqrt{9 - x^2}$

Answers for question 1:

(a) $y = \frac{x}{2}$

(b) $y = x^2$

(c) $y = \sqrt{x}$

(d) $y = \tan x$ (DEG)

(e) $y = \tan x$ (RAD)

(f) $y = \frac{x}{100}$

(g) $y = \log x$

(h) $y = \frac{1}{x}$

(i) $y = e^x$

(j) $y = x!$

(k) $y = x^3$

(l) $y = \sqrt[3]{x}$

WORKSHEET 18

FUNCTIONS II

Your calculator is capable of performing a large number of functions. These functions can be combined to form a great variety of other functions. This process is known as composition. Consider, as an example, the function $y=\sqrt{x^2-3}$ This is composed of three basic calculator functions – they are the "square" function, the "subtract 3" function and the "square root" function. Identifying the functions involved is simplified by planning the keystrokes necessary to calculate values of the function – the entry/display pairs. In this example the keystrokes are as follows, using O for the input and Δ for the display :

$$O\ \underbrace{\boxed{x^2}}_{1}\ \underbrace{\boxed{-}\ 3}_{2}\ \boxed{=}\ \underbrace{\boxed{\sqrt{\ }}}_{3}\ \boxed{=}\ \Delta$$

1. Identify the calculator functions that go to make up each of the following:

(a) $y = x^2 + 3x - 1$

(b) $y = \sqrt{4x^2+1}$

(c) $y = \dfrac{x+1}{x-2}$

Calculators exist so that we can enter numbers into them, process them in a meaningful way and obtain a display which also has some significance. The question we are asking is " If I enter this number, what will the display be?"

Suppose we reverse the question. "Here is the display –

What number must be entered to give this display?" The calculator can also be "reversed" to deal with this type of problem. We do this by applying the appropriate inverse operation.

For example, "a certain number +2 is 5." Reversing the calculation, we get "5 −2 is the original number." So the inverse of +2 is −2.

2. Check through the keys of your calculator and make a list of all the pairs of functions and their inverses that you can find. Check to see that they all work. For example, any number $\boxed{\sqrt{}}$ $\boxed{x^2}$ should give the original number. (This is one way of checking the quality of any calculator.)

WORKSHEET 19

WHAT IS A LOGARITHM?

The logarithm of a number is defined as the power to which a particular base must be raised to give that number. The important thing to remember from the definition is that a <u>logarithm is an index</u>.

Your calculator is designed to work with two sets of logarithms – one where the base is 10 $\boxed{\text{log}}$ and the other where the base is a very special number, e $\boxed{\text{ln}}$. To begin with, we will concentrate on logs with the base 10.

If you enter 2 in your calculator and press $\boxed{\text{log}}$, your display reads 0,30103. What this means is that $\log 2 = 0,30103$. In other words, 0,30103 is the power to which 10 must be raised to give 2. So the equation $\log 2 = 0,30103$ could also be written in the form

$$10^{0,30103} = 2 .$$

You can verify this using the $\boxed{y^x}$ key:

the keystrokes are 10 $\boxed{y^x}$ 0,30103 $\boxed{=}$

An alternative method using fewer keystrokes and the $\boxed{10^x}$ key is 0,30103 $\boxed{10^x}$

From the above you will have noticed that we have two ways of presenting the same information. The first is called <u>LOG FORM</u> and the second <u>INDEX FORM</u>. It is important

Page 2

that you be familiar with both of these forms and be able to convert readily from one form to the other.

To practise this, use your calculator to find the log of each of the following numbers and write equations interpreting the display in both log form and index form.

NUMBER	log	LOG FORM	INDEX FORM
2	0,30103	log 2 = 0,30103	$10^{0,30103} = 2$
5			
0,8			
14,5			
−18,2			

Can you explain why your calculator gives an error display when you enter −18,2 and press $\boxed{\log}$?

Check your equations in the right hand column by making use of the $\boxed{10^x}$ key.

Carry out the following sequence of keystrokes and then explain the results:

23,4 $\boxed{\log}$ $\boxed{x \to M}$. The display is 1,3692159.

Now enter 1,3692159 and press $\boxed{10^x}$. The display now reads 23,400002. Yet if we press $\boxed{RM}$ $\boxed{10^x}$ the display is 23,4. What is happening?

There are some numbers whose logarithms we should be able to work out without even using our calculators. Complete the following table: (NOTE that in some examples the base is no longer 10.)

Page 3

NUMBER	BASE	LOG	LOG FORM	INDEX FORM
100	10		log 100 = …………	$10^{\ldots} = 100$
1	10			
0,001	10			
10000	10			
0,00001	10			
9	3		$\log_3 9 = \ldots$	$3^{\ldots} = 9$
125	5		$\log_5 \ldots = \ldots$	
16	4			
8	$\frac{1}{2}$			
$\frac{1}{27}$	3			

Note that when we write 'log' we understand that the base is 10. If we are working with a base other than 10, then it is filled in as a subscript, for example $\log_2$ means the base is 2.

A useful test of understanding is whether or not you can give a general formula which sums up the examples in the above table:

Write the general equation $a^b = c$ in log form:

…………………………………………………………

WORKSHEET 20

CHANGING THE BASE

The previous worksheet concluded with a general statement of an equation written in both log and index forms:

$$\log_a c = b \quad \text{and} \quad a^b = c$$

Up to now we have been concentrating mainly on logs with the base 10 but in the example above "a" could actually have any positive value. (Why positive?) Having repeated the definition of a logarithm to yourself at least 5 times, find the value of the following: (Remember that when the base is not written, it is assumed to be 10.)

(1)	$\log_2 4$	(6)	$\log 0.01$
(2)	$\log_5 125$	(7)	$\log_5 0,04$
(3)	$\log_2 1$	(8)	$\log_4 2$
(4)	$\log_4 (\frac{1}{4})$	(9)	$\log_{25} 125$
(5)	$\log_3 (\frac{1}{27})$	(10)	$\log_{27} 81$

If you find difficulty with these expressions, especially the last few, it might help if you make equations of them: let the expression equal x and then rewrite the new equation in index form. Use your knowledge of exponential equations to find the value of x.

Example: find the value of $\log_{32} 16$

$$\text{let } \log_{32} 16 = x$$
$$\therefore 32^x = 16$$
$$\therefore 2^{5x} = 2^4$$
$$\therefore \underline{x = \frac{4}{5}}$$

$\longrightarrow$

Page 2

(This technique of converting from log form to index form will be used later to solve log equations.)

Consider the expression $\log_3 4$. Can we find its value using a similar technique to that employed in the last example? Where does the method break down?

Examine closely the final statement in that previous example.

We have $x = \log_{32} 16 = \frac{4}{5}$

Can you think of logarithmic expressions which can be written in place of "4" and "5"? In other words can you write an equation in the form

$$\log_{32} 16 = \frac{\log_{\dots} \dots\dots}{\log_{\dots} \dots\dots} \quad \text{which is equivalent to } \log_{32} 16 = \frac{4}{5}\ ?$$

Careful consideration of the numbers 16 and 32 should reveal what base you should choose.

Complete the following:

$$\log_9 27 = \frac{\log_{\dots} \dots\dots}{\log_{\dots} \dots\dots} = \frac{\dots}{\dots}$$

$$\log_8 \left(\tfrac{1}{16}\right) = \frac{\log_{\dots} \dots\dots}{\log_{\dots} \dots\dots} = \dots\dots$$

Can you now deal with $\log_3 4$? You will need the help of your calculator: (The answer you are looking for is 1,2618595.)

Find the values of $\log_6 9$ and $\log_{19} 14$.

WORKSHEET 21

INTRODUCING THE LAWS OF LOGARITHMS

The laws of logarithms are the basis of much of the theory of logs. Here is a practical introduction to the laws. Use your calculator to complete the following table:

a	b	$\log ab$	$\log \frac{a}{b}$	$\log a^b$	$\log a + \log b$	$\log a - \log b$	$b \log a$

Study the table carefully. You should be able to identify three sets of relationships from the information recorded. These are the three laws of logarithms. Write them down:

LAW I : $\log ab$ = ...

LAW II : $\log \frac{a}{b}$ = ...

LAW III : $\log a^b$ = ...

To complete the table we used logs with the base 10. The question now is do these relationships still hold if some other base is used? Complete a similar table using logs with the base e, $\boxed{ln}$.

What do you find?

It would seem reasonable to assume that the laws apply for logs with any base and this is in fact the case. However, mathematicians need a water-tight general proof before they will be convinced and this is a job for algebra.

