FTD-TT65-289

TRANSLATION

COLLECTION OF PROBLEMS ON THE THEORY OF AUTOMATIC CONTROL

By

V. A. Besekerskiy, I. P. Pal'tov, et al.

FOREIGN TECHNOLOGY DIVISION

AIR FORCE SYSTEMS COMMAND
WRIGHT-PATTERSON AIR FORCE BASE
OHIO

This translation was made to provide the users with the basic essentials of the original document in the shortest possible time. It has not been edited to refine or improve the grammatical accuracy, syntax or technical terminology.

FTD-TT- 65-289/1+2

UNEDITED ROUGH DRAFT TRANSLATION

COLLECTION OF PROBLEMS ON THE THEORY OF AUTOMATIC CONTROL

BY: V. A. Besekerskiy, I. P. Pal'tov, et al.

English pages: 375

PREPARED BY:

TRANSLATION DIVISION
FOREIGN TECHNOLOGY DIVISION
WP-AFB, OHIO.

SBORNIK ZADACH PO TEORII AVTOMATICHESKOGO REGULIROVANIYA

V. A. Besekerskiy, I. P. Pal'tov, Ye. A. Fabrikant,
S. M. Federov and P. I. Chinayev

Pod Redaktsiyey

V. A. Besekerskogo

Gosudarstvennoye Izdatel'stvo

Fiziko-Matematicheskoy Literatury

Moskva 1963

FTD-TT-65-289/1+2

Published by the Ministry of Higher and Intermediate Special Education of the RSFSR as a Textbook for Students in Departments of Power and Electrical Engineering and Higher Educational Institutions

TABLE OF CONTENTS

PREFACE

This collection is designed for students specializing in the field of automatic control and servosystems, but it may also be used by students in other specialities when studying automatic control theory. Since there is at present no one single text in automatic control theory, this collection is not a supplement to any one book, but is oriented toward several books used in higher educational institutions; a list of these books has been given at the end of the collection.

The authors also considered it desirable to give a few problems whose solution requires reference to the journal literature. These problems are indicated by an asterisk.

The problem area covered by this collection includes topics found, as a rule, in the automatic control theory syllabus used in many higher educational institutions. They include such topics as "Equations of Motion of Automatic Control Systems (s.a.r.)," "Construction of Frequency-Response Curves," "Stability Studies," "Construction of Transient-Process Curves," "Judging Control Quality," " Random Processes in s.a.r.," "Nonlinear Systems," and, etc. In addition, the collection contains problems dealing with design methods for automatic control systems; they are intended to aid the student in independent performance of assignments, as well as for course and diploma design projects. These problems are directed toward the topics "Choice of s.a.r. Parameters from Required Accuracy," "Choice s.a.r. Parameters from Required Dynamic Properties," "s.a.r. Correction," etc.

The collection also contains problems from several new branches of

BLANK PAGE

automatic control theory: "Discrete Control Systems," "Linked Systems," and "Adaptive Systems."

Chapters 5, 6, 7, 8 and 13 were written by V.A. Besekerskiy, Chapters 3 and 9 by I.P. Pal'tov, Chapters 2 and 4 by Ye.A. Fabrikant, Chapters 10, 11, and 12 by S.M. Fedorov, and Chapters 1 and 14 by P.I. Chinayev.

The authors will be grateful for all critical comments on the contents of this collection.

The Authors

Manuscript Page No.	[Transliterated Symbols]
1	c.a.p = s.a.r = sistema avtomaticheskogo regulirovaniya = automatic control system

Part I

ANALYSIS OF AUTOMATIC CONTROL SYSTEMS

Chapter 1

EQUATIONS OF MOTION OF AUTOMATIC CONTROL SYSTEMS

§1. SETTING UP THE INITIAL DIFFERENTIAL EQUATIONS AND THEIR LINEARIZATION

1. Set up the linearized equation of motion for the centrifugal sensing element whose diagram is shown in Fig. 1, and find its transfer function.

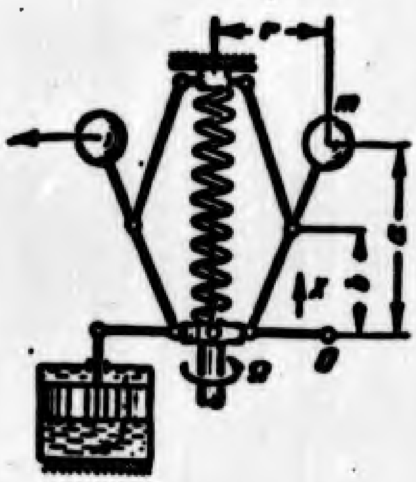


Fig. 1. Centrifugal sensing element.

Solution. The input variable of the centrifugal device is the angular velocity Ω, and the output variable the translation X of the coupling.

Let us set up the equation for the forces acting on the coupling. The moving force is the centrifugal force acting on the weights. It is opposed by: the elastic force of the spring, damping forces, and inertial forces. Let us express these forces in terms of the input and output variables, their velocities, and their accelerations. At the same time, we shall express all forces in terms of the coupling center of mass:

1) $F = 2\,\frac{a}{b}\,rm\Omega^2$ is the centrifugal force due to the weights, reduced to the coupling center of mass;

2) $F_p = cX$ is the spring force;

3) $F_u = m_u \frac{d^2X}{dt^2}$ is the inertial force due to the translating masses;

4) $F_d = S(P_1 - P_2)$ is the force developed by the hydraulic damper.

On the basis of the law of equilibrium of forces, we have

$$m_p \frac{d^2X}{dt^2} + S(P_1 - P_2) + cX = 2m \frac{a}{b} r\Omega^2. \qquad (1)$$

The quantities contained in this equation have the following meanings: m_p is the mass of the translating parts, reduced to the coupling center of mass, $\underline{m}$ is the mass of the weight; S is the effective piston area; P_1 and P_2 are the pressures of the liquid above and below the piston; $\underline{r}$ is the distance of the weight center of mass from the rotational axis; $\underline{a}$ and $\underline{b}$ are the distances of the points of application of the centrifugal forces, real and reduced, from the point O.

In the case considered, the relative rate of motion of the liquid through the vent is proportional to the velocity of the coupling. Thus the damping force may be written as

$$F_g = S(P_1 - P_2) = \frac{SR\mu}{d} \frac{dX}{dt},$$

where R is the Reynolds number, μ is the dynamic viscosity, and $\underline{d}$ is the vent diameter.

The centrifugal force F entering into Eq. (1) is a nonlinear function of X and Ω.

We linearize, letting $\Omega = \Omega_0 + \Delta\Omega$, $F = F_0 + \Delta F$, and $X = X_0 + x$, where Ω_0 is the initial velocity, and X_0 is so chosen that for an initial velocity $\Omega = \Omega_0$, the condition $x = 0$ is satisfied; we then obtain

$$\Delta F = 2m \frac{a}{b} \Omega_0^2 \Delta r + 4m \frac{a}{b} r_0 \Omega_0 \Delta\Omega.$$

Taking into account the kinematic link between $\underline{r}$ and X (Fig. 2), we obtain $\Delta r = \frac{h_2}{h_1} x$.

Thus,

$$\Delta F = 2m \frac{a}{b} \Omega_0^2 \frac{h_2}{h_1} x + 4m \frac{a}{b} r_0 \Omega_0 \Delta\Omega.$$

Going over in the initial equation to deviations and substituting the force increments found, we obtain the equation of the centrifugal sensing element in deviations:

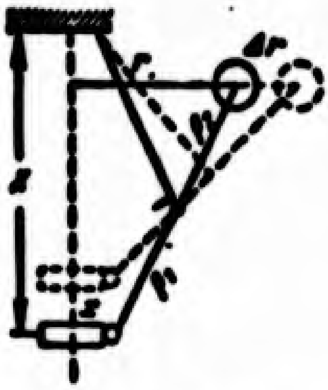


Fig. 2. Determining the geometric relationships in a centrifugal sensing element.

$$m_0 \frac{d^2x}{dt^2} + \frac{SR_p}{d}\frac{dx}{dt} +$$
$$+\left(c - 2m\frac{a}{b}\Omega_0^2\frac{r_0}{r_1}\right)x = 4m\frac{a}{b}r_0\Omega_0\Delta\Omega.$$

Letting

$$T^2 = \frac{m_0}{c - 2m\frac{a}{b}\Omega_0^2\frac{r_0}{r_1}},$$
$$\xi = \frac{SR_p}{2dT\left(c - 2m\frac{a}{b}\Omega_0^2\frac{r_0}{r_1}\right)},$$
$$k = \frac{4m\frac{a}{b}r_0\Omega_0}{c - 2m\frac{a}{b}\Omega_0^2\frac{r_0}{r_1}}$$

and going over to the transforms, we finally obtain

$$(T^2p^2 + 2\xi Tp + 1)x = k\Delta\Omega. \quad (2)$$

The transfer function is

$$W(p) = \frac{x}{\Delta\Omega} = \frac{k}{1 + 2\xi Tp + T^2p^2}. \quad (3)$$

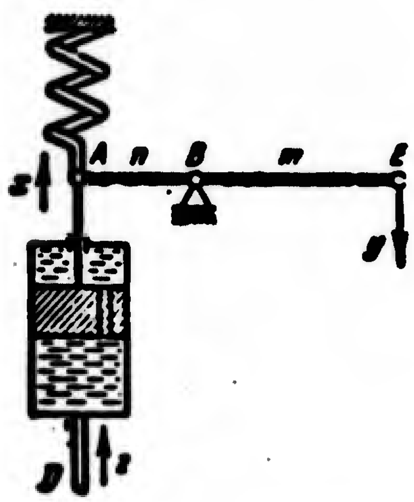


Fig. 3. Hydraulic differentiating device.

2. Set up an operator equation for the motion of a hydraulic differential device (Fig. 3) and find its transfer function under the following initial conditions: spring stiffness c = 5 kg/cm; lever length n = 8 cm and m = 16 cm, vent diameter d = 0.06 cm, piston diameter D = 6 cm, rod diameter δ = 0.6 cm, dynamic viscosity of the liquid μ = 0.9 kg·sec/m^2, Reynolds number R = 2000.

Solution. The following forces act on the piston:

1) a damping force

$$F_g = S(P_2 - P_1),$$

where $S = \frac{\pi D^2}{4}\left(1 - \frac{\delta^2}{D^2}\right)$ is the effective piston area, and $(P_2 - P_1)$ is the difference in the pressures above and below the piston.

2) a spring force

$$F_s = cx.$$

3) an inertial force whose effect we neglect.

From the Bernoulli equation, as in our solution to Problem 1, we write

$$F_a = \frac{R_0 S}{d}(v_2 - v_1),$$

where v_1 is the rate at which point D moves:

$$v_1 = \frac{dx}{dt},$$

while v_2 is the rate at which point A moves:

$$v_2 = \frac{dz}{dt}.$$

Let us now write the law governing the equilibrium of forces acting on the piston:

$$F_a + F_s = 0.$$

After substituting the values of the forces and dividing by the spring stiffness, we have

$$\frac{R_0 S}{cd}\frac{dx}{dt} + x = \frac{R_0 S}{cd}\frac{dz}{dt}.$$

If we take the translation of point E as the output variable:

$$y = \frac{m}{n}x = kx,$$

then the equation of motion is obtained in the form

$$(Tp + 1)y = kTpz, \qquad (1)$$

where the time constant of the device is

$$T = \frac{\pi D^2}{4}\frac{R_0\left(1 - \frac{d^2}{D^2}\right)}{cd}.$$

After substitution of the numerical values, we obtain T = 1.7 sec and k = 2. The transfer function is

$$W(p) = \frac{y}{z} = \frac{kTp}{1 + Tp} = \frac{3.4p}{1 + 1.7p}.$$

3. Set up the linearized equation for the carbon-pile voltage regulator shown in Fig. 4. In this figure, 1 is the carbon pile, 2 are laminated springs, 3 is the armature, 4 is the coil, 5 is the core. For the input variable we take the voltage across the coil 4, and as the output variable the change in the resistance of the carbon pile 1.

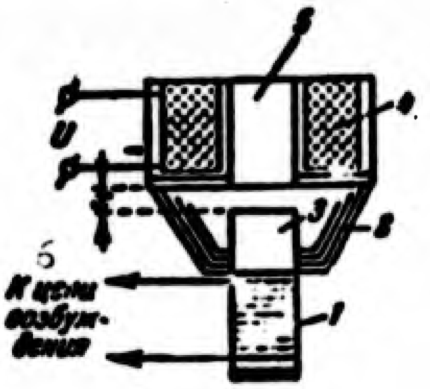


Fig. 4. Carbon-pile voltage regulator. 6) To excitation circuit.

Solution. We set up the equation for the electrical part of the regulator. On the basis of the second Kirchoff law, for the circuit of coil 4

$$L\frac{dI}{dt}+RI=U,$$

where L is the inductance of the coil circuit. R is the impedance of the coil circuit, U is the applied voltage, and I the coil current.

From this we obtain

$$(Tp+1)I=\frac{U}{R}.$$

The time constant of the coil circuit is $T = L/R$. This equation will also hold for deviations from some steady operating regime; they are determined by the expressions $I = I_0 + i$ and $U = U_0 + u$, i.e.,

$$(Tp+1)i=\frac{u}{R}. \qquad (1)$$

We now find the equation for the mechanical part of the carbon-pile regulator. The carbon pile is compressed by the force

$$F=F_n-k_1I^2,$$

where F_n is the initial compressing force, while k_1 is a coefficient of proportionality. Linearizing this equation, we obtain

$$\Delta F=\frac{\partial F}{\partial I}i=-2k_1I_0i \qquad (2)$$

for small increments.

Figure 5 shows the carbon-pile resistance as a function of the

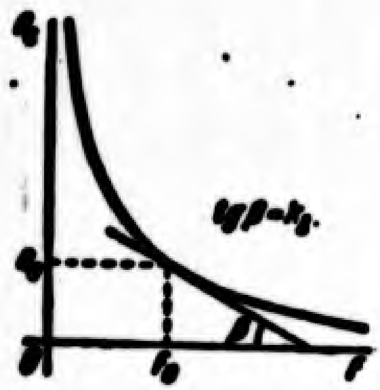


Fig. 5. Characteristic curve of carbon pile.

compressive force. For small deviations from a certain steady state ($F = F_0$ and $R_s = R_0$), we can write

$$\Delta R_s = -k_2 \Delta F, \qquad (3)$$

where k_2 is the absolute value of the slope of the characteristic curve at the point $F = F_0$.

Taking into account (1), (2), and (3), we finally have

$$(Tp + 1)\Delta R_s = ku, \qquad (4)$$

where

$$k = \frac{2k_1 k_2 l_0}{R}.$$

The transfer function is

$$w(p) = \frac{\Delta R_s}{u} = \frac{k}{1 + Tp}.$$

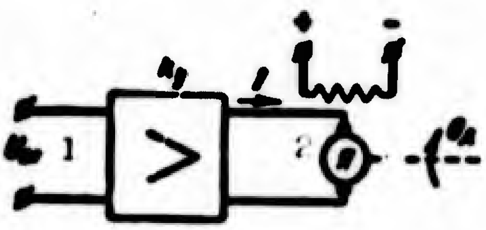


Fig. 6. Control circuit of armature-controlled motor. 1) U_{vkh}; 2) armature.

4. Set up the equation of motion of a DC electric motor armature-controlled with the aid of an amplifier having a voltage gain k_y. The circuit is shown in Fig. 6.

<u>Answer.</u> When the load torque equals zero, the equation of motion may be written in the form

$$(T_y T_M p^2 + T_M p + 1)p\vartheta_d = kU_{vkh}. \qquad (1)$$

The electromagnetic time constant of the armature circuit is

$$T_y = \frac{L_{ya}}{R_{ya}},$$

where L_{ya} and R_{ya} are the resultant inductance and armature-circuit resistance, with allowance for the final stage of the amplifier.

The electromechanical time constant of the motor is

$$T_M = J\frac{\Omega_0}{M_0} = \frac{GD^2}{375}\frac{n_0}{M_0} = J\frac{R_y}{C_e C_m}.$$

where J[kg·m·sec^2] is the reduced moment of inertia, GD^2[kg·m^2] is the reduced moment of gyration, $\Omega_0 = \pi n_0/30$ [1/sec] is the ideal no-load angular velocity, M_0[kgm] is the motor starting torque, $C_e = U_n/\Omega_0$ is the back emf coefficient, equaling the ratio of the motor nominal voltage to the ideal no-load speed, $C_m = M_n/I_n$ is the motor torque coefficient, equal to the ratio of the nominal torque to the nominal armature current.

The nominal torque may be found from the motor rating plate:

$$M_n = 0{,}975\frac{P_n}{n_n}\ [\text{kgm}],$$

where P_n(watts) is the rated power of the motor, n_n [rpm] is the nominal speed of the motor.

The over-all transfer function of the motor together with the amplifier is

$$k = \frac{k_y}{C_e}\ \text{rad/v·sec}.$$

The motor transfer function is

$$W(p) = \frac{\Omega_{\partial}}{U_{вх}} = \frac{k}{p(1 + T_м p + T_э T_м p^2)}. \qquad (2)$$

5. Set up an operator equation of motion for a field-winding controlled DC electric motor using an amplifier with voltage gain k_u. The circuit is shown in Fig. 7.

<u>Answer</u>. For small deviations, the equation of motion may be represented as

$$(T_в p + 1)(T_э T_м p^2 + T_м p + 1)p\varphi_{\partial} = k_y[k_1(T_в p + 1) - k_2]U_{вх}. \qquad (1)$$

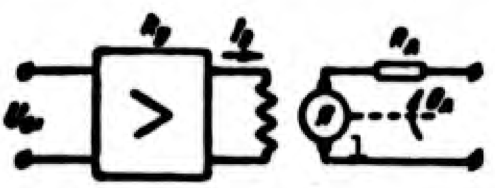


Fig. 7. Control circuit of field-winding controlled motor. 1) Armature.

The electromagnetic time constant of the field circuit is

$$T_в = \frac{L_в}{R_в},$$

where L_v and R_v are the resultant inductance and resistance of the field circuit, taking into account the final stage of the amplifier.

The electromagnetic time constant of the armature circuit is

$$T_a = \frac{L_{ya}}{R_{ya} + R_d},$$

where L_{ya} is the armature-circuit inductance, R_{ya} is the armature-circuit resistance, R_d is the series resistance.

The electromechanical time constant is

$$T_m = J\frac{R_{ya} + R_d}{C'_e C'_m I_{vo}^2},$$

where J is the reduced moment of inertia; I_{vo} is the field current corresponding to the steady-state regime from which small deviations are measured; $C'_e = C_e/I_0$ is a coefficient of proportionality between the back emf and the product of the angular velocity by the field current; $C'_m = C_m/I_0$ is a coefficient of proportionality between the torque and the product of the armature and field-circuit currents.

The transfer functions are

$$k_1 = \frac{(R_{ya} + R_d) I_{yao}}{C'_e R_v I_{vo}},$$

$$k_2 = \frac{\Omega_0}{R_v I_{vo}},$$

where I_{yao} is the armature current corresponding to the steady-state regime from which small deviations are measured; Ω_0 is the steady-state angular velocity.

When $k_1 > k_2$, an increase in the input voltage corresponds to an increase in the output speed, while when $k_1 < k_2$, the converse is true.

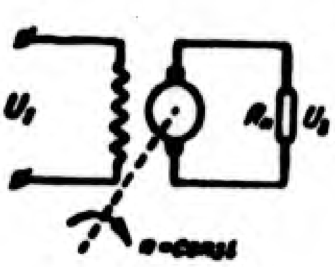


Fig. 8. DC generator circuit.

The motor transfer function is

$$W(p) = \frac{\Omega_0}{U_{vx}} = \frac{k_2 [k_1 (1 + T_a p) - k_2]}{p(1 + T_m p + T_a T_m p^2)}. \qquad (2)$$

6. Set up an operator equation and determine the transfer function of a DC generator turning at constant speed and working into a pure resistance

R_n (Fig. 8). As the input variable we take the change in voltage across the field winding, and as the output variable the change in voltage across the load resistance.

<u>Answer</u>. The equation for deviations may be represented in the form

$$(T_a p+1)(T_v p+1)U_2=kU_1. \qquad (1)$$

The armature-circuit time constant is

$$T_a=\frac{L_a}{R_a+R_n}.$$

The field-circuit time constant is

$$T_v=\frac{L_v}{R_v}.$$

The inductance L_v should be found from the magnetization curve at the point corresponding to a steady-state regime ($U_1 = U_{10}$ and $U_2 = U_{20}$) from the expression

$$L_v=\frac{d\Psi}{dI_v}.$$

where Ψ is the field-winding flux linkage, I_v is the field current.

The transfer function is

$$k=\frac{R_n}{R_a+R_n}\frac{PNn}{a\cdot 60}\frac{c}{R_v}.$$

where P is the number of pole pairs, N is the number of active armature conductors, <u>a</u> is the number of pairs of parallel armature-winding branches, <u>n</u> is the armature speed in rpm, $c = d\Phi/dI_v$ is the slope of the magnetization curve at a point corresponding to the steady-state regime, Φ is the excitation flux.

The slope of the magnetization curve is associated with the field-circuit inductance by the relationship

$$L_v=2Pw_v\sigma c,$$

where w_v is the number of field-winding turns per pole, $\sigma > 1$ is the pole leakage factor.

The generator transfer function is

$$\Phi(p) = \frac{U_3}{U_1} = \frac{k}{(1+T_1p)(1+T_2p)}. \qquad (2)$$

7. Set up the equation and find the transfer function of the cross-field amplidyne whose basic circuit is shown in Fig. 9. We assume that the armature-reaction flux is completely compensated and the amplifier operates into a pure resistance.

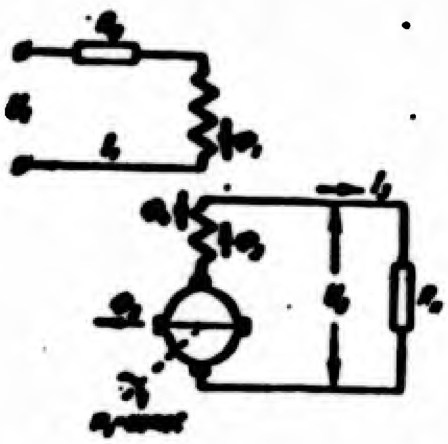


Fig. 9. Circuit of cross-field amplidyne.

Resistance. The second Kirchoff law for the control circuit yields

$$L_1 \frac{dI_1}{dt} + (R_v + R_1) I_1 = U_1, \qquad (1)$$

where L_1 and R_1 are the control-winding inductance and resistance, and R_v is the output resistance of the input-voltage source.

The second Kirchoff law yields

$$L_2 \frac{dI_2}{dt} + R_2 I_2 = E_2 = c_1 I_1, \qquad (2)$$

for the quadrature circuit, where L_2 and R_2 are the quadrature-circuit inductance and resistance, E_2 is the quadrature-circuit emf, c_1 is the coefficient of proportionality between the emf and control current.

The emf in the direct circuit is proportional to the current in the quadrature circuit, i.e., $E_3 = c_2 I_2$. The voltage across the load resistance is

$$U_3 = R_n I_3 = \frac{R_n}{R_n + R_3} E_3, \qquad (3)$$

where R_3 is the direct-circuit resistance.

From Eqs. (1), (2), and (3) we find

$$(T_1 p + 1)(T_2 p + 1) U_3 = k U_1. \qquad (4)$$

The time constants are

$$T_1 = \frac{L_1}{R_v + R_1},$$
$$T_2 = \frac{L_2}{R_2};$$

The voltage gain is

$$k = \frac{a_1 a_2 R_2}{(R_0 + R_1) R_2 (R_0 + R_2)}.$$

The transfer function is

$$W(p) = \frac{U_2}{U_1} = \frac{k}{(1 + T_1 p)(1 + T_2 p)}.$$

8. Find an approximate expression for the transfer function of a magnetic amplifier working into a purely resistive load with the following initial conditions: load resistance R_2 = 100 ohms, load current I_2 = 0.5 amp, input current I_1 = 0.01 amp, primary-circuit resistance R_1 = 1000 ohms, power-supply frequency f = 500 cps.

<u>Solution</u>. The power gain is

$$k_p = \frac{R_2 I_2^2}{R_1 I_1^2} = \frac{100 \cdot 0{,}5^2}{1000 \cdot 0{,}01^2} = 250.$$

The voltage gain is

$$k_u = \frac{R_2 I_2}{R_1 I_1} = \frac{100 \cdot 0{,}5}{1000 \cdot 0{,}01} = 5.$$

With a secondary efficiency η = 0.9, the rough value of the amplifier time constant will be

$$T = \frac{k_p}{4f\eta} = \frac{250}{4 \cdot 500 \cdot 0{,}9} = 0{,}14 \text{ sec}$$

The magnetic-amplifier transfer function is

$$W(p) = \frac{U_2}{U_1} = \frac{k_u}{1 + Tp} = \frac{5}{1 + 0{,}14p}.$$

9. Solve the preceding problem where positive selffeedback is used in the amplifier. The feedback factor is $k_{o.s}$ = 0.95.

<u>Solution.</u> Since the same input and output variables are used, the power and voltage gains remain as before. The time constant of the magnetic amplifier will be

$$T = \frac{(1 - k_{o.s}) k_p}{4f\eta} = \frac{(1 - 0{,}95)\, 250}{4 \cdot 500 \cdot 0{,}9} = 0{,}007 \text{ sec}$$

The transfer function is

$$\Phi(p)=\frac{U_2}{U_1}=\frac{k_U}{1+Tp}=\frac{5}{1+0{,}001p}.$$

10. Find the transfer function of a two-phase variable-speed induction motor together with its supply amplifier if the low-speed mechanical characteristics can be approximated by parallel straight-line segments.

Answer. The transfer function is

$$\Phi(p)=\frac{\theta_d}{U_{vkh}}=\frac{k}{p(1+Tp)},$$

where θ_d is the angle of rotation of the motor shaft, U_{vkh} is the amplifier input voltage.

The electromechanical time constant of the motor is

$$T=\frac{J}{\gamma},$$

where J is the reduced moment of inertia, γ[g·cm·sec/rad] is the slope of the mechanical-characteristic curves.

In many cases we may assume in approximation that

$$\gamma\approx\frac{M_0-M_n}{\Omega_n}\approx\frac{M_n}{\Omega_n}\ \text{for}\ M_n\approx\frac{M_0}{2},$$

where M_n and M_0 are the rated and starting torques of the motor when supplied from the final amplifier stage, Ω_n is the rated angular velocity at which the motor turns.

The transfer constant is

$$k=k_m\frac{k_u}{\gamma},$$

where k_m[g·cm/v] is the slope of the motor starting-torque curve, k_u is the amplifier voltage gain.

§2. DYNAMIC ELEMENTS

11. Find the transfer function for a type one element where its reaction to unit input (transient response) has the form shown in Fig. 10.

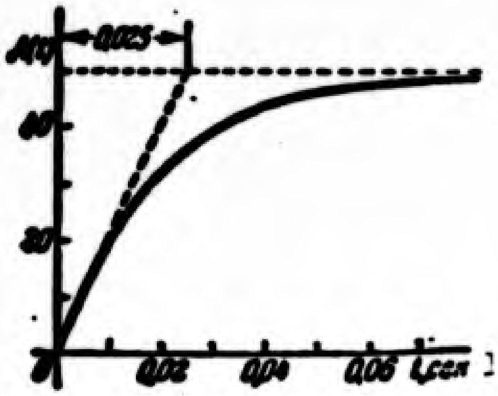


Fig. 10. Transfer-response curve for Problem 11. 1) Sec.

Answer.

$$w(p) = \frac{k}{1+Tp} = \frac{50}{1+0{,}025p}.$$

12. Find the transfer function for a type two element, if its transient response has the form shown in Fig. 11.

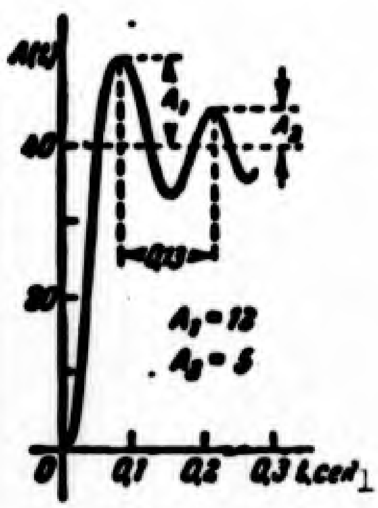


Fig. 11. Transient-response curve for Problem 12. 1) Sec.

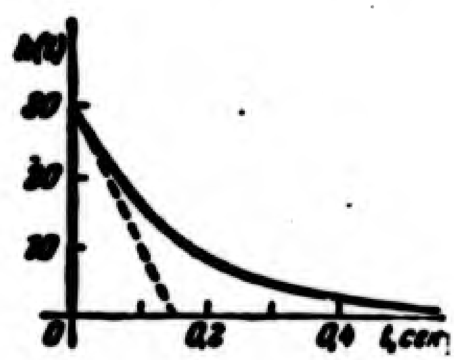


Fig. 12. Weighting function for Problem 13. 1) Sec.

Answer.

$$w(p) = \frac{k}{1+2\xi Tp+T^2p^2} =$$

$$= \frac{40}{1+2\cdot 0{,}135\cdot 0{,}02p+0{,}02^2p^2}.$$

13. Find the transfer function for a type one element if its unit impulse response (weight function) has the form shown in Fig. 12.

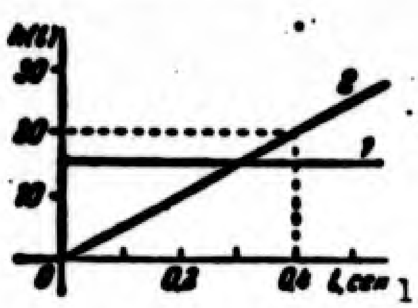


Fig. 13. Weighting function for Problem 14. 1) Sec.

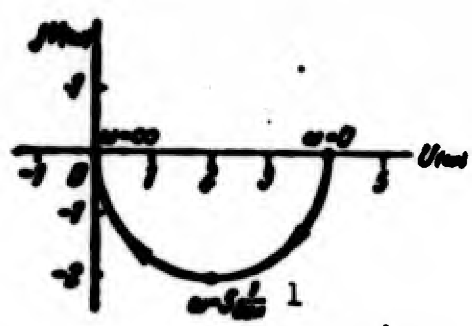


Fig. 14. Gain-phase characteristic for Problem 15. 1) Sec.

Answer.

$$w(p) = \frac{k}{1+Tp} = \frac{4{,}5}{1+0{,}15p}.$$

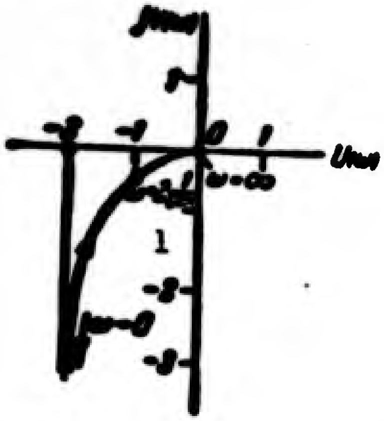


Fig. 15. Gain-phase characteristic for Problem 16. 1) Sec.

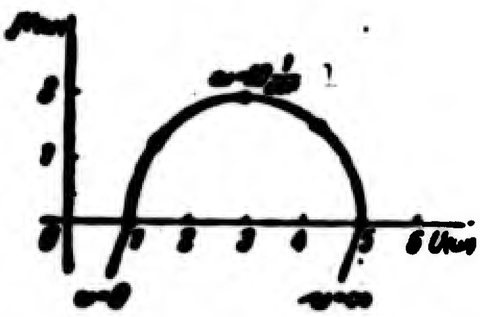


Fig. 16. Gain-phase characteristic for Problem 17. 1) Sec.

14. Do the same for the weighting functions 1 and 2 shown in Fig. 13.

Answer.

$$w_1(p)=\frac{k_1}{p}=\frac{15}{p},$$

$$w_2(p)=\frac{k_2}{p^2}=\frac{50}{p^2}.$$

15. Find the transfer function of an element if its gain-phase characteristic, constructed for positive frequencies, has the form of a semicircle (Fig. 14).

Answer.

$$w(p)=\frac{k}{1+Tp}=\frac{4}{1+0.2p}.$$

16. Find the transfer function for a type two element, if for positive frequencies its gain-phase characteristic has the form shown in Fig. 15.

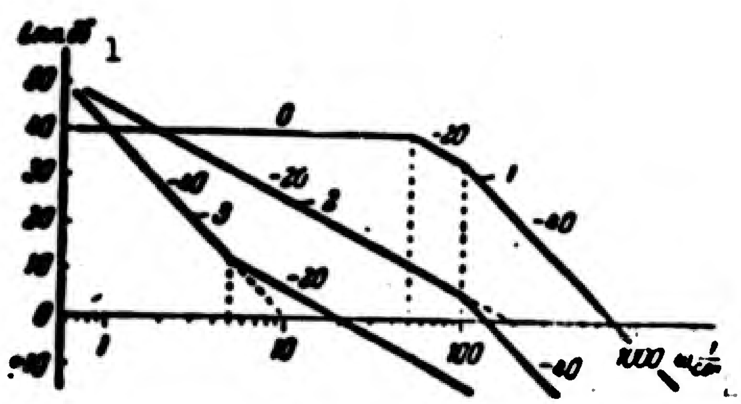


Fig. 17. Logarithmic amplitude characteristic for Problem 18. 1) db; 2) sec.

Answer.

$$w(p)=\frac{k}{p(1+Tp)}=\frac{4}{p(1+0.5p)}.$$

17. Find the transfer function of an element if its gain-phase characteristic is a semicircle for positive frequencies (Fig. 16).

Answer.

$$w(p)=\frac{1+T_1p}{1+T_2p}=\frac{1+0{,}5p}{1+0{,}1p}.$$

18. Figure 17 shows the asymptotic logarithmic amplitude characteristics (l.a.kh.) for phase-minimum elements. Determine their transfer functions.

Answer.

$$w_1(p)=\frac{k_1}{(1+T_1p)(1+T_2p)}=\frac{100}{(1+0{,}02p)(1+0{,}01p)},$$

$$w_2(p)=\frac{k_2}{p(1+T_2p)}=\frac{200}{p(1+0{,}01p)},$$

$$w_3(p)=\frac{k_3(1+T_3p)}{p^2}=\frac{100(1+0{,}2p)}{p^2}.$$

19. Find the transfer function of the passive integrating series compensating network illustrated in Fig. 18.

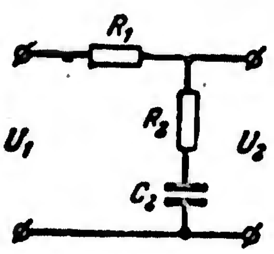


Fig. 18. Passive integrating network.

Solution. Representing the network as a voltage divider, we obtain the transfer function

$$w(p)=\frac{z_2(p)}{z_1(p)+z_2(p)}.$$

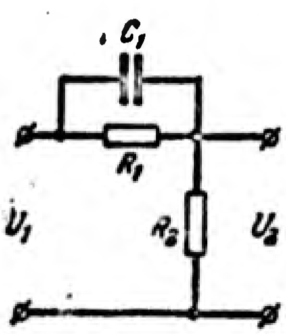


Fig. 19. Passive differentiating network.

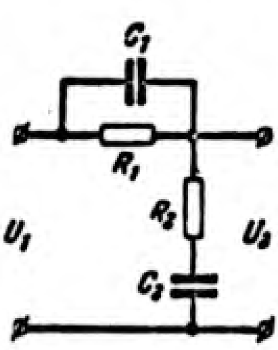


Fig. 20. Passive lead-lag network.

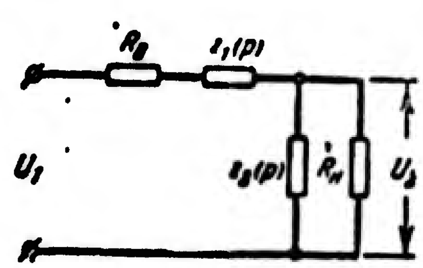


Fig. 21. Generalized diagram of series compensating network.

The expressions for the divider resistances in operator form are:

$$z_1(p)=R_1,$$

$$z_2(p)=R_2+\frac{1}{pC_2}=\frac{1+R_2C_2p}{pC_2}.$$

After substitution of the expressions for the resistances, we obtain

$$w(p)=\frac{1+T_2p}{1+T_1p},$$

where $T_1 = (R_1 + R_2)C_2$ and $T_2 = R_2C_2$.

20. Find the transfer function of the passive differentiating compensating network of Fig. 19.

Answer.

$$w(p)= G_0\frac{1+T_1p}{1+T_2p},$$

where $T_1=R_1C_1, \quad T_2=\frac{R_1R_2}{R_1+R_2}C_1, \quad G_0=\frac{R_2}{R_1+R_2}.$

21. Find the transfer function of the passive lead-lag network of Fig. 20.

Answer.

$$w(p)=\frac{(1+T_1p)(1+T_2p)}{(1+T_3p)(1+T_4p)},$$

where $T_1 = R_1C_1$, $T_2 = R_2C_2$, $T_3T_4 = T_1T_2$, and $T_3 + T_4 = T_1 + T_2 + R_1C_2$.

22. Find the transfer functions of the passive compensating networks using the values of Problems 19, 20 and 21 for the case in which we cannot neglect the effect of the output resistance R_v of the preceding stage and the load (input) resistance R_n of the following stage. The generalized network circuit for this case is shown in Fig. 21.

Answer.

1) For the passive integrating network

$$w(p)=G_0\frac{1+T_2p}{1+T_1p},$$

where

$$G_0=\frac{R_n}{R_v+R_1+R_n}, \quad T_2=R_2C_2,$$

$$T_1=\left[R_2+\frac{R_n(R_1+R_v)}{R_v+R_1+R_n}\right]C_2.$$

2) For the passive differentiating network

$$w(p) = O_0 \frac{1+T_1 p}{1+T_2 p},$$

where

$$O_0 = \frac{R_n R_2}{(R_n + R_2)(R_0 + R_1) + R_n R_2} = \frac{R_2}{\left(1+\frac{R_2}{R_n}\right)(R_0 + R_1) + R_2},$$

$$T_1 = R_1 C_1, \quad T_2 = \frac{R_1 + \left(1+\frac{R_2}{R_n}\right) R_0}{\left(1+\frac{R_2}{R_n}\right)(R_0 + R_1) + R_2} T_1.$$

3) For the passive lead-lag network

$$w(p) = O_0 \frac{(1+T_1 p)(1+T_2 p)}{(1+Ap+Bp^2)},$$

where

$$O_0 = \frac{R_n}{R_n + R_1 + R_0}, \quad T_1 = R_1 C_1, \quad T_2 = R_2 C_2,$$

$$A = \frac{(R_0 + R_n)(R_1 C_1 + R_2 C_2) + (R_0 R_n + R_n R_1 + R_1 R_0) C_2}{R_n + R_1 + R_0},$$

$$B = \frac{R_1 C_1 R_2 C_2 (R_0 + R_n) + R_0 R_n R_1 C_1 C_2}{R_n + R_1 + R_0}.$$

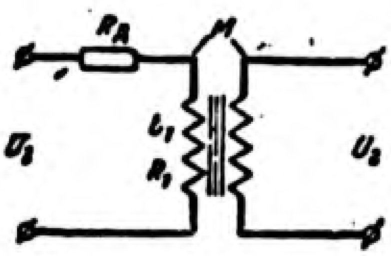


Fig. 22. Differentiating transformer.

23. Find the transfer function of the differentiating transformer whose circuit is shown in Fig. 22.

Answer.

$$w(p) = \frac{Mp}{R_д + R_1 + L_1 p} = \frac{w_2}{w_1} \frac{T_1 p}{1+T_1 p},$$

where $T_1 = L_1/(R_d + R_1)$, L_1 is the inductance of the primary winding, $M = (w_2/w_1)L_1$ is the mutual inductance, w_1 and w_2 are the number of primary and secondary turns.

24. Show that an oscillating circuit with transfer function

$$w(p) = \frac{1}{1+2\xi T p + T^2 p^2}$$

may be represented as series-connected aperiodic and integrating networks included in a proportional unit negative-feedback loop.

Solution. The transfer functions of the aperiodic and integrating networks are, respectively,

$$w_1(p)=\frac{k_1}{1+T_1p} \text{ and } w_2(p)=\frac{k_2}{p}.$$

If these networks are connected in series and have a unit feedback loop, the resultant transfer function will be

$$w(p)=\frac{w_1(p)\,w_2(p)}{1+w_1(p)\,w_2(p)}=\frac{1}{1+\frac{1}{k_1k_2}p+\frac{T_1}{k_1k_2}p^2}.$$

Satisfying the conditions

$$k_1k_2=\frac{1}{2\xi T} \text{ and } \frac{T_1}{k_1k_2}=T^2,$$

we obtain a transfer function coinciding with the given transfer function of the oscillating circuit.

25. For the case in which a passive integrating compensating network is introduced into the control circuit in series with an instantaneously responding amplifier (Fig. 18), find the transfer function for the equivalent negative feedback loop taken around an instantaneously responding amplifier with voltage gain equal to k.

Solution. The transfer function of the equivalent feedback loop is found from the formula [3]

$$w_{\text{о.с}}(p)=\frac{1-w_{\text{п.з}}(p)}{w_{c}(p)\,w_{\text{п.з}}(p)}. \qquad (1)$$

In our case, the transfer function of the network around which the feedback loop is taken will be $w_s(p) = k$, while the transfer function of the series integrating network is

$$w_{\text{п.з}}(p)=\frac{1+T_2p}{1+T_1p}.$$

Substitution in (1) yields

$$w_{\text{о.с}}(p)=\frac{(T_1-T_2)p}{k(1+T_2p)}=\frac{T_1-T_2}{kT_2}\,\frac{T_2p}{1+T_2p}.$$

Such a feedback loop can be realized, for example, by employing a differentiating capacitor and voltage divider (Fig. 23).

The equivalence conditions (for $R \gg \frac{r_1r_2}{r_1+r_2}$) are

$$\frac{r_2}{r_1 + r_2} = \frac{T_1 - T_2}{kT_2} \quad \text{and} \quad RC = T_2.$$

We can obtain a similar result by using a differentiating transformer (Fig. 22) in the feedback loop.

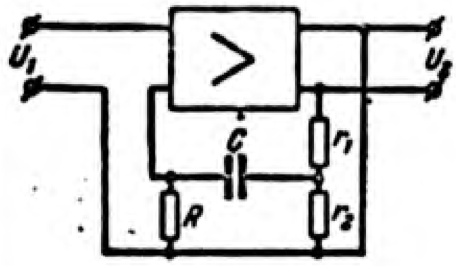


Fig. 23. Feedback loop equivalent to passive integrating network.

26. Do the same for the passive differentiating network of Fig. 19.

<u>Answer.</u> The equivalent feedback loop is shown in Fig. 24. A type-one aperiodic element should be introduced into the feedback loop; it should have a transfer function

$$w_{o.c}(p) = \frac{k_{o.c}}{1 + T_2 p} = \frac{T_1 - T_2}{kT_2} \, \frac{1}{1 + T_2 p}.$$

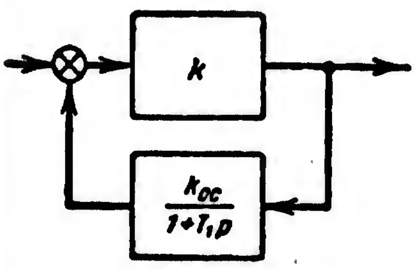


Fig. 24. Feedback loop equivalent to passive differentiating network.

27. Do the same for the passive lead-lag network of Fig. 20.

<u>Answer</u>. The feedback loop should contain a series-connected real differentiating element (capacitor or transformer) and a series-connected type-one aperiodic network with overall transfer function

$$w_{o.c}(p) = \frac{(T_3 + T_4 - T_1 - T_2)p}{(1 + T_2 p)(1 + T_4 p)k}.$$

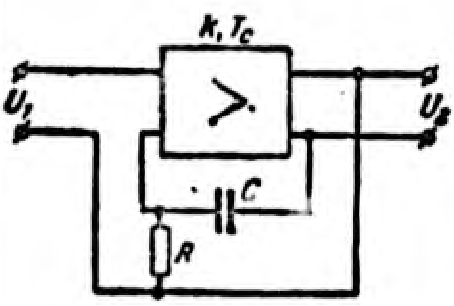


Fig. 25. Feedback loop for Problem 28.

28. Figure 25 shows an amplifier with finite response time having a transfer function

$$w_c(p) = \frac{k}{1 + T_c p},$$

and forming part of a negative-feedback loop containing a differentiating capacitor with transfer function

$$w_{o.c}(p) = \frac{Tp}{1 + Tp}.$$

where T = RC. Find the series compensating network equivalent to this feedback loop.

Solution. From the formula by which we go over to the equivalent series network [3], we have

$$w_{п.з}(p)=\frac{1}{1+w_c(p)\,w_{о.с}(p)}=\frac{(1+T_cp)(1+Tp)}{(1+T_cp)(1+Tp)+kTp}.$$

The transfer function corresponds to a passive lead-lag network (Fig. 20). It may be represented in the following form:

$$w_{п.з}(p)=\frac{(1+T_1p)(1+T_2p)}{(1+T_3p)(1+T_4p)},$$

where $T_1=T_c$, $T_2=T$, $T_3T_4=T_1T_2$ and $T_3+T_4=T_1+T_2+kT$.

§3. TRANSFER FUNCTIONS OF AUTOMATIC CONTROL SYSTEMS

29. Figure 26 shows electromechanical and block diagrams of a remote-readin g servosystem with sine-cosine magslips (SKVT). In Fig. 26 θ_1, θ_2 are the rotation angles of the command and final-control shafts, $\theta = \theta_1 - \theta_2$ is the error, RM is the working mechanism (object), R is a reducing gear, D is the motor, TG is a tachometer generator, k_1[v/rad] is the transfer constant of the sensing element (SKVT) on the linear part of the characteristic curve, k_2 and k_3 are the amplifier voltage gains, k_4[rad/v·sec] is the transfer constant of the final-control motor, $k_5 = 1/n$ is the transfer constant of the reducing gear, n is the gear ratio, k_6[v·sec/rad] is the tachometer generator transfer constant, k_7[rad/g·cm·sec] is the slope coefficient for the motor mechanical characteristic curve, T_1 and T_2 are the time constants of the amplifier and motor, T = RC is the time constant of the differentiating capacitor.

We are to determine the system open-loop transfer function, the system closed-loop transfer function (principal operator), the error transfer function for the manipulated variable, and the error transfer function for the disturbance variable (load torque).

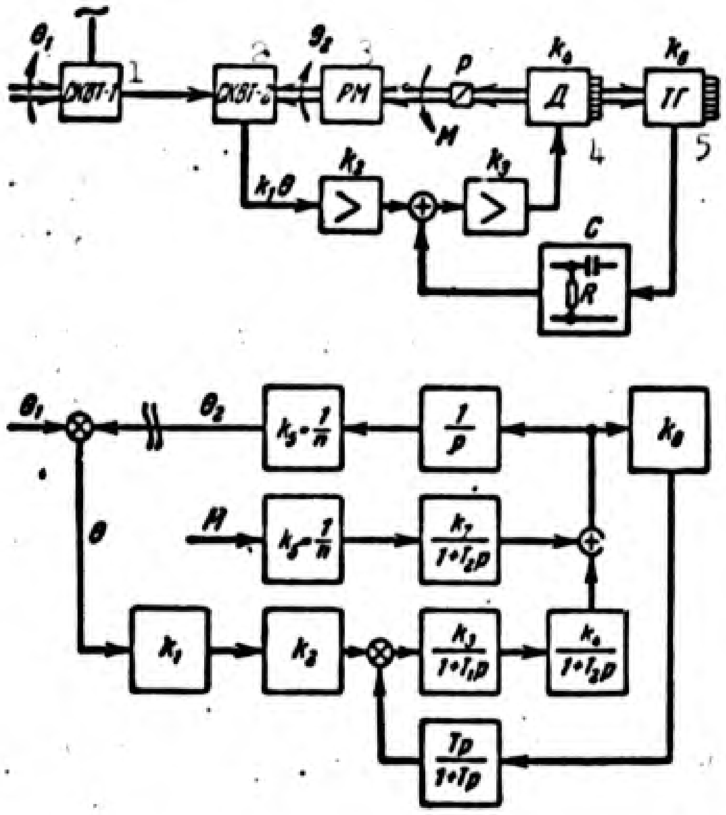


Fig. 26. Servosystem. 1) SKVT-1; 2) SKVT-2; 3) working mechanism; 4) motor; 5) tachometer generator.

Solution. We first determine the equivalent transfer function for the final amplifier together with the motor where the negative-feedback loop includes the tachometer generator and differentiating capacitor:

$$w_e(p) = \frac{\dfrac{k_3k_4}{(1+T_1p)(1+T_2p)}}{1+\dfrac{k_3k_4k_6Tp}{(1+T_1p)(1+T_2p)(1+Tp)}} =$$

$$= \frac{k_3k_4(1+Tp)}{(1+T_1p)(1+T_2p)(1+Tp)+k_3k_4k_6Tp}.$$

Next, for the open-loop system as shown in Fig. 26, we find the transfer function, which equals the product of the transfer functions of the series-connected elements:

$$W(p) = \frac{\theta_2(p)}{\theta(p)} = k_1k_2w_e(p)\,\frac{1}{p}\,k_5.$$

After substituting $w_e(p)$, we have

$$W(p) = \frac{K(1+Tp)}{p\left[(1+T_1p)(1+T_2p)(1+Tp)+k_3k_4k_6Tp\right]},$$

where the velocity figure of merit (the ratio of the constant tracking velocity to the steady-state error) is

$$K = k_1 k_2 k_3 k_4 k_5 \text{ [1/sec]}.$$

The open-loop system transfer function may also be written in the following form:

$$W(p) = \frac{K(1+Tp)}{p(1+A_1 p + A_2 p^2 + A_3 p^3)},$$

where

$$A_1 = (1 + k_3 k_4 k_6) T + T_1 + T_2,$$
$$A_2 = T_1 T_2 + T_1 T + T_2 T,$$
$$A_3 = T T_1 T_2.$$

The closed-loop system transfer function is

$$\Phi(p) = \frac{\theta_2(p)}{\theta_1(p)} = \frac{W(p)}{1+W(p)}.$$

Substituting W(p), we obtain

$$\Phi(p) = \frac{K(1+Tp)}{p[(1+T_1 p)(1+T_2 p)(1+Tp) + k_3 k_4 k_6 Tp] + K(1+Tp)} = \frac{b_0 p + b_1}{a_0 p^4 + a_1 p^3 + a_2 p^2 + a_3 p + a_4},$$

where

$$a_0 = T_1 T_2 T, \quad a_1 = T_1 T_2 + T_1 T + T_2 T,$$
$$a_2 = T_1 + T_2 + (1 + k_3 k_4 k_6) T,$$
$$a_3 = 1 + KT, \quad a_4 = K,$$
$$b_0 = KT, \quad b_1 = K.$$

The error transfer function for the manipulated variable is

$$\Phi_0(p) = \frac{\theta(p)}{\theta_1(p)} = \frac{1}{1+W(p)} = 1 - \Phi(p).$$

Substitution of W(p) or Φ(p) yields

$$\Phi_0(p) = \frac{p[(1+T_1 p)(1+T_2 p)(1+Tp) + k_3 k_4 k_6 Tp]}{p[(1+T_1 p)(1+T_2 p)(1+Tp) + k_3 k_4 k_6 Tp] + K(1+Tp)} = \frac{a_0 p^4 + a_1 p^3 + a_2 p^2 + (a_3 - b_0) p}{a_0 p^4 + a_1 p^3 + a_2 p^2 + a_3 p + a_4}.$$

The error transfer function for the disturbance variable in the open-loop system with an open local feedback loop is

$$W_M(p) = \frac{\theta_2(p)}{M(p)} = \frac{k_1 k_2'}{p(1+T_2 p)}.$$

When the local feedback loop is closed

$$W_m'(p) = \frac{W_m(p)}{1 + \frac{k_3 k_4 k_6 Tp}{(1+T_1p)(1+T_2p)(1+Tp)}} =$$

$$= \frac{k_7 k_5^2 (1+T_1p)(1+Tp)}{p[(1+T_1p)(1+T_2p)(1+Tp) + k_3k_4k_6Tp]}.$$

When the main feedback loop is closed, the error transfer function for the disturbance variable will be

$$\Phi_m'(p) = \frac{\theta(p)}{M(p)} = \frac{W_m'(p)}{1 + W(p)}.$$

Substitution of $W_m'(p)$ and $W(p)$ yields

$$\Phi_m(p) = \frac{k_7 k_5^2 (1+T_1p)(1+Tp)}{p[(1+T_1p)(1+T_2p)(1+Tp) + k_3k_4k_6Tp] + K(1+Tp)} =$$

$$= \frac{c_0p^2 + c_1p + c_2}{a_0p^4 + a_1p^3 + a_2p^2 + a_3p + a_4},$$

where

$$c_0 = k_7 k_5^2 T_1 T, \quad c_1 = k_7 k_5^2 (T_1 + T), \quad c_2 = k_7 k_5^2.$$

The torque figure of merit (the ratio of the load torque at the system final-control shaft to the steady-state positional error) is

$$K_m = \frac{K}{k_7 k_5^2} = \frac{Kn^2}{k_7} = \frac{k_1k_2k_3k_4n}{k_7} \text{ [g·cm/rad]}.$$

30. For the preceding problem, find the numerical values of the coefficients contained in the open-loop system transfer function with the following initial conditions: slope of sensing-element characteristic k_1 = 1 v/degree = 57.3 v/rad, amplifier gains k_2 = 2.5 and k_3 = 80, nominal motor voltage U_n = 110 v, no-load speed $n_{kh.kh}$ = 9000 rpm, starting torque M_p = 55 g·cm, reduced moment of inertia of motor together with object J = 0.01 g·cm·sec^2, gear ratio of reduction gear n = = 1000, transfer constant of tachometer generator k_6 = 0.001 v·min/rpm = = $9.8 \cdot 10^{-3}$ v·sec/rad, amplifier time constant T_1 = 0.01 sec, differentiating-capacitor time constant T = 0.14 sec.

Solution. The motor transfer constant is

$$k_4 = \frac{\Omega_{kh.kh}}{U_n} = \frac{\pi n_{kh.kh}}{30U_n} = \frac{3,14 \cdot 9000}{30 \cdot 110} = 8,6 \text{ rad/v·sec}.$$

The slope coefficient of the mechanical characteristic curve is

$$k_7=\frac{\Omega_{x.x}}{M_n}=\frac{\pi n_{x.x}}{30M_n}=\frac{3,14\cdot 9000}{30\cdot 55}=17,2 \text{ rad/g}\cdot\text{cm}\cdot\text{sec}.$$

The motor time constant is

$$T_2=J\frac{\Omega_{x.x}}{M_n}=Jk_7=0,01\cdot 17,2=0,172 \text{ sec}$$

The system velocity figure of merit is

$$K=k_1k_2k_4k_5k_3=\frac{57,3\cdot 2,5\cdot 80\cdot 8,6}{1000}\approx 100 \text{ 1/sec}$$

We next determine the coefficients.

$$A_1=(1+80\cdot 8,6\cdot 0,0098)\,0,14+0,01+0,172=1,18 \text{ sec}$$

$$A_2=0,01\cdot 0,172+0,01\cdot 0,14+0,172\cdot 0,14=0,027 \text{ sec}^2$$

$$A_3=0,14\cdot 0,01\cdot 0,172=0,00024 \text{ sec}^3$$

The system open-loop transfer function is

$$W(p)=\frac{100(1+0,14p)}{p(1+1,18p+0,027p^2+0,00024p^3)}.$$

Rearranging the denominator of this last expression into factors, the system open-loop transfer function may be represented in the following form:

$$W(p)=\frac{K(1+Tp)}{p(1+T_3p)(1+2\xi T_4p+T_4^2p^2)},$$

where T_3 = 1.16 sec, T_4 = 0.0145 sec, and ξ = 0.8.

The torque figure of merit is

$$K_M=\frac{Kn^2}{k_7}=\frac{100\cdot 1000^2}{17,2}=5,8\cdot 10^6 \text{ g}\cdot\text{cm/rad} =$$

$$=1700 \text{ g}\cdot\text{cm/ang}\cdot\text{min}.$$

31. Figure 27 shows the basic diagram of an automatic speed-regulating system for a heat engine. Here the sensing element is a centrifugal mechanism. When the rate of rotation changes, the centrifugal forces cause the weights to move apart, moving the coupling. The force due to a spring acts on the other side of the coupling; in this system, the spring is the element that sets the operating point. The magnitude

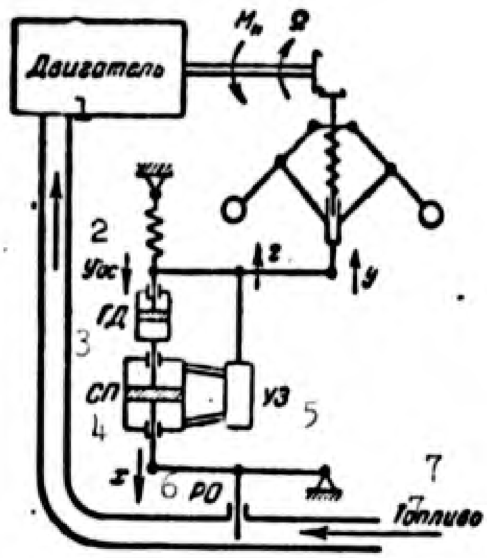


Fig. 27. Speed-control system. 1) Engine; 2) Y axis; 3) hydraulic damper; 4) power piston; 5) control slide valve; 6) final control element; 7) fuel.

of the output variable of the sensing element is transmitted to the control slide valve UZ of the actuating element. The power piston SP of the actuating element is connected to the final control element RO and through the hydraulic damper GD to the control slide valve. The hydraulic damper (proportional-integral element) provides elastic feedback.

The linearized equations for the system elements are:

1) The engine

$$(T_0p+1)\Omega = -k_1x - k_2M_n \quad (1)$$

where Ω is the engine angular velocity, $\underline{x}$ is the slide-valve displacement, M_n is the load torque, and k_1 and k_2 are coefficients;

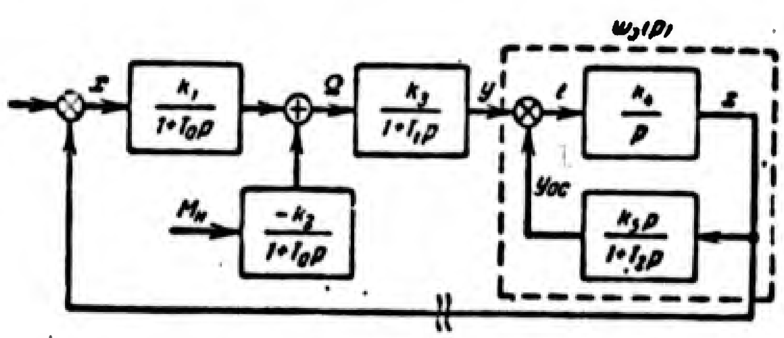


Fig. 28. Block diagram of speed-control system. 1) Y axis.

2) The sensing element

$$(T_1p+1)y = k_3\Omega, \quad (2)$$

where $\underline{y}$ is the displacement of the centrifugal-device coupling, T_1 is the time constant, and k_3 is a coefficient;

3) The slide valve

$$z = y - y_{o.s}, \qquad (3)$$

where z is the slide-valve displacement, $y_{o.s}$ is the displacement of the feedback link;

4) The actuating element

$$px = k_4 z, \qquad (4)$$

where k_4 is a coefficient;

5) The feedback element

$$(T_2 p + 1) y_{o.s} = k_5 px, \qquad (5)$$

where T_2 is the time constant of the hydraulic differentiating device, and k_5 is a coefficient. Set up the block diagram for this system, find the system open-loop transfer function and the error transfer function for the disturbance variable (load torque).

Solution. Figure 28 shows the block diagram in accordance with Eqs. (1)-(5).

We first find the transfer function of the element around which the feedback is taken:

$$w_e(p) = \frac{k_4}{p\left(1 + \frac{k_4}{p}\,\frac{k_5 p}{1 + T_2 p}\right)} = \frac{k_4(1 + T_2 p)}{p(1 + k_4 k_5 + T_2 p)} = \frac{k_6(1 + T_2 p)}{p(1 + T_3 p)},$$

where $k_6 = \frac{k_4}{1 + k_4 k_5}$ and $T_3 = \frac{T_2}{1 + k_4 k_5}$.

We next write the system open-loop transfer function as the product of the transfer functions of the series-connected elements:

$$W(p) = \frac{k_1}{1 + T_0 p}\,\frac{k_2}{1 + T_1 p}\,\frac{k_6(1 + T_2 p)}{p(1 + T_3 p)} = \frac{K(1 + T_2 p)}{p(1 + T_0 p)(1 + T_1 p)(1 + T_3 p)},$$

where the over-all gain is

$$K = k_1 k_2 k_6 = \frac{k_1 k_2 k_4}{1 + k_4 k_5} \text{ [1/sec]}$$

The error transfer function for the disturbance variable in the open-loop system is

$$W_{в}(p)=\frac{-\Omega(p)}{M(p)}=\frac{k_2}{1+T_0 p}$$

and in the closed-loop system is

$$\Phi_{в}(p)=\frac{W_{в}(p)}{1+W(p)}=\frac{k_2(1+T_1 p)(1+T_2 p)p}{p(1+T_0 p)(1+T_1 p)(1+T_2 p)+K(1+T_2 p)}.$$

32. For two coupled control systems (Fig. 29), find the system closed-loop transfer functions if $W = W(p)$ is the open-loop transfer function for one system, and a_{11}, a_{22}, a_{12} and a_{21} are coefficients of proportionality.

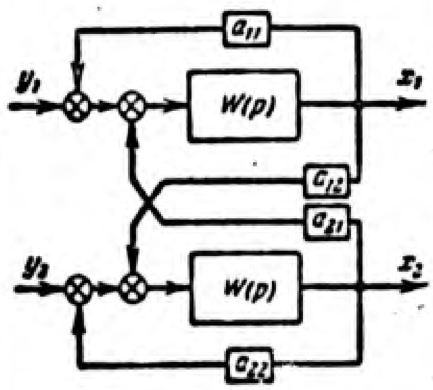


Fig. 29. Coupled controlled systems.

Solution. In accordance with the block diagram shown, we have the control-system equations of motion:

$$x_1=W(y_1-a_{11}x_1)-a_{21}Wx_2,$$
$$x_2=W(y_2-a_{22}x_2)-a_{12}Wx_1.$$

Solving them simultaneously, we obtain

$$x_1=\frac{(1+a_{22}W)Wy_1-a_{21}W^2y_2}{(1+a_{11}W)(1+a_{22}W)-a_{12}a_{21}W^2},$$
$$x_2=\frac{-a_{12}W^2y_1+(1+a_{11}W)Wy_2}{(1+a_{11}W)(1+a_{22}W)-a_{12}a_{21}W^2}.$$

From this we obtain four system closed-loop transfer functions

$$\Phi_{11}(p)=\frac{x_1(p)}{y_1(p)}=\frac{(1+a_{22}W)W}{(1+a_{11}W)(1+a_{22}W)-a_{12}a_{21}W^2},$$
$$\Phi_{21}(p)=\frac{x_1(p)}{y_2(p)}=\frac{-a_{21}W^2}{(1+a_{11}W)(1+a_{22}W)-a_{12}a_{21}W^2},$$
$$\Phi_{22}(p)=\frac{x_2(p)}{y_2(p)}=\frac{(1+a_{11}W)W}{(1+a_{11}W)(1+a_{22}W)-a_{12}a_{21}W^2},$$
$$\Phi_{12}(p)=\frac{x_2(p)}{y_1(p)}=\frac{-a_{12}W^2}{(1+a_{11}W)(1+a_{22}W)-a_{12}a_{21}W^2}.$$

33. Do the same for the system whose block diagram is shown in Fig. 30.

Answer.

$$\Phi_{11}(p)=\frac{x_1(p)}{y_1(p)}=\frac{(1+W)W-a^2W}{(1+W)^2+a^2W^2},$$
$$\Phi_{21}(p)=\frac{x_1(p)}{y_2(p)}=\frac{aW(1+2W)}{(1+W)^2+a^2W^2},$$
$$\Phi_{22}(p)=\frac{x_2(p)}{y_2(p)}=\frac{(1+W)W+a^2W^2}{(1+W)^2+a^2W^2}.$$

$$\Phi_{12}(p) = \frac{x_2(p)}{y_1(p)} = \frac{-aW(1+2W)}{(1+W)^2 + a^2W^2}.$$

34.* For a system with antisymmetric cross-coupling (Fig. 30) find the equivalent block diagram for one system and the equivalent closed-loop and open-loop transfer functions for the systems.

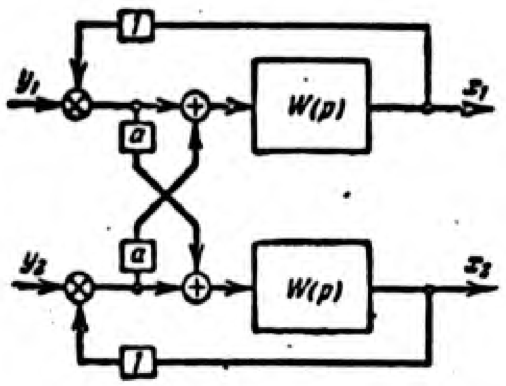


Fig. 30. System with antisymmetric cross-coupling.

Solution. For the diagram shown in Fig. 30, we have as initial equations

$$x_1 = W(y_1 - x_1) + aW(y_2 - x_2), \qquad (1)$$

$$x_2 = W(y_2 - x_2) - aW(y_1 - x_1), \qquad (2)$$

where W = W(p) is the system open-loop transfer function without the cross-coupling. We let [16]

$$x = x_1 + jx_2 \text{ and } y = y_1 + jy_2.$$

Multiplying Eq. (2) by j and combining it with (1), we have

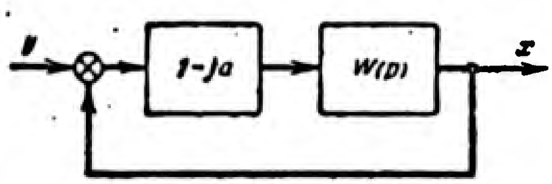


Fig. 31. Equivalent block diagram.

$$x = W(p)(y - x) - jaW(p)y + jaW(p)x,$$

from which we obtain

$$x = \frac{W(p)(1 - ja)}{1 + W(p)(1 - ja)} y = \Phi(p, ja)y,$$

where $\Phi(p, ja)$ is the equivalent system closed-loop transfer function. The equivalent system open-loop transfer function is

$$W(p, ja) = \frac{\Phi(p, ja)}{1 - \Phi(p, ja)} = W(p)(1 - ja).$$

The block diagram is shown in Fig. 31.

Manuscript Page No.	[Footnote]
30	To solve problems with asterisks, reference to the journal literature is required.

Manuscript Page No.	[Transliterated Symbols]
3	п = p = pruzhina = spring
3	м = m = massa = mass
3	д = d = dempfer = damper
7	н = n = nachal'nyy = initial
8	с = s = stolb = pile
8	вх = vkh = vkhod = input
8	я = ya = yakor' = armature
8	м = m = mekhanicheskiy = mechanical
8	д = d = dvigatel' = motor
9	н = n = nominal'nyy = nominal, rated
9	в = v = vozbuzhdeniye = field
10	д = d = dobavochnyy = in-series
10	н = n = nagruzka = load
13	о.с = o.s = obratnaya svyaz' = feedback
14	у = u = usilitel' = amplifier
17	л.а.х. = l.a.kh. = logarifmicheskaya amplitudnaya kharakteristika = logarithmic amplitude characteristic
20	п.з = p.z = posledovatel'noye zveno = series element
20	с = s [not identified]
22	СКВТ = SKVT = sinusno-kosinusnyy vrashchayushchiy transformator = sinecosine magslip
22	РМ = RM = rabochiy mekhanizm = working mechanism
22	Р = R = reduktor = reduction gear
22	Д = D = dvigatel' = motor
22	ТГ = TG = takhogenerator = tachometer generator
24	м = m = mestnyy = local

25 х.х = kh.kh = kholostoy khod = no load

25 п = p = pusk = starting

26 м = m = moment = torque

27 УЗ = UZ = upravlyayushchiy zolotnik = control slidevalve

27 СР = SP = silovoy porshen' = power piston

27 РО = RO = reguliruyushchiy organ = control element

27 ГД = GD = gidravlicheskiy dempfer = hydraulic damper

Chapter 2

FREQUENCY CHARACTERISTICS OF AUTOMATIC CONTROL SYSTEMS

§4. GAIN-PHASE CHARACTERISTICS

35. Construct the gain-phase characteristic of an integrating element with transfer function

$$w(p)=\frac{k}{p}.$$

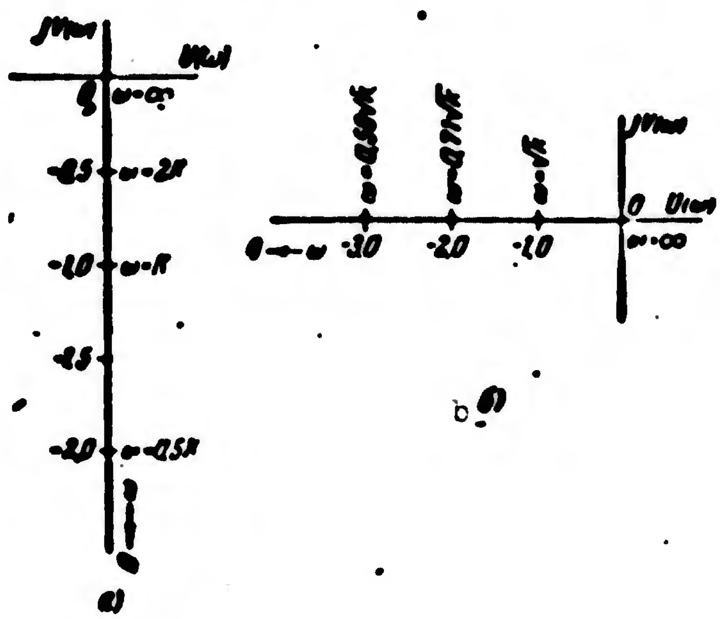


Fig. 32. Gain-phase characteristic of first-order integrating elements (a) and second-order integrating elements (b).

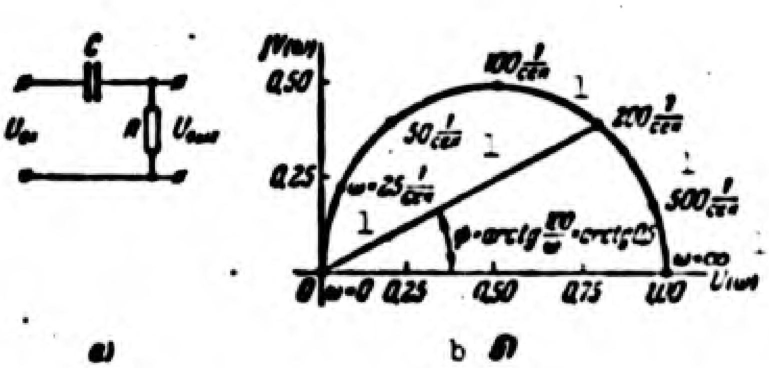


Fig. 33. Gain-phase characteristic of differentiating element, case 1. 1) Sec.

Answer. The gain-phase characteristic coincides with the negative imaginary semiaxis, as shown in Fig. 32a.

36. Construct the gain-phase characteristic for an element with transfer function

$$w(p)=\frac{k}{p^2}.$$

<u>Answer</u>. The gain-phase characteristic coincides with the negative real semiaxis (Fig. 32b).

37. Construct the gain-phase characteristic for the RC circuit shown in Fig. 33a; R = 1 kohm, C = 10 μf.

<u>Solution</u>. The circuit frequency transfer function equals

$$w(j\omega)=\frac{j\omega T}{1+j\omega T}, \qquad (1)$$

where

$$T=RC=10^3\cdot 10^{-5}=10^{-2} \text{ sec}$$

We transform Expression (1) so that it becomes a complex number in algebraic form:

$$w(j\omega)=u(\omega)+jv(\omega)=\frac{\omega^2T^2}{1+\omega^2T^2}+j\frac{\omega T}{1+\omega^2T^2}= \\ =\frac{10^{-4}\omega^2}{1+10^{-4}\omega^2}+j\frac{10^{-2}\omega}{1+10^{-4}\omega^2}. \qquad (2)$$

Given individual values of ω, we can use Formula (2) to compute several pairs of values u(ω) and v(ω); on this basis we plot the gain-phase characteristic of the circuit.

Analysis of (2) shows, however, that this characteristic is determined by the equation

$$u^2(\omega)+v^2(\omega)=u(\omega)$$

and for all positive frequencies may be plotted simply as a semicircle in the upper half plane with center at the point (0.5, j0) and radius 0.5 (Fig. 33b).

It is clear from (2) that when ω = 0, w(jω) = 0 + j0, while when ω = ∞, w(jω) = 1 + j0. Points corresponding to these frequencies, as well as to certain intermediate frequencies, are shown in Fig. 33b;

Fig. 34. Gain-phase characteristic of differentiating element, case 2. 1) Sec.

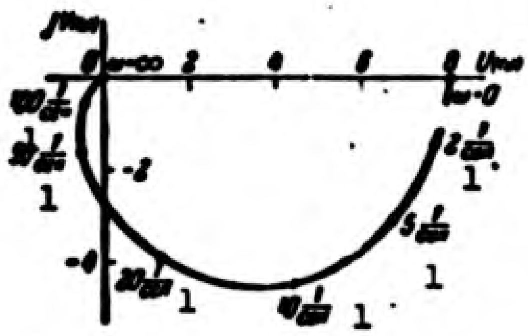


Fig. 35. Gain-phase characteristic of two aperiodic elements connected in series. 1) Sec.

here and in all the subsequent figures, the frequencies are given in rad/sec. Frequencies corresponding to intermediate points in the curve may be found as follows.

The argument of the complex number (2) equals

$$\varphi = \arg w(j\omega) = \operatorname{arctg} \frac{1}{\omega T} = \operatorname{arctg} \frac{100}{\omega}; \qquad (3)$$

Thus a ray drawn from the origin at an angle φ to the <u>x</u> axis will intersect the gain-phase characteristic at a point at which the quantity ω is determined in terms of φ, in accordance with (3). One such ray is shown in the figure.

38. Find the equation for the curve representing the gain-phase characteristic of the differentiating network shown in Fig. 34a. Construct the gain-phase characteristic of the network for the case in which R_1 = 40 kohm, R_2 = 10 kohm, C = 2.5 μf.

Answer. The equation of the curve takes the form

$$u^2(\omega)+v^2(\omega)=(\rho+1)u(\omega)-\rho, \qquad (1)$$

where

$$\rho=\frac{R_0}{R_1+R_0}.$$

In accordance with (1), the gain-phase characteristic for positive frequencies is a semicircle located in the upper halfplane with center at the point $\left(\frac{\rho+1}{2}, j0\right)$ and radius $\frac{1-\rho}{2}$. Figure 34b shows this characteristic plotted for the values indicated.

39. Construct the gain-phase characteristic of a system having a transfer function of the form

$$W(p)=\frac{K}{(1+T_1p)(1+T_2p)},$$

when K = 8, T_1 = 80 msec, T_2 = 12 msec.

Answer. See Fig. 35.

40. An automatic control system has the block diagram shown in Fig. 36; ChE is the sensing element, D is the motor, R is the reduction gear. The system open-loop transfer function equals

$$W(p)=\frac{K}{p(1+T_1p)(1+T_2p)}.$$

Construct the gain-phase characteristic of the system when K = 400 1/sec, T_1 = 80 msec, T_2 = 12 msec.

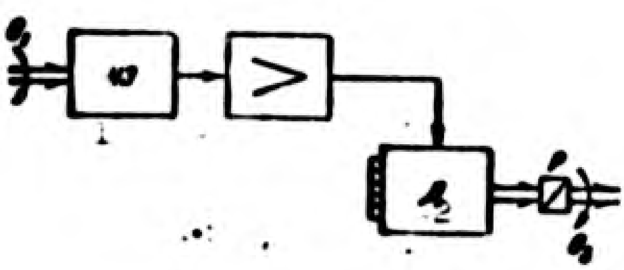


Fig. 36. Block diagram for Problems 40 and 41. 1) Sensing element; 2) motor.

Answer. The gain-phase characteristic may be plotted from the absolute values of A(ω) given in the table and the argument ψ(ω) of the frequency transfer function $W(j\omega) = A(\omega)e^{j\psi(\omega)}$.

ω, 1/sec [1]	0	2	5	10	20	50	100	300	∞
$A(\omega)$	∞	196	74	31	10,3	1,66	0,319	0,015	0
$\psi(\omega)$	$-90°$	$-100°$	$-115°$	$-135°$	$-162°$	$-197°$	$-223°$	$-252°$	$-270°$

1) ω, 1/sec.

41. An automatic control system has the block diagram shown in Fig. 36. The system open-loop transfer function takes the form

$$W(p) = \frac{K}{p(1+T_1 p)(1+T_2 p)}. \qquad (1)$$

Find a method of representing the gain-phase characteristic that will permit us to cover cases involving various combinations of the system parameters K, T_1, T_2.

Solution. We represent Expression (1) in the form

$$W(p) = \frac{KT_1}{T_1 p(1+T_1 p)(1+aT_1 p)}, \qquad (2)$$

where

$$a = \frac{T_2}{T_1}.$$

The frequency transfer function corresponding to Expression (2) will take the form

$$W(j\omega) = \frac{KT_1}{jT_1\omega(1+jT_1\omega)(1+jaT_1\omega)} = KT_1 W_0(jT_1\omega) = \\ = KT_1 U_0(T_1\omega) + jKT_1 V_0(T_1\omega). \qquad (3)$$

Given a sequence of nearby values $a = T_2/T_1$ from $a = 0$ to $a = 1$, we can construct a family of gain-phase characteristics which for all practical purposes will cover all possible types of systems having the transfer function (1).

In Fig. 37 we have plotted a family of such universal gain-phase characteristics for a = 0, 0.2, 0.4, 0.6, 0.8, 1. The construction is carried out on the basis of Expression (3) using its absolute value and argument for the various frequency values; along the coordinate axes we

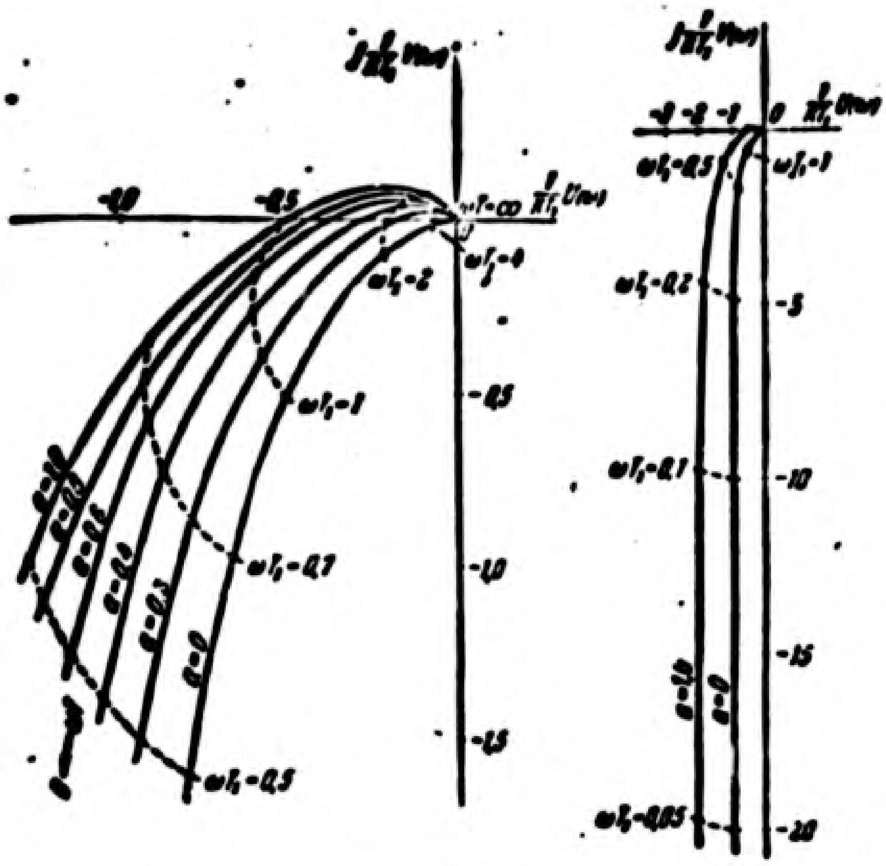


Fig. 37. Universal gain-phase characteristics for Problem 41.

have plotted the quantities

$$U_0(\omega)=(KT_1)^{-1}\,U(\omega) \text{ and } V_0(\omega)=(KT_1)^{-1}\,V(\omega).$$

In order to go to a characteristic corresponding to a definite value of KT_1, we need only make a simple change in the scale of the universal graph, i.e., multiply the numbers plotted along the coordinate axes by the quantity KT_1.

By interpolation, we can easily obtain the gain-phase characteristics of systems for which the values of $a = T_2/T_1$ are other than those shown in Fig. 37.

42. Construct the gain-phase characteristics of two systems which have the following open-loop transfer functions.

$$1.\ W_1(p)=\frac{20}{p(1+0{,}1p)}.$$

$$2.\ W_2(p)=\frac{200}{p(1+0{,}05p)(1+0{,}02p)}.$$

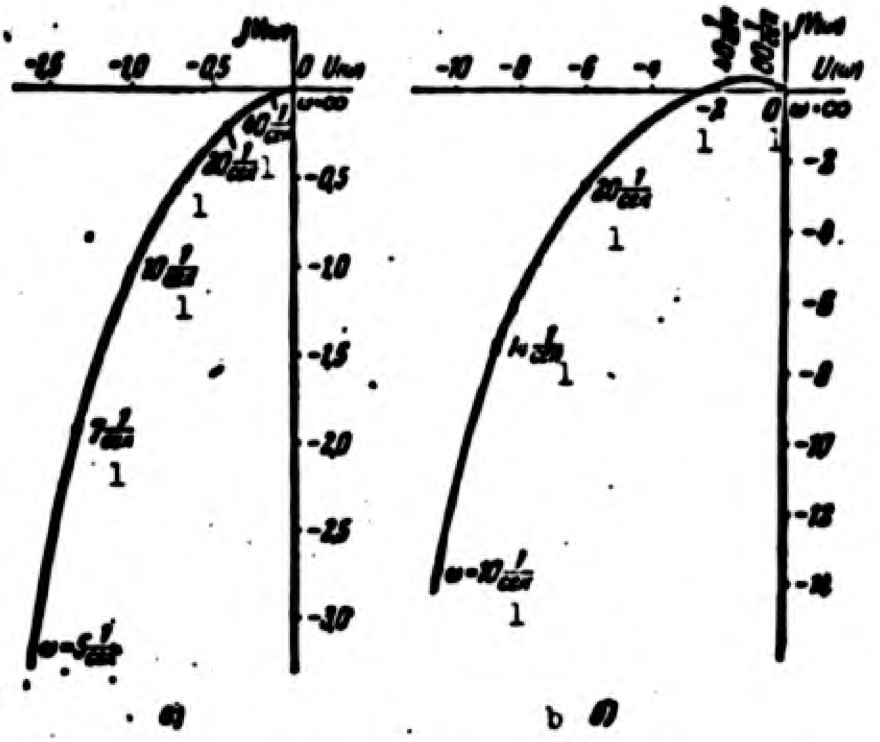


Fig. 38. Gain-phase characteristics for Problem 42. a) Curve for first system; b) curve for second system. 1) Sec.

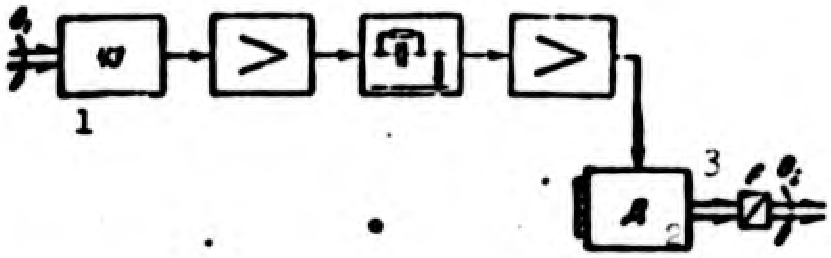


Fig. 39. Block diagram for Problem 43. 1) Sensing element; 2) motor; 3) reduction gear.

