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INTERPOLATED LATTICE GAUGE FIELDS AND CHIRAL FERMIONS IN THE SCHWINGER MODEL

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ABSTRACT

The effective action induced by fermions in the chiral Schwinger model with charges (3,4,5) is investigated. Pauli-Villars regularization is combined with momentum cut-off for the evaluation of the fermion determinants on continuum gauge fields interpolated between lattice points. The convergence and gauge variance are studied numerically on gauge configurations taken from quenched updating.

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1 Introduction

The non-perturbative definition of chiral gauge theories is a long-standing problem of quantum field theory on the lattice (for reviews see, for instance, [1]–[4]). Recently there have been several interesting new proposals trying to circumvent the obstacles represented by the Nielsen-Ninomiya theorem [5] in different ways [6]–[9] (for further references see these papers).

In the present letter the idea put forward by 't Hooft [8] is applied to the simple test case of the two-dimensional chiral Schwinger model. It is based on the interpolation of the lattice gauge field and the definition of the chiral fermion determinant on the obtained continuum gauge field, by exploiting the knowledge accumulated in continuum approaches (for a review see [10]). Similar ways of defining chiral gauge theories on the lattice were discussed for some time [11, 12], and have been recently further developed in refs. [13, 14].

The aim of the present paper is to study numerically the definition of the effective action induced by chiral fermions on the interpolated lattice gauge field. As a first step, the two-dimensional massless chiral Schwinger model is considered here. Since this model is well known and exacly soluble (in the extensive literature see, for instance, the papers in ref. [15]), the questions are mainly oriented towards the qualitative behaviour of the calculation of the effective action along the line of refs. [8, 13, 14]. The methods used will be such that they can be extended to four dimensions in a straightforward way. In the next section the interpolation of the U(1) gauge field is discussed. This is followed by a short discussion of some useful numerical algorithms. In section 4 the convergence of the chiral fermion determinant is considered by removing the momentum cut-off. The variation with respect to gauge transformations is numerically investigated and discussed.

2 Gauge field interpolation

The lattice gauge field is defined by the parallel transporters on the discrete links of the lattice, which is chosen in the present paper, for simplicity, to be hypercubic with periodic boundary conditions. To extend the gauge field connection into the meshes of the lattice is highly arbitrary. In order to reduce this arbitrariness some guiding principles must be respected, such as smoothness and some minimality principle which chooses among the different possibilities. Since in chiral gauge theories the anomaly plays an important rôle, one can also connect the gauge field interpolation to the geometrical definition of the topological charge [16, 17]. Both this interpolation and the piece-wise linear one minimizing the Euclidean action, which has been proposed in [8], have the important property that in momentum space the support of the Fourier transform is concentrated near the momenta allowed for the gauge field on the lattice. This has to be required as a condition for any reasonable interpolation: the momentum cut-off imposed on the gauge field by the lattice has to be approximately maintained by the interpolated gauge field too.

Let us denote the U(1) gauge link variables on the lattice by

$$U_{x\mu} = \exp(igA_{x\mu}) \ . \tag{1}$$

Here x denotes lattice points: $x = (x_1, x_2)$ with integer x_μ satisfying $0 \le x_\mu \le L_\mu - 1$, $(\mu = 1, 2)$. The lattice extensions are denoted by L_μ, g is the bare gauge coupling and the number of lattice points will be denoted by $\Omega = L_1 L_2$. Note that throughout this paper the lattice spacing of the lattice for the gauge field is set to be $a = 1$. In other words, every dimensional quantity, as for instance the gauge field $A_{x\mu}$, is measured in lattice units of the gauge field lattice. The Fourier transformation to momentum space is defined, as usual, by

$$\tilde{A}_{k\mu} \equiv \sum_x e^{-ik\cdot x - \frac{i}{2}k_\mu} A_{x\mu} \ . \tag{2}$$