APPENDIX E

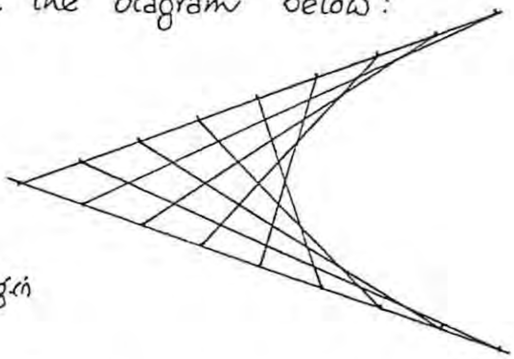
The Calculus Worksheets

WORKSHEET C_1

An encounter with limits:

Your first encounter with limits - in the mathematical sense - might have been via any one of a number of activities including curve-stitching or the graphs of hyperbolas or the sum to infinity of a geometric series.

A common first exercise in curve-stitching is when we take a large V, mark off equal intervals on each arm and join them as indicated in the diagram below:

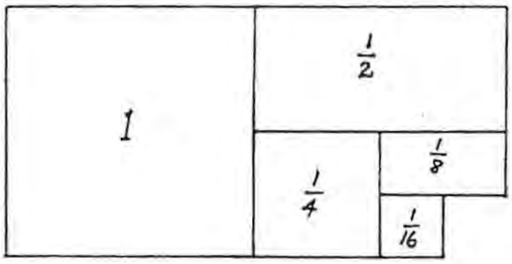


fig(i)

The inner "curve" is made up of a number of short segments of straight lines. The smaller the intervals we mark off on the arms of the V, the smaller will be the lengths of the segments, the greater will be their number and the better they will approximate to a true curve. (To explore this theme further, see Worksheet C_2.)

You will have met the geometric sequence $1; \frac{1}{2}; \frac{1}{4}; \frac{1}{8}; \frac{1}{16}; \ldots$ Let us regard these terms as representing area. On a sheet of graph paper draw a square enclosing 16 2cm blocks. Let this represent the first term of the sequence. Using fig(ii) as a guide, add a rectangle measuring 4x2 blocks and representing the second term of the sequence, $\frac{1}{2}$. Then add a square of 2×2 blocks representing $\frac{1}{4}$. Continue for as long as you can with reasonable accuracy. You will notice that the further you continue with this process, the closer the area of the shape you are building up will approximate to 2.

fig (ii)



You can check this with your calculator. The rectangle you have been building up represents the geometric series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$ with a common ratio of $\frac{1}{2}$ (i.e. 0.5). This sequence of keystrokes makes use of the memory and the automatic constant facility:

$$\underbrace{1\ \boxed{x\to M}}_{\text{Puts } T_1 \text{ in memory}}\ \underbrace{0.5\ \boxed{M+}}_{\text{Adds } T_2 \text{ to memory}}\ \underbrace{\boxed{\times}\ \boxed{=}}_{\text{Finds } T_3}\ \underbrace{\boxed{M+}}_{\text{Adds it to memory}}\ \underbrace{\overset{*}{\boxed{=}}\ \boxed{M+}}_{\text{Finds } T_4 \text{ and adds it to memory}}\qquad \text{Go to } *$$

Use these keystrokes to complete the table:

No. of terms	Sum
5	
10	
15	
20	
25	
$S_\infty = \frac{a}{1-r}$	

Notice how the sum keeps increasing, edging towards 2 by ever-decreasing amounts.

The series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$ approached its limit, 2, from just one direction. Sometimes we make use of a sand-

-wiching' approach to arrive at a limit from two directions. This is how we find square roots using trial and error. Say, for example, we had to find $\sqrt{5}$. Working methodically we would attempt to find values on either side of $\sqrt{5}$ (looking at the problem from a number line point of view) and then try to increase the accuracy of our guesses with subsequent attempts. For starters we know that

$2 < \sqrt{5} < 3$ ($\sqrt{5}$ lies in a slice of number line with a thickness of 1.)

Working to one decimal place, $2.2^2 = 4.84$ and $2.3^2 = 5.29$

So $2.2 < \sqrt{5} < 2.3$ ($\sqrt{5}$ lies in a slice of number line with a thickness of 0.1.)

Then with two decimal places, $2.23^2 = 4.9729$ and $2.24^2 = 5.0176$

So $2.23 < \sqrt{5} < 2.24$ ($\sqrt{5}$ lies in a slice 0.01 thick.)

And so working with all the accuracy that our calculator can provide, we can narrow down the slice of number line in which $\sqrt{5}$ resides. (Remember that when your calculator tells you that $\sqrt{5}$ is 2.236068 what it is really saying is that $2.236067 < \sqrt{5} < 2.236069$ and though the number in the display is accurate enough for all practical purposes, it is NOT the exact value of $\sqrt{5}$ in decimal notation.)

Exercise: Using the trial-and-error method, narrow down the range in which each of the following occur to a slice 0.001 thick (i.e. to 3 decimal places). Use $\boxed{\sqrt{\ }}$ to check your answers. (a) $\sqrt{11}$ and (b) $\sqrt{29}$

There is a method for finding square roots which converges far more rapidly. It is known as the Newton method and uses the idea that if y is an approximate value of $\sqrt{x}$, then the average, $\frac{1}{2}(y + \frac{x}{y})$ will be a better approximation. Using this method to find $\sqrt{40}$:

$\sqrt{40}$ lies between 6 and 7, rather closer to 6 than

than to 7. So let's take 6.2 as our first guess and then apply the formula with $x = 40$ and $y = 6.2$. The results supplied by the calculator are :

Guess : 6.2
1st trial : 6.3258065
2nd trial : 6.3245554
3rd trial : 6.3245553
4th trial : 6.3245553

The method has yielded $\sqrt{40}$ as accurately as possible with the calculator after only 3 trials. Even if we had used a hopelessly inaccurate first guess, the results would still have converged very rapidly : For example

Guess : 2
1st trial : 11
2nd trial : 7.3181818
3rd trial : 6.3920102
4th trial : 6.3249112
5th trial : 6.3245553

Exercise : Using your calculator and the Newton method find the following as accurately as possible :

(a) $\sqrt{15}$ (b) $\sqrt{57}$ (c) $\sqrt{33.4}$

An aside : can you prove that after the first trial the approximations are always going to DECREASE towards the limit ? In other words, prove that

$$\tfrac{1}{2}\left(y + \tfrac{x}{y}\right) \geqslant \sqrt{x} .$$

The trial-and-error method was based on sandwiching the square root between successive guesses. Here is an example of a sequence which does its own sandwiching :

A 13th century Italian, Leonardo of Pisa, nicknamed Fibonacci, devised a sequence which has applications in biology and art. In this sequence, each term is the sum of the two previous terms. The sequence is 1; 1; 2; 3; 5; 8 ... Find the first 21 terms of the Fibonacci sequence. (As a check $T_{22} = 17711$.) If you now calculate the ratios of consecutive terms (that is $\frac{T_2}{T_1}$; $\frac{T_3}{T_2}$; $\frac{T_4}{T_3}$; ...), you will get the sequence 1; 2; 1.5; 1.666...; 1.6; .... and so on. Notice that the odd numbered terms form an increasing sequence while the even terms are decreasing. Work out the ratios of consecutive terms for all 21 terms of the sequence that you found and record them in a table:

$\frac{T_2}{T_1} = 1$	$\frac{T_3}{T_2} = 2$
$\frac{T_4}{T_3} = 1.5$	$\frac{T_5}{T_4} = 1.6666667$
$\frac{T_6}{T_5} = 1.6$	$\frac{T_7}{T_6} = ---$
⋮	⋮
$\frac{T_{20}}{T_{19}} = ---$	$\frac{T_{21}}{T_{20}} = ---$

Notice that the ratios in each column converge. The ratio towards which they are converging can be represented by $\frac{1+\sqrt{5}}{2}$ (use your calculator to check). It is known as the Golden Ratio and a rectangle with sides in this ratio is supposed to be the most satisfying aesthetically. The golden rectangle occurs in art and architecture to an extent that cannot be accounted for by pure coincidence.

Continue with Worksheet C_3.

WORKSHEET C_2

Stitching curves and curves of pursuit

The idea of a sequence of small steps approximating to a limit is at the heart of calculus. It can also be an avenue to attractive patterns. We can produce these patterns either by careful drawing or by stitching with thread on a piece of card or threading it around panel pins hammered into a piece of wood.