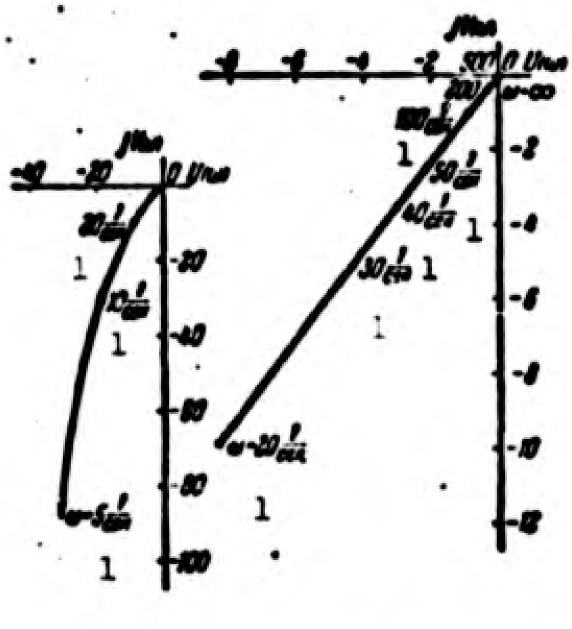


Fig. 40. Gain-phase characteristic for Problem 43. 1) Sec.

Hint. It is possible to use the curves available from the preceding problem.

Answer. See Fig. 38.

43. Construct the gain-phase characteristic of the system whose block diagram is shown in Fig. 39; here ChE is the sensing element, D is the motor, R is the reduction gear. The system open-loop transfer function has the form

$$W(p)=\frac{K(1+T_2p)}{p(1+T_1p)(1+T_3p)}=\frac{300(1+0.03p)}{p(1+0.1p)(1+0.0\text{[illegible]}p)}.$$

Answer. See Fig. 40.

44. Find the equation for the curve representing the gain-phase characteristic of the system having the transfer function

$$W(p)=\frac{K(1+Tp)}{p^2}.$$

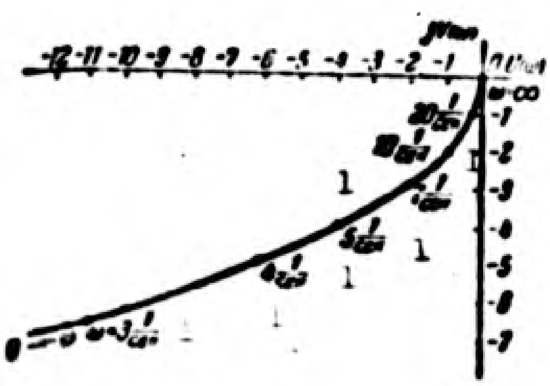


Fig. 41. Gain-phase characteristic in parabolic form for Problem 44. 1) Sec.

Construct the gain-phase characteristic for the case in which K = 100 1/sec^2 and T = 0.2 sec.

Solution. The frequency transfer function equals

$$W(j\omega)=\frac{K(1+jT\omega)}{-\omega^2}=U(\omega)+jV(\omega),$$

where

$$U(\omega)=-\frac{K}{\omega^2},\quad V(\omega)=-\frac{KT}{\omega}. \quad (1)$$

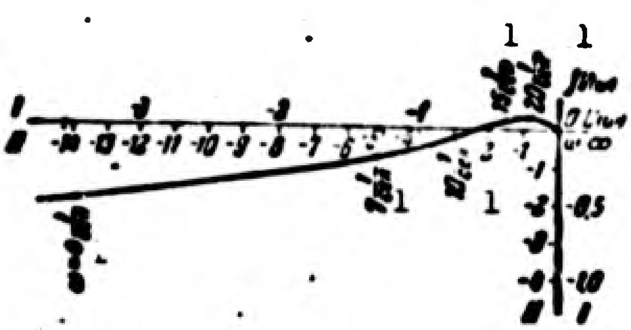


Fig. 42. Gain-phase characteristics for Problem 45. Scale I, K = 50 1/sec^2, scale II, K = 200 1/sec^2. 1) Sec.

From (1) we obtain

$$U(\omega) = -\frac{1}{KT^2} V^2(\omega). \qquad (2)$$

According to (1) and (2), the gain-phase characteristic for positive frequencies takes the form of a branch of a parabola lying in the third quadrant of the complex plane.

The point on the gain-phase characteristic corresponding to any frequency ω is easily found as the point at which the parabola intersects a ray drawn from the origin and forming the angle

$$\varphi = \operatorname{arctg} \frac{V(\omega)}{U(\omega)} = -T\omega.$$

with the real axis.

A gain-phase characteristic has been plotted for given parameters in Fig. 41.

45. Construct the gain-phase characteristic for a system with the following open-loop transfer function:

$$W(p) = \frac{K(1+0.15p)}{p^2(1+0.05p)}$$

where $K = 50\ 1/sec^2$ and $K = 200\ 1/sec^2$.

Answer. See Fig. 42.

46. Construct the gain-phase characteristic for a system with the following open-loop transfer function:

$$W(p) = \frac{K(1+0.2p)^2}{p^3(1+0.04p)}$$

for $K = 200\ 1/sec^3$ and $K = 100\ 1/sec^3$.

Answer. See Fig. 43.

47. Figure 44 shows a tachometer feedback circuit with passive compensating networks; TG is a tachometer generator. Construct the gain-phase characteristic for this circuit if its transfer function equals

$$W(p) = \frac{Kp^2}{(1+T_1p)^2(1+T_2p)}.$$

$K = 4$ v·sec^2/deg, $T_1 = 0.5$ sec, $T_2 = 0.1$ sec.

Answer. See Fig. 45, where the numbers plotted along the axes have the dimensions of v/degree.

48. Construct the gain-phase characteristic for a circuit having the transfer function

$$\Phi(p) = \frac{k}{p(-1+Tp)} = \frac{100}{p(-1+0.1p)}.$$

Answer. See Fig. 46.

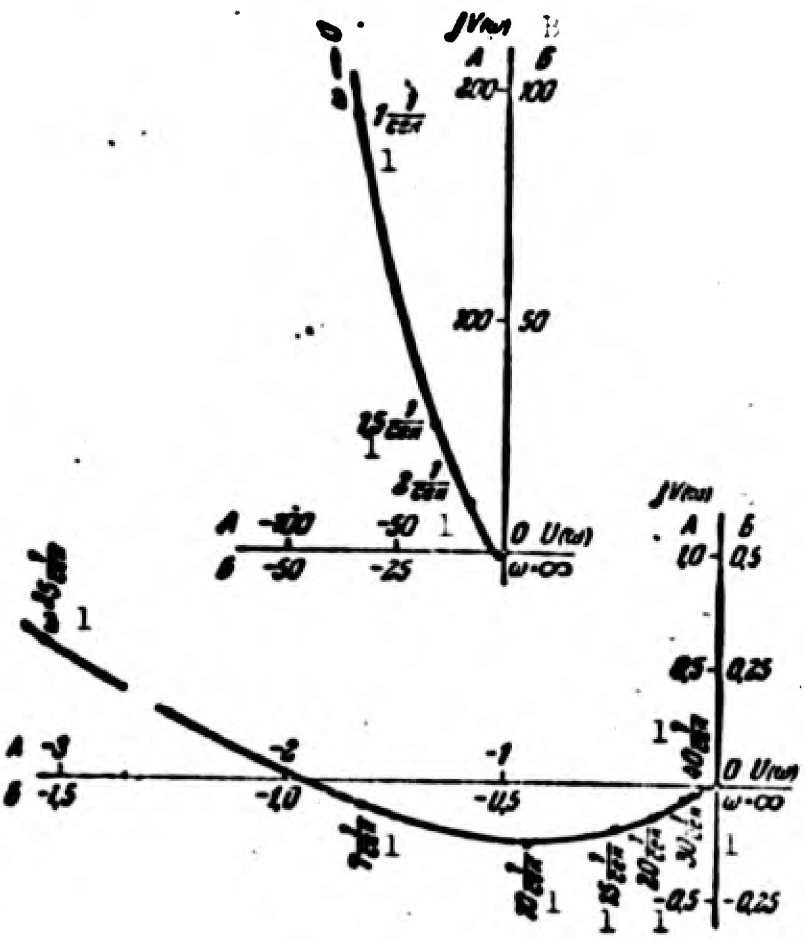


Fig. 43. Gain-phase characteristics for Problem 46. Scale A, K_3 = 200 1/sec^3, scale B, K = 100 1/sec^3. 1) Sec.

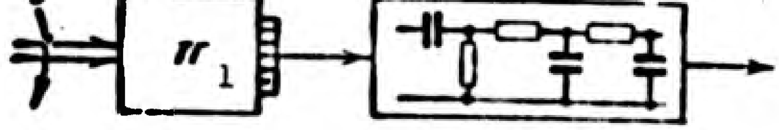


Fig. 44. Diagram for Problem 47. 1) Tachometer generator.

49. Construct the gain-phase characteristics of systems having the transfer functions

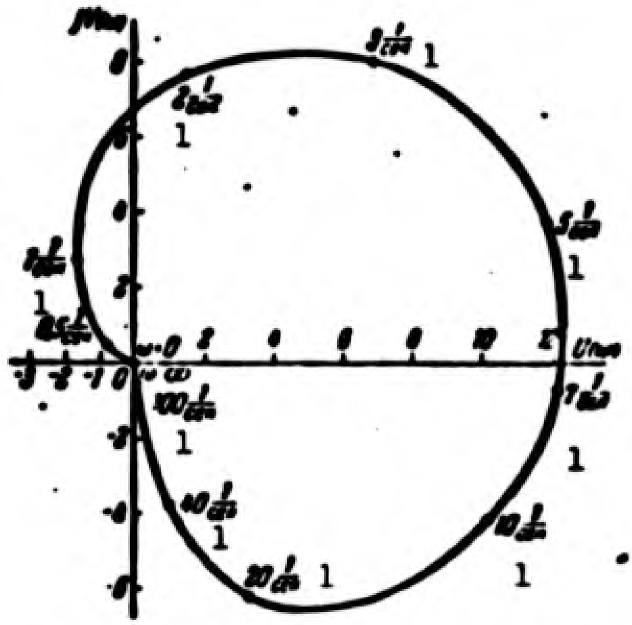


Fig. 45. Gain-phase characteristic for tachometer feedback circuit of Problem 47. 1) Sec.

(A) $W(p)=\frac{K}{(-1+2T_1p+T_2^2p^2)(1+T_3p)}$;

(B) $W(p)=\frac{K(1+T_4p)}{(-1+2T_1p+T_2^2p^2)(1+T_3p)(1+T_4p)}$,

for the case in which K = 5, T_1 = 0.1 sec, T_2 = 0.05 sec, T_3 = 0.03 sec, T_4 = 0.006 sec.

<u>Answer</u>. See Fig. 47.

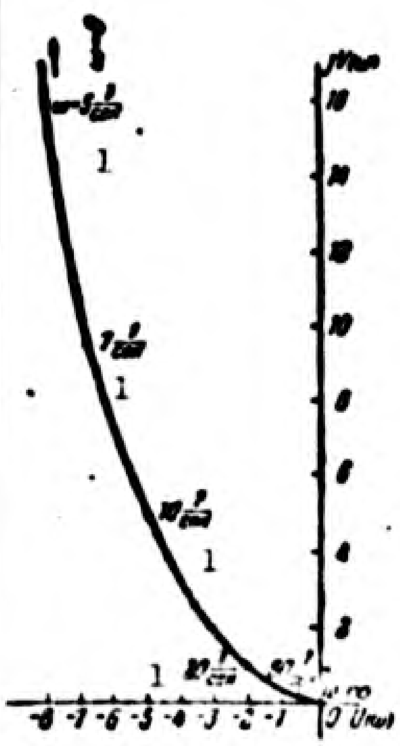


Fig. 46. Gain-phase characteristic for system with unstable element. 1) Sec.

50. Construct the amplitude-phase characteristic of a stable oscillating element with transfer function

$$w(p)=\frac{k}{1+2\xi Tp+T^2p^2}$$

when k = 1, ξ = 0.15, T = 0.02 sec.

<u>Answer</u>. See Fig. 48.

51. The block diagram of a gyro-stabilization system open at the input of the precession-angle pickoff may be represented [3] in the form shown in Fig. 49; DUP is the precession-angle pickoff, D is the motor, R the reducing gear, G the gyroscope. Where an amplifier with zero response

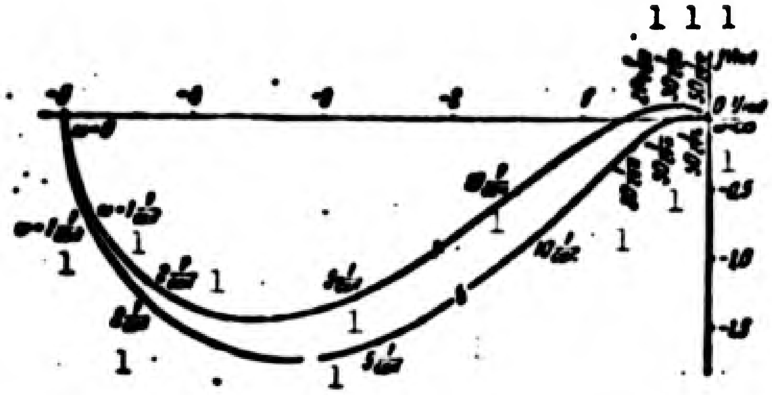


Fig. 47. Gain-phase characteristics for Problem 49. 1) Sec.

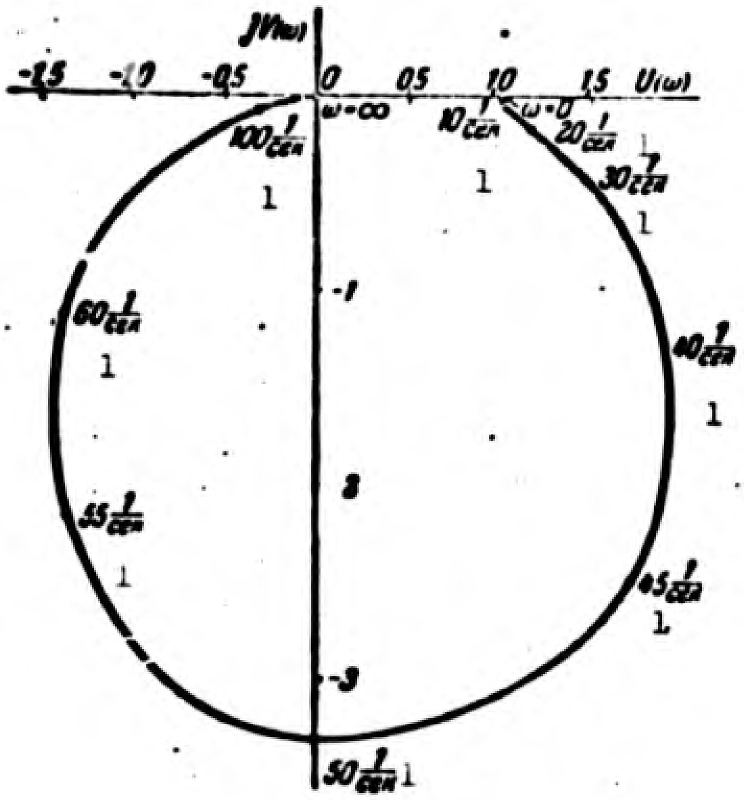


Fig. 48. Gain-phase characteristic of stable oscillating element. 1) Sec.

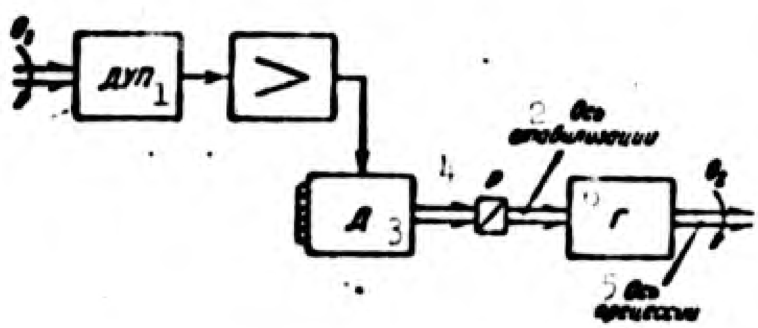


Fig. 49. Block diagram of gyro-stabilization system; see Problems 51 and 52. 1) Precession-angle pickoff; 2) stabilization axis; 3) motor; 4) reduction gear; 5) precession axis; 6) gyroscope.

time is used, the system open-loop transfer function may be written in the form

$$W(p)=\frac{K}{p(1+2\xi Tp+T^2p^2)}.$$

Construct the gain-phase characteristic of this system for $K = 20$ 1/sec, $\xi = 0.15$, $T_2 = 0.02$ sec.

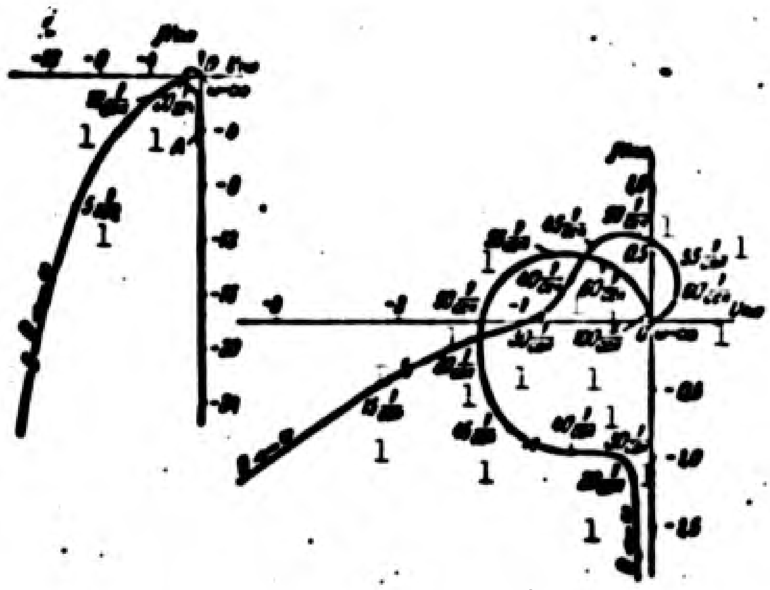


Fig. 50. Gain-phase characteristics of gyro systems.

Answer. See curve A of Fig. 50.

52. The gyro-stabilization system whose block diagram is shown in Fig. 49 (see also Problem 51), has the following open-loop transfer function when an amplifier with finite response time is used:

$$W(p)=\frac{K}{p(1+T_1p)(1+2\xi T_2p+T_2^2p^2)}.$$

Construct the gain-phase characteristic of this system for $K = 20$ 1/sec, $T_1 = 0.2$ sec, $T_2 = 0.02$ sec, $\xi = 0.15$.

Answer. See curve B of Fig. 50.

§5. REAL FREQUENCY CHARACTERISTICS

53. Construct the real frequency characteristic $P(\omega)$ of a closed-loop automatic control system. The system open-loop transfer function is

$$W(p)=\frac{K}{p(1+Tp)}. \qquad (1)$$

K = 20 1/sec, T = 0.1 sec.

Solution. The real frequency characteristic is constructed from points. These points may be found by various methods.

1. The real frequency characteristic $P(\omega)$ may be plotted from its analytic expression

$$P(\omega)=\mathrm{Re}[\Phi(j\omega)], \tag{2}$$

where $\Phi(j\omega)$ is the system closed-loop transfer function, equal to

$$\Phi(j\omega)=\frac{W(j\omega)}{1+W(j\omega)}. \tag{3}$$

From (3) and (1), we obtain

$$\Phi(j\omega)=\frac{K(K-T\omega^2)}{(K-T\omega^2)^2+\omega^2}-j\frac{K\omega}{(K-T\omega^2)^2+\omega^2}. \tag{4}$$

From (4) and (2), we obtain

$$P(\omega)=\frac{K(K-T\omega^2)}{(K-T\omega^2)^2+\omega^2}=\frac{20(20-0,1\omega^2)}{(20-0,1\omega^2)^2+\omega^2}. \tag{5}$$

Substituting various values of ω into (5), we obtain Table 1, which we use to plot $P(\omega)$.

TABLE 1

ω, 1/sec	0	5	7	10	15	19	20	25	30	40	50	60	∞
$P(\omega)$ [1]	1,00	1,06	1,04	1,00	−0,21	−0,2[illegible]	−0,30	−0,35	−0,21	−0,15	−0,09	−0,05	0

1) ω, 1/sec.

The real frequency characteristic is plotted in Fig. 51a on the basis of the data of Table 1.

2. If for several values of the frequency ω, we have the coordinates $U(\omega)$ and $V(\omega)$ of points on a system open-loop gain-phase characteristic (Table 2), then the corresponding values of $P(\omega)$ may be found from the formula

$$P(\omega)=\frac{U^2(\omega)+V^2(\omega)+U(\omega)}{[1+U(\omega)]^2+V^2(\omega)}. \tag{6}$$

For the case in which the coordinates of the points on the gain-phase characteristic are given in the form of the absolute value $a(\omega)$

TABLE 2

ω, 1/сек (1)	0	5	7	10	20	40	∞
$U(\omega)$	−∞	−1,60	−1,34	−1,00	−0,41	−0,13	0
$V(\omega)$	−∞	−3,18	−1,93	−1,00	−0,21	−0,02	0.

1) ω, 1/sec.

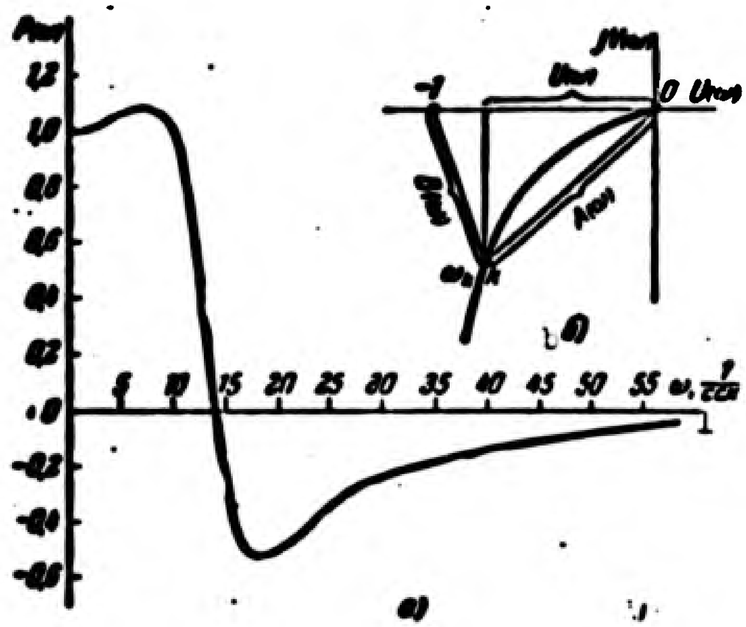


Fig. 51. Real frequency characteristic for Problem 53. 1) Sec.

and the argument $\psi(\omega)$ (Table 3) of the frequency transfer function $W(j\omega)$, the characteristic $P(\omega)$ may be plotted from the formula

$$P(\omega)=\frac{A^2(\omega)+A(\omega)\cos\psi(\omega)}{A^2(\omega)+2A(\omega)\cos\psi(\omega)+1}, \tag{7}$$

obtained from Formula (6) by the substitution

$$U^2(\omega)+V^2(\omega)=A^2(\omega) \text{ and } U(\omega)=A(\omega)\cos\psi(\omega).$$

TABLE 3

ω, 1/сек (1)	0	5	7	10	20	30	40	50	60	∞
$A(\omega)$	∞	3,56	2,34	1,41	0,118	0,211	0,121	0,078	0,054	0
$\psi(\omega)$, град (2)	−90	−116	−125	−135	−154	−162	−166	−169	−170	−180

1) ω, 1/sec; 2) $\psi(\omega)$, degrees.

3. If we have the gain-phase characteristic for the open-loop system, then it is convenient to use the formula

$$P(\omega)=\frac{A^2(\omega)+U(\omega)}{B^2(\omega)}, \qquad (8)$$

to plot $P(\omega)$; this formula is obtained from (6) by the substitutions $A^2(\omega) = U^2(\omega) + V^2(\omega)$ and $B^2(\omega) = [1 + U(\omega)]^2 + V^2(\omega)$. The values of $A(\omega)$ and $B(\omega)$ for each given frequency ω are easily obtained from the gain-phase characteristic, since $A(\omega)$ is the absolute value of the vector $W(j\omega)$, i.e., the distance from the origin to the given point $\underline{k}$ on the characteristic, while $B(\omega)$ it is the distance from the point (−1, j0) to the point $\underline{k}$ (see Fig. 51b).

The numbers entering into Formula (8) and needed for plotting of the real frequency characteristic given in Fig. 51a may be obtained from the gain-phase characteristic of the system with transfer function (1) shown in Fig. 38a.

4. In plotting the real frequency characteristic for a system when we have the gain-phase characteristic (Fig. 38a), we may use a nomogram, called a real circle diagram. Such a nomogram is given in Appendix 3.

54. Construct the real frequency characteristic $P(\omega)$ for a closed automatic control system if the system open-loop transfer function is

$$W(p)=\frac{500(1+0{,}03p)}{p(1+0{,}1p)(1+0{,}006p)}.$$

In plotting $P(\omega)$, we may use the gain-phase characteristic of the system shown in Fig. 40, or the following table of absolute values of $A(\omega)$ and the argument $\psi(\omega)$ of the system frequency transfer function.

ω, 1/сек 1)	0	10	20	30	40	50	100	200	500	∞
A(ω)	∞	36,8	13,0	6,85	4,6	3,38	1,35	0,488	0,093	0
ψ(ω), град 2)	−90	−122	−130	−130	−130	−130	−133	−136	−165	−180

1) ω, 1/sec; 2) ψ(ω), degrees.

<u>Answer</u>. See Fig. 52.

55. Construct the real frequency characteristic $P(\omega)$ for a closed system. The open-loop gain-phase characteristic of the system is shown

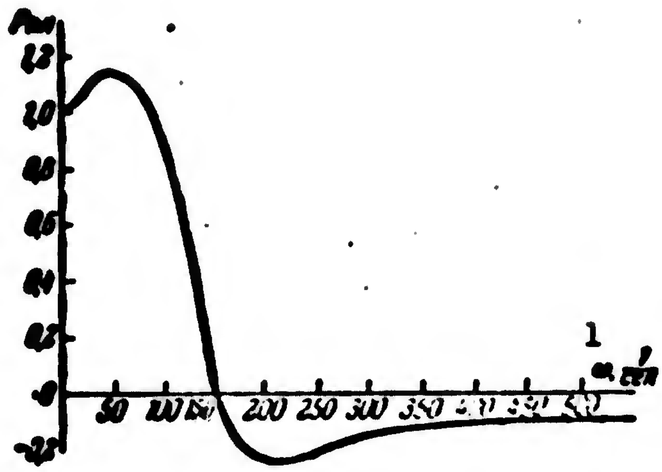


Fig. 52. Real frequency characteristic for Problem 54. 1) Sec.

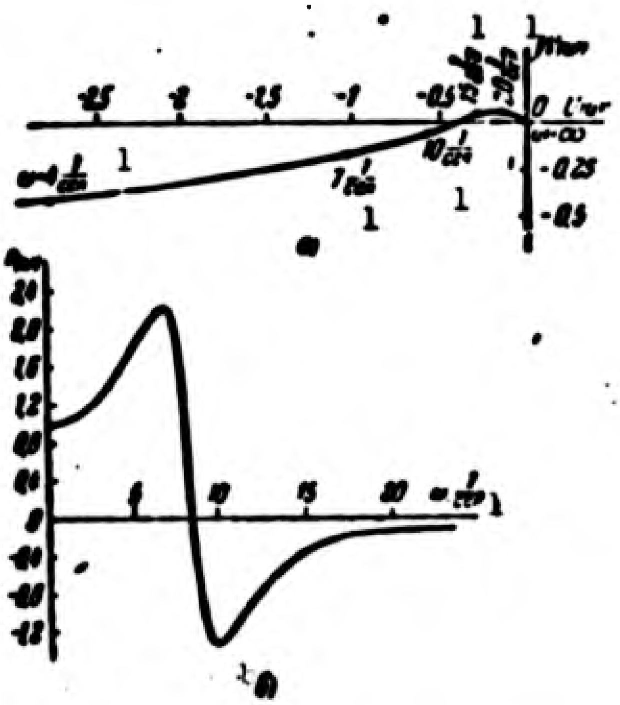


Fig. 53. Gain-phase and real frequency characteristic for Problem 55. 1) Sec.

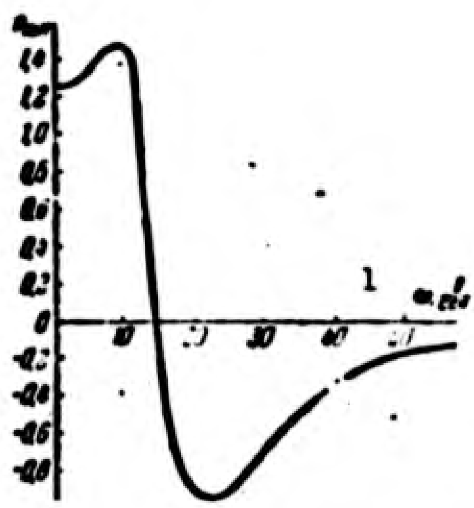


Fig. 54. Real frequency characteristic for Problem 56. 1) Sec.

in Fig. 53a. The data of the following table may be used in the construction:

ω, 1/sec [1]	0	2	4	7	10	15	20	∞
$A(\omega)$	∞	10,32	2,80	1,05	0,58	0,28	0,16	0
$\psi(\omega)$, degree [2]	−180	−175	−172	−172	−177	−188	−199	−180

1) ω, 1/sec; 2) $\psi(\omega)$, degree.

Answer. See Fig. 53b.

56. Construct the real frequency characteristic of a static closed-loop system. The open-loop gain-phase characteristic of the system is given by curve B of Fig. 47.

Fig. 55. Real frequency characteristic for Problem 57. 1) Sec.

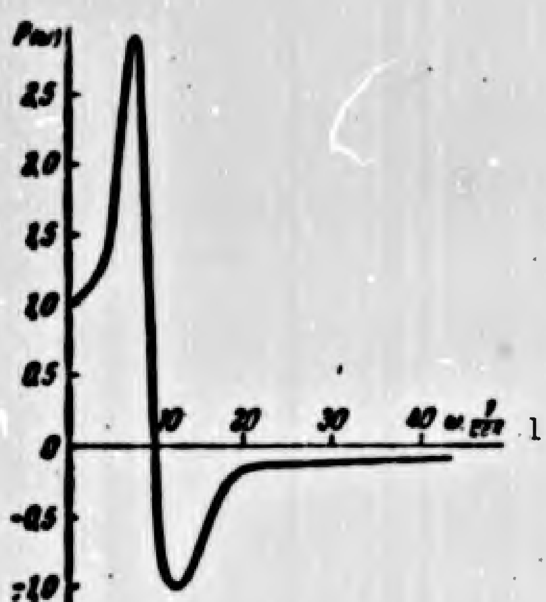


Fig. 56. Real frequency characteristic for Problem 58. 1) Sec.

Answer. See Fig. 54.

57. Construct the closed-loop real frequency characteristic for a static system. The system open-loop gain-phase characteristic is shown in Fig. 35.

Answer. See Fig. 55.

58. Construct the closed-loop real frequency characteristic of a system with third-order astatism. The system open-loop gain-phase characteristic is given in Fig. 43 (scale A).

Answer. See Fig. 56.

§6. LOGARITHM CHARACTERISTICS

59. Construct the logarithmic amplitude characteristic $L(\omega) = 20 \log |w(j\omega)|$ and the logarithmic phase characteristic $\psi(\omega)$ for an aperiodic element with transfer function

$$w(p)=\frac{k}{1+Tp} \qquad (1)$$

for the following two cases: 1) in universal form, suitable for any k or T; 2) for k = 100, T = 50 msec.

Solution. The logarithmic amplitude characteristic corresponding to Expression (1) equals

$$L(\omega)=20\lg|W(j\omega)|=20\lg\frac{k}{\sqrt{1+(\omega T)^2}}. \qquad (2)$$

The asymptotic logarithmic amplitude characteristic corresponding to (2) is plotted in Fig. 57a; the quantity ωT is plotted along the x axis in a logarithmic scale, and $L(\omega)$ along the y axis in decibels.

The asymptotic logarithmic amplitude characteristic (l.a.kh.) has, according to (2) an abrupt bend at the point at which $\omega T = 1$. To the left of this bend, it is horizontal and placed at a height of 20 log k; to the right of the bend it has a slope of −20 db/decade. The point at which the characteristic curve intersects the frequency axis, i.e., the cutoff frequency ω_s is found from the condition

$$L(\omega_s)\approx 20\lg\frac{k}{\omega_s T}=0 \text{ or } \omega_s=\frac{k}{T}.$$

The maximum departure of the asymptotic characteristic from the exact curve occurs at the point at which $\omega T = 1$; the deviation amounts to 3 db, as can be determined from Expression (2). For $\omega T = 0.5$ and $\omega T = 2$, the difference between the asymptotic characteristic and the exact curve equals roughly 1 db, while outside the limits of the section $\omega T = 1 \pm 1$ octave, this difference is negligible.

The phase characteristic of the element is determined, in accordance with (1), by the expression

$$\psi(\omega)=\arg W(j\omega)=-\operatorname{arctg}\omega T. \qquad (3)$$

For low frequencies, $\psi(\omega) \to 0$, for high frequencies $\psi(\omega) \to -90^\circ$; for $\omega T = 1$, $\psi(\omega) = -45^\circ$. From Expression (3), it also follows that the phase characteristic is symmetric about the point $\omega T = 1$, $\psi = -45^\circ$.

The phase characteristic of an aperiodic element with transfer

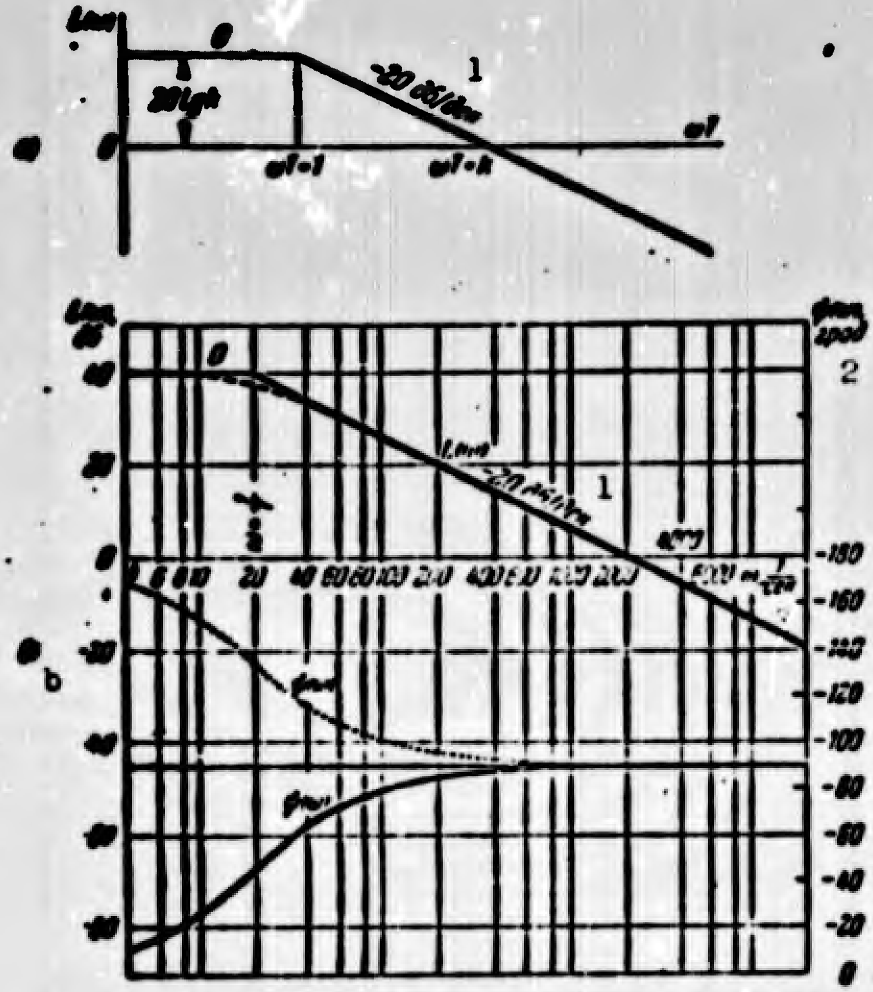


Fig. 57. Logarithmic characteristics of stable and unstable aperiodic elements; see Problems 59 and 60. 1) db/decade; 2) degree; 3) 1/sec.

function (1) has been plotted from (3) in the form $\psi(\omega T)$ in Appendix 7.

The following table was used in the construction:

ωT	0	0.05	0.1	0.2	0.5	1	2	5	10	20	∞
$\psi(\omega T)$ [1]	0	-2°50'	-5°40'	-11°20'	-26°30'	-45°	-63°30'	-78°40'	-84°20'	-87°10'	-90°

1) $\psi(\omega T)$, degrees.

The logarithmic amplitude and phase characteristics of an element with transfer function

$$w(p) = \frac{100}{1+0.05p} \qquad (4)$$

have been plotted in Fig. 57b; the dashed line indicates the exact amplitude characteristic plotted from Formula (2). The frequency ω has been plotted along the x axis in logarithmic units, while the y axis is in decibels and degrees.

60. Construct the logarithmic amplitude and phase characteristics of an unstable aperiodic element with transfer function

$$W(p)=\frac{100}{-1+0.05p}.$$

Answer. The amplitude characteristic $L(\omega)$ is the same as for a stable element with transfer function (4) of the preceding example (see Fig. 57b).

The phase characteristic $\psi(\omega)$ is given in Figs. 57b by the plotted curve.

61. An automatic control system has the block diagram shown in Fig. 36; ChE is the sensing element, D is the motor, R is the reduction gear. If the amplifier is an element with zero response time, the system open-loop transfer function will take the form

$$W(p)=\frac{K}{p(1+Tp)}.$$

Construct the logarithmic amplitude characteristic $L(\omega)$ and the logarithmic phase characteristic $\psi(\omega)$ of the system when $K = 400$ 1/sec for the following three cases: 1) $T = 25$ msec, 2) $T = 5$ msec, 3) $T = 2.5$ msec.

Hint. In constructing the phase characteristic, it is desirable to make use of Appendix 7.

Answer. See Fig. 58; the subscripts on $L(\omega)$ and $\psi(\omega)$ indicate the case numbers. For the first case ($T = 25$ msec), the dashed line shows the exact amplitude characteristic.

62. Construct the logarithmic amplitude and phase characteristics for the system having the transfer function

$$W(p)=\frac{40}{1+0.12p+0.002p^2}. \qquad (1)$$

Solution. In order to construct the logarithmic characteristics, we

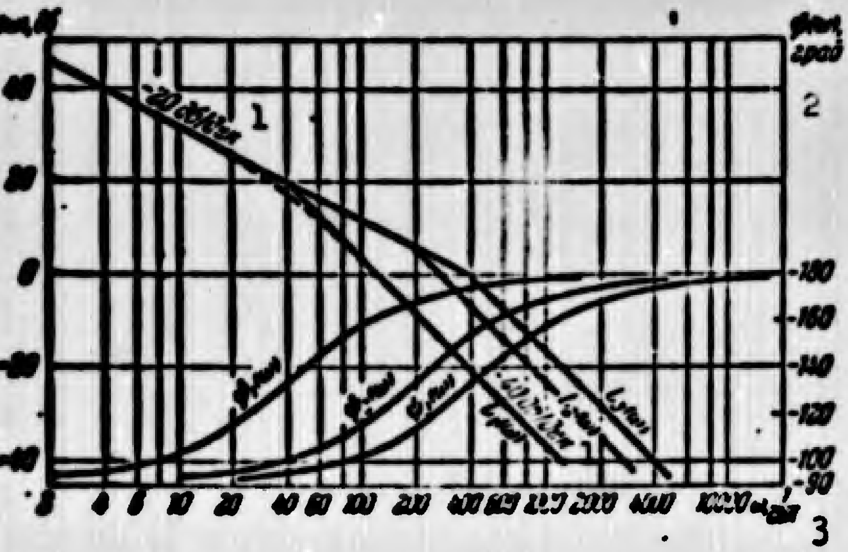


Fig. 58. Logarithmic characteristics for Problem 61. 1) db/decade; 2) degrees; 3) 1/sec.

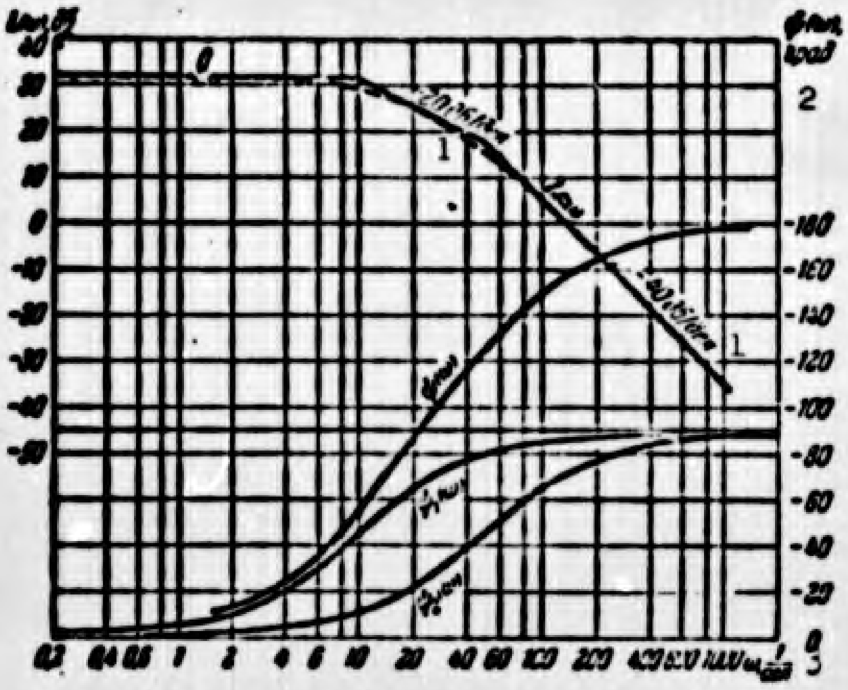


Fig. 59. Logarithmic characteristics for Problem 62. 1) db/decade; 2) degrees; 3) 1/sec.

must separate the denominator of (1) into two factors.

To do this, we find the roots of the denominator, which prove equal to −10 1/sec and −5 1/sec, and represent (1) in the form

$$W(p)=\frac{40}{(1+T_1p)(1+T_2p)}=\frac{40}{(1+0.1p)(1+0.02p)}. \quad (2)$$

From this we obtain the logarithmic amplitude characteristic for the system

$$L(\omega)=20\lg\left|\frac{40}{(1+j0{,}1\omega)(1+j0{,}02\omega)}\right|=$$
$$=20\lg\frac{40}{\sqrt{[1+(0{,}1\omega)^2][1+(0{,}02\omega)^2]}}. \qquad (3)$$

From Expression (2) or (3) it follows that the asymptotic l.a.kh. has two abrupt bends. At the points $\omega_1 = 1/T_1 = 10$ 1/sec and $\omega_2 = 1/T_2 = 50$ 1/sec. The characteristic will consist of three segments: a horizontal segment at a height of 20 log 40 = 32 db, a segment with a slope of −20 db/decade, and a segment with a slope of −40 db/decade. This asymptotic characteristic is shown in Fig. 59.

Since the ratio $T_1/T_2 = 5$, i.e., is greater than two octaves, it then follows from the solution of Problem 59 that the difference between the asymptotic amplitude characteristic and the exact curve in the area of each bend will have the same form as for the aperiodic element, and the deviation will not exceed 3 db.

The phase characteristic will have the form

$$\psi(\omega)=-\operatorname{arctg}0{,}1\omega-\operatorname{arctg}0{,}02\omega. \qquad (4)$$

This last expression enables us to construct $\psi(\omega)$ on the basis of the points; it is simpler, however, to plot $\psi(\omega)$ as the sum of the y-axis values of the phase characteristics $\psi_1(\omega)$ and $\psi_2(\omega)$ for two aperiodic elements with time constants $T_1 = 0.1$ sec and $T_2 = 0.02$ sec, since each of these characteristics is easily plotted with the aid of the graphs of Appendix 2.

The phase characteristic $\psi(\omega)$ of the systems given in Fig. 59.

63. Construct the logarithmic amplitude and phase characteristics of a system having the transfer function

$$W(p)=\frac{K}{(1+Tp)^3}=\frac{300}{(1+0{,}02p)^3}.$$

Answer. See Fig. 60. It is clear from the figure that in plotting logarithmic characteristics we need not construct a logarithmic frequency grid; it is sufficient to make appropriate markers on the fre-

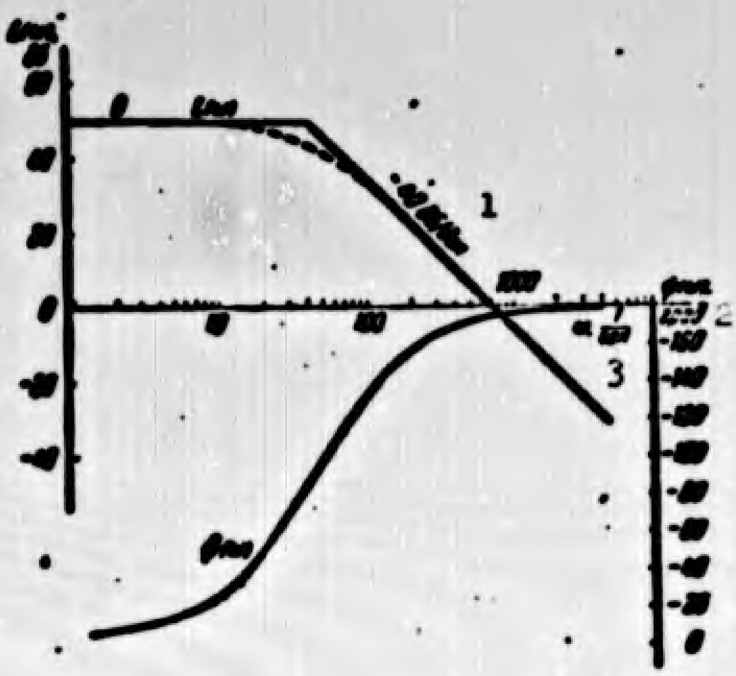


Fig. 60. Logarithmic characteristics for Problem 63. 1) db/decade; 2) degree; 3) 1/sec.

quency axis.

A slide-rule scale is normally used to make these markers; the <u>k</u> scale of a small sliderule is convenient to use.

64. Construct the logarithmic amplitude and phase characteristics of a stable oscillating element with transfer function

$$W(p)=\frac{k}{1+2\xi Tp+T^2p^2}. \qquad (1)$$

Consider the following cases:

1) the characteristics $L(\omega T)$ and $\psi(\omega T)$ for $k = 1$ and $\xi = 0.05$, 0.10, ..., 0.8, 1.0;

2) the characteristics $L(\omega)$ and $\psi(\omega)$ for $k = 30$, $\xi = 0.2$, $T = 50$ msec.

<u>Solution</u>. 1) The frequency transfer function corresponding to (1) for $k = 1$ is

$$W(j\omega)=\frac{1}{(1-T^2\omega^2)+j2\xi T\omega}. \qquad (2)$$

From (2) we find the logarithmic amplitude characteristic:

$$L(\omega T)=20\lg\frac{1}{\sqrt{[1-(T\omega)^2]^2+4\xi^2(T\omega)^2}}\qquad(3)$$

and the logarithmic phase characteristic:

$$\psi(\omega T)=-\operatorname{arctg}\frac{2\xi\omega T}{1-(\omega T)^2}.\qquad(4)$$

Using Formulas (3) and (4), we plot the amplitude and phase characteristics, using various values of ξ from 0.05 to 1.0. These characteristic curves are given in Appendix 8.

The amplitude characteristic (3) has two asymptotes:

$$\left.\begin{aligned}L'(\omega T)&=20\lg 1=0 \text{ for } \omega T<1\\ L''(\omega T)&=-20\lg(\omega T)^2 \text{ for } \omega T>1.\end{aligned}\right\}\qquad(5)$$

and

The asymptotic amplitude characteristic found from Expressions (5) is plotted in Fig. 61a.

For an oscillating element, the asymptotic amplitude characteristic may depart sharply from the exact characteristic, as we can see by comparing Fig. 61a with the figure given in Appendix 8. Thus for an oscillating element it is usual to construct an exact amplitude characteristic. This construction is conveniently made by summing the y-axis values of the asymptotic characteristic and the y-axis values of the curve representing the difference $\Delta L(\omega)$ between the asymptotic characteristic and the exact characteristic; such a curve is given in Appendix 9.

2) The characteristics $L(\omega)$ and $\psi(\omega)$ an element with transfer function

$$W(p)=\frac{30}{1+2\cdot 0{,}2\cdot 0{,}05p+0{,}0025p^2}=\frac{30}{1+0{,}02p+0{,}0025p^2},\qquad(6)$$

are plotted with the aid of Appendices 8 and 9; they are shown in Fig. 61b.

65. Construct the logarithmic amplitude and phase characteristics

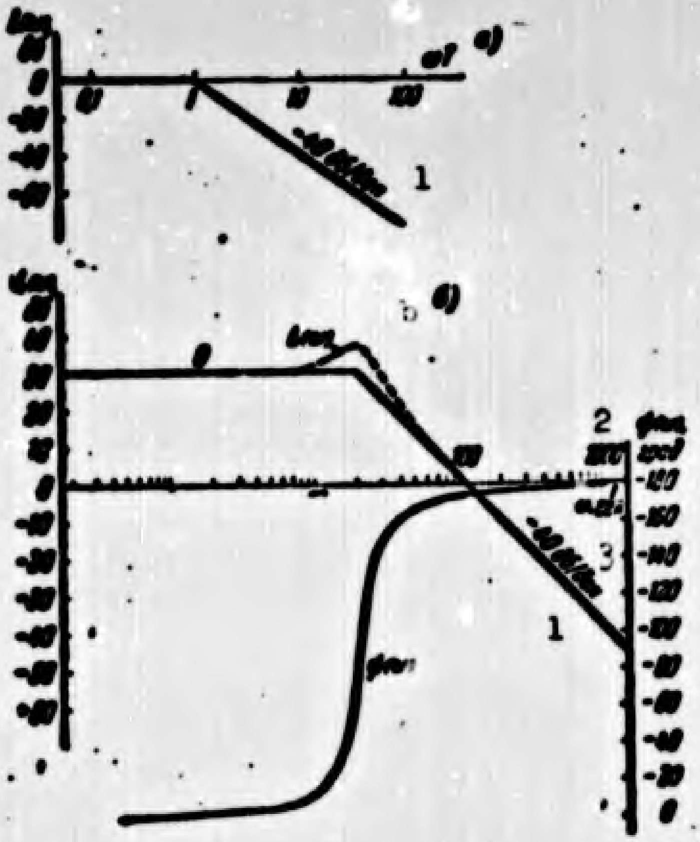


Fig. 61. Logarithmic characteristics of oscillating element. 1) db/decade; 2) degrees; 3) 1/sec.

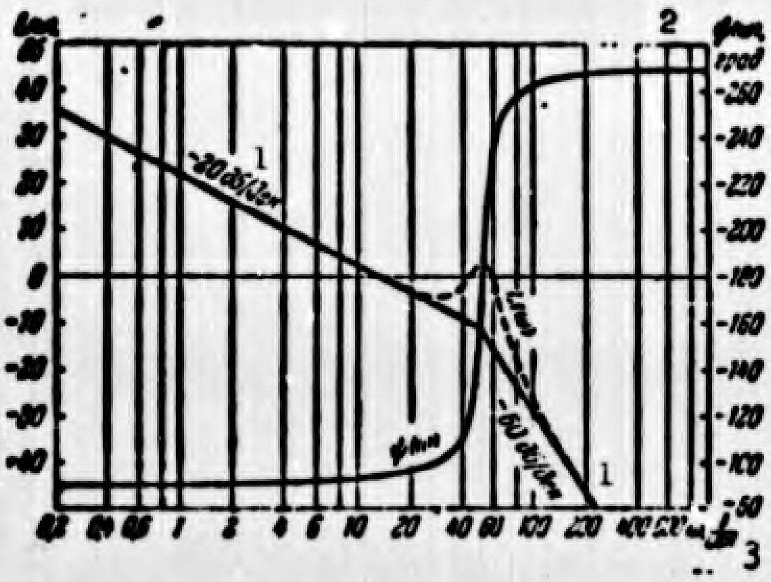


Fig. 62. Logarithmic characteristics for Problem 66. 1) db/decade; 2) degrees; 3) 1/sec.

for an unstable oscillating element with transfer function

$$W(p)=\frac{k}{1-2\xi Tp+T^2p^2}$$

for k = 30, T = 50 msec, ξ = 0.2.

<u>Answer</u>. The amplituce characteristic coincides with L(ω) for the

stable oscillating element of the preceding example, which has transfer function (6) (see Fig. 61b). The phase characteristic differs from $\psi(\omega)$ for the element with transfer function (6) only in the sign.

66. Construct the logarithmic amplitude and phase characteristics for a system with transfer function

$$W(p)=\frac{12{,}5}{p(1+0{,}004p+0{,}0004p^2)}.$$

<u>Hint.</u> The transfer function should be reduced to a form convenient for plotting of logarithmic characteristics, i.e., we should determine whether or not the quadratic polynomial in the denominator corresponds to two aperiodic elements or whether it corresponds to an oscillating element, and we must then find the necessary parameters of these elements.

<u>Answer</u>. See Fig. 62.

67. An automatic control system whose block diagram is set up in accordance with the specimen shown in Fig. 39 has the open-loop transfer function

$$W(p)=\frac{K(1+T_1p)}{p(1+T_2p)(1+T_3p)(1+T_4p)}=$$
$$=\frac{K(1+0{,}017p)}{p(1+0{,}05p)(1+0{,}0025p)(1+0{,}001p)}. \qquad (1)$$

Construct the logarithmic asymptotic amplitude and logarithmic phase characteristics of the system for two values of gain: K = 500 1/sec and K = 2000 1/sec.

<u>Solution</u>. The frequency transfer function corresponding to (1) takes the form

$$W(j\omega)=\frac{K(1+j0{,}017\omega)}{j\omega(1+j0{,}05\omega)(1+j0{,}0025\omega)(1+j0{,}001\omega)}. \qquad (2)$$

From Expression (2) or from Expression (1) it is clear that the asymptotic amplitude characteristic takes the form of a broken line with segments having negative slopes of 20, 40, 20, 40, 60 db/decade,

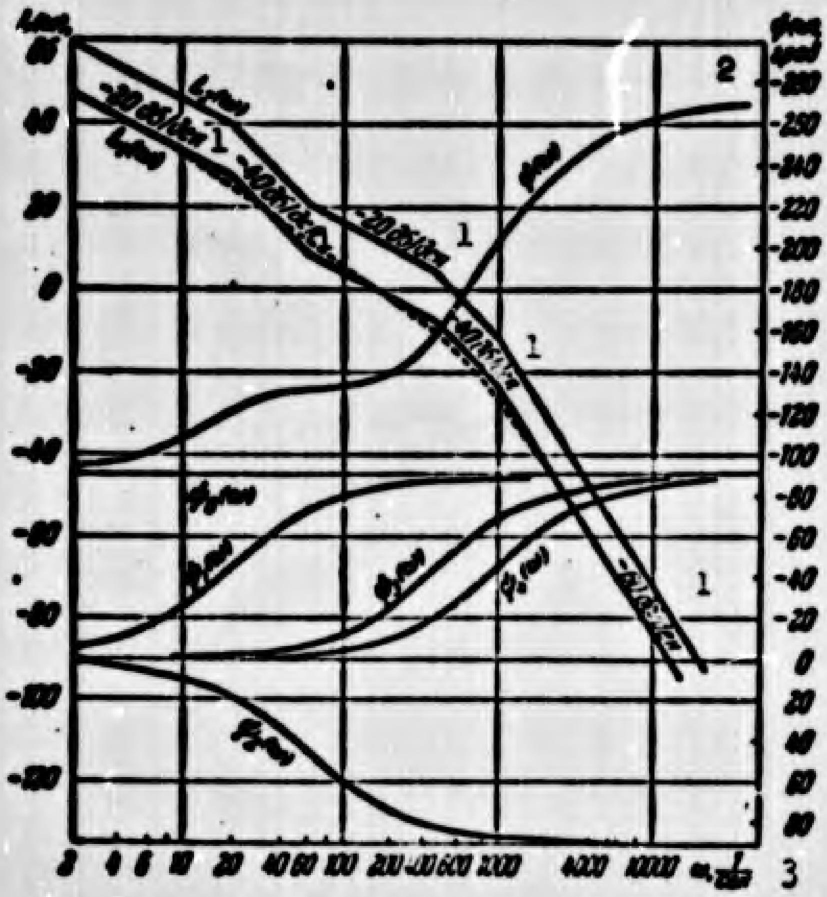


Fig. 63. Logarithmic characteristics for Problem 67. 1) db/decade; 2) degree; 3) 1/sec.

ılth abrupt bends at the points $\omega_1 = 1/T_1 = 20$ 1/sec, $\omega_2 = 1/T_2 = 60$ 1/sec, $\omega_3 = 1/T_3 = 400$ 1/sec, $\omega_4 = 1/T_4 = 1000$ 1/sec; the first part of the characteristic curve is a straight-line segment with slope −20 db/decade cutting the frequency axis at the point $\omega = K$.

The asymptotic amplitude characteristics $L_1(\omega)$ for the case in hich K = 500 1/sec and $L_2(\omega)$ for the case in which K = 2000 1/sec are hown in Fig. 63.

The phase characteristic is the same for both cases and, from (1) r (2), can be found as the sum of the y-axis values of the phase characteristic $\psi_0(\omega)$ of an ideal integrating element, the phase characterstics $\psi_1(\omega)$, $\psi_3(\omega)$, and $\psi_4(\omega)$ of aperiodic elements with time constants T_1, T_3, and T_4, and $\psi_2(\omega)$ – for a differentiating element with time onstant T_2.

These phase characteristics for the elements and the resultant phase characteristic $\psi(\omega)$ for the entire system are plotted in Fig. 63.

68. Construct the logarithmic asymptotic amplitude characteristic $L(\omega)$ and the logarithmic phase characteristic $\psi(\omega)$ for a system with transfer function

$$W(p)=\frac{K(1+T_1 p)}{p^2(1+T_2 p)(1+T_3 p)}$$

where $K = 75$ 1/sec^2, $T_1 = 200$ msec, $T_2 = 25$ msec, $T_3 = 6$ msec.

Answer. See Fig. 64.

69. Construct the logarithmic amplitude and phase characteristics of the system with transfer function

$$W(p)=\frac{K(1+T_1 p)^2}{p^3(1+T_2 p)(1+T_3 p)}=\frac{K(1+0.25p)^2}{p^3(1+0.025p)(1+0.005p)}$$

for the following three cases: 1) $K = 250$ 1/sec^3, 2) $K = 75$ 1/sec^3, 3) $K = 1000$ 1/sec^3.

Answer. Figure 65 shows the asymptotic amplitude characteristics $L_1(\omega)$, $L_2(\omega)$, and $L_3(\omega)$; here the subscripts indicate the case number; for case 1, the dashed line gives the exact amplitude characteristic. The phase characteristic $\psi(\omega)$ is exactly the same for all cases.

70. Construct the logarithmic amplitude and phase characteristics of a system with transfer function

$$W(p)=\frac{20}{p(1+0.101p+0.0005p^2+0.0001p^3)}.$$

Hint. The transfer-function denominator must be factored so as to reduce W(p) to a form convenient for plotting of the logarithmic characteristics.

Answer. The amplitude characteristic $L(\omega)$ and phase characteristic $\psi(\omega)$ are plotted in Fig. 66.

71. Construct the logarithmic amplitude and phase characteristics for a system with transfer function

$$W(p)=\frac{Kp}{(1+T_1 p)(1+2\xi T p+T^2p^2)}$$

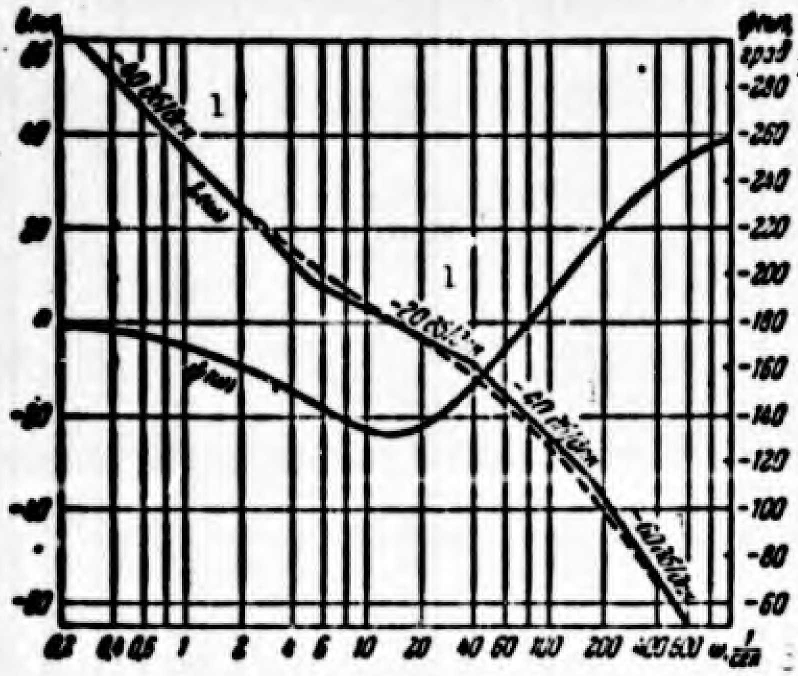


Fig. 64. Logarithmic characteristics for Problem 68. 1) db/decade; 2) degree; 3) 1/sec.

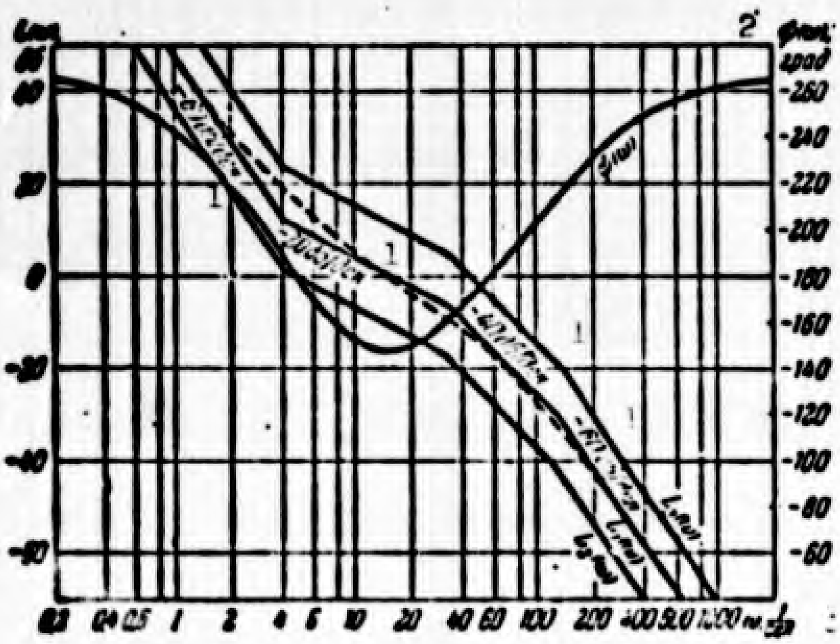


Fig. 65. Logarithmic characteristics for Problem 69. 1) db/decade; 2) degree; 3) 1/sec.

where K = 0.0645 sec, T_1 = 30 msec, T_2 = 7 msec, ξ = 0.2.

<u>Answer.</u> See Fig. 67.

72. Construct the logarithmic gain-phase characteristics for a system with transfer function

$$W(p)=\frac{K(1+T_{[illegible]}p)}{p(1+T_{[illegible]}p)(1+T_{[illegible]}p)(1+T_{[illegible]}p)}=$$

$$= \frac{K(1 + 0.017p)}{p(1 + 0.05p)(1 + 0.0025p)(1 + 0.001p)}$$

for the following two cases: 1) $K = K_1 = 500$ 1/sec, 2) $K = K_2 = 2000$ 1/sec.

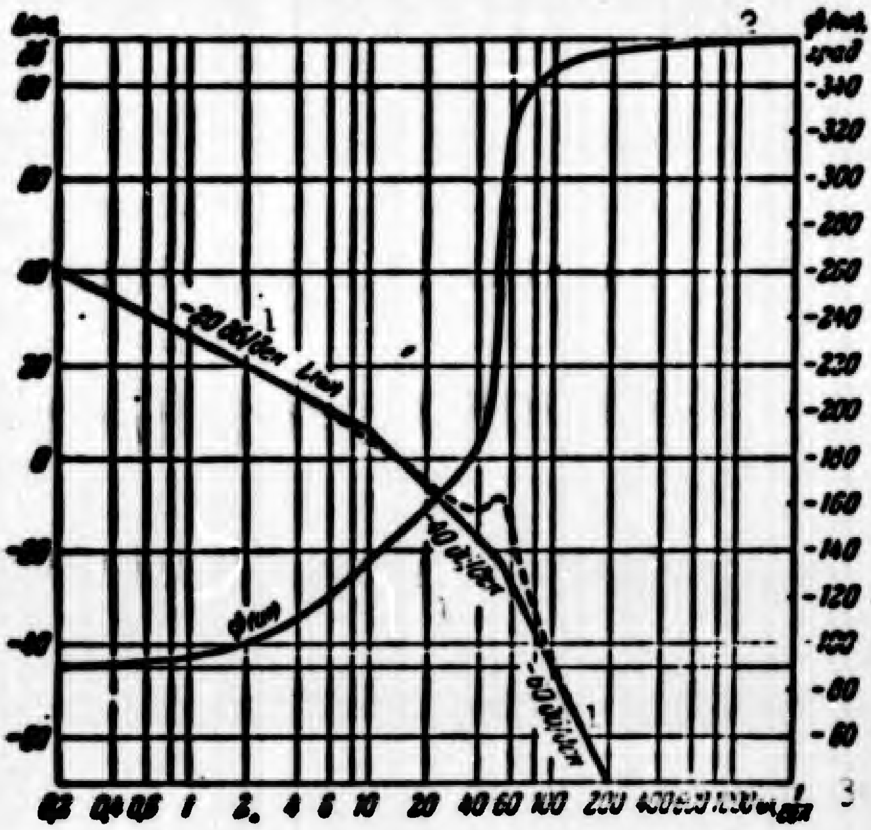


Fig. 66. Logarithmic characteristics for Problem 70. 1) db/decade; 2) degree; 3) 1/sec.

<u>Solution.</u> In order to plot the logarithmic gain-phase characteristic $20 \log |W(j\omega)| = f[\psi(j\omega)]$, we first construct the logarithmic amplitude and phase characteristics for the system. Using these characteristics $L_1(\omega)$ and $\psi(\omega)$, which are shown in Fig. 63 (see Problem 67), we make use of the points to construct the logarithmic gain-phase characteristic for the case in which $K = K_1 = 500$ 1/sec. This characteristic is shown in Fig. 68 (Curve 1). The numbers near the markers on the curve indicate the corresponding values of the frequency ω in 1/sec.

The high-frequency part of the curve for which $\psi(\omega) < -180^\circ$ is replaced by its mirror image with respect to the y-axis. For this part of the curve, shown in the figure by dashed lines, an additional scale of angles running from -180° to -280° is provided on the x-axis. The

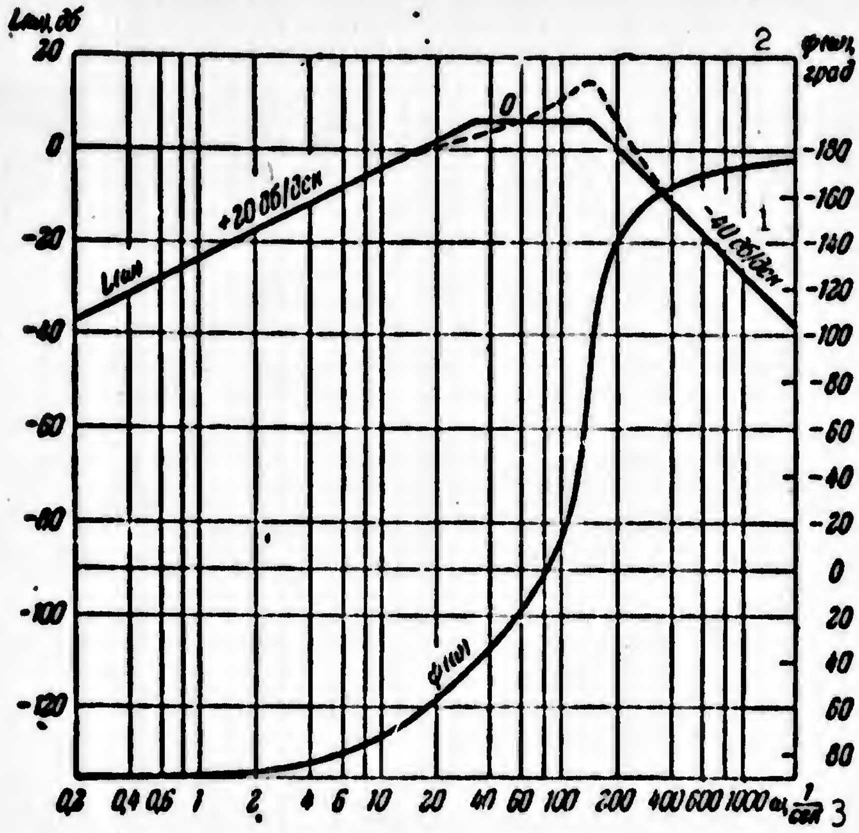


Fig. 67. Logarithmic characteristics for Problem 71. 1) db/decade; 2) degree; 3) 1/sec.

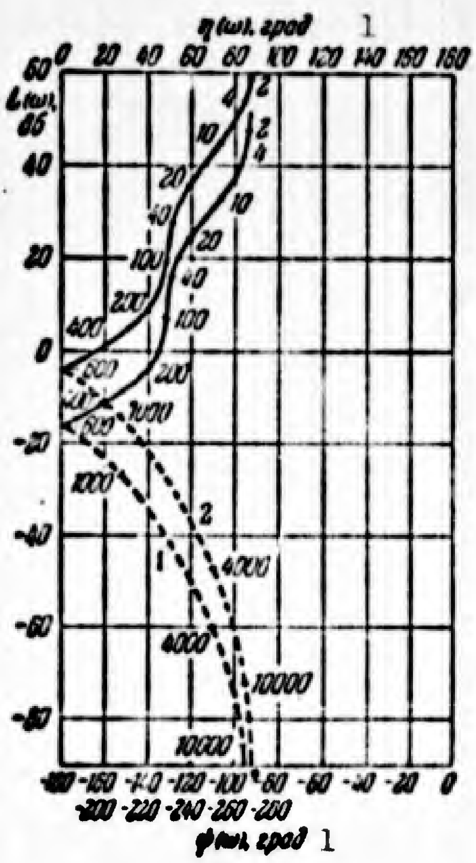


Fig. 68. Logarithmic gain-phase characteristics for Problem 72. 1) Degrees.

figure also has a scale for the phase margin, which equals $\eta(\omega) = \psi(\omega) \pm 180^\circ$.

A similar curve for the case in which $K = K_2 = 2000$ 1/sec may be plotted by shifting all points on curve 1 upward by 12 db ($20 \log K_2/K_1 = 12$ db); see Fig. 68, curve 2.

73. Construct the logarithmic gain-phase characteristics for a system with transfer function

$$W(p) = \frac{K(1+T_1 p)}{p^2(1+T_2 p)(1+T_3 p)} = \frac{K(1+0{,}2p)}{p^2(1+0{,}025p)(1+0{,}008p)}$$

for the following two cases: 1) $K = 75$ 1/sec^2, 2) $K = 400$ 1/sec^2.

Hint. It is possible to make use of the solution for Problem 68.

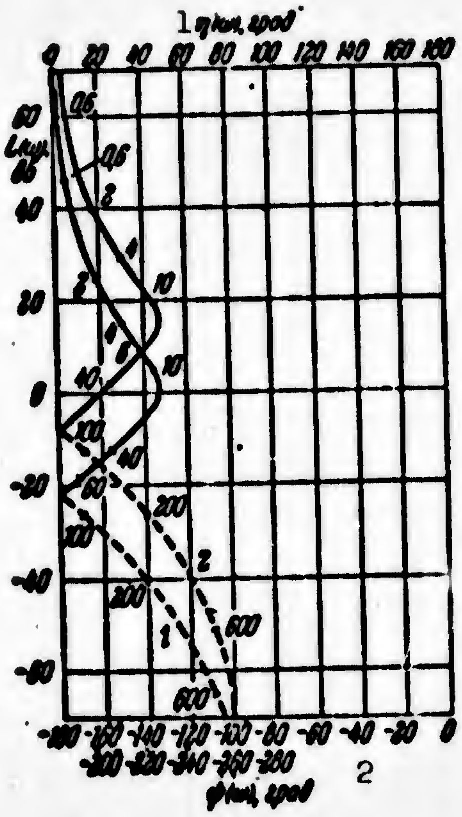


Fig. 69. Logarithmic gain-phase characteristics for Problem 73. 1) Degrees.

<u>Answer</u>. See Fig. 69, where curve 1 refers to the first case and curve 2 to the second.

Manuscript Page No.	[Transliterated Symbols]
31	вх = vkh = vkhodnoy = input
31	вйх = vykh = vykhodnoy = output
34	ЧЭ = ChE = chuvstvitel'nyy element = sensing element
34	Д = D = dvigatel' = motor
34	Р = R = reduktor = reduction gear
39	ТГ = TG = takhogenerator = tachometer generator
41	ДУП = DUP = datchik ugla pretsessii = precession angle pick-off
41	Г = G = giroskop = gyroscope
49	с = s = srez = cutoff
53	л.а.х. = l.a.kh. = logarifmicheskaya amplitudnaya kharakteristika = logarithmic amplitude characteristic

Chapter 3

STABILITY DETERMINATIONS FOR LINEAR AUTOMATIC SYSTEMS

§7. STABILITY INVESTIGATIONS USING THE ALGEBRAIC VYSHNEGRADSKIY AND HURWITZ CRITERIA.

74. The differential equation for an automatic control system has the form

$$[(T_0p+1)(T_1p+1)(T_2p+1)+k_0k_r]x = \\ =(T_1p+1)(T_2p+1)f(t).$$

The parameters have the following values: T_0 = 0.02 sec, T_1 = 0.01 sec, T_2 = 0.05 sec, k_0 = 20, k_r = 0.2. Determine the stability of the automatic system.

<u>Solution</u>. In accordance with the differential equation, the characteristic equation will be

$$a_0p^3+a_1p^2+a_2p+a_3=0,$$

where

$$a_0=T_0T_1T_2=10^{-5},\quad a_1=T_0T_1+T_0T_2+T_1T_2=1,7\cdot10^{-3},$$
$$a_2=T_0+T_1+T_2=8\cdot10^{-2},\quad a_3=1+k_0k_r=5.$$

Employing the Vyshnegradskiy stability criterion, we have for all positive coefficients

$$a_0>0,\quad a_1>0,\quad a_2>0,\quad a_3>0,$$
$$a_1a_2=1,7\cdot10^{-3}\cdot8\cdot10^{-2}=1,36\cdot10^{-4},$$
$$a_0a_3=10^{-5}\cdot5=0,5\cdot10^{-4},$$

i.e.,

$$a_1a_2>a_0a_3,$$

and as a consequence, the system is stable.

75. We are given the characteristic equation of a fourth-order au-

tomatic system

$$a_0p^4+a_1p^3+a_2p^2+a_3p+a_4=0.$$

The equation coefficients have the following values:

$$a_0=2\cdot10^{-9},\ a_1=2\cdot10^{-6},\ a_2=3\cdot10^{-3},$$
$$a_3=1{,}3\cdot10^{-1},\ a_4=100.$$

Determine system stability.

Solution. Using the Hurwitz stability criterion, we see whether or not the determinants of the matrix of characteristic-equation coefficients are

$$\begin{vmatrix} a_1 & a_3 & 0 & 0 \\ a_0 & a_2 & a_4 & 0 \\ 0 & a_1 & a_3 & 0 \\ 0 & a_0 & a_2 & a_4 \end{vmatrix}$$

are positive.

Evaluating the determinants, we obtain:

$$\Delta_1=a_1=2\cdot10^{-6}>0,$$
$$\Delta_2=\begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix}=a_1a_2-a_0a_3=2\cdot10^{-6}\cdot3\cdot10^{-3}-2\cdot10^{-9}\cdot1{,}3\cdot10^{-1}=$$
$$=6\cdot10^{-9}-0{,}026\cdot10^{-9}=5{,}074\cdot10^{-9}>0,$$
$$\Delta_3=\begin{vmatrix} a_1 & a_3 & 0 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix}=a_1a_2a_3-a_0a_3^2-a_1^2a_4=$$
$$=a_3(a_1a_2-a_0a_3)-a_1^2a_4=$$
$$=1{,}3\cdot10^{-1}(2\cdot10^{-6}\cdot3\cdot10^{-3}-2\cdot10^{-9}\cdot1{,}3\cdot10^{-1})-4\cdot10^{-12}\cdot10^{2}=$$
$$=7{,}47\cdot10^{-10}-40\cdot10^{-10}=-32{,}53\cdot10^{-10}<0,$$

and, as a consequence, the system is unstable.

It should be remembered that the determinant Δ_2 is contained as a factor in the positive part of the determinant Δ_3 and that the latter can be positive when $a_3 > 0$ only where $\Delta_2 > 0$. Thus, for a fourth-order system, there is no reason to test to see that Δ_2 is positive. Nor is there any need to check the last Δ_n of the determinant for systems of any order to see if they are positive, since $\Delta_n = a_n\Delta_{n-1}$ and when $a_n > 0$, it is sufficient to check all determinants up to Δ_{n-1} to see that they are positive. Thus, where the coefficients of the characteristic equation for fourth-order system are positive, we need only check to see that the inequality

$$a_3(a_1a_2-a_0a_3)-a_1^2a_4>0.$$

is satisfied.

For a fifth-order system with positive coefficients, the following two inequalities should be satisfied:

$$a_1a_2-a_0a_3>0,$$
$$(a_1a_2-a_0a_3)(a_3a_4-a_2a_5)-(a_1a_4-a_0a_5)^2>0.$$

76. Using the Vyshnegradskiy criterion, determine the stability of an automatic system whose free-motion equation has the form

$$(a_0p^3+a_1p^2+a_2p+a_3)x=0,$$

where

$$\begin{aligned}
&\text{a) } a_0=0{,}02;\ a_1=0{,}4;\ a_2=1{,}3;\ a_3=25.\\
&\text{b) } a_0=0{,}02;\ a_1=0{,}4;\ a_2=1{,}3;\ a_3=30.\\
&\text{c) } a_0=0{,}01;\ a_1=0{,}2;\ a_2=1{,}5;\ a_3=60.
\end{aligned}$$

<u>Answer</u>: a) The system is stable. b) The system is unstable. c) The system is unstable.

77. Using the Hurwitz criterion, determine the stability of an automatic system whose free-motion equation has the form

$$(a_0p^4+a_1p^3+a_2p^2+a_3p+a_4)x=0,$$

where

$$\begin{aligned}
&\text{a) } a_0=0{,}001;\ a_1=0{,}05;\ a_2=0{,}4;\ a_3=1;\ a_4=20.\\
&\text{b) } a_0=0{,}001;\ a_1=0{,}05;\ a_2=0{,}4;\ a_3=1;\ a_4=100.
\end{aligned}$$

<u>Answer</u>: a) The system is stable. b) The system is unstable.

78. Determine the stability of an automatic system whose characteristic equation has the form

$$a_0p^5+a_1p^4+a_2p^3+a_3p^2+a_4p+a_5=0$$

where the coefficients have the following values:

a) $a_0 = 0{,}001$; $a_1 = 0{,}03$; $a_2 = 0{,}25$; $a_3 = 1$; $a_4 = 10$; $a_5 = 60$.
b) $a_0 = 0{,}001$; $a_1 = 0{,}02$; $a_2 = 0{,}5$; $a_3 = 4$; $a_4 = 10$; $a_5 = 40$.

<u>Answer.</u> a) The system is unstable. b) The system is stable.

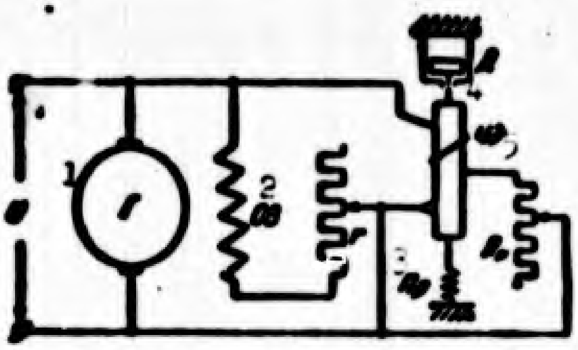


Fig. 70. Voltage regulation system. 1) Generator; 2) OV; 3) spring; 4) damper; 5) sensing element.

79. Using the Vyshnegradskiy and Hurwitz criteria, determine the stability of automatic systems where the free equations of motion for the systems have the form

$$[(T_1p+1)(T_2p+1)(T_3p+1)+K]x=0,$$
$$[(T_0p+1)(T_2^2p^2+T_1p+1)(T_3p+1)+K]x=0,$$

for the following four cases:

a) $T_1 = 1$ sec; $T_2 = 0{,}2$ sec; $T_3 = 0{,}05$ sec; $K = 25$.
b) $T_1 = 0{,}8$ sec; $T_2 = 0{,}25$ sec; $T_3 = 0{,}04$ sec; $K = 100$.
c) $T_0 = 1$ sec; $T_1 = 0{,}5$ sec; $T_2 = 0{,}8$ sec; $T_3 = 0{,}2$ sec; $K = 10$.
d) $T_0 = 1$ sec; $T_1 = 0{,}5$ sec; $T_2 = 0{,}8$ sec; $T_3 = 0{,}2$ sec; $K = 2$.

<u>Answer</u>: a) The system is stable. b) The system is unstable. c) The system is unstable. d) The system is stable.

80. For the voltage-regulation system of Fig. 70, the characteristic equation has the form

$$(T_0p+1)(T_1p+1)(T_2^2p^2+T_3p+1)+k_0k_r=0,$$

where the parameters have the following values:

the time constant of the manipulated element (generator field circuit) is T_0 = 0.2 sec; the time constant of the sensing-element electrical circuit is T_1 = 0.05 sec; the time constants of the mechanical portion of the sensing element are T_3 = 1 sec, T_2^2 = 0.1 sec^2; the transfer constant of the manipulated element is k_0 = 0.5 v/ohm; the transfer constant of the regulator is k_r = 5 ohm/v. Determine system stability.

<u>Answer.</u> The system is stable.

81. For the system of Problem 80, determine the value of the trans-

fer constant of the regulator $k_{r.gr}$, corresponding to the limit of stability.

<u>Answer</u>.

$$k_{r.gr} = 21{,}4 \text{ ohm/v.}$$

82. For the same system (Problem 80), determine the stability if the time constant of the system T_0 is increased to 0.5 sec.

<u>Answer</u>. The system is stable.

83. The closed-loop transfer function of a system has the form

$$\Phi(p) = \frac{1 + 0{,}3p}{1 + 0{,}3p + 1{,}4p^2 + 0{,}1p^3}.$$

Determine system stability.

<u>Answer</u>. The system is stable.

84. Determine the stability of two interconnected automatic systems representable by the block diagram of Fig. 71 if the parameters have the following values:

$$T_{11} = 0{,}01 \text{ sec}, \; T_{12} = 0{,}05 \text{ sec}, \; K_1 = k_{11}k_{12} = 100 \text{ 1/sec},$$
$$K_2 = 20 \text{ 1/sec}; \; a_{12} = 0{,}5, \quad a_{21} = 0{,}4.$$

<u>Solution.</u> The transfer functions for the separate open-loop systems will be

$$W_1(p) = \frac{K_1}{p(1 + T_{11}p)(1 + T_{12}p)}, \qquad (1)$$
$$W_2(p) = \frac{K_2}{p}.$$

In accordance with the block diagram, we have the following system equations of motion

$$x_1 = W_1(p)(y_1 - x_1) - a_{12}W_1(p)x_2, \qquad (2)$$
$$x_2 = W_2(p)(y_2 - x_2) - a_{21}W_2(p)x_1.$$

Solving Eqs. (2) simultaneously, we obtain

$$x_1 = \frac{[1 + W_2(p)]W_1(p)y_1 - a_{12}W_1(p)W_2(p)y_2}{[1 + W_1(p)][1 + W_2(p)] - a_{12}a_{21}W_1(p)W_2(p)}, \qquad (3)$$
$$x_2 = \frac{[1 + W_1(p)]W_2(p)y_2 - a_{21}W_1(p)W_2(p)y_1}{[1 + W_1(p)][1 + W_2(p)] - a_{12}a_{21}W_1(p)W_2(p)}.$$

As a consequence, for the associated systems, the characteristic equation will take the form

$$[1+W_1(p)][1+W_2(p)]-a_{12}a_{21}W_1(p)W_2(p)=0. \qquad (4)$$

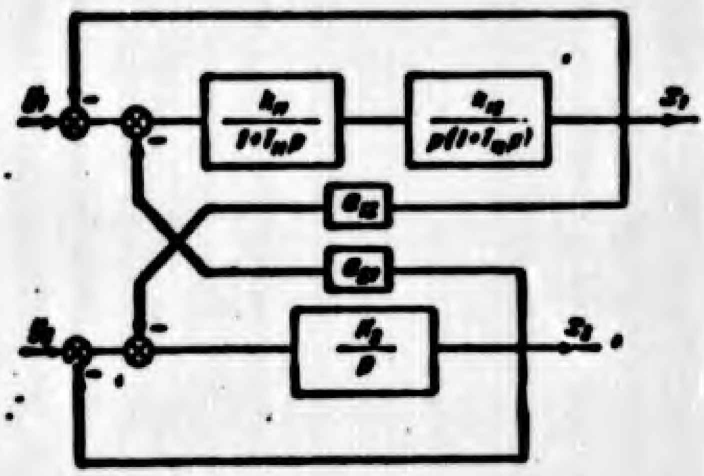


Fig. 71. Block diagram of two interconnected automatic systems.