The inverse relation is

$$A_{x\mu} = \frac{1}{\Omega} \sum_k e^{ik\cdot x} \tilde{A}_{k\mu} \ , \tag{3}$$

where the sum is running, of course, on the points of the Brillouin zone

$$k_\mu = \frac{2\pi}{L_\mu} \nu_\mu \ , \qquad \nu_\mu = -\mathrm{int}(L_\mu/2), -\mathrm{int}(L_\mu/2)+1, \ \ldots \ , +\mathrm{int}(L_\mu/2) \ . \tag{4}$$

The above discussed condition on the interpolation means that, if the Fourier transformation of the interpolated gauge field is performed on the continuous torus, the Fourier coefficients are approximately the same as in (2). This suggests the introduction of eq. (3) as the definition of the interpolation by simply extending its validity to continuous x. This means that the interpolated gauge field on the continuum is, as a function of the continuous x_c:

$$A_\mu(x_c) \equiv \sum_x D_\mu(x_c - x) A_{x\mu} \tag{5}$$

with the *interpolation kernel*

$$D_\mu(x_c - x) \equiv \frac{1}{\Omega} \sum_k e^{ik\cdot(x_c - x) - \frac{i}{2}k_\mu} \ . \tag{6}$$

This is a very smooth interpolation indeed, since the result is infinitely many times differentiable (an entire function for complex x_c) satisfying, for integer x:

$$A_\mu(x + \hat{\mu}/2) = A_{x\mu} \ . \tag{7}$$

Here, as usual, $\hat{\mu}$ denotes the unit vector in direction μ.

Before going further, let us make a short technical remark. The Fourier coefficients in eq. (2) have the periodicity properties

$$\tilde{A}_{k+2\pi\hat{\mu},\mu} = -\tilde{A}_{k,\mu} \ , \qquad \tilde{A}_{k+2\pi\hat{\nu},\mu} = \tilde{A}_{k,\mu} \ , \tag{8}$$

where $\nu = 3 - \mu$. This allows us, for instance, to choose the symmetric interval for momenta in (4). Of course, for the summation in the definition of D_μ in (6) one has to specify the interval. In fact, for even L_μ it is advantageous to stick to an exactly symmetric definition by dividing the Fourier coefficient at $\nu_\mu = L_\mu/2$ into equal halfs at $\nu_\mu = L_\mu/2$ and $\nu_\mu = -L_\mu/2$. (For an odd L_μ the points in (4) are automatically symmetric.) In the numerical study discussed in section 4 this symmetric definition was always taken.

It must be emphasized that the interpolation defined by eqs. (5) and (6) is only one example among many others. It satisfies the important condition that the momentum cut-off for the gauge field be transferred from the lattice to the continuum. This is the only condition which will be exploited in what follows. This implies that the conclusions from the numerical study in section 4 will be qualitatively valid also for the interpolations defined in refs. [17, 8].

In fact, from the point of view of gauge covariance the above definition satisfying (7) is not optimal. Instead of it one can also require the alternative condition

$$\int_x^{x+\hat{\mu}} dy A_\mu(y) = A_{x\mu} \ . \tag{9}$$

This can be achieved, as one easily sees, by changing the definition of the continuation kernel in (6) to

$$D_\mu(x_c - x) \equiv \frac{1}{\Omega} \sum_k \frac{k_\mu}{\hat{k}_\mu} e^{ik\cdot(x_c - x) - \frac{i}{2}k_\mu} \ , \tag{10}$$

where $\hat{k}_\mu \equiv 2\sin(k_\mu/2)$. The consequence of eq. (9) is gauge covariance. Performing the gauge transformation on the lattice by the U(1) elements $\Lambda_x = \exp(i\alpha_x)$ and continuing α_x to a function $\alpha(x_c)$ on the continuum in such a way that, for integer $x_c = x$,

$$\alpha(x) = \alpha_x \ , \tag{11}$$

we obtain the relation for the gauge-transformed links

$$U^{(\Lambda)}_{x\mu} = \Lambda^{-1}_{x+\hat{\mu}} U_{x\mu} \Lambda_x = \exp\left\{ ig \int_x^{x+\hat{\mu}} dy A^{(\Lambda)}_\mu(y) \right\} \ . \tag{12}$$