The most basic design is when one marks off equal intervals on the arms of a V:

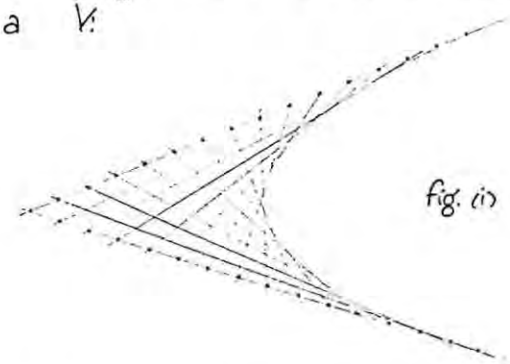


fig. (i)

You could try a similar idea using the sides of a square or a rhombus or a rectangle (if you had the same number of divisions on adjacent sides). There is no reason why the sides of your basic diagram need be straight lines - experiment with circles and ellipses and other curves or combine curves with straight lines.

Your imagination is the limit!

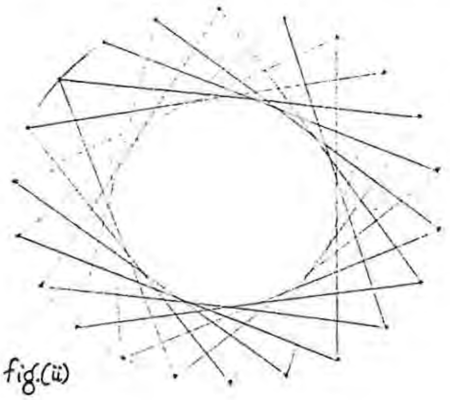
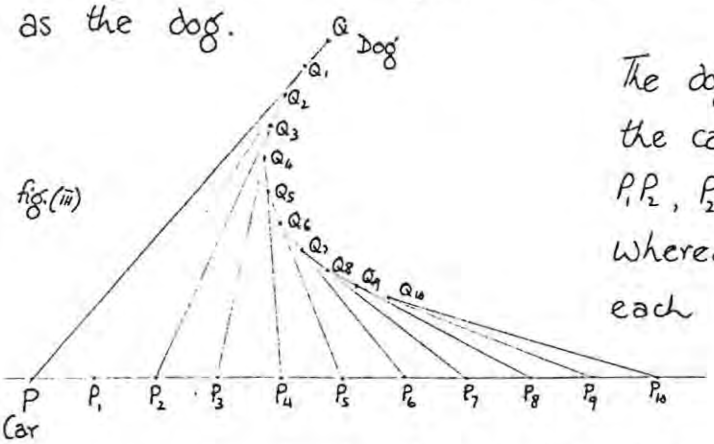


fig.(ii)

Curves of pursuit

You have probably seen a dog chase a car or a bicycle or even sea-gulls on the beach. The dog always aims at the position of whatever he is chasing at that particular instant. Consider the case of dog vs car. Assume that they are both moving at constant speeds and that the car is twice as fast as the dog.

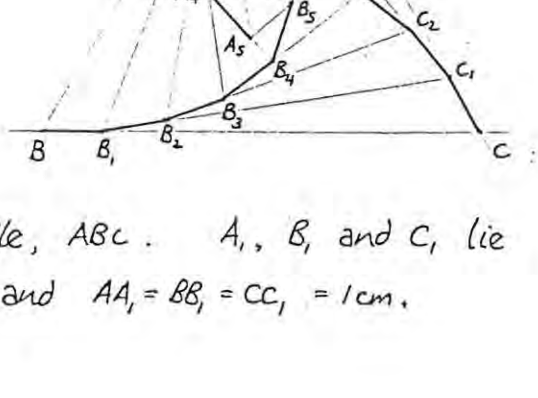
fig.(iii)



The dog is at Q when he spots the car at P. The intervals PP_1, P_1P_2, P_2P_3, ... are each 10mm whereas QQ_1, Q_1Q_2, Q_2Q_3 --- are each 5mm since the car is twice as fast as the dog. Experiment by changing the speed of the dog relative to whatever he is chasing. What if the object pursued is moving along some sort of curve?

A variation on this theme is the problem of the three beetles with cannibalistic tendencies. The beetles are at three points A, B and C, the vertices of an equilateral triangle. A wants to eat B, B fancies C and C wants to get his teeth into A. They move at the same speed. The accompanying diagram shows the path followed by each in his quest for dinner. To reproduce this sketch, start with the equilateral triangle, ABC. A_1, B_1 and C_1 lie on AB, BC and CA respectively and $AA_1 = BB_1 = CC_1 = 1cm$.

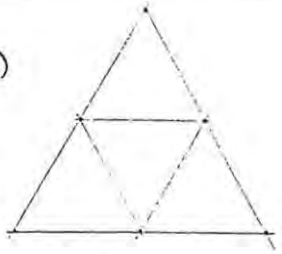
fig.(iv)



A_2, B_2 and C_2 lie on A_1B_1, B_1C_1 and C_1A_1 respectively and $A_1A_2 = B_1B_2 = C_1C_2 = 1cm$. The positions of A_3, B_3 and C_3 are found in a similar way.

A possible variation would be to start with four beetles at the corners of a square. Another possibility is illustrated in fig.(v). It represents fig.(iv) repeated four times. With careful drawing some most attractive patterns can be created.

fig.(v)



These curves are generated whenever we have something homing in on something else. These situations range from a dog chasing a cat to a missile homing in on an aircraft.

In all the above examples we are trying to approximate actual curves by means of "curves" made up of segments of straight lines. The smaller the segments, the better will be the fit but the more you will have to concentrate to produce an accurate drawing.

WORKSHEET C_3

Some more limits

Consider the function defined by $f(x) = 1 + \frac{1}{x}$. Sketch its graph for positive values of x.

Notice that the larger we make the value of x, the smaller becomes the value of $\frac{1}{x}$. This means that the value of $1+\frac{1}{x}$ can be made as close as we please to 1 by taking x sufficiently large. Here $1 + \frac{1}{x}$ has a limit in the everyday sense of the word, that is, a boundary which cannot be passed.

Mathematically, the concept of limit is not restricted to a boundary which is approached from one side only. Consider the sequence defined by $T_n = \frac{n + (-1)^n}{n}$.

Complete the table:

n	1	2	3	4	5	6	……	99	100	101	102	……	999	1000
T_n														

What is the limiting value which T_n is approaching? Notice how the terms home in on this limit alternately from above and below and how we can make T_n as close as we like to this limit by taking sufficiently large values of n.

Exercises: Given below are the general terms defining various sequences. Examine the behaviour of each as n becomes very large. For each sequence, is there a limit and if

So, what is that limit?

(a) $T_n = 2 - \frac{1}{n^2}$

(b) $T_n = 3n - 2$

(c) $T_n = 3$ for all values of n

(d) $T_n = \frac{n(n+1)}{4}$

(e) $T_n = \frac{3n-2}{n}$

(f) $T_n = \frac{2n^2-n-1}{n^2}$

(g) $T_n = (1 + \frac{1}{n})^n$

The important idea is that a sequence approaches a limit if we can get as close as we please to this limit by going far enough with the sequence. When using a calculator you will find that after a while it starts giving the same answer (as with the Newton method for finding square roots). Remember that the number in the display is not necessarily the exact value of the limit. Its accuracy is limited to that of the calculator's display or, if you wish to do some digging, its internal capacity.

For further examples on sequences which tend to limits, see Worksheet C_4 on recurrence relations.

WORKSHEET C_4

Recurrence Relations - another way of generating sequences

In Worksheet C_3 sequences were generated by means of a formula for T_n, the general term. Another method of generating a sequence is by means of a recurrence relation. This tells us how to calculate the next term of a sequence from the term preceding it. This idea was used in the Newton method for finding square roots in Worksheet C_1.

For example, if a sequence is formed by adding 1 to the previous term, we would say that $T_{n+1} = T_n + 1$. However, this is not much good to us unless we have a starting point. (In the Newton method we had to begin with a guess at the square root we were trying to find.) So, to go back to our example, if we were told that $T_1 = 5$, then we would be in a position to find the first few terms of the sequence: 5; 6; 7; 8; ....

> Recurrence relations are of great importance to the designers of electronic calculators because they are the means by which calculators compute much of the information they supply. Efficient calculator algorithms are usually the manufacturer's carefully guarded secrets.

Exercises:

(1.) A sequence is defined by $T_1 = 0.1$ and $T_{n+1} = T_n(2 - 3T_n)$
Calculate T_2, T_3, T_4, T_5 and T_6.
The more terms you calculate, the smaller will become the

the difference between successive terms as the sequence approaches its limit and we can get as close as we please to this limit by taking n sufficiently large, that is by taking enough terms. You will probably be able to guess the limit by inspecting the terms you calculated but it is possible to find it algebraically.