As we can see from (3) and (4), the order of the differential equations and the degree of the characteristic equation equal the sum of the orders and degrees of the equations for the separate systems. In order to make a stability determination for this case, we may use either of the criteria.

Substituting the values of $W_1(p)$ and $W_2(p)$ into (4), we obtain

$$T_{11}T_{12}p^4+(T_{11}+T_{12}+K_1T_{11}T_{12})p^3+[1+K_2(T_{11}+T_{12})]p^2+$$
$$+(K_1+K_2)p+(1-a_{12}a_{21})K_1K_2=0,$$

which yields, when the values of the parameters are considered,

$$0{,}0003p^4+0{,}07p^3+2{,}2p^2+120p+1600=0.$$

Using the Hurwitz stability criterion, we check to see whether or not the inequality $a_3(a_1a_2-a_0a_3)>a_1^2a_4$: is satisfied:

$$a_3(a_1a_2-a_0a_3)=120\,(0{,}07\cdot 2{,}2-0{,}0003\cdot 120)=11{,}3,$$
$$a_1^2a_4=0{,}07^2\cdot 1600=7{,}83.$$

The inequality holds and, as a consequence, the systems are stable.

85. Determine the stability of the interconnected systems given in

Problem 84, where the numerical values of the parameters are the same, but one of the links a_{12} or a_{21} is made positive.

Answer. The systems are unstable.

§8. STABILITY DETERMINATIONS USING THE MIKHAYLOV CRITERION

86. Determine the stability of an automatic system by means of the Mikhaylov criterion where the characteristic equation has the form

$$a_0p^3 + a_1p^2 + a_2p + a_3 = 0,$$

where

$$a_0 = 0.04,$$
$$a_1 = 0.5,$$
$$a_2 = 2,$$
$$a_3 = 10.$$

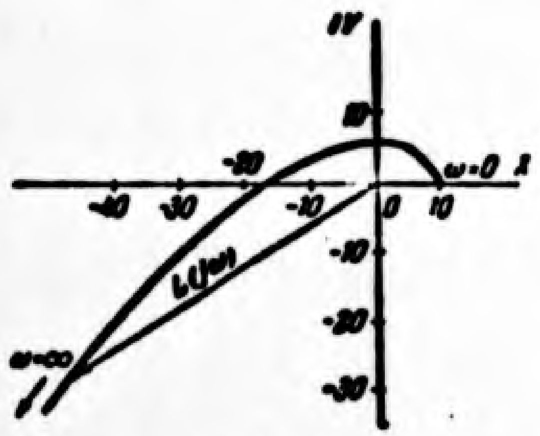


Fig. 72. Mikhaylov curve for Problem 86.

Solution. Making the substitution $p = j\omega$ in the left side of the characteristic equation and separating the real and imaginary parts, we obtain

$$L(j\omega) = X(\omega) + jY(\omega),$$

where

$$X(\omega) = a_3 - a_1\omega^2 = 10 - 0.5\omega^2,$$
$$Y(\omega) = \omega(a_2 - a_0\omega^2) = \omega(2 - 0.04\omega^2).$$

Given values $0 \leq \omega \leq \infty$, we construct the Mikhaylov curve (Fig. 72).

Since the resultant angle of rotation of the vector $L(j\omega)$ when $0 \leq \omega \leq \infty$ is

$$\varphi = n\frac{\pi}{2} = 3\frac{\pi}{2},$$

the system is stable.

87. Determine the stability of an automatic system using the Mikhaylov criterion if we are given the characteristic equation

$$0.0014p^4 + 0.022p^3 + 0.7p^2 + 1.6p + 5 = 0.$$

Answer. The Mikhaylov curve $L(j\omega) = X(\omega) + jY(\omega)$ has the value $X(\omega)$ and $Y(\omega)$ given in the following table:

ω	0	1	2	3	4	5	8	10	20	30	∞
$X(\omega)$	5,00	4,30	2,22	−1,17	−5,83	−11,6	−31,0	−51,0	−51,0	508	∞
$Y(\omega)$	0	1,58	3,02	4,21	4,99	5,25	1,55	−6,00	−144	−546	$-\infty$

The system is stable.

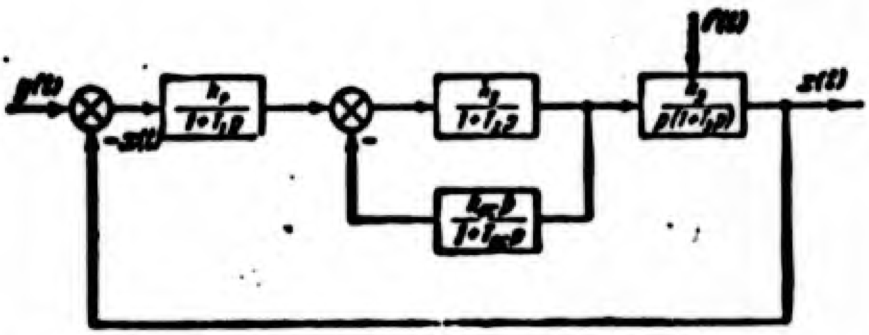


Fig. 73. Block diagram of automatic system for Problem 88.

88. Using the Mikhaylov criterion, determine the stability of the automatic system whose block diagram and element transfer functions are represented in Fig. 73. The system parameters have the following values:

$$k_1 = 60,\quad k_2 = 5,\quad k_3 = 0{,}1 \text{ 1/sec},\quad k_{o.c} = 0{,}2,$$
$$T_1 = 0{,}02 \text{ sec},\quad T_2 = 0{,}05 \text{ sec},\quad T_3 = 0{,}1 \text{ sec},\quad T_{o.c} = 0{,}2 \text{ sec}.$$

Answer. The system is unstable.

89. The characteristic equation of an automatic system takes the form

$$0{,}004p^5 + 0{,}1p^4 + 1{,}05p^3 + 2{,}8p^2 + 4{,}3p + 1{,}6 = 0.$$

Find the stability of the system using the Mikhaylov criterion.

Answer. The system is stable.

90. For an automatic system whose free motion is described by differential equation

$$(a_0p^4 + a_1p^3 + a_2p^2 + a_3p + 1 + K)x_1 = 0,$$

use the Mikhaylov stability criterion to determine the transfer constant corresponding to the limit of stability if we are given the coefficients

$$a_0 = 0{,}02,\quad a_1 = 0{,}25,\quad a_2 = 1,\quad a_3 = 5.$$

Solution. In the left side of the characteristic equation

$$a_0p^4 + a_1p^3 + a_2p^2 + a_3p + 1 + K = 0$$

we substitute the imaginary quantity $j\omega$ for p, obtaining

$$L(j\omega) = X(\omega) + jY(\omega),$$

where

$$X(\omega) = 1 + K - a_2\omega^2 + a_0\omega^4,$$
$$Y(\omega) = a_3\omega - a_1\omega^3.$$

For the case in which undamped oscillations appear on the stability boundary, the value of the transfer constant K_{gr} can be determined if the real and imaginary parts are set equal to zero: $X(\omega) = 0$ and $Y(\omega) = 0$ when $\omega \neq 0$, i.e.,

$$1 + K - a_2\omega^2 + a_0\omega^4 = 0,$$
$$a_3\omega - a_1\omega^3 = 0.$$

From the second equation we have $\omega^2 = a_3/a_1$. Substituting the value of ω^2 into the first equation, we obtain

$$K_{gr} = \frac{a_2a_3}{a_1} - \frac{a_0a_3^2}{a_1^2} - 1,$$

which, when we take into account the numerical values of the coefficients, yields $K_{gr} = 11$. The other two conditions on the stability limit $a_n = 0$ and $a_0 = 0$, are meaningless here, since $a_0 \neq 0$ by hypothesis, while the condition $a_n = 1 + K = 0$ yields $K_{gr} = -1$, corresponding to an illegitimate connection of system elements.

91. Using the Mikhaylov criterion, find the limiting value of the transfer constant of a regulator k_r if the system characteristic equation has the form

$$a_0p^4 + a_1p^3 + a_2p^2 + a_3p + 1 + k_0k_r = 0,$$

where $a_0 = 0.004$, $a_1 = 0.35$, $a_2 = 4.2$, $a_3 = 30$, $k_0 = 2$.

<u>Answer</u>.

$$k_{r.gr} = 165.$$

92. Using the Mikhaylov criterion, determine the stability of an automatic system whose free-motion equation takes the form

$$(a_0p^5+a_1p^4+a_2p^3+a_3p^2+a_4p+a_5)x=0,$$

where

$$a_0=6\cdot 10^{-4},\quad a_1=10^{-2},\quad a_2=0{,}2,\quad a_3=1,$$
$$a_4=1{,}8,\quad a_5=2.$$

The problem is solved by using the root alternation condition.

Solution. Making the substitution p = $j\omega$ in the left side of the characteristic equation, we obtain

$$L(j\omega)=X(\omega)+jY(\omega),$$

where

$$X(\omega)=a_5-a_3\omega^2+a_1\omega^4,\quad Y(\omega)=a_4\omega-a_2\omega^3+a_0\omega^5.$$

Setting X(ω) and Y(ω) equal to zero, we find the roots of the equations

$$1{,}8\omega-0{,}2\omega^3+6\cdot 10^{-4}\omega^5=0,\quad 2-\omega^2+10^{-4}\omega^4=0.$$

From the first equation we have

$$\omega_1=0,\quad \omega_3=3{,}03\ 1/\sec\ \omega_5=18\ 1/\sec$$

From the second equation, we have

$$\omega_2=2{,}1\ 1/\sec\ \omega_4=6{,}76\ 1/\sec$$

Since $\omega_1 < \omega_2 < \omega_3 < \omega_4 < \omega_5$, i.e., the roots alternate, the system is stable.

93. Using the Mikhaylov criterion, determine the stability of an automatic system using the root-alternation condition, where the characteristic equation takes the form

$$3\cdot 10^{-4}p^5+10^{-2}p^4+0{,}12p^3+0{,}8p^2+4p+100=0.$$

Answer. The system is unstable.

94. Using the Mikhaylov stability criterion, determine the stability of an automatic system whose free motion is described by the differential equation

$$(10^{-8}p^6+4\cdot 10^{-6}p^5+1{,}5\cdot 10^{-3}p^4+0{,}12p^3+0{,}5p^2+2p+1{,}7)x=0.$$

Answer. The system is stable.

95. Using the Mikhaylov criterion, determine the stability of an

automatic system having the characteristic equation

$$10^{-?}p^7+2\cdot10^{-?}p^6+6\cdot10^{-?}p^5+5\cdot10^{-?}p^4+$$
$$+10^{-?}p^3+8\cdot10^{-?}p^2+7p+50=0.$$

Answer. The system is unstable.

§9. STABILITY DETERMINATIONS USING THE NYQUIST CRITERION

96. Using the Nyquist criterion, by constructing the gain-phase frequency characteristic, determine the stability of a closed-loop automatic system if the open-loop transfer function has the form

$$W(p)=\frac{K}{p(1+T_1p)(1+T_2p)}, \qquad (1)$$

where

$$K=86\ 1/\text{sec}\ T_1=0{,}02\ \text{sec}\quad T_2=0{,}03\ \text{sec}$$

Solution. Making the substitution $p = j\omega$ in the transfer function, we obtain an analytic expression for the frequency transfer function

$$W(j\omega)=\frac{K}{j\omega(1+T_1j\omega)(1+T_2j\omega)}. \qquad (2)$$

The modulus of the reduced expression equals

$$A(\omega)=\frac{K}{\omega\sqrt{(1+T_1^2\omega^2)(1+T_2^2\omega^2)}},$$

while the phase is found as the argument of the same expression:

$$\psi(\omega)=-(90^\circ+\operatorname{arctg}T_1\omega+\operatorname{arctg}T_2\omega).$$

After substituting the parameter values, we have working formulas for plotting the gain-phase characteristic:

$$A(\omega)=\frac{86}{\omega\sqrt{(1+4\cdot10^{-4}\omega^2)(1+9\cdot10^{-4}\omega^2)}},$$
$$\psi(\omega)=-(90^\circ+\operatorname{arctg}0{,}02\omega+\operatorname{arctg}0{,}03\omega).$$

The results of a calculation using the formulas obtained are given in this table:

ω, 1/sec 1	0	10	20	30	50	100	∞
$A(\omega)$	∞	2,35	2,71	1,34	0,49	0,1	0
$\psi(\omega)$, degrees 2	−90°	−118°	−143°	−163°	−191°	−226°	−270°

1) ω, 1/sec; 2) ψ(ω), degrees.

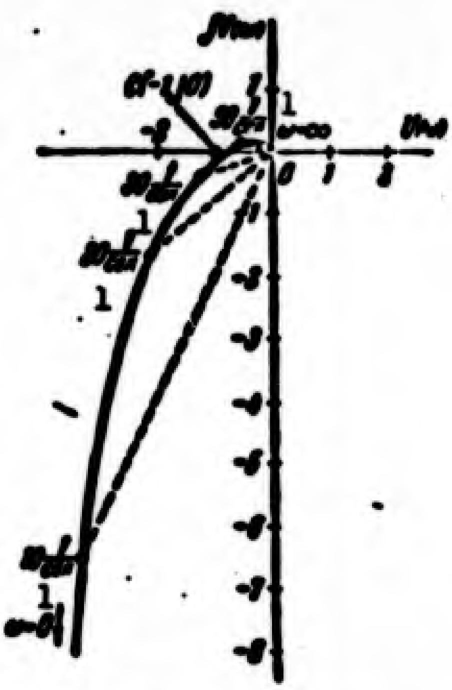


Fig. 74. Open-loop gain-phase characteristic for Problem 96. 1) Sec.

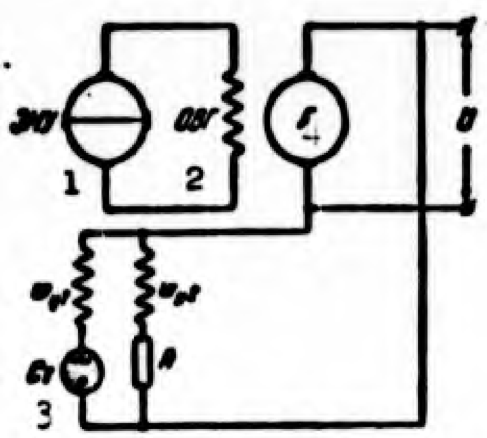


Fig. 75. Voltage-regulation system using VR tube. 1) Amplidyne; 2) OVG; 3) VR tube; 4) generator.

From the data of the table, we construct the gain-phase characteristic (Fig. 74), which as we can see does not envelope the point C(−1, j0) on the complex plane for a variation $0 \leq \omega \leq \infty$. Taking into account the fact that an open-loop system is neutral (there is one zero root in the denominator of the transfer function and two other negative roots), we conclude that the closed-loop system is stable.

Note. A stability determination using the Nyquist criterion for third-order systems need not involve construction of the frequency characteristic. Here it is necessary to use the condition that the absolute value of A should be less than unity for a phase $\psi = -\pi$.

If the analytic expression (2) is written in the form

$$W(j\omega)=\frac{K}{j\omega[1-T_1T_2\omega^2+j\omega(T_1+T_2)]}$$

and, correspondingly, the formula for the phase characteristic in the form

$$\psi(\omega)=-90^\circ-\operatorname{arctg}\frac{\omega(T_1+T_2)}{1-T_1T_2\omega^2},$$

then after the resulting expression has been set equal to $-\pi$, we have

$$\operatorname{arctg}\frac{\omega(T_1+T_2)}{1-T_1T_2\omega^2}=90^\circ$$

and, as a consequence,

$$1-T_1T_2\omega^2=0,\quad \omega^2=\frac{1}{T_1T_2}.$$

Substituting the value of ω into the formula for the absolute value

$$A(\omega)=\frac{K}{\omega\sqrt{(1-T_1T_2\omega^2)^2+\omega^2(T_1+T_2)^2}}$$

and requiring that the condition $A(\omega) < 1$ be satisfied, we obtain a relationship connecting the parameters that determine the stability of a closed-loop astatic third-order system:

$$K<\frac{1}{T_1}+\frac{1}{T_2}.$$

97. Determine the stability of the automatic system considered in Problem 96 using the Nyquist criterion if the time constant T_2 is increased to 0.05 sec.

Answer. The system is unstable.

98. The automatic voltage-regulation system for a DC generator using an amplidyne as an exciter and a comparison circuit with a VR tube (Fig. 75) will have an open-loop transfer function under no-load conditions of the following form:

$$W(p)=\frac{K}{(1+T_{у}p)(1+T_{кз}p)(1+T_{в}p)},$$

where $K = k_{s.s}k_{EMU}k_g$ are the system open-loop transfer constants, $k_{s.s}$ is the transfer constant of the comparison circuit, k_{EMU} is the transfer constant of the amplidyne, k_g is the transfer constant of the generator, T_u, T_{kz}, T_g are, respectively, the total time constant of the amplidyne control-winding circuits, the time constant of the amplidyne short-circuit loop, and the time constant of the generator field-winding circuit. Determine the system closed-loop stability using the Nyquist criterion when the parameters have the following values:

$$T_{у}=0{,}07\text{ sec},\ T_{кз}=0{,}035\text{ sec},\ T_{в}=0{,}18\text{ sec},$$
$$k_{эму}k_{сс}=20,\quad k_{в}=0{,}5.$$

Answer. The system is stable.

99. The system open-loop transfer function has the form

$$W(p)=\frac{K}{(1+T_1p)(1+T_2p)(1+T_3p)(1+T_4p)},$$

where the parameters have the following values:

$$K=200,\ T_1=0{,}5\ \text{sec},\ T_2=0{,}2\ \text{sec},\ T_3=0{,}03\ \text{sec},$$
$$T_4=0{,}004\ \text{sec}.$$

Determine the system closed-loop stability using the Nyquist criterion.

<u>Answer</u>. The system is unstable.

100. The automatic system consists of three structural units (Fig. 76). The equations for the first and third units take the form

$$(T_1p+1)x_1=k_1x,\quad (T_3p+1)px_3=k_3x_2.$$

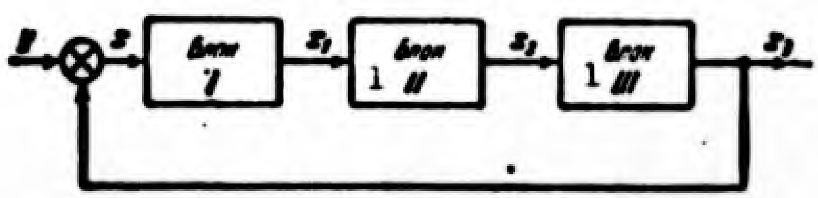


Fig. 76. Block diagram of automatic system for Problem 100. 1) Unit.

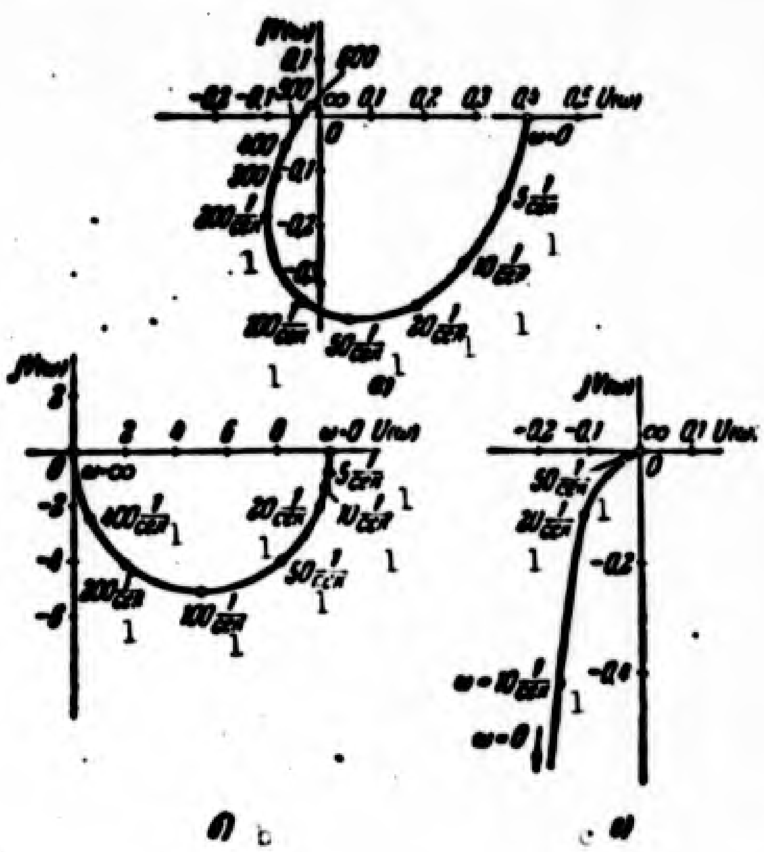


Fig. 77. Gain-phase characteristics for units of Problem 100. 1) Sec.

For unit II, owing to the difficulty of setting up the differential equation, we have used an empirical gain-phase frequency characteristic, which is shown in Fig. 77a. Determine the stability of the system using the Nyquist criterion. The parameters for the first and third units have the following values:

$$T_1 = 0{,}01 \text{ sec}, \; k_1 = 10, \; T_3 = 0{,}05 \text{ sec}, \; k_3 = 5 \text{ 1/sec}.$$

Solution. To determine the system closed-loop stability, we construct its open-loop gain-phase characteristic. The gain-phase characteristic of the entire system is found as the product of the characteristics of the series-connected elements (units):

$$W(j\omega) = w_1(j\omega)\, w_2(j\omega)\, w_3(j\omega).$$

From the equations for units I and III, their gain-phase characteristics are plotted from the analytic expressions

$$w_1(j\omega) = \frac{k_1}{1 + T_1 j\omega}, \quad w_3(j\omega) = \frac{k_3}{j\omega(1 + T_3 j\omega)}.$$

For the given values of the parameters k_1, k_3, T_1, T_3 these characteristics take the form of the curves shown in Figs. 77b and c.

In plotting the $W(j\omega)$ characteristic, we used the relationships among the absolute values and phases:

$$A = A_1 A_2 A_3,$$
$$\varphi = \varphi_1 + \varphi_2 + \varphi_3.$$

Carrying out the calculations and constructions for the various values of ω in the range $0 \leq \omega \leq \infty$, we obtain the gain-phase frequency characteristic of the system (Fig. 78). As we can see, the point $C(-1, j0)$ is enclosed by the gain-phase frequency characteristic.

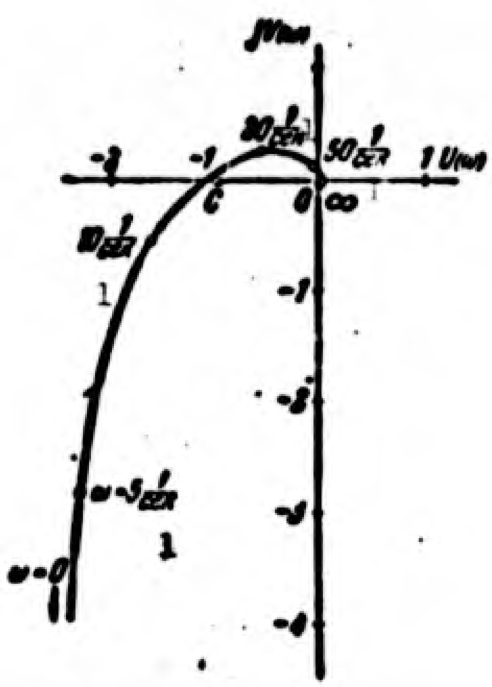


Fig. 78. Gain-phase characteristic of system of Problem 100. 1) Sec.

As a consequence, the closed-loop system is unstable.

101. Determine the stability of the system given in Problem 100, using the Nyquist criterion, if we take the transfer constant of the

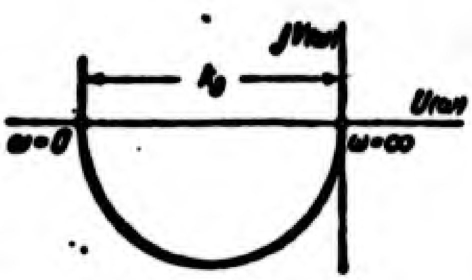


Fig. 79. Gain-phase characteristic for unstable aperiodic element.

first unit to be $k_1 = 8$ and the time constant of the third unit to be $T_3 = 0.03$ sec.

Answer. The system is stable.

102. Determine the closed-loop stability of an automatic system containing an unstable element, using the Nyquist criterion, if its open-loop transfer function has the form

$$W(p)=\frac{K(1+T_2p)}{p(1+T_1p)(-1+T_3p)}, \qquad (1)$$

and the parameters have the following values:

$$T_1=0{,}01 \text{ sec}, T_2=0{,}1 \text{ sec}, T_3=0{,}05 \text{ sec}, K=20 \text{ 1/sec}.$$

Solution. As we can see from (1), the denominator for W(p) has one root in the left half plane, one root in the right half plane, and one zero root. The open-loop system is unstable.

Before constructing the gain-phase frequency characteristic, let us see whether or not the necessary condition for closed-loop stability is satisfied. The closed-loop characteristic equation for the system is

$$1+W(p)=0,$$

which when the expression for W(p) is taken into account yields

$$T_1T_3p^3+(T_3-T_1)p^2+(KT_2-1)p+K=0.$$

Since for the given parameter values

$$T_3-T_1=0{,}05-0{,}01=0{,}04>0,$$
$$KT_2-1=20\cdot 0{,}1-1=1>0,$$

then all the coefficients of the characteristic equation are positive and, as a consequence, the necessary condition for stability is satisfied.

In order to construct the gain-phase frequency characteristic, in the expression for the transfer function we replace p by the imaginary value $j\omega$ and find the formulas for calculating the absolute value and phase.

From (1) we have

$$A(\omega)=\frac{K\sqrt{1+T_2^2\omega^2}}{\omega\sqrt{1+T_1^2\omega^2}\cdot\sqrt{1+T_3^2\omega^2}}. \tag{2}$$

for the absolute value.

In order to write a formula for the phase, we must consider the form of the gain-phase frequency characteristic of the unstable aperiodic element (Fig. 79) and its phase characteristic

$$\varphi_3(\omega)=-180^\circ+\operatorname{arctg} T_3\omega.$$

The open-loop phase characteristic of the entire system is determined by the formula

$$\varphi(\omega)=\operatorname{arctg} T_2\omega-90^\circ-\operatorname{arctg} T_1\omega-180^\circ+\operatorname{arctg} T_3\omega$$

or

$$\varphi(\omega)=-270^\circ+\operatorname{arctg} T_2\omega-\operatorname{arctg} T_1\omega+\operatorname{arctg} T_3\omega. \tag{3}$$

The gain-phase frequency characteristic, computed from Formulas (2) and (3) is shown in Fig. 80. As we can see, for a frequency variation $0 \leq \omega \leq \infty$, allowing for a rotation through $-\pi/2$, the vector $1 + W(j\omega)$, which runs to one zero root in the denominator of the transfer function, turns through an angle $\psi = \pi$ around the point $C(-1, j0)$. Where there is a single positive root in the open-loop characteristic equation of the system, the closed-loop system is stable.

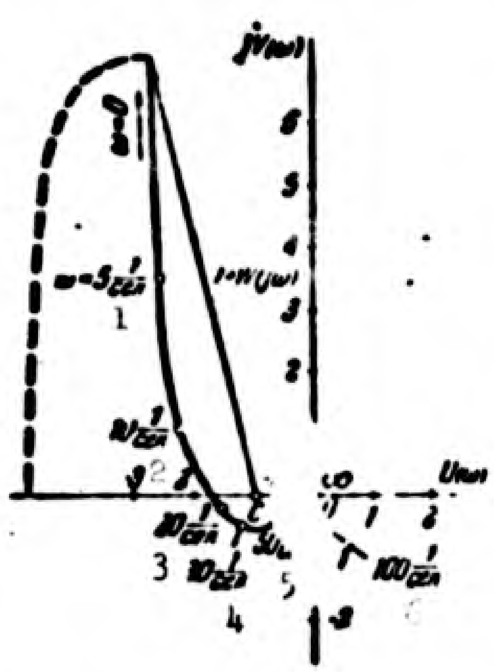


Fig. 80. Open-loop gain-phase frequency characteristic for system of Problem 102. 1) sec.

We see that in accordance with (2) as the open-loop over-all transfer coefficient K of the system decreases, the absolute value of the frequency characteristic will decrease, while the phase relationships remain as before. Here as K decreases, the closed-loop system loses its stability margin, and at a certain value becomes unstable. This is characteristic of systems that are unstable when in the open-loop condition, or so-called conditionally stable systems.

103. Using the Nyquist criterion, determine the closed-loop stability of the automatic system considered in Problem 102 if the open-loop gain is decreased to K = 11 1/sec.

Answer. The system is unstable.

104. Using the Nyquist criterion, determine the closed-loop stability of an automatic system if its open-loop transfer function has the form

$$W(p)=\frac{K(1+T_1p)}{(1+T_2p)(1+T_3p+T_4^2p^2)(-1+T_5p)},$$

and the parameters have the values: T_1 = 0.5 sec, T_2 = 0.2 sec, T_3 = 0.1 sec, T_4 = 0.33 sec, T_5 = 0.4 sec, K = 10.

Answer. The system is unstable.

105. Using the Nyquist criterion, investigate the closed-loop stability of an automatic system having two unstable elements if we are given the circuit open-loop transfer function

$$W(p)=\frac{K(1+T_1p+T_2^2p^2)}{(1+T_3p)(T_4p+1)(-1+T_5p)(-1+T_6p)}$$

and the following parameter values: T_1 = 0.1 sec, T_2 = 0.02 sec, T_3 = = 1 sec, T_4 = 0.33 sec, T_5 = 0.8 sec, T_6 = 0.5 sec, K = 20.

Answer. The system is stable.

106. The open-loop transfer function of an automatic system has the form

$$W(p)=\frac{K}{(1+T_1p)(1+T_2p)(1+T_3p)},$$

where the parameters have the following values: k = 25, T_1 = 0.02 sec, T_2 = 0.1 sec, T_3 = 0.5 sec. Determine system closed-loop stability from the inverse phase-amplitude characteristic.

<u>**Solution.**</u> The inverse transfer function will be

$$E(p)=\frac{1}{W(p)},$$

where

$$W(p)=\frac{K}{1+(T_1+T_2+T_3)p+(T_1T_2+T_2T_3+T_1T_3)p^2+T_1T_2T_3p^3},$$

and, as a consequence,

$$E(p)=\frac{1}{K}+\frac{T_1+T_2+T_3}{K}p+$$
$$+\frac{T_1T_2+T_1T_3+T_2T_3}{K}p^2+\frac{T_1T_2T_3}{K}p^3.$$

Substituting in the parameter values, we obtain

$$E(p)=0{,}04+0{,}0248p+0{,}00248p^2+0{,}00004p^3.$$

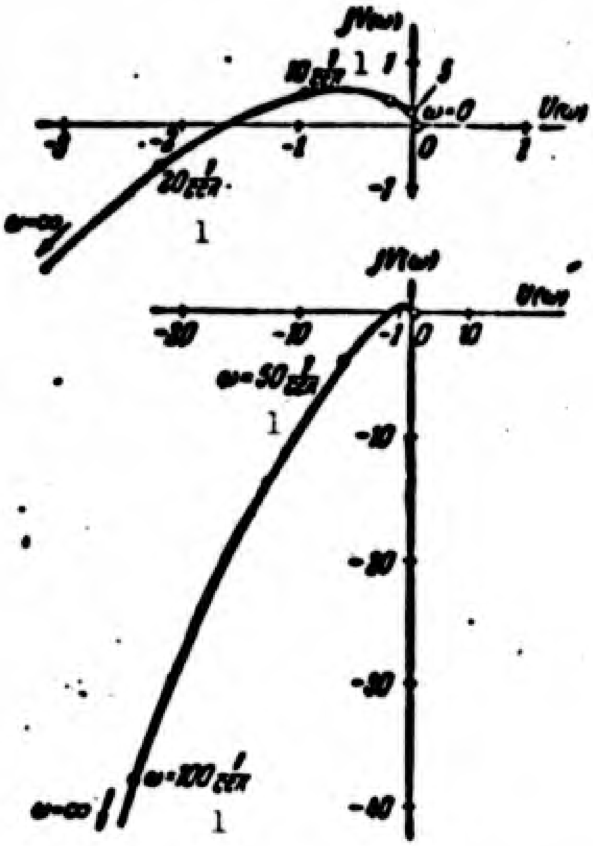


Fig. 21. Inverse phase-amplitude characteristic for Problem 106. 1) Sec.

The expression for the inverse phase-amplitude characteristic will be

$$E(j\omega)=U(\omega)+jV(\omega).$$

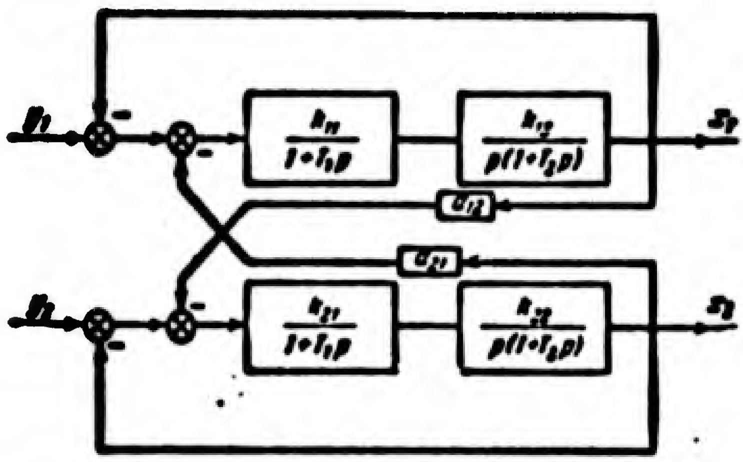


Fig. 82. Block diagram for interconnected systems.

where

$$U(\omega) = 0{,}04 - 0{,}00248\omega^2, \quad V(\omega) = 0{,}0248\omega - 0{,}00004\omega^3.$$

Calculating U(ω) and V(ω), we tabulate them:

ω	0	5	10	20	30	50	100	∞
U(ω)	0,04	−0,022	−0,208	−0,952	−2,192	−6,16	−24,76	−∞
V(ω)	0	0,119	0,208	0,176	−0,336	−3,76	−37,52	−∞

The hodograph of E(jω) of Fig. 81 has been plotted from the data of the table. As we can see, the hodograph of E(jω) encloses the point (−1, j0) in the positive direction for the angle $\varphi = 3\pi/2$, which corresponds to stability of a third-order closed-loop system.

107. Determine the closed-loop stability of an automatic system using the inverse phase-amplitude characteristic if the system open-loop transfer function has the form

$$W(p) = \frac{K}{(1 + T_1 p)(1 + T_2 p)(1 + T_3 p)(1 + T_4 p)},$$

where T_1 = 1 sec, T_2 = 0.05 sec, T_3 = 0.1 sec, T_4 = 0.002 sec, K = 170.

Answer. The system is unstable.

108. Determine the closed-loop stability of an automatic system using the inverse phase-amplitude characteristic if the system open-loop transfer function has the form

$$W(p)=\frac{K}{T_1p(1+T_2p)(1+T_3p)(1+T_4p)},$$

where T_1 = 0.5 sec, T_2 = 1 sec, T_3 = 0.1 sec, T_4 = 0.03 sec, K = 50.

Answer. The system is unstable.

109. Using the Nyquist criterion, determine the stability of the interconnected automatic systems whose block diagrams are shown in Fig. 82 where the parameters have the following values: T_1 = 0.02 sec, T_2 = = 0.03 sec, $K_1 = k_{11}k_{12}$ = 10 1/sec, $K_2 = k_{21}k_{22}$ = 20 1/sec, a_{12} = 0.5, a_{21} = 0.8.

Solution. The transfer functions for the isolated systems will be

$$W_1(p)=\frac{K_1}{p(T_1p+1)(T_2p+1)},\qquad W_2(p)=\frac{K_2}{p(T_1p+1)(T_2p+1)}. \tag{1}$$

In accordance with the block diagram (Fig. 82), the system equations of motion will take the form

$$x_1=W_1(p)(y_1-x_1)-a_{12}W_1(p)x_2,\qquad x_2=W_2(p)(y_2-x_2)-a_{21}W_2(p)x_1. \tag{2}$$

Simultaneous solution of Eqs. (2) yields

$$x_1=\frac{[1+W_2(p)]W_1(p)y_1-a_{12}W_1(p)W_2(p)y_2}{[1+W_1(p)][1+W_2(p)]-a_{12}a_{21}W_1(p)W_2(p)},\qquad x_2=\frac{[1+W_1(p)]W_2(p)y_2-a_{21}W_1(p)W_2(p)y_1}{[1+W_1(p)][1+W_2(p)]-a_{12}a_{21}W_1(p)W_2(p)}. \tag{3}$$

In accordance with (3), the characteristic equation for the coupled systems will take the form

$$[1+W_1(p)][1+W_2(p)]-a_{12}a_{21}W_1(p)W_2(p)=0. \tag{4}$$

Since the transfer functions $W_1(p)$ and $W(p)$ of the isolated systems differ only in the transfer constants, each of them may be written in the form

$$W_1(p)=K_1W(p),\quad W_2(p)=K_2W(p),$$

where

$$W(p)=\frac{1}{p(T_1p+1)(T_2p+1)}.$$

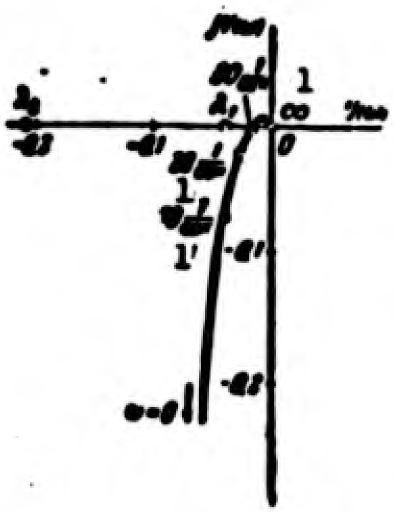


Fig. 83. Gain-phase frequency characteristic for investigating the stability of interconnected automatic systems. 1) Sec.

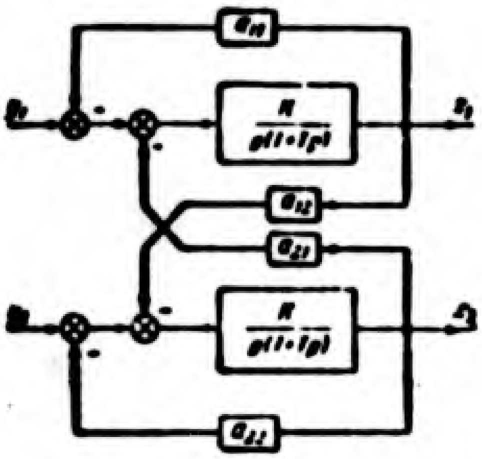


Fig. 84. Block diagram of interconnected automatic systems for Problem 110.

Then Eq. (4) may be written as

$$K_1K_2(1-a_{12}a_{21})W^2(p)+(K_1+K_2)W(p)+1=0. \qquad (5)$$

From (5) we find the roots λ_1 and λ_2 for the variable W(p):

$$\lambda_{1,2}=\frac{-(K_1+K_2)\pm\sqrt{(K_1+K_2)^2-4K_1K_2(1-a_{12}a_{21})}}{2K_1K_2(1-a_{12}a_{21})}.$$

Substituting in the numerical values of the parameters, we have

$$\lambda_1=-0.0396,\quad \lambda_2=-0.21.$$

Now in order to determine the stability of the coupled systems [21], we need only plot the gain-phase characteristic $W(j\omega)$ and see that it does not encompass the critical points λ_1 and λ_2. The given characteristic, constructed from the expression

$$W(j\omega)=\frac{1}{j\omega(1+T_1j\omega)(1+T_2j\omega)},$$

is shown in Fig. 83. As we can see, the points λ_1 and λ_2 are not encompassed by the gain-phase frequency characteristic and, as a consequence, the systems are stable.

110. For the interconnected automatic systems having the block diagram shown in Fig. 84, determine the stability using the Nyquist criterion where the parameters have the following values: T = 0.2 sec, K = 10 1/sec, a_{11} = 0.87, a_{22} = 87, a_{12} = = − 5, a_{21} = 0.5.

Answer. The systems are stable.

111. For the interconnected automatic systems considered in Problem 110, use the Nyquist criterion to determine the stability for the

following parameter values: T = 0.2 sec, K = 10 1/sec, a_{11} = 0.87, a_{22} = = 8.7, a_{12} = 5, a_{21} = 2.

Answer. The systems are unstable.

112. For the interconnected automatic systems of Problem 110, determine the stability using the Nyquist criterion for the following parameter values: T = 0.2 sec, K = 10 1/sec, a_{11} = 0.17, a_{22} = 1.7, a_{12} = = − 9.8, a_{21} = 0.98.

Answer. The systems are unstable.

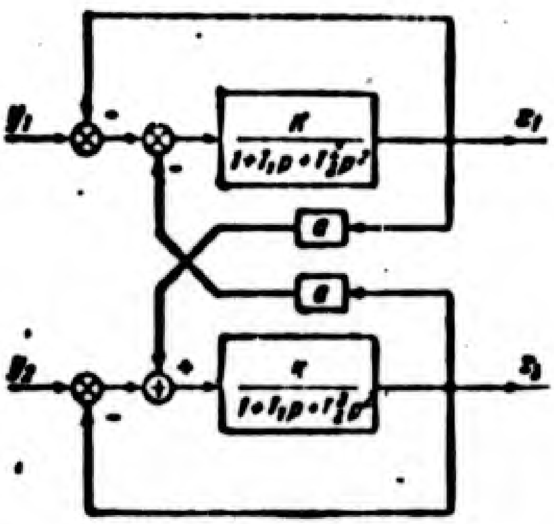


Fig. 85. Block diagram of interconnected automatic systems for Problem 113.*

113.* For interconnected automatic systems with crossed antisymmetric coupling (Fig. 85), determine the stability by the complex transfer-function method [16] if the parameters have the following values: T_2^2 = 0.1 sec^2, T_1 = 0.4 sec, K = 5, a = 0.5.

Solution. Letting W(p) be the open-loop transfer functions of the isolated systems, from the block diagram (Fig. 85), we write the equations of motion:

$$x_1 = W(p)(y_1 - x_1) + aW(p)(y_2 - x_2). \qquad (1)$$

$$x_2 = W(p)(y_2 - x_2) - aW(p)(y_1 - x_1). \qquad (2)$$

We introduce the complex variables:

$$x = x_1 + jx_2, \quad y = y_1 + jy_2.$$

Multiplying Eq. (2) by j and combining it with (2), we have

$$x = W(p)(y - x) - jaW(p)y + jaW(p)x,$$

and from this

$$x = \frac{W(p)(1 - ja)}{1 + W(p)(1 - ja)}\,y = \Phi(p, ja)y,$$

where Φ(p, ja) is the equivalent system closed-loop transfer function.

The equivalent system open-loop transfer function is found by the usual rules:

$$W(p, ja) = \frac{\Phi(p, ja)}{1-\Phi(p, ja)} = W(p)(1-ja).$$

The interconnected systems will be stable if and only if the equivalent system with open-loop transfer function $W(p, ja)$ is stable and the gain-phase frequency characteristic does not encircle the point $(-1, j0)$ on the complex plane. For the case considered,

$$W(p, ja) = \frac{K(1-ja)}{T_2^2 p^2 + T_1 p + 1} = \frac{5(1-j0.5)}{0.1p^2 + 0.4p + 1},$$

and the equivalent system is stable in the open-loop state. From $W(p, ja)$ we obtain when $p = j\omega$ an analytic expression for the frequency transfer function:

$$W(j\omega) = \frac{5(1-j0.5)}{1-0.1\omega^2 + 0.4j\omega},$$

from which we obtain formulas for computing the absolute value and phase:

$$A = \frac{5.6}{\sqrt{(1-0.1\omega^2)^2 + 0.16\omega^2}},$$

$$\varphi = -27^\circ - \operatorname{arctg} \frac{0.4\omega}{1-0.1\omega^2}.$$

The results of the computations are given in the following table:

ω, 1/sec 1)	0	1	2	3	5	10	50	100	∞
A	5,6	5,7	5,6	4,3	2,24	0,37	0,02	0,006	0
φ, degrees 2)	−27°	−51°	−80°	−113°	−154°	−183°	−203°	−205°	−207°

1) ω, 1/sec; 2) φ, degrees.

From the tabular data, we construct the equivalent system open-loop gain-phase frequency characteristic (Fig. 86), which does not encircle the point $C(-1, j0)$ and, as a consequence, the interconnected automatic systems are stable.

114. For the interconnected automatic systems considered in Problem 113, determine the stability by the method of complex transfer functions for the following parameter values: $T_2^2 = 0.2\ \text{sec}^2$, $T_1 = 0.5$ sec,

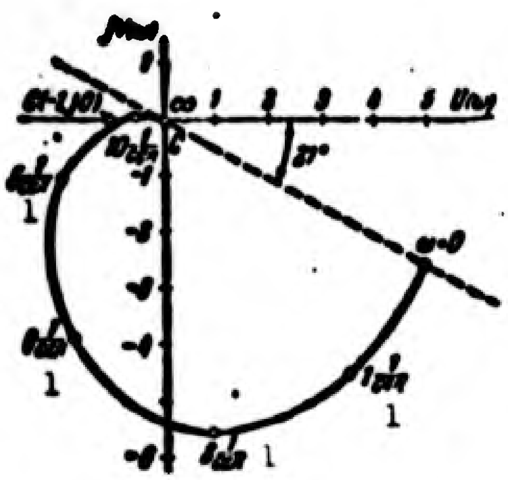


Fig. 86. Gain-phase frequency characteristic of equivalent open-loop system. 1) Sec.

$K = 10$, $a = 0.7$.

Answer. The systems are unstable.

115. Determine the stability of a servo system by the logarithmic-characteristic method if we are given the open-loop transfer function

$$W(p)=\frac{K_\Omega}{p(1+T_1p)(1+T_2p)}, \qquad (1)$$

and parameters with the following values: $K_\Omega = 50$ 1/sec, $T_1 = 0.04$ sec, $T_2 = 0.01$ sec. If the system is stable, then determine the system amplitude and phase stability margin.

Solution. To construct the logarithmic amplitude and phase characteristics, we determine the frequencies at which the asymptotes of the asymptotic logarithmic amplitude characteristic bend:

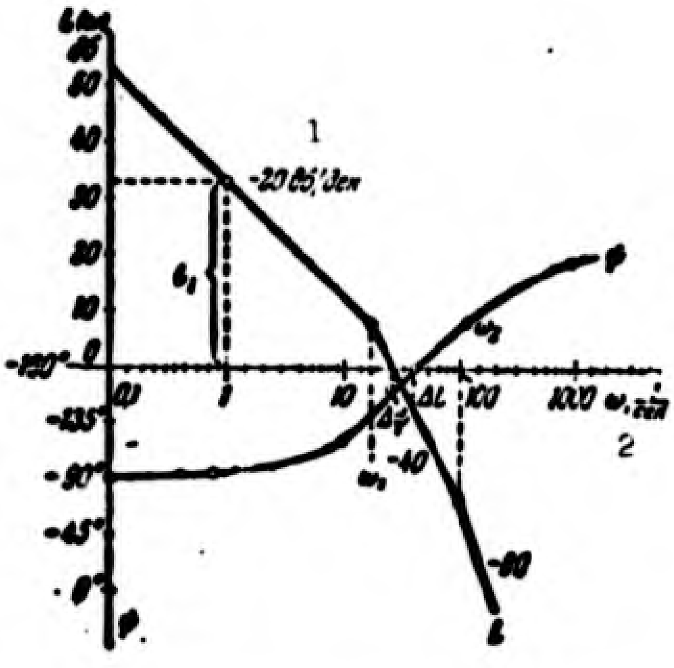


Fig. 87. Logarithmic characteristics for Problem 115. 1) db/decade; 2) sec.

$$\omega_1=\frac{1}{T_1}=\frac{1}{0.04}=25 \text{ 1/sec}$$

$$\omega_2=\frac{1}{T_2}=\frac{1}{0.01}=100 \text{ 1/sec}$$

Let us find the slope of the logarithmic amplitude characteristic at a frequency of $\omega = 1$ 1/sec, $L_1 = 20 \log K_\Omega = 20 \log 50 = 34$ db.

On the basis of the data obtained and the form of the transfer function (1), we construct the logarithmic amplitude characteristic (Fig. 87).

To determine the phase characteristic we have from (1):

$$\varphi(\omega) = -90^\circ - \operatorname{arctg} T_1\omega - \operatorname{arctg} T_2\omega$$

or

$$\varphi(\omega) = -90^\circ - \operatorname{arctg} 0{,}04\omega - \operatorname{arctg} 0{,}01\omega. \quad (2)$$

Taking values of ω, we calculate the phase values as summarized in the table:

ω, 1/sec [1]	0,1	1	10	60	100	1000
$\varphi(\omega)$, degrees [2]	-90°	-93°	-118°	-180°	-211°	-263°

1) ω, 1/sec; 2) $\psi(\omega)$, degrees.

From the tabular data, we construct the phase characteristic.

As we can see from Fig. 8, the logarithmic amplitude characteristic (l.a.kh.) takes on negative values before the phase characteristic reaches $\psi = -180^\circ$ and, as a consequence, the closed-loop system is stable. Here the amplitude-stability margin is $\Delta L \approx 8$ db and the phase-stability margin $\Delta\psi \approx 20^\circ$.

116. For the servo system of Problem 115, use the logarithmic-characteristics to determine the stability with the following parameter values: $K_\Omega = 90$ 1/sec, $T_1 = 0.05$ sec, $T_2 = 0.02$ sec.

Answer. The system is unstable.

117. Determine the closed-loop stability of an automatic system, using the logarithmic-characteristics method, if its open-loop transfer function has the form

$$W(p) = \frac{K(1 + T_1 p)}{(1 + T_2 p)(1 + T_3 p)(1 + T_4 p)(1 + T_5 p)},$$

and the parameters have the following values: $K = 100$, $T_1 = 1.2$ sec,

$T_2 = 0.8$ sec, $T_3 = 0.2$ sec, $T_4 = 0.05$ sec, $T_5 = 0.04$ sec.

Answer. The closed-loop system is stable.

118. Use the method of logarithmic characteristics to determine the closed-loop stability of a system with the following open-loop transfer function

$$W(p) = \frac{K}{p(1+T_1p)(1+T_2p)(1+T_3p)},$$

where $T_1 = 1$ sec, $T_2 = 0.2$ sec, $T_3 = 0.05$ sec, $K = 350$ 1/sec.

Answer. The closed-loop system is unstable.

119. Using the method of logarithmic characteristics, determine the closed-loop stability of a system having the open-loop transfer function

$$W(p) = \frac{K}{(1+T_1p)(1+T_2p)(1+T_3p)}$$

Prior to the introduction of a compensating network and after introduction of a series compensating network with the transfer function

$$W_{ск}(p) = \frac{1+T_4p}{1+T_5p},$$

if the parameters have the following values: $T_1 = 0.5$ sec, $T_2 = 0.02$ sec, $T_3 = 0.0025$ sec, $T_4 = 0.1$ sec, $T_5 = 0.4$ sec, $K = 500$.

Answer. Without the compensating network, the closed-loop system is unstable. With the compensating network, the closed-loop system is stable.

§10. DETERMINATION OF STABILITY REGIONS. D-DECOMPOSITION

120. For the closed-loop automatic system with differential equation

$$[(T_1p+1)(T_2p+1)(T_3p+1)+K]x=0$$

determine the stability region with respect to the parameter T_1 if the other parameters have the following values: $T_2 = 0.5$ sec, $T_3 = 0.05$ sec, $K = 25$.

Solution. The system characteristic equation will be

$$T_1T_2T_3p^3 + (T_1T_2 + T_1T_3 + T_2T_3)p^2 + (T_1 + T_2 + T_3)p + 1 + K = 0.$$

Solving the characteristic equation for T_1, we obtain

$$T_1 = -\frac{T_2T_3p^2 + (T_2 + T_3)p + 1 + K}{T_2T_3p^3 + (T_2 + T_3)p^2 + p}.$$

Substituting the imaginary value $p = j\omega$ into the expression for T_1 and separating the real and imaginary parts, we have

$$T_1 = A(\omega) + jB(\omega),$$

where

$$A(\omega) = \frac{K(T_2 + T_3)}{T_2^2T_3^2\omega^4 + (T_2^2 + T_3^2)\omega^2 + 1},$$

$$B(\omega) = \frac{T_2^2T_3^2\omega^4 + (T_2^2 + T_3^2 - KT_2T_3)\omega^2 + 1 + K}{\omega[T_2^2T_3^2\omega^4 + (T_2^2 + T_3^2)\omega^2 + 1]}.$$

After substitution of the numerical parameter values, we obtain

$$A(\omega) = \frac{22\,000}{\omega^4 + 404\omega^2 + 1600},$$

$$B(\omega) = \frac{\omega^4 - 396\omega^2 + 4100}{\omega[\omega^4 + 404\omega^2 + 1600]}.$$

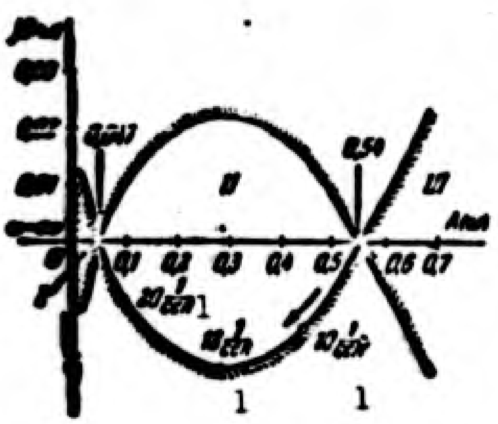


Fig. 88. Curves for determining the stability region with respect to one parameter, Problem 120. 1) Sec.

Taking values $-\infty \leq \omega \leq \infty$, we construct the curve of Fig. 88. The curve is hatched to the left if we go in the direction from $\omega = -\infty$ to $\omega = \infty$. Here the stability regions may be regions I and III.

In the general case, it is necessary to investigate system stability under some criterion for values of the parameter T_1 corresponding to the regions I and III. For the problem under consideration, the system is stable when $T_1 = 0$, since it is described by a second-order equation with positive coefficients, while for very large values of T_1, the system is clearly stable by the Vyshnegradskiy criterion. As a consequence, regions I and III are stable regions for

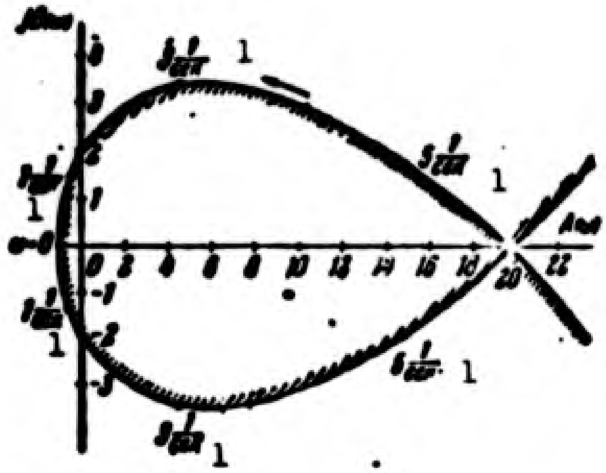


Fig. 89. Curve for determining stability region with respect to one parameter, Problem 121. 1) Sec.

values $0 < T_1 < 0.047$ sec and $T_1 > 0.54$ sec.

121. For the closed-loop automatic system considered in Problem 120, determine the stability region with respect to the transfer constant K for time-constant values of

$$T_1=0{,}5 \text{ sec}, \quad T_2=0{,}1 \text{ sec}, \quad T_3=1 \text{ sec}$$

and determine the value of the transfer constant corresponding to the boundary of stability.

Answer. Figure 89 shows the stability-region curve. The value of the transfer constant corresponding to the limit of stability is $K_{gr} = 19.8$.

122. For an automatic control system whose proper motion is described by the differential equation

$$[(T_1p+1)(T_2^2p^2+T_3p+1)+k_1k_p]x=0,$$

construct the stability region on a plane of two parameters: T_2^2 and the regulator transfer constant k_r if the other parameters have the values

$$T_1=0{,}06 \text{ sec}, \quad T_3=0{,}1 \text{ sec}, \quad k_1=1.$$

Solution. According to the given differential equation, the characteristic equation will be

$$T_1T_2^2p^3+(T_1T_3+T_2^2)p^2+(T_1+T_3)p+1+k_1k_p=0.$$

The stability condition for positive coefficients of the characteristic equation is determined from the Vyshnegradskiy criterion by means of the inequality

$$(T_1T_3+T_2^2)(T_1+T_3)>T_1T_2^2(1+k_1k_r). \qquad (1)$$

The stability boundaries are determined by three conditions: the equation obtained from (1) when the inequality is changed to an equation, the vanishing of the last coefficient of the characteristic equation, and the vanishing of the first coefficient, i.e.,

$$(T_1T_3+T_2^2)(T_1+T_3)=T_1T_2^2(1+k_1k_r). \qquad (2)$$

$$1+k_1k_r=0, \qquad (3)$$

$$T_1T_2^2=0. \qquad (4)$$

Solving (2) for k_r, we obtain

$$k_r=\frac{T_3(T_1+T_3)}{k_1T_2^2}+\frac{T_3}{k_1T_1}.$$

Substituting in the numerical parameter values, we have

$$k_r=\frac{0{,}016}{T_2^2}+1{,}67.$$

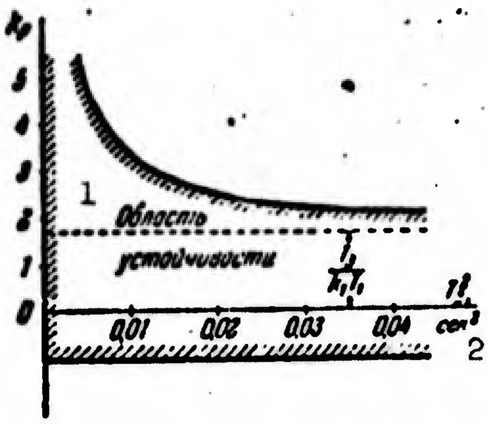


Fig. 90. Limits for determination of stability region with respect to two parameters, Problem 122. 1) Stability region; 2) sec^2.

Taking values of T_2^2 and computing k_r, we construct the hyperbola of Fig. 90.

From Condition (3), we have

$$k_r=-\frac{1}{k_1}=-1,$$

which gives us a line parallel to the T_2^2 axis.

Condition (4) yields $T_2^2 = 0$ when $T_1 \neq 0$, corresponding to a line coinciding with the k_r axis.

It is clear from Inequality (1) that an increase in k_r will lead to system instability and, as a consequence, the stability region lies below the hyperbola. In a case where this is not evident, it is possible to determine the location of the stability region with the aid of the

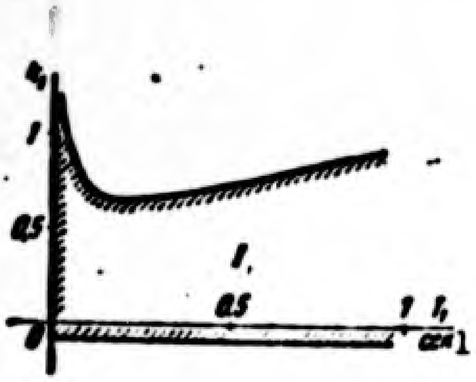


Fig. 91. Limit for determination of stability region with respect to two parameters, Problem 123. 1) Sec.

stability criterion; to do this, we substitute into the criterion values of the parameters corresponding to the region investigated.

123. Construct the stability region for the closed-loop automatic system having the characteristic equation

$$(T_k p+1)(T_2 p+1)\times \times(T_3 p+1)+k_1 k_2 k_3=0$$

on the plane of parameters T_k and k_1 if the other parameters have the following values: T_2 = 0.1 sec, T_3 = 0.5 sec, $k_2 k_3$ = 20.

Answer. The curves for the stability boundary are shown in Fig. 91. The stability region is region I.

Manuscript Page No.	[Transliterated Symbols]
67	ОВ = OV = obmotka vozbuzhdeniya = field winding
67	Г = G = generator = generator
67	пр = pr = pruzhina = spring
67	Д = D = dempfer = damper
67	ЧЭ = ChE = chuvstvitel'nyy element = sensing element
67	н = n = nagruzka = load
67	р = r = regulyator = regulator
68	гр = gr = granitsa = limit, boundary
71	о.с = o.s = obratnaya svyaz' = feedback
75	ЭМУ = EMU = elektromekhanicheskiy usilitel' = amplidyne
75	Ст = St = stabilitron = VR tube
76	сс = ss = skhema sravneniya = comparison circuit
76	г = g = generator = generator

76 у = u = usilitel' = amplifier

76 кз = kz = korotkoye zamykaniye = short circuit

89 л.а.х. = l.a.kh. = logarifmicheskaya amplitydnaya kharakteristika = logarithmic amplitude characteristic

90 п.к = p.k = posledovatel'nyy kontur = series network

Chapter 4

CONSTRUCTING TRANSIENT-PROCESS CURVES FOR AUTOMATIC CONTROL SYSTEMS

§11. THE CLASSICAL METHOD OF SOLVING DIFFERENTIAL EQUATIONS

124. Find the law governing the voltage u_C across a capacitance C if when t = 0 it is discharged through a resistance R (Fig. 92). Before the circuit is closed, $u_C = U_0$.

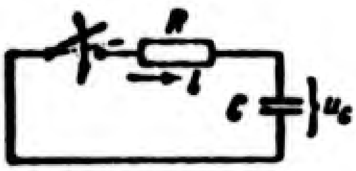


Fig. 92. Capacitor-discharge circuit.

Solution. According to the second Kirchhoff law

$$Ri + u_C = 0. \quad (1)$$

Since the current equals $i = C\frac{du_C}{dt}$, we have

$$RC\frac{du_C}{dt} + u_C = 0. \quad (2)$$

The root of the circuit characteristic equation is

$$RCp + 1 = 0 \quad (3)$$

which equals

$$p_1 = -\frac{1}{RC}.$$

According to (2) the function sought equals

$$u_C = Ae^{-\alpha t} = Ae^{-\frac{t}{RC}} = Ae^{-\frac{t}{T}}, \quad (4)$$

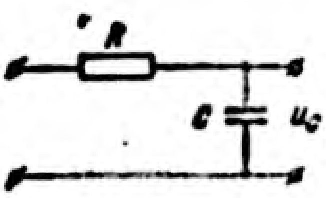


Fig. 93. Aperiodic network.

where α is the absolute value of a root of Eq. (3), and T = RC is the circuit time constant.

The constant of integration A is found from the initial condition, which in this case may be written in the form

$$(u_C)_{t=0} = U_0. \quad (5)$$

From (4) and (5) we find $A = U_0$; then

BLANK PAGE

$$u_C = U_0 e^{-\frac{t}{T}}. \qquad (6)$$

125. Find the voltage u_C across the output of an aperiodic network (Fig. 93) when a voltage in the form of a unit step function 1(t) is applied across the network input. The capacitor has previously been charged to a voltage U_0. Find the step response and weighting function for the network.

Solution. In contrast to the equation of the preceding problem, the network differential equation has the right side

$$T\frac{du_C}{dt} + u_C = 1(t). \qquad (1)$$

Here T = RC and it is assumed that 1(t) has the dimensions of voltage.

The relationship between the initial conditions holding when t = + 0, i.e., directly following application of the step function, and the initial conditions that t = − 0, i.e., directly before application of the step function is given for the general case in Appendix 10.

In the given case

$$U_{+0} = U_{-0} = U_0 \qquad (2)$$

The solution of Eq. (1) has the form

$$u_C = 1(t) + Ae^{-\frac{t}{T}}. \qquad (3)$$

From this and (2) we obtain 1(t) + A = U_0, i.e., A = U_0 − 1(t). Then

$$u_C = (1 - e^{-\frac{t}{T}})\,1(t) + U_0 e^{-\frac{t}{T}}. \qquad (4)$$

The step response A(t) is the reaction of the network to the function 1(t) under the zero-conditions. From (4) we have

$$A(t) = (1 - e^{-\frac{t}{T}})\,1(t). \qquad (5)$$

The weighting function w(t) is the reaction of the network to the unit impulse function δ(t) under the zero-conditions; $w(t) = \frac{d}{dt}A(t)$. From (5) we obtain

$$w(t) = \frac{1}{T} e^{-\frac{t}{T}} 1(t). \qquad (6)$$

126. Find the output variable x(t) of the system described by the equation

$$T \frac{d}{dt} x(t) + x(t) = y(t).$$

for the following two cases.

1. A manipulated variable governed by the harmonic law

$$y(t) = Y_m \sin \omega t;$$

is fed into the system input; the initial condition is $x(0) = X_0$.

2. Under steady-state conditions corresponding to a manipulated variable $y(t) = Y_m \sin \omega t$, an abrupt shift in supply-voltage phase by $+90^{\circ}$ occurs; the shift takes place at the time when $\omega t = n2\pi$, with n being an integer.

Answer.

1. $$x(t) = X_m \sin(\omega t - \varepsilon) + (X_0 + X_m \sin \varepsilon) e^{-\frac{t}{T}},$$

$$X_m = \frac{Y_m}{\sqrt{1 + (\omega T)^2}}, \quad \varepsilon = \operatorname{arctg} \omega T.$$

2. $$x(t) = X_m \cos(\omega t - \varepsilon) - X_m (\sin \varepsilon + \cos \varepsilon) e^{-\frac{t}{T}}.$$

127. We are given the servosystem shown in Fig. 94. The difference between the manipulated variable y and the output variable x is applied to the input of amplifier 1. In addition, the first derivative y' of the manipulated variable is applied to the amplifier. Unit 2 includes the motor, reducing gear, and final control element. The system is described by the equation

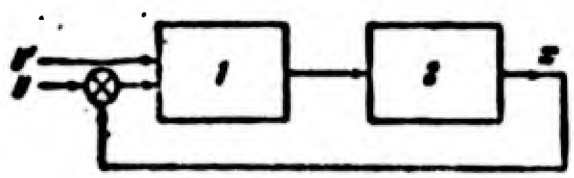


Fig. 94. Servosystem for Problem 127.

$$(Tp^2 + p + K) x(t) = (Kkp + K) y(t) \qquad (1)$$

The time constant T = 5 msec, the manipulated-variable gain is K = 40 1/sec, the gain for the manipulated-variable derivative is Kk = 0.8. Find the law governing the change in the output variable $\underline{x}$ for the following two cases:

1. The system is activated after an initial error of X_0 in the absence of control input.

2. The control input takes the form of a unit step function $1(t)$ with zero-conditions $X_{-0} = X'_{-0} = 0$.

<u>Solution.</u> 1. The system differential equation for the first case has the form

$$(Tp^2+p+K)x(t)=0 \text{ or } (0{,}005p^2+p+40)\,x(t)=0. \qquad (2)$$

The characteristic equation

$$0{,}005p^2+p+40=0 \qquad (3)$$

has two real roots: $p_1 = -55.3$ 1/sec, $p_2 = -144.7$ 1/sec.

For the case of real roots, the solution to Eq. (2) has the form

$$x(t)=A_1e^{-\alpha_1 t}+A_2e^{-\alpha_2 t}, \qquad (4)$$

where α_1 and α_2 are the absolute values of the roots of the characteristic equation.

The initial conditions are:

$$\left.\begin{aligned} \text{for } t=0 \quad & x=X_0 \\ t \quad & x'=X'_0=0. \end{aligned}\right\} \qquad (5)$$

From (4) and (5) we obtain

$$\left.\begin{aligned} A_1+A_2&=X_0 \\ -\alpha_1A_1-\alpha_2A_2&=0. \end{aligned}\right\} \qquad (6)$$

From (6) we find

$$A_1=\frac{\alpha_2X_0}{\alpha_2-\alpha_1},\quad A_2=\frac{\alpha_1X_0}{\alpha_1-\alpha_2}. \qquad (7)$$

The solution to the problem for the first case, i.e., for the error-correcting process, has the following form in accordance with (4) and (7):

$$x(t)=\frac{X_0}{\alpha_2-\alpha_1}(\alpha_2e^{-\alpha_1 t}-\alpha_1e^{-\alpha_2 t})\,1(t)$$

or

$$x(t) = X_0(1.619e^{-55.3t} - 0.619e^{-144.7t})1(t). \quad (8)$$

Expression (8) may also be obtained directly from the problem data if we make use of Appendix 11, which gives solutions for first-, second-, and third-degree homogeneous equations for both real and complex roots.

2. The system differential equation for the second case may be written, in accordance with (1), in the form

$$(a_0p^2 + a_1p + a_2)x(t) = (b_0p + b_1)y(t). \quad (9)$$

where $a_0 = T = 0.005$ sec, $a_1 = 1$, $a_2 = K = 40$ 1/sec, $b_0 = Kk = 0.8$, $b_1 = K = 40$ 1/sec.

We first find the initial conditions that apply directly before the unit step function is applied to the system.

To do this, it is convenient to make use of Appendix 10. According to this procedure, we find from (9) n = 2, m = 1, and obtain

$$X_{+0} = X_{-0} = 0,$$
$$X'_{+0} = X'_{-0} + \frac{b_0}{a_0}1(t) = 0 + \frac{0.8}{0.005}1(t) = 160\,1(t) \text{ 1/sec} \quad (10)$$

It is convenient to reduce the solution of Eq. (9) to the solution of a homogeneous equation with the same coefficients, going over to the new variable

$$z(t) = x(t) - X_{\text{уст}}, \quad (11)$$

where

$$X_{\text{уст}} = \frac{b_m}{a_n}1(t) = \frac{b_1}{a_2}1(t) = 1(t) \quad (12)$$

is a particular solution of Eq. (9), i.e., the steady value of the output variable <u>x</u>. Thus, we obtain in place of (9) the equation

$$(a_0p^2 + a_1p + a_2)z(t) = 0 \quad (13)$$

and the initial conditions

$$Z_{+0} = X_{+0} - X_{\text{уст}}, \quad Z'_{+0} = X'_{+0} \quad (14)$$

These relationships are obtained from (11).

The solution to (13) has the form

$$z(t) = A_1 e^{-\alpha_1 t} + A_2 e^{-\alpha_2 t}, \qquad (15)$$

where, in accordance with the first case, α_1 = 55.3 1/sec, α_2 = 144.7 1/sec.

In order to find the constants of integration A_1 and A_2, we obtain from (15), by (10), (12), and (14), the equations

$$\left.\begin{aligned} & A_1 + A_2 = Z_{+0} = X_{+0} - X_{ycr} \\ \text{or} \quad & A_1 + A_2 = -1(t), \\ & -\alpha_1 A_1 - \alpha_2 A_2 = Z_{+0} = X_{+0} \\ \text{or} \quad & -\alpha_1 A_1 - \alpha_2 A_2 = 160\,1(t). \end{aligned}\right\} \qquad (16)$$

From (16) we obtain

$$\left.\begin{aligned} A_1 &= \frac{-\alpha_2 + 160}{\alpha_2 - \alpha_1}\,1(t) = 0.171\,1(t), \\ A_2 &= \frac{-\alpha_1 + 160}{\alpha_1 - \alpha_2}\,1(t) = -1.171\,1(t). \end{aligned}\right\} \qquad (17)$$

We note that a solution to Eq. (13) may be obtained by using Appendix 11.

We obtain from (15), in accordance with (11), (12), and (17)

$$x(t) = z(t) + X_{ycr} = (0.171 e^{-55.3t} - 1.171 e^{-144.7t})\,1(t) + 1(t).$$

Thus, when the unit step function 1(t) is applied to the system, the output variable is governed by the law

$$x(t) = [1 + 0.171 e^{-55.3t} - 1.171 e^{-144.7t}]\,1(t). \qquad (18)$$

Equation (8) has been used to construct curve 1 of Fig. 95, and Eq. (18) for curve 2.

128. Solve Problem 127 for the following values:

T = 0.005 sec, K = 200 1/sec, Kk = 0.8.

Answer.

1. When the system is in adjustment, its law of motion is

$$x(t) = 1.155\,X_0 e^{-100t} \sin(173t + 60^\circ)$$

(Curve 1 of Fig. 96).

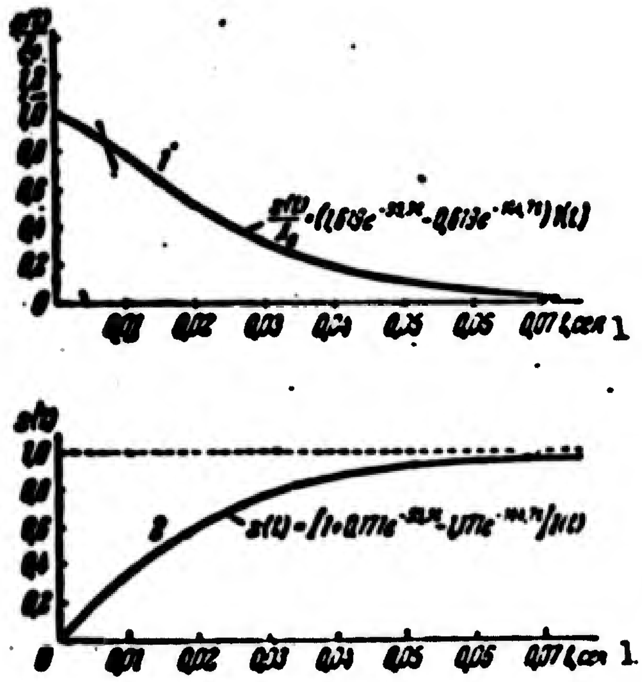


Fig. 95. Curves for the transient processes of Problem 127; 1) System in adjustment; 2) system reaction to step input. 1) Sec.

2. When the system is acted upon by a unit step function

$$x(t)=[1+1{,}059e^{-100t}\sin(173t-70^{\circ}50')]\,1(t)$$

(Curve 2 of Fig. 96).

129. Find the weighting function $w(t)$: 1) for the system of Problem 127; 2) for the system of Problem 128.

Hint. It is possible to use the step responses obtained in Problems 127 and 128 for these systems.

Answer.

1) $w(t)=(169.2e^{-141.7t}-9.45e^{-55.3t})\,1(t)$;
2) $w(t)=212e^{-100t}\cos(173t-40^{\circ}50')\,1(t)$.

130. Find the step response $A(t)$ and the weighting function $w(t)$ for the system described by the equation

$$(a_0p^2+a_1p+a_2)\,x(t)=b_0y(t).$$

All of the equation coefficients are positive: $b_0 = a_2$; $a_1^2 > 4a_0a_2$.

Answer.

$$A(t) \doteq \left(1 - \frac{\alpha_2 e^{-\alpha_1 t} - \alpha_1 e^{-\alpha_2 t}}{\alpha_2 - \alpha_1}\right) 1(t).$$

$$w(t) = \frac{\alpha_1 \alpha_2}{(\alpha_2 - \alpha_1)} \left(e^{-\alpha_1 t} - e^{-\alpha_2 t}\right) 1(t).$$

where α_1 and α_2 are the absolute values of the roots of the system characteristic equation.

131. We are given the static automatic control system described by the equation

$$(a_0 p^2 + a_1 p + a_2) x(t) = b_0 y(t).$$

where $a_0 = 0.002\ \text{sec}^2$, $a_1 = 0.12$ sec, $a_2 = 5$, $b_0 = 4$.

Find the system response to the unit step input $y(t) = 1(t)$.

Answer.

$$x(t) = [0.8 - e^{-30t} \sin(40t + 53°10')] 1(t).$$

132. The automatic control system is described by the equation

$$(a_0 p^3 + a_1 p^2 + a_2 p + a_3) x(t) = (b_1 p + b_0) y(t). \qquad (1)$$

where $a_0 = 5 \cdot 10^{-4}\ \text{sec}^2$, $a_1 = 0.105$ sec, $a_2 = 2.16$, $a_3 = b_1 = 65.3$ 1/sec, $b_0 = 1.16$.

Find the transient response for the following two cases:

1. When the system is actuated after an initial error of X_0.

2. When the manipulated variable takes the form of the unit step function $y(t) = 1(t)$ and the zero-conditions are $X_{-0} = X'_{-0} = X''_{-0} = 0$.

Solution. 1. The characteristic equation corresponding to (1) has the following form for the given coefficients:

$$0.0005 p^3 + 0.105 p^2 + 2.16 p + 65.3 = 0. \qquad (2)$$

The roots of Eq. (2) may be found by any of the well-known methods. These roots equal

$$\left.\begin{aligned} p_1 &= -\alpha = -180\ 1/\text{sec}, \\ p_{2,3} &= -\gamma \pm j\lambda = -10 \pm j25\ 1/\text{sec} \end{aligned}\right\} \qquad (3)$$

Adjustment of a system whose characteristic equation has one real root

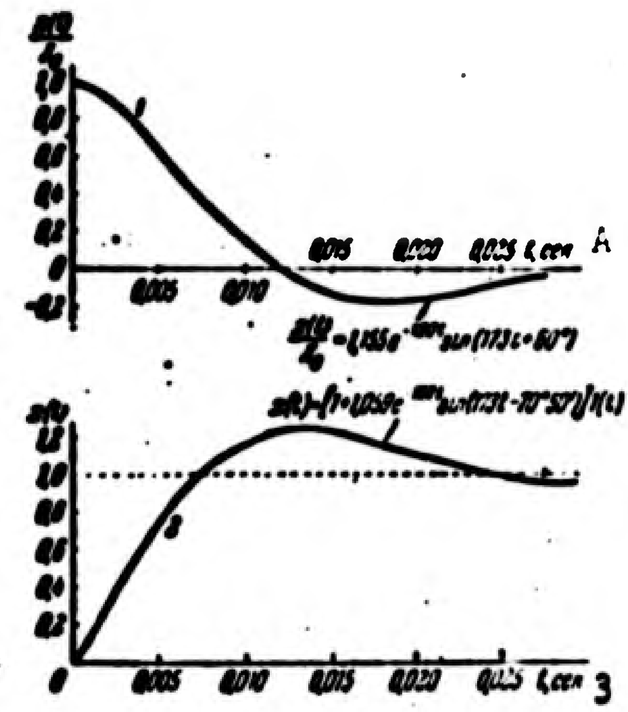


Fig. 96. Transient-process curves for Problem 128; 1) System in adjustment; 2) system step response; A) sec.

and a pair of complex roots will take place in accordance with the law

$$x(t) = Ae^{-\alpha t} + Be^{-\gamma t}\sin(\lambda t + \beta). \qquad (4)$$

The initial conditions equal

$$x(0) = X_0,\quad x'(0) = 0,\quad x''(0) = 0. \qquad (5)$$

From (4) we find

$$\left.\begin{aligned} x'(t) &= -\alpha Ae^{-\alpha t} + Be^{-\gamma t}[\lambda\cos(\lambda t+\beta) - \gamma\sin(\lambda t+\beta)],\\ x''(t) &= \alpha^2 Ae^{-\alpha t} + Be^{-\gamma t}[(\gamma^2-\lambda^2)\sin(\lambda t+\beta) -\\ &\quad - 2\gamma\lambda\cos(\lambda t+\beta)]. \end{aligned}\right\} \qquad (6)$$

From Expressions (4)-(6) we obtain a system of equations for determining the constants of integration A, B, β:

$$\left.\begin{aligned} A + B\sin\beta &= X_0,\\ -\alpha A + B\lambda\cos\beta - \gamma B\sin\beta &= 0,\\ \alpha^2 A + B(\gamma^2-\lambda^2)\sin\beta - 2\gamma\lambda\cos\beta &= 0. \end{aligned}\right\} \qquad (7)$$

Using (3), after substitution of the quantities, α, γ, λ, we obtain by solving System (7)

$$A = 0{,}0246\,X_0,\quad B = 1{,}13'X_0,\quad \beta = 59°50'. \qquad (8)$$

Substitution of (8) into (4) yields the solution to the problem

$$x(t) = X_0[0{,}0246e^{-180t} + 1{,}13e^{-10t}\sin(25t + 59°50')].$$

This result may be obtained directly from (2) and (5), if we make use of Appendix 11.

2. <u>Hint</u> for the solution of the problem in the second case: the initial conditions obtaining directly after the step input has been applied may be found with the aid of Appendix 10.

<u>Answer</u>.

$$x(t) = [1 + 0{,}0341e^{-180t} - 1{,}0341e^{-10t}\sin(25t + 88°15')]1(t).$$

133. Find the transient response of the system given in the preceding problem where the manipulated variable increases in accordance with the linear law

$$y(t) = at\,1(t).$$

<u>Hint.</u> Look for a particular solution to the system differential equation, i.e., for a forced component of the transient response, in the form

$$x_0(t) = b + ct.$$

<u>Answer.</u>

$$x(t) = [at - 0{,}000302e^{-180t} - \\ - 0{,}0392ae^{-10t}\sin(25t - 23°30') - 0{,}01532a]1(t).$$

134. An automatic control system is described by the equation

$$(a_0 p + a_1)x(t) = b_0 p y(t). \qquad (1)$$

Find the system transient response using the Duhamel integral for two types of control inputs:

$$1)\ y(t) = at\,1(t), \qquad (2)$$

$$2)\ y(t) = b(e^{-\alpha t} - e^{-\beta t})1(t) \qquad (3)$$

for zero-conditions.

<u>The solution</u> for the case $y(t) = at\ 1(t)$.

The Duhamel integral may be written in the form

$$x(t) = A(t)\,y(0) + \int_0^t y'(\tau)\,A(t-\tau)\,d\tau, \tag{4}$$

where A(t) is the system step response.

In order to determine A(t), we find the system response to a unit step input, i.e., we solve the equation

$$(a_0 p + a_1)\,x(t) = b_0 p\,1(t) \tag{5}$$

under the zero-conditions.

In accordance with Eq. (5), we find

$$X_{\text{уст}} = 0. \tag{6}$$

Using Appendix 10, we also find

$$X_{+0} = X_{-0} + \frac{b_0}{a_0}\,1(t) = \frac{b_0}{a_0}\,1(t). \tag{7}$$

Taking (6) and (7) into account, the solution to Eq. (5) will take the form

$$x(t) = Ae^{-\frac{t}{T}} + X_{\text{уст}} = Ae^{-\frac{t}{T}} = 1(t)\frac{b_0}{a_0}\,e^{-\frac{t}{T}}, \tag{8}$$

where $T = a_0/a_1$.

Thus, the system step response is

$$A(t) = \frac{b_0}{a_0}\,e^{-\frac{t}{T}}\,1(t). \tag{9}$$

For a linear control input (2) we have

$$y'(t) = a. \tag{10}$$

We substitute (9) and (10) into (4):

$$x(t) = \int_0^t a\,\frac{b_0}{a_0}\,e^{-\frac{t-\tau}{T}}\,d\tau. \tag{11}$$

Integrating Eq. (11), we obtain an answer for the first case of this problem:

$$x(t) = a\,\frac{b_0}{a_1}\left(1 - e^{-\frac{t}{T}}\right)1(t).$$

<u>Answer</u>. For the case of an aperiodic control input (3):

$$x(t) = b\frac{b_0}{a_0}\,\frac{(r - qrT)e^{-qt} - (q - qrT)e^{-rt} + (q - r)e^{-\frac{t}{T}}}{T\left(\frac{1}{T} - q\right)\left(\frac{1}{T} - r\right)}\,1(t).$$

135. Find the transient response of the system described by the equation

$$(a_0 p + a_1)x(t) = b_0 y(t).$$

for a damped oscillatory control input

$$y(t) = ce^{-\varepsilon t}\sin\omega t$$

and the zero-conditions.

Hint. It is suggested that the Duhamel integral be used.

Answer.

$$x(t) = c\frac{b_0}{a_0}\,\frac{\left(\frac{1}{T} - \varepsilon\right)e^{-\varepsilon t}\sin\omega t - \omega e^{-\varepsilon t}\cos\omega t + \omega e^{-\frac{t}{T}}}{T\left[\left(\frac{1}{T} - \varepsilon\right)^2 + \omega^2\right]}\,1(t).$$

where $T = a_0/a_1$.