Here $A^{(\Lambda)}_\mu(x)$ is the gauge-transformed continuum field

$$A^{(\Lambda)}_\mu(x) = A_\mu(x) - g^{-1}\partial_\mu \alpha(x) \ . \tag{13}$$

A convenient interpolation satisfying eq. (11) is given by

$$\alpha(x_c) \equiv \sum_x D(x_c - x)\alpha_x \tag{14}$$

with the continuation kernel for scalar fields, analogous to D_μ in (6) or (10),

$$D(x_c - x) \equiv \frac{1}{\Omega} \sum_k e^{ik\cdot(x_c - x)} \ . \tag{15}$$

Even if the gauge covariance relation in (12) is satisfied, the uniqueness of the gauge-field interpolation is still not guaranteed until it is not specified in which *reference gauge* the relation in (5) holds. (The gauge transformation with respect to this gauge is then given by (13), (14).) At this point maximal smoothness can be taken as a guiding principle, which suggests that we take the *Landau gauge* to be the reference. This gauge can be defined on the lattice by requiring

$$f_2[U] \equiv \sum_x \sum_{\mu=1}^{2} A^2_{x\mu} \tag{16}$$

to be minimal with respect to gauge transformations. For U(1) gauge fields there are efficient algorithms to find this absolute minimum, for instance the one discussed in [18], which will be used in the present paper.

The advantage of the gauge-field interpolation given by eqs. (5), (10) is its simplicity and the direct relation to momentum space, which will be useful for the evaluation of the determinant in momentum basis. Concerning topological charge, it defines a continuous gauge field on the torus which has a total classical topological charge zero. On large volumes this is not a serious constraint, because the parts of the volume can have any topological charge. In fact, the interpolated gauge field can be used to define a topological charge density operator with the appropriate renormalization procedure for composite operators. (See [19] and references therein.)

3 Computing the determinant

The Euclidean action for massless chiral fermions in the U(1) background gauge field $A_\mu(x)$ is given in the continuum by

$$S = \int d^2x \left\{ \overline{\psi}(x)\gamma_\mu \partial_\mu \psi(x) - igA_\mu(x)\overline{\psi}(x)\gamma_\mu \left(P_R Q_R + P_L Q_L\right)\psi(x) \right\} . \tag{17}$$

Here $P_R \equiv (1+\gamma_3)/2$ and $P_L \equiv (1-\gamma_3)/2$ are the chiral projectors for right-handed and left-handed fermions, respectively. We shall use the Pauli matrices for the γ-matrices in two dimensions: $\gamma_\mu \equiv \sigma_\mu$, $(\mu = 1,2,3)$. Q_R and Q_L are the charges of the chiral fermion components. Note that in (17) we implicitly adopt the "doubling trick" [20, 21]: even if one of the charges $Q_{R,L}$ is zero, we use a Dirac fermion field. In this way the chiral fermion determinant is always a *determinant* indeed.

The fermion matrix in momentum space corresponding to eq. (17) is

$$M^{Q_R,Q_L}_{k_2 k_1} = \Omega \delta_{k_2 k_1} i\gamma \cdot k_1 - ig\gamma_\mu \left(P_R Q_R + P_L Q_L\right) \tilde{A}_{k_2-k_1,\mu} . \tag{18}$$

After multiplication by the fermion propagator we obtain

$$N^{Q_R,Q_L}_{k_2 k_1} \equiv M^{Q_R,Q_L}_{k_2 k_1} \frac{(-i\gamma \cdot k_1)}{\Omega k_1^2} \equiv \delta_{k_2 k_1} - K^{Q_R,Q_L}_{k_2 k_1} . \tag{19}$$