Theoretically, when we have reached this limit, T_{n+1} will be equal to T_n. So substitute x for both T_{n+1} and T_n in the recurrence relation and solve:

$$x = x(2-3x)$$
$$\therefore x = 2x - 3x^2$$
$$\therefore 3x^2 - x = 0$$
$$\therefore x(3x-1) = 0$$
$$\therefore x = 0 \text{ or } \tfrac{1}{3}.$$

In the context of this example, we can disregard the first root. So the limit is $\frac{1}{3}$ or 0,333...

(2.) Basing your reasoning on the previous example, can you devise a recurrence relation to give the decimal expansion of $\frac{1}{7}$?

(3.) Here are some more recurrence relations which tend to limits as you make n larger. Each one has to be "seeded" with a guess before you can start and before you can make a guess, you need to know what they are designed to find. To discover this, use the algebraic method described in exercise (1.) to find the value of N, the limit towards which they will converge. Experiment with the relations. See how quickly they will converge. Maybe you can design some recurrence relations yourself.

Before you try some examples, here is one by way of explanation.

$$T_{n+1} = \sqrt{\sqrt{N.T_n}}$$

The theory is that when the limit is reached, if that were possible T_{n+1} would equal T_n. So we write:

$$x = \sqrt{\sqrt{N.x}} \qquad (\text{letting } T_{n+1} = x)$$

$$\therefore \; x^2 = \sqrt{N.x}$$

$$\therefore \; x^4 = Nx$$

$$\therefore \; N = x^3$$

In other words the relation is designed to find a number x which when cubed gives N. Its purpose is to give cube roots. So we would put the number whose cube root we wish to find in place of N and then make a guess at it to get the sequence going. Say we wanted $\sqrt[3]{10}$ and used 2 as our first guess:

$T_{n+1} = \sqrt{\sqrt{10.T_n}}$ Guess : 2

1st trial 2.1147425
2.1444423
2.1519322
2.1538088
2.1542782
2.1543956
2.1544249
2.1544322
2.1544341
2.1544345
11th trial 2.1544347
2.1544347

You would actually have to repeat the process a few more times before cubing the display would give 10. Can you explain why?

Here are some for you to play with:

(a) $T_{n+1} = 1 + \frac{N-1}{1+T_n}$ (b) $T_{n+1} = \sqrt{\sqrt{\frac{N}{T_n}}}$

(c) $T_{n+1} = 2.T_n - N(T_n)^2$

Recurrence relations also provide a useful method for solving equations. This was dealt with in an earlier worksheet.

WORKSHEET C_5

Rates of change

Averages tend to obscure detail. For example, the fact that your average for the last exams was 65% does not make clear that you almost failed Afrikaans while getting 90% for Maths...

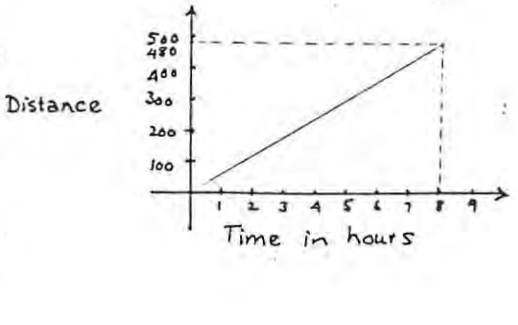
Long distance runners like having their times recorded at intervals during a race. They are greatly interested in knowing their speed in minutes/kilometre. Here are the times recorded for a runner in a standard marathon (42.2 km).

DISTANCE (in km)	8	16	21.1	32	42.2
TIME (in minutes)	32.9	67	88.9	141.9	198.4

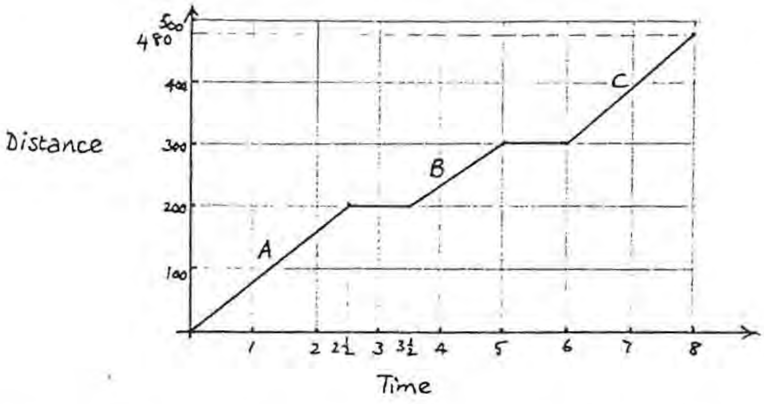
Using this information, find his average speed in minutes/km for
- (a) the whole race
- (b) the first half
- (c) the second half
- (d) each section of the race.

Notice how as we work out the average speed over shorter intervals, we get a better insight into the runner's performance through the various stages of the race.

Consider the illustration of a journey by car. Let us take the case of a driver who completes a journey of 480 km in 8 hours. We could illustrate this information by means of a graph. Notice that the gradient is $\frac{\text{Change in D}}{\text{Change in T}} = \frac{480}{8} = 60$



which is the average speed over the whole journey. This is a very simplified view of the situation. Let us put in a bit more detail. Say the driver stopped for one hour after he had covered 200 km in $2\frac{1}{2}$ hours. After driving for $1\frac{1}{2}$ hours, he again stopped for an hour before completing the remaining 180 km of his trip. Transferring this information to a graph:



Find his average speed over each stage of the journey by calculating the gradient of the graph:

A : $\frac{\text{Change in D}}{\text{Change in T}} = \frac{\quad}{} =$ km/h

B : $\frac{\text{Change in D}}{\text{Change in T}} = \frac{\quad}{} =$ km/h

C : $\frac{\text{Change in D}}{\text{Change in T}} = \frac{\quad}{} =$ km/h

Here is an example for you to try:

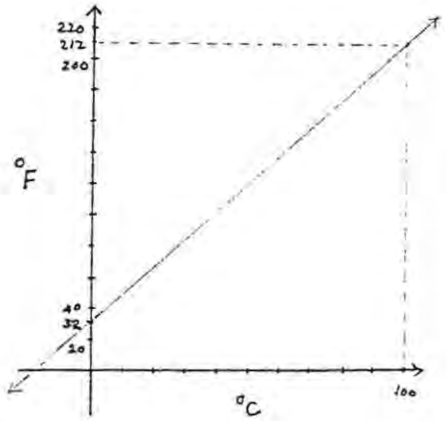
Draw a Distance/Time graph to illustrate this information. A man sets off on a journey of 620 km. He stops for half an hour after having travelled 150 km in an hour and a half. His next break comes 350 km and 4 hours later. He stops for an hour. He completes the remainder of the journey in an hour and a half. Find the gradient of the graph and hence the driver's average speed on each of the three stages.

Note that this is still a simplification of the true story. It assumes that the car is moving at a constant speed from the moment it starts until it stops. It does not take into account all those conditions of the road that cause one to slow down or accelerate. There is only one way to get an idea of the car's speed at any particular moment on the journey and that is to look at the speedometer. The speedometer is designed to tell you the speed at any particular instant in time. And this is a function it shares with the differential calculus.

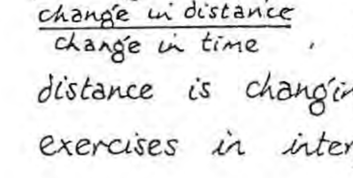
The gradient m of the straight line $y=mx+c$ is $\frac{\text{change in } y}{\text{change in } x}$. Getting down to practical examples, x and y can be used to represent concepts such as time and distance. So in the car journey examples above, the gradient, $\frac{\text{change in } y}{\text{change in } x}$, becomes $\frac{\text{change in distance}}{\text{change in time}}$, which is speed or velocity, the rate at which distance is changing with respect to time. Here are some more exercises in interpreting the gradient of graphs:

Exercises:

(1.) The sketch represents the relationship between °c and °F. What is the equation of the line? What does the gradient of the graph represent?



(2.)