§12. USING LAPLACE AND KARSON-HEAVISIDE TRANSFORMS

136. The open-loop transfer function of an automatic control system is

$$W(p) = \frac{K}{p(1 + Tp)} = \frac{20}{p(1 + 0{,}1p)}. \qquad (1)$$

Find the step response $A(t)$ and the weighting function $w(t)$ for the closed-loop system.

Solution. The system closed-loop transfer function, taking (1) into account, is

$$\Phi(p) = \frac{W(p)}{1 + W(p)} = \frac{K}{Tp^2 + p + K} = \frac{20}{0{,}1p^2 + p + 20}. \qquad (2)$$

The step response $A(t)$ is the system response to a unit step input $1(t)$.

The transform $X(p)$ of the output variable $x(t)$ of a closed-loop system for a control input $y(t)$ whose transform is $Y(p)$ will be, under zero-conditions, the product

$$X(p) = \Phi(p)\,Y(p).$$

The Karson-Heaviside transform of a unit step function will be 1, and the Laplace transform 1/p. Thus the step response A(t) of the system may be obtained as the result of inverse Karson-Heaviside transformation of the system closed-loop transfer function, i.e., Expression (2), or as a result of inverse Laplace transformation of the product

$$\frac{1}{p}\Phi(p)=\frac{20}{p(0{,}1p^2+p+20)}. \qquad (3)$$

In going from transform (2) and (3) to the sought preimage A(t), it is necessary to factor the denominator of the transform. To do this, we set the denominator of (2) equal to zero:

$$Tp^2+p+K=0 \quad \text{or} \quad 0{,}1p^2+p+20=0 \qquad (4)$$

and find the roots of the resulting equation (4):

$$\begin{aligned} p_1&=-k+j\lambda=-5+j13{,}2 \quad 1/\text{sec}\\ p_2&=-k-j\lambda=-5-j13{,}2 \quad 1/\text{sec}\end{aligned} \qquad (5)$$

We next can write the denominator of Expression (2) in the form

$$\begin{aligned} 0{,}1p^2+p+20&=0{,}1(p-p_1)(p-p_2)=\\ &=0{,}1[p-(-k+j\lambda)][p-(-k-j\lambda)]=\\ &=0{,}1[(p+k)^2+\lambda^2]=0{,}1[(p+5)^2+(13{,}2)^2]. \end{aligned} \qquad (6)$$

Now in place of (2) we obtain

$$\Phi(p)=\frac{20}{0{,}1[(p+5)^2+(13{,}2)^2]}=\frac{200}{(p+5)^2+(13{,}2)^2}. \qquad (7)$$

From tables of Karson-Heaviside transforms, we select a formula corresponding to Expression (7):

$$\frac{1}{(p+k)^2+\omega^2}\doteqdot\frac{1}{k^2+\omega^2}+\frac{1}{\omega\sqrt{k^2+\omega^2}}e^{-kt}\sin(\omega t-\varphi), \qquad (8)$$

$$\varphi=\operatorname{arctg}\frac{\omega}{-k}.$$

When such a tabulated formula is selected, it should be remembered that in the available books these formulas are given in order of increasing degree of the polynomial in p in the transform denominator.

For the case of real roots and for complex roots, separate formulas are always provided. Thus, if the roots of the denominator of the transform (2) turn out to be real, the tabulated formula

$$\frac{1}{(p+\alpha_1)(p+\alpha_2)} \risingdotseq \frac{1}{\alpha_1\alpha_2} + \frac{1}{\alpha_1-\alpha_2}\left(\frac{1}{\alpha_1}e^{-\alpha_1 t} - \frac{1}{\alpha_2}e^{-\alpha_2 t}\right).$$

should be taken in place of Formula (8); here α_1 and α_2 are the absolute values of the roots.

Comparing (7) and (8), we obtain the preimage of Expression (7), i.e., the step response A(t) of the system:

$$A(t) \risingdotseq \Phi(p) \risingdotseq \left[\frac{200}{5^2+(13{,}2)^2} - \frac{200}{13{,}2\sqrt{5^2+(13{,}2)^2}}e^{-5t} \times \sin(13{,}2t + 69^\circ 15')\right] 1(t)$$

or

$$A(t) = [1 - 1{,}068e^{-5t}\sin(13{,}2t + 69^\circ 15')]\, 1(t). \qquad (9)$$

Remark. Care should be taken in computing the angle ψ from Formula (8), since the signs in the formulas for ψ that are typical for the expressions chosen are distinctively written. The sign of the numerator in the expression for the tangent of ψ is the sign of the sine of ψ, while the sign of the denominator is the sign of the cosine of ψ. Thus, the formula for ψ contains an indication of the quadrant in which this angle lies. This enables us to avoid the ambiguity in the value for ψ caused by the fact that the tangents of two angles differing by π will be the same.

In the case under consideration, where $\tan\psi = -13.2/5 = -2.64$, of the two possible values of ψ equaling $-69^\circ 15'$ and $+110^\circ 45'$, we should take the second value, since the expression $\psi = \arctan \omega/-k = \arctan 13.2/-5$ indicates that the angle ψ lies in the second quadrant.

As a result we obtain from Formula (8)

$$\sin(\omega t - \psi) = \sin(13{,}2t - 110^\circ 45') = -\sin(13{,}2t + 69^\circ 15'),$$

which is also considered in Expression (9).

The system weighting function w(t) can be found as the derivative of the step response (9) with respect to time.

The weighting function may also be found directly from the transfer function (2) by the inverse Laplace transformation

$$w(t)=L^{-1}[\Phi(p)]=L^{-1}\left[\frac{200}{(p+5)^2+(13{,}2)^2}\right] \tag{10}$$

or by the inverse Karson-Heaviside transformation of the product

$$p\Phi(p)=\frac{20p}{0{,}1p^2+p+20}. \tag{11}$$

From tables of Laplace transforms of functions, we choose a formula corresponding to (10):

$$\frac{1}{(p+a)^2+\omega^2} \doteqdot \frac{1}{\omega}e^{-at}\sin\omega t. \tag{12}$$

In accordance with (7), (10), and (12) we obtain the system weighting function

$$w(t)=15{,}15e^{-5t}\sin 13{,}2t\,1(t). \tag{13}$$

137. For the closed-loop automatic control system given in the preceding problem, find the law of motion governing the adjusting process, i.e., the law governing the variation in the output variable x(t) in the absence of a control input, with an initial error $x(0) = X_0$, and zero initial velocity.

Solution. According to Eq. (2) of the first problem, the closed-system differential equation has the form

$$(Tp^2+p+K)x(t)=Ky(t), \tag{1}$$

where y(t) is the control input. In order to obtain the transform of the output variable x(t) from (1), it is necessary to make use of operator expressions for the derivatives, taking the initial conditions into account. We write these expressions, using the Karson-Heaviside transforms on the assumption that X(p) is the transform of the function x(t):

$$\begin{aligned} px(t)&=x'(t)\doteqdot pX(p)-px(0),\\ p^2x(t)&=x''(t)\doteqdot p^2X(p)-p^2x(0)-px'(0). \end{aligned} \tag{2}$$

Here x(0) and x'(0) are the initial values of the output variable and its derivative. From (1) and (2), since y(t) = 0, we obtain

$$Tp^2X(p) - Tp^2x(0) - Tpx'(0) + pX(p) - px(0) + KX(p) = 0,$$

or

$$X(p) = \frac{p(Tp+1)x(0) + Tpx'(0)}{Tp^2+p+K}. \tag{3}$$

Substituting in the values of the initial conditions $x(0) = X_0$ and $x'(0) = 0$ and of the equation coefficients $T = 0.1$ sec and $K = 20$ 1/sec, we obtain the Karson-Heaviside representation of the system adjustment law:

$$X(p) = \frac{p(0,1p+1)X_0}{0,1p^2+p+20} = \frac{p(0,1p+1)X_0}{0,1[(p+5)^2+13,2^2]} = \frac{p(p+10)X_0}{(p+5)^2+13,2^2}. \tag{4}$$

The appropriate tabular formula (Karson-Heaviside) is:

$$\frac{p(p+a_0)}{(p+k)^2+\omega^2} \doteqdot \frac{1}{\omega}\sqrt{(a_0-k)^2+\omega^2}\,e^{-kt}\sin(\omega t+\varphi), \tag{5}$$

$$\varphi = \operatorname{arctg}\frac{\omega}{a_0-k}.$$

From Transform (4), on the basis of Formula (5) we obtain the law governing system adjustment:

$$x(t) = X_0\frac{\sqrt{(10-5)^2+13,2^2}}{13,2}e^{-5t}\sin(13,2t+69°15'),$$

or

$$x(t) = X_0\,1,068e^{-5t}\sin(13,2t+69°15').$$

Remarks. Utilization of tabular formulas of the type (5) is not the only way of going from the transform of a function to its preimage. It is possible, for example, to make use of a decomposition theorem.

Before going to the preimage $x(t)$, we may check the legitimacy of the transform $X(p)$ by several rules. It is possible, in particular, to check the transform in terms of its dimensions. The Karson-Heaviside transform of any function, for example $x(t)$:

$$X(p) = p\int_0^\infty x(t)e^{-pt}\,dt, \tag{6}$$

will have the same dimensions as the preimage $x(t)$. This is clear, for example, from the fact that the Karson-Heaviside transform of a step function will equal the function itself, i.e., A1(t) $\doteq$ A when t $\geq$ 0. It follows from Expression (6) that the argument p of the transform will have the dimensions of frequency, i.e., $(\text{time})^{-1}$. The dimensions of a Laplace transform for a function

$$X(p)=L[x(t)]=\int_0^{\infty} x(t)\, e^{-pt}\, dt \qquad (7)$$

will equal the dimensions of the preimage multiplied by time, i.e., will differ from the Karson-Heaviside transform dimensions by a factor (time).

Let us use these dimenional considerations to check the Karson-Heaviside transform (3) for the x coordinate of the investigated system. The right side of Expression (3) should have the dimensions of the x coordinate. Since the dimensions of p are $(\text{time})^{-1}$, we find that the dimensions of all terms in the numerator are (coordinate) $\times$ $(\text{time})^{-1}$, and of the denominator $(\text{time})^{-1}$. Thus, this check gives a positive result.

Let us go to other types of tests for transforms.

From Expression (3), we can directly find the initial value of the preimage

$$x(0)=\lim_{p\to\infty} X(p). \qquad (8)$$

Applying (8) to (3), we obtain $x(0)=\frac{T}{T}x(0)$, i.e., this test also produces convergence.

From Expression (3) we can also find the limit of the preimage x(t) when $t \to \infty$ (if this limit exists) from the formula

$$\lim_{t\to\infty} x(t)=\lim_{p\to 0} X(p). \qquad (9)$$

Applying (9) to (3), we find $x(\infty)=\frac{0}{K}=0$, which is clearly legitimate, since the considered system is from Expression (1) astatic; thus the steady-state error equals zero. Formula (9) is applicable provided X(p) has poles only in the left half plane of the complex variable p, i.e., provided the real parts of all roots of the denominator for function $X(p)$ are negative.

These types of test on the transform obtained yield only the necessary conditions for correctness of the result; in practice, however, these conditions are frequently also sufficient.

138. The open-loop transfer function of a system equals

$$W(p)=\frac{K}{(1+T_1p)(1+T_2p)}=\frac{3}{(1+0{,}2p)(1+0{,}01p)}.$$

Find the system closed-loop step response.

<u>Answer.</u>

$$h(t)=(0{,}750+0{,}341\,e^{-80t}-1{,}091\,e^{-25t})\,1(t).$$

139. For the system of the preceding problem find the law governing motion in the absence of a control input under initial conditions $x(0) = X_0$ and $x'(0) = X_0'$.

<u>Hint.</u> The solution will be the sum of two terms one proportional to X_0 and the other to X_0'; it is convenient to find these terms separately and combine the results.

<u>Answer.</u>

$$x(t)=X_0[1{,}455^{-25t}-0{,}455e^{-80t}]+0{,}0182\,X_0'[e^{-25t}-e^{-80t}].$$

140. For a servosystem with transfer function (see Problem 136)

$$\Phi(p)=\frac{20}{0{,}1p^2+p+20}$$

find the law of motion where there is a control input in the form of the step function $b\,1(t)$ under the initial conditions $x(0) = X_0$ and $x'(0) = 0$.

<u>Answer.</u>

$$x(t)=b[1-1{,}068e^{-5t}\sin(13{,}2t+69^\circ 15')]+$$
$$+X_0\,1{,}068e^{-5t}\sin(13{,}2t+69^\circ 15').$$

141. For a servosystem having an open-loop transfer function

$$W(p)=\frac{K}{p(1+Tp)}=\frac{24}{p(1+0{,}0067p)}\cdot$$

find the output variable x(t) where there is a control input in the form of the step function $y(t) = b\ 1(t)$ under the initial conditions $x(0) = X_0$, $x'(0) = X_0'$.

<u>Answer</u>

$$x(t)=b[1-1{,}333e^{-30t}+0{,}333e^{-120t}]+$$
$$+X_0[1{,}333e^{-30t}-0{,}333e^{-120t}]+0{,}0111X_0'[e^{-30t}-e^{-120t}].$$

142. Find the law governing the variation in the output variable x(t) of a closed-loop servosystem where there is a step control input 1(t) under the zero-conditions. The system open-loop transfer function is

$$W(p)=\frac{K(1+T_2p)}{p(1+T_1p)(1+T_3p)}=\frac{500(1+0{,}03p)}{p(1+0{,}1p)(1+0{,}006p)}. \qquad (1)$$

<u>Solution.</u> We find the system closed-loop transfer function:

$$\Phi(p)=\frac{W(p)}{1+W(p)}=\frac{K(1+T_2p)}{p(1+T_1p)(1+T_3p)+K(1+T_2p)}=$$
$$=\frac{15p+500}{0{,}0006p^3+0{,}106p^2+16p+500}. \qquad (2)$$

The Karson-Heaviside transform for the unknown system response to a step input will have the form

$$X(p)=\Phi(p). \qquad (3)$$

Next, without regard to the suggested method for going from (3) to the preimage, we must find the roots of the denominator of the transform (2), i.e., the roots of the equation

$$0{,}0006p^3+0{,}106p^2+16p+500=0. \qquad (4)$$

As a result of calculations which we shall not show here, the following roots are obtained for Eq. (4):

$$\left.\begin{aligned} p_1&=-39{,}2\ 1/\text{sec},\\ p_2&=-68{,}8+j\,128{,}5\ 1/\text{sec}\\ p_3&=-68{,}8-j\,128{,}5\ 1/\text{sec}\end{aligned}\right\} \qquad (5)$$

If we now consider (5) and represent the denominator of Transform

(2) as the product

$$0{,}0006p^3+0{,}106p^2+16p+500=$$
$$=0{,}0006\,(p+39{,}2)\,[(p+68{,}8)^2+128{,}5^2],$$

we can obtain the preimage by means of tables of transforms.

Here we shall use another method for obtaining the preimage — a decomposition theorem. Let the unknown function x(t) have the following Karson-Heaviside transform:

$$X(p)=\frac{B(p)}{D(p)}=\frac{b_0p^m+b_1p^{m-1}+\ldots+b_{m-1}p+b_m}{a_0p^n+a_1p^{n-1}+\ldots+a_{n-1}p+a_n}, \qquad (6)$$

where $m \leq n$ and the equation $D(p) = 0$ has neither zeroes nor multiple roots. Then from the decomposition theorem, the preimage x(t) may be found from the formula

$$x(t)=\frac{B(0)}{D(0)}+\sum_{k=1}^{n}\frac{B(p_k)}{p_kD'(p_k)}e^{p_kt}, \qquad (7)$$

where $p_1, \ldots, p_k, \ldots, p_n$ are the roots of the equation and $D'(p)=\frac{d}{dp}D(p)$. In accordance with (2) and (3), we write

$$X(p)=\frac{15p+500}{0{,}0006p^3+0{,}106p^2+16p+500}=$$
$$=\frac{25\,000\,(p+33{,}3)}{p^3+176{,}6p^2+26\,650p+833\,000}. \qquad (8)$$

Comparing (6) and (8), we obtain

$$\left.\begin{array}{ll} B(p)=25\,000\,(p+33{,}3), & B(0)=833\,000, \\ D(p)=p^3+176{,}6p^2+26\,650p+833\,000, & \\ D(0)=833{,}000, & D'(p)=3p^2+353p+26\,650. \end{array}\right\} \qquad (9)$$

According to decomposition theorem (7), we obtain

$$x(t)=1+\sum\frac{25\,000\,(p_k+33{,}3)}{p_k\,(3p_k^2+353p_k+26\,650)}e^{p_kt}. \qquad (10)$$

We separately calculate the terms contained in (10) following the summation sign. When $p_1 = -39.2$ 1/sec, we obtain

$$\frac{B(p_1)}{p_1D'(p_1)}e^{p_1t}=\frac{-147\,500}{-39{,}2\cdot17\,430}e^{-39{,}2t}=0{,}216\,e^{-39{,}2t}, \qquad (11)$$

when $p_2 = -68.8 + j\,128.5$ 1/sec $= 146e^{j118^\circ10'}$ 1/sec, we obtain

$$\frac{B(p_2)}{p_2 D'(p_2)} e^{p_2 t} = \frac{2{,}5 \cdot 10^6 \cdot 133{,}5\, e^{j103^\circ 25'}}{146\, e^{j118^\circ 10'} \cdot 2{,}4 \cdot 10^6\, e^{-j166^\circ 35'}}\, e^{(-68{,}8 + j128{,}5)t} =$$

$$= 0{,}672\, e^{j(151^\circ 50')}\, e^{-68{,}8t}\, e^{j128{,}5t} = 0{,}672\, e^{j(128{,}5t + 151^\circ 50')}\, e^{-68{,}8t}. \qquad (12)$$

when $p_3 = -68.8 - j128.5$ 1/sec $= 146e^{-j118^\circ 10'}$ 1/sec, we obtain

$$\frac{B(p_3)}{p_3 D'(p_3)} e^{p_3 t} = 0{,}672\, e^{-j(128{,}5t + 151^\circ 50')}\, e^{-68{,}8t}. \qquad (13)$$

Expression (13) was written without calculation, directly from inspection of Expression (12), since the roots p_2 and p_3 are conjugate, while the coefficients of Expression (10) are purely real. Under these conditions, the complex expressions (12) and (13) must be conjugate.

If all the roots of Eq. (4) had been real, then Expressions (11)-(13) would have contained no complex numbers, and the calculation could have been finished by substituting these expressions into Formula (10).

For the case given, Expressions (12) and (13) are complex so that they must be converted. Applying the Euler formula

$$\frac{e^{ja} + e^{-ja}}{2} = \cos a,$$

to the sum of the conjugate expressions (12)and (13), we obtain

$$\frac{B(p_2)}{p_2 D'(p_2)} e^{p_2 t} + \frac{B(p_3)}{p_3 D'(p_3)} e^{p_3 t} =$$

$$= 0{,}672\, e^{-68{,}8t} \left[e^{j(128{,}5t + 151^\circ 50')} + e^{-j(128{,}5t + 151^\circ 50')}\right] =$$

$$= 1{,}345\, e^{-68{,}8t} \cos(128{,}5t + 154^\circ 8') =$$

$$= -1{,}345\, e^{-68{,}8t} \cos(128{,}5t - 23^\circ 32'). \qquad (14)$$

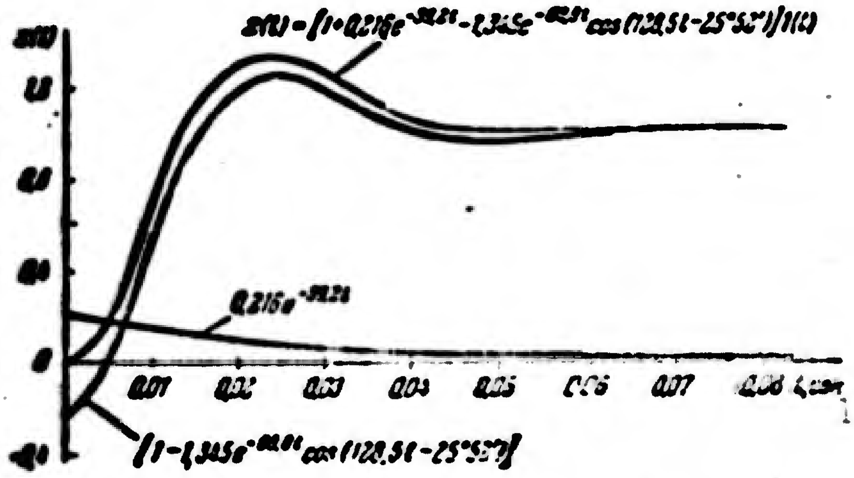


Fig. 97. Transient response of a servosystem with first-order astatism in the presence of a step control input. 1) Sec.

Substitution of Functions (11) and (14) into Formula (10) yields the final result, i.e., the system response to the step input 1(t):

$$x(t) = [1 + 0{,}216e^{-39{,}3t} - 1{,}345e^{-69{,}9t}\cos(128{,}5t - 25°52')]\,1(t). \qquad (15)$$

The separate terms of this expression and the curve for x(t) have been plotted on Fig. 97.

143. For the closed-loop servosystem given in the preceding problem, find in general form the Laplace and Karson-Heaviside transforms X(p) of the output variable x(t) in terms of the transform Y(p) of the control input for the nonzero initial conditions $x(0) = X_0$, $x'(0) = X'_0$, and $x''(0) = X''_0$.

Answer. The Laplace transform is

$$X(p) = L[x(t)] = \frac{(15p + 500)Y(p) + (6\cdot10^{-4}p^2 + 0{,}106p + 16)X_0 + (6\cdot10^{-4}p + 0{,}106)X'_0 + 6\cdot10^{-4}X''_0}{6\cdot10^{-4}p^3 + 0{,}106p^2 + 16p + 500}.$$

The Karson-Heaviside transform is

$$X(p) = \frac{(15p + 500)Y(p) + p(6\cdot10^{-4}p^2 + 0{,}106p + 16)X_0 + p(6\cdot10^{-4}p + 0{,}106)X'_0 + 6\cdot10^{-4}pX''_0}{6\cdot10^{-4}p^3 + 0{,}106p^2 + 16p + 500}.$$

144. Find the law governing motion of the system given in Problems 142 and 143 in the absence of a control input with the initial conditions $x(0) = X_0$, $x'(0) = 0$, $x''(0) = 0$.

Answer.

$$x(t) = X_0[1{,}221e^{-39{,}3t} + 0{,}335e^{-69{,}9t}\sin(128{,}5t - 41°45')].$$

145. The servosystem given in Problem 142 has an open-loop transfer function

$$W(p) = \frac{K(1 + T_3p)}{p(1 + T_1p)(1 + T_2p)} = \frac{500(1 + 0{,}03p)}{p(1 + 0{,}1p)(1 + 0{,}006p)}.$$

Find the output variable x(t) for the closed-loop servosystem where there is a control input in the form of the impulse function $A\delta(t)$.

Answer.

$$x(t)=A[-8,46e^{-30,2t}+196,4e^{-41,4t}\sin(128,5t+2°30')].$$

146. For the closed-loop servosystem whose open-loop transfer function is

$$W(p)=\frac{K}{p(1+Tp)}=\frac{24}{p(1+0,0067p)},$$

find the output variable x(t) for a control input that increases in accordance with the linear law y(t) = at 1(t), under zero-conditions.

Solution. The system closed-loop transfer function is

$$\Phi(p)=\frac{W(p)}{1+W(p)}=\frac{3600}{p^2+150p+3600}. \qquad (1)$$

The Laplace transform of the control input is

$$Y(p)=\frac{a}{p^2}. \qquad (2)$$

From (1) and (2), the Laplace transform of the output variable will equal

$$X(p)=\Phi(p)Y(p)=\frac{3600a}{p^2(p^2+150p+3600)}. \qquad (3)$$

To find the preimage of Expression (3), we may make use of a convolution theorem according to which

$$x(t)=\int_0^t x_1(\tau)x_2(t-\tau)d\tau, \qquad (4)$$

if

$$X(p)=X_1(p)X_2(p) \qquad (5)$$

and

$$x_1(t)\doteqdot X_1(p), \qquad (6)$$

$$x_2(t)\doteqdot X_2(p). \qquad (7)$$

In accordance with (5), Transform (3) should be decomposed into two factors so that the product of their preimages can easily be integrated. We choose these factors as follows:

$$X(p)=\frac{3600}{p(p^2+150p+3600)}\frac{a}{p}.$$

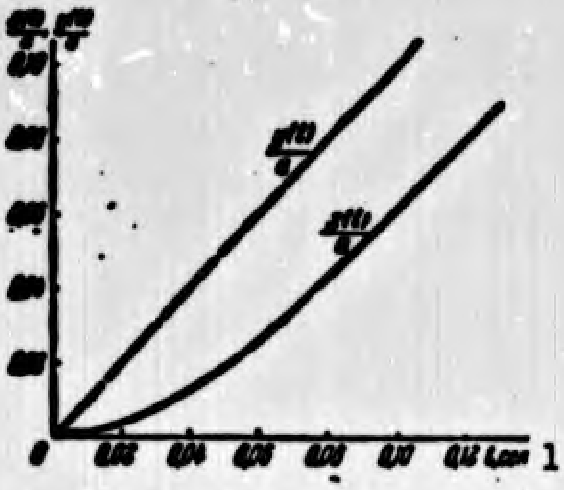


Fig. 98. Transient response of servosystem with first-order astaticism under a linear control input. 1) Sec.

i.e.,

$$X_1(p)=\frac{3600}{p(p^2+150p+3600)}=\frac{3600}{p(p+30)(p+120)}. \qquad (8)$$

$$X_2(p)=\frac{a}{p}. \qquad (9)$$

The denominator of Expression (8) is factored by a standard method. For Expressions (8) and (9), we select suitable formulas from a table of Laplace transforms:

$$\frac{1}{p(p+\alpha_1)(p+\alpha_2)}\doteqdot\frac{1}{\alpha_1\alpha_2}+\frac{\frac{1}{\alpha_1}e^{-\alpha_1 t}-\frac{1}{\alpha_2}e^{-\alpha_2 t}}{\alpha_1-\alpha_2}. \qquad (10)$$

$$\frac{1}{p}\doteqdot 1(t). \qquad (11)$$

We now find from (6)-(11)

$$x_1(t)=(1-1{,}333e^{-30t}+0{,}333e^{-120t})\,1(t). \qquad (12)$$

$$x_2(t)=a\,1(t). \qquad (13)$$

We substitute the preimages (12) and (13) into Formula (4) of the convolution theorem:

$$x(t)=\int_0^t[1-1{,}333e^{-30\tau}+0{,}333e^{-120\tau}][a\,1(t-\tau)]\,d\tau. \qquad (14)$$

We integrate (14):

$$x(t)=a[t+0{,}0445e^{-30t}-0{,}00277e^{-120t}].$$

From this we obtain the desired solution to the problem

$$x(t)=[at+a(0{,}0445e^{-30t}-0{,}00277e^{-120t})-0{,}0417a]\,1(t).$$

The manipulated variable y(t) and the output variable x(t) are plotted on Fig. 98.

147. The open-loop transfer function of a servosystem equals

$$W(p)=\frac{K}{p(1+Tp)}=\frac{24}{p(1+0{,}0067p)}.$$

Find the servo-system closed-loop error $\varepsilon(t) = y(t) - x(t)$ under zero-conditions for two types of control input:

1) for a step input $y(t) = b\,1(t)$;

2) for a control input that increases in accordance with a linear law $y(t) = at\,1(t)$.

Hint. The servo-system error transfer function equals

$$\Phi_\varepsilon(p)=\frac{1}{1+W(p)}.$$

Answer.

1) $\varepsilon(t)=b(1{,}333e^{-30t}-0{,}333e^{-120t})\,1(t)$;

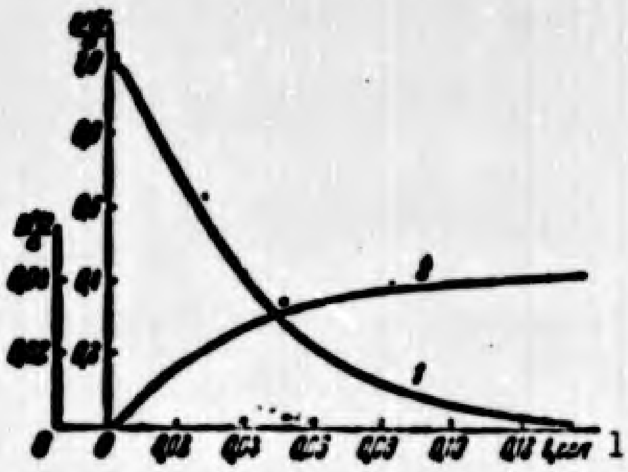


Fig. 99. Graphs showing error of servosystem with first-order astatism for step (curve 1) and linear (curve 2) control inputs. 1) Sec.

2) $\varepsilon(t)=a(0{,}0417-0{,}0445e^{-30t}-0{,}00277e^{-120t})\,1(t)$.

The errors for both cases have been plotted in Fig. 99.

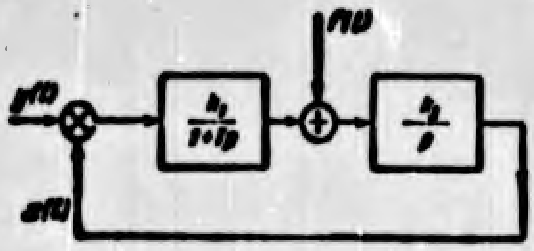


Fig. 100. Block diagram of servosystem for Problems 148 and 150.

148. The servosystem whose diagram is shown in Fig. 100 has an open-loop transfer function

$$W(p)=w_1(p)w_2(p)=\frac{k_1}{1+Tp}\frac{k_2}{p}.$$

The system consists of two elements between which a disturbance f(t) is applied. Find the output variable x(t) for a step disturbance f(t) = d 1(t) in the absence of control input y(t) under zero-conditions; $K = k_1k_2 = 24$ 1/sec, T = 6.7 msec, $k_2 = 0.01$ 1/sec. The last coefficient is given on the assumption that the coordinates x(t) and y(t) are dimensionless, while the input variable of the second element, including the disturbance f(t) has the dimensions of voltage.

Answer

$$x(t)=10^{-4}d\,[4,17-4,45e^{-30t}+0,278e^{-120t}]\,1(t).$$

149. A servosystem consists of the two elements shown in Fig. 100;

$$W(p)=\frac{k_1k_2}{p(1+Tp)}=\frac{100}{p(1+0,025p)}.$$

At the input of the second element there acts a disturbance taking the form of the impulse function f(t) = gδ(t); the control input y(t) is absent, zero initial conditions obtain. Find the output variable x(t) of the closed-loop servosystem.

Answer.

$$x(t)=k_2g\,1,053e^{-20t}\sin(60t+71^\circ34')\,1(t).$$

150. The servo-system open-loop transfer function is

$$W(p)=\frac{K(1+Tp)}{p^2},$$

where K = 4000 1/sec^2, T = 0.01 sec. Find the output variable x(t) of the closed-loop system under a step control input y(t) = b 1(t) and zero-conditions.

Answer

$$x(t) = b[1 + 1{,}053e^{-20t}\sin(60t - 71°34')]\,1(t).$$

151. We are given two servosystems having the following open-loop transfer functions:

1) $$W_1(p) = \frac{K_1}{p(1 + T_1 p)},$$

2) $$W_2(p) = \frac{K_2(1 + T_2 p)}{p^2},$$

where K_1 = 100 1/sec, T_1 = 25 msec, K_2 = 4000 1/sec^2, T_2 = 10 msec. Find the output variables x(t) and errors $\varepsilon(t) = y(t) - x(t)$ of the closed-loop servosystems for a control input that increases in accordance with a linear law $y(t) = at\,1(t)$ under zero-conditions. Plot the error curves for both systems on one graph.

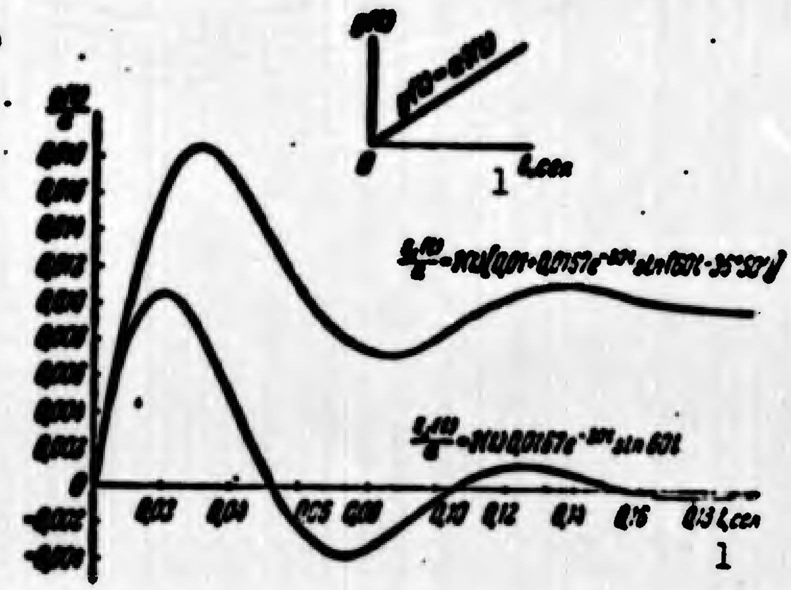


Fig. 101. Errors for control input $y(t) = at\,1(t)$, servosystem with first-order astatism $\varepsilon_1(t)$ and second-order astatism $\varepsilon_2(t)$. 1) Sec.

Answer

$$x_1(t) = a[t - 0{,}01 - 0{,}0167e^{-20t}\sin(60t - 36°50')]\,1(t),$$
$$\varepsilon_1(t) = a[0{,}01 + 0{,}0167e^{-20t}\sin(60t - 36°50')]\,1(t),$$
$$x_2(t) = a[t - 0{,}0167e^{-20t}\sin 60t]\,1(t),$$
$$\varepsilon_2(t) = a\,0{,}0167e^{-20t}\sin 60t\,1(t).$$

The curves for $\varepsilon_1(t)$ and $\varepsilon_2(t)$ are plotted in Fig. 101.

152. A closed-loop automatic control system is described by the equation

$$(0{,}1479p^4 + 3{,}7p^3 + 15{,}61p^2 + 17{,}9p + 20)x(t) = (17{,}9p + 20)y(t)$$

Find the output variable x(t) for the step control input $y(t) = b\,1(t)$ under zero-conditions.

Answer.

$$x(t) = b[1 + 1{,}456e^{-0{,}5t}\sin(1{,}2t - 72^\circ) + 0{,}398e^{-4t} - 0{,}019e^{-20t}]1(t)$$

153. Find the output variable x(t) for the system given in the preceding problem in the absence of control input, with initial conditions $x(0) = X_0$, $x'(0) = X_0'$, $x''(0) = X_0''$ and $x'''(0) = X_0'''$.

Answer

$$\begin{aligned} x(t) = {} & X_0[1{,}202e^{-0{,}5t}\cos(1{,}2t - 45^\circ) + 0{,}155e^{-4t} - 0{,}003e^{-20t}] + \\ & + X_0'[1{,}112e^{-0{,}5t}\sin(1{,}2t - 4^\circ 50') + 0{,}099e^{-4t} - 0{,}003e^{-20t}] + \\ & + X_0''[0{,}283e^{-0{,}5t}\sin(1{,}2t - 19^\circ 30') + 0{,}096e^{-4t} - 0{,}002e^{-20t}] + \\ & + X_0'''[0{,}288e^{-0{,}5t}\sin(1{,}2t - 22^\circ 15') + 0{,}114e^{-4t} - 0{,}004e^{-20t}]. \end{aligned}$$

154. Find the step response A(t) and weighting function w(t) for the system whose transfer function equals

$$\Phi(p) = \frac{K}{(p+\alpha)^n},$$

where n is a positive integer.

Hint. The convolution theorem should be used.

Answer

$$A(t) = K\left[\frac{1}{\alpha^n} - e^{-\alpha t}\sum_{k=0}^{n-1}\frac{t^k}{\alpha^{n-k}k!}\right].$$

$$w(t) = \frac{K}{(n-1)!}t^{n-1}e^{-\alpha t}.$$

§13. APPROXIMATE METHOD FOR DETERMINATION OF TRANSIENT RESPONSE

A. Investigation of Real Frequency Characteristics

155. Using the real frequency characteristic $P(\omega)$ for the control system (Fig. 102a), construct the transient-response curve under a unit step input and zero initial conditions.

The curve $P(\omega)$ is replaced, approximately, by several trapezoidal curves in order that the sum of the y-axis values of the trapezoids equal the y-axis value of the real frequency characteristic $P(\omega)$. In the given case, it is possible to take four trapezoids, as shown in Fig. 102b; one of them is positive and the rest negative. Each trapezoid should have the typical form shown in Fig. 102c; then it will determined completely by three numbers: the cutoff frequency ω_s, the slope $\kappa = \omega_d/\omega_s$, and the height $\underline{r}$. The trapezoids of Fig. 102b have the parameters shown in Table 1.

TABLE 1

1 № трапеции	1	2	3	4
$\kappa = \frac{\omega_d}{\omega_s}$	0,82	0,31	0,46	0,70
ω_s, 1/сек 2	16	3,5	61,5	28,5
r	1,02	−0,09	−0,24	−0,29

1) Trapezoid number; 2) ω_s, 1/sec.

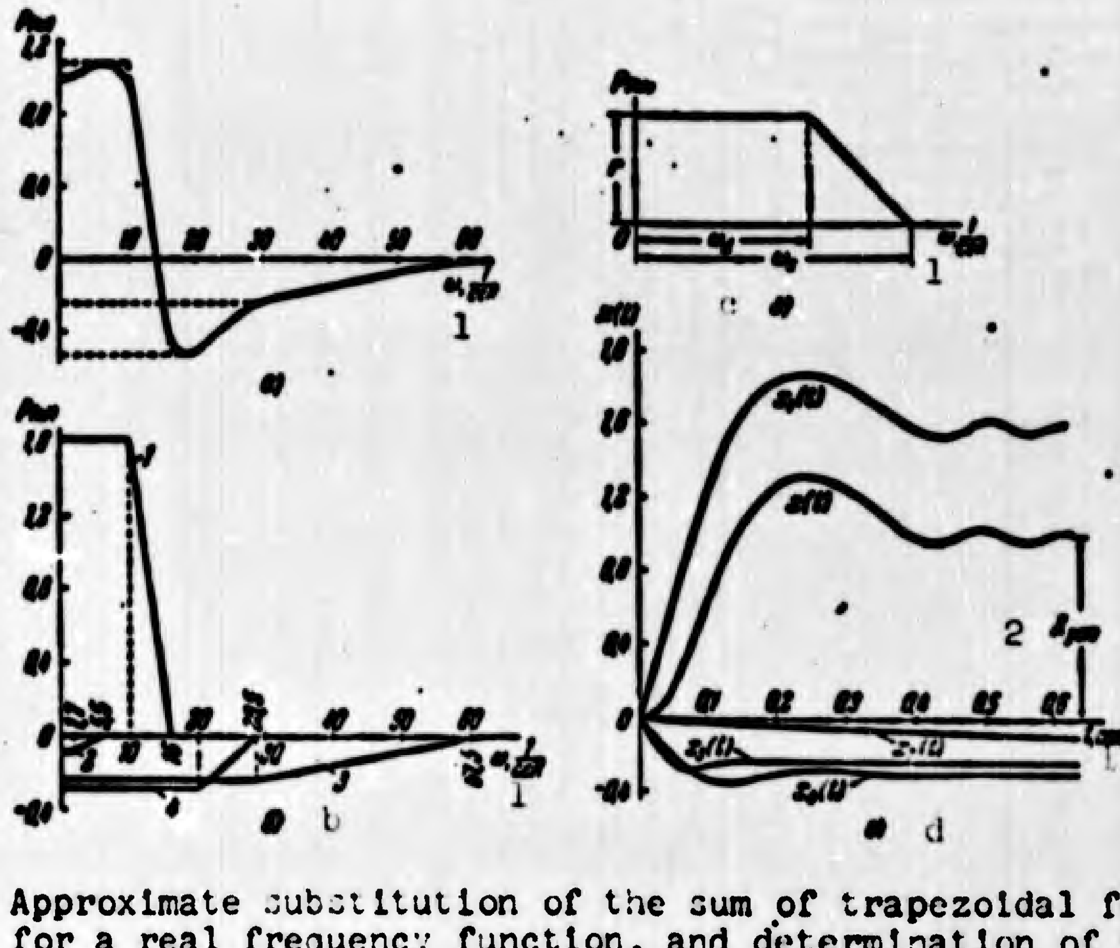


Fig. 102. Approximate substitution of the sum of trapezoidal frequency functions for a real frequency function, and determination of transient-response curve. 1) Sec; 2) X_{ust}.

We next should make use of tables of the function $h(t_0)$.

The function $h(t_0)$ is the transient-response curve of a system whose real frequency characteristic is a unit trapezoid having $r = +1$ and $\omega_s = 1$ 1/sec. The function $h(t_0)$ has been tabled for various slopes $0 \leq \kappa \leq 1$; interpolation can be carried out if κ lies between two tabular values. An abridged table of such functions is given in Appendix 23.

We shall use a table of the $h(t_0)$-function for $\kappa = 0.62$ (the slope of trapezoid 1), and shall write several values of the time t_0 and the function $h(t_0)$ (see the first two lines of Table 2). In order to obtain points on the transient-response curve $x(t)$ corresponding to a trapezoid other than the unit trapezoid, each value of the function $h(t_0)$ must be multiplied by the height of the trapezoid r while the time t_0 is divided by the cutoff frequency ω_s, i.e.,

$$x(t) = rh\left(\frac{t_0}{\omega_s}\right).$$

The third and fourth lines of Table 2 give the numbers t and $x_1(t)$ for trapezoid 1.

In like manner, we obtain $x_2(t)$, $x_3(t)$ and $x_4(t)$ for the remaining trapezoids (see Tables 3-5). From the data of Tables 2-5, we have plotted the curves $x_1(t)$, $x_2(t)$, $x_3(t)$, and $x_4(t)$ on Fig. 102d. Adding the y-axis values of these curves with allowance for their signs, we obtain on Fig. 102d the curve $x(t)$ representing the transient response of the given system to a unit step function. The figure also shows the quantity $X_{yct} = \lim_{t \to \infty} x(t)$.

For a nonunit step input $y(t) = b\,1(t)$, the y-axis values of the curve $x(t)$ should be multiplied by b.

156. From the real frequency response curve $P(\omega)$ of a control system (Fig. 103a), construct the transient-response curve $x(t)$ for the control input $y(t) = b\,1(t)$ under zero initial conditions.

Answer. The curve for $P(\omega)$ may be replaced by two trapezoids, shown

TABLE 2

1 Трапеция 1

t_0	0	0,2	0,4	0,8	1,0	1,6	2,6	3,0	4,0	4,4	4,8	5,4	6,0	7,0	7,8	9,0	10
$h(t_0)$	0	0,10	0,20	0,40	0,50	0,73	1,04	1,11	1,16	1,15	1,12	1,07	1,01	0,95	0,94	0,96	1,00
2 t, сек	0	0,0125	0,025	0,050	0,0625	0,100	0,162	0,188	0,250	0,275	0,300	0,337	0,375	0,438	0,488	0,562	0,625
$x_1(t)$	0	0,17	0,33	0,65	0,81	1,21	1,68	1,80	1,88	1,86	1,82	1,73	1,64	1,54	1,52	1,56	1,62

1) Trapezoid 1; 2) t, sec.

TABLE 3

1 Трапеция 2

2 t, сек	0	0,109	0,218	0,364	0,546	0,728	0,822	1,09	1,27
$x_2(t)$	0	−0,022	−0,043	−0,067	−0,086	−0,096	−0,098	−0,096	−0,094

1) Trapezoid 2; 2) t, sec.

TABLE 4

1 Трапеция 3

2 t, сек	0	−0,0065	0,0163	0,026	0,0325	0,0488	0,065	0,0813	0,0976	0,114	0,13
$x_3(t)$	0	−0,048	−0,108	−0,163	−0,194	−0,25	−0,271	−0,269	−0,254	−0,242	−0,235

1) Trapezoid 3; 2) t, sec.

TABLE 5

1 Трапеция 4

2 t, сек	0	0,014	0,028	0,042	0,070	0,105	0,133	0,176	0,210	0,246	0,281	0,316	0,351	0,386	0,456
$x_4(t)$	0	−0,064	−0,122	−0,467	−0,267	−0,328	−0,339	−0,314	−0,284	−0,27	−0,27	−0,284	−0,296	−0,302	−0,290

1) Trapezoid 4; 2) t, sec.

on Fig. 103a by the dashed line. The values for trapezoid 1 are: $\kappa = 0.78$, $\omega_s = 79$ 1/sec, $r = 0.688$; the values for trapezoid 2 are: $\kappa = 0.84$, $\omega_s = 95$ 1/sec, $r = 0.2$.

The curves for $x_1(t)$ and $x_2(t)$ of Fig. 103b were plotted from these trapezoids; from the same figure, we have given the sought function $x(t)$ for the case in which $b = 1$. When $b \neq 1$, the y-axis values for the curve $x(t)$ should be multiplied by <u>b</u>.

157. Construct the transient-process curve $x(t)$ for a closed-loop system with the control input $y(t) = 1(t)$ under the zero initial conditions. The system open-loop transfer function is

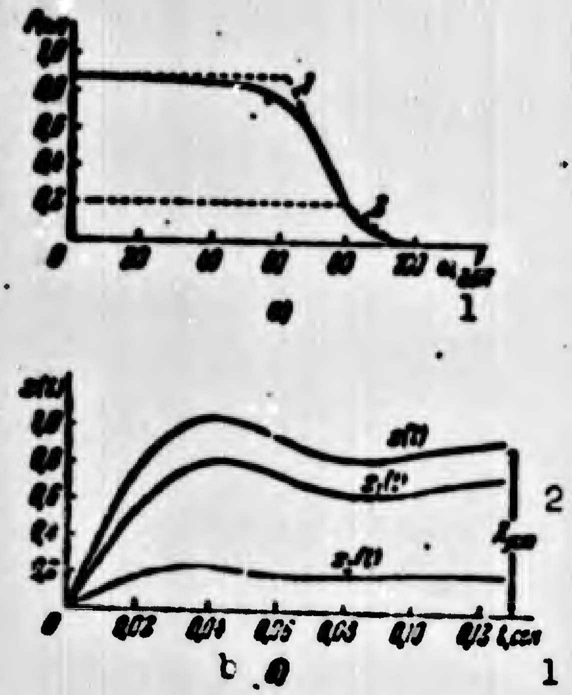


Fig. 103. Real frequency response $P(\omega)$ and transient-response curves $x(t)$ for Problem 156. 1) Sec; 2) X_{ust}.

$$W(p)=\frac{K(1+T_4p)}{(-1+2\xi Tp+T^2p^2)(1+T_1p)(1+T_2p)}=$$
$$=\frac{3(1+0{,}03p)}{(-1+0{,}2p+0{,}01p^2)(1+0{,}05p)(1+0{,}006p)}.$$

Hint. It is possible to make use of the results obtained in solving Problems 49 (B) and 56.

Answer. See Fig. 104b. Curves $x_{1,2,3,4}(t)$ have been plotted from the four trapezoids shown in Fig. 104a.

158. Construct the curve $x(t)$ for the closed-loop transient response of a system under the control input $y(t) = 1(t)$ and zero-conditions.

The system open-loop transfer function is

$$W(p)=\frac{K(1+T_4p)}{p(1+T_1p)(1+T_2p)}=\frac{300(1+0{,}03p)}{p(1+0{,}1p)(1+0{,}006p)}.$$

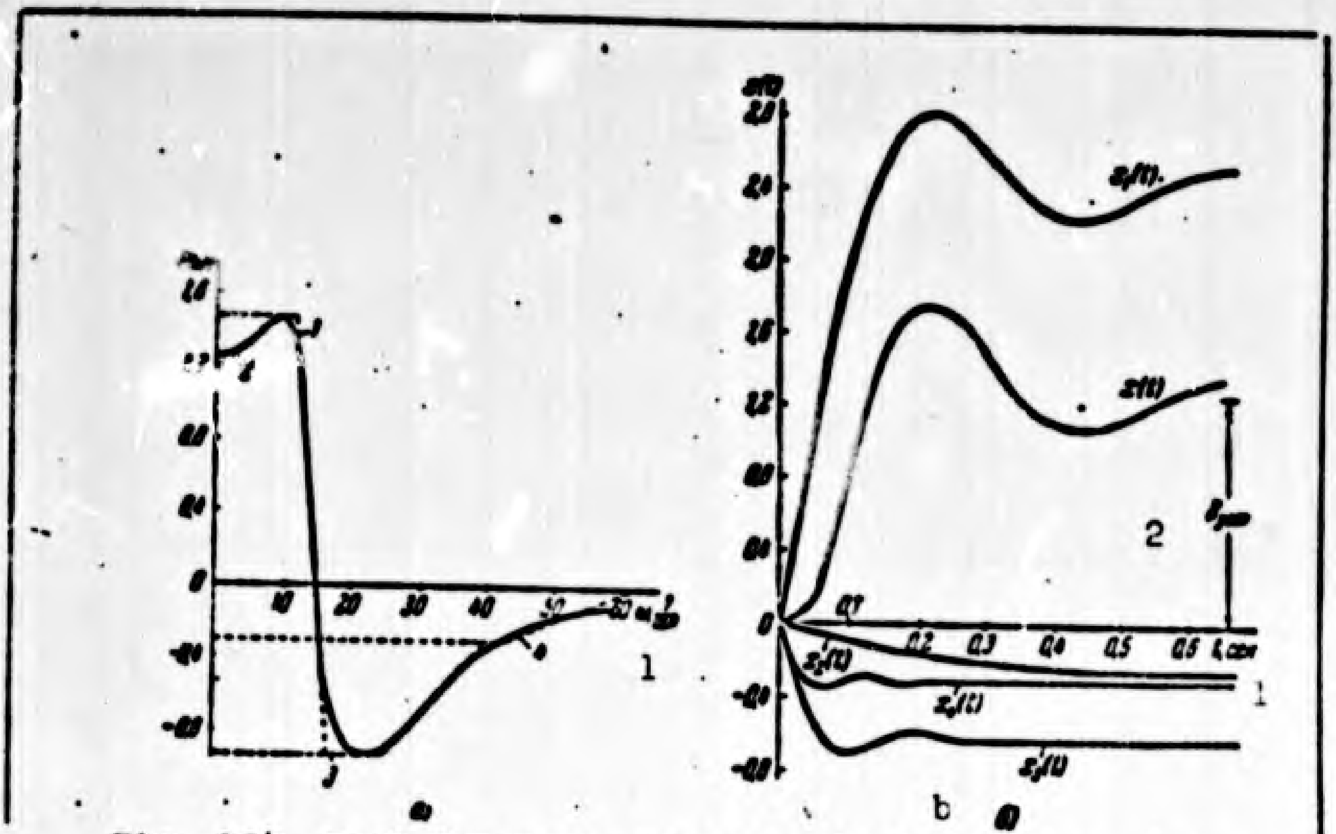


Fig. 104. Real frequency characteristic and transient-response curve for Problem 157. 1) Sec; 2) X_{ust}.

Hint. It is possible to use the results obtain in solving Problems

43 and 54.

Answer. See Fig. 105b. The curves $x_{1,2,3}(t)$ have been plotted from the three trapezoids shown in Fig. 105a.

B. Using the Conjugating Frequencies of the Logarithmic Amplitude Characteristic

159. The open-loop transfer function of a system equals

$$W(p)=\frac{K(1+T_2p)}{p(1+T_1p)(1+T_3p)}, \qquad (1)$$

where $K = 500$ 1/sec, $T_1 = 0.1$ sec, $T_2 = 0.025$ sec, $T_3 = 0.0025$ sec. Construct the approximate system error curve $\varepsilon(t) = y(t) - x(t)$ for a unit step input $y(t) = 1(t)$ and zero-conditions. The construction is carried out on the basis of the conjugating frequencies of the logarithmic frequency characteristic (l.a.kh.).

Solution. The l.a.kh. of the system is plotted in Fig. 106a. This l.a.kh. satisfies a condition requiring that the length of its segment intersecting the frequency axis with slope −20 db/decade should be no more than one decade; thus it is possible to plot the desired curve from the conjugate frequencies of the l.a.kh.

We determine the l.a.kh. cutoff frequency directly from the l.a.kh. or from the formula $\omega_s = K\,T_2/T_1$, using the figure; $\omega_s \approx 125$ 1/sec.

In accordance with the method employing the conjugating frequencies of the l.a.kh., we discard from the l.a.kh. all of it lying to the right of the cutoff frequency, replacing it with the horizontal segment coinciding with the frequency axis. This new transformed l.a.kh. corresponds to the transfer function

$$W^*(p)=\frac{K(1+T_2p)\left(1+\frac{1}{\omega_c}p\right)}{p(1+T_1p)} \qquad (2)$$

or

$$W^*(p)=\frac{KT_2\frac{1}{\omega_c}\left(\frac{1}{T_2}+p\right)(\omega_c+p)}{T_1p\left(\frac{1}{T_1}+p\right)}=\frac{(p+\omega_2)(p+\omega_c)}{p(p+\omega_1)}=$$

$$=\frac{(p+40)(p+125)}{p(p+10)}. \qquad (2')$$

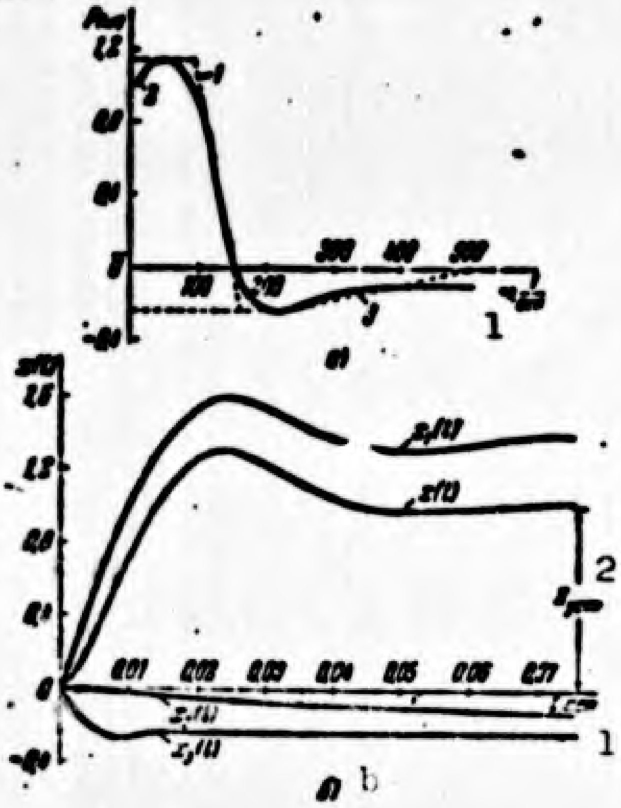


Fig. 105. Real frequency response and transient-response curve for Problem 158. 1) Sec; 2) X_{ust}.

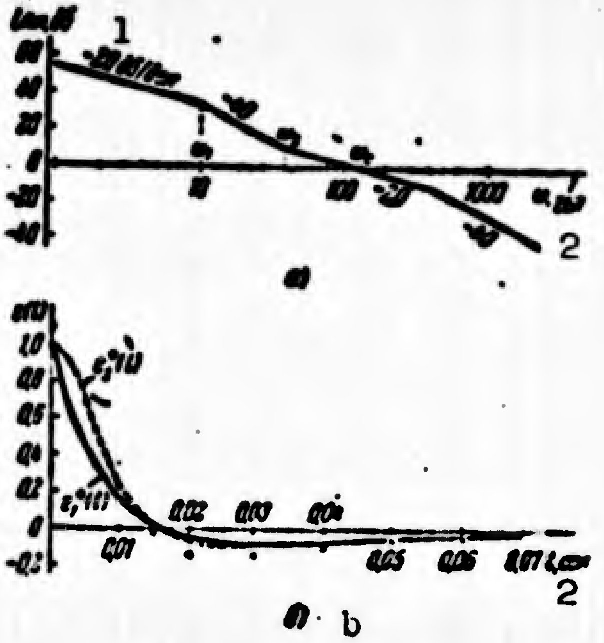


Fig. 106. Logarithmic amplitude characteristic and transient-response curve for Problem 159. 1) db/decade; 2) sec.

where $\omega_1 = 1/T_1$, $\omega_2 = 1/T_2$.

Formulas (2) and (2') correspond to the transformed error transfer function of the system

$$\Phi^*(p)=\frac{1}{W^*(p)}=\frac{p(p+10)}{(p+40)(p+125)}. \tag{3}$$

Taking into account the Laplace transform Y(p) = 1/p for the input y(t) = 1(t), we find the Laplace transform for the first approximation $\varepsilon_1^*(t)$ of the function $\varepsilon(t)$:

$$E(p)=\Phi^*(p)Y(p)=\frac{p+10}{(p+40)(p+125)}. \tag{4}$$

From tables of Laplace transforms, we find an appropriate formula:

$$\frac{p+a_0}{(p+\alpha)(p+\gamma)} \doteq \frac{(a_0-\alpha)e^{-\alpha t}-(a_0-\gamma)e^{-\gamma t}}{\gamma-\alpha}. \tag{5}$$

Formulas (4) and (5) give the answer for the first approximation of the system error:

$$\varepsilon_1^*(t)=1{,}353e^{-125t}-0{,}353e^{-40t}. \tag{6}$$

This function is plotted by the solid line on Fig. 106b.

To obtain the second approximation $\varepsilon_2^*(t)$ to the desired solution, we must multiply the y-axis values of the curve $\varepsilon_1^*(t)$ by the correction coefficient ρ in the range $T_3 < t < T_2$, i.e., 0.0025 sec < < t < 0.025 sec. This coefficient is found from the formula

$$\rho = \left| \frac{W^*(p)}{1 + W(p)} \right|_{p = j\omega_c}$$

or, from (1) and (2'),

$$\rho = \left| \frac{\dfrac{(p+40)(p+125)}{p(p+10)}}{1 + \dfrac{500(1+0.025p)}{p(1+0.1p)(1+0.0025p)}} \right|_{p=j125} =$$

$$= \left| \frac{(j125+40)(j125+125)(j125+400)}{50\,000(j125+40)+j125(j125+10)(j125+400)} \right| = 1.485.$$

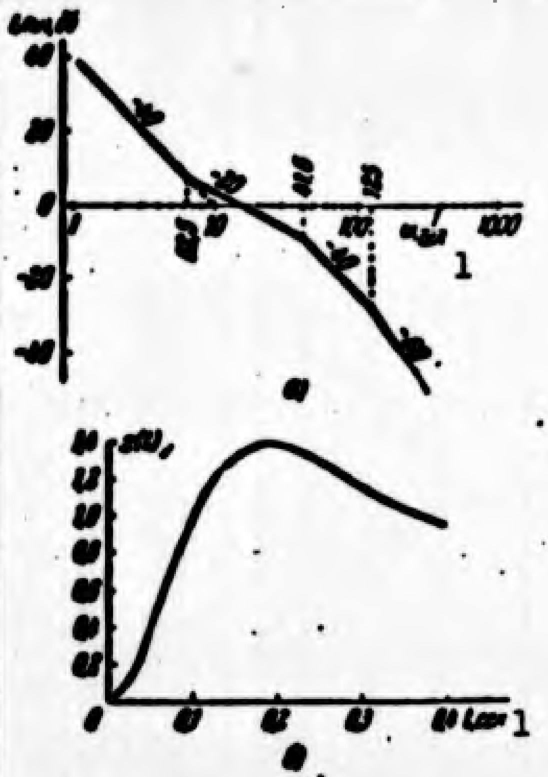


Fig. 107. Logarithmic amplitude characteristic and transient-response curve for Problem 160. 1) Sec.

The second approximation to the solution is plotted in Fig. 106b (dashed line). On the same figure, the crosses indicate points on the exact solution.

C. <u>Using Universal Curves for Phase-Minimum Systems with Standard Logarithmic Amplitude Characteristics</u>

160. The open-loop transfer function of a servosystem equals

$$W(p) = \frac{K(1+T_1p)}{p^2(1+T_2p)(1+T_3p)} = \frac{100(1+0.160p)}{p^2(1+0.024p)(1+0.008p)}.$$

Construct the graph of the output variable $x(t)$ under a unit step control function $y(t) = 1(t)$ and the zero-conditions.

<u>Solution.</u> We plot the logarithmic amplitude characteristic of the given system (Fig. 107a). From the formulas given in Appendix 16, we find the base frequency of the l.a.kh.: $\omega_0 = \sqrt{K} = \sqrt{100} = 10$ 1/sec, with the following relative time constants: $\tau_2 = T_1\omega_0 = 0.160 \cdot 10 = 1.6$, $\tau_3 = T_2\omega_0 = 0.024 \cdot 10 = 0.24$, and the relatively small time constant $\tau_m = T_3\omega_0 = 0.008 \cdot 10 = 0.08$, and the length of the segment of the l.a.kh.

having slope −20 db/decade: $h = \tau_2/\tau_3 = 6.67$.

In accordance with the computed numbers and the form of the normalized l.a.kh. shown in Appendix 16, we find that this l.a.kh. may be reduced to a normalized curve with variability index M lying between 1.5 and 1.7.

Thus the sought transient-response curve x(t) should be plotted so as to lie between the curves $x(\omega_0 t)$ given in Appendix 17 for M = 1.5 and M = 1.7. In going from the normalized curve $x(\omega_0 t)$ for the transient response to x(t), the x-axis values of the normalized curve should be divided by $\omega_0 = 10$ 1/sec.

As a result, we obtain the curve x(t) plotted in Fig. 107b.

161. The open-loop transfer function of a system equals

$$W(p) = \frac{250(1 + 0{,}024p)}{(1 + 0{,}2p)(1 + 0{,}00125p)^2(1 + 0{,}0005p)(1 + 0{,}0001p)}.$$

Construct the transient-response curve x(t) for the closed-loop system with the control input y(t) = 1(t) and the zero-conditions.

<u>Answer.</u> The transient-response curve x(t) is close to the normalized curve $x(\omega_0 t)$ for an index of variability M = 1.3; the base frequency $\omega_0 = 79$ 1/sec.

Fig. 108. Transient-response curves for Problem 162, plotted by three methods: from the normalized curve $x(\omega_0 t)$ – solid line, from the exact solution – crosses, and from the real frequency characteristic – circles.

162. The open-loop transfer function of a system equals

$$W(p) = \frac{500(1 + 0{,}03p)}{p(1 + 0{,}1p)(1 + 0{,}005p)}.$$

1. Use the normalized transient-response curves to construct the graph of the output variable x(t) for the closed-loop system under a unit step control input and zero initial conditions.

2. Solve the same problem exactly (by the classical or operator

methods), and also make use of the real frequency characteristic of the system.

Plot all three solutions on one graph.

Hint. In the second part of the problem, the solutions obtained for Problems 142 and 158 may be used.

Answer. On Fig. 108 we have plotted the curve x(t) found from the normalized transient-response curve with $\omega_0 = 70.7$ 1/sec and $M \approx 1.6$. Points belonging to the exact solution are indicated by crosses and points obtained from the real frequency characteristic by circles.

163. The open-loop transfer function of a system equals

$$W(p)=\frac{400(1+0.04p)}{p(1+0.1p)(1+0.003p)(1+0.0005p)}.$$

Construct the transient-process curve x(t) for the closed-loop system under a unit step control input and zero-conditions.

Answer. The curve x(t) may be plotted from the normalized curve $x(\omega_0 t)$ for $M = 1.3$ with a base frequency $\omega_0 = 63.2$ 1/sec.

D. Graphical Determination of Transient-Process Curve by the Method of D.A. Bashkirov

164. Construct the graph of the output variable x(t) for the system described by the equation

$$a_0\frac{dx}{dt}+a_1x=b_0y(t)$$

or

$$2\frac{dx}{dt}+20x=8y(t), \qquad (1)$$

where the control input y(t) is given graphically by Fig. 109a. x(t) and y(t) have the same dimensions. The initial condition is $x(0) = -2$.

Solution. We write (1) in the form

$$T\frac{dx}{dt}+x=y_1(t), \qquad (2)$$

where the time constant is

$$T = \frac{a_0}{a_1} = \frac{2}{20} = 0,1 \text{ sec}$$

and the disturbance variable is

$$y_1(t) = \frac{b_0}{a_1} y(t) = 0,25y(t). \qquad (3)$$

On Fig. 109b, we plot the two coordinate systems t, x(t) and t, $y_1(t)$, having the same scale; the time axes of both systems are the same, but the origin O_1 from which $y_1(t)$ is measured is shifted to the right with respect to the origin O from which x(t) is measured by the amount T. Using Formula (3) and Fig. 109a, we plot the function $y_1(t)$.

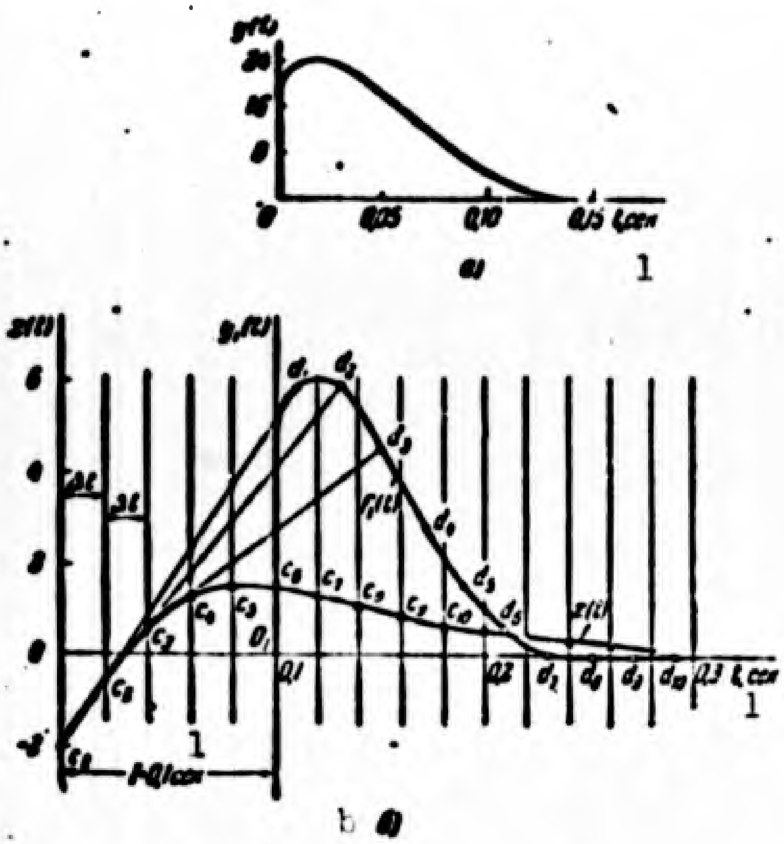


Fig. 109. Construction of transient-process curve x(t) by the method of D.A. Bashkirov. 1) Sec.

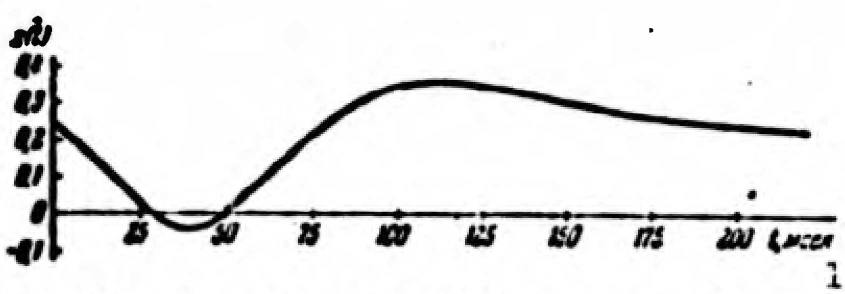


Fig. 110. Transient-process curve for Problem 165. 1) msec.

We use an element of integration $\Delta t = T/n$, where n is an integer.

We take $\Delta t = 0.020$ sec and devide the graph of Fig. 109b into sections of 0.020 sec each. On the graph of the function $y_1(t)$, we let d_1, d_2, d_3, ... denote the values of these functions at the center of each section. On the graph of $x(t)$, we plot the initial value $x(0) = -2$ and join the resulting point c_1 to point d_1 by a straight line. The intersection of line c_1d_1 and the abscissa of the end of the first section yields the second point c_2 on the curve sought. Drawing line c_2d_2, we obtain point c_3 at the intersection of this line with the abscissa of the end of the second section, etc. The desired function $x(t)$ is found from the smooth curve connecting points c_1, c_2, c_3, ...

165. Construct the graph of the output variable $x(t)$ for the system described by the equation

$$a_0\frac{dx}{dt}+a_1x=b_0y(t)$$

or

$$\frac{dx}{dt}+20x=12y(t)$$

where

$$x(0)=0.25$$

and the control variable $y(t)$ is given in the form of the following table (the control input has the dimensions of the output variable):

t, msec 1)	0	5	10	15	20	25	30	35	45	50
y(t)	0	−0.200	−0.456	−0.541	−0.646	−0.617	−0.530	−0.350	+0.300	+0.575
t, msec 1)	60	65	75	85	100	115	125	140	160	∞
y(t)	+0.500	+0.960	+1.000	+0.950	+0.710	+0.561	+0.172	+0.347	+0.380	+0.330

1) t, msec.

<u>Answer.</u> See Fig. 110.

166. Draw the graph of the output variable $x(t)$ for the system described by the equation

$$a_0\frac{d^2x}{dt^2}+a_1\frac{dx}{dt}+a_2x=b_0y(t)$$

or

$$0{,}048\frac{d^2x}{dt^2}+0{,}4\frac{dx}{dt}+10x=5y(t)$$

where

$$x(0)=-1{,}5,\quad x'(0)=75\ 1/\text{sec}$$

and the control input y(t) is shown in Fig. 111a.

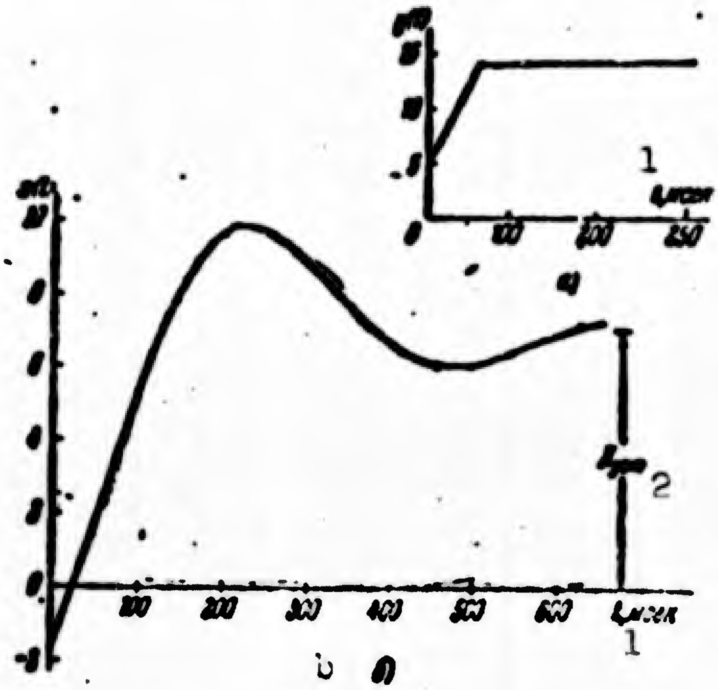


Fig. 111. Control input y(t) and output variable x(t) for the system of Problem 166. 1) msec.

Hint. The given equation should be reduced to the form

$$T_1T_2\frac{d^2x}{dt^2}+T_2\frac{dx}{dt}+x=y_1(t).$$

Answer. See Fig. 111b.

Manu-script Page No.	[Transliterated Symbols]
99	уст = ust = ustanovivshiysya = steady-state
123	с = s = srez = cutoff
127	л.а.х. = l.a.kh. = logarifmicheskaya amplitudnaya kharakteristika = logarithmic amplitude characteristic

Chapter 5

EVALUATING STEADY-STATE CONTROL PERFORMANCE

§14. DETERMINING ACCURACY IN THE PRESENCE OF A CONTROL INPUT

167. The closed-loop transfer function of a servosystem has the form

$$\Phi(p)=\frac{b_0 p^m + b_1 p^{m-1} + \dots + b_{m-1} p + b_m}{a_0 p^n + a_1 p^{n-1} + \dots + a_{n-1} p + a_n}.$$

Under what conditions will we have: 1) zero-order astatism; 2) first-order astatism; 3) second-order astatism?

<u>Answer.</u>

1) $b_m \neq a_n$;
2) $b_m = a_n$, $b_{m-1} \neq a_{n-1}$;
3) $b_m = a_n$, $b_{m-1} = a_{n-1}$, $b_{m-2} \neq a_{n-2}$.

168. The open-loop transfer function of a servosystem (Fig. 112) has the form

$$W(p)=\frac{A_0 p^m + A_1 p^{m-1} + \dots + A_{m-1} p + A_m}{B_0 p^n + B_1 p^{n-1} + \dots + B_{n-1} p + B_n}.$$

Fig. 112. Servosystem.

Under what conditions will we have: 1) zero-order astatism; 2) first-order astatism; 3) second-order astatism?

<u>Answer.</u>

1) $B_n \neq 0$;
2) $B_n = 0$;
3) $B_n = 0$ and $B_{n-1} = 0$.

169. The open-loop transfer function of a servosystem (Fig. 112) has the form

$$W(p)=\frac{K}{p(1+T_1 p)(1+T_2 p)}.$$

Find the first three error coefficients as well as the velocity figure of merit.

Solution. We find the error transfer function:

$$\Phi_x(p)=\frac{1}{1+W(p)}=\frac{p(1+T_1p)(1+T_2p)}{p(1+T_1p)(1+T_2p)+K}.$$

Dividing the numerator by the denominator, we expand this expression into a series:

$$\begin{array}{l|l} p+(T_1+T_2)p^2+T_1T_2p^3 & K+p+(T_1+T_2)p^2+T_1T_2p^3 \\ \cline{2-2} \quad p+\frac{1}{K}p^2+\frac{T_1+T_2}{K}p^3+\frac{T_1T_2}{K}p^4 & \frac{1}{K}p+\frac{1}{K}\left(T_1+T_2-\frac{1}{K}\right)p^2+\dots \\ \cline{1-1} \left(T_1+T_2-\frac{1}{K}\right)p^2+\left(T_1T_2-\frac{T_1+T_2}{K}\right)p^3-\frac{T_1T_2}{K}p^4 & \end{array}$$

From this we obtain the error coefficients:

$$C_0=0,\quad C_1=\frac{1}{K}\ [\text{sec}],\quad \frac{C_2}{2}=\frac{1}{K}\left(T_1+T_2-\frac{1}{K}\right)\ [\text{sec}^2]$$

The velocity figure of merit is

$$K_\Omega=\frac{1}{C_1}=K\ [1/\text{sec}].$$

170. For the preceding problem, determine the numerical values of the error coefficients if K = 100 1/sec, T_1 = 0.01 sec and T_2 = 0.005 sec.

Answer.

$$C_1=0{,}01\ \text{sec and}\ \frac{C_2}{2}=0{,}00005\ \text{sec}^2$$

171. Determine the steady-state error for the preceding problem where the servosystem moves with a velocity Ω = 12 degree/sec.

Answer.

$$\theta_{уст}=\frac{\Omega}{K_\Omega}=C_1\Omega=0{,}01\cdot 12=0{,}12^\circ=7{,}2'.$$

172. The closed-loop transfer function of a system has the form

$$\Phi(p)=\frac{5p+200}{0{,}001p^3+0{,}502p^2+6p+200}.$$

Find the steady-state error (after damping of a transient) when the input variable is governed by the law

$$Y(t)=5+20t+10t^2.$$

Solution. We find the error transfer function:

$$\Phi_e(p) = 1 - \Phi(p) = \frac{0.001p^3 + 0.502p^2 + p}{0.001p^3 + 0.502p^2 + 6p + 200}.$$

We divide the numerator by the denominator (see Problem 169), and find the error coefficients:

$$C_0 = 0, \quad C_1 = \frac{1}{200} \text{ sec and } \frac{C_2}{2} = 0.00236 \text{ sec}^2$$

We next find the derivatives:

$$r'(t) = 20 + 20t,$$
$$r''(t) = 20.$$

The expression for the error is:

$$\varepsilon(t) = C_0 r(t) + C_1 r'(t) + \frac{C_2}{2} r''(t) =$$
$$= \frac{20 + 20t}{200} + 20 \cdot 0.00236 = 0.1472 + 0.1t.$$

173. The open-loop transfer function of a system (Fig. 112) has the form

$$W(p) = \frac{50(1 + 0.15p)}{p^2(1 + 0.02p)}.$$

Find the first three error coefficients as well as the velocity figure of merit and the acceleration figure of merit.

Answer.

$$C_0 = 0, \quad C_1 = 0, \quad \frac{C_2}{2} = 0.02 \text{ sec}^2$$

The velocity figure of merit $K_\Omega \to \infty$, the acceleration figure of merit $K_\varepsilon = 50$ 1/sec^2.

174. In a static control system (Fig. 113a), the open-loop transfer function has the form

$$W(p) = \frac{K}{(1 + T_1 p)(1 + T_2 p)}.$$

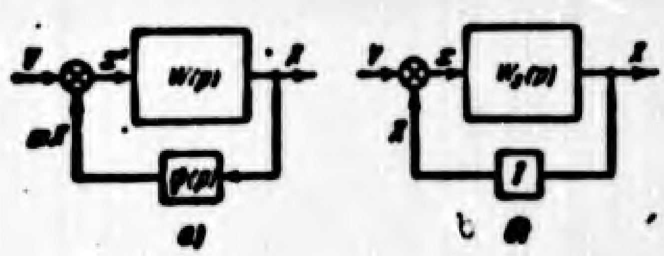


Fig. 113. Static system with non-unity feedback.

Determine the transfer constant $\psi(p) = m$ for nonunity feedback for which the system will acquire first-order astatism, and the open-loop transfer function of the equivalent system with unity feedback (Fig. 113b).

Solution. The closed-loop transfer function of the system is

$$\Phi(p)=\frac{W(p)}{1+mW(p)}=\frac{K}{T_1T_2p^2+(T_1+T_2)p+1+mK}. \qquad (1)$$

The condition for the absence of static error is: $K = 1 + mK$; from this we obtain

$$m=\frac{K-1}{K}=1-\frac{1}{K}.$$

Then the system closed-loop transfer function will take on the form

$$\Phi(p)=\frac{K}{T_1T_2p^2+(T_1+T_2)p+K}, \qquad (2)$$

and the equivalent open-loop transfer function of the system with unity feedback will be

$$W_e(p)=\frac{\Phi(p)}{1-\Phi(p)}=\frac{K}{(T_1+T_2)p+T_1T_2p^2}=\frac{K_v}{p(1+T_ep)},$$

where the velocity figure of merit is

$$K_v=\frac{K}{T_1+T_2}$$

and the equivalent time constant is

$$T_e=\frac{T_1T_2}{T_1+T_2}.$$

175. For the preceding problem, find the first two error coefficients for the following two cases:

1) the over-all gain of the direct circuit is stable (K = const);

2) the over-all gain of the direct circuit is unstable ($K \neq$ const).

Solution. For the case in which K = const, we have from (2) the error transfer function

$$\Phi_x(p)=1-\Phi(p)=\frac{T_1T_2p^2+(T_1+T_2)p}{T_1T_2p^2+(T_1+T_2)p+K}.$$

Dividing the numerator by the denominator so as to expand it into series (see Problem 169), we find the error coefficients:

$$C_0=0 \quad \text{and} \quad C_1=\frac{T_1+T_2}{K}.$$

Where $K \neq$ const, we use $K = K_0 + \Delta K$ [we shall assume that $\Delta K/K_0 < 1$ and that the feedback-loop transfer constant $m = 1 - (1/K_0)$]. The closed-loop transfer function of the system (1) in this case will take the form

$$\Phi(p)=\frac{K_0+\Delta K}{T_1T_2p^2+(T_1+T_2)p+K_0+\Delta K-\frac{\Delta K}{K_0}}.$$

The error transfer function is

$$\Phi_\varepsilon(p)=1-\Phi(p)=\frac{T_1T_2p^2+(T_1+T_2)p-\frac{\Delta K}{K_0}}{T_1T_2p^2+(T_1+T_2)p+K_0+\Delta K-\frac{\Delta K}{K_0}}.$$

Expanding it into series, we obtain

$$C_0=-\frac{\Delta K}{K_0\left(K_0+\Delta K-\frac{\Delta K}{K_0}\right)}\approx-\frac{\Delta K}{K_0^2},$$

$$C_1=\frac{(T_1+T_2)(K_0+\Delta K)}{\left(K_0+\Delta K-\frac{\Delta K}{K_0}\right)^2}\approx\frac{T_1+T_2}{K_0}.$$

176. Find the transfer function for nonunity feedback $\psi(p)$ such that in a static control system, positional and velocity errors are eliminated. The block diagram of the control system with nonunity feedback is shown in Fig. 113a. The transfer function is

$$W(p)=\frac{K}{(1+T_1p)(1+T_2p)}.$$

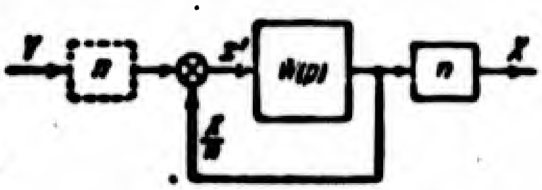


Fig. 114. Static system with scaling in direct circuit.

Solution. The closed-loop transfer function of a system with nonunity feedback in the general case will have the form

$$\frac{X}{Y}=\Phi(p)=\frac{W(p)}{1+\psi(p)W(p)}=\frac{A_mp^m+A_{m-1}p^{m-1}+\ldots+A_1p+A_0}{B_np^n+B_{n-1}p^{n-1}+\ldots+B_1p+B_0}.$$

The positional error vanishes when

$$A_0=B_0.$$

When the additional condition

$$A_1=B_1$$

is satisfied, the velocity error in the system will vanish.

For the case under consideration, elimination of the positional and velocity errors may be achieved by introducing a filter with transfer function

$$\varphi(p)=\frac{k_{oc}}{1+\tau_0 p}.$$

into the feedback circuit. Then the closed-loop system transfer function will have the form

$$\frac{X}{Y}=\Phi(p)=\frac{W(p)}{1+\varphi(p)W(p)}=$$

$$=\frac{K(1+\tau_0 p)}{T_1T_2\tau_0 p^3+(T_1T_2+T_1\tau_0+T_2\tau_0)p^2+(T_1+T_2+\tau_0)p+1+Kk_{oc}}.$$

when

$$1+Kk_{oc}=K,$$

$$k_{oc}=\frac{K-1}{K}$$

and

$$K\tau_0=T_1+T_2+\tau_0$$

$$\tau_0=\frac{T_1+T_2}{K-1}$$

The system will possess second-order astatism. The positional and velocity errors will then vanish.

177. For the system shown in Fig. 113, determine the first two error coefficients if T_1 = 1 sec, T_2 = 0.02 sec, and $K = 1000 \pm 50$.

<u>Solution.</u> On the basis of the formulas obtained in Problem 175, we have

$$C_0\approx-\frac{\Delta K}{K_0^2}=\mp\frac{50}{1000^2}=5\cdot10^{-5},$$

$$C_1\approx\frac{T_1+T_2}{K_0}=\frac{1+0.02}{1000}=1{,}02\cdot10^{-3}\text{ sec.}$$

178. In a static control system (Fig. 114), the system open-loop transfer function has the form

$$W(p)=\frac{K}{(1+T_1p)(1+T_2p)(1+T_3p)}.$$

Find the transfer constant <u>n</u> of the scaler in the input or output circuit that will give the system first-order astatism with respect to the control input.

Solution. With allowance for the scaler, the closed-loop system transfer function will be

$$\Phi(p)=\frac{aK}{T_3T_4T_5p^3+(T_3T_4+T_3T_5+T_4T_5)p^2+(T_3+T_4+T_5)p+1+K}.$$

The condition for first-order astatism is:

$$aK=1+K,$$

from which we obtain

$$a=\frac{1+K}{K}.$$

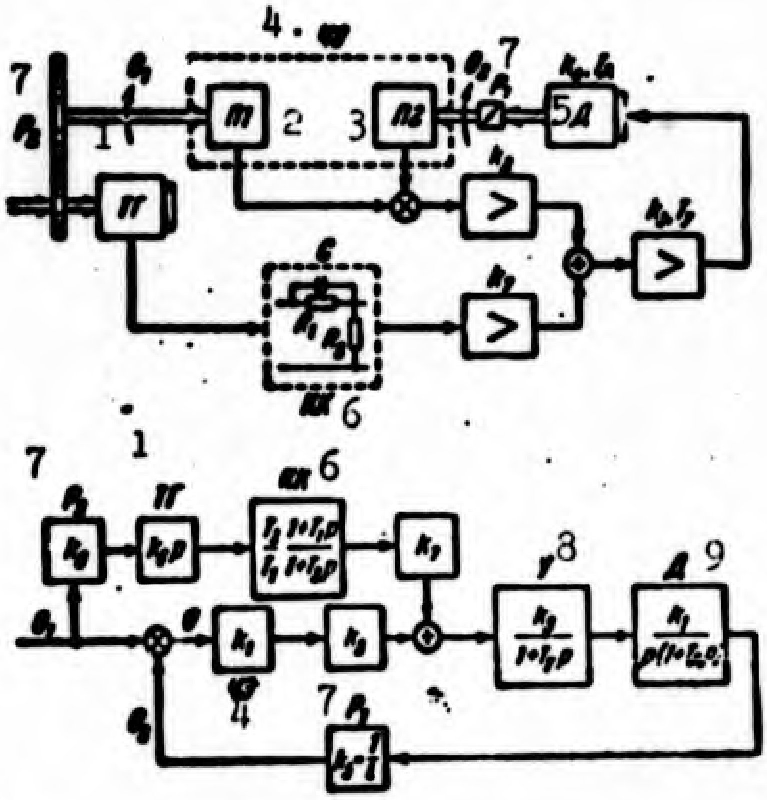


Fig. 115. Combination-control system. 1) Tachometer generator; 2) potentiometer 1; 3) potentiometer 2; 4) sensing element; 5) motor; 6) compensating network; 7) reduction gear; 8) amplifier; 9) motor.