This has the following matrix elements in spinor indices:

$$\begin{aligned}
N^{Q_R,Q_L}_{k_2 k_1}(1,1) &= \delta_{k_2 k_1} - Q_L(a_{k_2-k_1,1} - ia_{k_2-k_1,2})(k_{1,1} + ik_{1,2})/k_1^2 , \\
N^{Q_R,Q_L}_{k_2 k_1}(1,2) &= 0 , \\
N^{Q_R,Q_L}_{k_2 k_1}(2,1) &= 0 , \\
N^{Q_R,Q_L}_{k_2 k_1}(2,2) &= \delta_{k_2 k_1} - Q_R(a_{k_2-k_1,1} + ia_{k_2-k_1,2})(k_{1,1} - ik_{1,2})/k_1^2 .
\end{aligned} \tag{20}$$

Here the explicit representation of the γ-matrices and the short notation $a \equiv g\tilde{A}/\Omega$ is used. For a massive vector-like fermion with mass m and $Q \equiv Q_R = Q_L$, which will be used for Pauli-Villars fields, the matrix elements corresponding to (20) are:

$$N^{Q(m)}_{k_2 k_1}(1,1) = \delta_{k_2 k_1} - Q(a_{k_2-k_1,1} - ia_{k_2-k_1,2})(k_{1,1} + ik_{1,2})/(m^2 + k_1^2) ,$$

$$N^{Q(m)}_{k_2 k_1}(1,2) = -imQ(a_{k_2-k_1,1} - ia_{k_2-k_1,2})/(m^2+k_1^2) \; ,$$
$$N^{Q(m)}_{k_2 k_1}(2,1) = -imQ(a_{k_2-k_1,1} + ia_{k_2-k_1,2})/(m^2+k_1^2) \; ,$$
$$N^{Q(m)}_{k_2 k_1}(2,2) = \delta_{k_2 k_1} - Q(a_{k_2-k_1,1} + ia_{k_2-k_1,2})(k_{1,1} - ik_{1,2})/(m^2+k_1^2) \; . \tag{21}$$

For the computation of the determinants of these matrices in momentum basis an appropriate algorithm is the LU (lower-upper triangular) decomposition (see, for instance, [22]). It turned out to be both robust and sufficiently fast on the lattices considered. It can also be used for the computation of the full inverse matrix, and the algorithm can be organized in such a way that the matrix has to be stored only once. Of course, storing these large matrices even only once is the main limiting factor of the computation. For very large matrices also the time requirement is growing dangerously: it behaves as the third power of the matrix extension.

In order to extend the range of feasible lattice sizes one can exploit some additional iterative procedures. Before describing them let us discuss the momentum cut-off scheme used. Since, according to the previous section, the Fourier components of the gauge field are constrained to the points (4) of the Brillouin zone belonging to the gauge field lattice, it is natural to use a momentum cut-off for the calculation of the determinants of the infinite-dimensional matrices in (19)–(21). One can imagine to make the lattice finer for the fermions by adding more points to the gauge field lattice. In this case, however, the periodicity in momentum components is maintained, which introduces some non-zero elements also near the upper right and lower left corner, besides the ones near the main diagonal, for which $a_{k_2-k_1,\mu} \neq 0$. This makes the effect of the cut-off stronger, therefore it is more advantageous to abandon periodicity and drop the extra non-zero elements. In this way, for momentum cut-offs much larger than π (in gauge field lattice units), the matrix has a band structure.