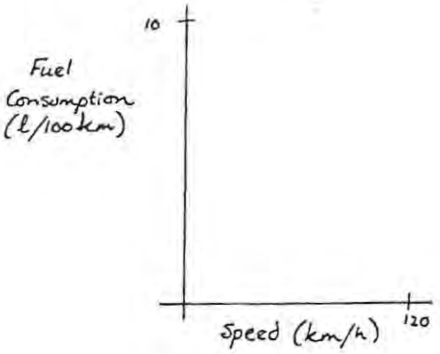


It takes 30 hours to fill a swimming-pool with a capacity of 75000 l. What does the gradient of the graph represent?

(3.) A motoring magazine gives the following figures for the fuel consumption (in litres/100 km) for a certain Japanese sedan.

Speed	Fuel Consumption
60	4.78
70	5.24
80	5.70
90	6.25
100	6.89
110	7.61
120	8.90

Format for graph:



Illustrate this information on graph paper, joining the points by means of straight lines. Find the gradients of the line segments between each pair of points (i.e. from 60 to 70, 70 to 80 ..... and 110 to 120). What does the gradient represent? What are these gradients telling us? Are straight lines the most accurate way of joining the points?

(4.) The perimeter of a rectangle is 16 units. This means that the sum of the length and breadth is 8 units. Complete the following table which gives values to the length, finds the breadth and calculates the area:

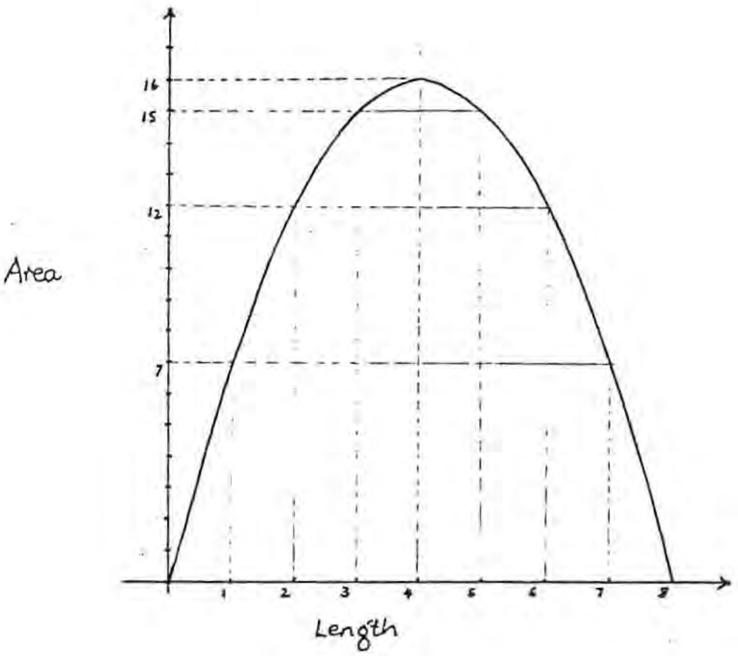
LENGTH	0	1	2	3	4	5	6	7	8
BREADTH	8	7							
AREA	0	7							

Translating this table into algebraic notation, if we let the length be x, the breadth is $8-x$ and the area, y, is length × breadth.

So $y = x(8-x)$

Representing this information graphically, we get the parabola:

<u>Page 5</u>



When dealing with straight lines, the question of gradient was simple because it remained constant. With a curve, however, it is changing all the time. Think back to curve-stitching and imagine a curve as being composed of very many infinitely small segments of straight line.

(a) What does the gradient of the above curve represent?

(b) What is happening to the gradient when the length is between 0 and 4?

(c) What is the gradient when the length is 4?

(d) What is happening to the gradient when the length is between 4 and 8?

WORKSHEET C_6

The gradient of a tangent at any point on a curve

An important point made in the previous worksheet was that the gradient of a straight line gives us the rate at which the relationship described by the line is changing. For many relationships a straight line is not the best model. The graphs representing car journeys, for instance, were at best only models that helped us find average speeds. In reality their graphs should actually be meandering lines, steep as the car accelerates and flattening as it changes gear, slows down and stops. On those few occasions where one is able to maintain a steady speed, it may be a straight line. As was pointed out, if you want to know the speed of the car at any particular instant, you have to look at the speedometer or, thinking in terms of graphs, find the gradient of the tangent at that particular point on an accurate graph. As a first encounter with tangents to curves, try this example:

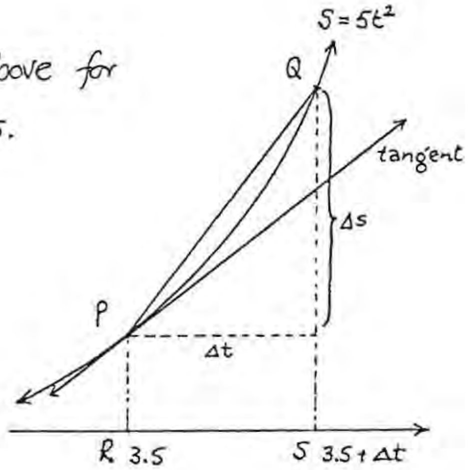
The distance S (in metres) travelled by a falling object is given by the formula $S = 5t^2$ where t is the time in seconds. Draw an accurate graph on graph paper to depict this relationship. for t from 0 to 4 seconds. Use your calculator to find extra intermediate points to make the curve as accurate as possible (careful drawing is VERY important in this exercise).

(a) What does the gradient of this curve represent?

(b) Use your ruler to draw the tangents to the curve when t is (i) 1 sec. (ii) 2 secs. and (iii) 3.5 secs.

By carefully examining the lines you have drawn, work out the gradient of each as accurately as you can and interpret the result.

The method set out in the above example is obviously not the most efficient way of finding the gradient of the tangent to a curve. We need to be able to calculate the gradient exactly and to do this we use the idea of limits, aided by our calculators acting as a sort of mathematical magnifying glass.



Let us examine the graph mentioned above for values of t in the region of 3.5 seconds. The task was to find the gradient of the tangent at P. Let Q be a point on the graph. The theory is that the closer we bring Q to P, the closer the gradient of PQ will come to the gradient of the tangent at P.

In the context of this graph, the gradient of PQ is $\frac{\text{change in } s}{\text{change in } T}$ or using mathematical symbols, $\frac{\Delta s}{\Delta t}$. (Note that we would no more cancel the words "change in" in the first fraction than we would cancel "Δ" in the second. Δs does NOT mean Δ multiplied by s - it means the change or increment in s.)

Before starting our calculations, notice that Δs is $QS - PR$. This is the value of the function at $3.5 + \Delta t$ minus the value of the function at 3.5. Using symbolic language,

$$\Delta s = f(3.5 + \Delta t) - f(3.5)$$

Complete the following table for the function $f(t) = 5t^2$:

Δt	$f(3.5+\Delta t)$	$f(3.5)$	$\Delta s = f(3.5+\Delta t) - f(3.5)$	$\frac{\Delta s}{\Delta t}$
0.5	80	61.25	18.75	37.5
0.4				
0.3				
0.2				
0.1				
0.01				
0.001				

From the table it appears reasonable to assume that the limit towards which $\frac{\Delta s}{\Delta t}$ tends as Δt tends to zero is 35. (How did this compare with your answer in the example on page 1?) In fact, using words we have used before, we can make $\frac{\Delta s}{\Delta t}$ as close as we please to 35 by making Δt sufficiently small.

Exercises:

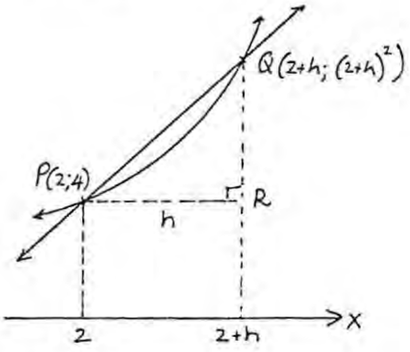
(1.) A sandwiching manoeuvre might help to reinforce the idea of the gradient of the tangent at a point on a curve as being a limit. Complete the table to find the gradient of the tangent to the graph of $y = x^2$ when $x = 2$

Δx	$f(2+\Delta x)$	$f(2)$	$\Delta y = f(2+\Delta x) - f(x)$	$\frac{\Delta y}{\Delta x}$
0.3	5.29	4	1.29	4.3
0.2				
0.1				
0.01				
0.001				
− 0.001				
−0.01				
− 0.1	3.61	4	− 0.39	3.9

(2.) Using the same method as outlined above and taking $\Delta x = 0.3 ; 0.2 ; 0.1 ; 0.01 ; 0.001 ; 0.0001$, find the gradient of the tangent to the curve defined by $y = x^2 + 3x$ when $x = 1$.