179. For the preceding problem, find the open-loop transfer function of the equivalent system without scaler.

Answer.

$$W_0(p)=\frac{\Phi(p)}{1-\Phi(p)}=\frac{K_0}{p(1+ap+bp^2)}.$$

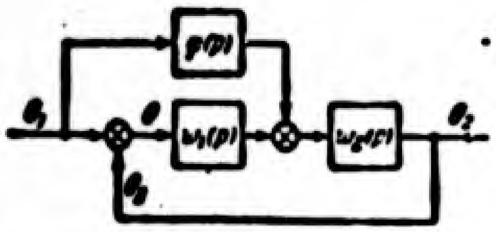


Fig. 116. Transformed block diagram of combination-control system.

where the equivalent velocity figure of merit is

$$K_v = \frac{K}{T_0 + T_1 + T_2} \text{ [1/sec]}$$

The coefficients are:

$$a = \frac{T_0 T_1 + T_0 T_2 + T_1 T_2}{T_0 + T_1 + T_2},$$

$$b = \frac{T_0 T_1 T_2}{T_0 + T_1 + T_2}.$$

180. For a combination-control system (Fig. 115), determine the conditions for third-order astatism and the error coefficient C_3. In Fig. 115, ChE is the sensing element, which consists of the two potentiometers P1 and P2 for the transmitting and receiving shafts, D is the actuating motor, R_1 and R_2 are reduction gears, TG is the tachometer generator, KK is the compensating network, θ_1 is the turn angle of the transmitting shaft, θ_2 is the angle through which the receiving (actuating) shaft rotates, and $\theta = \theta_1 - \theta_2$ is the error. The initial conditions are: k_1 = 1 v/degree = 57.3 v/rad is the slope of the sensing element characteristic curve, k_2 = 25 is the voltage gain of the main-circuit preamplifier, k_3 = 4 is the voltage gain of the final amplifier, k_4 = 27.3 revolutions/v·min = 2.86 rad/v·sec is the transfer constant of the actuating motor, $k_5 = 1/i_1$ = 1/1000 is the transfer constant of the reducing gear R_1, k_6 = 0.055 v·min/revolution = 0.525 v·sec/rad is the transfer constant of the tachometer generator, k_7 is the voltage gain of the compensating-network preamplifier, $k_8 = i_2$ = = 500 is the transfer constant of reducing gear R_2, T_u = 0.005 sec is the amplifier time constant, T_d = 0.1 sec is the time constant of the actuating motor, $T_1 = R_1C$ and $T_2 = R_1R_3\,C/(R_1 + R_2)$ are the time constants of the passive differentiating network. Here k_7, T_1, and T_2 are the unknown parameters.

<u>Solution.</u> The transformed block diagram for the system considered is shown in Fig. 116. The transfer functions for the elements of the main

circuit are

$$w_1(p)=k_1k_2$$
$$w_2(p)=\frac{k_3k_4k_5}{p(1+T_yp)(1+T_Mp)}.$$

The compensating-circuit transfer function is

$$\varphi(p)\doteq k_6k_7k_8p\frac{T_2(1+T_1p)}{T_1(1+T_2p)}. \qquad (1)$$

The closed-loop transfer function of the system is

$$\Phi(p)=\frac{\theta_2(p)}{\theta_1(p)}=\frac{W(p)+\varphi(p)w_1(p)}{1+W(p)}, \qquad (2)$$

where the transfer function of the initial open-loop system is

$$W(p)=w_1(p)w_2(p)=\frac{K}{p(1+T_yp)(1+T_Mp)}. \qquad (3)$$

The over-all gain is

$$K=k_1k_2k_3k_4k_5=\frac{57.3\cdot 25\cdot 4\cdot 2.86}{1000}=16.4 \text{ 1/sec.}$$

The error transfer function is

$$\Phi_\theta(p)=\frac{\theta(p)}{\theta_1(p)}=1-\Phi(p)=\frac{1-\varphi(p)w_1(p)}{1+W(p)}. \qquad (4)$$

Substitution of (1) and (3) yields

$$\Phi_\theta(p)=\frac{b_0p^4+b_1p^3+b_2p^2+b_3p}{(1+T_2p)[T_yT_Mp^3+(T_y+T_M)p^2+p+K]}, \qquad (5)$$

where

$$b_0=T_2T_yT_M$$
$$b_1=T_yT_M+T_yT_2+T_MT_2$$
$$b_2=T_y+T_M+T_2-k_3k_4k_5k_6k_7k_8T_2$$
$$b_3=1-k_3k_4k_5k_6k_7k_8\frac{T_2}{T_1}.$$

The conditions for obtaining second-order astatism are:

$$b_2=0 \text{ and } b_3=0.$$

From this we obtain the two equations

$$k_7\frac{T_2}{T_1}=\frac{1}{k_3k_4k_5k_6k_8}, \qquad (6)$$

$$T_2(k_3k_4k_5k_6k_7k_8-1)=T_y+T_M \qquad (7)$$

The three unknowns k_7, T_1, and T_2 enter into the two equations (6) and (7). The missing third equation may be obtained on the basis of restrictions imposed on the value of the error coefficients following after C_0, C_1, and C_2, which equals zero since the system has second-order astatism. If there are no such restrictions on the subsequent error coefficients, the calculation may be carried out on the basis of the following considerations.

For passive differentiating circuits, the time-constant ratio T_2/T_1 will normally equal roughly 10. Setting $T_1/T_2 = 10$ in (6), we obtain the required value of amplifier gain for the compensating circuit:

$$k_2 = \frac{\frac{T_1}{T_2}}{k_0 k_1 k_3 k_4 k_5} = \frac{10 \cdot 1000}{4 \cdot 2{,}86 \cdot 0{,}525 \cdot 500} = 3{,}34.$$

From (7), we obtain the required time-constant value:

$$T_2 = \frac{T_y + T_д}{k_0 k_1 k_2 k_3 k_4 k_5 - 1} = \frac{0{,}005 + 0{,}1}{4 \cdot 2{,}86 \cdot 0{,}5 \cdot 0{,}525 \cdot 3{,}34 - 1} = \frac{0{,}105}{10 - 1} =$$
$$= 0{,}0117 \text{ sec}$$

In addition, we find

$$T_1 = 10 T_2 = 0{,}117 \text{ sec}$$

When Conditions (6) and (7) are satisfied, the error transfer function (5) will take the form

$$\Phi_x(p) = \frac{b_4 p^4 + b_3 p^3}{(1 + T_2 p)[T_y T_д p^3 + (T_y + T_д) p^2 + p + K]}. \qquad (8)$$

Dividing the numerator by the denominator in (8), we find the error coefficient with respect to the third derivative of the control input:

$$\frac{C_3}{3!} = \frac{b_3}{K} = \frac{T_y T_д + T_y T_2 + T_д T_2}{K}. \qquad (9)$$

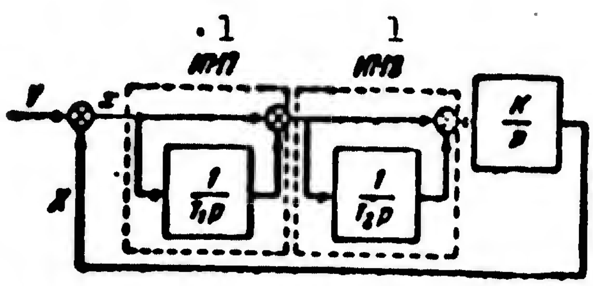


Fig. 117. System with PI mechanisms. 1) PI mechanism.

Substitution of the numerical values yields

$$\frac{C_3}{6} = \frac{0{,}005 \cdot 0{,}1 + 0{,}005 \cdot 0{,}0117 + 0{,}1 \cdot 0{,}0117}{16{,}4} = 1{,}3 \cdot 10^{-4}\ \mathrm{sec}^3$$

Equation (9) is the missing equation, and it may be used for simultaneous solution of Eqs. (6) and (7).

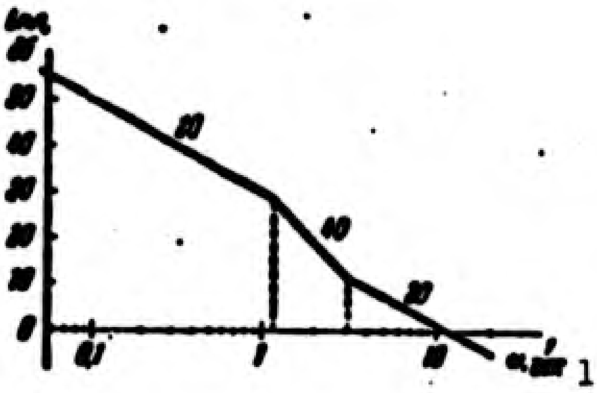


Fig. 118. Logarithmic amplitude characteristic of servosystems. 1) Sec.

181. In a control system (Fig. 117), two proportional-plus-floating mechanisms PI1 and PI2 are introduced in order to increase the order of astatism. Find the first five error coefficients.

Answer.

$$C_0 = 0,\ C_1 = 0,\ C_2 = 0,\ \frac{C_3}{6} = \frac{T_1 T_2}{K},\ \frac{C_4}{24} = \frac{T_1 T_2 (T_1 + T_2)}{K}.$$

182. The open-loop transfer system of a servosystem has the form

$$W(p) = \frac{K}{p(1 + T_1 p)(1 + T_2 p)}.$$

The parameter values are: K = 20 1/sec, T_1 = 0.02 sec, and T_2 = = 0.03 sec. A harmonic input is applied to the system with amplitude $\theta_{1max} = 10^{\circ}$ and period T_k = 7 sec. Find the error amplitude.

Solution. 1) For the exact solution, we find the error transfer function:

$$\Phi_1(p) = \frac{1}{1 + W(p)} = \frac{p(1 + T_1 p)(1 + T_2 p)}{p(1 + T_1 p)(1 + T_2 p) + K}.$$

After combining like terms and substituting in the values of the parameters, we obtain

$$\Phi_1(p) = \frac{6 \cdot 10^{-4} p^3 + 5 \cdot 10^{-2} p^2 + p}{6 \cdot 10^{-4} p^3 + 5 \cdot 10^{-2} p^2 + p + 20}.$$

The error amplitude is

$$\theta_{max} = |\Phi_1(j\omega)|\,\theta_{1max}.$$

Find the absolute value of the frequency transfer function with respect to the error when $\omega = \omega_k = 2\pi/T_k$ = 0.9 1/sec:

$$|\Phi_\varepsilon(j\omega_k)| = \left|\frac{6\cdot10^{-4}(j\omega_k)^3 + 5\cdot10^{-3}(j\omega_k)^2 + j\omega_k}{6\cdot10^{-4}(j\omega_k)^3 + 5\cdot10^{-3}(j\omega_k)^2 + j\omega_k + 20}\right| =$$

$$= \left|\frac{-0{,}004 + j0{,}9}{20 + j0{,}9}\right| = \sqrt{\frac{0{,}004^2 + 0{,}9^2}{20^2 + 0{,}9^2}} = 0{,}045.$$

We next find

$$\theta_{max} = 0{,}045\cdot 10 = 0{,}45^\circ = 27'.$$

2) For an approximate solution, we find the absolute value of the frequency transfer function of the open-loop system when $\omega = \omega_k$:

$$A(\omega_k) = |W(j\omega_k)| = \frac{20}{0{,}9\sqrt{1+0{,}9^2\cdot0{,}02^2}\sqrt{1+0{,}9^2\cdot0{,}03^2}} = 22{,}2.$$

The error amplitude is

$$\theta_{max} = \frac{\theta_{1max}}{A(\omega_k)} = \frac{10}{22{,}2} = 0{,}45^\circ = 27'.$$

183. For a servosystem, we are given the system open-loop logarithmic amplitude characteristic (Fig. 118). Determine the error amplitude if the input variable is governed by the law $\theta_1 = \theta_{1max} \sin \omega_k t$, where $\theta_{1max} = 15^\circ$ and $\omega_k = 0.2$ 1/sec.

Solution. From the logarithmic amplitude characteristic shown in Fig. 118, we determine the absolute value in decibels at a frequency $\omega = \omega_k = 0.2$ 1/sec:

$$L(\omega_k) = 20 \lg A(\omega_k) = 45 \text{ db}$$

We next find log $A(\omega_k) = 2.25$. From the logarithm, we find the absolute value:

$$A(\omega_k) = 10^{2.25} = 168.$$

The error amplitude is

$$\theta_{max} = \frac{\theta_{1max}}{A(\omega_k)} = \frac{15}{168} = 0{,}089^\circ = 5{,}3'.$$

184. Solve the preceding problem if:

1) $\theta_{1max} = 5^\circ$, $\omega_k = 0{,}1$ 1/sec;
2) $\theta_{1max} = 10^\circ$, $\omega_k = 0{,}8$ 1/sec;
3) $\theta_{1max} = 30^\circ$, $\omega_k = 0{,}4$ 1/sec.

Answer: 1) 0.88'; 2) 14.2'; 3) 21.2'.

185. The open-loop transfer function of a servosystem has the form

$$W(p)=\frac{K(1+T_3 p)}{p(1+T_1 p)(1+T_2 p)},$$

where K = 200 1/sec, T_1 = 0.5 sec, T_2 = 0.1 sec, and T_3 = 0.01 sec. Determine the phase error in the reproduction of a harmonic input signal with amplitude θ_{1max} = 20° and period T_k = 1 sec.

<u>Solution.</u> 1) For the exact solution, we find the frequency transfer function of the closed-loop system when $\omega = \omega_k = 2\pi/T_k$ = 6.28 1/sec:

$$\Phi(j\omega_k)=\frac{W(j\omega_k)}{1+W(j\omega_k)}=$$

$$=\frac{K(1+j\omega_k T_3)}{j\omega_k(1+j\omega_k T_1)(1+j\omega_k T_2)+K(1+j\omega_k T_3)}=$$

$$=\frac{K(1+j\omega_k T_3)}{K-\omega_k^2(T_1+T_2)+j[\omega_k(1+KT_3)-\omega_k^3 T_1 T_2]}.$$

Substitution of the numerical parameter values yields

$$\Phi(j\omega_k)=\frac{200+j133}{180+j130}=1{,}09-j0{,}0325.$$

The phase error is

$$\varphi=-\operatorname{arctg}\frac{0{,}0325}{1{,}09}=-\operatorname{arctg}0{,}03\approx-1{,}7^{\circ}.$$

2) For an approximate solution, we make use of the fact that in the frequency region of the input, the system open-loop frequency transfer function has the form

$$W(j\omega)\approx\frac{K}{j\omega(1+j\omega T_1)}.$$

For a value $\omega = \omega_k$, the error transfer function may be taken as

$$\Phi_\varepsilon(j\omega_k)\approx\frac{1}{W(j\omega_k)}=\frac{j\omega_k(1+j\omega_k T_1)}{K}.$$

The phase error is

$$\varphi\approx-\operatorname{Im}\frac{1}{W(j\omega_k)}=-\frac{\omega_k}{K}=-\frac{6{,}28}{200}=-0{,}0314\ \text{rad}=-1{,}8^{\circ}.$$

186. Determine the phase error for the preceding problem if

$$1)\ T_k=10\ \text{sec},\quad 2)\ T_k=2\ \text{sec}.$$

<u>Answer</u>

$$1)\ -0{,}18^{\circ},\quad 2)\ -0{,}9^{\circ}.$$

§15. DETERMINING ACCURACY IN THE PRESENCE OF A DISTURBANCE

187. For the system used to stabilize the speed of a heat motor (Problem 31), determine the steady-state error when a constant load torque M_n = const is applied.

Answer.

$$\Omega = 0.$$

188. For the servosystem shown in Fig. 26 (Problems 29 and 30), determine the steady-state torque error if the load torque on the actuating shaft amounts to M = 2000 g·cm, and the reducing-gear efficiency is 0.8.

Solution. The torque figure of merit of the considered servosystem (see Problem 30) is K_m = 1700 g·cm/ang.min. From this we obtain the torque error:

$$\theta_m = \frac{M_s}{K_m} = \frac{M}{\eta K_m} = \frac{2000}{0,8 \cdot 1700} = 1,47';$$

189. Solve the preceding problem if we are given the load torque at the motor shaft M_{nd} = 5 g·cm.

Solution. We determine the torque figure of merit with respect to the motor shaft:

$$K_{m.d} = \frac{K_m}{i} = \frac{1700}{1000} = 1,7 \text{ g·cm/ang.min}$$

where i = 1000 is the gear ratio of the reduction gear.

The error torque is

$$\theta_m = \frac{M_{n.d}}{K_{m.d}} = \frac{5}{1,7} = 2,95'.$$

190. Determine the torque errors for servosystems having first-order astatism under the following initial conditions:

1. The velocity figure of merit is K_Ω = 200 1/sec, the gear ratio of the reduction gear is i = 500, the no-load motor speed is $n_{kh.kh}$ = = 6000 rpm, the starting torque is M_p = 100 g·cm, the load torque applied to the motor shaft is $M_{n.d}$ = 30 g·cm.

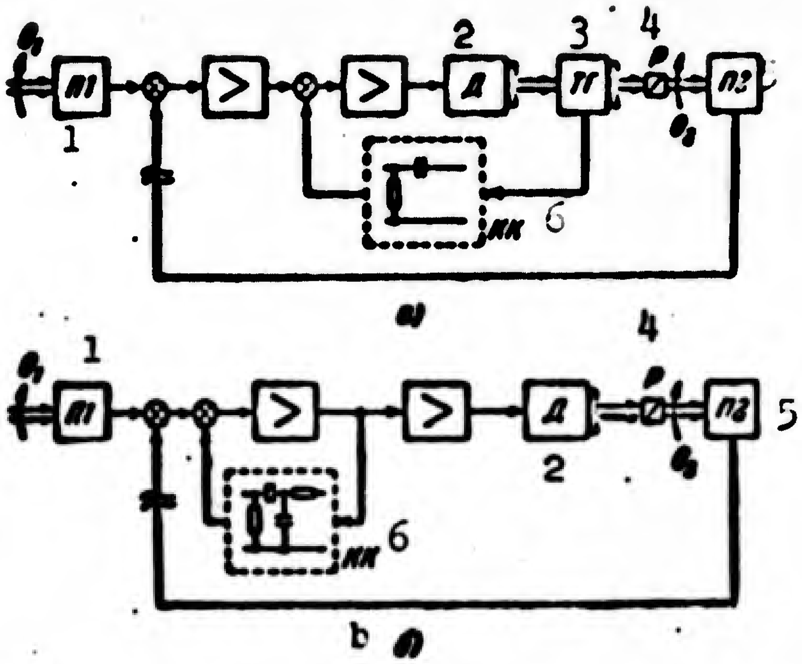


Fig. 119. Servosystems for Problem 191. 1) Command-shaft potentiometer; 2) motor; 3) tachometer generator; 4) reduction gear; 5) actuating-shaft potentiometer; 6) compensating network.

2. K_Ω = 500 1/sec, i = 10,000, $n_{kh.kh}$ = 7500 rpm, M_p = 300 g·cm, $M_{n.d}$ = 150 g·cm.

Answer.

$$1.\ \theta_a = \frac{3440\pi n_{kh.kh}}{30 i K_\Omega} \frac{M_{n.d}}{M_p} = \frac{3440 \cdot 3{,}14 \cdot 6000}{30 \cdot 500 \cdot 200} \frac{30}{100} = 6{,}5'.$$

$$2.\ \theta_a = \frac{3440 \cdot 3{,}14 \cdot 7500}{30 \cdot 10\,000 \cdot 300} \frac{150}{300} = 0{,}27'.$$

191. Figure 119 shows the electromechanical diagrams of two servosystems using different compensating devices. The system open-loop transfer functions for both diagrams, taking into account the compensating networks, are the same and equal to

$$W(p) = \frac{K_\Omega (1 + T_2 p)}{p(1 + T_1 p)(1 + T_3 p)},$$

where K_Ω = 400 1/sec, T_1 = 0.25 sec, T_2 = 0.05 sec, and T_3 = 0.01 sec. In Fig. 119, P1 and P2 are the command- and actuating-shaft potentiometers, D is the motor, R is the reduction gear, TG is the tachometer generator, KK is the compensating network, θ_1 and θ_2 are the angles through which the command and actuating shafts turn. In both arrangements, the compensating network in the linear representation is equivalent in effect to a series integrating-differentiating element with transfer function

$$W_{k.z}(p)=\frac{(1+T_2 p)(1+T_3 p)}{(1+T_1 p)(1+T_4 p)},$$

where T_d is the motor time constant.

Determine the maximum error in transmission of a harmonic input signal with amplitude $\theta_{1max} = 15^{\circ}$ and period T = 8 sec if the torque error caused by dry-friction forces is $\theta_m = 2'$.

Solution. 1. For the arrangement shown in Fig. 119a, by appropriate use of feedback around the actuating motor, the maximum error may be found by summing the amplitude of the error of the control input θ_a and the torque error θ_m:

$$\theta_{max}=\theta_a+\theta_m. \qquad (1)$$

The first term of (1)

$$\theta_a\approx\frac{\theta_{1max}}{|W(j\omega_k)|}\approx\frac{\theta_{1max}}{K_v}\sqrt{1+\omega_k^2T_1^2}. \qquad (2)$$

where $\omega_k = 2\pi/T = 6.28/8 = 0.785$ 1/sec is the input angular frequency. Substituting the numerical coefficient values into (2), we obtain

$$\theta_a=\frac{15}{400}\sqrt{1+0{,}785^2\cdot0{,}25^2}=0{,}038^{\circ}=2{,}3'.$$

The maximum error (1) is

$$\theta_{max}=2{,}3+2=4{,}3'.$$

2. For the arrangement shown in Fig. 119b, with appropriate use of feedback that does not include the actuating motor and which introduces a long time constant $T_1 = 0.25$ sec into the channel ahead of the motor terminals, the maximum error may be computed from the approximate formula [3]

$$\theta_{max}=\sqrt[3]{15{,}5\varepsilon_{1max}T_1^2\theta_m^2}-\theta_m. \qquad (3)$$

where $\varepsilon_{1max} = \omega_k^2\theta_{1max} = 0.785^2\cdot15 = 9.3$ degree/sec^2 = 550 ang.min/sec^2.

Substition of numerical values in (3) gives

$$a_{max} = \sqrt[3]{15,5 \cdot 550 \cdot 0,25^2 \cdot 2^3} - 2 = 14,2.$$

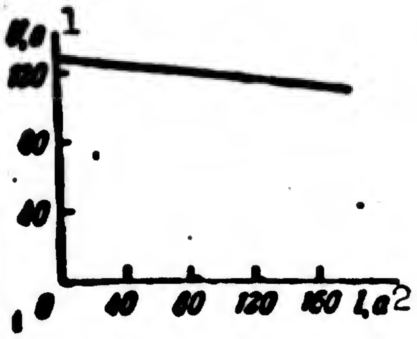


Fig. 120. Generator external characteristic. 1) U, v; 2) I, amp.

192. The external characteristic of a generator (the voltage across its terminals as a function of the load current) is shown in Fig. 120. The slope of the characteristic curve is $\beta = 0.1$ v/amp. The generator has a static voltage-regulating system with an over-all open-loop gain of K = 200. Determine the steady-state error under a load surge $\Delta I_n = 100$ amp.

Solution.

$$\Delta U = \frac{\beta \Delta I_n}{1+K} = \frac{0,1 \cdot 100}{1+200} \approx 0,05 \text{ v.}$$

193. In a furnace temperature-stabilization system, a thermocouple is used as the sensing element. With the control system disconnected, an external disturbance causes a temperature deviation from the set value of $\Delta\tau_0 = 200°C$. Determine the steady-state temperature deviation if a control system having an open-loop transfer function

$$W(p) = \frac{K}{(1+T_1 p)(1+T_2 p)},$$

is used; here K = 500.

Solution.

$$\Delta\tau = \frac{\Delta\tau_0}{1+K} = \frac{200}{1+500} \approx 0,4°C.$$

Manuscript Page No.	[Transliterated Symbols]
136	уст = ust = ustanovivshiysya = steady-state
138	э = e = ekvivalentnyy = equivalent
140	о.с = o.s = obratnaya svyaz' = feedback
141	ТГ = TG = takhogenerator = tachometer generator

141 П = P = potentsiometr = potentiometer

141 ЧЭ = ChE = chuvstvitel'nyy element = sensing element

141 Д(д) = D(d) = dvigatel' = motor

141 КК = KK = korrektiruyushchiy kontur = compensating network

141 Р = R = reduktor = reduction gear

141 У = U = usilitel' = amplifier

144 ИМ = IM = izodromnyy mekhanizm = PI mechanism (proportional-plus-integral)

148 н = n = nagruzka = load

148 м = m = moment = torque

148 х.х = kh.kh = kholostoy khod = no load

148 п = p = pusk = starting

150 п.з = p.z = posledovatel'noye zveno = series element

150 а = a = amplituda = amplitude

BLANK PAGE

Chapter 6

EVALUATING DYNAMIC CONTROL PERFORMANCE

§16. ROOT METHODS

194. Given the following control-system characteristic equations:

1) $p^3+14p^2+53p+130=0$;
2) $p^3+11p^2+51p+41=0$;
3) $p^3+2,5p^2+27p+13=0$;
4) $p^4+7p^3+418p^2+1220p+808=0$;
5) $p^4+3p^3+5,5p^2+6p+2,5=0$.

Determine the roots of the equation, and the system degree of stability h, oscillation μ, and damping η.

Answer.

1) $p_1=-10$ 1/sec $p_{2,3}=-2\pm j3$ 1/sec $h=2$ 1/sec

$$\mu=\frac{3}{2}=1,5,\quad \eta=1-e^{-\frac{2\pi}{\mu}}=98,5\%;$$

2) $p_1=-1$ 1/sec $p_{2,3}=-5\pm j4$ 1/sec $h=1$ 1/sec

$$\mu=0,8,\quad \eta=99,96\%;$$

3) $p_1=-0,5$ 1/sec $p_{2,3}=-1\pm j5$ 1/sec $h=0,5$ 1/sec

$$\mu=5,\quad \eta=71,5\%;$$

4) $p_1=-1$ 1/sec $p_2=-2$ 1/sec $p_{3,4}=-2\pm 20$ 1/sec,

$$h=1 \text{ 1/sec}\quad \mu=10,\quad \eta=47\%;$$

5) $p_1=-1$ 1/sec $p_2=-1$ 1/sec $p_{3,4}=-0,5\pm j1,5$ 1/

$$h=0,5 \text{ 1/sec}\quad \mu=3,\quad \eta=88\%.$$

195. Given the control-system characteristic equations

$$\begin{aligned}&1)\quad p^3+4p^2+41p+64=0;\\&2)\quad p^3+14p^2+144p+1000=0.\end{aligned}\qquad (1)$$

Using a Vyshnegradskiy diagram, determine system damping and degree of stability without finding the roots.

<u>Solution.</u> 1) We make the substitution $p = \sqrt[3]{64}\, q = 4q$. Then Eq. (1) after division by 64 will take the form

$$q^3 + q^2 + \frac{41}{16} q + 1 = 0.$$

The Vyshnegradskiy parameters are: $A = 1$ and $B = 41/16 = 2.56$. From a Vyshnegradskiy diagram on which lines of equal damping have been plotted (Appendix 1), we find $\eta = 70\%$. From a Vyshnegradskiy diagram on which lines of equal degree of stability have been plotted (Appendix 2), we find the relative degree of stability $h_0 = 0.25$. We next determine the absolute degree of stability: $h = 4h_0 = 1$ 1/sec.

2) $\eta = 75\%$, $h = 2$ 1/sec.

196. We are given the open-loop transfer function of a system with first-order astatism

$$W(p) = \frac{K_\Omega}{p(1 + Tp)}. \qquad (1)$$

Find the relationship between the velocity figure of merit K_Ω and the time constant T for which the damping over one cycle will be no less than a given value η.

<u>Solution.</u> We find the characteristic equation of the system

$$1 + W(p) = 0$$

or, after substitution of (1),

$$p^2 + \frac{1}{T} p + \frac{K_\Omega}{T} = 0. \qquad (2)$$

The roots of this equation are

$$p_{1,2} = -\frac{1}{2T} \pm j\sqrt{\frac{K_\Omega}{T} - \frac{1}{4T^2}} = -\alpha \pm j\beta, \qquad (3)$$

where

$$\alpha = \frac{1}{2T} \quad \text{and} \quad \beta = \sqrt{\frac{K_\Omega}{T} - \frac{1}{4T^2}}.$$

The oscillation is connected with the damping over one period by the relationship

$$\mu = \frac{\beta}{\alpha} = \frac{2\pi}{\ln\frac{1}{1-\eta}}. \qquad (4)$$

We next find

$$\frac{\beta}{\alpha} = \sqrt{4K_v T - 1}. \qquad (5)$$

Simultaneous solution of (4) and (5) yields the desired condition

$$K_v T < \frac{\pi^2}{\left(\ln\frac{1}{1-\eta}\right)^2} + 0{,}25. \qquad (6)$$

197. In a system with open-loop transfer function

$$W(p) = \frac{K_v}{p(1+Tp)}$$

the time constant is T = 0.1 sec. Determine the admissible velocity figure of merit corresponding to single-period dampings η = 90%, η = 95%, η = 98%, and η = 100%. (See the last problem).

Answer.

$$K_v = 31{,}1 \text{ 1/sec} \quad K_v = 13{,}6 \text{ 1/sec} \quad K_v = 8{,}7 \text{ 1/sec} \quad K_v = 2{,}5 \text{ 1/sec}$$

198. The open-loop transfer function of a system with second-order astatism has the form

$$W(p) = \frac{K_\varepsilon(1+Tp)}{p^2}. \qquad (1)$$

Determine the relationship between the acceleration figure of merit K_ε and the time constant T for which the single-period acceleration will not be less than a prescribed value η.

Solution. We find the characteristic closed-loop system equation

$$1 + W(p) = 0$$

or

$$p^2 + K_\varepsilon T p + K_\varepsilon = 0. \qquad (2)$$

The roots of this equation are

$$p_{1,2} = -\frac{K_\varepsilon T}{2} \pm j\sqrt{K_\varepsilon - \frac{K_\varepsilon^2 T^2}{4}} = -\alpha \pm j\beta. \qquad (3)$$

The oscillation is

$$\mu = \frac{\beta}{\alpha} = \sqrt{\frac{4}{T^2K_\varepsilon} - 1}. \qquad (4)$$

Using the relationship between the oscillation and the damping

$$\mu = \frac{2\pi}{\ln \frac{1}{1-\eta}}, \qquad (5)$$

we finally find

$$\frac{1}{K_\varepsilon T^2} < \frac{\pi^2}{\left(\ln \frac{1}{1-\eta}\right)^2} + 0,25 \qquad (6)$$

or

$$K_\varepsilon T^2 > \frac{1}{\frac{\pi^2}{\left(\ln \frac{1}{1-\eta}\right)^2} + 0,25}. \qquad (7)$$

199. In a system with the open-loop transfer function

$$W(p) = \frac{K_\varepsilon(1 + Tp)}{p^2}$$

the acceleration figure of merit is $K_\varepsilon = 100\ 1/\text{sec}^2$. Determine the minimum time constant T corresponding to single-period damping of $\eta = 90\%$, $\eta = 95\%$, $\eta = 98\%$, and $\eta = 100\%$. (See the preceding problem.)

Answer.

T = 0.069 sec, T = 0.086 sec, T = 0.107 sec, T = 0.20 sec.

200. In a static control system, the open-loop transfer function takes the form

$$W(p) = \frac{K}{(1 + T_0p)(1 + T_1p)}.$$

The time constants equal $T_0 = 1$ sec and $T_1 = 0.5$ sec. Determine the admissible over-all gain K for which the single-period damping will be at least $\eta = 90\%$.

Answer.

$$K < \frac{(T_0 + T_1)^2}{T_0T_1}\left[\frac{\pi^2}{\left(\ln \frac{1}{1-\eta}\right)^2} + 0,25\right] - 1 =$$
$$= \frac{(1 + 0,5)^2}{1 \cdot 0,5}\left[\frac{3,14^2}{\left(\ln \frac{1}{1-\eta}\right)^2} + 0,25\right] - 1 = 8,5.$$

§17. TRANSIENT-RESPONSE EVALUATION

201. A closed-loop control system is described by the differential equation

$$(a_0 p^2 + a_1 p + 1) X = (a_2 p + 1) Y. \qquad (1)$$

Determine the overshoot on the assumption that the characteristic equation has the complex roots $p_{1,2} = -\alpha \pm j\beta$, where there is no control input, $Y = 0$. The initial conditions are $X = X_0$ and $\dot{X} = 0$ for $t = 0$.

Answer. The transient is determined by the expression

$$X = X_0 e^{-\alpha t}\left(\cos \beta t + \frac{\alpha}{\beta} \sin \beta t\right) = X_0 \frac{\sqrt{\alpha^2+\beta^2}}{\beta} e^{-\alpha t} \sin\left(\beta t + \operatorname{arctg} \frac{\beta}{\alpha}\right). \qquad (2)$$

An extremum investigation yields the first value:

$$X_m = -X_0 e^{-\frac{\alpha}{\beta}\pi} = -X_0 e^{-\frac{\pi}{\mu}}. \qquad (3)$$

From which we obtain the unknown overshoot

$$\sigma = \frac{|X_m|}{X_0} = e^{-\frac{\alpha}{\beta}\pi} = \exp\left[-\frac{\pi}{\sqrt{4\frac{a_0}{a_1^2} - 1}}\right]. \qquad (4)$$

202. For the preceding problem, determine the condition for abscence of overshoot.

Answer. This is $\beta = 0$, corresponding to satisfaction of the condition $a_0 = 0.25 a_1^2$.

203. For Problem 201, determine the relationship of the coefficients such that the overshoot will be $\sigma = 10\%$, $\sigma = 20\%$, $\sigma = 50\%$.

Answer

$$a_0 = 0{,}72 a_1^2, \quad a_0 = 1{,}22 a_1^2, \quad a_0 = 5{,}25 a_1^2.$$

204. For the control system whose differential equation (1) was given in Problem 201, determine the overshoot when a unit step function $Y(t) = 1(t)$ is applied to the input if prior to the application of the input signal the system has been at rest.

Answer. The transient is determined by the expression

$$X = 1 - e^{-\alpha t}\left(\cos\beta t - \frac{\alpha}{\beta}\sin\beta t\right).$$

A maximum study yields

$$X_m = 1 + e^{-\frac{1}{\mu}\operatorname{arctg}\frac{2\mu}{1-\mu^2}}.$$

From this we determine the overshoot:

$$\sigma = \frac{X_m - 1}{1} = \exp\left[-\frac{\operatorname{arctg}\frac{2\mu}{1-\mu^2}}{\mu}\right],$$

where

$$\mu = \frac{\beta}{\alpha} = \sqrt{4\frac{a_0}{a_1^2} - 1}.$$

205. For the relationship of coefficients a_0 and a_1 corresponding to overshoots of $\sigma = 0\%$, $\sigma = 10\%$, $\sigma = 20\%$, and $\sigma = 50\%$ for adjustment from a fixed position (see Problems 202 and 203), determine the overshoot when the unit step signal $Y(t) = 1(t)$ as applied to the input, and compare the overshoot values.

Answer. The overshoot values are given in the table:

1 Вид движения	$a_0 = 0.25\, a_1^2$	$a_0 = 0.72\, a_1^2$	$a_0 = 1.32\, a_1^2$	$a_0 = 3.26\, a_1^2$
2 Согласование из неподвижного положения	0%	10%	20%	50%
3 Отработка единичного ступенчатого воздействия	13.5	27%	32%	55%

1) Type of motion; 2) adjustment from fixed positions; 3) transmission of unit step input.

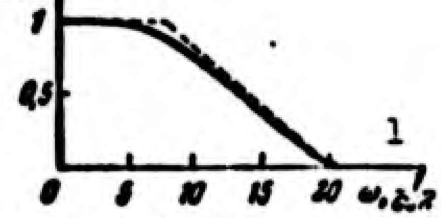


Fig. 121. Trapezoidal real frequency characteristic. 1) sec.

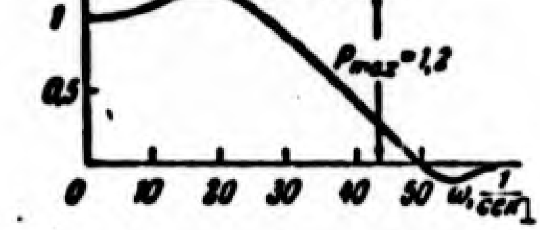


Fig. 122. Real characteristic with peak. 1) sec.

206. The closed-loop transfer function of a control system has the form

$$\Phi(p)=\frac{a_1 p+1}{a_3 p^3+a_2 p^2+a_1 p+1}.$$

A unit step function 1(t) is applied to the system input. By plotting the transient curve, determine the overshoot and transient time for the following coefficients:

1) a_1 = 0.33 sec, a_2 = 0.01 sec^2, a_3 = $1.58 \cdot 10^{-4}$ sec^3;

2) a_1 = 0.415 sec, a_2 = 0.04 sec^2, a_3 = 0.002 sec^3;

3) a_1 = 0.087 sec, a_2 = 0.0025 sec^2, a_3 = $0.435 \cdot 10^{-4}$ sec^3.

Answer

1) $\sigma = 13.8\%$, $t_p = 0.775$ sec;
2) $\sigma = 26.5\%$, $t_p = 1.17$ sec;
3) $\sigma = 37.2\%$, $t_p = 0.27$ sec.

207. Figure 121 shows the real frequency response of a closed-loop system. Determine the rough values of overshoot and transient time.

Solution. The interval of real frequencies for the real characteristic is ω_s = 20 1/sec. This gives a transient time of

$$\frac{\pi}{\omega_s}<t_p<\frac{4\pi}{\omega_s},$$

or

$$0.157 \text{ sec} < t_p < 0.628 \text{ sec}.$$

The overshoot $\sigma < 18\%$.

For a more accurate calculation, it is necessary to turn to the curves given in Appendix 4. The slope of the real response curve (Fig. 121) is $\kappa = 0.4$. This yields $\sigma = 10\%$ and $t_p = 7/20 = 0.35$ sec.

208. Determine the overshoot and transient time for the real frequency response curve shown in Fig. 122.

Solution. The high-frequency part of the characteristic corresponding to $P(\omega) < 0$ may be started, since $P_{min} < 0.2$. Then the system overshoot will be

$$\sigma < \frac{1,18P_{max} - P(0)}{P(0)} = \frac{1,18 \cdot 1,2 - 1}{1} = 0,41 = 41\%.$$

The transient time is

$$t_n > \frac{\pi}{\omega_c} = \frac{3,14}{50} = 0,0628 \text{ sec.}$$

For a more accurate calculation, it is necessary to turn to the curves of Appendix 5. By using them, we find

$$\sigma = 23\% \text{ and } t_n = \frac{3\pi}{\omega_c} = \frac{3 \cdot 3,14}{50} = 0,18 \text{ sec.}$$

§18. INTEGRAL ESTIMATES

209. The open-loop transfer function of a servosystem with first-order astatism has the form

$$W(p) = \frac{K_\Omega}{p(1+T_1p)(1+T_2p)}.$$

For values of the time constants of $T_1 = 0.02$ sec and $T_2 = 0.04$ sec, find the velocity figure of merit corresponding to the minimum integral square estimate for transmission of a unit step input.

<u>Solution</u>. The closed-loop system transfer function is

$$\Phi(p) = \frac{W(p)}{1+W(p)} = \frac{K_\Omega}{K_\Omega + p + (T_1+T_2)p^2 + T_1T_2p^3}.$$

The Laplace transform of the output variable is

$$X(p) = \frac{K_\Omega}{K_\Omega + p + (T_1+T_2)p^2 + T_1T_2p^3} \frac{1}{p} =$$

$$= \frac{b_0}{a_0 + a_1p + a_2p^2 + a_3p^3} \frac{1}{p}.$$

In accordance with Appendix 14, we find the integral estimate:

$$I = \frac{B_0\Delta_0}{2a_0^2\Delta},$$

where $B_0 = b_0^2 = K_\Omega^2$, $a_0 = K_\Omega$. The values of the determinants are:

$$\Delta = \begin{vmatrix} a_0 & -a_2 & 0 \\ 0 & a_1 & -a_3 \\ 0 & -a_0 & a_2 \end{vmatrix} = a_0(a_1a_2 - a_0a_3),$$

$$\Delta_3 = \begin{vmatrix} a_1 & -a_3 & 0 \\ a_0 & a_2 & -a_4 \\ 0 & -a_1 & a_3 \end{vmatrix} = a_1^2 a_4 - a_1 a_2 a_3 + a_0 a_3^2.$$

As a result, we have

$$I = \frac{1}{2K_\Omega} + \frac{1}{2}\frac{(T_1 + T_2)^2}{T_1 + T_2 - K_\Omega T_1 T_2}.$$

In order to obtain the minimum value of the integral estimate, we set the derivative to zero:

$$\frac{dI}{dK_\Omega} = \frac{1}{2}\left[-\frac{1}{K_\Omega^2} + \frac{T_1 T_2 (T_1 + T_2)^2}{(T_1 + T_2 - K_\Omega T_1 T_2)^2}\right] = 0,$$

from which the optimum figure of merit will be

$$K_\Omega = \frac{T_1 + T_2}{T_1 T_2 + (T_1 + T_2)\sqrt{T_1 T_2}}.$$

Substitution of the numerical values for the time constants yields

$$K_\Omega = \frac{0.06}{8 \cdot 10^{-4} + 6\sqrt{8} \cdot 10^{-4}} = 24 \ 1/\text{sec}.$$

210. The open-loop transfer function of a system has the form

$$W(p) = \frac{K_\Omega + K_1 p}{p(1 + Tp)}.$$

For fixed values T = 0.1 sec and velocity figure of merit K_Ω = 20 1/sec, determine the value of the coefficient K_1 which determines the signal level with respect to the first derivative, corresponding to the minimum integral square error when a control signal in the form of the unit impulse function Y(t) = δ(t) is applied to the system input.

Solution. The system closed-loop transfer function is

$$\Phi(p) = \frac{W(p)}{1 + W(p)} = \frac{K_\Omega + K_1 p}{K_\Omega + (1 + K_1)p + Tp^2}.$$

The transform of the control input is Y(p) = 1. The transform of the output variable is

$$X(p) = \Phi(p)\,Y(p) = \frac{K_\Omega + K_1 p}{K_\Omega + (1 + K_1)p + Tp^2}.$$

The value of the integral square estimate (see Appendix 14) is

$$I = \frac{B_1\Delta_1 + B_2\Delta_2}{2a_0^2\Delta}.$$

The coefficients equal:

$$B_1 = b_1^2 = K_2^2, \quad B_2 = b_0^2 = K_1^2,$$
$$a_0 = K_2, \quad a_1 = 1 + K_1 \text{ and } a_2 = T.$$

The values of the determinants are:

$$\Delta = \begin{vmatrix} a_0 & -a_2 & 0 \\ 0 & a_1 & 0 \\ 0 & -a_0 & a_2 \end{vmatrix} = a_0a_1a_2 = K_2(1 + K_1)T,$$

$$\Delta_1 = \begin{vmatrix} a_0 & a_2 & 0 \\ 0 & a_0 & 0 \\ 0 & 0 & a_2 \end{vmatrix} = a_0^2a_2 = K_2^2T,$$

$$\Delta_2 = \begin{vmatrix} a_0 & -a_2 & a_1 \\ 0 & a_1 & a_0 \\ 0 & -a_0 & 0 \end{vmatrix} = a_0^2 = K_2^2.$$

We next find

$$I = \frac{K_2^2K_2^2T + K_1^2K_2^2}{2K_2^2K_2(1 + K_1)T} = \frac{K_2T + K_1^2}{2(1 + K_1)T}.$$

In order to find the minimum of I, we set the derivative equal to zero: $dI/dK_1 = 0$. As a result we have

$$K_1^2 + 2K_1 - K_2T = 0.$$

The solution of this equation for K_1 is

$$K_1 = -1 + \sqrt{1 + K_2T}.$$

Substitution of the numerical values yields

$$K_1 = -1 + \sqrt{1 + 20 \cdot 0.1} = 0.73.$$

211. The open-loop transfer function of a system takes the form

$$W(p) = \frac{K_2}{p(1 + Tp)}.$$

For a fixed value of the time constant T = 0.2 sec, determine the optimum velocity figure of merit corresponding to the minimum improved integral estimate

$$I = \int_0^\infty (x^2 + \tau^2\dot{x}^2)\,dx \qquad (1)$$

for the unit step input Y(t) = 1(t) for values of the extremum time constant $\tau = 0$, $\tau = 0.1$ sec, $\tau = 0.5$ sec, and $\tau = 1$ sec.

Solution. We separate Integral (1) into two integrals:

$$I = I_1 + I_2 = \int_0^\infty x^2 dt + \tau^2 \int_0^\infty \dot{x}^2 dt.$$

We find the system closed-loop transfer function

$$\Phi(p) = \frac{K_0}{K_0 + p + Tp^2}.$$

The transform of the output variable for Y(p) = 1/p:

$$X(p) = \Phi(p) Y(p) = \frac{K_0}{K_0 + p + Tp^2} \frac{1}{p}.$$

In accordance with Appendix 14, we find

$$I_1 = \frac{B_0 \Delta_0}{2a_0^2 \Delta}. \tag{2}$$

The values of the coefficients are:

$$B_0 = b_0^2 = K_0^2,\quad a_0 = K_0,\quad a_1 = 1 \text{ and } a_2 = T.$$

The values of the determinants are:

$$\Delta = \begin{vmatrix} a_0 & -a_2 \\ 0 & a_1 \end{vmatrix} = a_0 a_1 = K_0,$$

$$\Delta_0 = \begin{vmatrix} a_1 & -a_2 \\ a_0 & a_1 \end{vmatrix} = a_1^2 + a_0 a_2 = 1 + K_0 T.$$

We substitute the values found into (2):

$$I_1 = \frac{K_0^2 (1 + K_0 T)}{2K_0^2 K_0} = \frac{1 + K_0 T}{2K_0}.$$

In order to find I_2, we determine the transform of the rate at which the output variable changes:

$$pX_1(p) = \frac{K_0}{K_0 + p + Tp^2}.$$

In accordance with Appendix 14, we find

$$I_2 = \tau^2 \frac{B_1 \Delta_1}{2a_0^2 \Delta}, \tag{3}$$

where $B_1 = b_1^2 = K_{\Omega}^2$, while the determinant

$$\Delta_2 = \begin{vmatrix} a_0 & a_1 \\ 0 & a_0 \end{vmatrix} = a_0^2 = K\tau.$$

We next have

$$I_2 = \frac{K_\Omega \tau^2}{2}.$$

The resulting value for the improved integral estimate is

$$I = I_1 + I_2 = \frac{1 + K_\Omega T}{2K_\Omega} + \frac{K_\Omega \tau^2}{2}. \qquad (4)$$

In order to find the optimum value of K_Ω, we set the first derivative (4) equal to zero:

$$\frac{dI}{dK_\Omega} = 0.$$

After differentiation we have

$$-\frac{1}{K_\Omega^2} + \tau^2 = 0,$$

so that the optimum velocity figure of merit is $K_\Omega = 1/\tau$. The numerical values are: $K_\Omega \to \infty$, $K_\Omega = 10$ 1/sec, $K_\Omega = 2$ 1/sec and $K_\Omega = 1$ 1/sec.

§19. FREQUENCY ESTIMATES

212. Figure 123 shows the gain-frequency characteristic for a closed-loop system. Determine the magnitude ratio.

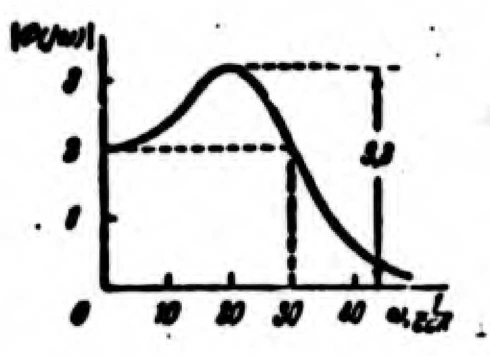


Fig. 123. Gain-frequency curve for closed-loop system. 1) sec.

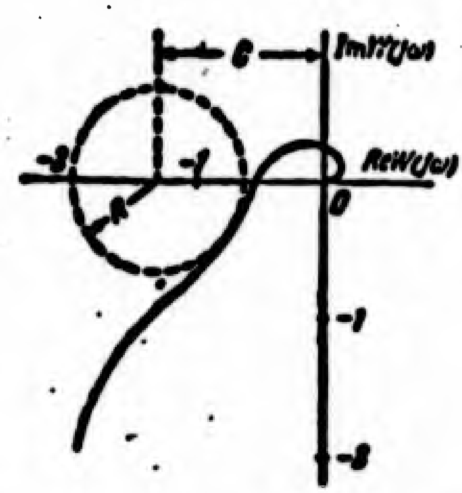


Fig. 124. Gain-phase characteristic for open-loop system.

Answer

$$M=\frac{|\Phi(j\omega)|_{max}}{|\Phi(0)|}=\frac{3{,}2}{2}=1{,}6.$$

213. Figure 124 shows the gain-phase characteristic of an open-loop servosystem. It may be plotted from the table on this page. Determine the system closed-loop magnitude ratio.

$\mathrm{Re}\,W(j\omega)$	−2	−1,75	−1,5	−1,25	−1	−0,75	−0,5	−0,25	0
$\mathrm{Im}\,W(j\omega)$	−1,95	−1,8	−1,75	−1,6	−1,4	−1,05	−0,55	−0,65	−0,55

Solution. In order to find the magnitude ratio it is necessary to determine the parameters of the circle tangent to the gain-phase characteristic. The parameters of the circle are connected with the magnitude ratio by the formulas

$$R=\frac{M}{M^2-1} \quad \text{and} \quad C=\frac{M^2}{M^2-1}.$$

where R is the radius of the circle and C the displacement of the center of the circle to the left of the origin. As result of our choice, we find that the tangent circle corresponds to

$$M=2, \quad R=\frac{2}{3} \quad \text{and} \quad C=\frac{4}{3}.$$

The construction is given by the dashed line of Fig. 124.

214. The open-loop transfer function of a servosystem has the form

$$W(p)=\frac{K_\Omega}{p(1+Tp)}.$$

Determine the relationship between the velocity figure of merit K_Ω and the time constant for which the system will have a magnitude ratio not exceeding a given value M.

Solution. The system closed-loop transfer function is

$$\Phi(p)=\frac{W(p)}{1+W(p)}=\frac{K_\Omega}{K_\Omega+p+Tp^2}.$$

The system closed-loop frequency transfer function is

$$\Phi(j\omega)=\frac{K_\Omega}{K_\Omega+j\omega-\omega^2 T}.$$

The modulus is

$$|\Phi(j\omega)|=\frac{K_\Omega}{\sqrt{(K_\Omega-\omega^2 T)^2+\omega^2}}.$$

A maximum investigation of this expression yields a value for the magnitude ratio of

$$|\Phi(j\omega)|_{max}=\frac{2K_\Omega T}{\sqrt{4K_\Omega T-1}}=M \quad (\text{for } K_\Omega T > 0.5).$$

From this last expression we obtain

$$K_\Omega T=\frac{M^2+M\sqrt{M^2-1}}{2}.$$

215. Solve the preceding problem if the system open-loop transfer function has the form

$$W(p)=\frac{K_\varepsilon(1+\tau p)}{p^2},$$

where K_ε is the acceleration figure of merit and τ is the compensating-network time constant.

Answer

$$K_\varepsilon\tau^2=2\frac{M^2-M\sqrt{M^2-1}}{M^2-1}.$$

216. Figure 125 shows the logarithmic frequency amplitude and phase characteristic (l. a. kh. and l. f. kh.) for an open-loop system. Determine the system closed-loop magnitude ratio.

Solution. In order to find the magnitude ratio we must construct the forbidden band for the phase characteristics so that the phase characteristic touches this zone. The forbidden zone is plotted in accordance with Appendix 24, which gives the necessary phase margins as a function of the modulus in decibels for various values of the magnitude ratio. As a result of our choice, we find that the magnitude ratio

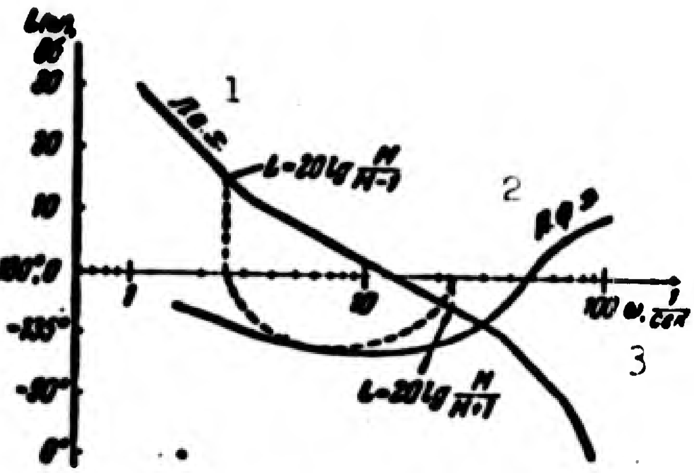


Fig. 125. Logarithmic amplitude characteristic and logarithmic phase characteristic for open-loop systems. 1) Logarithmic amplitude characteristic; 2) logarithmic phase characteristic; 3) sec.

is M = 1.2. The forbidden-zone construction is shown by the dashed line of Fig. 125.

217. Construct the logarithmic amplitude characteristic and logarithmic phase characteristic and determine the magnitude ratio if the open-loop transfer function of a system has the form:

1) $W(p)=\frac{100(1+0{,}173p)}{p^2(1+0{,}035p)}$;

2) $W(p)=\frac{25(1+0{,}66p)}{p^2(1+0{,}03p)}$;

3) $W(p)=\frac{400(1+0{,}1p)}{p(1+p)(1+0{,}013p)}$;

4) $W(p)=\frac{1000(1+0{,}05p)}{p(1+0{,}4p)(0{,}013p)}$.

Answer: 1) M = 1.5; 2) M = 1.1; 3) M = 1.3; 4) M = 1.7.

218. The open-loop transfer function of a system has the form

$$W(p)=\frac{K_v}{p\prod_{i=1}^{n}(1+T_i p)}.$$

Determine the condition under which the magnitude ratio for the closed-loop system will not exceed unity provided the number of time constants is arbitrary, i.e., n is an arbitrary integer.

Answer

$$K_0 \sum_{i=1}^{n} T_i = \frac{1}{2}.$$

219. For the amplitude-frequency characteristic of a closed-loop system (Fig. 123), determine the system passband.

Answer

$$\omega_p = 30 \text{ 1/sec} \quad f_p = 4.8 \text{ cps.}$$

220. For for the logarithmic amplitude characteristic shown in Fig. 125, determine the rough size of the passband.

Answer.

In first approximation the passband of a closed-loop system may be taken to equal the cutoff frequency of the open-loop logarithmic amplitude characteristic. As a result, we have $\omega_p \approx \omega_{sr} = 13$ 1/sec or $f_p = 2.1$ cps.

Manuscript Page No.	[Transliterated Symbols]
158	п = p = perekhodnyy = transient
158	с = s = sushchestvennyy = real, essential
165	л.а.х. = l.a.kh. = logarifmicheskaya amplitudnaya kharakteristika = logarithmic amplitude characteristic
165	л.ф.х. = l.f.kh. = logarifmicheskaya fazovaya kharakteristika = logarithmic phase characteristic
167	п = p = propuskaniye = pass
167	ср = sr = srez = cutoff

Chapter 7

RANDOM PROCESSES IN AUTOMATIC SYSTEMS

§20. CALCULATING CORRELATION FUNCTIONS AND SPECTRAL DENSITIES

221. Determine the correlation function $R(\tau)$ and spectral density $S(\omega)$ for a quantity varying in accordance with the harmonic law

$$x = A \sin(\beta t + \varphi).$$

Verify that integration of the spectral density over all frequencies, as well as the value $R(0)$, yield the mean square (in this case it equals the variance) of the quantity considered. The amplitude $A = 10$ and the angular frequency $\beta = 2$ 1/sec.

Solution. The correlation function is

$$R(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} x(t)\, x(t+\tau)\, dt =$$
$$= \frac{1}{T_0} \int_0^{T_0} A^2 \sin(\beta t + \varphi) \sin(\beta t + \beta\tau + \varphi)\, dt = \frac{A^2}{2} \cos\beta\tau,$$

where $T_0 = 2\pi/\beta$. Substitution of the initial conditions yields $R(\tau) = 50 \cos 2\tau$, and $R(0) = 50$.

The spectral density may be computed from the Fourier integral

$$S(\omega) = \int_{-\infty}^{+\infty} R(\tau)\, e^{-j\omega\tau}\, d\tau = \frac{\pi A^2}{2} [\delta(\omega - \beta) + \delta(\omega + \beta)],$$

where $\delta(\omega - \beta)$ and $\delta(\omega + \beta)$ are unit impulse functions located at frequencies $\omega = \beta$ and $\omega = -\beta$.

Integration of the spectral density over all frequencies gives

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} S(\omega)\, d\omega = \frac{A^2}{4} \int_{-\infty}^{+\infty} [\delta(\omega - \beta) + \delta(\omega + \beta)]\, d\omega.$$

The integrals of the unit impulse functions are equal to unity:

$$\int_{-\infty}^{+\infty}\delta(\omega-\beta)\,d\omega=\int_{-\infty}^{+\infty}\delta(\omega+\beta)\,d\omega=1.$$

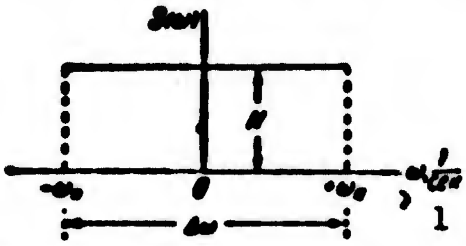


Fig. 126. White spectrum in limited frequency band. 1) sec.

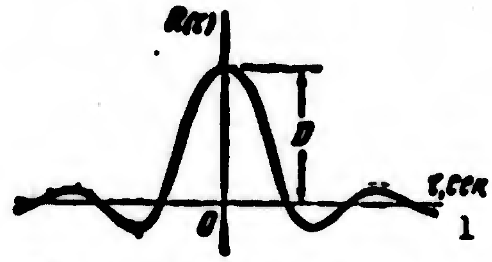


Fig. 127. Correlation function for Problem 222. 1) sec.

Thus we obtain as a result

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}S(\omega)\,d\omega=\frac{A^2}{2}=\frac{10^2}{2}=50.$$

222. For a stationary random process having a white spectrum in the band from $-\omega_p$ to $+\omega_p$ (Fig. 126), compute the mean value (mathematical expectation), mean square (second-order moment), and variance, and also find an analytic expression and plot a graph for the correlation function.

Solution. The mean value of a random quantity equals zero, $\bar{x} = 0$, since the spectral density contains no singularities of the impulse-function (delta-function) type $\Omega = 0$. As a result, the variance equals the mean square of the random quantity:

$$D=\overline{x^2}-\bar{x}^2=\overline{x^2}=\sigma^2,$$

where σ is the mean-square deviation. We next find

$$\overline{x^2}=D=\frac{1}{2\pi}\int_{-\infty}^{+\infty}S(\omega)\,d\omega=\frac{1}{2\pi}\int_{-\omega_p}^{+\omega_p}N\,d\omega=\frac{N\Delta\omega}{2\pi},$$

where $\Delta\omega = 2\omega_p$ is the band of angular frequencies in radians). This last formula may also be written in the following form:

$$\overline{x^2}=D=N\Delta f,$$

where $\Delta f = \Delta\omega/2\pi$ is the frequency band (in cycles per second).

The mean-square value of the random quantity is

$$x_{ms}=\sigma=\sqrt{N}\sqrt{\Delta f}.$$

The correlation function may be found from the transformed Fourier integral

$$R(\tau)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}S(\omega)e^{j\omega\tau}\,d\omega=\frac{1}{\pi}\int_{0}^{\infty}S(\omega)\cos\omega\tau\,d\omega$$

or

$$R(\tau)=\frac{1}{\pi}\int_{0}^{\omega_p}N\cos\omega\tau\,d\omega=\frac{N}{\pi\tau}\sin\omega_p\tau.$$

The graph of the correlation function is shown in Fig. 127. The value of the correlation function at $\tau = 0$ is

$$R(0)=\lim_{\tau\to 0}\frac{N}{\pi\tau}\sin\omega_p\tau=\frac{N\omega_p}{\pi}=D.$$

223. For the preceding problem, determine the normalized spectral density and correlation function.

Answer. The normalized spectral density for $-\omega_p < \omega < \omega_p$ is

$$\sigma(\omega)=\frac{S(\omega)}{D}=\frac{2\pi}{\Delta\omega}=\frac{1}{\Delta f}.$$

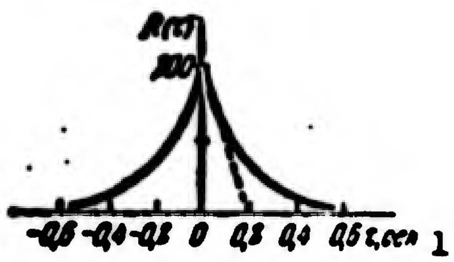


Fig. 128. Correlation function of exponential type. 1) sec.

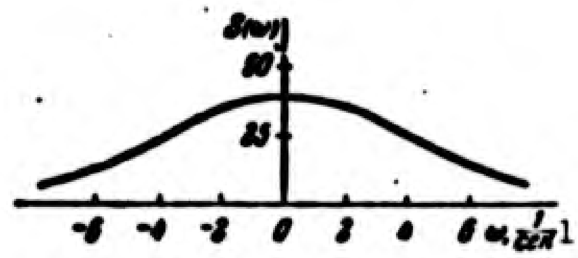


Fig. 129. Spectral density corresponding to correlation function of exponential type. 1) sec.

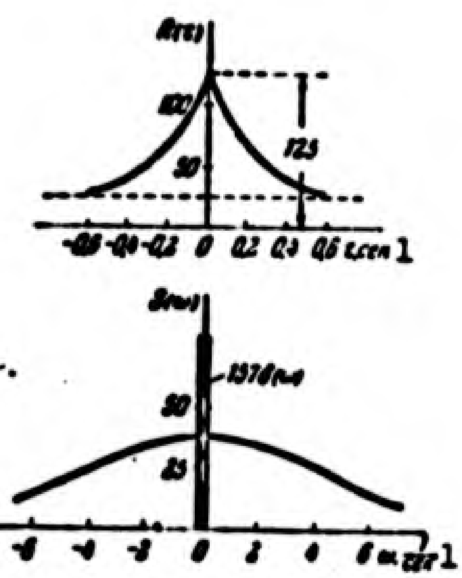


Fig. 130. Correlation function and spectral density for Problem 225. 1) sec.

The normalized correlation function is

$$\rho(\tau)=\frac{R(\tau)}{D}=\frac{\sin\omega_p\tau}{\omega_p\tau}.$$

The value of $\rho(\tau)$ at $\tau = 0$ is

$$\rho(0)=\lim_{\tau\to 0}\frac{\sin\omega_p\tau}{\omega_p\tau}=1.$$

224. By processing an oscillogram for a stationary random process with mathematical expectation (mean value) of zero, we obtain an expression for the correlation function

$$R(\tau)=De^{-\mu|\tau|},$$

where D = 100 is the variance and μ = 5 1/sec is the damping factor. The correlation function is plotted in Fig. 128. Find the spectral density and plot it.

Solution. The spectral density may be found from the Fourier integral

$$S(\omega)=\int_{-\infty}^{+\infty}R(\tau)e^{-j\omega\tau}d\tau=\int_{-\infty}^{+\infty}De^{-\mu|\tau|}e^{-j\omega\tau}d\tau.$$

This last integral for the sake of convenience should be separated into two integrals:

$$S(\omega)=D\left[\int_{-\infty}^{0}e^{(\mu-j\omega)\tau}d\tau+\int_{0}^{+\infty}e^{-(\mu+j\omega)\tau}d\tau\right]=\frac{2\mu D}{\mu^2+\omega^2}=\frac{2TD}{1+\omega^2T^2},$$

where

$$T=\frac{1}{\mu}=0.2 \text{ sec.}$$

Substitution of the numerical values yields

$$S(\omega)=\frac{40}{1+0.04\omega^2}.$$

The spectral density is plotted in Fig. 129.

225. Solve the preceding problem if the stationary random process considered has a mean value (mathematical expectation) $\bar{x}$ = 5. Plot the correlation function and spectral density.

Answer. The mean square of the random quantity is

$$\overline{x^2}=D+\bar{x}^2=$$
$$=100+5^2=125.$$

The correlation function is

$$R(\tau)=De^{-\mu|\tau|}+\bar{x}^2=$$
$$=100e^{-5|\tau|}+25.$$

The spectral density is

$$S(\omega)=2\pi\bar{x}^2\delta(\omega)+\frac{2TD}{1+\omega^2T^2}=157\,\delta(\omega)+\frac{40}{1+0.04\omega^2},$$

where $\delta(\omega)$ is the unit impulse function. The graphs are plotted in

Fig. 130.

226. By processing of an oscillogram for a stationary random process of irregular swinging type with mathematical expectation of zero, an expression for the correlation function is obtained:

$$R(\tau) = De^{-\mu|\tau|}\cos\beta\tau,$$

where D = 40 is the variance, μ = 0.5 1/sec is the damping factor, and β = 2 1/sec is the resonant frequency. The correlation function is shown in Fig. 131. Find an analytic expression for the spectral density, and plot it.

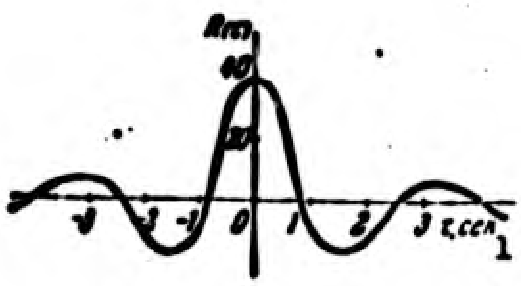


Fig. 131. Correlation function for irregular swinging. 1) sec.

Answer. The spectral density is

$$S(\omega) = \mu D\left[\frac{1}{\mu^2 + (\beta-\omega)^2} + \frac{1}{\mu^2 + (\beta+\omega)^2}\right].$$

After substitution of numerical values

$$S(\omega) = \frac{20}{0,25 + (2-\omega)^2} + \frac{20}{0,25 + (2+\omega)^2}.$$

The graph of the spectral density is shown in Fig. 132.

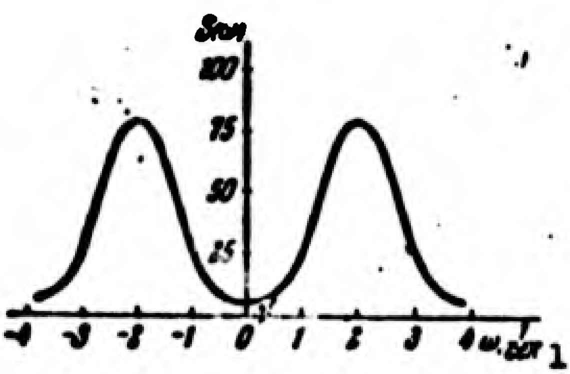


Fig. 132. Spectral density for irregular swinging. 1) sec.

227. For approximation of the formula for a correlation function on the basis of the initial data of the preceding problem, the more accurate expression

$$R(\tau) = De^{-\mu|\tau|}\left(\cos\beta\tau + \frac{\mu}{\beta}\sin\beta|\tau|\right).$$

is used. Find the spectral density for this case.

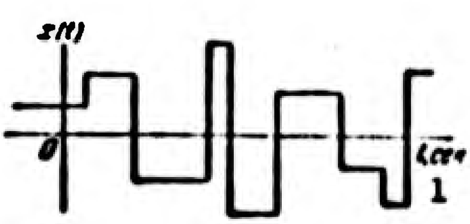


Fig. 133. Typical input signal for servosystem. 1) sec.

Answer

$$S(\omega)=\frac{\alpha}{\pi}D\left[\frac{2\beta-\omega}{\alpha^2+(\beta-\omega)^2}+\frac{2\beta+\omega}{\alpha^2+(\beta+\omega)^2}\right]=$$
$$=10\left[\frac{4-\omega}{0{,}25+(2-\omega)^2}+\frac{4+\omega}{0{,}25+(2+\omega)^2}\right].$$

228. A stationary random process has the form of a typical servo-system input signal (Fig. 133). The mean-square value of the considered variable $x_{sk} = 2$. The mean length of the segment x = const is T = 10 sec. Find the correlation function and spectral density.

Solution. The correlation function may be found from the expression

$$R(\tau)=\overline{x^2}P_1+\bar{x}^2P_2 \qquad (1)$$

where P_1 is the probability for finding the multiplied y-axis values of the random process in one interval, where x = const, i.e., the probability that there will be no change of rate on the time segment τ; $P_2 = 1 - P_1$ is the probability for the presence of a change of rate on the time segment τ.

Since for the process under consideration $\bar{x} = 0$, then $\overline{x^2} = D$ and Formula (1) takes the form

$$R(\tau)=DP_1. \qquad (2)$$

The probability for a change in the considered random quantity over the small time interval $\Delta\tau$ may be assumed to be proportional to the value of $\Delta\tau$ and equal to $\Delta\tau/T$. The probability that there will be no change in the random quantity will be $1 - (\Delta\tau/T)$. The probability that there will be no change on the time segment τ equals the product of the probabilities

$$P=\left(1-\frac{\Delta\tau}{T}\right)^{\frac{\tau}{\Delta\tau}}. \qquad (3)$$

The unknown probability P_1 may be found as the limit of Expression (3) when $\Delta\tau \to 0$:

$$P_1 = \lim_{\Delta\tau \to 0}\left(1 - \frac{\Delta\tau}{T}\right)^{\frac{\tau}{\Delta\tau}} = e^{-\frac{\tau}{T}}.$$

Since $P_1(\tau) = P_1(-\tau)$, as a result we obtain the correlation function in the form

$$R(\tau) = De^{-\frac{|\tau|}{T}} = 4e^{-0.1|\tau|}.$$

The spectral density is

$$S(\omega) = \int_{-\infty}^{+\infty} R(\tau) e^{-j\omega\tau} d\tau = \frac{2TD}{1+\omega^2 T^2} = \frac{80}{1+100\omega^2}.$$

Fig. 134. Graph of process for Problem 229. 1) sec.

Fig. 135. Pulse train. 1) sec.

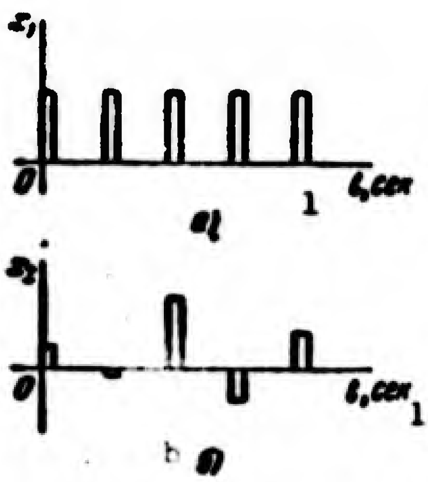


Fig. 136. Components of pulse train. 1) sec.

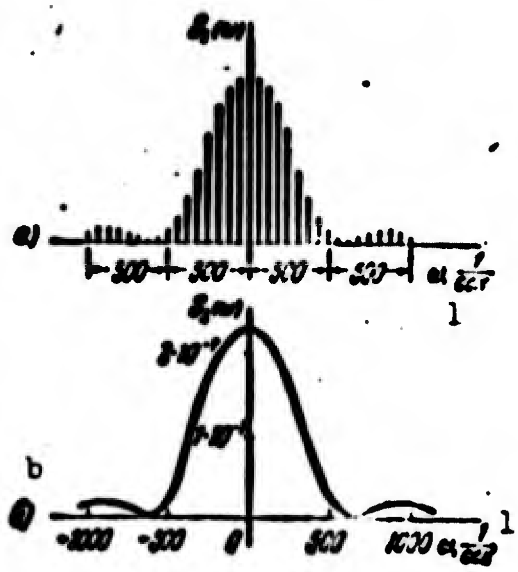


Fig. 137. Spectral-density components for Problem 230. 1) sec.

229. Solve the preceding problem on the assumption that we know that the segments $x > 0$ and $x < 0$ alternate and a change in value will always be accompanied by a change in sign. The graph of such a process

is shown in Fig. 134.

Answer

$$R(\tau)=De^{-2\frac{|\tau|}{T}}=4e^{-0.2|\tau|},$$
$$S(\omega)=\frac{TD}{1+\frac{T^2}{4}\omega^2}=\frac{40}{1+25\omega^2}.$$

230. Determine the spectral density of a train of equidistant positive pulses having the same width and random amplitudes (Fig. 135) under the following initial condition: the pulse spacing is T = 0.1 sec, the pulse width is γT = 0.01 sec, corresponding to a pulse duty cycle γ = 0.1; the mean pulse amplitude $\bar{x}$ = 20; the mean-square value of the pulse amplitude $\sqrt{\bar{x^2}}=x_{ef}=25.$

Solution. We represent the function x(t) as the sum of a periodic component $x_1(t)$ consisting of a sequence of pulses with constant amplitude of $\bar{x}$ (Fig. 136a) and a fluctuating component $x_2(t)$ consisting of a sequence of pulses with random amplitude and mean value of zero (Fig. 136b).

We expand the periodic component into a Fourier exponential series:

$$x_1(t)=\sum_{k=-\infty}^{+\infty} C_k e^{j\frac{2\pi kt}{T}}, \qquad (1)$$

where C_k is a complex number.

The amplitudes of the harmonics

$$A_k=A_{-k}=|C_k|=\left|\frac{\bar{x}}{k\pi}\sin k\pi\gamma\right|, \qquad (2)$$

so that when the initial conditions are substituted, we arrive at the equation

$$A_k=A_{-k}=\left|\frac{6{,}4}{k}\sin 0{,}314\,k\right|.$$

This gives the following values for the amplitudes of the harmonics:

$A_0 = 2,$	$A_6 = 1,$	$A_{12} = 0{,}31,$
$A_1 = 1{,}9,$	$A_7 = 0{,}73,$	$A_{13} = 0{,}39,$
$A_2 = 1{,}86,$	$A_8 = 0{,}46,$	$A_{14} = 0{,}43,$
$A_3 = 1{,}7,$	$A_9 = 0{,}21,$	$A_{15} = 0{,}42,$
$A_4 = 1{,}51,$	$A_{10} = 0,$	$A_{16} = 37$
$A_5 = 1{,}27,$	$A_{11} = 0{,}17,$	and etc.

The spectral density for the periodic component (1) may be written in the form (see Problem 221)

$$S_1(\omega) = 2\pi \frac{A_k^2}{4} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right) \qquad (3)$$

and is a line spectrum. It is shown in Fig. 137a; the area of the impulse function, equal to $2\pi\, A_k^2/4$ is arbitrarily shown as the amplitude of a pulse of finite height.

The value of the amplitudes of the harmonics (2) may also be found from the expression for the envelope of the amplitudes plotted as a function of frequency; this can be obtained from the Fourier transform of a single pulse of height $\bar{h}$ and length γT. The Fourier transform for such a pulse is

$$F_1(j\omega) = \int_0^{\gamma T} \bar{h} e^{-j\omega t}\, dt = \bar{h} \frac{1 - e^{-j\omega\gamma T}}{j\omega}.$$

The modulus of this expression is

$$|F_1(j\omega)| = \left| \frac{\bar{h}\, 2 \sin \frac{\omega\gamma T}{2}}{\omega} \right|. \qquad (4)$$

The amplitude of the kth harmonic may be obtained from (4) for a frequency ω_k by the substitution $\omega = 2\pi k/T$ and division of the resulting value by the pulse spacing T:

$$A_k = \frac{\left| F_1\left(j \frac{2\pi k}{T}\right) \right|}{T} = \left| \frac{\bar{h}}{k\pi} \sin k\pi\gamma \right|.$$

This expression agrees with (2).

The spectral density of a fluctuating component may be found from the general expression for the spectral density of a random quantity

$$S(\omega) = \lim_{T_0 \to \infty} \frac{1}{2T_0} |F(j\omega)|^2,$$

which in the case considered becomes the expression

$$S_2(\omega) = \frac{1}{T} |F_2(j\omega)|^2,$$

where $F_2(j\omega)$ is the Fourier transform of a single pulse whose root-mean-square value is

$$a = \sqrt{\overline{x^2} - \bar{x}^2}.$$

In like manner, Formula (4) may be written

$$|F_2(j\omega)| = \left| \frac{2a \sin \frac{\omega T}{2}}{\omega} \right|. \qquad (5)$$

From this we find the spectral density of the fluctuating component

$$S_2(\omega) = \frac{4a^2 \sin^2 \frac{\omega T}{2}}{T\omega^2}. \qquad (6)$$

Substitution of the numerical values yields

$$S_2(\omega) = \frac{9000 \sin^2 0{,}003\,\omega}{\omega^2}.$$

The spectrum is continuous. It is shown in Fig. 137b. In shape, it resembles the envelope of a line spectrum, since the spectral-density values are also proportional to the square of the modulus of the transform for the single pulse (4).

231. The input signal of a servosystem takes the form of the typical signal for a servosystem (Fig. 133), whose spectral density, written for the input velocity, may be represented in the form

$$S_1(\omega) = \frac{2TD_\Omega}{1 + \omega^2 T^2}. \qquad (1)$$

where $D_\Omega = \Omega^2_{sk}$ is the mean-square velocity. The load torque on the actuating shaft is constant in magnitude ($M = M_n = \text{const}$), while its sign changes together with the change in sign of the actuating-shaft velocity. We assume an approximation that the sign of the torque changes together with the sign of the input velocity; determine the correlation function for the low torque $S_2(\omega)$, as well as the cross-correlation functions for the input velocity and load torque $S_{12}(\omega)$ and $S_{21}(\omega)$. The input-velocity distribution may be assumed to be normal.

Solution. The spectral density of the load torque may be obtained from the velocity spectral density of the input signal (1), if in it we replace the mean-square velocity by the mean-square torque, $M^2 = M_n^2$:

$$S_2(\omega)=\frac{2TM_n^2}{1+\omega^2T^2}.$$

The cross-spectral density may be computed from the cross-correlation function, determined as the time average for ensemble average:

$$R_{12}(\tau)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T}\Omega(t+\tau)M(t)dt=\overline{\Omega(t+\tau)M(t)}.$$

The probability for finding $\Omega(t + \tau)$ and $M(t)$ in the same interval (see Problem 228) is

$$P_1=e^{-\frac{|\tau|}{T}},$$

or the probability for finding them in different intervals is

$$P_2=1-P_1=1-e^{-\frac{|\tau|}{T}}.$$

When the velocity and torque are found in different intervals, the average of their product equals zero.

Where Ω and M are found in the same interval, the sign of the torque equals the sign of the velocity. The product of the velocity by the torque will then always be positive. Here, since the magnitude of the torque is constant, the torque need not remain under the averaging bar:

$$\overline{\Omega(t+\tau)M(t)} = M_a\overline{\Omega(t+\tau)} = M_a\Omega_c,$$

where Ω_s is the average absolute value of the velocity. For a normal distribution

$$\Omega_c = \Omega_{cq}\sqrt{\frac{2}{\pi}} = 0{,}8\,\Omega_{cq}.$$

Thus, we have the cross-correlation function

$$R_{12}(\tau) = M_a\Omega_c P_1 = M_a\Omega_c e^{-\frac{|\tau|}{T}} = 0{,}8\,M_a\Omega_{cq} e^{-\frac{|\tau|}{T}}. \qquad (2)$$

The spectral density is found as the Fourier transform of (2):

$$S_{12}(\omega) = \frac{2TM_a\Omega_c}{1+\omega^2T^2} = \frac{1{,}6TM_a\Omega_{cq}}{1+\omega^2T^2}. \qquad (3)$$

In like manner, we can find that $R_{21}(\tau) = R_{12}(\tau)$ and $S_{21}(\omega) = S_{12}(\omega)$.

§21. TRANSMISSION OF A STATIONARY RANDOM SIGNAL THROUGH A LINEAR SYSTEM

232. A star-tracking system consists of a phototube, instant-response amplifier, filter (a first-order aperiodic element), and an actuating device in the form of a gyroscope for a tachometer drive (ideal integrating element). We assume that the noise at the phototube output is white noise with spectral density $S(\omega) = N$. Show that the mean-square fluctuation error for the system is independent of the filter time constant.

Solution. The system open-loop transfer function has the form

$$W(p) = \frac{K}{p(1+Tp)},$$

where K [1/sec] is the velocity figure of merit and T is the filter time constant.

The closed-loop transfer function is

$$\Phi(p) = \frac{W(p)}{1+W(p)} = \frac{K}{Tp^2+p+K}.$$

The error spectral density is

$$S_\varepsilon(\omega)=|\Phi(j\omega)|^2 S(\omega)=\frac{K^2N}{|T(j\omega)^2+j\omega+K|^2}.$$

Integration of the error spectral density over all frequencies (see Appendix 15) yields the mean-square error

$$D=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{K^2N\,d\omega}{|T(j\omega)^2+j\omega+K|^2}=\frac{KN}{2}=\Delta fN,$$

where the equivalent white-noise passband is

$$\Delta f=\frac{K}{2}\,[\text{cps}].$$

As we can see from the resulting expressions, the mean-square error does not depend on the filter time constant.

233. For a star-tracking system (see the preceding problem), the mean-square noise voltage of the phototube is U_{sk} = 6 v in the frequency band Δf = 10,000 cps ($\pm$5000 cps). The slope of the phototube characteristic curve is k_{fe} = 10 mv/ang.min. Determine the permissible over-all gain (velocity figure of merit K) for which the mean-square fluctuation error will not exceed one minute of arc.

<u>Solution</u>. We calculate the phototube noise voltage in terms of the equivalent mean-square angle signal at the input:

$$\Theta_{sk}=\frac{U_{sk}}{k_{fe}}=\frac{6}{10\cdot 10^{-3}}=600'.$$

The white-noise level at the input is

$$S_0(\omega)=N'=\frac{\Theta_{sk}^2}{\Delta f}=\frac{600'^2}{10\,000}=36\ (\text{ang.min})^2/\text{cps}.$$

In Problem 232 it was found that the root-mean-square error is

$$\varepsilon=\sqrt{\frac{KN}{2}}.$$

From this we find the value of the over-all gain:

$$K<\frac{2\varepsilon^2}{N}=\frac{2\cdot 1}{36}=0{,}055\ \ 1/\text{sec}.$$

234. We are given the open-loop transfer functions of a control system with first-order astatism:

1) $W(p)=\frac{K}{p}$;

2) $W(p)=\frac{K}{p(1+T_1p)}$;

3) $W(p)=\frac{K}{p(1+T_1p)(1+T_2p)}$.

Calculate the equivalent white-noise passband of the closed-loop system if the velocity figure of merit is K = 10 1/sec and the time constants are T_1 = 0.1 sec and T_2 = 0.05 sec.

Answer

1) $\Delta f=\frac{K}{2}=\frac{10}{2}=5$ cps;

2) $\Delta f=\frac{K}{2}=\frac{10}{2}=5$ cps;

3) $\Delta f=\frac{K}{2\left(1-\frac{KT_1T_2}{T_1+T_2}\right)}=\frac{10}{2\left(1-\frac{10\cdot 0{,}1\cdot 0{,}05}{0{,}1+0{,}05}\right)}=7{,}5$ cps.

235. We are given the open-loop transfer function of a control system with second-order astatism:

1) $W(p)=\frac{K(1+\tau p)}{p^2}$;

2) $W(p)=\frac{K(1+\tau p)}{p^2(1+Tp)}$.

Calculate the equivalent closed-loop white-noise passband of the system if the acceleration figure of merit K = 10 1/sec^2, and the time constants are τ = 1 sec and T = 0.5 sec.

Answer

1) $\Delta f=\frac{1+K\tau^2}{2\tau}=\frac{1+10\cdot 1^2}{2\cdot 1}=5{,}5$ cps;

2) $\Delta f=\frac{1+K\tau^2}{2(\tau-T)}=\frac{1+10\cdot 1^2}{2(1-0{,}5)}=11$ cps.