As a consequence of this band structure, one can effectively apply the iterative algorithm previously used for the numerical hopping parameter expansion in QCD [23]. For this one determines the traces of the powers of the *hopping matrix* K (here in momentum space). Having these traces one can use either the usual infinite expansion

$$\det(1-K) = \exp\left\{ -\sum_{j=1}^{\infty} \frac{\mathrm{Tr}\, K^j}{j} \right\} \; , \tag{22}$$

or the finite *polymer representation* [24]

$$\det(1-K) = 1 + \sum_{\nu=1}^{n} \sum_{r=1}^{\nu} \frac{(-1)^r}{r!} \sum_{\rho_1=1}^{\nu-r+1} \ldots \sum_{\rho_r=1}^{\nu-r+1} \delta_{\nu,\rho_1+\ldots+\rho_r} \frac{\mathrm{Tr}\, K^{\rho_1}}{\rho_1} \frac{\mathrm{Tr}\, K^{\rho_2}}{\rho_2} \ldots \frac{\mathrm{Tr}\, K^{\rho_r}}{\rho_r} \; . \tag{23}$$

This latter is always convergent because it is a sum of a finite number of terms. In practical calculations one can go without any problems to $j_{\max} \simeq 100$ in eq. (22) or to $n_{\max} \simeq 40$ in eq. (23). As a consequence of the band structure of the matrices K in (19)–(21), the storing of the full matrices is not necessary, and the computational load is growing as the second power of the matrix extensions times $j^2_{\max}$ or $n^2_{\max}$.

Inspection of the matrices in (19)–(21) shows that only the traces of even powers are non-zero. One can also easily see that

$$\left[\mathrm{Tr} \left(K^{Q_R,Q_L} \right)^{\rho} \right]^* = \mathrm{Tr} \left(K^{Q_L,Q_R} \right)^{\rho} \; . \tag{24}$$

This corresponds to the relation

$$\left[\det N^{Q_R,Q_L}\right]^* = \det N^{Q_L,Q_R} \ . \tag{25}$$

Therefore the determinant of the vector-like fermion $\det N^{Q(m)}$ is real. (One can also easily prove that it is non-negative.)

Concerning the practical convergence of the *trace expansions* in (22) and/or (23) in the chiral Schwinger model with charges (3,4,5) the experience is negative. Typically neither of them converges, because in the calculated range the contributions rapidly increase. This is mainly the consequence of the large value of the charges (see next section). One can, however, easily save their advantages by calculating the determinant and inverse of the matrices N truncated to a smaller sublattice (typically of the same size as the lattice for the gauge field), and then use

$$\det N = \det N_{\text{small}} \det[N_{\text{small}}^{-1}(1-K)] = \det N_{\text{small}} \det(1-K_{\text{new}}) \ , \tag{26}$$

with

$$K_{\text{new}} \equiv 1 - N_{\text{small}}^{-1}(1-K) \ . \tag{27}$$

It turns out that the expansions in the traces of the powers of K_{new} converge rapidly. The omitted higher-order terms could always be kept smaller than 10^{-8}, relative to the result.

4 Convergence and gauge variance

The commonly considered example of an anomaly-free chiral Schwinger model has fermion charges $(Q_R = 3, Q_L = 0)$, $(Q_R = 4, Q_L = 0)$ and $(Q_R = 0, Q_L = 5)$. (The anomaly is cancelled if the sum of squared charges of the right-handed fermions is equal to those of the left-handed ones.) In order to regulate the ultraviolet divergence of the two-point function in a gauge-invariant way, one can introduce a Pauli-Villars vector-like fermion field with charge $Q = 5$ [25]. In this way the effective action S_{eff} induced by the fermions is given by

$$\exp\{-S_{\text{eff}}(M_{PV})\} \equiv E_{\text{eff}}(M_{PV}) = \det N^{3,0} \det N^{4,0} \det N^{0,5} / \det N^{5(M_{PV})} \ . \tag{28}$$

Here M_{PV} is the mass of the Pauli-Villars regulator field in units of the gauge field lattice. As argued in ref. [14], in the continuum limit M_{PV} should be kept finite, for instance of order 1. This is necessary in order to maintain the possibility of a simple renormalization.