(3.) The relationship between the volume and radius of a sphere is given by $V = \frac{4\pi r^3}{3}$. Find the rate at which the volume of a spherical balloon is changing with respect to its radius when $r = 2$, giving the answer correct to 3 decimal places. Use $\Delta r = \pm 0.1 ; \pm 0.01 ; \pm 0.001 ;$ and ± 0.0001. (Think carefully about how you will carry out the calculations on your calculator. Keep the value of $f(2)$ in memory. Maybe you can dispense with intermediate results?)

Let's look at exercise (1.) again. The problem was to find the gradient of the tangent to $y = x^2$ at $x = 2$.
The co-ordinates of point P are $(2;4)$.
If we increase the value of x by a small amount, h, the co-ordinates of point Q are $(2+h; (2+h)^2)$.



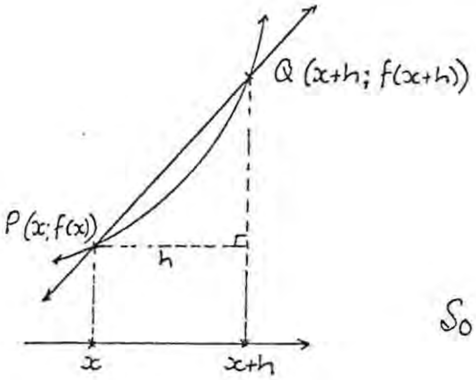
$$\text{The gradient } \frac{QR}{PR} = \frac{\Delta y}{\Delta x} = \frac{\text{change in } y}{\text{change in } x}$$
$$= \frac{(2+h)^2 - 4}{h}$$
$$= \frac{4 + 4h + h^2 - 4}{h}$$
$$= \frac{4h + h^2}{h}$$
$$\underrightarrow{= 4 + h}$$

Now consider what happens as we bring Q closer to P, in other words, we make h smaller and smaller. As h tends towards 0, the value of $\frac{QR}{PR}$ will tend towards 4, the gradient of the tangent at P.

Putting this in its proper mathematical dress, we would say

$$\lim_{h \to 0} \frac{(2+h)^2 - 4}{h} = 4$$

"the limit as h tends to zero of $\frac{(2+h)^2-4}{h}$ is 4." What this is telling us is that the gradient of the tangent to $y = x^2$ at $x = 2$ is 4. Our next step is to arrive at a general statement of what is meant by the gradient of the tangent to the curve $y = f(x)$ at the point $(x; f(x))$:



Q $(x+h; f(x+h))$ is a point on $y = f(x)$ which comes closer to P as $h \to 0$.

So the gradient of

$$PQ = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

WORKSHEET C_7

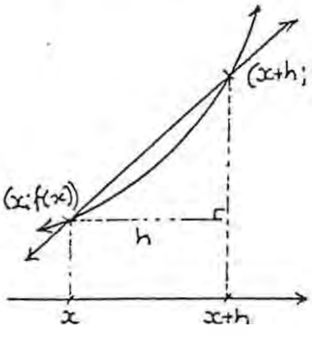
Derivatives from first principles

We have defined the gradient of the tangent (if it exists) at the point $(x; f(x))$ on the curve $y = f(x)$ as $\lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

This limit is called "the derivative of the function $y = f(x)$ with respect to x." It is denoted by any one of a number of symbols. For the moment we will call it $f'(x)$, which is read "f prime of x".

In the previous worksheet we found the gradient of tangents to curves at specific points on the curve. For every point on the curve there was a whole procedure to go through. When faced with the prospect of doing the same thing over and over again, mathematicians, being lazy, invariably look for a general method which will sum up all the calculations concerned. Here is how we would find $f'(x)$, the gradient of the tangent to the curve $y = x^2$ at any point $(x; f(x))$.

First we calculate the value of $\frac{f(x+h) - f(x)}{h}$ which represents the gradient of the line passing through the point $(x; f(x))$ and another point $(x+h; f(x+h))$ a little bit further along the graph:



$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{(x+h)^2 - x^2}{h} \\ &= \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \frac{2xh + h^2}{h} \\ &= 2x + h\end{aligned}$$

By definition, the gradient is

$$f'(x) = \lim_{h \to 0} (2x + h) .$$

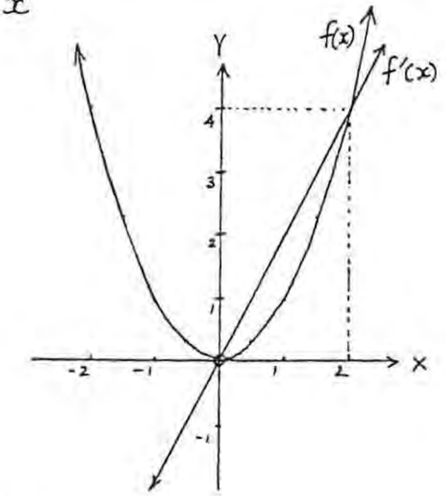
Applying the limiting process and consider what will happen to $2x+h$ as we make h smaller and smaller, letting it tend towards zero. It is the old story - we can make $2x+h$ as close as we like to $2x$ by making h sufficiently close to zero. And so the conclusion is that when

$f(x) = x^2$, $\qquad f'(x) = 2x$.

Exercise (1) in Worksheet C_6 required us to find the gradient of the tangent to the curve $y = x^2$ at the point where $x = 2$. Using the formula we found above, the answer is simply

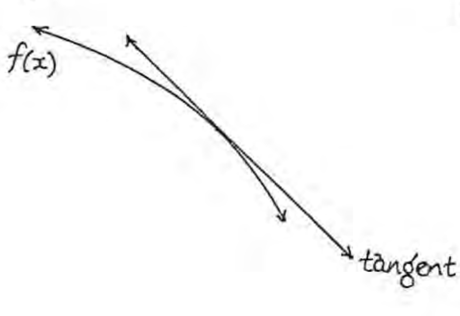
$$\underrightarrow{f'(2) = 2(2) = 4}.$$

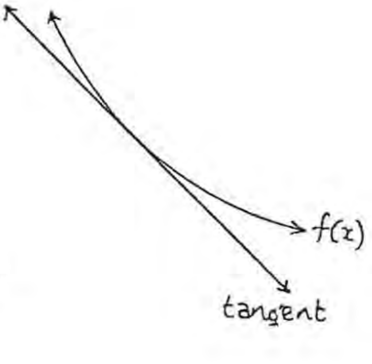
Before looking at another example, it is interesting to compare the graphs of $f(x) = x^2$ and $f'(x) = 2x$



Remembering that $f'(x)$ tells us about the gradient of $f(x)$, do you see the relationship between the two graphs?

When $f'(x)$ is negative, $f(x)$ is decreasing. $f'(x) < 0$ tells us that the tangent has a negative slope. As far as $f(x)$ is concerned, there are two possibilities - it is getting smaller either concavely or convexly:

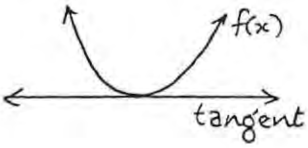


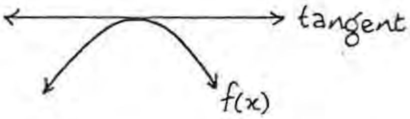


Page 3

A similar situation holds when $f'(x)$ is positive except, of course, that $f(x)$ is increasing.

When $f'(x) = 0$, the tangent is parallel to the x-axis and the function $f(x)$ is neither increasing or decreasing and one possible interpretation is that it is at a turning point:





Here are some more worked examples in finding derivatives from first principles:

(1.) If $f(x) = x^2 - x + 2$, find the value of $f'(x)$:

<u>Working</u>:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{[(x+h)^2 - (x+h) + 2] - [x^2 - x + 2]}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x - h + 2 - x^2 + x - 2}{h}$$

$$= \lim_{h \to 0} \frac{2xh - h + h^2}{h}$$

$$= \lim_{h \to 0} (2x - 1 + h)$$

$$\therefore f'(x) = 2x - 1$$

(2.) If $y = \frac{1}{x}$, find $f'(x)$.

<u>Working</u>:

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h \to 0} \frac{x - (x+h)}{hx(x+h)}$$

$$f'(x) = \lim_{h \to 0} \frac{x-x+h}{hx(x+h)}$$

$$= \lim_{h \to 0} \frac{h}{hx(x+h)}$$

$$= \lim_{h \to 0} \frac{1}{x(x+h)}$$

$$= \lim_{h \to 0} \frac{1}{x^2+xh}$$

$$\therefore f'(x) = \frac{1}{x^2}$$

It is most important that you become thoroughly familiar with this process. Here are some examples for you to practise on. The answers are given at the end of the exercise and examples marked with a * are for HG only.