236. Noise with spectral density

$$S_n(\omega)=\frac{2T_n\sigma_n^2}{1+\omega^2T_n^2}.$$

appears at the control-system input. Determine the system smoothing coefficient, which equals the ratio of the mean-square noise at the input to the mean-square error:

$$K_{eq}=\frac{\varepsilon_0}{\varepsilon}.$$

and the mean-square error ε. The system open-loop transfer function is

$$W(p)=\frac{K}{p}.$$

The numerical values of the coefficients are:

$$K=0{,}5\ 1/\text{sec},\quad \varepsilon_0=10,\quad T_0=0{,}1\ \text{sec}.$$

Answer

$$K_{eq}=\sqrt{1+\frac{1}{KT_0}}=\sqrt{1+\frac{1}{0{,}5\cdot 0{,}1}}=\sqrt{21}=4{,}6,$$
$$\varepsilon=\frac{\varepsilon_0}{K_{eq}}=\frac{10}{4{,}6}=2{,}18.$$

237. Solve the preceding problem; here we assume that the open-loop transfer function of the system is

$$W(p)=\frac{K}{p(1+T_1 p)},$$

where T_1 = 1 sec.

Answer

$$K_{eq}=\sqrt{\frac{T_0}{T_1+T_0}+\frac{1}{KT_0}}=\sqrt{\frac{0{,}1}{1+0{,}1}+\frac{1}{0{,}5\cdot 0{,}1}}=$$
$$=\sqrt{20{,}1}=4{,}5,$$
$$\varepsilon=\frac{\varepsilon_0}{K_{eq}}=\frac{10}{4{,}5}=2{,}22.$$

238. A useful signal whose velocity varies in accordance with Fig. 133 appears at the input of a servosystem. The spectral density, in terms of velocity, has the form

$$S_\Omega(\omega)=\frac{2TD_\Omega}{1+\omega^2T^2},$$

where $D_\Omega = \Omega_{sk}$ is the velocity variance. The mean-square velocity Ω_{sk} = = 2 degrees/sec. The mean length of one time interval T = 1 sec. Find the mean-square error if the system open-loop transfer function has the form

$$W(p)=\frac{K}{p(1+T_1 p)}.$$

The velocity figure of merit is K = 25 1/sec, and the time constant is T = 0.05 sec.

<u>Solution</u>. The error transfer function is

$$\Phi_x(p) = \frac{1}{1+W(p)} = \frac{p(1+Tp)}{Tp^2+p+K}.$$

The error spectral density is

$$S_x(\omega) = |\Phi_x(j\omega)|^2 \frac{S_g(\omega)}{\omega^2} = \frac{2T_1D_g(1+\omega^2T^2)}{(1+\omega^2T_1^2)|T(j\omega)^2+j\omega+K|^2}.$$

We reduce it to a form convenient for integration (see Appendix 15):

$$S_x(\omega) = 2T_1D_g \frac{-T^2(j\omega)^2+1}{|TT_1(j\omega)^3+(T+T_1)(j\omega)^2+(1+KT_1)j\omega+K|^2}.$$

Integration over all frequencies gives the mean-square error

$$\overline{\varepsilon^2} = 2T_1D_gI_3,$$

where the integral

$$I_3 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{[-T^2(j\omega)^2+1]\,d\omega}{|TT_1(j\omega)^3+(T+T_1)(j\omega)^2+(1+KT_1)j\omega+K|^2}$$

equals, in accordance with Appendix 15,

$$I_3 = \frac{-a_2b_0+a_0b_1-\frac{a_0a_1b_2}{a_3}}{2a_0(a_0a_3-a_1a_2)}.$$

The values of the coefficients are:

$$a_0 = TT_1 \qquad b_0 = 0,$$
$$a_1 = T+T_1 \quad b_1 = -T^2,$$
$$a_2 = 1+KT_1 \quad b_2 = 1.$$
$$a_3 = K.$$

As a result, we have

$$I_3 = \frac{b_1-\frac{a_1b_2}{a_3}}{2(a_0a_3-a_1a_2)} = \frac{T+T_1+KT^2}{2K(T+T_1+KT_1^2)}.$$

Finally,

$$\sigma_x = \sqrt{\frac{T_1D_g(T+T_1+KT^2)}{K(T+T_1+KT_1^2)}} =$$
$$= \sqrt{\frac{1\cdot 4(0{,}05+1+25\cdot 0{,}05^2)}{25(1+0{,}05+25\cdot 1^2)}} = \sqrt{0{,}0068} = 0{,}082^\circ \approx 5'.$$

The approximate expression for the mean-square error is:

$$\sigma_{ош}=\sqrt{\frac{TDT}{KK_1T}}=\frac{\sigma_{ош}}{K}=\frac{2}{25}=0{,}08^\circ\approx 4{,}8'.$$

239. Noise of the irregular swinging type appears with a correlation function

$$R_0(\tau)=D_0e^{-\mu|\tau|}\left(\cos\beta\tau+\frac{\mu}{\beta}\sin\beta|\tau|\right)$$

and a spectral density

$$S_0(\omega)=D_0\frac{\mu}{\beta}\left[\frac{2\beta-\omega}{\mu^2+(\omega-\beta)^2}+\frac{2\beta+\omega}{\mu^2+(\omega+\beta)^2}\right].$$

The numerical values of the coefficients are:

$$D_0=\sigma_0^2=100,\quad \mu=0{,}4 \text{ 1/sec and } \beta=5 \text{ 1/sec}.$$

Determine the smoothing coefficient, which equals the ratio of the mean-square input noise to the mean-square error at the system output:

$$K_{сгл}=\frac{\sigma_0}{\sigma},$$

and the mean-square error σ. The system open-loop transfer function is

$$W(p)=\frac{K}{p},$$

where the system velocity figure of merit is K = 0.1 1/sec.

<u>Answer</u>. The smoothing coefficient is

$$K_{сгл}=\frac{\beta^2+\mu^2}{\beta K\sqrt{1+\frac{2\mu(\beta^2+\mu^2)}{\beta^2K}-\frac{2\mu^2}{\beta^2}}}=$$

$$=\frac{5^2+0{,}4^2}{5\cdot0{,}1\sqrt{1+\frac{2\cdot0{,}4(5^2+0{,}4^2)}{5^2\cdot0{,}1}-\frac{2\cdot0{,}4^2}{5^2}}}=16{,}7.$$

The mean-square error is

$$\sigma=\frac{\sigma_0}{K_{сгл}}=\frac{10}{16{,}7}=0{,}6.$$

240. To approximate irregular swinging, two formulas for the correlation function are used:

$$R(\tau)=De^{-\mu|\tau|}\cos\beta\tau, \qquad (1)$$

$$R(\tau)=De^{-\mu|\tau|}\left(\cos\beta\tau+\frac{\mu}{\beta}\sin\beta|\tau|\right). \qquad (2)$$

These correlation functions correspond to the following spectral densities:

for Formula (1)

$$S(\omega)=\mu D\left[\frac{1}{\mu^2+(\omega-\beta)^2}+\frac{1}{\mu^2+(\omega+\beta)^2}\right]. \qquad (3)$$

for Formula (2)

$$S(\omega)=\frac{\mu}{\beta}D\left[\frac{2\beta-\omega}{\mu^2+(\omega-\beta)^2}+\frac{2\beta+\omega}{\mu^2+(\omega+\beta)^2}\right]. \qquad (4)$$

Determine the variance for the rate of irregular swinging for Formulas (1) and (2).

Answer: 1) $D_{\dot{\omega}}\to\infty$; 2) $D_{\dot{\omega}}=(\mu^2+\beta^2)D$.

§22. CALCULATIONS FOR MINIMUM MEAN-SQUARE ERROR

241. The open-loop transfer function of a control system has the form

$$W(p)=\frac{K(1+\tau p)}{p^2},$$

where K = 100 1/sec^2 is the over-all open-loop gain, and τ is the time constant of the compensating device. A useful control signal of the form $Y = at + bt^2/2$ acts at the system input; where a = 100 degree/sec and b = 10 degree/sec^2, and the noise is white noise with a spectral density $S_p(\omega) = N = 0.2$ degree^2/cps. Determine the compensating-device constant corresponding to minimum steady-state mean-square error, as well as the mean-square error.

Solution. The steady-state error for the useful signal is

$$x_s=C_1\dot{Y}+\frac{C_2}{2}\ddot{Y}=C_1(a+bt)+\frac{C_2}{2}b,$$

where C_1 and C_2 are error coefficients. On the basis of the expansion of the error transfer function

$$\Phi_x(p)=\frac{1}{1+W(p)}=\frac{p^2}{p^2+K\tau p+K}$$

into a power series, we have $D_1 = 0$ and $C_2/2 = 1/K$. As a result, the regular error component is

$$x_0 = \frac{\dot{v}}{K};$$

or

$$x_0^2 = \frac{\dot{v}^2}{K^2}. \qquad (1)$$

The mean-square fluctuation error (see Appendix 15) is

$$\overline{x_2^2} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\Phi(j\omega)|^2 N\, d\omega =$$
$$= \frac{K^2 N}{2\pi}\int_{-\infty}^{+\infty} \frac{|\tau(j\omega)+1|^2 d\omega}{|(j\omega)^2 + K\tau j\omega + K|^2} = \frac{(1+K\tau^2)N}{2\tau}. \qquad (2)$$

Mean-square resultant error is

$$\overline{x^2} = x_0^2 + \overline{x_2^2} = \frac{\dot{v}^2}{K^2} + \frac{(1+K\tau^2)N}{2\tau}. \qquad (3)$$

In order to find the minimum of the last expression, we set the first derivative with respect to the time constant of the compensating device equal to zero:

$$2K\tau^2 - (1 + K\tau^2) = 0,$$

from which we obtain

$$\tau = \frac{1}{\sqrt{K}} = \frac{1}{\sqrt{100}} = 0{,}1 \text{ sec.}$$

The mean-square error is found from (3):

$$x_{ck} = \sqrt{\frac{10^2}{100^2} + \frac{1 + 100 \cdot 0{,}1^2}{2 \cdot 0{,}1} \cdot 0{,}2} = 1{,}41^\circ.$$

242. Solve the preceding problem if the system open-loop transfer function has the form

$$W(p) = \frac{K(1+\tau p)}{p^2(1+Tp)},$$

where $K = 100\ 1/\text{sec}^2$ and $T = 0.05$ sec.

Answer

$$\tau = T + \sqrt{T^2 + \frac{1}{K}} = 0{,}05 + \sqrt{0{,}05^2 + 0{,}01} = 0{,}16 \text{ sec.}$$

The mean-square error is

$$x'_{ms} = \sqrt{\frac{F}{K^2} + \frac{(1+K\tau^2)N}{2(\tau - T_1)}} =$$
$$= \sqrt{\frac{10^2}{100^2} + \frac{(1+100 \cdot 0{,}16^2)\,0{,}2}{2(0{,}16-0{,}05)}} = 1{,}36°.$$

243. Solve Problem 241 on the assumption that it is possible to vary both the time constant of the compensating device τ and the over-all gain K.

Solution. Differentiating Expression (3) in Problem 241 with respect to τ and with respect to K and setting the partial derivatives equal to zero, we obtain two minimum conditions:

$$\tau = \frac{1}{\sqrt{K}}, \qquad (1)$$

$$-\frac{2F}{K^3} + \frac{\tau N}{2} = 0. \qquad (2)$$

Substituting (1) into (2) and solving the latter equation, we have

$$K_{opt} = \sqrt[5]{\frac{16F^2}{N^2}} = \sqrt[5]{\frac{16 \cdot 10^4}{0{,}2^2}} = 21 \text{ 1/sec.}$$

The time constant of the compensating network is

$$\tau = \frac{1}{\sqrt{K}} = \frac{1}{\sqrt{21}} = 0{,}218 \text{ sec.}$$

The mean-square error is found from (3) of Problem 241:

$$x_{ms} = \sqrt{\frac{10^2}{21^2} + \frac{(1+21 \cdot 0{,}218^2)\,0{,}2}{2 \cdot 0{,}218}} = 1{,}07°.$$

244. The open-loop transfer function of a control system has the form

$$W(p) = \frac{K}{p(1+T_1 p)},$$

where K is the over-all gain and T_1 is the time constant. The system closed-loop transfer function is

$$\Phi(p) = \frac{W(p)}{1+W(p)} = \frac{K}{T_1 p^2 + p + K}.$$

White noise with a spectral density $S_p(\omega) = N$ appears at the system input together with a useful signal having a spectral density

$$S_s(\omega)=\frac{2T_sD}{1+\omega^2T_s^2}.$$

There is no correlation between the noise and signal. The initial conditions are T_1 = 0.1 sec, T_s = 20 sec, D = 100 degree^2, and N = 0.01 degree^2/cps. Find the optimum value of over-all gain K_{opt} corresponding to minimum mean-square error, and the mean-square error for $K = K_{opt}$.

Solution. The mean-square-error component due to the noise (see Appendix 15) is

$$\overline{\varepsilon_n^2}=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{K^2N\,d\omega}{|T_1(j\omega)^2+j\omega+K|^2}=\frac{KN}{2}. \qquad (1)$$

The mean-square-error component due to the useful signal at the input (see Appendix 15) is

$$\overline{\varepsilon_s^2}=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\omega^2(1+\omega^2T_1^2)}{|T_1(j\omega)^2+j\omega+K|^2}\,\frac{2T_sD}{1+\omega^2T_s^2}\,d\omega=2T_sD\times$$
$$\times\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{[T_1^2(j\omega)^4-(j\omega)^2]\,d\omega}{|T_1T_s(j\omega)^3+(T_1+T_s)(j\omega)^2+(1+KT_s)j\omega+K|^2}=$$
$$=D\frac{T_1+T_s+KT_1T_s}{T_1+T_s+KT_s^2}. \qquad (2)$$

The resultant mean-square error is

$$\overline{\varepsilon^2}=\overline{\varepsilon_n^2}+\overline{\varepsilon_s^2}=\frac{KN}{2}+D\frac{T_1+T_s+KT_1T_s}{T_1+T_s+KT_s^2}. \qquad (3)$$

In order to study the minimum for the mean-square error it is necessary to set the derivative of the last expression with respect to the gain equal to zero. As a result we obtain

$$\frac{N}{2}-\frac{DT_s(T_s^2-T_1^2)}{(T_1+T_s+KT_s^2)^2}=0.$$

The solution of the last equation yields the optimum gain:

$$K_{opt}=\sqrt{\frac{2D(T_s^2-T_1^2)}{NT_s^3}}-\frac{T_s+T_1}{T_s^2}. \qquad (4)$$

We find the numerical value of the optimum gain:

$$K_{opt}=\sqrt{\frac{2\cdot100\,(20^2-0.1^2)}{0.01\cdot20^3}}-\frac{20+0.1}{20^2}\approx30\ 1/\text{sec}.$$

From (3), the mean-square error is

$$\varepsilon_m=\sqrt{\frac{20\cdot 0.01}{2}+100\,\frac{0.1+20+30\cdot 0.1\cdot 20}{0.1+20+30\cdot 20^2}}=0.9^\circ.$$

245. For the last problem, determine the control-system tranfer function corresponding to the theoretical minimum mean-square error, and determine the value of the latter.

<u>Solution</u>. Provided the control system is physically feasible, the unknown frequency transfer function for the closed-loop system may be represented in the form [25]

$$\Phi(j\omega)=\frac{B(j\omega)}{\Psi(j\omega)}. \tag{1}$$

The denominator of (1) is found from the equation

$$\Psi(j\omega)\Psi^*(j\omega)=S_s(\omega)+S_n(\omega), \tag{2}$$

where $\Psi^*(j\omega)$ is the complex conjugate of $\Psi(j\omega)$. For our case,

$$S_s(\omega)+S_n(\omega)=\frac{2T_cD}{1+\omega^2T_c^2}+N=\frac{2T_cD+N(1+\omega^2T_c^2)}{1+\omega^2T_c^2}.$$

We expand the last expression in complex conjugate factors:

$$\frac{2T_cD+N(1+\omega^2T_c^2)}{1+\omega^2T_c^2}=A\,\frac{(1+j\alpha\omega)(1-j\alpha\omega)}{(1+jT_c\omega)(1-jT_c\omega)}.$$

From this we obtain

$$\Psi(j\omega)=\sqrt{A}\,\frac{1+j\alpha\omega}{1+jT_c\omega}, \tag{3}$$

$$\Psi^*(j\omega)=\sqrt{A}\,\frac{1-j\alpha\omega}{1-jT_c\omega}, \tag{4}$$

where

$$A=2T_cD+N,$$

$$\alpha^2=\frac{NT_c^2}{2T_cD+N}=\frac{NT_c^2}{A}.$$

We next obtain the expression

$$\frac{S_s(\omega)}{\Psi^*(\omega)}=\frac{2T_cD(1-jT_c\omega)}{(1+\omega^2T_c^2)\sqrt{A}(1-j\alpha\omega)}=\frac{2T_cD}{\sqrt{A}}\,\frac{1}{(1+jT_c\omega)(1-j\alpha\omega)}.$$

We expand the last expression into simple fractions:

$$\frac{S_1(\omega)}{\Psi^*(j\omega)}=\frac{2T_cD}{\sqrt{A}}\left[\frac{T_c}{T_c+a}\frac{1}{1+jT_c\omega}+\frac{a}{T_c+a}\frac{1}{1-ja\omega}\right].$$

The function B(jω) is determined by the terms of the expansion in simple fractions corresponding to the poles $S_s(\omega)$ lying in the upper half plane. As a result we have

$$B(j\omega)=\frac{2T_cD}{\sqrt{A}}\frac{T_c}{T_c+a}\frac{1}{1+jT_c\omega}. \tag{5}$$

The desired closed-loop frequency transfer function for the system (1) is

$$\Phi(j\omega)=\frac{B(j\omega)}{\Psi(j\omega)}=\frac{2T_cD}{A}\frac{T_c}{T_c+a}\frac{1}{1+ja\omega}. \tag{6}$$

We find the numerical values of the coefficients:

$$a=T_c\sqrt{\frac{N}{2T_cD+N}}=20\sqrt{\frac{0{,}01}{2\cdot 20\cdot 100+0{,}01}}=0{,}032 \text{ sec},$$
$$\frac{2T_c^2D}{A(T_c+a)}=\frac{2T_c^2D}{(2T_cD+N)(T_c+a)}\approx\frac{2T_c^2D}{2T_c^2D}=1.$$

The final expression for the system closed-loop transfer function will be

$$\Phi(p)=\frac{1}{1+Tp}, \tag{7}$$

where T = 0.032 sec. This transfer function corresponds to the open-loop transfer function

$$W(p)=\frac{\Phi(p)}{1-\Phi(p)}=\frac{1}{Tp}=\frac{K}{p}, \tag{8}$$

where K = 1/T = 31 1/sec is the over-all open-loop gain (velocity figure of merit).

The error transfer function is

$$\Phi_1(p)=1-\Phi(p)=\frac{Tp}{1+Tp}. \tag{9}$$

The error spectral density is

$$S_1(\omega)=|\Phi_1(j\omega)|^2S_c(\omega)+|\Phi(j\omega)|^2S_n(\omega)=$$
$$=\frac{T^2\omega^2}{1+\omega^2T^2}\frac{2T_cD}{1+\omega^2T_c^2}+\frac{1}{1+\omega^2T^2}N. \tag{10}$$

Integrating (10) over all frequencies, we obtain the mean-square error

$$\overline{\theta^2} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_\theta(\omega)\,d\omega = \frac{TD}{T_c+T} + \frac{N}{2T}. \qquad (11)$$

The mean-square error is

$$\theta_{sk} = \sqrt{\frac{TD}{T_c+T} + \frac{N}{2T}} = \sqrt{\frac{0{,}032\cdot 100}{20+0{,}032} + \frac{0{,}01}{2\cdot 0{,}032}} = 0{,}56^\circ.$$

Manu-script Page No.	[Transliterated Symbols]
169	п = p = propuskaniye = pass
169	ск = sk = srednekvadratichnyy = root-mean-square
177	н = n = nagruzka = load
179	с = s = sredniy = average or signal = signal or sistema = = system
180	э = e = equivalentnyy = equivalent
180	ф = f = fotoelement = phototube
181	п = p = pomekha = noise
182	сгл = sgl = sglazhivaniye = smoothing
187	опт = opt = optimal'nyy = optimum

Chapter 8

SPECIAL LINEAR SYSTEMS

§23. CONSTANT-LAG SYSTEMS

246. The open-loop transfer function of a control system is

$$W_0(p) = \frac{K}{p},$$

where K = 10 1/sec is the over-all open-loop gain. A dead-time element is connected in series into the control channel; it has a transfer function of the form $e^{-\tau p}$, where τ is the lag. Determine the critical lag τ_k corresponding to the system closed-loop stability limit.

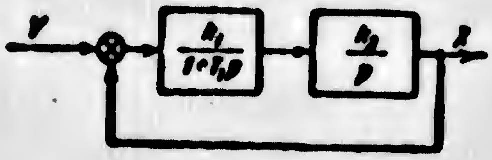


Fig. 138. Block diagram for Problem 247.

Solution. We find the system open-loop frequency transfer function:

$$W_0(j\omega) = \frac{K}{j\omega}. \qquad (1)$$

The cutoff frequency for which (1) will have a modulus of unity will be $\omega_{sr} = K$. The phase shift at this frequency is $\psi = -\pi/2$. The phase stability margin is $\eta = \pi + \psi = +\pi/2$. From this we find the critical lag

$$\tau_k = \frac{\eta}{\omega_{sr}} = \frac{\pi}{2}\frac{1}{\omega_{sr}} = \frac{1,57}{10} = 0,157 \text{ sec.}$$

247. Solve the preceding problem for the control system whose block diagram is shown in Fig. 138.

The values of the coefficients are

$$k_1 k_2 = K = 10 \text{ 1/sec and } T_1 = 0,2 \text{ sec.}$$

Answer

$$\tau_0 = \frac{\frac{\pi}{2} - \operatorname{arctg} \frac{1}{\sqrt{2}}\sqrt{\sqrt{1+4K^2T_1^2}-1}}{\frac{1}{\sqrt{2}}\sqrt{\sqrt{1+4K^2T_1^2}-1}} T_1 = 0.107 \text{ sec.}$$

Fig. 139. Gain-phase characteristic for open-loop system.
1) sec.

Fig. 140. Logarithmic amplitude characteristic and logarithmic phase characteristic for Problem 249.
1) sec.

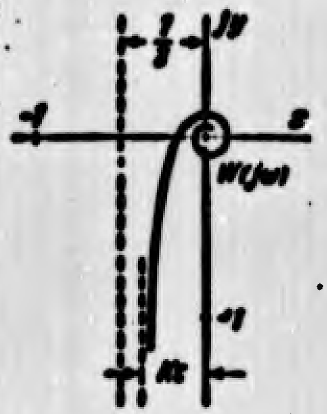


Fig. 141. Gain-phase characteristic for Problem 250.

248. Figure 139 shows the gain-phase characteristic of an open-loop system. Determine the critical time lag corresponding to the limit of stability if the dead-time element is connected in series into the control channel and has a transfer function $e^{-\tau p}$.

Solution. On the complex plane, from the origin we draw a circle with radius unity. The circle is shown in Fig. 139 by the dashed line.

At the point at which the circle intersects the gain-phase characteristic, we determine the cutoff frequency: ω_{sr} = 12 1/sec and the phase stability margin: $\eta = 35^{\circ} = 0.61$ rad. From this we find the critical time:

$$\tau_k = \frac{1}{\omega_{cp}} = \frac{0{,}61}{12} = 0{,}051 \text{ sec.}$$

249. Solve the preceding problem for the logarithmic amplitude characteristic and logarithmic phase characteristic shown in Fig. 140.

Answer

$$\tau_k = \frac{1}{\omega_{cp}} = \frac{0{,}88}{50} = 0{,}0176 \text{ sec.}$$

250. The open-loop transfer function of a control system has the form

$$W_0(p) = \frac{K}{p},$$

where K = 20 1/sec is the over-all gain. A dead-time element with transfer function $e^{-\tau p}$ is connected in series into the control channel. Find the maximum permissible lag τ for which the closed-loop magnitude ratio will not exceed M = 1, as well as the critical value τ_k corresponding to the stability limit.

Solution. The resultant open-loop transfer function is

$$W(p) = W_0(p)e^{-\tau p} = \frac{Ke^{-\tau p}}{p}. \qquad (1)$$

The frequency transfer function is

$$W(j\omega) = \frac{Ke^{-j\omega\tau}}{j\omega} = \frac{K(\cos\omega\tau - j\sin\omega\tau)}{j\omega}. \qquad (2)$$

In order to have M = 1 it is necessary for the open-loop gain-phase characteristic constructed from Expression (2) not go outside the line x = − 1/2 corresponding to the forbidden zone for M = 1 (Fig. 141).

The transfer function (2) is represented as the sum of the real and imaginary parts:

$$W(j\omega) = x + jy = -\frac{K\sin\omega\tau}{\omega} - j\frac{K\cos\omega\tau}{\omega}. \qquad (3)$$

The maximum value of the real part in (3) will occur when $\omega \rightarrow 0$:

$$x_{max} = \lim_{\omega \to 0}\left[-\frac{K \sin \omega\tau}{\omega}\right] = -K\tau.$$

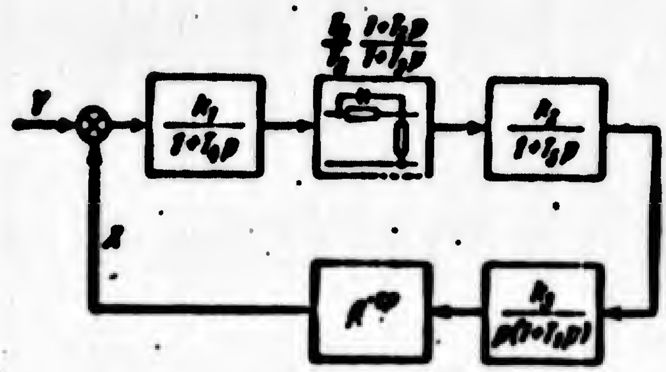


Fig. 142. Block diagram for Problem 251.

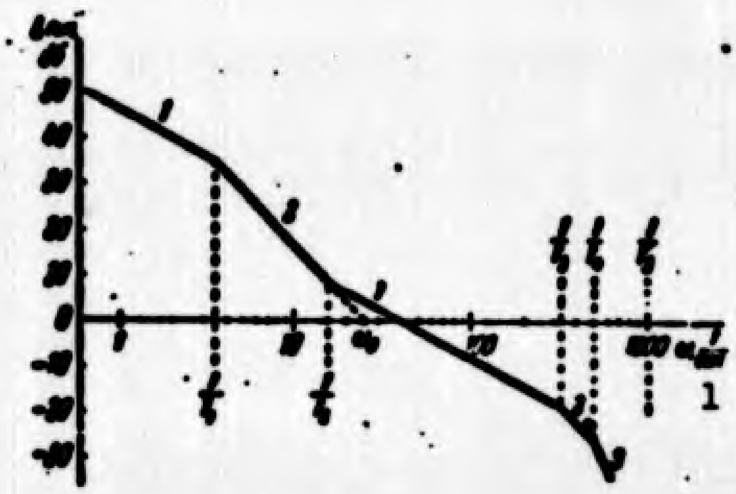


Fig. 143. Logarithmic amplitude characteristic for Problem 251. 1) sec.

The condition that must be satisfied if the forbidden zone is determined by the line x = – 1/2 is to remain inviolate is

$$K\tau < \frac{1}{2}.$$

From this we obtain permissible time lag:

$$\tau < \frac{1}{2K} = \frac{1}{2 \cdot 20} = 0{,}025 \text{ sec.}$$

The critical lag, in accordance with the solution to Problem 246, will be

$$\tau_k = \frac{1{,}57}{20} = 0{,}078 \text{ sec.}$$

251. For the control system whose block diagram is shown in Fig. 142, determine the parameters of the passive differentiating network T_2 and T_3 that correspond to a system closed-loop magnitude ratio $M = 1.3$. The initial conditions are as follows: over-all gain $K = 200$ 1/sec, $T_1 = 0.3$ sec, $T_4 = 0.001$ sec, $T_5 = 0.002$ sec and the dead time is $\tau = 0.003$ sec.

Solution. The open-loop transfer function is

$$W(p)=\frac{K(1+T_2p)e^{-p\tau}}{p(1+T_1p)(1+T_3p)(1+T_4p)(1+T_5p)}.$$

The asymptotic logarithmic amplitude characteristic is shown in Fig. 143.

The value of the logarithmic amplitude characteristic base frequency [3] is

$$\omega_0=\sqrt{\frac{K}{T_1}}=\sqrt{\frac{200}{0,3}}=26 \text{ 1/sec.}$$

If a given magnitude ratio is to be obtained, the algebraic sum of the conjugating frequencies to the left of the cutoff frequency must satisfy a particular condition, which in the given case reduces to the inequality

$$\frac{1}{T_2}-\frac{1}{T_1}<\omega_0\sqrt{\frac{M-1}{M}}=26\sqrt{\frac{1,3-1}{1,3}}=12,4 \text{ 1/sec.}$$

From this we find the constant T_2 of the differentiating network:

$$T_2=\frac{1}{12,4+\frac{1}{T_1}}=\frac{1}{12,4+\frac{1}{0,3}}=0,064 \text{ sec.}$$

In addition, in the high-frequency region (to the right of the cutoff frequency), the condition

$$T_3+T_4+T_5+\tau<\frac{1}{\omega_0}\,\frac{\sqrt{M(M-1)}}{M+1}=\frac{1}{26}\,\frac{\sqrt{1,3(1,3-1)}}{1,3+1}=0,0092 \text{ sec.}$$

should be satisfied.

From this we obtain the required time constant T_1 of the passive differentiating network:

$$T_1 = 0.0092 - (T_2 + T_3 + \tau) = 0.0092 - (0.001 + 0.002 + 0.003) = 0.0032 \text{ sec.}$$

Thus, the transfer function of the differentiating network should have the form

$$W_{d.n}(p) = \frac{T_1}{T_0}\frac{1+T_0 p}{1+T_1 p} = \frac{1}{20}\frac{1+0.064p}{1+0.0032p}.$$

252. Construct the transient response of the control system considered in Problem 250 when a unit step function Y (t) = 1(t) is applied to the system as a control input. The values of the coefficients are: over-all gain K = 20 1/sec and lag τ = 0.025 sec.

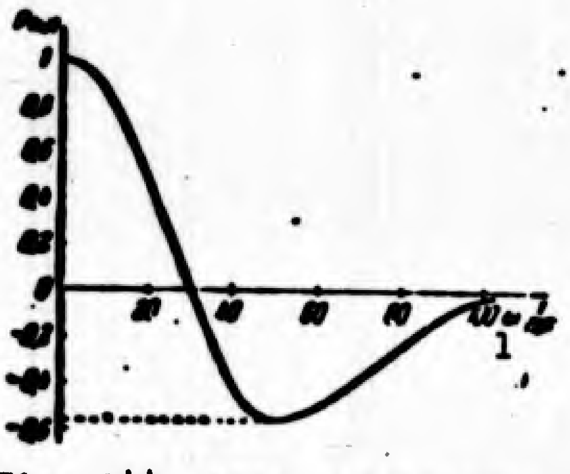


Fig. 144. Real characteristic for Problem 252. 1) sec.

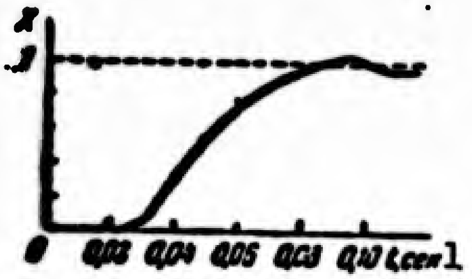


Fig. 145. Transient for Problem 252. 1) sec.

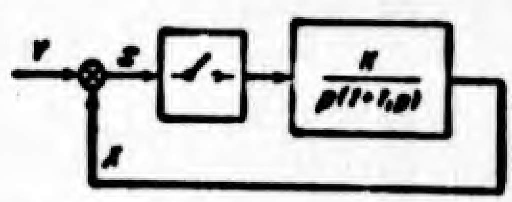


Fig. 146. Block diagram of sampled data system.

Solution. The closed-loop transfer function is

$$\Phi(p) = \frac{W(p)}{1+W(p)} = \frac{Ke^{-\tau p}}{p+Ke^{-\tau p}}.$$

The frequency transfer function is

$$\Phi(j\omega)=\frac{K(\cos\omega\tau-j\sin\omega\tau)}{j\omega+K(\cos\omega\tau-j\sin\omega\tau)}.$$

The real characteristic is

$$P(\omega)=\frac{K^2-K\omega\sin\omega\tau}{K^2+\omega^2-2K\omega\sin\omega\tau}.$$

Substitution of the numerical values yields

$$P(\omega)=\frac{400-20\omega\sin 0{,}025\omega}{400+\omega^2-40\omega\sin 0{,}025\omega}.$$

The real characteristic is plotted in Fig. 144. From the real frequency characteristic, we plot the transient response (Fig. 145) by the trapezoid (h-function) method.

§24. DISCRETE SYSTEMS

253. Determine the open-loop and closed-loop z-transfer function for the sampled-data control system whose block diagram is shown in Fig. 146. We assume that the sampler generates square pulses with duty cycle γ = 0.01. The numerical values of the coefficients are: over-all gain of continuous part K = 100 1/sec, time constant T_1 = 0.25 sec, and pulse spacing of sampler T_0 = 1 sec.

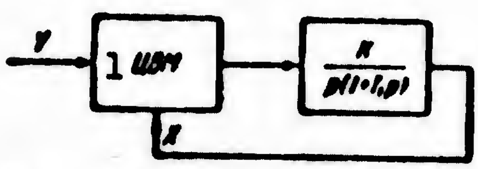


Fig. 147. Block diagram of control system using digital computer.

Solution. The system open-loop transfer function equals the product of the sampler transfer function and the pulse-filter transfer function (the continuous section at whose input the discrete function generated by the sampler appears).

The sampler transfer function may be taken to equal its duty cycle:

$$w_1(z)=\gamma=0{,}01. \qquad (1)$$

The pulse-filter transfer function is found from the following general rule:

$$w_1(z) = T_0 \sum_{k=0}^{\infty} w(kT_0) z^{-k} = T_0 F(z). \qquad (1)$$

where $w(kT_0)$ is the weighting function of the continuous part at the discrete times ($k = 0, 1, 2, \ldots$) while $F(z)$ is the z-transform of the waiting function.

The transfer function of the continuous part is

$$w_0(p) = \frac{K}{p(1+T_1 p)} = \frac{K}{p} - \frac{KT_1}{1+T_1 p}. \qquad (3)$$

It corresponds to the waiting function

$$w_0(t) = K\left(1 - e^{-\frac{t}{T_1}}\right). \qquad (4)$$

This time function corresponds to the z-transform (See Appendix 13)

$$F(z) = K\left(\frac{z}{z-1} - \frac{z}{z-d}\right) = \frac{K(1-d)z}{(z-1)(z-d)}, \qquad (5)$$

where

$$d = e^{-\frac{T_0}{T_1}} = e^{-4} = 0{,}018.$$

In accordance with (2) the t ansfer function of the pulse filter is

$$w_1(z) = \frac{KT_0(1-d)z}{(z-1)(z-d)}. \qquad (6)$$

The open-loop transfer function is

$$W(z) = w_1(z) w_2(z) = \frac{\gamma K T_0 (1-d) z}{(z-1)(z-d)}. \qquad (7)$$

Substitution of the numerical values yields

$$W(z) = \frac{0{,}982z}{(z-1)(z-0{,}018)}.$$

The closed-loop transfer function is

$$\Phi(z) = \frac{W(z)}{1+W(z)} = \frac{\gamma K T_0 (1-d) z}{z^2 - [1 + d - \gamma K T_0 (1-d)] z + d}.$$

Substitution of the numerical values yields

$$\Phi(z) = \frac{0{,}982z}{z^2 - 0{,}036z + 0{,}018}.$$

254. Investigate the stability of the control system considered in the previous example.

Solution. The system characteristic equation is

$$z^2-0{,}036z+0{,}018=0.$$

The stability condition is

$$|z|<1,$$

i.e., the roots of the characteristic equation should be less than unity in absolute value. In a second-order characteristic equation

$$z^2+Az+B=0$$

this will occur under three conditions:

$$1+A+B>0,\quad 1-A+B>0,\quad B<1.$$

For the equation under investigation, we obtain three conditions:

$$1-0{,}036+0{,}018>0,\quad 1+0{,}036+0{,}018>0,$$
$$0{,}018<1.$$

All three conditions are satisfied; as a consequence, the system is stable.

255. A control system has a digital computer (TsVM) in its circuit. The system block diagram is shown in Fig. 147. Find the system open-loop and closed-loop z-transfer function on the assumption that there is no delay in the digital computer and that we may neglect the effect of level quantization, i.e., we can consider the linear problem. The numerical values of the coefficients are as follows: over-all gain K = 10 1/sec, time constant T_1 = 0.05 sec, and computer sampling period T_0 = 0.1 sec.

Solution. The transfer function of the continuous element is

$$W(p)=\frac{K}{p(1+T_1p)}. \qquad (1)$$

The system open-loop transfer function W(z) may be found from the general rule [25]:

$$W(z)=\frac{z-1}{z}\sum_{k=0}^{\infty}A(kT_0)z^{-k}=\frac{z-1}{z}F(z), \tag{2}$$

where $A(kT_0)$ is the step response of the continuous element at the discrete times (k = 0, 1, 2, ...), while $F(z)$ is the z-transform of this function.

The step response for (1) takes the form

$$A(t)=K[t-T_1(1-e^{-\frac{t}{T_1}})]. \tag{3}$$

From a table of z-transformations we find

$$F(z)=K\left[\frac{T_0 z}{(z-1)^2}-T_1\frac{(1-d)z}{(z-1)(z-d)}\right], \tag{4}$$

where

$$d=e^{-\frac{T_0}{T_1}}=e^{-2}=0{,}135.$$

Next we use (2) to find the system open-loop transfer function

$$W(z)=K\left[\frac{T_0}{z-1}-\frac{T_1(1-d)}{z-d}\right]=$$

$$=\frac{KT_0\left[z-d-\frac{T_1}{T_0}(1-d)(z-1)\right]}{(z-1)(z-d)}. \tag{5}$$

Substitution of the numerical values yields

$$W(z)=\frac{0{,}568z+0{,}297}{(z-1)(z-d)}.$$

The system closed-loop transfer function is

$$\Phi(z)=\frac{W(z)}{1+W(z)}=\frac{0{,}568z+0{,}297}{z^2-0{,}567z+0{,}432}.$$

256. Solve the preceding problem where the continuous-element transfer function is

$$W(p)=\frac{K}{p(1+T_1p)(1+T_2p)}.$$

<u>Answer</u>

$$W(z)=K\left[\frac{T_0}{z-1}+\frac{T_1^2}{T_2-T_1}\,\frac{1-d_1}{z-d_1}+\frac{T_2^2}{T_1-T_2}\,\frac{1-d_2}{z-d_2}\right]$$

where

$$d_1 = e^{-\frac{T_0}{T_1}} \text{ and } d_2 = e^{-\frac{T_0}{T_2}}.$$

257. The open-loop transfer function of a control system with a computer has the form

$$W(z) = \frac{KT_0}{z-1}. \qquad (1)$$

Construct the system closed-loop transient response when a unit step function $Y(t) = 1(t)$ as applied to the input for $KT_0 = 1$, $KT_0 = 0.5$, and $KT_0 = 1.5$.

<u>Solution</u>. Consider the case $KT_0 = 1$. The system closed-loop transfer function is

$$\Phi(z) = \frac{W(z)}{1+W(z)} = \frac{KT_0}{z-1+KT_0} = \frac{1}{z}. \qquad (2)$$

The transform of the input variable (see Appendix 13) is

$$Z[1(t)] = F_Y(z) = \frac{z}{z-1}. \qquad (3)$$

The transform of the output variable is

$$F_X(z) = \Phi(z)F_Y(z) = \frac{1}{z}\cdot\frac{z}{z-1} = \frac{1}{z-1}. \qquad (4)$$

We expand the last expression into a Series by dividing the numerator by the denominator:

$$\frac{1}{z-1} = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \ldots \qquad (5)$$

This gives the following values of X at the output at the discrete times: when $t = 0$, $X = 0$; when $t = T_0$, $X = 1$; when $t = 2T_0$, $X = 1$; when $t = 3T_0$, $X = 1$, and moreover $X = 1$ for all values $t = kT_0$. A graph of this function is shown in Fig. 148 (curve 1).

Between the discrete values of the time function at the output, we draw straight lines, since the transfer function (1) corresponds to an ideal integrating element whose step response will be a linear relationship.

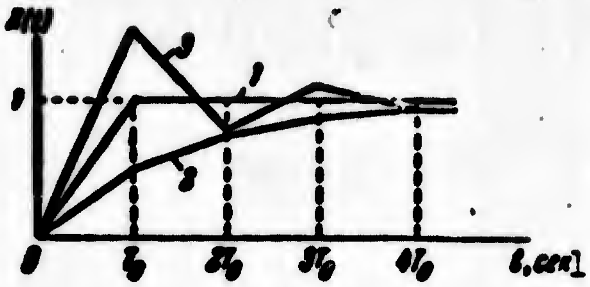


Fig. 148. Transient responses for Problem 257. 1) sec.

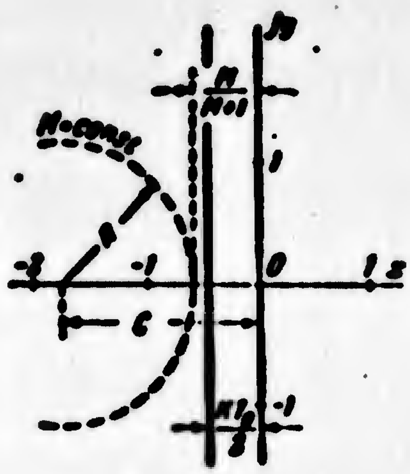


Fig. 149. Gain-phase characteristic for Problem 258.

In like manner, we obtain the Laurent expansion for $KT_0 = 0.5$:

$$F_x^*(z) = 0{,}5\,\frac{1}{z} + 0{,}75\,\frac{1}{z^2} + 0{,}875\,\frac{1}{z^3} + 0{,}9375\,\frac{1}{z^4} + \ldots$$

The transient curve is shown in Fig. 148 (curve 2).

For the case in which $KT_0 = 1.5$, we obtain

$$F_x(z) = 1{,}5\,\frac{1}{z} + 0{,}75\,\frac{1}{z^2} + 1{,}125\,\frac{1}{z^3} + 0{,}8375\,\frac{1}{z^4} + \ldots$$

The transient curve for this case is shown in Fig. 148 (curve 3).

258. The transfer function of the continuous portion of a system with a computer has the form

$$W(p) = \frac{K}{p}, \tag{1}$$

where $K = 50$ 1/sec is the over-all gain. Determine the permissible sampling cycle T_0 for the computer for which the system closed-loop magnitude ratio will not exceed $M = 1.5$. The computer has zero delay, and the effect of level quantization may be neglected.

Solution. The open-loop transfer function of the system with the computer may be found from the general rule

$$W(z) = \frac{z-1}{z}\sum_{k=0}^{\infty} A(kT_0)\,z^{-k} = \frac{z-1}{z}F(z), \tag{2}$$

where $A(kT_0)$ is the step response of the continuous portion of (1) while F(z) is the z-transform of this function. As in Problem 255, we obtain the system open-loop transfer function

$$W(z)=\frac{KT_0}{z-1}. \qquad (3)$$

We construct the gain-phase characteristic of the open-loop system from Expression (3). We make the substitution

$$z=e^{j\omega T_0}=\cos\omega T_0+j\sin\omega T_0.$$

As a result, we obtain the frequency transfer function

$$W(j\omega)=\frac{KT_0}{\cos\omega T_0-1+j\sin\omega T_0}=-\frac{KT_0}{2}-j\frac{KT_0}{2}\operatorname{ctg}\frac{\omega T_0}{2}.$$

It is not difficult to see that the gain-phase characteristic is a straight line parallel to the imaginary axis and a distance $KT_0/2$ away from it (Fig. 149).

If the magnitude ratio is not to exceed a given value, the gain-phase characteristic must not enter the circle forming the forbidden zone. It is indicated on Fig. 149 by the dashed line. From this we obtain the condition

$$\frac{KT_0}{2}<\frac{M}{M+1}. \qquad (4)$$

The permissible sampling cycle length is

$$T_0<\frac{2}{K}\,\frac{M}{M+1}.$$

For the given numerical values

$$T_0<\frac{2}{50}\,\frac{1{,}5}{1{,}5+1}=0{,}024 \text{ sec.}$$

259.* The transfer function of the continuous element of a control system with a computer has the form

$$W(p)=\frac{K(1+\tau p)}{p^2}, \qquad (1)$$

where $K = 100\ 1/\sec^2$ is the over-all gain of the open-loop control system while τ is the time constant of the compensating device. Determine

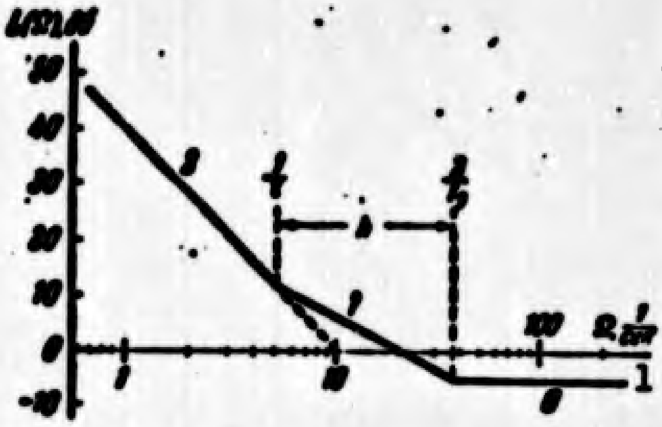


Fig. 150. Logarithmic amplitude characteristic and gain-phase characteristic for Problem 259. 1) sec.

the permissible sampling cycle T_0 for the computer and the required compensating-device time constant such that the magnitude ratio will not exceed $M = 1.3$ if the computer has zero delay and the effect of level quantization may be neglected.

<u>Solution</u>. We determine the open-loop transfer function of the system together with the computer:

$$W(z)=\frac{z-1}{z}\sum_{k=0}^{\infty}A(kT_0)z^{-k}=\frac{z-1}{z}F(z), \qquad (2)$$

where $A(kT_0)$ is the step response of the continuous part of (1), while $F(z)$ is the z-transform of this function. For (1) we have

$$A(t)=\frac{Kt^2}{2}+K\tau t. \qquad (3)$$

In accordance with Appendix 13

$$F(z)=\frac{KT_0^2z(z+1)}{2(z-1)^3}+\frac{KT_0\tau z}{(z-1)^2}. \qquad (4)$$

Next we obtain from (2)

$$W(z)=\frac{KT_0^2(z+1)}{2(z-1)^2}+\frac{KT_0\tau}{z-1}. \qquad (5)$$

Turning to the w-transformation, we make the substitution

$$z=\frac{1+w}{1-w}. \qquad (6)$$

As a result we obtain

$$W(w)=\frac{KT_0^2}{4}\frac{\left(1+2\frac{\tau}{T_0}w\right)(1-w)}{w^2}. \tag{7}$$

We now obtain the frequency transfer function by the substitution [6]

$$w=j\frac{T_0}{2}\Omega, \tag{8}$$

where Ω is the absolute pseudofrequency. Using the substitution (8), we have from (7)

$$W(j\Omega)=\frac{K(1+j\tau\Omega)\left(1-j\frac{T_0}{2}\Omega\right)}{(j\Omega)^2}. \tag{9}$$

The modulus of the system open-loop frequency transfer function is

$$|W(j\Omega)|=\frac{K\sqrt{1+\tau^2\Omega^2}\sqrt{1+\frac{T_0^2}{4}\Omega^2}}{\Omega^2}, \tag{10}$$

and the phase is

$$\varphi=-180^\circ+\operatorname{arctg}\tau\Omega-\operatorname{arctg}\frac{T_0\Omega}{2}. \tag{11}$$

From Expression (10), we have plotted the logarithmic amplitude characteristic of Fig. 150. On the basis of the phase characteristic (11), this case reduces to a logarithmic amplitude characteristic of type C [3]*. As a result, we obtain the following formulas for the calculations: the base pseudofrequency of the logarithmic amplitude characteristic is

$$\Omega_0=\sqrt{K}=10 \text{ 1/sec};$$

the required compensating-element time constant is

$$\tau=\frac{1}{\Omega_0}\sqrt{\frac{M}{M-1}}=\frac{1}{10}\sqrt{\frac{1.3}{1.3-1}}=0.21 \text{ sec};$$

the required length of the section of logarithmic amplitude characteristic with slope of 20 db/decade is

$$b=\frac{M+1}{M-1}=\frac{1,3+1}{1,3-1}=7,7;$$

the permissible sampling cycle length is

$$\frac{T_0}{2}<\frac{\tau}{b}=\frac{0,21}{7,7}=0,027 \text{ sec},$$

where $T_0 \leq 0.054$ sec.

260. For the sampled-data control system whose block diagram is shown in Fig. 146 (see Problem 253), determine the first two error coefficients.

<u>Solution</u>. The system open-loop transfer function has been obtained in Problem 253:

$$W(z)=\frac{\gamma K T_0 (1-d) z}{(z-1)(z-d)}.$$

The error transfer function is

$$\Phi_e(z)=\frac{1}{1+W(z)}=\frac{(z-1)(z-d)}{(z-1)(z-d)+\gamma K T_0 (1-d) z}.$$

Substituting z = 1 into this expression (or p = 0 into the expression $z = e^{pT_0}$), we obtain the coefficient $C_0 = 0$.

In order to obtain the coefficient C_1, we find the first derivative:

$$\frac{d\Phi_e(e^{pT_0})}{dp}=\frac{\gamma K T_0^2 (1-d)(z^2-dz)}{[(z-1)(z-d)+\gamma K T_0 (1-d) z]^2}.$$

The substitutions z = 1 (p = 0) yields the error coefficients

$$C_1=\frac{1}{\gamma K}=\frac{1}{0,01\cdot 100}=1 \text{ sec},$$

as well as the velocity figure of merit

$$K_v=\frac{1}{C_1}=1 \text{ 1/sec}.$$

261.* Design a servosystem with first-order astatism containing a computer in the circuit. The initial conditions are: 1) maximum input velocity Ω_{max} = 10 degree/sec; 2) maximum input acceleration ε_{max} = 5 degree/sec^2; 3) maximum permissible error θ_{max} = 2'; 4) continuous element contains the time constants T_1 = 0.01 sec, T_2 = 0.002 sec, and T_3 = 0.001 sec; 5) the permissible magnitude ratio is M = 1.5; 6) the load torque equals zero; 7) there is no delay in the computer. It is necessary to determine the parameters of the series compensating element connected into the continuous portion, the permissible computer repetition cycle, and to plot the transient resulting from a unit step input. The effect of level quantization may be neglected.

Solution. To the left of the cutoff frequency, the logarithmic amplitude characteristic of the system and computer coincides with the logarithmic amplitude characteristic of the continuous portion, while the absolute pseudofrequency $\Omega = 2w/jT_0$ (see Problem 259) coincides with the real frequency ω. Thus the normal methods may be used to obtain the logarithmic amplitude characteristic to the left of the cutoff frequency [3].

We construct the forbidden zone for the logarithmic amplitude characteristic from the accuracy conditions (Fig. 151). The control frequency is

$$\Omega_k = \frac{\varepsilon_{max}}{\Omega_{max}} = \frac{5}{10} = 0{,}5 \ \text{1/sec.}$$

The modulus of the system open-loop transfer function with $\Omega = \Omega_k$ is

$$|W(j\Omega)| = \frac{\Omega^2_{max}}{\varepsilon_{max}\theta_{max}} = \frac{10^2 \cdot 60}{5 \cdot 2} = 600 = 55{,}6 \ \text{db.}$$

Using this data, we plot the control point A_k and the forbidden zone formed from lines with slopes 20 db/decade and 40 db/decade on Fig. 151 (sloping sections 1 and 2).

The desired logarithmic amplitude characteristic in the low-frequency region is formed so that it passes above point A_k by 3 db, corresponding to an increase in gain by a factor of $\sqrt{2}$. It consists of

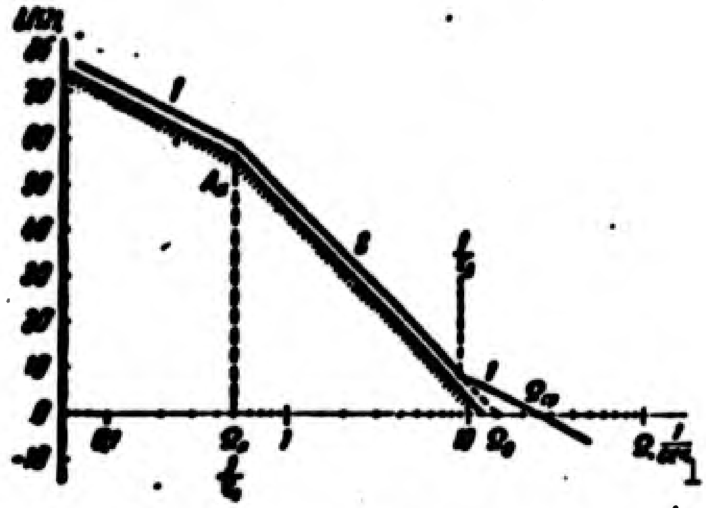


Fig. 151. Low-frequency portion of logarithmic amplitude characteristic for Problem 261. 1) sec.

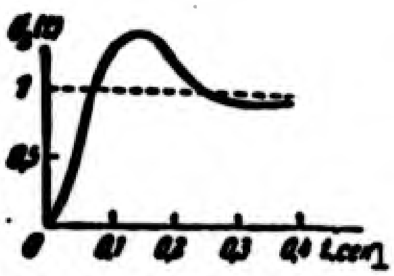


Fig. 152. Transient for Problem 261. 1) sec.

line segments with slopes 1-2-1. In the low-frequency region, the system open-loop frequency transfer function will have the form

$$W(j\Omega)=\frac{K(1+j\tau_1\Omega)}{j\Omega(1+j\tau_2\Omega)}.$$

We determine the parameters of the desired logarithmic amplitude characteristic for the system open-loop transfer function in the low-frequency section. The base frequency of the logarithmic characteristic is

$$\Omega_0=\sqrt{\sqrt{2}\frac{\varepsilon_{max}}{\theta_{max}}}=\sqrt{1.41\frac{5\cdot 60}{2}}=14.5\ 1/\text{sec}.$$

The time constant of the compensating element forming the first bend in the logarithmic characteristic is

$$\tau_1=\frac{1}{\Omega_1}=\frac{1}{0.5}=2\ \text{sec}.$$

In order to obtain the given magnitude ratio, it is necessary to satisfy the inequality [3, 6]

$$\frac{1}{\tau_2}-\frac{1}{\tau_1}<\Omega_0\sqrt{\frac{M-1}{M}}.$$

From this we obtain the minimum value of the second compensating-element time constant:

$$\tau_2=\frac{1}{\Omega_c\sqrt{\frac{M-1}{M}}+\frac{1}{\tau_1}}=\frac{1}{14{,}5\sqrt{\frac{1{,}5-1}{1{,}5}}+\frac{1}{2}}=0{,}112 \text{ sec.}$$

The transfer function of the series compensating element is

$$W_{ck}(p)=\frac{1+\tau_2 p}{1+\tau_1 p}\quad(\tau_1>\tau_2)$$

which corresponds to a passive integrating-type element.

We next determine the required over-all gain:

$$K=\sqrt{2}\,\frac{\Omega_{max}}{\theta_{max}}=$$
$$=1{,}41\,\frac{10\cdot 60}{2}=420 \text{ 1/sec}$$

and the cutoff frequency for the logarithmic amplitude characteristic:

$$\Omega_{cp}=\frac{K\tau_2}{\tau_1}=$$
$$=\frac{420\cdot 0{,}112}{2}=23{,}5 \text{ 1/sec.}$$

In accordance with the requirements imposed on the logarithmic amplitude characteristic in the high-frequency region [6] we have

$$\frac{T_0}{2}+T_1+T_2+T_3<\frac{1}{\Omega_{cp}}\frac{M}{M+1},$$

from which we obtain the permissible computer sampling cycle length:

$$T_0<2\left[\frac{1}{\Omega_{cp}}\frac{M}{M+1}-T_1-T_2-T_3\right]=$$
$$=2\left[\frac{1}{23{,}5}\frac{1{,}5}{1{,}5+1}-0{,}01-0{,}002-0{,}001\right]=0{,}012 \text{ sec.}$$

The transient curve constructed by expanding the output variable into a Laurent series is shown in Fig. 152.

Manu-
script
Page
No.

[Footnote]

206 See also Appendix 16.

Manu-script Page No.	[Transliterated Symbols]
192	к = k = kriticheskiy = critical
192	ср = sr = srez = cutoff
198	ЦВМ = TsVM = tsivrovaya vychislitel'naya mashina = digital computer
208	:: = k = kontrol'nyy = control
210	к.з = k.z = korrektiruyushcheye zveno = compensating element

BLANK PAGE

Chapter 9

NONLINEAR SYSTEMS

§25. GRAPHICAL CONSTRUCTION OF PROCESSES IN NONLINEAR AUTOMATIC SYSTEMS

262. Make a graphical plot of the transient process for free motion of a nonlinear automatic system with the block diagram of Fig.

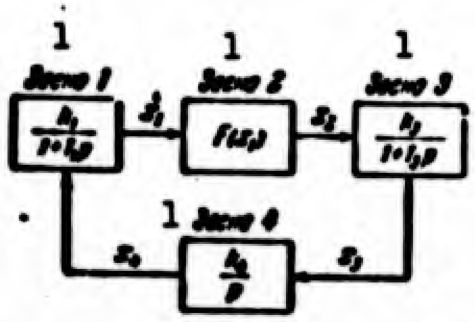


Fig. 153. Block diagram of nonlinear automatic system for Problem 262. 1) Element.

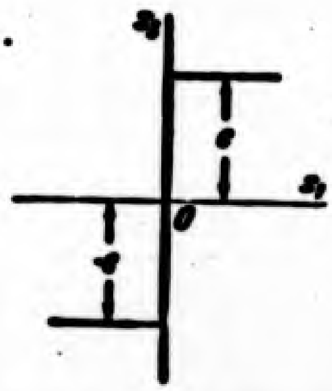


Fig. 154. Static characteristic of nonlinear element for Problem 262.

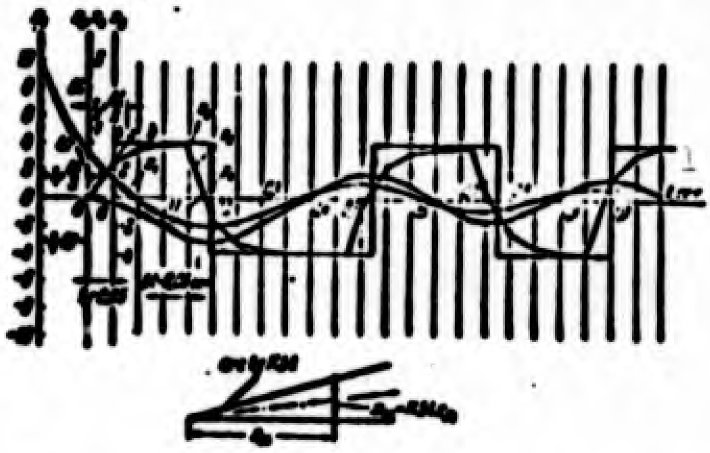


Fig. 155. Transient-process plot for Problem 262. 1) sec.

153 if the parameters have the following values: $T_1 = 0.1$ sec, $T_3 = 0.05$ sec, $k_1 = 5$, $k_3 = 10$, $k_4 = 0.1$ 1/sec, and the nonlinear function

$F(x_1)$ corresponds to an ideal static relay characteristic (Fig. 154) with $c = 4$. The initial conditions are: $x_1^0 = 10$, $x_3^0 = 0$, $x_4^0 = 0$.

<u>Solution</u>. We use the Bashkirov method to plot the transient response. We take a solution element $\Delta t = T_3 = 0.05$ sec and plot the time origin for all variables (Fig. 155). By shifting the time origin for x_3 with respect to the time origin for x_2 by the amount $T_3 = 0.05$ sec, we are able to allow for the response time of the third element. In like manner, by shifting the time origin for x_1 with respect to the time origin for x_4 by the amount $T_1 = 0.1$ sec, we allow for the response time of the first element.

We give the transfer constants k_1 and k_3 in terms of the integrating element, i.e., for the integrating element we assume the transfer constant is $K = k_1 k_3 k_4 = 5 \cdot 10 \cdot 0.1 = 5$ 1/sec. This has no effect on the final solution, but only changes the scale of the graph.

The responses for aperiodic elements are plotted by the secant method while the method of mean ordinates is used for integration.

For an integrating element, we have

$$x_4 = \frac{K}{p} x_3 = K \int_0^t x_3\,dt \approx K\Delta t \sum_{i=1}^{n} x_{3i} =$$
$$= 5 \cdot 0{,}05 \sum_{i=1}^{n} x_{3i} = 0{,}25 \sum_{i=1}^{n} x_{3i}.$$

Multiplication of the mean ordinates x_{3i} by a coefficient equal to $K\Delta t = 0.25$ is most conveniently performed with the aid of an oblique line. To do this, we draw an oblique line at an angle arctan $K\Delta t$ (Fig. 155, bottom). If we now use dividers to lay off x_{3i} on the lower side of the angle from the vertex then, moving one leg of the divider from the angle vertex to the second side at a point on the perpendicular erected at the point on the lower side corresponding to the end of the segment x_{3i}, we obtain a segment equal to the result of integration by one element:

$$x_4 = K\Delta t x_3.$$

We begin construction of the response curve by plotting the initial value $x_1^0 = 10$. This value, according to the static nonlinear characteristic curve for the second element, corresponds to a value $x_2 = c = 4$. For x_2, we draw a line at the level of the indicated positive value. From the known value of x_2, we find a graphical solution for x_3 in the interval of the first solution element. To do this, from the time origin for x_3 we draw a ray to point a on $x_2 = c = 4$, displaced an amount $T_3 + (\Delta t/2)$ from the time origin for x_2. The segment of this ray falling within the first solution element will then be the desired solution. The variable x_3, known for the first solution element, is then integrated by the mean-ordinate method; we obtain a value for x_4, which we take with reversed sign (in virtue of the operating principle of closed-loop systems), i.e., we plot the result at the end of the first solution element with allowance for the minus sign.

Having the solution for x_4 for the first solution element, we can a solution within the first solution element for x_1. To do this, from the point corresponding to the initial value $x_1^0 = 0$, we draw a ray to a point on x_4 and displaced an amount $T_1 + (\Delta t/2)$ away from the x_1 time origin.

After this, we turn to the solution for the second solution element. To do this, from the end of the solution segment for x_3 corresponding to the first solution element, we draw a ray to a point b on x_2 and displace by one more time element. The segment of this ray will then be the x_3 solution for the second solution element. We next integrate x_3 over the second solution element and take the result of integration with allowance for the allowance sign and add it to the value of x_4 at the end of the second solution element. From the known value of x_4, we find x_1 in the second solution element. In like manner, we obtain a solution for the third solution element, etc.

In the ensuing discussion, it should be remembered that when the variable x_1 changes sign (in our example this occurs at the end of the fourth solution element), x_2 will abruptly change its value, taking on a value $x_2 = - c = - 4$. The same procedure is followed in continuing the solution with allowance for the new value of x_2. If the change in the value of x_2 occurs between integral values of solution elements, then the solution for the first instant after switching should be obtained in terms of a fraction of the solution element, just as for an integral element. Here, in the integrating circuit there will be a change in the coefficient $K\Delta t$; this means that the oblique line should be drawn at a smaller angle. Such an oblique line is shown on the Figure for a case in which the solution element decreases by one half. The solution should then be continued in terms of the integral solution element.

From the result of the solution represented in Fig. 155 it is clear that in the system selfoscillation will occur; the variable x_2 at the output of the nonlinear circuit element will oscillate with rectangular waveform; the variable x_3 will exhibit oscillations made up of exponential segments; the variables x_1 and x_4 will exhibit nearly harmonic oscillations..

In order to determine the oscillation amplitudes for each variable, it is necessary to use the steady-state equations to find the scale factor for each variable. Thus, for the variable x_3 we have

$$x_3 = k_3 x_2,$$

so that allowing for the values $x_2 = c = 4$ and $k_3 = 10$, we have

$$x_3 = 10 \cdot 4 = 40.$$

i.e., the level $x_2 = 4$ corresponds to the value $x_3 = 40$.

For the first circuit element under steady conditions we have

$$x_1 = k_1 x_4,$$

so that for values $x_1 = 10$, $k_1 = 5$ we have

$$x_4 = \frac{x_1}{k_1} = \frac{10}{5} = 2,$$

i.e., the value $x_1 = 10$ corresponds to the value $x_4 = 2$.

From the curves representing the variation in all variables we can determine their oscillation amplitudes. Thus, for x_1 we have $A_{x_1} \approx \approx 0.8$. In like manner, we find $A_{x_2} \approx A_{x_3} \approx 40$, $A_{x_4} \approx 0.28$. The oscillation periods and frequencies will be the same for all variables and, as we can see from the graph, will be $T = 0.45$ sec, $\omega \doteq 2\pi/T = 14$ 1/sec.

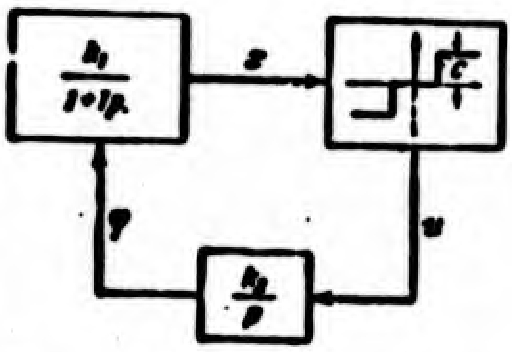


Fig. 156. Block diagram of nonlinear automatic system for Problem 263.

It should be noted that the graphical method ensures construction of the transient response for all variables simultaneously.

263. For the nonlinear system having the block diagram shown in Fig. 156, construct graphically the free-motion tran-

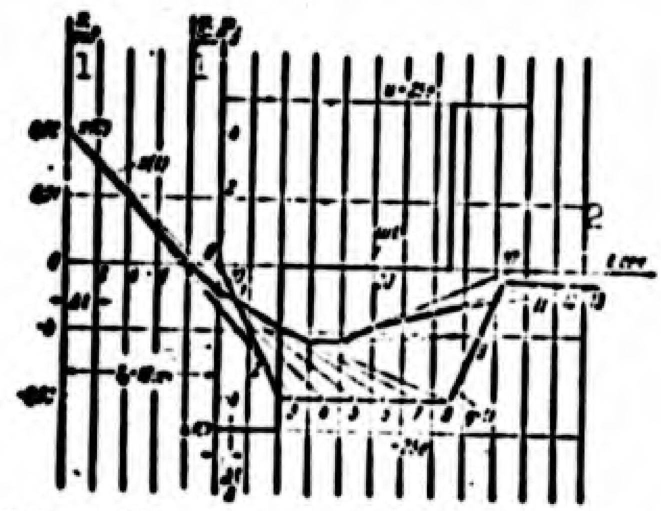


Fig. 157. Construction of transient response for Problem 263. 1) rad; 2) sec.

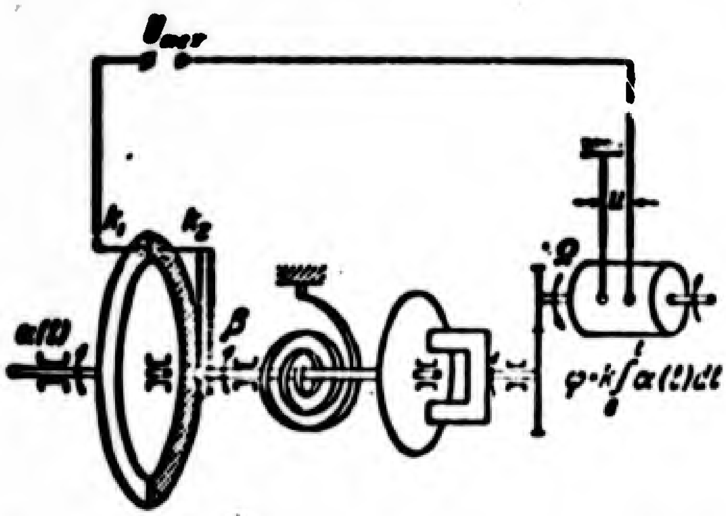


Fig. 158. Diagram of relay integrating device for Problem 264.

sient response and determine the stability if we are given the following parameters and initial values of the variables: $T_1 = 10$ sec, $k_1 = 0.25$, $k_2 = 0.4$ 1/sec·v, $c = 25$ v, $b = 0.01$ rad, $x(0) = 0.02$ rad, $\varphi(0) = 0$.

Answer. The nonlinear system is stable and the process converges to equilibrium. The transient response is shown in Fig. 157.

264. For a relay integrating device (Fig. 158), plot the transient response graphically. The system element equations have the forms

$$(T_1 p + 1)\Omega = k_1 u,$$

$$(T_2 T_3 p^2 + T_3 p + 1)\beta = k_2 \Omega,$$

$$u = \begin{cases} U_{max} & \text{for } \beta < \beta_0, \\ 0 & \text{for } \beta > \beta_0, \end{cases}$$

where Ω is the speed at which the electric motor turns, β is the angle through which the contact lever of the magnetic tachometer rotates, u is the voltate applied through the tachometer contacts to the control winding of the electric motor. The parameters of the device are as follows: $T_1 = 0.33$ sec is the electromechanical time constant of the motor; $T_2 = 0.1$ sec, $T_3 = 0.017$ sec are the time constants of the magnetic tachometer; $U_{max} = 30$ v is the voltage across the motor control winding when the tachometer contacts are closed; β_0 is the value of β corresponding to the given speed; in our example $\beta_0 = 0.5\beta_{max}$; $\beta_{max} = k_2\Omega_{max}$; $\Omega_{max} = k_1U_{max}$.

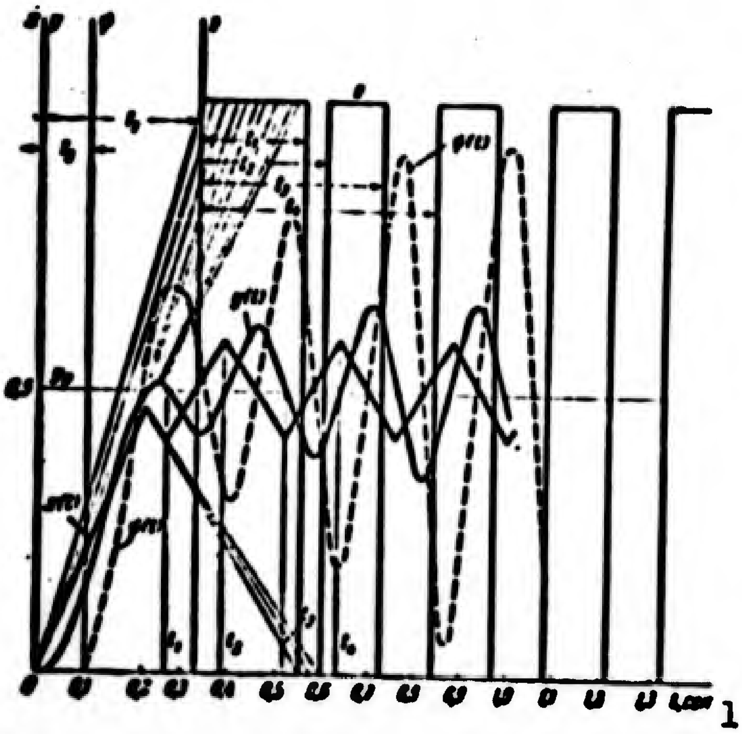


Fig. 159. Construction of transient response for Problem 264. 1) sec

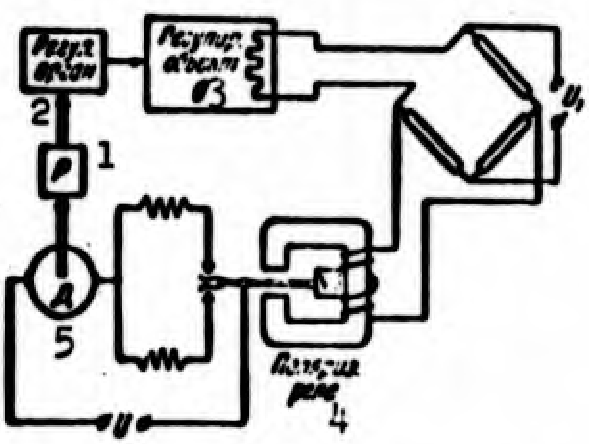


Fig. 160. Basic diagram of nonlinear temperature-regulation system for Problem 265. 1) Reducing gear; 2) control element; 3) controlled system; 4) polar relay; 5) motor.

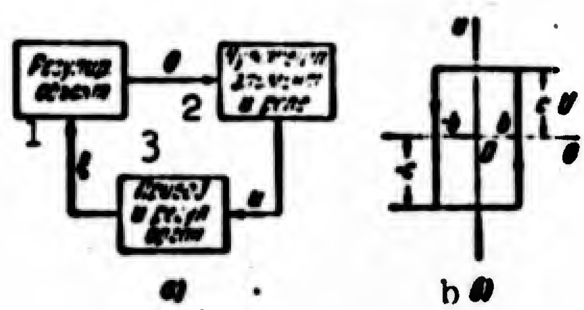


Fig. 161. Block diagram and static characteristic of nonlinear element of temperature-regulation system for Problem 265. 1) Controlled system; 2) sensing element and relay; 3) drive mechanism and control element.

Remark. Letting $x=\frac{\Omega}{\Omega_{max}}$, $y=\frac{\beta}{\beta_{max}}$, $z=\frac{u}{U_{max}}$, we obtain the following system of equations:

$$(T_1p+1)x=z,\quad (T_2T_3p^2+T_3p+1)y=x,$$
$$z=\begin{cases}1, & y<0.5,\\ 0, & y>0.5,\end{cases}$$

which we use as the initial system for the graphical construction.

The second-order equation for the oscillatory element may be represented as two equations with the additional variable ψ, i.e.,

$$(T_3p+1)y=\psi,\quad T_2p\psi=x-y.$$

Then the construction may be carried out as for a system consisting of two aperiodic first-order elements, one integrating element and one nonlinear element.

<u>Answer</u>. The transient-response plot is shown in Fig. 159. The system will go into self-sustained oscillation under weak excitation.

§26. INVESTIGATION OF NONLINEAR AUTOMATIC SYSTEMS BY THE PHASE-PATH METHOD

265. The phase-path method is used to investigate the temperature-regulation process in a nonlinear system with a two-position polar relay (Fig. 160). The system parameters have the following values: T_0 = = 10 sec, k_0 = 10 degree/rad, k_1 = 0.01, rad/sec·v. The static characteristic of the nonlinear element is shown in Fig. 161b, where c = 20 v, b = 2°.

<u>Solution</u>. From the given basic circuit, we set up the block diagram (Fig. 161a) and the equations of the system elements.

The equation of the controlled system will be

$$(T_0p+1)\theta=-k_0\lambda$$

The equation for the drive mechanism and final-control element, neglecting the motor time constant, will take the form

$$p\lambda=k_1U,$$

where U is the voltage across the motor.

For the sensing element together with the bridge circuit and relay, the values U = F(θ) are determined from the static characteristic (Fig.

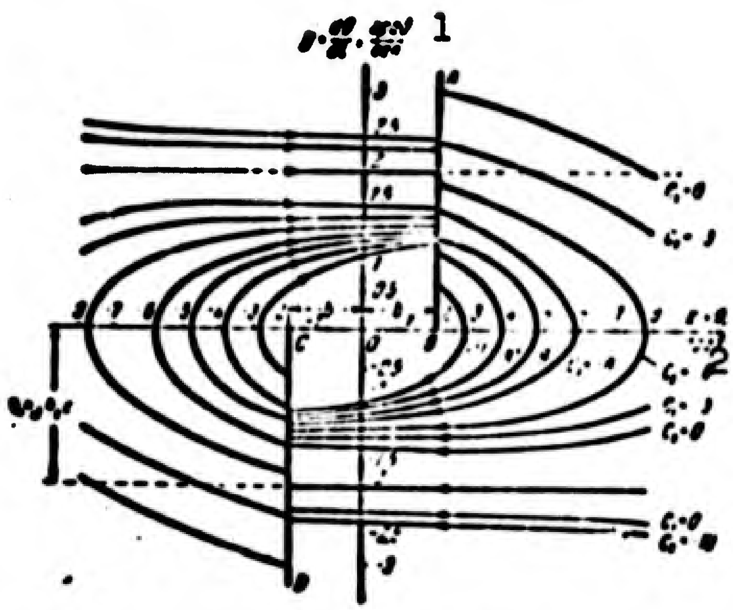


Fig. 162. Phase paths for nonlinear temperature-regulation system of Problem 265. 1) Degree/sec; 2) degree.

161b).

In accordance with the equations for the linear elements and the static characteristic of the nonlinear element, the equations for the entire system will take the form:

1) for the case of rising temperatures $d\theta/dt > 0$

$$\left.\begin{aligned}(T_0p+1)p\theta &= k_0k_1c \quad \text{for} \quad \theta < b,\\ (T_0p+1)p\theta &= -k_0k_1c \quad \text{for} \quad \theta > b;\end{aligned}\right\} \tag{1}$$

2) for the case of decreasing temperatures $d\theta/dt < 0$

$$\left.\begin{aligned}(T_0p+1)p\theta &= -k_0k_1c \quad \text{for} \quad \theta > -b,\\ (T_0p+1)p\theta &= k_0k_1c \quad \text{for} \quad \theta < -b.\end{aligned}\right\} \tag{2}$$

We plot the coordinate axes of the phase plane $x = \theta$ and $y = d\theta/$ $/dt$ (Fig. 162). For the half plane lying to the right of the polygon ABCD, the equation of the system from (1) and (2) will take the form

$$T_0\frac{d^2\theta}{dt^2}+\frac{d\theta}{dt}=-k_0k_1c, \tag{3}$$

while for the half plane to the left of the polygon ABCD, the system equation will be

$$T_0\frac{d^2\theta}{dt^2}+\frac{d\theta}{dt}=k_0k_1c. \tag{4}$$

We find the phase-path equation for the right half plane. Taking $x = \theta$ and $y = d\theta/dt$ into account, Eq. (3) may be written as

$$T_0 \frac{dy}{dt} + y = -k_0 k_1 c. \tag{5}$$

Eliminating the time from Eq. (5) and dividing it by $dx/dt = y$, we obtain

$$\frac{dy}{dx} = -\frac{1}{T_0} - \frac{k_0 k_1 c}{T_0 y},$$

which after separation of variables yields

$$dx = -\frac{T_0 y \, dy}{y + k_0 k_1 c}. \tag{6}$$

Integrating Eq. (6), we obtain

$$x = T_0 k_0 k_1 c \ln(y + k_0 k_1 c) - T_0 y + C_1. \tag{7}$$

Carrying out the same operations for Eq. (4) we obtain for the left half plane

$$x = -T_0 k_0 k_1 c \ln(y - k_0 k_1 c) - T_0 y + C_1,$$

where C_1 is an arbitrary constant of integration.

It is clear from (7) that when $y = -k_0 k_1 c$, we have $x = -\infty$. As a consequence, all phase paths corresponding to different values of the arbitrary constant C_1 will have a common asymptote $y = -k_0 k_1 c$ to which they tend when $x \to -\infty$. In addition, from the expression for the derivative dy/dx it is clear that when $y = 0$, $dy/dx = -\infty$. This means that the phase paths are perpendicular to the x axis at its points of intersection with them. The paths for the left half plane will be symmetric with respect to the paths in the right half plane about the origin.

Using the properties indicated and taking various values for the arbitrary constant C_1 with known system parameters, we construct the phase paths. The working formulas for the phase paths in accordance with the given parameter values will have the form

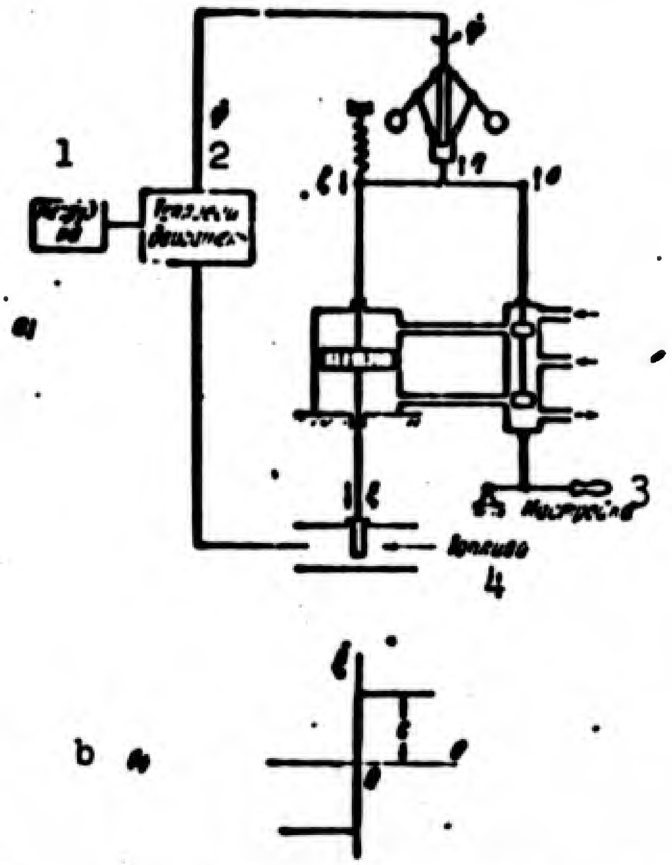


Fig. 163. Basic diagram and characteristic of nonlinear element in speed-control system for thermal engine of Problem 266. 1) Load; 2) thermal engine; 3) adjustment; 4) fuel.

$$x = 20 \ln(y + 2) - 10y + C_1,$$
$$x = -20 \ln(y - 2) - 10y + C_1.$$

Paths calculated from the formulas obtained are shown in Fig. 162.

It is clear from Fig. 162 that for small initial values of temperature deviation $x = \theta$ and velocity $y = d\theta/dt$, the process in the system will be divergent, and for large values convergent. Thus, there will be one phase path for a value of the arbitrary constant $C_1 \approx -9$ in the family of integral curves corresponding to a stable limit cycle. As a consequence, for any initial deviations, after the system transient has occurred there will be self-oscillation. The temperature-deflection amplitudes and rates of change of temperature for self-oscillation are easily found from the limit cycle and are $A_\theta \approx 4.8^\circ$, $A_{\dot\theta} \approx 1.2$ degree/ /sec.

266. For an automatic system, the phase-path method is used to investigate the process of controlling the speed of a thermal engine (163a), where we assume, in simplified form: a) the equation of the controlled system, without self-regulation is

$$T\dot{\varphi} = \xi,$$

where φ is the relative deviation in angular velocity;

b) the regulator equation neglecting mass and damping with proportional feedback is

$$\delta\eta = -\varphi, \quad \sigma = \eta - \zeta, \quad \zeta = \xi$$

or

$$\sigma = -\frac{1}{\delta}\varphi - \xi,$$

where η, ζ, and ξ are the relative deviations.

Let the drive mechanism of the final-control element have constant speed with instantaneous switching (Fig. 163b) when the control element (slide valve) passes through the neutral position (σ = 0), i.e., it will be described by the equation

$$\dot{\xi} = c\,\mathrm{sign}\,\sigma.$$

We take the following parameter values: T = 2 sec, δ = 0.5, c = = 0.1 m/sec.

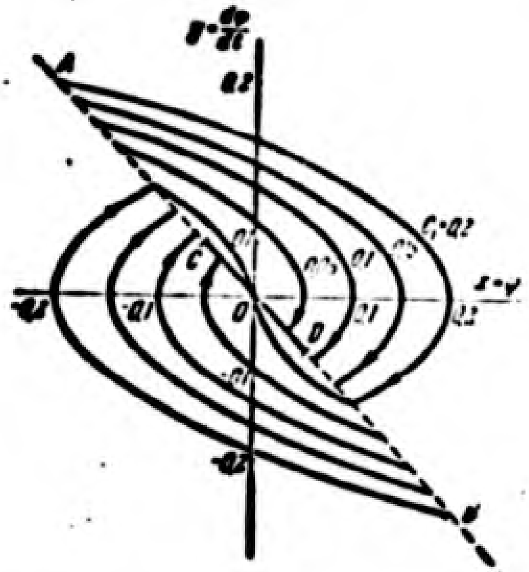


Fig. 164. Phase paths for thermal-engine speed-control system of Problem 266.