Some insight into the behaviour of the effective action defined by eq. (28) can be obtained by numerically evaluating the determinants on some typical gauge configurations taken from Monte Carlo updating. For this I took quenched updating by the usual compact U(1) gauge field Wilson action. The gauge coupling has been fixed by $\beta \equiv g^{-2} = 8$, which is a quite strong coupling for these fermions. Namely, the interaction strength is given by Qg, therefore weak couplings are beyond $\beta \simeq 25$. The gauge configurations were transformed to Landau gauge by the algorithm described in [18]. The gauge field lattices were either $4 \otimes 4$ or $10 \otimes 10$. The lattices defining the momentum cut-off for the evaluation of the determinants always had an odd number of points (see discussion after eq. (8)), and they went up to $61 \otimes 61$. This means that the momentum cut-offs went up roughly to 15π (in units of the gauge field lattice). For the Pauli-Villars mass values between $M_{PV} = \frac{1}{2}$ and $M_{PV} = \pi$ were tried.

A first important question is how fast the infinite cut-off limit is reached by the effective action. It turned out that for every considered gauge configuration, taken randomly from the updating and transformed to Landau gauge, a good convergence could be achieved with the above cut-offs, provided that M_{PV} was not too large. For an illustration on $4 \otimes 4$ lattice see fig. 1. The numerical results for nine configurations on $10 \otimes 10$ are shown in table 1.

It is interesting to investigate the gauge dependence of the determinants. The gauge transformation of the infinite-dimensional fermion matrix M in (18) is given by $\Lambda^\dagger M \Lambda$, where in momentum space

$$\Lambda_{k_2 k_1} = \frac{1}{\Omega} \int d^2x e^{ix\cdot(k_1-k_2)+i\alpha(x)} = \delta_{k_2 k_1} + \frac{i}{\Omega}\tilde{\alpha}_{k_2-k_1} + \frac{i^2}{\Omega^2}\sum_k \tilde{\alpha}_k \tilde{\alpha}_{k_2-k_1-k} + \dots\ , \qquad (29)$$

with $\tilde{\alpha}_k$ denoting the Fourier components of $\alpha(x)$. This infinite-dimensional unitary matrix is truncated by the momentum cut-off. Therefore, gauge invariance of the fermion determinant is lost even for vector-like fermions, which were gauge-invariant without truncation. The chiral fermion determinants remain gauge non-invariant also for infinite momentum cut-off.