Exercises: Find the derivative, $f'(x)$, for each of the following functions:

(1.) $y = 3$
(2.) $y = \frac{3}{x}$
* (3.) $y = \frac{2}{x^2}$
(4) $y = 3x$
(5.) $y = 5x-2$
(6.) $y = \frac{1}{2}x^2$
(7.) $y = -2x^2$
* (8.) $y = x^2 - 3x$
* (9.) $y = -3x^2 + 4x + 1$
* (10.) $y = \frac{1}{x+1}$
* (11.) $y = x^3$
* (12.) $y = 2x^3 - x^2 + 3x - 1$

ANSWERS: (1.) 0 ; (2.) $-\frac{3}{x^2}$; (3.) $-\frac{4}{x^3}$; (4) 3 ; (5.) 5 ; (6.) $6x$
(7.) $-4x$; (8.) $2x-3$; (9.) $-6x+4$; (10.) $\frac{-1}{(x+1)^2}$; (11.) $3x^2$; (12.) $6x^2-2x+3$

NOTATION: We have been using the symbol $f'(x)$ for the derivative of $f(x)$. There are, however, various other notations which are used for the derivative. Here are some you should know:

$D_x\, y$ or $D_x[f(x)]$; $\frac{d}{dx}y$ or $\frac{d}{dx}f(x)$; y' or $f'(x)$: $\frac{dy}{dx}$.

WORKSHEET C8

Some rules for differentiating

You will have noticed that there is still a fair bit of work involved in differentiating from first principles. Let us see what can be done about this. Most of the functions we have been working with have involved the function $f(x) = x^n$. Taking different values of n and finding $D_x[f(x)]$, try to discover a pattern which will enable you to differentiate the general function $f(x) = x^n$.

n	$f(x) = x^n$	$D_x[f(x)]$
-1	x^{-1}	
0	x^0	
1	x	
2	x^2	
3	x^3	
4	x^4	
5	x^5	
10	x^{10}	
n	x^n	

Hopefully you will have been able to find the pattern. If not, look again at the coefficient of x in the derivative and the value of the index and compare these with the index in the original function.

The general result is that if $f(x) = x^n$, then $D_x[f(x)] = n x^{n-1}$. Now the object of having rules to do mathematical procedures is to save us the necessity of having to work each example from first principles.

At this stage there are two more rules with which you must be familiar. These are:

(1.) $D_x\left[f(x) \pm g(x)\right] = D_x\left[f(x)\right] \pm D_x\left[g(x)\right]$

Be careful: this is NOT an example of the distributive property.

(2.) $D_x\left[k\, f(x)\right] = k\; D_x\left[f(x)\right]$ where k is a constant.

The first rule tells us that the derivative of the sum of two functions is equal to the sum of the derivatives of the individual functions. (We have already encountered examples of this type in the exercises in Worksheet C_7.) The second rule tells us that the derivative of a constant times a function is the constant times the derivative of the function. (We have also seen this in the previous worksheet.)

Example. Say we had to find the derivative of the function

$$y = 3x^2 + 2x - 1 .$$

Taking each term separately:

$$\left.\begin{aligned} D_x\left[3x^2\right] &= 3\,D_x(x^2) \\ &= 3(2x) \\ &= 6x \\ D_x\left[2x\right] &= 2\,D_x(x) \\ &= 2(1) \\ &= 2 \\ D_x\left[-1\right] &= 0 \end{aligned}\right\} \quad \begin{array}{l}\text{It is not necessary to show} \\ \text{this working in every example.}\end{array}$$

$$\underrightarrow{\therefore\ D_x\, y = 6x+2}$$

Look again at the examples at the end of Worksheet C_7 and this time try them by applying the rules.

Here is another opportunity to practise:

Exercises: Find the derivatives of the following functions

(1.) $y = \frac{3x}{2} + 1$

(2.) $y = \frac{3x+1}{2}$

(3.) $y = x^{\frac{3}{2}}$

(4.) $y = \sqrt{x}$

(5.) $y = 3 - x - 2x^2$

(6.) $y = x^{-2}$

(7.) $y = \frac{1}{x^3}$

(8.) $y = \frac{2}{x^3}$

(9.) $y = \frac{1}{2x^3}$

(10.) $y = x + \frac{1}{x}$

(11.) $y = \frac{1}{2}x^2 - 3x - 2$

(12.) $y = 2x(x-1)$

(13.) $y = -5x^3 + 3x^2 - 8x$

(14.) $y = 3x^3 - 5x^2 + 4x - 7$

(The answers are at the end of the worksheet.)

Unfortunately for pupils doing maths on the higher grade, they have to know the proofs for the two rules given above. These proofs are based on the definition, if $y = f(x)$, then

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = D_x\left[f(x)\right]$$

[Do you agree that $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ is the same as $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$?]

(1.) R.T.P. $D_x\left[f(x) \pm g(x)\right] = D_x\left[f(x)\right] \pm D_x\left[g(x)\right]$

Proof: Let $y = u \pm v$ where $u = f(x)$ and $v = g(x)$.

If x increases to $x + \Delta x$, u and v increase to $u + \Delta u$ and $v + \Delta v$ respectively and y to $y + \Delta y$

$\therefore \quad y + \Delta y = (u + \Delta u) \pm (v + \Delta v)$

but $\quad y = u \pm v$

and so by subtraction $\Delta y = \Delta u \pm \Delta v$.

Dividing by Δx $\quad \frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} \pm \frac{\Delta v}{\Delta x}$

and as $\Delta x \to 0$ $\quad D_x(y) = D_x(u) \pm D_x(v)$

$$\therefore \underline{D_x\left[f(x) \pm g(x)\right] = D_x\left[f(x)\right] \pm D_x\left[g(x)\right]}$$

(2.) R.T.P. $D_x[k\,f(x)] = k\,D_x[f(x)]$

Proof: Let $y = ku$ where k is a constant and $u = f(x)$

As u increases to Δu, y increases to $y + \Delta y$

$\therefore y + \Delta y = k(u + \Delta u)$

$\therefore y + \Delta y = ku + k\,\Delta u$

but $y = ku$

So by subtraction $\Delta y = k\,\Delta u$

dividing by Δx $\frac{\Delta y}{\Delta x} = k \cdot \frac{\Delta u}{\Delta x}$

and as $\Delta x \to 0$ $D_x(y) = k.\,D_x(u)$

$\therefore \underline{D_x[k\,f(x)] = k\,D_x[f(x)]}$

ANSWERS TO EXERCISES :

(1.) $\frac{3}{2}$ (2.) $\frac{3}{2}$ (3.) $\frac{3}{2}x^{\frac{1}{2}}$ (4.) $\frac{1}{2}x^{-\frac{1}{2}}$ (OR $\frac{1}{2\sqrt{x}}$)

(5.) $-1-4x$ (6.) $-2x^{-3}$ (or $-\frac{2}{x^3}$) (7.) $-3x^{-4}$ (OR $-\frac{3}{x^4}$)

(8.) $-6x^{-4}$ (OR $-\frac{6}{x^4}$) (9.) $\frac{-3x^{-4}}{2}$ (OR $-\frac{3}{2x^4}$) (10.) $1-\frac{1}{x^2}$

(11.) $x-3$ (12.) $4x-2$ (13.) $-15x^2+6x-8$ (14.) $9x^2-10x+4$

WORKSHEET C9

Tangents to curves

Using differentiation and the gradient-point formula for a straight line from co-ordinate geometry, we can now find the equation of any tangent to a given curve.

Example : Find the equations of the tangent and the normal (i.e. the line perpendicular to the tangent) to $y = x^2 - x + 2$ at $(2;4)$.

Working : First $f'(x) = 2x - 1$

$\therefore$ the gradient of the tangent at $(2;4)$ is $m = f'(2) = 3$

$\therefore$ the equation of the tangent is

$$y - 4 = 3(x-2)$$
$$\therefore y - 4 = 3x - 6$$
$$\therefore \underline{y = 3x - 2}$$

the gradient of the normal is $-\frac{1}{3}$ (since $m_T \times m_N = -1$)

$\therefore$ the equation of the normal is

$$y - 4 = -\tfrac{1}{3}(x-2)$$
$$\therefore y - 4 = -\tfrac{1}{3}x + \tfrac{2}{3}$$
$$\therefore \underline{y = -\tfrac{1}{3}x + \tfrac{14}{3}}$$

Exercises :

(1.) Find the equations of the tangent and the normal to $y = 2x^2 + 5x + 3$ at $(-1;0)$

(2.) Find the equations of the tangent and the normal to $y = \frac{1}{x}$ at $(2;\frac{1}{2})$

(3.) Find the equations of the tangent and the normal to

$y = x^3 - 2x^2 + 3x - 4$ at $(1; -2)$.