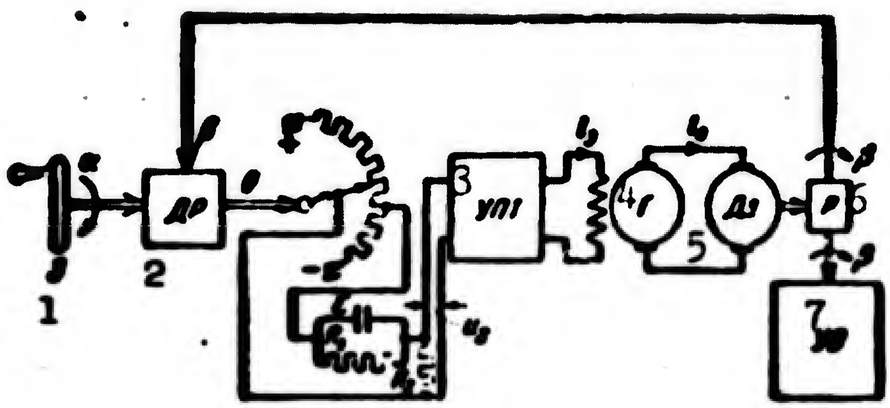


Fig. 165. Basic diagram for nonlinear servosystem of Problem 267. 1) Setting adjustment; 2) error-sensing element; 3) direct current amplifier; 4) generator; 5) motor; 6) reduction gear; 7) controlled system.

Answer. The phase paths are parabolas (Fig. 164) corresponding to the equation

$$x=-\frac{T}{2k}y^2+C_1=-10y^2+C_1.$$

Drive-mechanism switching ($\sigma = 0$) is determined by the line on the phase plane

$$x=-\delta T y=-y,$$

and segment CD of this line with the maximum ordinate

$$y_c=c\delta=0{,}1\cdot 0{,}5=0{,}05$$

is a special line. Upon entering segment CD, the representation point will aperiodically approach the steady state. As a consequence, there will first be an oscillatory transient degenerating at the end to a so-called "sliding" process.

267. The phase-path method is used to investigate the response of a nonlinear servosystem. Figure 165 shows the basic diagram of the servosystem; here Z is the setting adjustment, DR is the error-sensing element, UPT is a direct current amplifier, G is a generator, Dv is the motor, R is a reduction gear, UO is the controlled system. The equation for the motor, reduction gear, and controlled system as a nonlinear element is taken in the following form for the free motion of the system:

$$J\ddot{\beta} + c\,\mathrm{sign}\,\dot{\beta} = a i_a \text{ for } \dot{\beta} \neq 0 \text{ or for } \dot{\beta} = 0 \text{ and } |i_a| > \frac{c}{a},$$
$$\beta = \mathrm{const} \text{ for } \dot{\beta} = 0 \text{ and } |i_a| < \frac{c}{a}, \qquad (1)$$

i.e., we allow for dry friction in all moving elements driven by the electric motor. For the remaining system linear elements, provided the system setting adjustment is not moved, the equation will take the form

$$i_a = -k_1\dot{\beta} - k\beta. \qquad (2)$$

By (1) and (2), the general system equation will be

$$\ddot{\beta} + a_1\dot{\beta} + a_2\beta = -b_1\,\mathrm{sign}\,\dot{\beta} \text{ for } \dot{\beta} \neq 0 \text{ or } \dot{\beta} = 0 \text{ and } |\beta| > \frac{c}{ak},$$
$$\beta = \mathrm{const} \text{ for } \dot{\beta} = 0 \text{ and } |\beta| < \frac{c}{a_1 k}. \qquad (3)$$

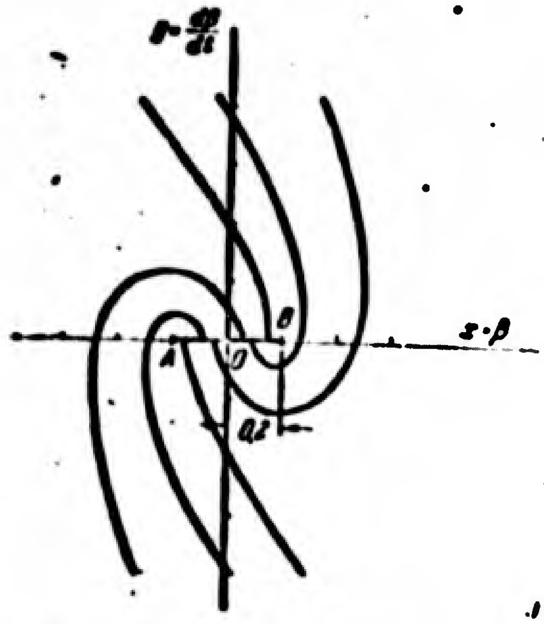


Fig. 166. Phase paths for nonlinear servosystem of Problem 267.

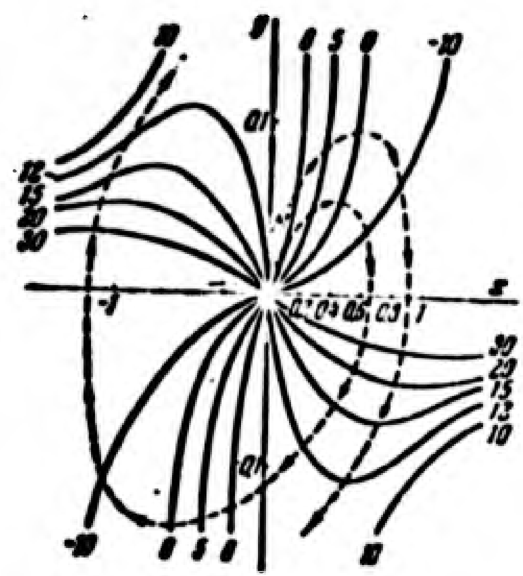


Fig. 167. Isoclines and phase paths for Problem 268.

where

$$a_1 = \frac{ak_1}{J}, \quad a_2 = \frac{ak}{J}, \quad b_1 = \frac{c}{J}.$$

The parameter numerical values are taken as

$$a_1 = 4 \text{ 1/sec}, a_2 = 5, \quad b_1 = 1.$$

Answer. In the coordinates $x = \beta$ and $y = d\beta/dt$, the phase paths will be represented as the curves shown in Fig. 166. The system displays a damped oscillatory transient response; the system equilibrium position is determined nonuniquely with respect to the coordinate $x = \beta$, i.e., the system may remain at any point on the critical segment AB.

268. The proper motion of the nonlinear system is described by the differential equation

$$\ddot{x} - 10(1 - x^2)\dot{x} + x = 0. \tag{1}$$

Investigate the stability of the automatic system using the isocline method.

Solution. We write Eq. (1) in the form

$$\frac{dy}{dt} = 10(1 - x^2)y - x, \quad y = \frac{dx}{dt}.$$

Making two equations from the first equation, we obtain the phase-path equation

$$\frac{dy}{dx} = 10(1 - x^2) - \frac{x}{y}.$$

Letting

$$\frac{dy}{dx} = C = \text{const},$$

in the resulting expression, we find the isocline equation

$$y = \frac{x}{10(1 - x^2) - C}. \tag{2}$$

From Eq. (2) we construct the isoclines (solid lines on Fig. 167) corresponding to various values of C.

Considering the properties of the phase paths, for different initial positions of the representation point M we draw the phase paths so that at the intersection of the appropriate isocline the slope of the phase path with respect to the axis of abscissas will equal arctan C (dashed curves of Fig. 167). As we can see from Fig. 167, for any initial position of the representation point, the latter will tend to move away from the phase-plane origin. As a consequence, the nonlinear system described by Eq. (1) is unstable.

§27. USING THE METHOD OF HARMONIC LINEARIZATION TO INVESTIGATE NONLINEAR AUTOMATIC SYSTEMS

269. Investigate the rheostat servosystem of Fig. 168 by the harmonic linearization method taking into account the saturation-type non-

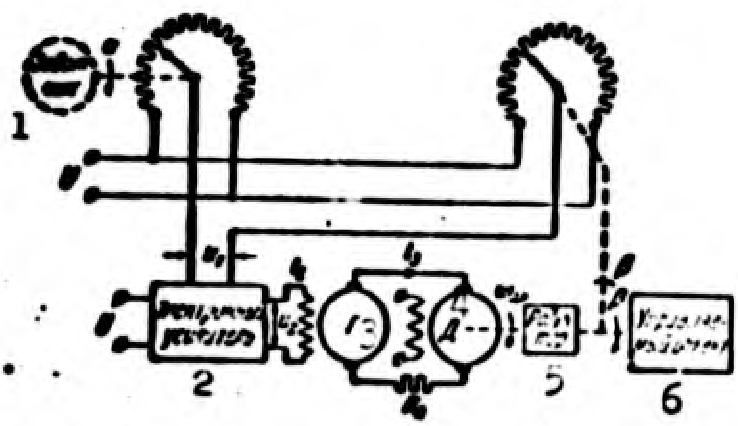


Fig. 168. Basic circuit of rheostat servosystem for Problem 269. 1) Setting adjustment; 2) electronic amplifier; 3) generator; 4) motor; 5) reduction gear; 6) controlled system.

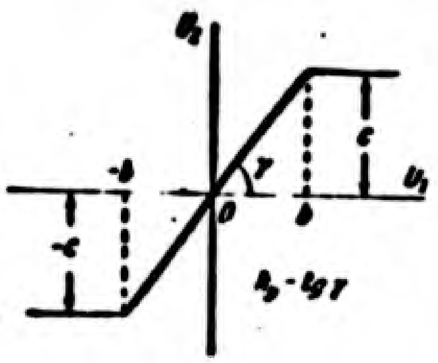


Fig. 169. Nonlinear static characteristic for electronic amplifier of Problem 269.

linearity of the chronic amplifier, if the element equations and parameters are given.

1. The equation of the error-sensing element is

$$u_1 = k_1\theta, \quad \theta = \alpha - \beta, \qquad (1)$$

where θ is the error angle, u_1 [v] is the voltage across the electronic-amplifier input, k_1[v/rad] is the transfer constant of the error-sensing element.

2. The equation of the generator excitation circuit is

$$(T_1 p + 1) i_1 = k_2 u_2, \qquad (2)$$

where T_1[sec] is the electrical time constant of the excitation circuit k_2[amp/v] is the transfer constant of the excitation circuit, equal to the winding conductance, u_2[v] is the voltage at the amplifier output with the load connected.

3. The equation for the generator-motor armature circuit, neglecting inductance, is

$$R_a I_a = e_g - e_m = k_g i_2 - C_e p \vartheta \qquad (3)$$

where the transfer constant of the generator k_g corresponds to the slope of the generator magnetization curve $e_g = f(i_2)$ at the origin, while the coefficient of proportionality C_e is found from the motor data plate:

$$C_e = \frac{\Omega_{kh.kh}}{U_n} [\text{v} \cdot \text{sec}].$$

Here $\Omega_{kh.kh}$ is the ideal no-load speed, U_n is the nominal armature voltage.

4. The equilibrium equation for the motor-shaft torques is

$$J p^2 \vartheta = C_m I_a \qquad (4)$$

where J is the reduced motor moment of inertia, $C_m = M_n/I_{n.ya}$ [kg·cm/ /amp] is the coefficient of proportionality between the motor torque and the armature current, equal to the ratio of the nominal torque to the nominal armature current.

Solving (3) and (4) simultaneously, we obtain the equation of motion of the motor together with the controlled device:

$$\left(J p^2 + \frac{C_e C_m}{R_a} p\right) \vartheta = \frac{C_m k_g}{R_a} i_2$$

or

$$(T_M p + 1) p \vartheta = k_2 i_2 \qquad (5)$$

where the electromechanical time constant is

$$T_M = \frac{J R_a}{C_e C_m} = J \frac{\Omega_{kh.kh}}{M_0}.$$

Here M_0 is the motor short-circuit torque and the transfer constant is

$$k_2 = \frac{k_g}{C_e}.$$

5. For a nonlinear element (electronic amplifier), we are given the static characteristic (Fig. 169). The zone of linearity is determined by the value b = 1 v.

The parameters for the linear zone are: over-all gain $k_l = k_1 k_2 k_3 = 20$ 1/sec while the time constants are $T_1 = 0.1$ sec and $T_2 = 1$ sec.

Solution. Combining the equations for the linear elements, we obtain the equation for the linear portion of the system (for $\alpha = 0$)

$$(T_1 p + 1)(T_2 p + 1) p u_1 = -k_l u_2. \tag{6}$$

Taking into account amplifier saturation, in accordance with the method of harmonic linearization [20], we replace the nonlinear characteristic by the linear relationship

$$u_2 = q(A) u_1, \tag{7}$$

where the harmonic linearization coefficient for the saturation characteristic (see Appendix 20) has the value

$$\left.\begin{array}{ll} q(A) k_2 & \text{for } A \leq b, \\ q(A) = \frac{2k_2}{\pi}\left(\arcsin\frac{b}{A} + \frac{b}{A}\sqrt{1 - \frac{b^2}{A^2}}\right) & \text{for } A \geq b. \end{array}\right\} \tag{8}$$

Substituting the value u_2 from (7) into (6), we obtain the linearized equation for the free motion of the system

$$[(T_1 p + 1)(T_2 p + 1) p + k_l q(A)] u_1 = 0, \tag{9}$$

corresponding to the characteristic equation

$$T_1 T_2 p^3 + (T_1 + T_2) p^2 + p + k_l q(A) = 0. \tag{10}$$

We shall seek a periodic solution for the variable u_1 in the form

$$u_1 = A \sin \Omega t.$$

Here in the characteristic polynomial L(p), we make the substitution $p = j\omega$, set the real and imaginary parts of $L(j\omega)$ equal to zero, and take into account the fact that in this case $\omega = \Omega$. As result, we obtain two equations for determining the periodic solution:

$$\left.\begin{array}{r} k_l q(A) - (T_1 + T_2)\Omega^2 = 0, \\ 1 - T_1 T_2 \Omega^2 = 0. \end{array}\right\} \tag{11}$$

From the second equation of (11), we obtain a formula for finding the frequency of the periodic solution in terms of the system parameters:

$$\Omega = \frac{1}{\sqrt{T_1 T_2}}. \qquad (12)$$

Substituting the values of Ω and $q(A)$ into the first equation of (11), we obtain a formula connecting the amplitude of the periodic solution and the system parameters:

$$\frac{2k_1k_2}{\pi}\left(\arcsin\frac{b}{A} + \frac{b}{A}\sqrt{1 - \frac{b^2}{A^2}}\right) = \frac{T_1 + T_2}{T_1 T_2}. \qquad (13)$$

In order to investigate the stability of the periodic solution, we employ an approximate criterion [20]. For this purpose, we write the values $X(a, \omega)$ and $Y(a, \omega)$ from the characteristic polynomial (10) for $p = j\omega$:

$$X(a, \omega) = k_1k_2 q(a) - (T_1 + T_2)\omega^2,$$
$$Y(a, \omega) = \omega - T_1 T_2 \omega^3,$$

where $\underline{a}$ and ω represent the amplitude and frequency near the periodic solution.

The periodic solution will be stable if the inequality

$$\left(\frac{\partial X}{\partial a}\right)^* \left(\frac{\partial Y}{\partial \omega}\right)^* - \left(\frac{\partial X}{\partial \omega}\right)^* \left(\frac{\partial Y}{\partial a}\right)^* > 0. \qquad (14)$$

holds. The asterisk indicates that after the derivatives have been taken it is necessary to substitute the amplitude and frequency values for the periodic solution $a = A$ and $\omega = \Omega$.

On the basis of the formula for $q(A)$, we first construct a graph (Fig. 170) and then find the corresponding derivatives. As a result, we obtain

$$\left(\frac{\partial X}{\partial a}\right)^{\circ} = k_c\left(\frac{\partial q}{\partial a}\right)^{\circ} < 0 \text{ for } A > b.$$
$$\left(\frac{\partial Y}{\partial a}\right)^{\circ} = 0.$$
$$\left(\frac{\partial X}{\partial \omega}\right)^{\circ} = -2(T_1 + T_2)\Omega < 0.$$
$$\left(\frac{\partial Y}{\partial \omega}\right)^{\circ} = 1 - 3T_1T_2\Omega^2.$$

and since from (12)

$$\Omega^2 = \frac{1}{T_1T_2}, \text{ then } \left(\frac{\partial Y}{\partial \omega}\right)^{\circ} = -2 < 0.$$

Employing Criterion (14), we have

$$\left(\frac{\partial X}{\partial a}\right)^{\circ}\left(\frac{\partial Y}{\partial \omega}\right)^{\circ} - \left(\frac{\partial X}{\partial \omega}\right)^{\circ}\left(\frac{\partial Y}{\partial a}\right)^{\circ} = -2k_c\left(\frac{\partial q}{\partial a}\right)^{\circ} > 0 \text{ for } A > b.$$

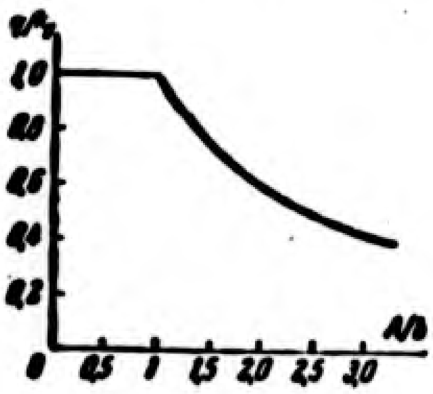


Fig. 170. Variation in harmonic-linearization coefficient with oscillation amplitude, Problem 269.

As a consequence, following the transient, the system will exhibit steady self-oscillation, since for the given parameter values, the condition A > B is satisfied. For values A < B, there will be no self-oscillation. This is understandable, since in this case the amplifier will operate on the linear portion of the static characteristic.

From Formulas (12) and (13), we construct the curves for the variation in self-oscillation amplitude and frequency with variation in each of the system parameters. It is clear from (12) that the self-oscillation frequency will depend solely on the time constants T_1 and T_2. Each of the time constants will affect the variation in self-oscillation amplitude and frequency, however. Thus the desired curves are plotted as functions of the parameters $k = k_\ell k_u$, T_2, and b.

Since Eq. (13) is transcendental in the amplitude, it is desirable to find expressions for the parameters in explicit form as a function of amplitude.

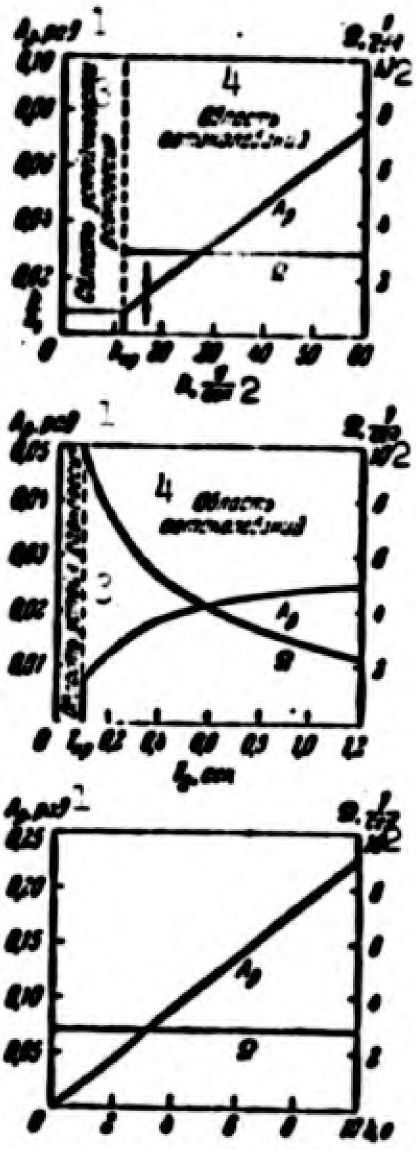


Fig. 171. Variations in self-oscillation amplitude and frequency as a function of system parameters, Problem 269. 1) rad; 2) sec; 3) region of stable equilibrium; 4) self-oscillation region.

In order to determine the effect of servo-system transfer constant on self-oscillation amplitude, we obtain from (13) the formula

$$k=\frac{\pi(T_1+T_2)}{2T_1T_2\left(\arcsin\frac{b}{A}+\frac{b}{A}\sqrt{1-\frac{b^2}{A^2}}\right)}. \quad (15)$$

It is clear from (15) that for values A = b, the transfer constant, which we shall call the critical constant, will equal

$$k_{kr}=\frac{T_1+T_2}{T_1T_2},$$

i.e., system self-oscillation occurs only in a well-defined region of values of the transfer constant $k \geq k_{kr}$. For values $A \leq B$, the servosystem may be considered to be linear and the transfer constant $k = k_l k_u$. Here the transfer constant found from the condition for limiting stability of the linear system will coincide with the value k_{kr}, i.e.,

$$k_{kr}=\frac{T_1+T_2}{T_1T_2}.$$

Thus, the investigation of the servosystem as a linear system gives the result that outside the stability region there lies a region of instability, and an investigation allowing for the saturation-type nonlinearity leads to the conclusion that outside the stable-equilibrium region there lies a region of stability with a steady self-oscillation process.

Formulas (12) and (13) were obtained with the determination of a periodic solution for the input variable of the nonlinear element, i.e., for the amplifier input voltage. It is of interest to obtain this solution for the angle β at the servo-system output.

The frequency of the periodic solution will be the same for any variable in this system. The value of the amplifier input-voltage amplitude is easily converted to the amplitude of angle β by means of the equation of the error-sensing element

$$u_1 = -k_1\beta,$$

and, as a consequence,

$$A = k_1 A_\beta$$

where A_β is the amplitude of the output-shaft oscillations for the servo-system reducing gear.

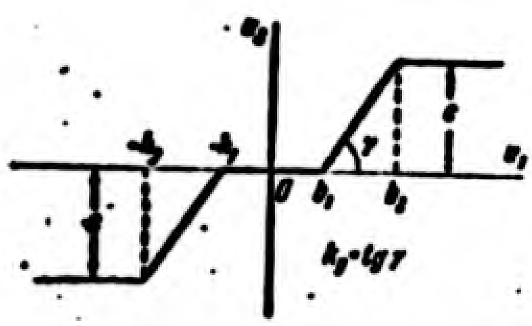


Fig. 172. Nonlinear static characteristic of electronic amplifier, Problem 270.

The calculations carried out for k_1 = 100 v/rad and the assumed parameter values are represented as curves in Fig. 171. The curves given show which parameters can be used to vary oscillation amplitude and frequency and to suppress (kill) self-oscillation.

For the selected parameter values, the oscillation frequency is Ω = 3.16 1/sec and the amplitude is A_β = 0.022 rad.

270. Use the method of harmonic linearization to investigate the servosystem of Problem 269 if the approximate amplifier static characteristic has the form shown in Fig. 172 and b_1 = 0.2 v, b_2 = 1 v, and k_u has its former value.

Answer. For the steady regime, there will be regions depending on the parameter values: a region of stable equilibrium and an oscillation region. There will be two periodic solutions in the oscillation region. Figure 173 shows the way in which the periodic-solution amplitudes and

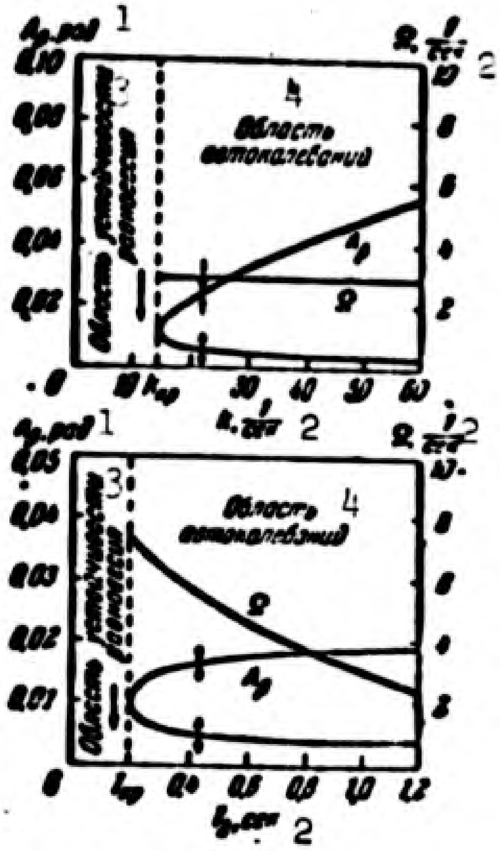


Fig. 173. Relationships for variation in amplitude and phase of periodic solution, Problem 270. 1) rad; 2) sec; 3) stable equilibrium region; 4) self-oscillation region.

the frequency depend on the transfer constand $\underline{k}$ and time constant T_2. The branches for the high-amplitude values belong to the stable periodic solution – the self-oscillations. The small-amplitude branches belong to the unstable periodic solution.

271. For the relay temperature-regulation system having the block diagram shown in Fig. 174, use the method of harmonic nonlinearity linearization to find the self-oscillation region and stable equilibrium region; construct curves for the variations in temperature-oscillation amplitude and frequency as a function of the following parameters: k_l, the trans-

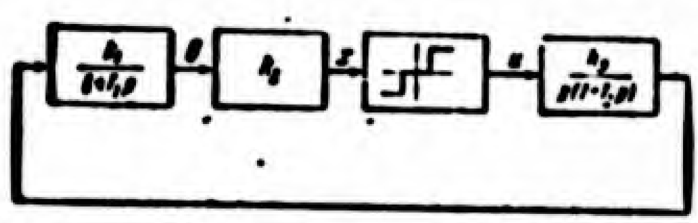


Fig. 174. Block diagram of temperature-regulation relay system, Problem 271.

fer constant of the system linear portion, and T_2, the time constant of the control-element drive mechanism. The nonlinear static characteristic of the relay amplifier is shown in Fig. 175. The system parameters have the following values: T_1 = 10 sec, T_2 = 0.1 sec, c = 25 v, b = 0.01 rad, $k_l = k_1 k_2 k_3$ = 0.01 1/sec·v, k_2 = 0.01 rad/degree.

Answer. The curves showing the variation in self-oscillation amplitude and frequency as a function of variation in k_l and T_2 are shown in

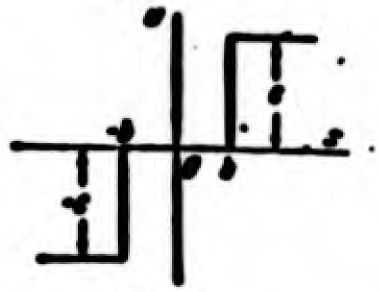


Fig. 175. Static characteristic of nonlinear element, Problem 271.

Fig. 176. In the self-oscillation region there are two periodic solutions: oscillation at large amplitudes and an unstable periodic solution for small amplitudes.

272. For the nonlinear system having the block diagram shown in Fig. 177, use the method of harmonic linearization to determine self-os-

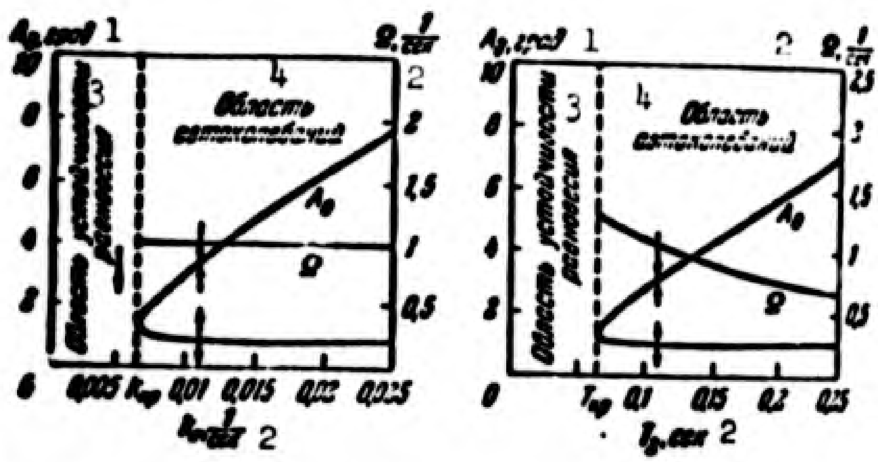


Fig. 176. Variations in periodic-solution amplitude and frequency as a function of system parameters, Problem 271. 1) Degree; 2) sec; 3) stable-equilibrium regions; 4) self-oscillation region.

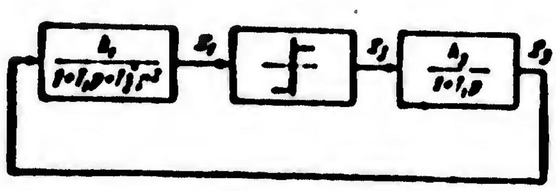


Fig. 177. Block diagram of nonlinear system, Problem 272.

cillation amplitude and frequency with the aid of frequency-response curves if the parameters of the linear elements are given: $T_2^2 = 0.001$ sec^2, $T_1 = 0.01$ sec, $k_1 = 10$, $T_3 = 0.02$ sec, $k_3 = 5$, and the static characteristic of the nonlinear element is that shown in Fig. 178.

Solution. In order to find the oscillations, we construct the amplitude-phase frequency response of the linear portion of the system $W_l(j\omega)$ and the locus of the harmonically linearized nonlinear element $-1/W_n(A)$. The point of intersection of these curves will also correspond to the periodic solution determining system self-oscillation for the variable x_1.

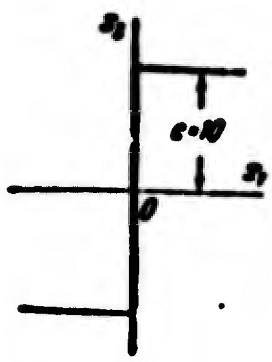


Fig. 178. Static characteristic of nonlinear element, Problem 272.

According to the block diagram, the transfer function of the linear portion of the system will be

$$W_l(p)=\frac{k_1k_2}{(1+T_1p+T_2^2p^2)(1+T_3p)}.$$

The formulas for determining the amplitude and phase of the frequency response for the linear portion will take the form

$$A=\frac{k_1k_2}{\sqrt{[(1-T_2^2\omega^2)^2+T_1^2\omega^2](1+T_3^2\omega^2)}},$$

$$\varphi=-\operatorname{arctg}\frac{T_1\omega}{1-T_2^2\omega^2}-\operatorname{arctg}T_3\omega.$$

After substitution of the parameter values

$$A=\frac{50}{\sqrt{[(1-0{,}001\omega^2)^2+0{,}0001\omega^2](1+0{,}0004\omega^2)}}, \qquad (1)$$

$$\varphi=-\operatorname{arctg}\frac{0{,}01\omega}{1-0{,}001\omega^2}-\operatorname{arctg}0{,}02\omega. \qquad (2)$$

The amplitude-phase frequency response $W_l(j\omega)$ calculated from Formulas (1) and (2) is shown in Fig. 179.

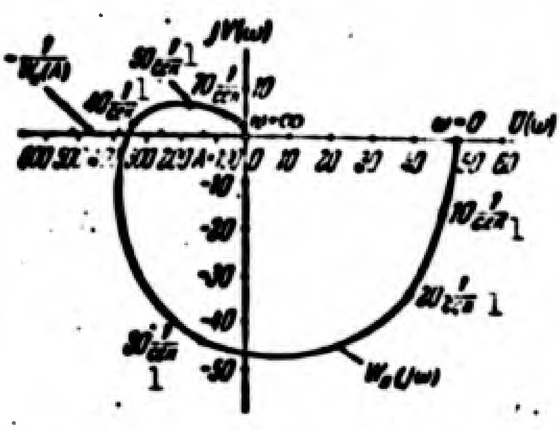


Fig. 179. Frequency characteristics for linear portion of system and nonlinear element, Problem 272. 1) sec.

To calculate the characteristic $-1/W_n(A)$, we take a value of the harmonic-linearization coefficient for the ideal relay characteristic

$$W_n(A)=q(A)=\frac{4c}{\pi A}.$$

We then obtain

$$-\frac{1}{W_n(A)}=-\frac{\pi}{4c}A,$$

which for a value c = 10 yields

$$-\frac{1}{W_n(A)}=-0{,}0785A. \qquad (3)$$

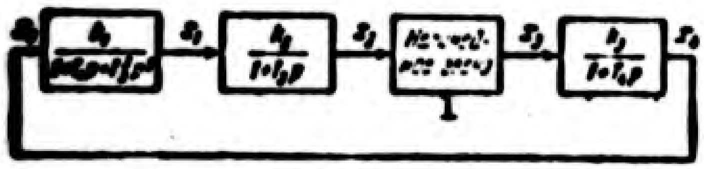


Fig. 180. Block diagram of nonlinear system, Problem 273. 1) Nonlinear element.

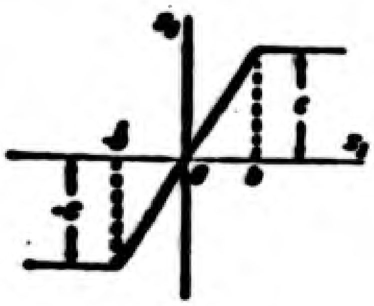


Fig. 181. Static characteristic of nonlinear element, Problem 273.

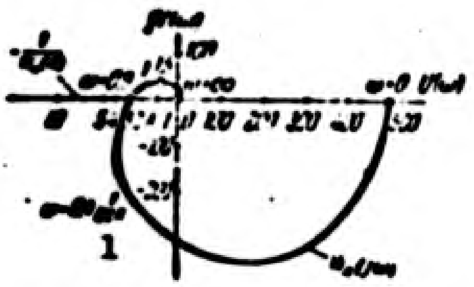


Fig. 182. Frequency characteristics for linear part of system and nonlinear element, Problem 273.

Taking values of A, we represent the characteristic $-1/W_n(A)$ on the complex plane (Fig. 179), which in the given case is a straight line coinciding with real axis.

The point of intersection of the characteristics permits us to use $W_l(j\omega)$ to find the frequency of the periodic solution $\Omega \approx 39$ 1/sec and to use $-1/W_n(A)$ to find the amplitude of the periodic solution $A \approx 370$.

The stability of the periodic solution is determined from the condition that in going from small amplitudes to large amplitudes on the characteristic $-1/W_n(A)$ we go from the interior to the exterior of the region covered by the characteristic $W_l(j\omega)$. As a consequence, for the problem considered, the periodic solution will be stable, i.e., there is self-oscillation.

273. Use the method of harmonic linearization with the gain-phase characteristics for the nonlinear system having the block diagram shown in Fig. 180 and with the static characteristic of the nonlinear element shown in Fig. 181 to determine whether or not steady self-oscillation

occurs in the system. If self-oscillation occurs in the system, determine the amplitude and frequency for the variable x_2. The system parameters have the values: T_2 = 1 sec, T_1 = 1.4 sec, T_3 = 1 sec, T_4 = 4 sec, $k_\ell = k_1k_2k_3 = 500$, b = 1, c = 50.

Answer. Self-oscillation with an amplitude A ≈ 2.7 and frequency Ω ≈ 0.8 1/sec will occur in the system. The gain-phase characteristics $W_\ell(j\omega)$ and $-1/W_n(A)$ are shown in Fig. 182.

274. A second-order relay system with time delay (Fig. 183a) is given. The linear portion is described by the equation

$$(T_1p+1)px_1=kx_2 \qquad (1)$$

while the nonlinear element is described by the equation

$$x_2=F_1(x)=e^{-\tau p}F(x), \qquad (2)$$

where F(x) is given in the form of an ideal static relay characteristic (Fig. 183b).

Determine the forced oscillations occuring under the sinusoidal external input

$$f(t)=B\sin\Omega_v t. \qquad (3)$$

We take the following parameter values: k = 10 1/sec, c = 10v, T_1 = 0.01 sec, τ = 0.01 sec, Ω_v = 10 1/sec, B = 20v.

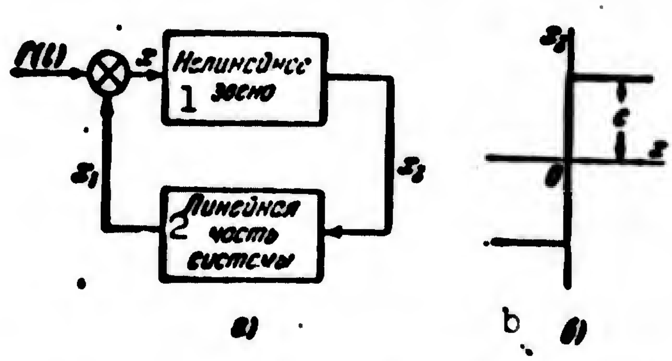


Fig. 183. Block diagram and static characteristic for nonlinear element of relay system, Problem 274. 1) Nonlinear element; 2) linear part of system.

Solution. We seek the forced oscillations in the input variable of the nonlinear element in the form

$$x = A_v \sin(\Omega_v t + \varphi). \qquad (4)$$

We determine the amplitude A_v and phase shift φ for the forced oscillations. Substituting (2) into (1) and allowing for the face that

$$x_1 = f(t) - x,$$

we obtain the system equation

$$(T_y p + 1) p x + k F_v(x) = (T_y p + 1) p f(t). \qquad (5)$$

The external sinusoidal input (3) is represented in the form

$$f(t) = B \sin[(\Omega_v t + \varphi) - \varphi] = $$
$$= B \cos\varphi \sin(\Omega_v t + \varphi) - B \sin\varphi \cos(\Omega_v t + \varphi).$$

Remembering that from (4)

$$p x = A_v \Omega_v \cos(\Omega_v t + \varphi),$$

we finally obtain

$$f(t) = \frac{B}{A_v}\left(\cos\varphi - \frac{\sin\varphi}{\Omega_v} p\right) x.$$

Substituting the value obtained for f(t) into Eq. (5), we have the system homogeneous nonlinear equation for the variable

$$(T_y p + 1)\left[1 - \frac{B}{A_v}\left(\cos\varphi - \frac{\sin\varphi}{\Omega_v} p\right)\right] p x + k F_v(x) = 0. \qquad (6)$$

Harmonic linearization of the ideal relay characteristic in accordance with the formula given in Appendix 20, with allowance for the time delay, yields

$$F_v(x) = \frac{4c}{\pi A_v} e^{-\tau p} x. \qquad (7)$$

The characteristic equation corresponding to Eq. (6), taking (7) into account, will take the form

$$(T_y p + 1)\left[1 - \frac{B}{A_v}\left(\cos\varphi - \frac{\sin\varphi}{\Omega_v} p\right)\right] p + \frac{4kc}{\pi A_v} e^{-\tau p} = 0.$$

Substituting the imaginary value $p = j\Omega_v$ into the obtained equation and remembering that

$$\cos\varphi - j \sin\varphi = e^{-j\varphi},$$

we obtain

$$A_v - \frac{4bce^{-j\varphi_0}}{\pi(T_1\Omega_0^2 - j\Omega_0)} = Re^{-j\varphi}.$$

after substitution of the numerical values taken for the parameters, we have

$$A_v - 2{,}5 - j12{,}3 = 20e^{-j\varphi}. \qquad (8)$$

We make use of the graphical method described in [20]. On the complex plane (Fig. 184), we draw a circle of radius R = 20 representing the right side of Eq. (8) and a line $z(A_v)$ corresponding to the left side of the same equation. On the line we plot the values of the forced-oscillation amplitudes A_v. The point of intersection of the circle and the line yields a solution for A_v and φ. We note that positive values of phase-shift angle are measured clockwise, since the right side of (8) is a vector rotating through the angle φ in a direction opposite to that usually used to measure positive angles. As we can see,

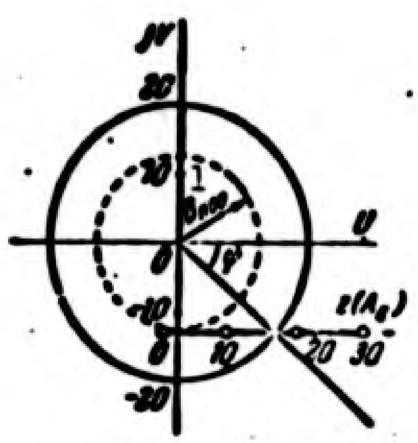


Fig. 184. Graphical construction for determining forced oscillations in nonlinear system, Problem 274. 1) B_{por}.

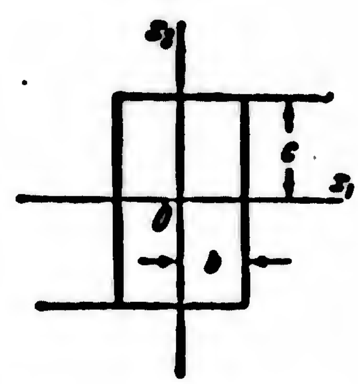


Fig. 185. Static characteristic of nonlinear element, Problem 275.

in our case A_v = 18.2 v, $\varphi = 38^o$. From the completed plot, we can conclude that for the frequency used Ω = 10 1/sec the minimum threshold amplitude for the external input B_{por} = 12.3 v, i.e., it equals the radius of the circle tangent to the line A_v.

275. For a third-order relay system with a linear portion described by the equation

$$(T_1p+1)(T_2p+1)px_1=kx_2,$$

and a relay element with coordinate delay having the static characteristic shown in Fig. 185, determine the amplitude and phase of forced oscillations if an external sinusoidal signal is applied to the system. The following numerical values are assumed for the system parameters and the external signal: k = 10 1/sec, c = 10v, b = 4v, T_1 = 0.01 sec, T_2 = 0.02 sec, B = 20v, Ω_v = 10 1/sec.

Answer. A_v = 21v, φ = 35°.

Manuscript Page No.	[Transliterated Symbols]
217	Р = R = reduktor = reducing gear
217	Д = D = dvigatel' = motor
223	ДР = DR = datchik rassoglasovaniy = error sensor
223	УПТ = UPT = usilitel' postoyannogo toka = dc amplifier
223	Г = G = generator = generator
223	Дв = Dv = dvigatel' = motor
223	УО = UO = upravlyayemyy ob"yekt = controlled system
226	я = ya = yakor' = armature
227	х.х = kh.kh = kholostoy khod = no load
227	н = n = nominal'nyy = nominal
228	л = l = lineynyy = linear
230	у = u = usilitel' = amplifier
231	кр = kr = kriticheskiy = critical
234	н = n = nelineynyy = nonlinear
238	в = v = vozdeystviye = input

239 в = v = vynuzhdennyy = forced

239 пор = por = porogovoy = threshhold

BLANK PAGE

Part 2

DESIGN METHODS FOR AUTOMATIC SYSTEMS

Chapter 10

SELECTION OF AUTOMATIC CONTROL SYSTEM PARAMETERS FROM REQUIRED ACCURACY IN TYPICAL MODES

§28. CALCULATIONS FOR VARIABLES GIVEN AS TIME FUNCTIONS

276. Determine the location of the logarithmic amplitude characteristic for an open-loop servosystem from the condition that the following error not exceed $\theta_{max} \leq 1.5'$ when the control input is governed by the harmonic law

$$\theta_1 = \theta_{1max} \sin \omega_k t,$$

where $\theta_{1max} = 25^{\circ}$,

$$\omega_k = \frac{2\pi}{T_k} = 6.28 \text{ 1/sec}.$$

The block diagram of the servosystem is shown in Fig. 186a.

<u>Solution</u>. The following error due to a variation in control input equals

$$\theta_{max} = \frac{1}{|1 + W(j\omega)|} \theta_{1max} \text{ for } \omega = \omega_k. \qquad (1)$$

where $W(j\omega)$ is the system open-loop frequency transfer function. Since, as a rule, in present-day servosystems $|W(j\omega_k)| \gg 1$, it is possible to use the approximate relationship

$$\theta_{max} \approx \frac{\theta_{1max}}{|W(j\omega_k)|}. \qquad (2)$$

Solving Expression (2) for $|W(j\omega_k)|$, we obtain the required value of the modulus for the system open-loop frequency transfer function at

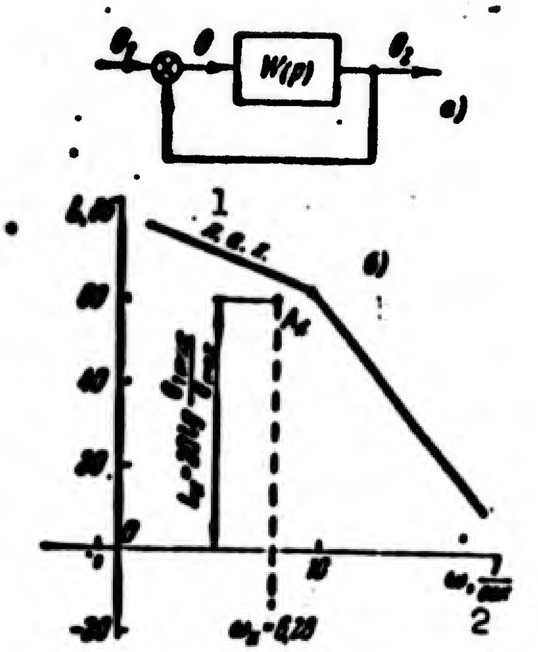


Fig. 186. a) Block diagram of servosystem; b) plotting of control point A_k. 1) Logarithmic amplitude characteristic; 2) sec.

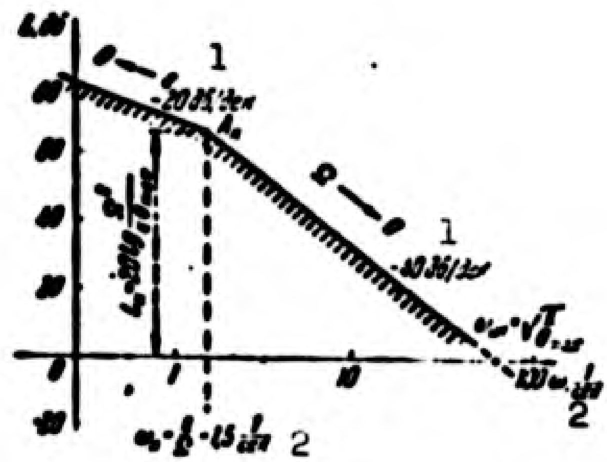


Fig. 187. Plotting of forbidden region. 1) db/decade; 2) sec.

the frequency ω_k:

$$|W(j\omega_k)| = \frac{\theta_{1max}}{\theta_{max}} \qquad (3)$$

or

$$L_k = 20\lg|W(j\omega)| = 20\lg\frac{\theta_{1max}}{\theta_{max}}. \qquad (4)$$

Using Formula (4) we plot the so-called control point A_k in a logarithmic coordinate system (Fig. 186b):

$$\omega_k = 6{,}28 \ 1/\text{сек}, \ 20\lg\frac{\theta_{1max}}{\theta_{max}} = 20\lg\frac{25\cdot 60}{1{,}5} = 60 \text{ db}.$$

The required following accuracy will be attained if the logarithmic amplitude characteristic of the system lies above point A_k; in the limit, it will intersect this point (Fig. 186b).

277. Determine the forbidden region for the logarithmic amplitude characteristic of an open-loop servosystem from the condition that the following error not exceed $\theta_{max} \leq 1.0'$ under a variation in control input having a maximum rate of Ω = 40 degree/sec and a maximum acceleration

$$\varepsilon = 60 \text{ degree/sec}^2.$$

Solution. Where the law governing the variation in the control input is not known, the calculations may be carried out for the equivalent sinusoidal input [3].

The parameters of the equivalent process are determined from the formulas

$$\omega_k = \frac{\varepsilon}{\Omega} = 1{,}5 \ 1/\text{sec},$$
$$\theta_{1max} = \frac{\Omega}{\omega_k} = \frac{\Omega^2}{\varepsilon} = \frac{40}{1{,}5} = 26{,}7^\circ. \qquad (1)$$

Here ω_k is the angular frequency of the oscillations in the equivalent sinusoidal input, θ_{1max} is the amplitude of the oscillations in the equivalent sinusoidal input.

The coordinates of the control point A_k (see the preceeding problem) will equal (Fig. 187)

$$\omega_k = \frac{\varepsilon}{\Omega} = 1{,}5 \ 1/\text{sec},$$
$$L_k = 20 \lg \frac{\theta_{1max}}{\theta_{max}} = 20 \lg \frac{\Omega^2}{\varepsilon\theta_{max}} = 20 \lg 1600 \approx 63 \text{db}.$$

In order to construct the entire forbidden region, we find the locus of the control points A_{ki} corresponding to the following two cases: 1) when Ω is a maximum, and when ε gradually drops to zero, and a) when ε is a maximum and Ω drops to zero. In the first case, the locus will be a straight line passing with slope -40 db/decade through point A_k. In the second case it will be a line with slope -20 db/decade (Fig. 187).

In order to obtain the required following accuracy, the logarithmic amplitude characteristic of the open-loop servosystem should not enter the forbidden zone bounded by these lines.

278. For the servosystem whose block diagram is shown in Fig. 186a, construct the low-frequency portion of the desired logarithmic amplitude characteristic and determine the required over-all gain from the condition by which the required following accuracy is ensured. The sys-

tem possesses first-order astatism. The requirements for following accuracy are the same as for Problem 277.

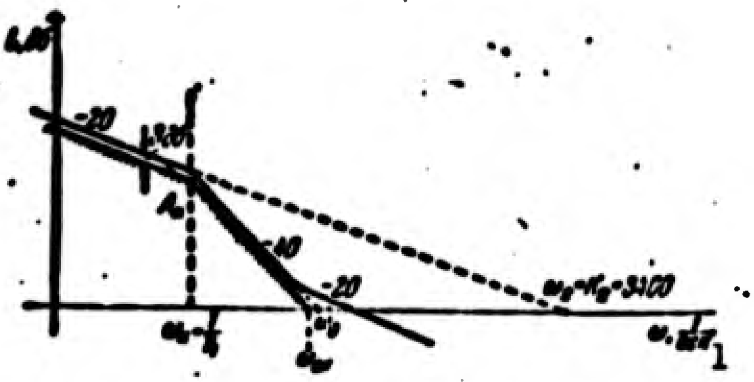


Fig. 188. Construction of local-frequency portion of desired logarithmic amplitude characteristic. 1) sec.

Solution. In order to simplify the problem of system damping, the logarithmic amplitude characteristic should be located as far as possible to the left. The maximum permissible shift of the logarithmic amplitude characteristic to the left is bounded by the forbidden zone with respect to accuracy. From this viewpoint it is desirable for the low-frequency branch of the desired logarithmic amplitude characteristic, with a slope of -40 db/decade, to pass as close as possible to the line bounding the forbidden zone (Fig. 188), i.e., for $\omega_0 = \omega_{0k} = 60$ 1/sec and $T_1 = 1/\omega_k$.

The first asymptote for the logarithmic amplitude characteristic, however, which has a slope of -20 db/decade, should pass above the boundary of the forbidden region by 3 db (Fig. 188).

If this asymptote is continued until it intersects the zero-decibel axis, the point of intersection ω_Ω will give the over-all system open-loop gain (velocity figure of merit K_Ω).

From Fig. 188 we have

$$K_\Omega = \sqrt{2}\frac{\Omega}{\theta_{max}} = 1{,}41\frac{40\cdot 60}{1} = 3400 \text{ 1/sec.}$$

279. Determine the required over-all open-loop gain of a servosys-

tem. The system has second-order astatism. The remaining data is the same as for Problem 278.

Answer. The over-all open-loop gain of the system (acceleration figure of merit) is

$$K_{\varepsilon} = 3600 \ 1/\text{sec}^2.$$

280. For a servosystem possessing first-order astatism, determin- the parameters of the low-frequency portion of the desired logarithmic amplitude characteristic from the condition that the required following accuracy be insured, with and without allowance for the load torque. The maximum following speed is Ω = 24 degree/sec, the maximum acceleration is ε = 20 degree/sec, the permissible error is θ_{max} = 0.1°. The load torque at the shaft of the actuating motor is M_n = 2g·cm. The inflexibility of the mechanical characteristic is $\beta = \Omega_0/M_0$ = 5000·6/ /57.3·10 = 52.3 1/g·cm·sec (where Ω_0 = 5000 rpm is the no-load speed, M_0 = 10g·cm is the motor starting torque). The gear ratio of the reduction gear is i = 1000.

Solution. 1. There is no load torque (see Problem 278). Then

$$T_1 = \frac{1}{\omega_\varepsilon} = \frac{\Omega}{\varepsilon} = 1{,}2 \ \text{sec}, \quad (1)$$

$$K_\Omega = \sqrt{2}\frac{\Omega}{\theta_{max}} = \frac{1{,}41 \cdot 24}{0{,}1} = 338 \ 1/\text{sec}. \quad (2)$$

2. The motor is loaded by a torque M_n = 2g·cm. The system is damped by the first method of [3], i.e., by the introduction of inverse feedback around the motor or of derivatives of the error angle. Then

$$T_1 = \frac{1}{\omega_\varepsilon} = \frac{\Omega}{\varepsilon} = 1{,}2 \ \text{sec}, \quad (3)$$

$$K_\Omega = \sqrt{2}\frac{\Omega}{\theta_{max}} + \frac{\beta M_n}{\theta_{max} i} = 338 + \frac{104{,}6 \cdot 57{,}3}{0{,}1 \cdot 1000} =$$

$$= 338 + 60 = 398 \ 1/\text{sec}, \quad (4)$$

where $\theta_{max} i$ is the error at the motor shaft.

3. The motor is loaded by a torque M_n = 2g·cm. The system is damped by the second method of [3], i.e., by the introduction of a time lag

in the amplifier channel ahead of the motor terminals. Then

$$T_1 < 0{,}236\,\frac{\sqrt{(\theta_{max} + \theta_m)^3}}{\theta_m \sqrt{i}}. \qquad ([illegible])$$

where $\theta_m = \beta M_n / K_\Omega i$ is the torque error in terms of the actuating shaft.

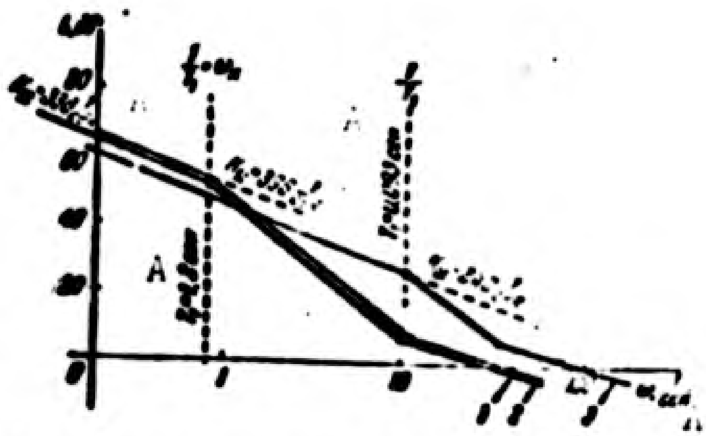


Fig. 189. Low-frequency branches of desired logarithmic amplitude characteristic: 1) With no allowance for load torque; 2) with allowance for load torque, first damping method; 3) allowance for load torque, second damping method. A) sec.

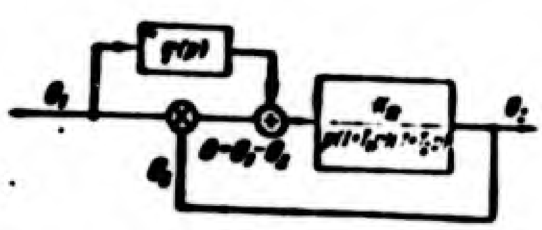


Fig. 190. Block diagram of combination-control servosystem.

If we take the over-all gain as

$$K_0 = \frac{\Omega}{\theta_{max}} = \frac{24}{0{,}1} = 240 \text{ 1/sec},$$

then

$$\theta_m = \frac{\beta M_n}{K_0 i} = \frac{104{,}6 \cdot 57{,}3}{240 \cdot 1000} = 0{,}025^\circ$$

and

$$T_1 < 0{,}236\,\frac{\sqrt{(0{,}1 + 0{,}025)^3}}{0{,}025\sqrt{20}} = 0{,}0935 \text{ sec}.$$

If the over-all gain is increased, for example, to

$$K_0 = 300 \text{ 1/sec},$$

then

$$\theta_m = \frac{\beta M}{K_0 i} = \frac{104{,}6 \cdot 57{,}3}{300 \cdot 1000} = 0{,}02^\circ$$

and

$$T_1 < 0{,}236\,\frac{\sqrt{(0{,}1 + 0{,}02)^3}}{0{,}02\sqrt{20}} = 0{,}11 \text{ sec}.$$

For the sake of illustration, on Fig. 189 we have plotted the logarithmic amplitude characteristic corresponding to the three cases considered.

281.* For a closed-loop combination-control system, determine the levels of the compensating signals with respect to the first and second derivatives of the control input such that a system possessing first-order astaticism will have no velocity error or acceleration-dependent error. The block diagram of the closed-loop combination control system is shown in Fig. 190. The compensating signals have the form

$$\varphi(p)\theta_1=(\tau_1 p+\tau_1\tau_2 p^2)\theta_1,$$

where τ_1 is the ratio obtained by dividing the signal slope with respect to the first derivative of θ_1 by the signal slope with respect to the error θ, while τ_2 is the ratio obtained by dividing the signal slope with respect to the second derivative by the signal slope with respect to the first derivative of θ_1.

Solution. In a combination-control system the output variable θ_2 is proportional not only to the error θ but also to the compensating signals $\varphi(p)\theta_1$, i.e.,

$$\theta_2=W(p)[\theta+\varphi(p)\theta_1],$$

where $W(p)=\frac{K_v}{p(1+T_1p)(1+T_2p)}$ is the system open-loop transfer function neglecting regulation with respect to the control input.

The system closed-loop error is

$$\theta=\frac{[1-W(p)\varphi(p)]\theta_1}{1+W(p)}.$$

Substituting the values of W(p) and φ(p), we obtain

$$\theta=\frac{[T_1T_2p^3+(T_1+T_2-K_v\tau_1\tau_2)p^2+(1-K_v\tau_1)p]\theta_1}{T_1T_2p^3+(T_1+T_2)p^2+p+K_v}.$$

When the condition

$$\tau_1=\frac{1}{K_v}$$

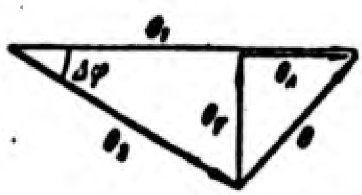


Fig. 191. Vector error diagram.

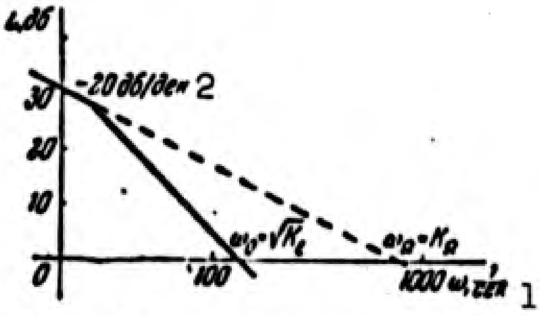


Fig. 192. Low-frequency section of desired logarithmic amplitude characteristic. 1) sec; 2) db/decade.

is satisfied, there will be no velocity error in the system. When the additional condition

$$T_1 + T_2 = K_\Omega \tau_1 \tau_2$$

or

$$\tau_2 = T_1 + T_2$$

is satisfied, the acceleration-dependent error will also vanish.

The system open-loop equivalent transfer function W(p) corresponds to a system with third-order astatism [4]

$$W_э(p) = \frac{W(p)[1 + \varphi(p)]}{1 - W(p)\varphi(p)} =$$
$$= \frac{K_\Omega (1 + \tau_1 p + \tau_1 \tau_2 p^2)}{T_1 T_2 p^3 + (T_1 + T_2 - K_\Omega \tau_1 \tau_2) p^2 + (1 - K_\Omega \tau_2) p} =$$
$$= \frac{K_\Omega / T_1 T_2 (1 + \tau_1 p + \tau_1 \tau_2 p^2)}{p^3}.$$

282. For a control system, construct the low-frequency portion of the desired logarithmic amplitude characteristic if we know that for a control input governed by the law $\theta_1 = \theta_{1max} \sin \omega_k t$, where

$$\theta_{1max} = 30°,$$
$$\omega_k = \frac{2\pi}{T_k} = 12.56 \text{ 1/sec},$$

the permissible following error should not exceed: in phase, $\Delta\varphi \leq 1°$, in amplitude $\Delta\theta/\theta_{1max} \leq 1\%$. The system possesses first-order astatism. In the low-frequency region, the system open-loop transfer function is approximated by the expression

$$W(j\omega)=\frac{K_0}{j\omega(1+j\omega T_1)}.$$

Solution. Figure 191 shows the vector error diagram. The following error is

$$\theta=\frac{\theta_1}{1+W(j\omega)}=(P+jS)\theta_1=\theta_A+j\theta_\varphi$$

where θ_A is the in-phase error component and θ_φ is the quadrature error component. The phase error is

$$\Delta\varphi=\operatorname{arctg}\frac{|\theta_\varphi|}{|\theta_1-\theta_A|},$$

while the relative amplitude error is

$$\frac{\Delta\theta}{\theta_{1max}}=\frac{|\theta_1|-|\theta_2|}{\theta_{1max}}.$$

If we assume that at the frequency ω_k the modulus $|W(j\omega_k)|\gg 1$, then the phase error may be computed from the approximate formula

$$\Delta\varphi\approx\frac{1}{\theta_{1max}}I_m\frac{\theta_{1max}}{W(j\omega_k)}=\frac{\omega_k}{K_0},$$

and the relative amplitude error from the formula

$$\frac{\Delta\theta}{\theta_{1max}}\approx\frac{\theta_A}{\theta_{1max}}\approx\frac{1}{\theta_{1max}}\operatorname{Re}\frac{\theta_{1max}}{W(j\omega_k)}=\frac{\omega_k^2T_1}{K_0}=\frac{\omega_k^2}{K_e}.$$

Having the phase and relative amplitude errors, we can determine the limiting left-hand positions of the first and second asymptotes of the logarithmic amplitude characteristic:

$$\omega_0=K_0=\frac{\omega_k}{\Delta\varphi}=\frac{12.56\cdot 57.3}{1}=720 \text{ 1/sec},$$

$$\omega_e=\sqrt{K_e}=\omega_k\sqrt{\frac{\theta_{1max}}{\Delta\theta}}=12.56\sqrt{\frac{1}{0.01}}=152 \text{ 1/sec}.$$

Figure 192 shows the low-frequency portion of the desired logarithmic amplitude characteristic.

283.* Determine the required gain for a constant-speed tachometer drive (Fig. 193). The admissible error in maintaining speed with a motor-shaft load torque $M_n = 0.2\ M_{k.z}$ should not exceed 0.1% of the no-

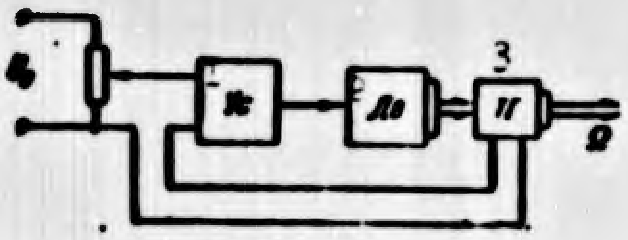


Fig. 193. Constant-speed tachometer drive. 1) Amplifier; 2) motor; 3) tachometer generator.

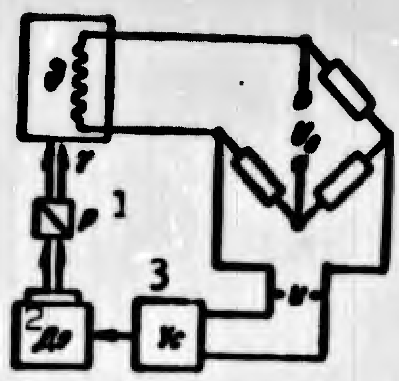


Fig. 194. Temperature regulation system. 1) Reduction gear; 2) motor; 3) amplifier.

load spead.

Solution. The regulation error $\Delta\Omega$ consists of two terms:

$$\Delta\Omega = \frac{1}{1+W(p)}\Omega_z \pm \frac{W_c(p)}{1+W(p)}M_n \qquad (1)$$

where Ω_z is the given drive speed, $W(p)$ is the system open-loop transfer function, $W_c(p)$ is the system open-loop transfer function with respect to the load torque. The first term corresponds to the error introduced by the selected control law, while the second term specifies the error component introduced by the effect of the load torque M_n.

If we take into account the amplifier time constant T_u and the electromechanical time constant of the motor T_m, then

$$W(p) = \frac{K}{(1+T_m p)(1+T_u p)}$$

and

$$W_c(p) = \frac{\frac{\Omega_0}{M_{k.z}}}{(1+T_m p)},$$

where $M_{k.z}$ is the short-circuit torque developed by the motor, Ω_0 is the motor no-load speed, and K is the over-all system open-loop gain.

The expression for the regulation error $\Delta\Omega$ takes the form

$$\Delta\Omega = \frac{[T_m T_u p^2 + (T_m + T_u)p + 1]\Omega_z}{T_m T_u p^2 + (T_m + T_u)p + 1 + K} \pm$$
$$\pm \frac{\frac{M_n}{M_{k.z}}\Omega_0(1+T_u p)}{T_m T_u p^2 + (T_m + T_u)p + 1 + K}. \qquad (2)$$

Usually, the system is so adjusted as to minimize the regulation error. This condition corresponds to adjustment such that no static error due to the control law appears in the static system. In order to eliminate the static error, the principal feedback transfer constant $K_{o.s}$ should differ from unity and will equal (see Problem 176)

$$K_{o.s}=\frac{K-1}{K}, \qquad (3)$$

i.e., there should be nonunity feedback in the control system [5].

In order to ensure the required accuracy of speed maintenance for a load torque $M_n = 0.2\ M_{k.z}$, the system open-loop gain K should be chosen on the basis of the condition

$$\Delta\Omega=\frac{\frac{M_n}{M_{k.z}}\Omega_0}{1+K}, \qquad (4)$$

from which we obtain

$$K=\frac{\frac{M_n}{M_{k.z}}-\frac{\Delta\Omega}{\Omega_0}}{\frac{\Delta\Omega}{\Omega_0}}=\frac{0.2-0.001}{0.001}=199. \qquad (5)$$

284. Determine the required over-all gain K for a temperature-regulation system (Fig. 194) from the condition ensuring the required regulation accuracy.

The deviation of the controlled variable ϑ is measured by means of a resistance thermometer connected into a bridge circuit. The voltage u from the bridge diagonal is applied to the balanced amplifier Us, which controls the motor Dv. The motor drives the control element through the reduction gear R. The control element acts on the controlled object by varying the manipulated variable γ.

The element equations have the form:

1. The equation of the controlled object is

$$(1+T_1 p)\vartheta=-k_1\gamma+k_0 F,$$

where T_1[sec] is the object time constant, k_1 and k_0 are transfer constants, F is a disturbance.

2. The equation of the sensing element — the bridge with resistance thermometer — is

$$u = k_2 \dot{\vartheta},$$

where k_2 [v/degree] is the transfer constant.

3. The equation of the drive mechanism together with the amplifier is

$$(1 + T_2 p) p \eta = k_3 u,$$

where T_2 [sec] is the electromechanical time constant and k_3 [1/sec] is the transfer constant.

Solution. The over-all system open-loop gain should be chosen on the basis of the condition

$$K = k_1 k_2 k_3 \geq \frac{k_2 pF}{\theta},$$

where pF is the log governing the variation in the disturbance derivative.

§29. CALCULATIONS INVOLVING RANDOM EFFECTS

285. Determine the parameters of the desired logarithmic amplitude characteristic of a servosystem when we know that the permissible mean-square error (s.k.o.) for following is $\theta_s \leq 0.1°$. The derivative of the output variable is a stationary random process whose spectral density equals $G(\omega) = \frac{2T_s \Omega_{sk}^2}{1 + \omega^2 T_s^2}$, where Ω_{sk} = 15 degree/sec is the mean-square rate of change of the input variable, T_s = 10 sec is the mean duration of constant speed at the input. The servosystem possesses second-order astatism.

Solution. As the desired logarithmic amplitude characteristic we take a characteristic having slopes -40, -20, -40 db/decade. The position and length of the section with slope -20 db/decade should be dictated completely by the need to ensure adequate stability margin [3].

Thus, for example, for a magnitude ratio $M = 1.3$ [3]

$$h = \frac{\omega_3}{\omega_2} = \frac{M+1}{M-1} = 7.67,$$

we have

$$\tau_2 = \omega_0 T_2 = \sqrt{\frac{M}{M-1}} = \sqrt{\frac{1.3}{1.3-1}} = 2.08,$$
$$\tau_3 = \omega_0 T_3 = \frac{\sqrt{M(M-1)}}{M+1} = \frac{\sqrt{1.3(1.3-1)}}{1.3+1} = 0.272,$$
$$\omega_0 = \sqrt{K_\varepsilon}$$

We find the required over-all gain, i.e., the acceleration figure of merit from the condition ensuring the given following accuracy

$$K_\varepsilon = \frac{\theta_{m}}{\varepsilon_m \sqrt{T_2 T_3}}.$$

Since the time constant

$$T_2 = \frac{\tau_2}{\omega_0} = \frac{\tau_2}{\sqrt{K_\varepsilon}},$$

we obtain

$$K_\varepsilon = \sqrt[3]{\frac{\theta_{m}^{2}}{\varepsilon_m^{2} \tau_2^{2}}} = \sqrt[3]{\frac{15^{2}}{2.08^{2}\cdot 10^{-4}\cdot 0.1^{2}}} = 105 \text{ 1/sec}^2.$$

From this we have

$$T_2 = \frac{\tau_2}{\sqrt{K_\varepsilon}} = \frac{2.08}{\sqrt{105}} = 0.203 \text{ sec},$$
$$T_3 = \frac{\tau_3}{\sqrt{K_\varepsilon}} = \frac{0.272}{\sqrt{105}} = 0.0265 \text{ sec}.$$

The logarithmic amplitude characteristic cutoff frequency is

$$\omega_c = \tau_2 \omega_0 = \tau_2 \sqrt{K_\varepsilon} = 2.08\sqrt{105} = 21.4 \text{ 1/sec}.$$

From the values computed, we have plotted the desired servo-system open loop logarithmic amplitude characteristic in Fig. 195.

Checking the mean-square error, we use an exact method:

$$\varepsilon_c = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{G(\omega)\,d\omega}{|1 + W(j\omega)|^2}}.$$

The system open-loop frequency transfer function corresponding to the desired logarithmic amplitude characteristic (Fig. 195) has the form

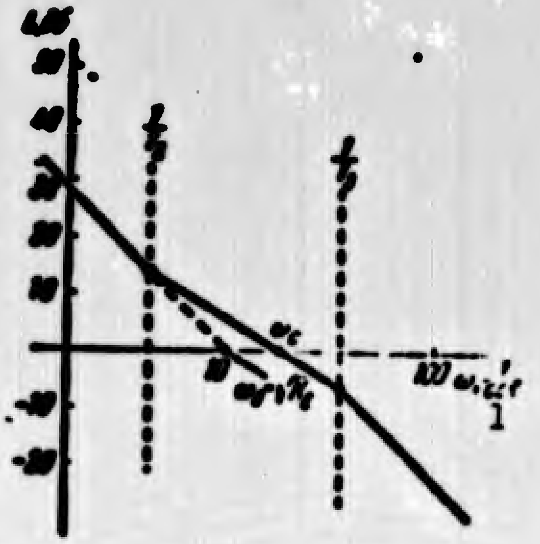


Fig. 195. Calculated logarithmic amplitude characteristic for Problem 285. 1) sec.

Fig. 196. Typical signal at servo-system input. 1) sec.

$$W(j\omega)=\frac{K_0(1+j\omega T_c)}{(j\omega)^2(1+j\omega T_3)}.$$

Then

$$\theta_s=\sqrt{\frac{2T_c\Omega_{cs}^2}{\beta\pi}\int_{-\infty}^{+\infty}\frac{[b_1(j\omega)^4+b_0(j\omega)^2]\,d(j\omega)}{|a_0(j\omega)^4+a_1(j\omega)^3+a_2(j\omega)^2+a_3j\omega+a_4|^2}}.$$

where

$$a_0=T_cT_3=0{,}265,\quad a_1=T_c+T_3=10{,}03,\quad a_2=1+K_0T_cT_3=224,$$
$$a_3=K_0(T_c+T_3)=1070,\quad a_4=K_0=103,$$
$$b_1=T_3^2=7\cdot10^{-4},\quad b_0=-1.$$

Finally, we have $\theta_s = 0.107^{\circ}$, i.e., the value obtained for the mean-square error does not exceed the given value for all practical purposes.

286. For the preceding example, determine the mean-square following error with allowance for the effect of the load torque. The load torque at the motor shaft is $M_n = 2$ g·cm, the moment of inertia of the moving masses for the motor shaft is $J = 1\cdot10^{-3}$ g·cm·sec^2. The gear ratio between the motor shaft and the actuating shaft is $i = 6000$.

<u>Solution</u>. The mean-square error of servosystems possessing second-order astatism when working into a load torque may be computed from the

following approximate formula [3]:

$$\theta_{sq} \approx \frac{1}{K_\varepsilon}\sqrt{\frac{\Omega^2_{sq}}{T_0 T_\Omega} + \frac{M^2_{sq}}{\rho^2 j^2}} = \sqrt{\theta_1^2 + \theta_2^2}$$

where $\theta_1 = \frac{\Omega_{sq}}{K_\varepsilon \sqrt{T_0 T_\Omega}}$ is the mean-square error when working with zero load torque, $\theta_2 = \frac{M_{sq}}{K_\varepsilon \rho j}$ is the torque error. For the values given

$$\theta_1 = \frac{15}{165\sqrt{10\cdot 0{,}083}} = 0{,}1°,$$

$$\theta_2 = \frac{2\cdot 57{,}3}{165\cdot 0{,}0388\cdot 10^{2}} = 0{,}179°,$$

$$\theta_{sq} \approx \sqrt{\theta_1^2 + \theta_2^2} = \sqrt{0{,}1^2 + 0{,}179^2} = 0{,}205°.$$

287. Determine the parameters of the desired logarithmic amplitude characteristic if we know that the spectral density of the input-variable derivative has the form

$$G_1(\omega) = \frac{2T_0\Omega^2_{sq}}{(1+\omega^2 T_0^2)(1+\omega^2 T_p^2)},$$

where T_p = 1 sec is the time constant for the exponential law governing the decrease in the difference between the instantaneous and asymptotic values of Ω. Figure 196 shows the nature of the change in the velocity of the input signal for the considered spectral density. The remaining initial conditions, as well as the following-accuracy requirements are the same as in Problem 285.

<u>Answer</u>

$$K_\varepsilon = 47.5 \text{ 1/sec}^2.$$

288. Determine the parameters of a servosystem from the condition that the required following accuracy be ensured. The input signal is of the irregular rolling type. The correlation function for the roll angle is approximated by the formula

$$R_\theta(\tau) = D_\theta e^{-\mu|\tau|}\left(\cos\beta_0\tau + \frac{\mu}{\beta_0}\sin\beta_0|\tau|\right),$$

where D_θ is the variance of the roll angle, θ_0 is the resonant frequency, and μ is the damping factor. The corresponding expression for the spectral density will take the form

$$S_\theta(\omega)=\frac{4\mu(\beta_0^2+\mu^2)D_\theta}{(\beta_0^2+\mu^2+\omega^2)^2-4\beta_0^2\omega^2}=$$

$$=\frac{\mu}{\beta_0}D_\theta\left[\frac{2\beta_0-\omega}{\mu^2+(\beta_0-\omega)^2}+\frac{2\beta_0+\omega}{\mu^2+(\beta_0+\omega)^2}\right].$$

The variances of the rate and angle of roll are associated by the relationship $D_\Omega=(\beta_0^2+\mu^2)D_\theta$ where $D_\Omega = \Omega_{sk}^2$, i.e., the variance of the roll rate equals its mean square (Ω_{sk} = 20 degree/sec). The roll resonant frequency β_0 = 1 1/sec, the damping factor is μ = 0.04 1/sec, the admissible mean-square following error is $\theta_s \leq 0.1^\circ$. The servosystem possesses second-order astatism, and its frequency transfer function for the low-frequency region takes the form

$$W(j\omega)=\frac{K_\varepsilon(1+T_2 j\omega)}{(j\omega)^2}.$$

Solution. Solution of the problem consists in finding the values of K_ε and T_2. We first determine the mean-square error:

$$\varepsilon^2=\frac{\mu(\beta_0^2+\mu^2)D_\theta}{\pi\beta_0K_\varepsilon^2}\int_{-\infty}^{+\infty}\frac{\omega^4d\omega}{(1+\omega^2T_2^2)[\mu^2+(\beta_0-\omega)^2]}=$$

$$=\frac{2\mu(\beta_0^2+\mu^2)D_\theta}{\beta_0K_\varepsilon^2}\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[-\frac{A\omega+B}{1+\omega^2T_2^2}+\frac{C\omega+D}{\mu^2+(\beta_0-\omega)^2}\right]d\omega, \qquad (1)$$

where the coefficients of the expansion equal, respectively;

$$A=\frac{1-T_2^2(\beta_0^2+\mu^2)}{1+2(\beta_0^2-\mu^2)T_2^2+(\beta_0^2+\mu^2)^2T_2^4}\approx 1,$$

$$B=\frac{2\beta_0}{1+2(\beta_0^2-\mu^2)T_2^2+(\beta_0^2+\mu^2)^2T_2^4}\approx 2\beta_0,$$

$$C=\frac{3\beta_0^2-\mu^2}{1+2(\beta_0^2-\mu^2)T_2^2+(\beta_0^2+\mu^2)^2T_2^4}\approx 3\beta_0^2-\mu^2,$$

$$D=\frac{2\beta_0(\beta_0^2+\mu^2)}{1+2(\beta_0^2-\mu^2)T_2^2+(\beta_0^2+\mu^2)^2T_2^4}\approx -2\beta_0(\beta_0^2+\mu^2).$$

The approximate values of A, B, C, and D are given on the assumption that $\beta_0T_2 \ll 1$ and $\mu < \beta_0$. Substituting the approximate values of A, B, C, and D into (1), we can find the ultimate expression for the mean-square error simply connecting the following error with the quantities K_ε and T_2:

$$q_0 \approx \frac{p\,\Omega_2^2 + p^3\eta D_0}{\lambda K_\varepsilon^2}\left(\frac{B}{T_0}+\frac{C\Omega}{p}+\frac{D}{p}\right)$$

or

$$q_0 = \frac{\Omega_m}{K_\varepsilon}\sqrt{K_\varepsilon^2 - 2p^2 + \frac{K_\varepsilon}{T_2}}.$$

The second additional equation relating K_ε and T_2 is taken in the form (see Problem 285)

$$T_2 = \frac{\xi}{\sqrt{K_\varepsilon}} = \frac{2,08}{\sqrt{K_\varepsilon}}.$$

Substituting in the numerical values, we have

$$K_\varepsilon = \frac{20}{0,1}\sqrt{1 - 0,0048 + 0,04\sqrt{K_\varepsilon}}$$

or

$$K_\varepsilon^2 \approx 40\,000(1 + 0,04\sqrt{K_\varepsilon}).$$

We let $\sqrt{K_\varepsilon} = x$; then

$$x^4 = 40\,000 + 1600x.$$

In approximation $x \approx 16$ and $K_\varepsilon = x^2 = 16^2 = 256$ 1/sec^2.

Manuscript Page No.	[Transliterated Symbols]
241	к = k = kolebaniye = oscillation or kontrol'nyy = control
245	н = n = nagruzka = load
246	м = m = moment = torque
249	к.з = k.z = korotkoye zamykaniye = short circuit
250	Дв = Dv = dvigatel' = motor
250	Ус = Us = usilitel' = amplifier
250	Р = R = reduktor = reduction gear
250	ТГ = TG = takhogenerator = tachometer generator
250	у = u = usilitel' = amplifier

250 м = m = mekhanicheskiy = mechanical

251 o.c = o.s = obratnaya svyaz' = feedback

252 с.к.о. = s.k.o. = srednekvadratichnaya oshibka = root-mean-square error

252 с = s = sredniy = mean

252 с = s = slezheniye = following, tracking

253 с = s = srez = cutoff

BLANK PAGE

Chapter 11

SELECTING AUTOMATIC CONTROL SYSTEM PARAMETERS ON THE BASIS OF THE REQUIRED DYNAMIC PROPERTIES

§30. COEFFICIENT METHODS FOR SELECTING AUTOMATIC CONTROL SYSTEM PARAMETERS

289. A carbon-stack voltage regulating system (Fig. 197) is described by the third-order equation

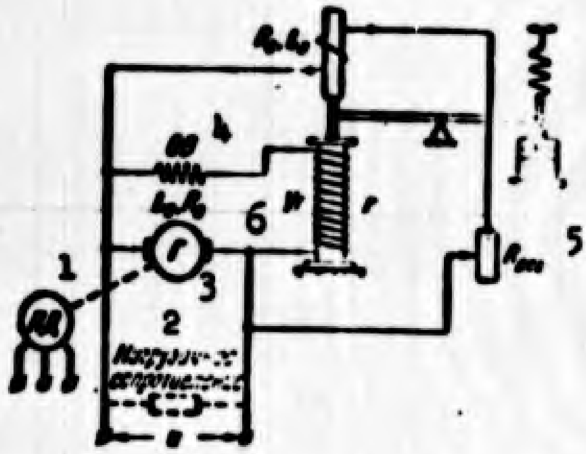


Fig. 197. Voltage-regulation system using carbon regulator. 1) PD; 2) load resistance; 3) generator; 4) OV; 5) R_{reg}; 6) carbon stack.

$$[(1+T_0p)(1+T_1p)(1+T_2p)+k_0k_r]\Delta u = -(T_1p+1)(1+T_2p)f(t),$$

where T_0 = 0.02 sec is the time constant of the generator (controlled object), k_0 = 36 v/ohm is the generator transfer constant, T_1 is the time constant of the sensing element (electromagnet winding), T_2 is the time constant of the regulator (carbon stack), k_r = 0.405 ohm/v is the regulator transfer constant.

Select the variable parameters of the control system T_1, T_2 so as to ensure a stability $h_0 \geq 0.4$ where the transient process is oscillatory in form.

<u>Solution</u>. We refer to the Vyshnegradskiy diagram (Appendices 1 and 2). The control-system characteristic equation has the form

$$a_0q^3 + a_1q^2 + a_2q + a_3 = 0, \tag{1}$$

where

$$a_0 = T_0T_1T_2, \quad a_1 = T_0T_1 + T_1T_2 + T_0T_2,$$
$$a_2 = T_0 + T_1 + T_2, \quad a_3 = 1 + k_0k_y.$$

We reduce it to normal form

$$q^3 + Aq^2 + Bq + 1 = 0, \tag{2}$$

where

$$A = \frac{a_1}{\sqrt[3]{a_0^2a_3}} = \frac{T_0T_1 + T_1T_2 + T_0T_2}{\sqrt[3]{T_0^2T_1^2T_2^2(1 + k_0k_y)}} \tag{3}$$

and

$$B = \frac{a_2}{\sqrt[3]{a_0a_3^2}} = \frac{T_0 + T_1 + T_2}{\sqrt[3]{T_0T_1T_2(1 + k_0k_y)^2}}$$

are the Vyshnegradskiy parameters.

It is possible to solve the problem by writing preliminary values for A and B (for example, A = 4 and B = 3) satisfying the requirements specified. This way of determining T_1 and T_2 involves solving a system of two cubic equations, however. It is simpler to obtain the values of T_1 and T_2 by the method of successive approximations, assigning numerical values and observing the path of the point with coordinates A and B on the Vyshnegradskiy diagram.

Substituting the given values of the parameters into (3), we obtain working formulas for calculating A and B:

$$A = \frac{0.2(T_1 + T_2) + 10T_1T_2}{1.84\sqrt[3]{(T_1T_2)^2}}, \quad B = \frac{0.02 + T_1 + T_2}{1.7\sqrt[3]{T_1T_2}}. \tag{4}$$

The results of calculations of A and B from Formulas (4) with T_2 varying and $T_1 = 0.01$ sec are tabulated on page 260.

T_2, сек[1]	0,1	0,2	0,5	0,8	1	2
A	1,8	2,1	2,8	3,3	3,5	3,8
B	0,8	1,1	1,9	2,4	2,8	4,4

1) T_2, sec.