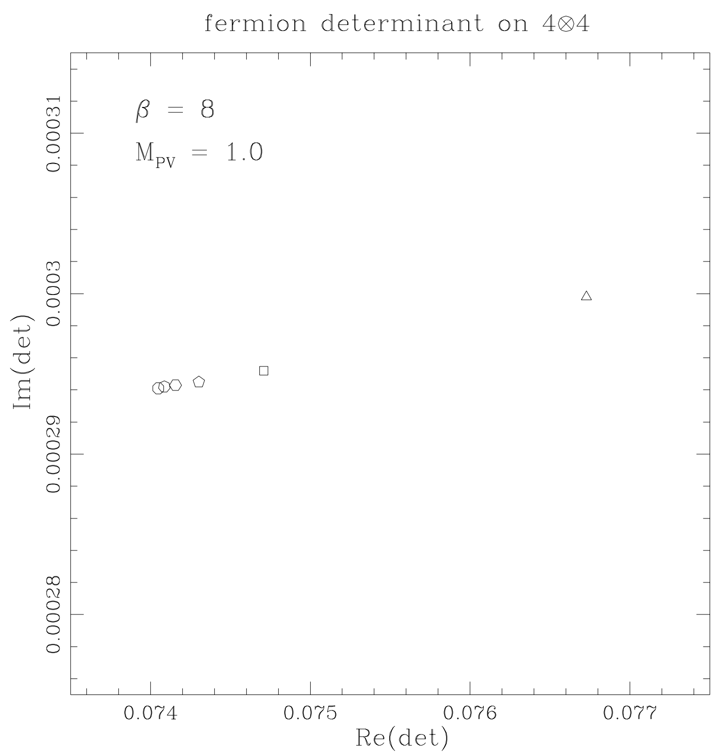

Figure 1: The values of $E_{\text{eff}}(1)$ in the complex plane on a $4 \otimes 4$ gauge field with momentum cut-off on $11 \otimes 11$ (triangle), $21 \otimes 21$ (quadrangle), etc., up to $61 \otimes 61$ (eight-angle).

The variation with gauge transformations is displayed in figure 2 for ten $\beta = 8$ gauge configurations on $10 \otimes 10$ lattice. The configurations were first transformed to Landau gauge and

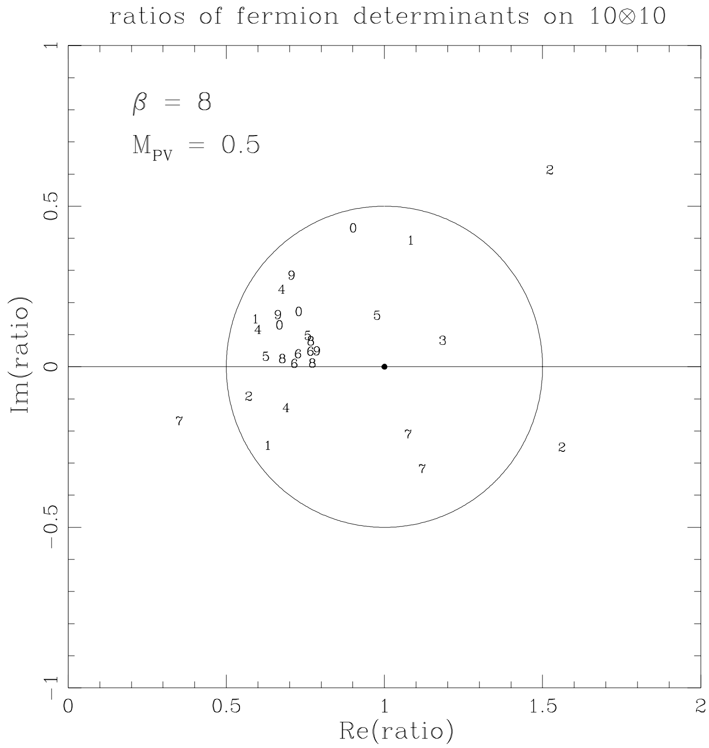

Figure 2: The residual gauge variance of the effective action shown by the (complex) ratios of $E_{\rm eff}(\frac{1}{2})$ atfer and before gauge transformation. The numbers are labeling different configurations. The circle with radius $\frac{1}{2}$ around the point (1,0) is drawn to guide the eyes.

then random gauge transformations were performed with parameters satisfying on the lattice points $-\pi/20 < \alpha_x < \pi/20$. The gauge transformations were interpolated in the continuum as described in section 2. The determinants were calculated on $31 \otimes 31$ (momentum cut-off $= 3.1\pi$). As is shown in the figure, the gauge variation is not very strong. In fact, both numerator and denominator of $E_{\rm eff}(\frac{1}{2})$ in (28) always change by 4–5 orders of magnitude, but the ratio remains close to 1. Performing random gauge transformations with larger magnitude (up to $-\pi < \alpha_x < \pi$) shows an ever-increasing change in numerators and denominators, such that it becomes difficult to keep the numbers in the computer, but the overwhelming part of the variation is cancelled in the ratio. The cancellation can be further improved by taking more Pauli-Villars fields with appropriately chosen larger masses.