(4.) Find the equations of the tangents to the following curves at the points indicated:

(a) $y = x^3 - 2x^2 + 1$ at $(2; 1)$

(b) $y = x + \frac{2}{x}$ at $(-2; -3)$

(c) $y = \sqrt{x}$ at $(4; 2)$

(5) Find the equation of the tangent to the curve $y = x^3$ at the point $(1; 1)$. Find the point at which this tangent meets the curve again.

* (6.) Find the equations of the tangents to $y = x^2 - x - 2$ <u>from</u> the point $(1; -3)$.

ANSWERS:

(1.) $y = x + 1$ and $y = -x - 1$

(2.) $y = -\frac{x}{4} + 1$ and $y = 4x - \frac{15}{2}$

(3.) $y = 2x - 4$ and $y = -\frac{x}{2} - \frac{5}{2}$

(4.) (a) $y = 4x - 7$

(b) $y = \frac{x}{2} - 2$

(c) $y = \frac{x}{4} + 1$

(5.) $y = 3x - 2$ and they intersect again at $(-2; -8)$

(6.) $y = -x - 2$ and $y = 3x - 6$

Referring back to (1.), it is possible to find the equation of the tangent to $y = x^2 - x + 2$ at $(2; 4)$ using the theory of quadratics and NO calculus. Can you do it?

WORKSHEET C_{10}

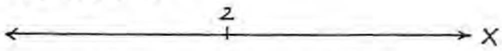
Curve Sketching

In Worksheet C_7 we compared the graph of a function and its derivative. It was pointed out that because the derivative gives the gradient of the tangent to the curve at any point on the curve, we can use it to deduce when a curve is increasing (i.e. has a positive gradient), decreasing (i.e. has a negative gradient) or is stationary (i.e. the gradient is zero).

Examples:

(1.) For a certain function, $y = f(x)$, $f'(x) = x - 2$. Describe the behaviour of the graph of $y = f(x)$.

$x - 2 = 0$ when $x = 2$. For values of x less than 2, $f'(x)$ is negative and for values of x greater than 2 it is positive. So, tabulating the conclusions:

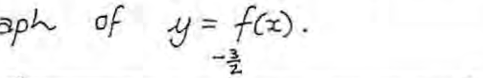


(A number line representing values of x.)

$f'(x)$	$-ve$	0	$+ve$
$\therefore$ $f(x)$ is	decreasing	stat.	increasing

Rough graph of $f(x)$

(2.) For a certain function, $y = f(x)$, $y' = -2x - 3$. Describe the behaviour of the graph of $y = f(x)$.



$f'(x)$	$+ve$	0	$-ve$
$\therefore f(x)$ is	increasing	stat.	decreasing

Rough graph of $f(x)$

(3.) For a certain function, $y = f(x)$, $D_x[f(x)] = x^2 - x - 6$. Describe the behaviour of the graph of $y = f(x)$.

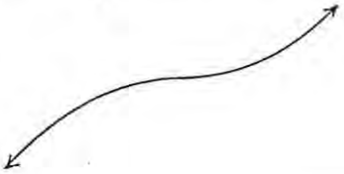
		-2		$+3$	X
$D_x[f(x)]$	+ve	0	−ve	0	+ve
$\therefore f(x)$ is	increasing	stat.	decreasing	stat.	increasing

Rough graph of $f(x)$

(4) For a certain function, $y = f(x)$, $f'(x) = x^2 - 2x + 1$. Describe the behaviour of the graph of $y = f(x)$.

		1	X
$f'(x)$	+ve	0	+ve
$\therefore f(x)$ is	increasing	stat.	increasing

Rough graph of $f(x)$



> Use your ruler to approximate the tangent to each of the above curves. Notice how its gradient corresponds to the value of the derivative.

Up to now we have only been concerned with what the derivative tells us about the general behaviour of the graph. Try the following exercises and then we will see about putting in some more detail.

Exercises: For each of the following derivatives, describe the behaviour of the graph of $y = f(x)$:

(1.) $f'(x) = x + 3$ (2.) $f'(x) = -x + 4$

(3.) $f'(x) = x^2 - 3x - 4$ (4.) $f'(x) = -x^2 + 6x - 9$

Page 3

We will now consider that extra detail that was mentioned. As our first example, let us take the graph of $y=x^2-2x-8$.

$f'(x) = 2x - 2$ which is zero when $x = 1$.

x		1		
$f'(x)$	$-ve$	0	$+ve$	
$\therefore f(x)$ is	decreasing	stat.	increasing	

So the graph is roughly:

We can see that the parabola has a minimum value when $f'(x)=0$, i.e. when $x=1$. To find the y-value at the turning point, calculate $f(1)$. This gives 9.

The y-intercept is -8 and the x-intercepts can be found in the usual way, giving $x = 4$ or -2.

This method is most useful for more complicated curves. Consider the function $y = x^3 - 7x - 6$.

Firstly $f'(x) = 3x^2 - 7$

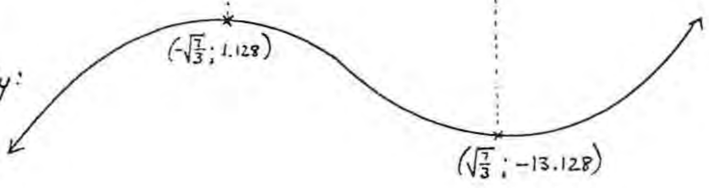
and $f'(x) = 0$ when $x = \pm\sqrt{\frac{7}{3}}$

Using a calculator, we can find the values of $f(\sqrt{\frac{7}{3}})$ and $f(-\sqrt{\frac{7}{3}})$ which turn out to be -13.128 and 1.128 respectively, accurate to 3 decimals.

Tabulating the implications of $f'(x)$ we get

x		$-\sqrt{\frac{7}{3}}$		$\sqrt{\frac{7}{3}}$	
$f'(x)$	$+ve$	0	$-ve$	0	$+ve$
$\therefore f(x)$ is	increasing	stat.	decreasing	stat.	increasing

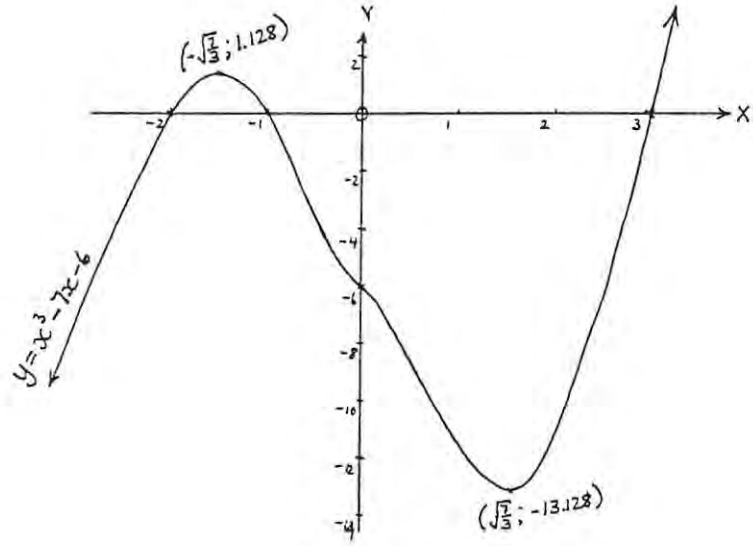
and so the graph is roughly:



Now for the intercepts with the axes:

Y-intercept : $f(0) = -6$, i.e. the graph cuts the y-axis at -6.

Roots: We can use the Factor Theorem to factorise the expression to $(x+1)(x+2)(x-3)$. So the graph cuts the x-axis at $-1, -2$ and 3. This gives us a reasonably full picture of the graph:



Exercises:

Sketch the graphs of the following functions indicating clearly where they cut the axes and the co-ordinates of any turning points:

(1.) $y = x^2 - x - 12$

(2.) $y = -\frac{1}{2}x^2 + x + 4$

(3.) $y = x^3 - 2x^2 + x$

(4.) $y = x^3 - 3x^2 + 3x - 2$

(5.) $y = -x^3 - 2x^2 + x + 2$

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