There are two possibilities for dealing with this residual gauge variation of the effective action. First, one can try to tolerate it, keeping the momentum cut-off finite in gauge field lattice units. Second, more radically, one can enforce exact gauge invariance by defining the effective action to be equal to its value in Landau gauge along the whole gauge transformation orbit. The hope is that at the end, in the continuum limit, both these procedures lead to the same well defined theory.

The gauge-field interpolation combined with momentum cut-off for the evaluation of the Pauli-Villars regulated determinants seems to work reasonably well in the (3,4,5) chiral Schwinger model. It can be expected that the effective action defined in this way leads to a well defined continuum limit. Of course, momentum cut-off is not the only possibility. Examples of other possibilities are, for instance, to take a finer lattice (in coordinate space) for fermions and to use the formalism of ref. [21] for the imaginary part of the effective action as suggested in [12], or to take on the finer fermion lattice the SLAC derivative, as proposed by ref. [7]. One has to see which one of these (or some other) approaches has the most conceptual and practical advantages.

Table 1: The values of the determinants on $10 \otimes 10$ gauge field configurations with momentum cut-off 5.1π. The first line for a given configuration label is the value on the "small" subspace $\det N_{\mathrm{small}}$, the second line the correction factor obtained by trace expansion. For the configurations above the double line N_{small} is with momentum cut-off 2.1π, below it with 3.1π. The complex numbers are given by pairs in parentheses.

	$\det N^{3,0}$	$\det N^{4,0}$	$\det N^{0,5}$	$\det N^{5(\frac{1}{2})}$
1	$(2.1481, -0.8303)$	$(0.3880, 0.5783)$	$(-68.1036, -40.6948)$	3.5545×10^{10}
	$(1.0216, -0.0055)$	$(1.0364, -0.0288)$	$(1.0541, 0.0308)$	1.3258
2	$(0.4173, -0.2723)$	$(0.2104, -0.2211)$	$(-0.0557, -0.1826)$	5.5351×10^{8}
	$(1.0374, -0.0053)$	$(1.0672, -0.0148)$	$(1.1058, 0.0367)$	1.4123
3	$(0.0987, 0.0139)$	$(0.0169, 0.0081)$	$(0.0020, -0.0020)$	5.6234×10^{4}
	$(1.0482, -0.0000)$	$(1.0892, -0.0040)$	$(1.1457, 0.0155)$	1.4466
4	$(0.6872, 0.2887)$	$(1.2979, 0.1028)$	$(1.2729, 0.0435)$	9.3387×10^{4}
	$(1.0303, 0.0009)$	$(1.0561, 0.0052)$	$(1.0929, -0.0166)$	1.3091
5	$(0.3724, 0.0593)$	$(0.1504, -0.0241)$	$(0.0073, 0.0931)$	1.3400×10^{4}
	$(1.0268, -0.0073)$	$(1.0458, -0.0117)$	$(1.0673, 0.0132)$	1.2518
6	$(0.2303, 0.0165)$	$(0.0764, 0.0111)$	$(0.0156, -0.0025)$	2.4780×10^{1}
	$(1.0271, -0.0008)$	$(1.0482, -0.0004)$	$(1.0751, -0.0007)$	1.2412
7	$(1.3350, -0.3054)$	$(2.0516, 0.3677)$	$(-1.1129, -17.2413)$	3.6316×10^{10}
	$(1.0474, 0.0084)$	$(1.0884, 0.0262)$	$(1.1481, -0.0658)$	1.5387
8	$(0.00956, -0.00109)$	$(0.000327, -0.000086)$	$(0.0000037, 0.0000019)$	5.920×10^{8}
	$(1.0402, -0.0048)$	$(1.0322, -0.6017)$	$(0.8859, -0.1220)$	1.3736
9	$(0.00945, 0.00077)$	$(0.000336, 0.000054)$	$(0.0000042, -0.0000024)$	1.600×10^{8}
	$(1.0404, -0.0009)$	$(1.0257, -0.0070)$	$(0.7122, 0.2096)$	1.3379

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