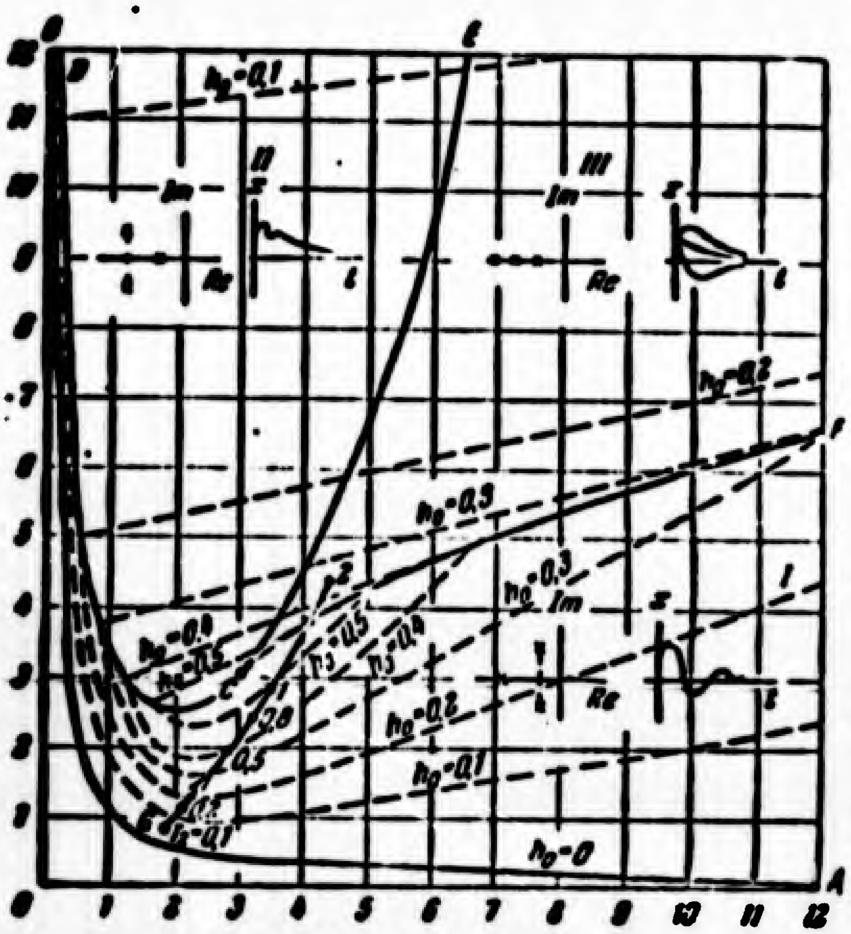


Fig. 198. Construction of path on Vyshnegradskiy diagram.

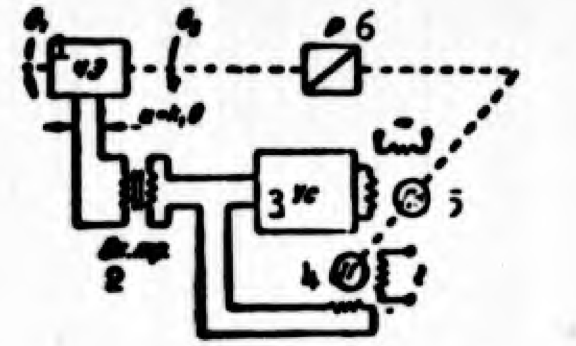


Fig. 199. Servosystem with tachometer feedback. 1) sensing element; 2) input transformer; 3) amplifier; 4) tachometer generator; 5) motor; 6) reduction gear.

On the Vyshnegradskiy diagram (Fig. 198), we plot the path of the point G(A, B). It follows from the graph that in order to meet the specified requirement $h_0 > 0.4$, it is sufficient if the condition

$$0,5 < T_2 < 1,8 \qquad (5)$$

is satisfied when $T_1 = 0.1$ sec. This condition may be satisfied by an appropriate adjustment of the regulator damper. If the path of G did not fall within the desired region of the Vyshnegradskiy diagram, it would be necessary to change the value of T_1 and find in similar manner a new shifted path passing through the required section of the Vyshnegradskiy diagram.

The parameters T_1 and T_2 should be varied in accordance with the technical feasibility of providing the values specified.

290. For the servosystem whose diagram is given in Fig. 199, determine the required amplifier gain k_u and feedback-loop transfer constant k_0 for the given values of system over-all gain K_Ω = 500 1/sec and transient-process damping factor λ = 98%. The system open-loop transfer function taking into account the tachometer feedback has the form

$$\frac{\theta}{\varepsilon} = W(p) = \frac{\frac{K}{1+k_0}}{p\left(1+\frac{T_m}{1+k_0}p\right)} = \frac{K_\Omega}{p\left(1+\frac{T_m}{1+k_0}p\right)},$$

where T_m = 0.03 sec is the electromechanical time constant of the motor $K = k_1 k_{tr} k_u k_{dv}$ is the over-all system gain neglecting the effect of feedback, k_1 = 0.1 v/rad is the slope of the sensing-element characteristic, k_{dv} = 1/0.55 rad/sec·v is the motor transfer constant, $k_0 = k_u k_{dv} k_{o.s}$ if the feedback-loop transfer constant, k_{tr} = 3 is the input-transformer voltage ratio, $k_{o.s}$ is the feedback-circuit transfer constant.

<u>Solution</u>. We find the system closed-loop error transfer function

$$\frac{\varepsilon}{\theta_1} = \Phi_\varepsilon(p) = \frac{p^2+\frac{1}{T_m}(1+k_0)}{p^2+\frac{1}{T_m}(1+k_0)p+\frac{K}{T_m}}. \qquad (1)$$

The system closed-loop characteristic equation will be

$$p^2 + B_1 p + B_2 = 0, \qquad (2)$$

where

$$B_1 = \frac{1}{T_m}(1+k_0) \text{ and } B_2 = \frac{K}{T_m}.$$

For a damping factor λ = 98%, the condition [23]

$$B_0 = \left(\frac{\pi^2+4}{16}\right) B_1^2 \qquad (3)$$

or

$$K = \frac{\pi^2+4}{16 T_м}(1+k_0)^2.$$

should hold, from which we obtain

$$k_y = \frac{(\pi^2+4)}{16 T_м k_1 k_{дм} k_{ср}}(1+k_0)^2.$$

The over-all system gain K_Ω is connected with the gain K by the relationship

$$K_\varepsilon = \frac{K}{1+k_0} = \frac{\pi^2+4}{16 T_м}(1+k_0). \qquad (4)$$

From this equation it follows that

$$k_0 = \frac{16 T_м}{\pi^2+4} K_\varepsilon - 1. \qquad (5)$$

We go to the numerical calculation:

$$k_0 = \frac{16 \cdot 0{,}03}{3{,}14^2+4} 500 - 1 = 16{,}3,$$
$$K = K_\varepsilon(1+k_0) = 500(1+16{,}3) = 8700 \text{ 1/sec},$$
$$k_y = \frac{K}{k_1 k_{дм} k_{ср}} = \frac{8700 \cdot 0{,}35}{1 \cdot 3} = 1617,$$
$$\dot{k}_{o.c} = \frac{k_0}{k_y k_{дм}} = \frac{16{,}3 \cdot 0{,}35}{1617} = 0{,}0037 \text{ v/rad/sec.}$$

291. Using the method of standard transient curves [39] (see Appendix 18), we select the control-system parameters so that the damping time for the transient t $\leq$ 1.5 sec, while the overshoot $\sigma \leq 10\%$. The system open-loop transfer function has the form

$$W(p) = \frac{K_\varepsilon(1+T_1 p)}{p^2(1+T_2 p)}.$$

where K_ε is the over-all system open-loop acceleration gain, and T_1 and T_2 are time constants.

Solution. The corresponding standard transfer function has the form (Appendix 18)

$$W(p)=\frac{6{,}3\omega_0^2p+\omega_0^3}{p^3+5{,}1\omega_0p^2}=\frac{\frac{\omega_0^2}{5{,}1}\left(1+\frac{6{,}3}{\omega_0}p\right)}{p^2\left(1+\frac{1}{5{,}1\omega_0}p\right)}.$$

Equating this to the given transfer function, we obtain the conditions for parameter selection

$$K_0=\frac{\omega_0^2}{5{,}1},\quad T_1=\frac{6{,}3}{\omega_0},\quad T_2=\frac{1}{5{,}1\omega_0}.$$

If the transient damping time is not to exceed the given value, it is necessary that

$$\omega_0=\frac{9}{\tau}=\frac{9}{1{,}5}=6 \text{ 1/sec},$$

where τ is the transient duration. Then

$$K_0=\frac{36}{5{,}1}=7{,}05 \text{ 1/sec}^2,$$
$$T_1=\frac{6{,}3}{6}=1{,}05 \text{ sec},$$
$$T_2=\frac{1}{5{,}1\cdot 6}=0{,}0326 \text{ sec}.$$

Thus, the system open-loop transfer function should have the form

$$W(p)=\frac{7{,}05(1+1{,}05p)}{p^2(1+0{,}0326p)}.$$

292. Using the method of standard transient curves (Appendix 18), select the parameters of a servosystem so that the over-all system open-loop acceleration gain will be $K_\varepsilon \geq 100$ 1/sec^2, while the overshoot will be $\sigma \leq 10\%$. In general form, the system open-loop transfer function is the same as that for Problem 291.

<u>Answer</u>

$$W(p)=\frac{100(1+0{,}28p)}{p^2(1+0{,}0087p)}.$$

§31. FREQUENCY METHODS FOR CHOOSING AUTOMATIC CONTROL SYSTEM PARAMETERS. DESIGN OF SERIES COMPENSATING NETWORKS

293.* Construct the desired logarithmic amplitude characteristic and select a series compensating device for an automatic control system

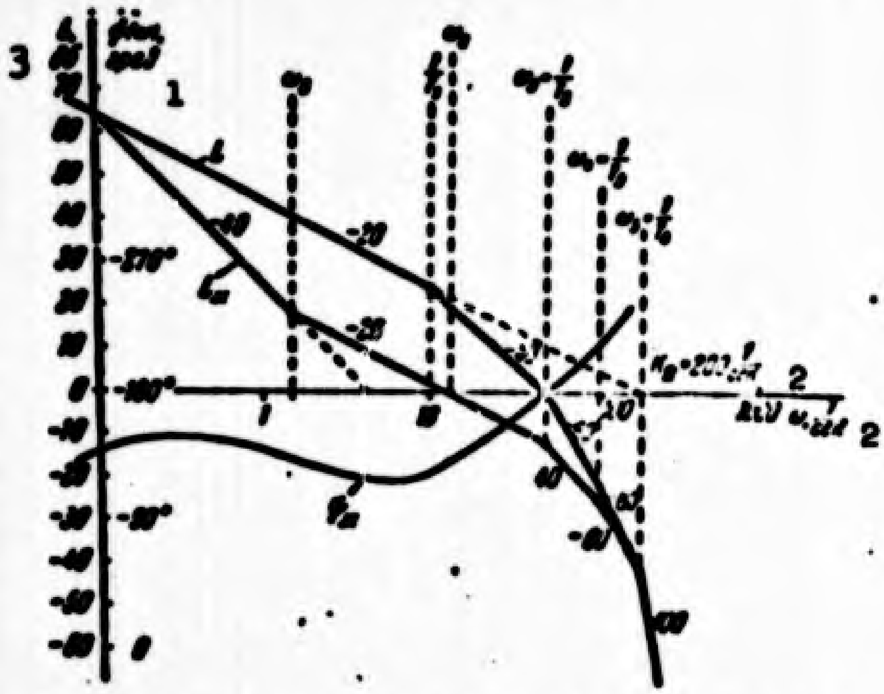


Fig. 200. Logarithmic amplitude and phase characteristics for Problem 293. 1) Degree; 2) sec; 3) db.

if the system open-loop transfer function in the absence of the compensating network has the form

$$W(p)=\frac{K_0}{p(1+T_1p)(1+T_2p)(1+T_3p)(1+T_4p)},$$

where T_1 = 0.1 sec, T_2 = 0.02 sec, T_3 = 0.01 sec, T_4 = 0.005 sec.

The control system should have first-order astatism and should meet the following performance characteristics: a) velocity error coefficient C_1 = 1/200 sec, b) acceleration error coefficient C_2 = 0.06 sec^2, c) overshoot σ for a unit step control input should not exceed 30%, d) transient duration t_m for a unit step control input should not exceed 0.8 sec for a number of oscillations not to exceed two.

Solution. In accordance with the method of [26], we plot on Fig. 200 the available logarithmic amplitude characteristic L (i.e., the characteristic of the uncompensated system) with gain K_Ω, equal to the required gain

$$K_0=\frac{1}{C_1}=200\ 1/\text{sec}.$$

Then, using the given performance characteristics, we construct the desired logarithmic amplitude characteristic L_{zh}. The first conjugating frequency of the desired logarithmic amplitude characteristic, in accordance with item b) is found from the following approximate expression:

$$\omega_1 \approx \frac{1}{C_0 K_0} = \frac{1}{0.05 \cdot 200} \approx 0.08 \ 1/\text{sec.}$$

In order to ensure that the condition of item c) is satisfied we need only ensure that the desired logarithmic amplitude characteristic L_{zh} has a stability margin of ± 16 db in modulus and 45° in phase (Fig. 201).

We now find the cutoff frequency ω_s. Using Fig. 263 (page 348), we find for $\sigma = 30\%$, corresponding to $P_{max} = 1.3$,

$$\omega_s \approx \frac{11.5}{t_0} \approx 14 \ 1/\text{sec.}$$

We draw a line through point ω_s with a slope of 20 db per decade. The intersection of this line and the second asymptote of the desired logarithmic amplitude characteristic with slope of 40 db per decade gives the second conjugating frequency $\omega_2 = 1.3$ 1/sec. In the example considered, $\omega_s/\omega_2 > 10$, which is fully permissible. Thus, the desired form of L_{zh} when $\omega < \omega_s$ has been found. We turn to the selection of the form of L_{zh} when $\omega > \omega_s$, concentrating on the fact that for each of these sections the slope of the desired logarithmic amplitude characteristic should deviate as little as possible from the slope of the initial logarithmic amplitude characteristic.

We shall attempt to satisfy the specified performance conditions, limiting the difference in slope between L_{zh} and L, not exceeding 20 db per decade. Then L_{zh} must have, as we can see from Fig. 200, conjugating frequencies $\omega_3 = 50$ 1/sec and $\omega_4 = 100$ 1/sec corresponding to the conjugating frequencies of the initial logarithmic amplitude characteristic. Beginning at a frequency $\omega_5 = 200$ 1/sec, the desired logarithmic

Fig. 201. Curves for selecting stability margin for modulus l and phase μ. 1) Degree.

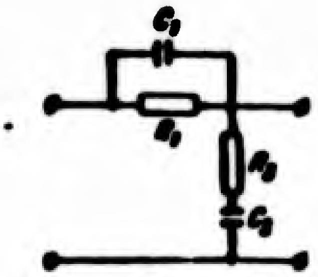


Fig. 202. Passive integral-differential network.

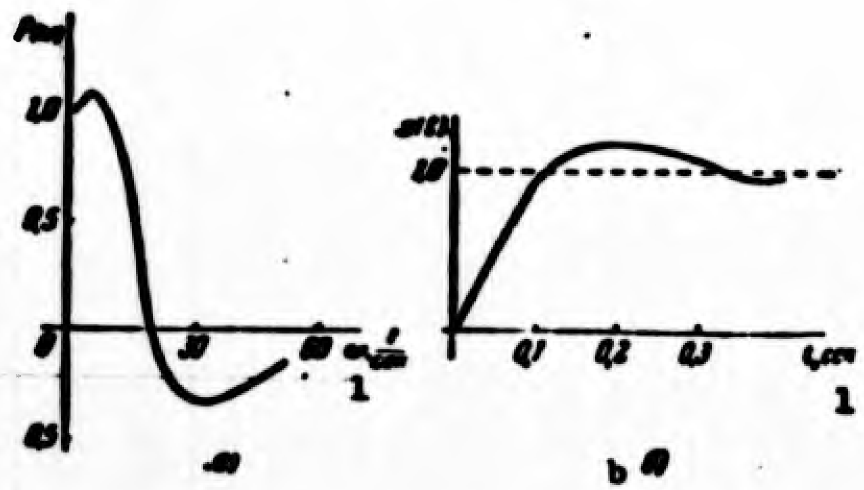


Fig. 203. a) Real closed-loop system frequency response; b) transient curve. 1) sec.

amplitude characteristic coincides with the initial characteristic. The desired transfer function has the form

$$W_{zh}(p)=\frac{K_0\left(1+\frac{p}{1.5}\right)}{p\left(1+\frac{p}{0.08}\right)\left(1+\frac{p}{50}\right)\left(1+\frac{p}{100}\right)\left(1+\frac{p}{200}\right)}.$$

The stability margin is determined by the shape of the logarithmic characteristic curves in the midfrequency region, i.e., in the range $\omega_2 \leq \omega \leq \omega_3$. Let us see whether or not the obtained logarithmic amplitude characteristic L_{zh} has the required phase stability margin when L_{zh} = 16 db ($\omega = \omega_2$), 0 db ($\omega = \omega_s$), and -14 db ($\omega = \omega_3$).

From Fig. 200 when L_{zh} = 16 db, ω = 2 1/sec and

$$\varphi(2) = -90 - \operatorname{arctg}\frac{2}{0{,}08} + \operatorname{arctg}\frac{2}{1{,}3} = -121°.$$

This corresponds to a phase margin of

$$\mu = 180 + \varphi = 180 - 121 = 59°.$$

When L_{zh} = -14 db, ω = 50 1/sec

$$\varphi(50) = -90 - \operatorname{arctg}\frac{50}{30} - \operatorname{arctg}\frac{50}{100} - 2\operatorname{arctg}\frac{50}{200} = -190°$$

and, correspondingly, μ = 180 − 190 = -10°.

When L_{zh} = 0, $\omega = \omega_s$ = 14 1/sec

$$\varphi(14) = -90 - \operatorname{arctg}\frac{14}{0{,}08} + \operatorname{arctg}\frac{14}{1{,}3} - \operatorname{arctg}\frac{14}{30} -$$

$$- \operatorname{arctg}\frac{14}{100} - 2\operatorname{arctg}\frac{14}{200} = -108°$$

and correspondingly,

$$\mu = 180 - 108 = 72°.$$

Of the three values obtained for $\mu(\omega)$ only the second does not fall within the required range. This may lead to a slight increase in the absolute value $|P_{min}|$ in comparison with that which we have taken ($|P_{min}| = P_{max} - 1 = 0.3$), which, as we know [26] is not important.

Thus the logarithmic amplitude characteristic L_{zh} found may serve as the initial characteristic for synthesizing compensating devices.

Subtracting the ordinates of L from the ordinates of the desired logarithmic amplitude characteristic L_{zh} (Fig. 200), we obtain the logarithmic amplitude characteristic of the series compensating device, which is not shown in Fig. 200. For the case under consideration, it is necessary to use a passive integral-differential element as the compensating network (Fig. 202), with a transfer function of the form

$$W_{k}(p) = \frac{\left(1+\frac{p}{1{,}3}\right)\left(1+\frac{p}{10}\right)}{\left(1+\frac{p}{0{,}08}\right)\left(1+\frac{p}{200}\right)} = \frac{(1+0{,}77p)(1+0{,}1p)}{(1+12{,}5p)(1+0{,}005p)}.$$

In order to check the results obtained, we construct the phase

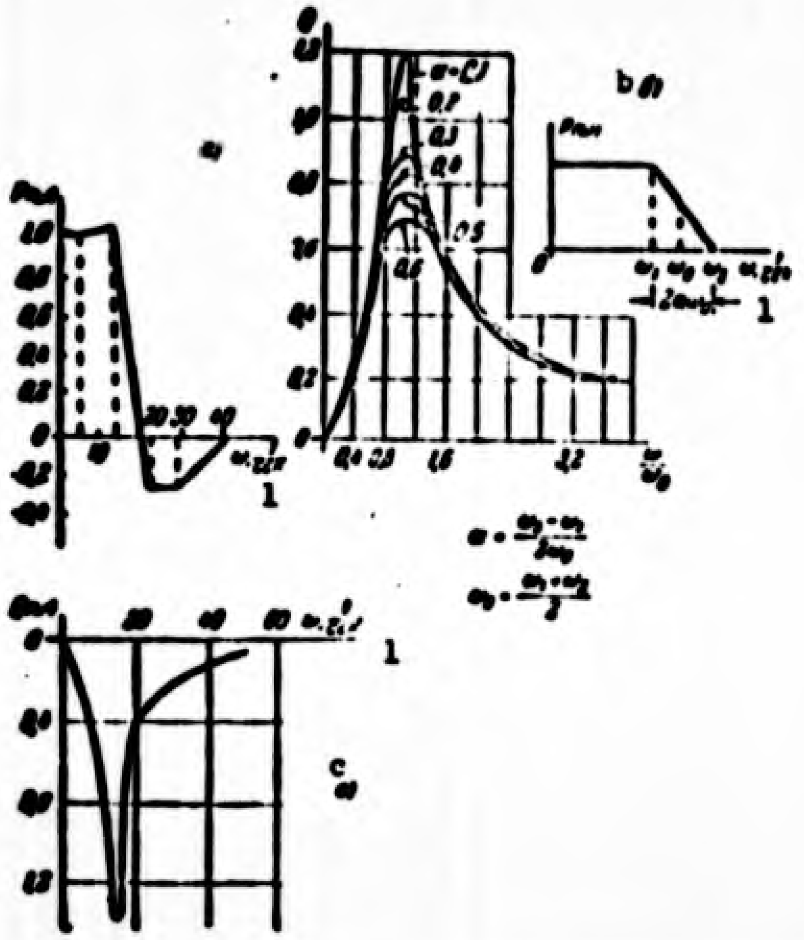


Fig. 204. a) Real frequency characteristic $P(\omega)$; b) graph of $Q = f(\omega/\omega_0)$; c) imaginary frequency characteristic $Q(\omega)$. 1) sec.

characteristic $\vartheta_{zh}(\omega)$ (Fig. 200) and also, using the nomogram of Appendix 3, determine the real frequency characteristic $P(\omega)$ for the closed-loop system (Fig. 203a). Using the method of trapezoidal characteristics, we construct the graph of the transient (Fig. 203b). The transient in the system satisfies the given performance criteria.

294. Select a series compensating element for an automatic control system. The open-loop transfer function of the uncompensated system has the form

$$W(p)=\frac{K}{(1+T_1p)(1+T_2p)(1+T_3p)},$$

where T_1 = 0.05 sec, T_2 = 0.1 sec, T_3 = 0.2 sec. The compensated system should provide the following performance characteristics with respect to the transient for a step control input: a) overshoot $\sigma \leq 20\%$; b) transient damping time $t_m \leq 0.6$ sec where the number of oscillations

$n \leq 3$; c) steady-state error Δ should not exceed 3%.

Solution. We select the compensating element with the aid of the gain-phase characteristics [1, 30]. In order to obtain a steady-state error of 3% it is necessary for the system transfer constant to be at least

$$K = \frac{1-\Delta}{\Delta} = \frac{1-0{,}03}{0{,}03} = 32.$$

In order to plot the gain-phase characteristic of the compensated system it is necessary to select the appropriate form of the real frequency response $P(\omega)$.

On the basis of the given performance characteristics, and a specified value of the slope $\kappa = 0.7$, we find with the aid of the nomogram of Fig. 264 (see Appendix 6) the values of the real frequency curve $P(\omega)$ that provide the required performance characteristics of the compensated system. For $\sigma = 20\%$ and $P_{max} = 1.0$, we find $P_{min} = 0.3$; the modulus stability margin $\Delta R = 55\%$, the phase margin is $\Delta\varphi = 40$, and $t_m = 3.8\pi/\omega_p$ as well. For a given regulation time, we obtain an interval of positive values

$$\omega_p = \frac{3{,}8\pi}{t_m} = \frac{3{,}8\pi}{0{,}6} \approx 20 \text{ 1/sec}.$$

On the basis of the value of ω_p and the parameters upon which the nomogram of Appendix 6 are based, we construct the real frequency curve $P(\omega)$ (Fig. 204a).

The initial ordinate $P(0) = \frac{K}{1+K} = \frac{32}{1+32} = 0{,}97$,

$$\omega_d = \kappa\omega_p = 0{,}7 \cdot 20 = 14 \text{ 1/sec},$$
$$\omega_b = \lambda\omega_p = 0{,}5 \cdot 20 = 10 \text{ 1/sec},$$
$$\omega_a = \kappa_a\omega_b = 0{,}5 \cdot 10 = 5 \text{ 1/sec},$$
$$\omega_1 = 25 \text{ 1/sec}, \omega_2 = \frac{\omega_1}{\lambda_1} = \frac{25}{0{,}6} = 42 \text{ 1/sec},$$
$$\omega_3 = \kappa_1\omega_2 = 0{,}7 \cdot 42 = 29 \text{ 1/sec}.$$

Using the curve $Q = f(\omega/\omega_0)$ drawn for a trapezoid with height unity (Fig. 204b), we have plotted the imaginary frequency characteristic for the closed-loop system (Fig. 204c).

Using the characteristic curves $P(\omega)$ and $Q(\omega)$, it is easy to plot the gain-frequency characteristic for the compensated system [30]. This curve has been plotted in Fig. 205a from the data of Table 1. In Fig. 205b, the dashed line shows the gain-phase characteristic for the uncompensated system (Table 2).

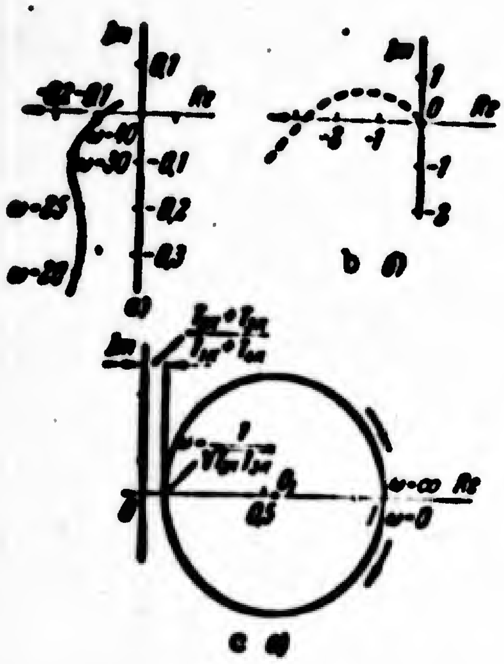


Fig. 205. Gain-phase characteristics: a) Compensated systems; b) uncompensated systems; c) compensating elements.

The modulus and argument of the gain-phase characteristic for the compensating element are obtained from the characteristics of the uncompensated and compensated systems

$$R_c(\omega)e^{i\theta_c(\omega)} = \frac{R_c(\omega)}{R(\omega)} e^{i[\psi_c(\omega) - \psi(\omega)]}.$$

TABLE 1

Gain-Phase Characteristic of Compensated System

ω	20	25	30	40	50
$R_c(\omega)$ $\psi_c(\omega)$	0,37 −116°	0,20 −130°	0,17 −150°	0,11 −160°	0,06 −181°

TABLE 2

Gain-Phase Characteristic of Uncompensated System

ω	15	20	25	30	40	50
$R(\omega)$ $\psi(\omega)$	4,6 −164°	2,3 −185°	1,15 −200°	0,99 −204°	0,5 −220°	0,23 −231°

The calculated data are given in Table 3.

TABLE 3
Gain-Frequency Characteristic of Compensating Element

ω	15	20	25	30	40	50
$R_k(\omega)$ $\psi_k(\omega)$	0,23 33°	0,16 69°	0,174 70°	0,17 55°	0,22 60°	0,26 50°

From the values found, it is possible to plot the gain-phase characteristic of the compensating element.

The rest of the solution for the problem requires us to select the type of compensating network whose gain-phase characteristic agrees most closely with the calculated gain-phase characteristic for the compensating element.

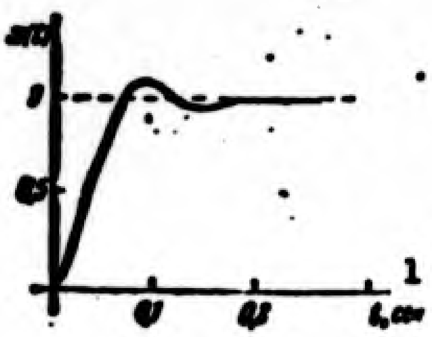


Fig. 206. Transient curve for Problem 295. 1) sec.

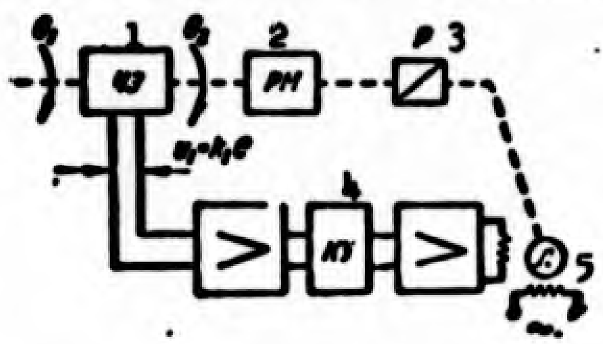


Fig. 207. Diagram of servosystem. 1) Sensing element; 2) working mechanism; 3) reduction gear; 4) compensating element; 5) motor.

We shall assume that in the low-and high-frequency ranges, the gain-phase characteristics of the corrected and uncorrected systems must coincide; then we must take as the compensating network a passive integral-differential network with transfer function

$$w_k(p)=\frac{(1+T_{1k}p)(1+T_{2k}p)}{(1+T_{3k}p)(1+T_{4k}p)}.$$

The gain-phase characteristic of this network is a circle with center at point O_1 (Fig. 205c). Taking values of the modulus R_k or phase ψ_k

for any four points, we find the time constants: $T_{1k} = 1.85$ sec, $T_{2k} =$ $= 0.18$ sec, $T_{3k} = 0.08$ sec, $T_{4k} = 0.02$ sec.

295. Determine the transfer constant of a series compensating element for a servosystem whose transfer function has the form

$$W(p) = \frac{K_0}{p^2(1+T_1p)(1+T_2p)(1+T_3p)},$$

where $T_1 = 0.04$ sec, $T_2 = 0.01$ sec, $T_3 = 0.002$ sec. The servosystem should have second-order astatism and should satisfy the following performance conditions: a) the over-all acceleration gain $K_\varepsilon \geq 100\ 1/\text{sec}^2$; b) the overshoot $\sigma \leq 30\%$; c) the transient damping time $t_m \leq 0.45$ sec.

Answer

$$W_k(p) = \frac{(1+0.25p)(1+0.01p)}{(1+0.0029p)(1+\text{[illegible]}p)}.$$

The transient curve is shown in Fig. 206.

296. Determine the series compensating elements and compute the required amplifier gain k_2 for the servosystem whose block diagram is shown in Fig. 207. On the diagram D is the motor, KU is the compensating element, R is the reduction gear, ChE is the sensing element determining the error, RM is the working mechanism, θ_1 and θ_2 are the angles through which the driving and actuating shafts turn.

The initial values are:

1) sensing-element characteristic slope

$$k_1 = 10 \text{ mv/ang.min} = 34.4 \text{ v/rad};$$

2) the gear ratio of the reduction gear is $i = 3500$;

3) the maximum following rate $\Omega = 5$ degree/sec $= 300$ ang.min/sec;

4) the maximum acceleration is

$$\varepsilon = 2 \text{ degree/sec}^2 = 120 \text{ ang.min/sec}^2;$$

5) the maximum error is $\theta_{max} = 1'$;

6) the maximum amplifier output voltage $U_{max} = 110$ v;

7) maximum motor speed with amplifier wide open $\Omega_{d\,max} = 6000$ rpm =

= 630 1/sec;

8) the starting torque with the amplifier wide open is M_0 = 100 g·cm; the mechanical characteristics of the motor together with the amplifier are reprsented by parallel straight lines;

9) the load torque at the motor shaft is M_n = 10 g·cm;

10) the moment of inertia at the motor shaft is J = 0.018 g·cm × × sec^2;

11) the amplifier time constant is T_u = 0.02 sec;

12) the magnitude ratio is $M \leq 1.5$;

13) the slope of the tachometer-generator characteristic is k_u = 0.05 v·sec.

<u>Solution</u>. The system open-loop transfer function in the absence of the compensating elements equals the product of the element transfer functions

$$W(p)=\frac{k_1k_2k_3\frac{1}{i}}{p(1+T_Mp)(1+T_yp)}=\frac{K_v}{p(1+T_Mp)(1+T_yp)}. \quad (1)$$

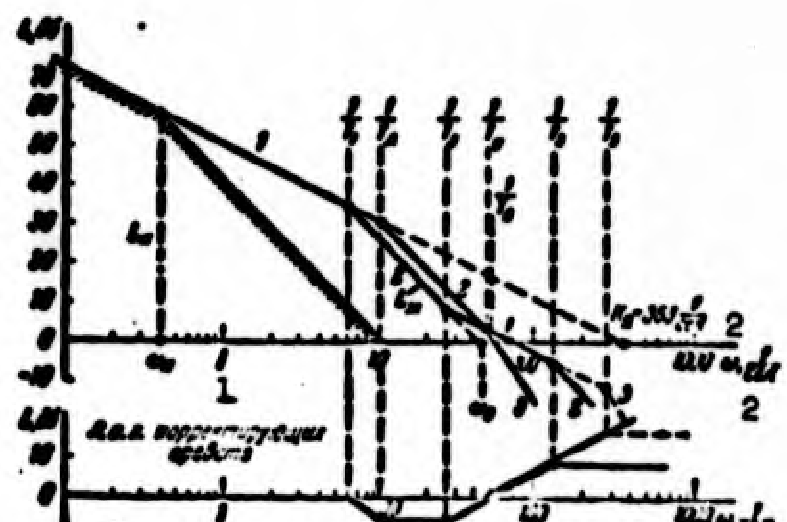


Fig. 208. Logarithmic amplitude characteristic for Problem 296. 1) Logarithmic amplitude characteristic of compensating element; 2) sec.

The motor transfer constant is

$$k_e = \frac{\Omega_{s\,max}}{U_{max}} = \frac{630}{110} = 5{,}73 \text{ 1/v·sec.}$$

The slope of the mechanical characteristics of the motor together with the reducing gear is

$$\beta = \frac{\beta_0}{i} = \frac{\Omega_{s\,max}}{iM_0} = \frac{630}{3500 \cdot 100} = 1{,}8 \cdot 10^{-3} \text{ 1/g·cm·sec} =$$
$$= 6{,}3 \text{ ang.min/g·cm·sec.}$$

The motor time constant is

$$T_s = \beta_0 J = \frac{630}{100} \cdot 0{,}018 \approx 0{,}1 \text{ sec.}$$

In order to determine the required over-all gain (figure of merit) with respect to velocity K_Ω, we construct the forbidden zone for the low-frequency of the logarithmic amplitude characteristic (see Problem 277). The control frequency is

$$\omega_k = \frac{\varepsilon}{\Omega} = \frac{2}{5} = 0{,}4 \text{ 1/sec.}$$

The ordinate of the control point is

$$L_k = 20 \lg \frac{\Omega^2 + \beta M_n \Omega}{\theta_{max}\varepsilon} = 20 \lg \frac{300^2 + 6{,}3 \cdot 10 \cdot 300}{1 \cdot 120} = 59 \text{ db.}$$

The limiting value of the velocity figure of merit is

$$K_\Omega = \frac{\Omega + \beta M_n}{\theta_{max}} = \frac{300 + 6{,}3 \cdot 10}{1} = 363 \text{ 1/sec.}$$

From this data, we construct the forbidden region (Fig. 208).

Let us see whether it is possible for the servosystem to operate without damping devices. Since the first conjugating frequency of logarithmic amplitude characteristic of the corresponding transfer function (1), equal to $\omega_1 = 1/T_d = 10$ 1/sec, is considerably higher than the control frequency $\omega_k = 0.8$ 1/sec, it is possible to use as the ultimate value the velocity figure of merit a value of 363 1/sec. The corresponding logarithmic amplitude characteristic of the 1-2-3 type is shown in Fig. 208.

The permissible sum of the time constants [3] is

$$\sum T = \frac{1}{K_{\Omega}} \frac{M^2 + M\sqrt{M^2 - 1}}{2} =$$

$$= \frac{1}{363} \frac{1.9^2 + 1.9\sqrt{1.9^2 - 1}}{2} = 0.0054 \text{ sec.}$$

In fact, the sum of the actual time constants is

$$\sum T = T_a + T_y = 0.10 + 0.02 = 0.12 \text{ sec.}$$

We thus see that without damping elements, the system will not have the required performance characteristics.

Let us consider a possible method of damping the system with aid of series elements [3].

If we introduce into the direct channel a passive element with a lagging time constant, it is necessary to form the desired logarithmic amplitude characteristic so that the peak error in the region of velocity sign reversal will not exceed the given maximum value θ_{max}. The value found for the velocity figure of merit K_{Ω} = 363 1/sec corresponds to a torque figure of merit

$$K_{м} = \frac{K_{\Omega}}{i} = \frac{363}{6.3} \approx 57.5 \text{ g·cm/ang. min.}$$

The torque error is

$$\theta_{м} = \frac{M_н}{K_м} = \frac{10}{57.5} \approx 0.174'.$$

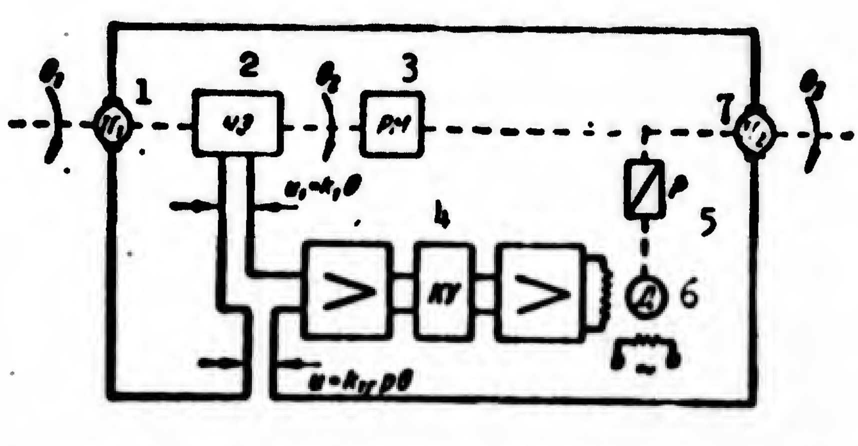


Fig. 209. Arrangement using tachometer generators to obtain element for pure differentiation. 1) Tachometer generator 1; 2) sensing element; 3) working mechanism; 4) compensating element; 5) reduction gear; 6) motor; 7) tachometer generator 2.

The permissible value of the first large time constant is

$$T_1 = 0{,}236\frac{\sqrt{(\theta_{max}+\theta_a)^2}}{\omega_k\sqrt{\varepsilon}} = 0{,}236\frac{\sqrt{(1+0{,}174)^2}}{0{,}174\sqrt{120}} = 0{,}16 \text{ sec.}$$

From the figure of merit $K\Omega$ = 363 1/sec and the time constant T_1 = 0.16 sec, we can construct the low-frequency portion of the logarithmic amplitude characteristic (Fig. 208). The base frequency for this characteristic is

$$\omega_0 = \sqrt{\frac{K_0}{T_1}} = \sqrt{\frac{363}{0{,}16}} = 47{,}5 \text{ 1/sec.}$$

Let us now form the low-frequency and high-frequency parts of the desired logarithmic amplitude characteristic, which is of the 1-2-1-2 type. From the base frequency we determine the required second time constant:

$$T_2 = \frac{1}{\omega_0}\sqrt{\frac{M}{M-1}} = \frac{1}{47{,}5}\sqrt{\frac{1{,}5}{1{,}5-1}} = 0{,}0365 \text{ sec.}$$

The third time constant is

$$T_3 = \frac{1}{\omega_0}\frac{\sqrt{M(M-1)}}{M+1} = \frac{1}{47{,}5}\frac{\sqrt{1{,}5(1{,}5-1)}}{1{,}5+1} = 0{,}0073 \text{ sec.}$$

From these values, we plot the entire desired logarithmic amplitude characteristic L_{zh}. The logarithmic amplitude characteristic of the compensating elements is obtained by substracting the ordinates of the initial logarithmic amplitude characteristic from the ordinates of the desired characteristic. This difference logarithmic amplitude characteristic is also shown in Fig. 208. From the shape of this curve it follows that the damping elements of series type should consist of: 1) a passive integral-differential element with transfer function

$$W_k(p) = \frac{(1+T_2p)(1+T_3p)}{(1+T_1p)(1+T_vp)},$$

where the time constant T_v is found from the well-known property of an integral-differential element

$$T_v = \frac{T_1 T_2}{T_3} = \frac{0{,}05 \cdot 0{,}0365}{4{,}16} = 0{,}0114 \text{ sec;}$$

2) a pure-differentiation element with transfer function

$$W_{kd}(p) = 1 + T_y p$$

and 3) a combination of a passive differentiating element and a linear amplifier with over-all transfer function

$$W_{ka}(p) = \frac{1 + T_v p}{1 + T_z p} = k_u \frac{\frac{T_z}{T_v}(1 + T_v p)}{1 + T_z p}.$$

The pure differentiation element can only be obtained by using tachometer generators on the main and actuating shafts, introducing the derivative of the error angle.

Where the signal from the tachometer generators is applied to the place at which the signal from the main sensing element is applied (Fig. 209), the required slope of the voltage characteristic for each tachometer generator will equal

$$k_{tg} = k_1 T_y = 10 \cdot 0{,}02 = 0{,}2 \text{ mv·sec/ang. min} =$$
$$= 0.37 \text{ v·sec/revolution} = 0.06 \text{ v·sec.}$$

The passive elements should be introduced into the amplifier direct channel and they may consist of RC networks.

The amplifier gain, allowing for the additional gain needed to operate the passive differentiating element $k_u = T_v/T_z$ is

$$k_y = \frac{T_v}{T_z} \frac{K_0 i}{k_1 k_2} = \frac{0{,}0114}{0{,}0073} \frac{363 \cdot 3700}{34{,}4 \cdot 5{,}75} = 9850.$$

Where tachometer generators cannot be used to introduce the derivative of the error angle, the type of compensating elements required may be changed.

As we can see from Fig. 208, the pure-differentiating element is obtained on the basis of the fact that the high-frequency asymptote of the initial logarithmic amplitude characteristic has a large slope as compared with the slope of the desired logarithmic amplitude characteristic. In order to eliminate this, it is possible to change the form of the desired logarithmic amplitude characteristic in the high-frequency section, going from a 1-2-1-2 type to a 1-2-1-3 type.

The high-frequency section of the latter logarithmic amplitude characteristic is shown on Fig. 208 by the dashed line. It corresponds to the transfer function

$$W(p)=\frac{K_0(1+T_2p)}{p(1+T_3p)(1+T_5p)^2}.$$

The time constant T_5 is found as

$$T_5=\frac{1}{2\omega_0}\sqrt{\frac{M(M-1)}{M+1}}=\frac{T_3}{2}=0{,}0036\ \text{sec.}$$

The logarithmic amplitude characteristic of the compensating elements in shown for this case in Fig. 208 by a dashed line as well. Looking at this curve we can see that the series-type compensating elements should consist of three passive elements: an integral-differential element and two passive differentiating elements combined with a linear amplifier to have an over-all transfer function

$$W_c(p)=\frac{(1+T_7p)}{(1+T_5p)}\frac{(1+T_8p)}{(1+T_5p)}.$$

The additional gain will be greater than in the preceding case:

$$k_7=\frac{T_7T_8}{T_5^2}=\frac{0{,}02\cdot 0{,}0114}{0{,}0036^2}=17{,}5.$$

The over-all amplifier gain also turns out to be considerably greater than where a pure differentiating element is used:

$$k_0=k_7\frac{K_0 l}{k_1k_6}=17{,}5\frac{353\cdot 2200}{34{,}4\cdot 3{,}74}=110\,000.$$

Another choice is possible for the high-frequency portion of the desired logarithmic amplitude characteristic; in particular, the corresponding transfer function may have the form

$$W(p)=\frac{K_0(1+T_2p)}{p(1+T_3p)(1+T_5p)(1+T_6p)}.$$

Here $T_4 \neq T_5$, but their sum, as before, should equal

$$T_4 + T_5 = \frac{1}{\omega_0} \frac{\sqrt{M(M-1)}}{M+1}.$$

Manuscript Page No.	[Transliterated Symbols]
258	э = e = ekvivalentnyy = equivalent
258	рэг = reg = regulyator = regulator
258	ОВ = OV = obmotka vozbuzhdeniya = field winding
258	в = v [= vozbuzhdeniye = field]
258	Уг = Ug = ugol'nyy = carbon
258	ПД = PD = privodnoy dvigatel' = drive motor
258	р = r = regulyator = regulator
260	ЧЭ = ChE = chuvstvitel'nyy element = sensing element
260	Р = R = reduktor = reduction gear
260	Вх.Тр = Vkh.Tr. = vkhodnoy transformator = input transformer
260	Ус = Us = usilitel' = amplifier
260	ТГ = TG = takhogenerator = tachometer generator
260	Дв = Dv = dvigatel' = motor
261	у = u = usilitel' = amplifier
261	м = m = mekhanicheskiy = mechanical
261	о.с = o.s = obratnaya svyaz' = feedback
265	ж = zh = zhelayemyy = desired
265	с = s = srez = cutoff
267	к = k = korrektiruyushchiy = compensating
269	п = p = polozhitel'nost' = positiveness
271	РМ = RM = rabochiy mekhanizm = working mechanism
271	КУ = KU = korrektiruyushcheye ustroystvo = compensating unit

271 Д = D = dvigatel' = motor

272 д = d = dvigatel' = motor

273 н = n = nagruzka = load

273 л.а.х. = l.a.kh. = logarifmicheskaya amplitudnaya kharakteristika = logarithmic amplitude characteristic

275 м = m = moment = torque

276 в = v [not identified]

BLANK PAGE

Chapter 12

COMPENSATION OF AUTOMATIC CONTROL SYSTEMS

§32. PARALLEL COMPENSATING DEVICES (FEEDBACK AND DIRECT COUPLING)

297. Select the proportional tachometer feedback parameters for the servosystem whose diagram is shown in Fig. 210. The initial data are the same as for Problem 296, except that $T_d = 0.05$ sec.

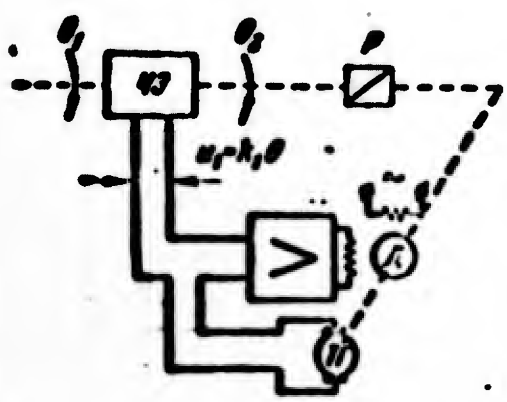


Fig. 210. Diagram of servosystem with proportional tachometer feedback.

Solution. The system open-loop transfer function taking into account the effect of the tachometer feedback will take the form

$$W(p)=\frac{K_{v\Omega}}{p(1+ap+bp^2)}, \qquad (1)$$

where $K_{v\Omega} = K_\Omega/1 + k_{o.s}$ is the new value of the over-all velocity gain (velocity figure of merit), $a=\frac{T_1+T_2}{1+k_{o.s}}$ and $b=\frac{T_1T_2}{1+k_{o.s}}$ are the coefficients of the equivalent second-order circuit, $k_{o.s} = k_2k_3k_{tg}$ is the feedback-loop gain, k_{tg} is the slope of the characteristic curve for the tachometer generator and scaler in the feedback circuit.

In order to provide the necessary stability margin, which is evaluated by means of the magnitude ratio, we must ensure that the inequality [3]

$$\frac{T_1+T_2}{1+k_{o.s}}=a<\frac{1}{K_{v\Omega}}\frac{M^2+M\sqrt{M^2-1}}{2}. \qquad (2)$$

is satisfied.

It is also necessary to take into account the fact that introduction of proportional tachometer feedback changes the slope of the mechanical characteristics of the actuating motor by a factor of $(1 + k_{o.s})$. The required velocity figure of merit, allowing for less flexible mechanical characteristics is

$$K_{v0} = \frac{\Omega_{max} + \frac{\beta M_n}{1 + k_{o.s}}}{\varepsilon_{max}}. \qquad (3)$$

Solving the last two equations simultaneously, we can determine the required feedback-loop gain:

$$k_{o.s} = \frac{\Omega_{max}(T_a + T_y)}{2\mu\varepsilon_{max}} - 1 + \sqrt{\frac{\Omega_{max}^2 (T_a + T_y)^2}{4\mu^2\varepsilon_{max}^2} + \frac{\beta M_n (T_a + T_y)}{\mu\varepsilon_{max}}} =$$

$$= \frac{300 \cdot 0{,}7}{2 \cdot 1{,}96 \cdot 1} - 1 + \sqrt{\frac{300^2 \cdot 0{,}07^2}{4 \cdot 1{,}96^2 \cdot 1} + \frac{6{,}3 \cdot 10 \cdot 0{,}07}{1{,}96 \cdot 1}} = 10,$$

where

$$\mu = \frac{M^2 + M\sqrt{M^2 - 1}}{2} = \frac{1{,}5^2 + 1{,}5\sqrt{1{,}5^2 - 1}}{2} = 1{,}96.$$

The required velocity figure of merit is from (3)

$$K_{v0} = \frac{300 + \frac{6{,}3 \cdot 10}{1 + 10}}{1} = 306 \text{ 1/sec.}$$

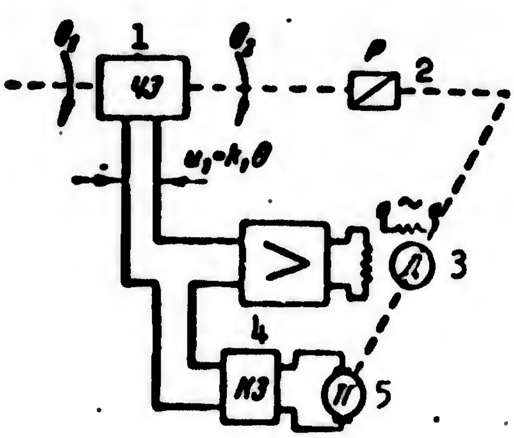


Fig. 211. Diagram of servosystem with elastic tachometer feedback. 1) Sensing element; 2) reduction gear; 3) motor; 4) compensating element; 5) tachometer generator.

The permissible sum of the time constants (2)

$$\Sigma T = \frac{1}{306} \cdot \frac{1{,}5^2 + 1{,}5\sqrt{1{,}5^2 - 1}}{2} = 0{,}0064 \text{ sec.}$$

We have an equivalent time constant

$$a = \frac{T_a + T_y}{1 + k_{o.s}} = \frac{0{,}07}{1 + 10} = 0{,}0064 \text{ sec.}$$

Thus, the problem of selecting the feedback-circuit parameters may be considered solved.

The open-loop transfer function

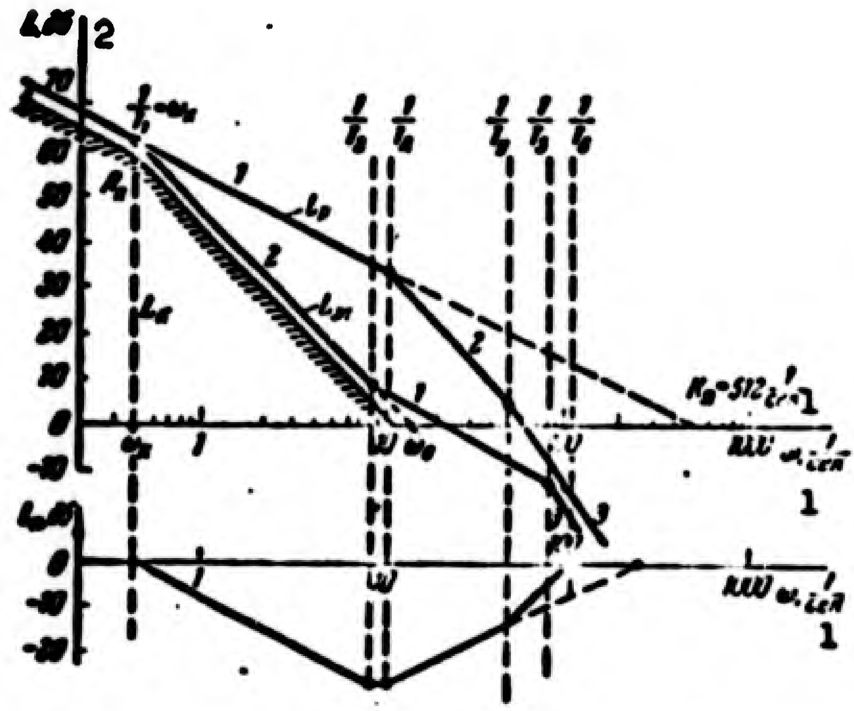


Fig. 212. Logarithmic amplitude characteristics for Problem 298. 1) sec; 2) db.

of the damped system will have the form

$$W(p)=\frac{K_v}{p(1+ap+bp^2)}=\frac{306}{p(1+6{,}4\cdot 10^{-3}p+9{,}1\cdot 10^{-6}p^2)}.$$

To conclude, we determine the required amplifier gain and required tachometer-generator characteristic slope. The over-all gain for the open-loop servosystem in the absence of feedback should be

$$K_{v0}=K_{ve}(1+k_{o.c})=306\cdot 11\approx 3360 \text{ 1/sec.}$$

The amplifer gain is

$$k_y=\frac{K_{v0}i}{k_1k_0}=\frac{3360\cdot 3500}{34{,}4\cdot 5{,}73}=59\,500.$$

The required slope of the tachometer-generator characteristic, allowing for the scaler, is

$$k_T=\frac{k_{o.c}}{k_yk_0}=\frac{15}{59\,500\cdot 5{,}73}\approx 3\cdot 10^{-5} \text{ v·sec.}$$

The low value of this transfer constant indicates that the signal from the tachometer generator may not be applied to the amplifier input but may be introduced somewhere within the amplifier channel.

The high amplifier gain is a drawback to this way of introducing feedback.

298. Select parameters for elastic tachometer feedback for the servosystem whose diagram is shown in Fig. 211. The initial data are the same as for Problem 296. The slope of the tachometer-generator curve $k_{tg} = 0.05$ v·sec.

Solution. In connection with the fact that damping is introduced by the first method of [3] (see Chapter 10), the desired logarithmic amplitude characteristic [l. a. kh.] L_{zh} may be formed so that its first bend coincides with the control frequency of point A_k (Fig. 212). In this case, the l.a.kh. should go above the forbidden zone at a distance of 3 db. The required velocity figure of merit will be

$$K_{vzh} = \sqrt{2}\,K_v = 1,41 \cdot 363 = 512 \text{ 1/sec.}$$

This makes it necessary to have an amplifier gain of

$$A_0 = \frac{K_v i}{k_1 k_2} = \frac{512 \cdot 3500}{34,4 \cdot 5,73} = 9100.$$

The base frequency of the desired l.a.kh. is

$$\omega_0 = \sqrt{\frac{K_{vzh}}{T_2}} = \sqrt{\omega_2 K_{vzh}} = \sqrt{0,4 \cdot 512} = 14,3 \text{ 1/sec.}$$

The second time constant of the desired l.a.kh. is

$$T_3 = \frac{1}{\omega_0}\sqrt{\frac{M}{M-1}} = \frac{1}{14,3}\sqrt{\frac{1,5}{1,5-1}} = 0,12 \text{ 1/sec.}$$

The permissible sum of the time constants corresponding to the conjugating frequencies to the right of the cutoff frequency [6] is

$$\Sigma T = \frac{1}{\omega_0}\frac{\sqrt{M(M-1)}}{M+1} = \frac{1}{14,3}\frac{\sqrt{1,5(1,5-1)}}{1,5+1} = 0,024 \text{ sec.}$$

We form the desired l.a.kh. so that its high-frequency asymptote has the same slope as the high-frequency asymptote of the original l. a.kh. L_r. In the given case, the slope is 60 db/decade. Then in the high-frequency part of the desired l.a.kh., there may be a double bend at the frequency $\omega_5 = 1/T_5$. The corresponding time constant should equal

$$T_5 = \frac{\Sigma T}{2} = \frac{0,024}{2} = 0,012 \text{ sec.}$$

In order to simplify the compensating elements, it is possible to continue the section having the same slope for the desired l.a.kh. until the high-frequency asymptotes L_{zh} and L_r coincide, which is shown in Fig. 212 by the dashed line. This yields a certain increase in the stability margin. The time constant determining the double bend in the desired l.a.kh. may be found by direct measurement of the conjugating frequency (Fig. 212). It equals $T_6 = 0.009$ sec.

The desired l.a.kh. constructed in this manner corresponds to a system open-loop transfer function

$$W(p)=\frac{K_{v}(1+T p)}{p(1+T p)(1+T_6 p)^2}=\frac{512(1+0.12p)}{p(1+2.5p)(1+0.009p)^2}.$$

The subsequent calculations shall be oriented to this simpler case.

In Fig. 212 we have plotted the l.a.kh. for series compensating elements L_k obtained by subtracting the ordinates of L_r from the ordinates of L_{zh}. It corresponds to series-connected integral-differential and differentiating elements with transfer functions

$$w_{p.z}(p)=\frac{(1+T p)(1+T p)(1+T p)}{(1+T p)(1+T_6 p)^2}.$$

The transfer function obtained is needed only as a preliminary result, since by the hypothesis of the problem, compensation of the system should be carried out by means of feedback rather than by series elements. Thus the transfer function obtained must be recomputed in terms of the equivalent feedback.

The transfer function of the compensating element in the tachometer-generator circuit may be found from the formula

$$w_{o.s}(p)=\frac{1-w_{p.z}(p)}{w_{p.z}(p)\,w_s(p)},$$

where $w_s(p)=\frac{k_1k_2k_{tt}}{(1+T p)(1+T p)}$ is the transfer function for the portion of the system enclosed by feedback.

As a result of substitution of the values $w_s(p)$ and $w_{p.z}(p)$ we have

$$W_{o.s}(p) = \frac{(T_0 + T_1 + T_a + T_y + 2T_2)p + (T_2^2 + T_1T_a + T_1T_y + T_aT_y)p^2 + T_1T_aT_y p^3}{k_y k_g k_u (1 + T_2 p)}.$$

Such an element cannot be realized physically, since the degree of the polynomial in the numerator is higher than the degree of the polynomial in the denominator. We may attempt, however, to employ a physically realizable element having a transfer function close to that desired. As a physically realizable transfer function, we may take, in first approximation, the function

$$W_{o.s}(p) = \frac{T_1 p}{k_y k_g k_u (1 + T_2 p)} = k_{o.s} \frac{T_2 p}{1 + T_2 p},$$

where

$$k_{o.s} = \frac{T_1}{T_2 k_y k_g k_u}.$$

This transfer function may be realized for a direct-current tachometer generator with the aid of a simple divider having transfer constant $k_{o.s}$ and a differentiating RC-network with time constant $RC = T_2 =$ $= 0.12$ sec.

The required divider transfer constant for the feedback circuit equals

$$k_{o.s} = \frac{25}{0.12 \cdot 9100 \cdot 5.73 \cdot 0.05} = 8.1 \cdot 10^{-3}.$$

It is possible to avoid the use of a special divider but in this case, the point at which the feedback is inserted in the amplifier should be so chosen that from this point to the amplifier output the voltage gain will be

$$k_s' = k k_{o.s} = 9100 \cdot 8.1 \cdot 10^{-3} = 73.$$

Thus, in first approximation, the transfer function of the compensating element in the feedback circuit should be

$$W_{o.s}(p) = 8.1 \cdot 10^{-3} \frac{0.12p}{1 + 0.12p}.$$

Let us now check to see whether this element can be used to obtain the required dynamic properties (for formation of the l.a.kh. of the desired shape). The system open-loop transfer function, taking into account elastic tachometer feedback, will take the form

$$W_{os}(p)=\frac{K_{10}(1+T_{o}p)}{p(1+a_1p+a_2p^2+a_3p^3)},$$

where

$$a_1=T_y+T_a+T_e+k_ek_sk_{u}k_{o.c}T_1=$$
$$=T_y+T_a+T_e+T_1=2{,}74\ \text{sec},$$
$$a_2=T_yT_a+T_yT_e+T_aT_e=1{,}64\cdot10^{-2}\ \text{sec}^2,$$
$$a_3=T_yT_aT_e=2{,}4\cdot10^{-4}\ \text{sec}^3.$$

Factoring the denominator of the obtained transfer function, we obtain

$$W_{os}(p)=\frac{K_{10}(1+T_{o}p)}{p(1+Tp)(1+ap+bp^2)}=$$
$$=\frac{512(1+0{,}12p)}{p(1+2{,}74p)(1+0{,}6\cdot10^{-2}p+0{,}38\cdot10^{-4}p^2)}.$$

In the low-frequency section, this transfer function will coincide, for all practical purposes, with the transfer function corresponding to the desired l.a.kh. L_{zh}. There is a small difference in the value of the time constant T_1 = 2.74 sec, forming the first bend in the l.a.kh. In the high-frequency region, the condition imposing a limit on the sum of the time constants is satisfied, since a = $0.6\cdot10^{-2}$ sec, while by hypothesis the permissible sum of the time constants is $\Sigma T = 2.4\times10^{-2}$ sec.

A check to see that the peak amplitude of the oscillating-element characteristic does not enter the forbidden region for the high-frequency portion of the l.a.kh., i.e.,

$$\mathrm{mod}\,|w(j\omega)|<\frac{M}{M+1},$$

also gives a positive result.

If it is desirable, better agreement may be obtained between the resulting transfer function and the desired transfer function in the low-frequency region, and the inequality $T \neq T_1$ may be avoided. To do this, we must compensate the coefficient $k_{o.s}$ and make it equal to

$$k'_{o.s} = \frac{T_1}{T} k_{o.s} = \frac{2.5}{2.74} \cdot 8.1 \cdot 10^{-3} = 7.4 \cdot 10^{-3}.$$

Then in like manner we can obtain the corrected open-loop system transfer function in the form

$$W_{os}(p) = \frac{512(1 + 0.12p)}{p(1 + 2.5p)(1 + 0.65 \cdot 10^{-2}p + 0.95 \cdot 10^{-4}p^2)}.$$

299. Determine the feedback form and parameters for the electrohydrolic servosystem whose block diagram is shown in Fig. 213. In Fig. 213 A and B are the halves of the hydraulic regulator, GU is the hydraulic amplifier, PD is the drive motor, RM is the working mechanism, UD is the control motor, ChE is the sensing element (tachometer generator), R is the reduction gear. The system open-loop transfer function will take the form

$$W(p) = \frac{K_\varepsilon}{p(1 + T_d p)(1 + T_{gm} p)},$$

where K_ε is the acceleration figure of merit, $T_d = 0.05$ sec is the electromechanical time constant of the control motor, $T_{gm} = 0.02$ sec is the hydromechanical time constant of the hydraulic regulator. The

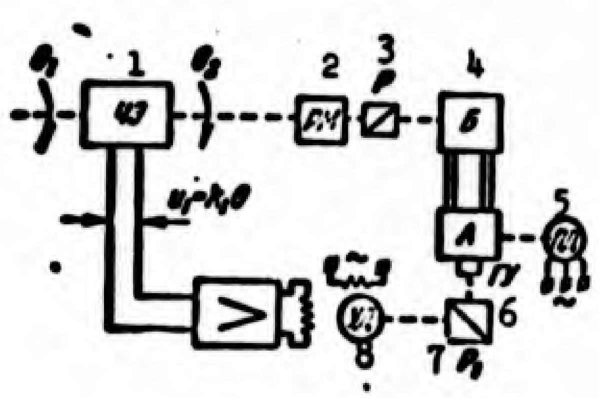


Fig. 213. Diagram of electrohydraulic servosystem. 1) Sensing element; 2) working mechanism; 3) reduction gear; 4) B; 5) drive motor; 6) hydraulic amplifier; 7) reduction gear 1; 8) control motor.

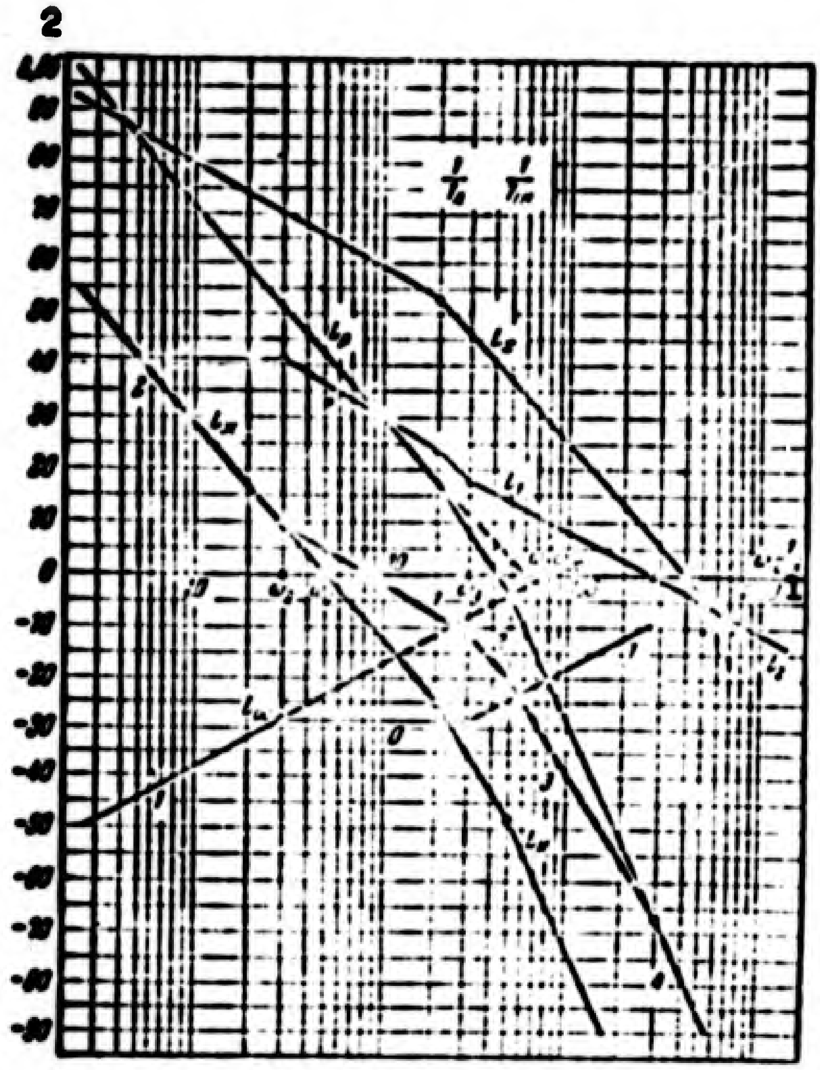


Fig. 214. Logarithmic amplitude characteristic for Problem 299. 1) sec; 2) db.

system should have an acceleration figure of merit $K_\varepsilon \geq 25$ 1/sec^2 and a magnitude ratio $M \leq 1.8$. Feedback is taken around the control motor and the amplifier.

Solution. In Fig. 214, we have plotted the l.a.kh. for the initial system L_1 for a value $K_\varepsilon = 25$ 1/sec^2. On the same figure, we have plotted the desired l.a.kh. L_{zh} corresponding to all required performance characteristics for the system.

Let us consider the sequence in which the feedback form and parameters are determined.

Taking into account the additional feedback, the system open-loop transfer function $W_{zh}(p)$ may be written in the form

$$W_{zh}(p)=\frac{W(p)}{1+W_2(p)W_{os}(p)}.$$

where $w_{o.s}(p)$ is the transfer function of the additional feedback circuit, $w_x(p)$ is the transfer function for the part of the system in series with the feedback loop. From the expression given it follows that the l.a.kh. of the feedback circuit $L_{o.s}$ may be found from the known l.a.kh. L_{zh} and L_i in the following manner [3, 31]:

1) from the l.a.kh. of the initial system L_i, we calculate the desired l.a.kh. L_{zh}, i.e., we find the l.a.kh. L_1 corresponding to the transfer function $1 + w_x(p)w_{o.s}(p)$;

2) from the shape of the l.a.kh. L_1, we construct l.a.kh. L_2 corresponding to the transfer function $w_x(p)w_{o.s}(p)$:

3) from the l.a.kh. L_2, we calculate the l.a.kh. for that part of the system in series with the feedback loop L_x; as a result, we determine the l.a.kh. for the feedback circuit $L_{o.s}$.

In the problem as solved, the difference l.a.kh. L_1 will lie completely above the 0-decibel axis, which violates the minimum-phase condition when we go to l.a.kh. L_2 [31]. Thus we should increase the gain of the initial system beforehand to a degree such that the difference l.a.kh. L_1 lies completely above the 0-decibel level or lies in the positive-decibel region.

In Fig. 214, we have plotted the l.a.kh. of the initial system with an increased gain $K'_\varepsilon = \omega'_0 = 3600\ 1/\sec^2$, which we designate by L_r. On the same figure we show the difference l.a.kh. L_1 obtained by subtracting the desired l.a.kh. L_{zh} from the initial l.a.kh. L_r. In order to determine the l.a.kh. L_2, we use a table of transformations of l.a. kh. (Appendix 19, transformation VII).

Since feedback is taken around the control motor and amplifier, then

$$w_x(p) = \frac{K_x}{p(1+T_x p)}.$$

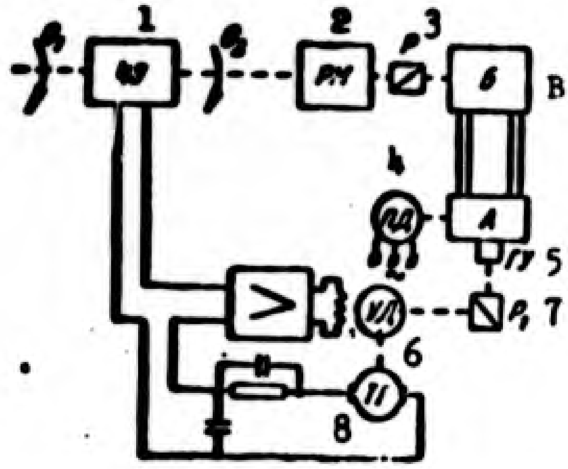


Fig. 215. Electrohydraulic servosystem with additional feedback circuit. 1) Sensing element; 2) working mechanism; 3) reduction gear; 4) drive motor; 5) hydraulic amplifier; 6) control motor; 7) reduction gear 1; 8) tachometer generator.

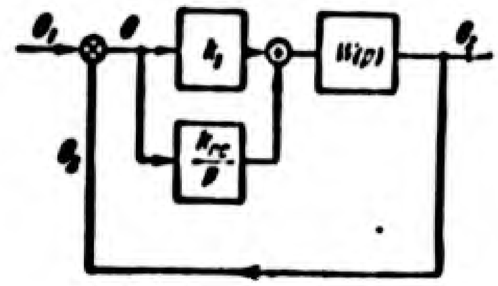


Fig. 216. Block diagram of system into which direct parallel coupling has been introduced.

The l.a.kh. L_x corresponding to this expression is plotted in Fig. 214.

Subtracting the l.a.kh. L_x from the l.a.kh. L_2, we construct the initial l.a.kh $L_{o.s}$; from its form, we can write an expression for the feedback circuit transfer function where

$$w_{oc}(p)=\frac{k_{oc}p(1+T_1p)}{1+T_2p},$$

where

$$k_{oc}=\frac{1}{\omega_{oc}},\quad T_1=\frac{1}{\omega_1},\quad T_2=\frac{1}{\omega_2}.$$

The transfer function obtained may easily be realized by connecting a tachometer generator and passive integrating element (Fig. 215) into the feedback circuit.

300. Determine feedback form and parameters for the system considered in Problem 299 on the assumption that the feedback circuit encloses part of the amplifier, i.e., $w_x(p) = k_x$. The remaining information is the same as for the preceding problem.

Answer. The feedback-circuit transfer function will have the form

$$W_{oc}(p)=\frac{K_{oc}(1+T_3 p)}{(1+T_1 p)(1+T_2 p)}.$$

This transfer function may be realized by connecting aperiodic (finite time-constant) and passive integrating elements in series.

301. Select direct parallel coupling parameters for the automatic control system whose block diagram is shown in Fig. 216. The open-loop transfer function of the initial system has the form

$$W(p)=\frac{K_\Omega}{p(1+T_d p)(1+T_u p)},$$

where K_Ω = 900 1/sec, T_d = 0.08 sec, T_u = 0.02 sec. After introduction of parallel direct coupling, the system should possess second-order astatism, should have an acceleration figure of merit K_ε = 100 1/sec^2, and a magnitude ratio $M \leq 1.5$.

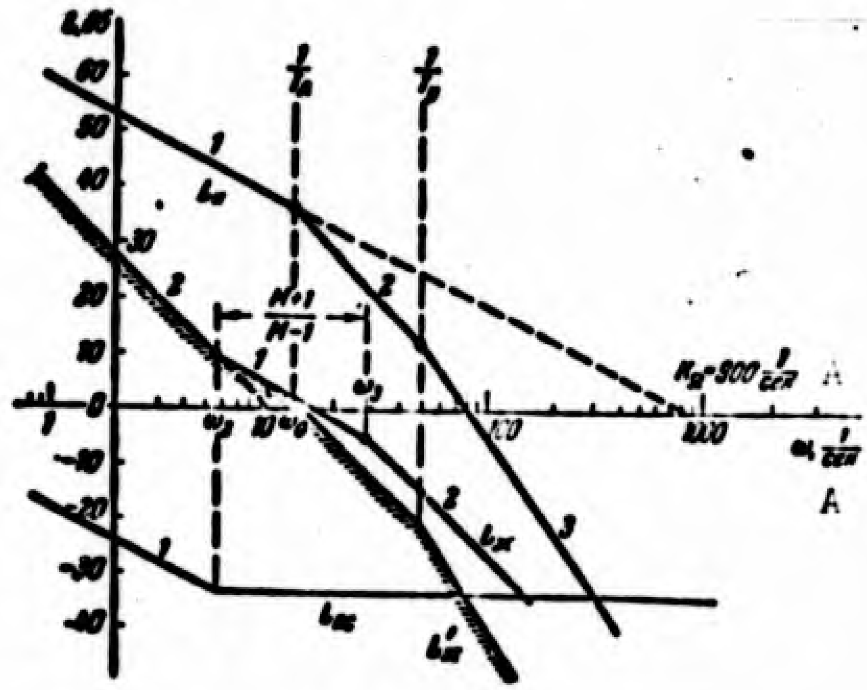


Fig. 217. Logarithmic amplitude characteristic for Problem 301. A) sec.

Solution. The system open-loop transfer function, taking into account introduction of the direct parallel coupling, is represented in the form

$$W_{ck}(p)=\frac{K_\Omega\left(1+\frac{k_1}{K_{oc}}p\right)}{p^2(1+T_d p)(1+T_u p)}.$$

where

$$K_0 = K_1 k_{p.s}$$

Direct coupling introducing a signal proportional to the integral of the error (misadjustment) can be realized with the aid of an integrating drive.

In Fig. 217 we have shown the l.a.kh. for the initial system L_1, the desired l.a.kh. L_{zh}, and the direct-coupling l.a.kh. $L_{p.s}$.

The required transfer constant for the direct-coupling circuit is found from the condition

$$K_0 = \omega_c = K_v k_{p.s}$$

or

$$k_{p.s} = \frac{K_c}{K_0} = \frac{100}{900} = 0{,}11 \text{ 1/sec.}$$

It is desirable to make the ratio $k_1/k_{p.s}$ equal to $1/\omega_2$. From this we obtain

$$k_1 = \frac{k_{p.s}}{\omega_2} = \frac{0{,}11}{6} \approx 0{,}018.$$

In order to satisfy this condition, we must deliberately reduce the transfer constant of the first element within the direct parallel-coupling loop. At the same time, the transfer constant of the second element in the amplifier main channel is increased by the same factor in order to keep the quantity K_Ω constant.

By introducing an integrating direct-coupling loop, it is possible to bring the l.a.kh. of the initial system L_1 close to the desired form L_{zh} only for the low and middle frequency ranges (L'_{zh}).

Final approximation of system l.a.kh. to the desired form may be obtained by compensating the l.a.kh. of the system in the middle and upper frequency ranges by using series differentiating elements or by means of equivalent direct coupling or feedback.

§33. DISTURBANCE CONTROL (COMBINATION CONTROL)

302.* Determine the required levels of compensating signals with respect to the first and second derivatives of the control variable for a combination-control servosystem (Fig. 218) with transfer functions

$$\varphi(p) = \tau_1 p + \frac{\tau_2 \tau_3 p^2}{1 + \tau_3 p},$$

$$W(p) = \frac{K_v}{p(1 + T_1 p)(1 + T_2 p)},$$

where T_1 = 0.05 sec and T_2 = 0.002 sec. The system should ensure following with an error $\theta_{max} \leq 0.1^\circ$ for a maximum following rate $\Omega_{max} = 150^\circ$/ /sec and a maximum acceleration ε_{max} = 750 degree/sec^2. The magnitude ratio $M \leq 1.5$.

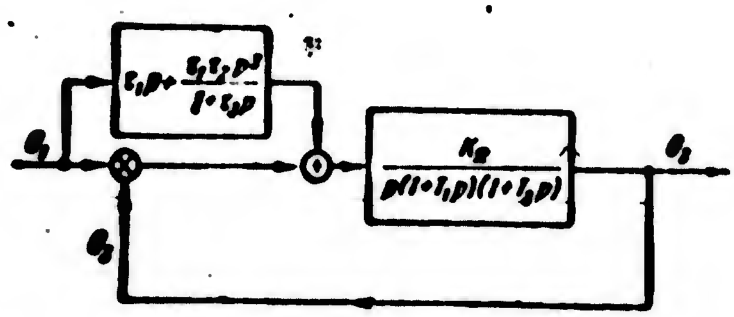


Fig. 218. Block diagram of combination-control system.

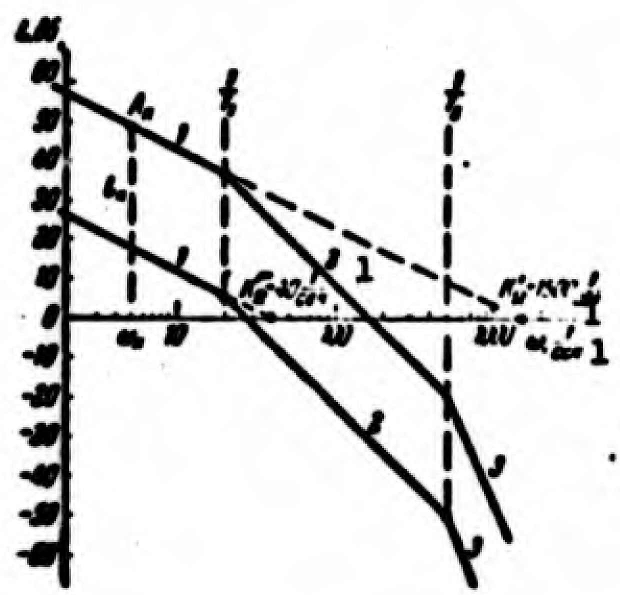


Fig. 219. Logarithmic amplitude characteristic for Problem 302. 1) sec.

<u>Solution</u>. In Fig. 219 we have plotted the control point A_k with coordinates

$$\omega_k = \frac{\varepsilon}{\Omega} = 5 \text{ 1/sec.}$$

and

$$L_k = 20 \lg \frac{\Omega^2_{max}}{\theta_{max}\varepsilon_{max}} = 20 \lg \frac{150^2}{0{,}1 \cdot 750} \approx 50 \text{ db.}$$

If we draw through this control point the low-frequency asymptote of the l.a.kh. corresponding to the transfer function of the initial system, then the required velocity figure of merit will be

$$K_\Omega = \frac{\Omega_{max}}{\theta_{max}} = \frac{150}{0{,}1} = 1500 \text{ 1/sec.}$$

We know, however, that for a given value of magnitude ratio M, the minimum permissible velocity figure of merit in the absence of any compensating devices will be [3]

$$K_\Omega = \frac{M^2 + M\sqrt{M^2-1}}{2(T_1+T_2)} = \frac{1{,}3^2 + 1{,}3\sqrt{1{,}3^2-1}}{2(0{,}05+0{,}02)} = 40 \text{ 1/sec.}$$

If we introduce a signal consisting of the first derivative of the control variable, the servosystem will acquire the properties of a system with second-order astatism.

The required acceleration figure of merit equals

$$K_\varepsilon = \frac{\varepsilon_{max}}{\theta_{max}} = \frac{750}{0{,}1} = 7500 \text{ 1/sec}^2.$$

Here the required velocity figure of merit for the initial system is found from the formula [4]

$$K''_\Omega = (T_1 \div T_2) K_\varepsilon = 0{,}052 \cdot 7500 = 390 \text{ 1/sec.}$$

i.e., K''_Ω is much less than K'_Ω. When the second derivative is also introduced, the required jerk figure of merit is

$$K_1 = \frac{\omega_k \varepsilon_{max}}{\theta_{max}} = \frac{5 \cdot 750}{0{,}1} = 37\,500 \text{ 1/sec}^3.$$

The required velocity figure of merit may be determined from the expression [4, 29]

$$K_{\Omega}''' = [T_1T_2 + \tau_2(T_1 + T_2)]K_{\Gamma}$$

Equating K_{Ω}''' and the velocity figure of merit, as may occur without compensating elements (K_{Ω} = 40 1/sec), we obtain the required value of the differentiating time constant:

$$\tau_2 = \frac{K_{\Omega} - T_1T_2K_1}{(T_1 + T_2)K_1} = \frac{40 - 0{,}05 \cdot 0{,}002 \cdot 37\,500}{(0{,}05 + 0{,}002) \cdot 37\,500} = 18{,}5 \cdot 10^{-3} \text{ sec.}$$

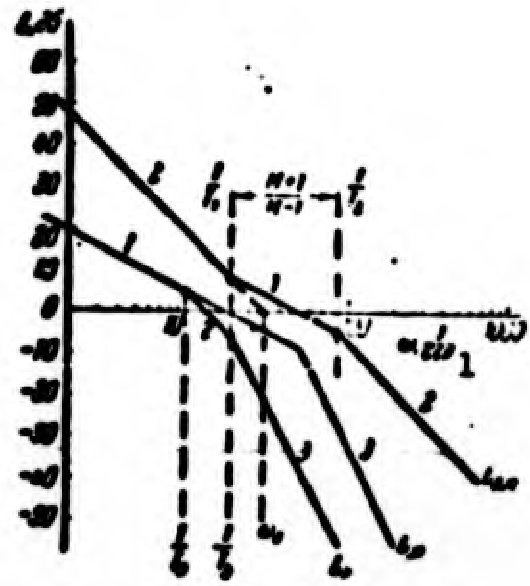


Fig. 220. Logarithmic amplitude characteristic for Problem 303. 1) sec.

The time constants determining the levels of the signals introduced are found from the compensation conditions [4, 29]

$$\tau = \frac{1}{K_{\Omega}} = \frac{1}{40} = 0{,}025 \text{ sec,}$$

$$\tau_1 = T_1 + T_2 + \tau_2 = 0{,}05 + 0{,}002 + 0{,}018 = 0{,}070 \text{ sec.}$$

Thus, the transfer function of the compensating circuit should have the form

$$\varphi(p) = 0{,}025p + \frac{0{,}025 \cdot 0{,}070p^2}{1 + 0{,}018p}.$$

The l.a.kh. of the system that corresponds to the parameters found is shown in Fig. 219 (lower l.a.kh.).

303. Determine the required level of compensating signal propor-

tional to the first derivative of the control variable

$$\varphi(p)=\tau p,$$

and compute the other required compensating elements for a servosystem whose open-loop transfer function has the form

$$W(p)=\frac{K_v}{p(1+T_d p)(1+T_u p)},$$

where T_d = 0.1 sec is the electromechanical time constant of the motor, T_u = 0.05 sec is the amplifier time constant. The system should have second-order astatism and should ensure following with an error $\theta_{max} \leq$ $\leq$ 2 ang.min for a maximum following rate of Ω_{max} = 30 degree/sec and a maximum acceleration of ε_{max} = 30 degree/sec^2. The stability margin is determined by the magnitude ratio M $\leq$ 1.5.

Solution. We first find the desired equivalent system open-loop transfer function.

The first asymptote of the l.a.kh. is a straight line with slope of 40 db/decade. Its position is determined by the base frequency (Fig. 220)

$$\omega_0=\sqrt{K_\varepsilon}=\sqrt{\frac{\varepsilon_{max}}{\theta_{max}}}=\sqrt{\frac{30\cdot 60}{2}}=30 \text{ 1/sec.}$$

In order to obtain a stability margin corresponding to the magnitude ratio M, the transfer function for middle frequencies should have the form [3]

$$W_{ek}(p)=\frac{\omega_0^2(1+T_1 p)}{p^2(1+T_2 p)},$$

where

$$\omega_0^2=K_\varepsilon=900 \text{ 1/sec}^2,$$

$$T_1=\frac{1}{\omega_0}\sqrt{\frac{M}{M-1}}=\frac{1}{30}\sqrt{\frac{1{,}5}{1{,}5-1}}=0{,}0575 \text{ sec},$$

$$T_2=\frac{M-1}{M+1}T_1=\frac{1{,}5-1}{1{,}5+1}0{,}0575=0{,}0115 \text{ sec}.$$

The desired system closed-loop transfer function is

$$\Phi_e(p)=\frac{W_{ek}(p)}{1+W_{ek}(p)}=\frac{K_\varepsilon(1+T_1 p)}{K_\varepsilon+K_\varepsilon T_1 p+p^2+T_2 p^3}.$$

When the compensating signal is introduced, the system closed-loop transfer function may be represented in the form [4, 29]

$$\Phi_{\text{ж}}(p) = \frac{W_{0\text{ж}}(p)}{1 + W_{0\text{ж}}(p)} = \frac{W_{\text{ж}}(p)[1 + \varphi(p)]}{1 + W_{\text{ж}}(p)} = \Phi_1(p) + \Phi_2(p).$$

Comparing the expressions given, we obtain

$$\varphi(p) = \tau_y p = T_y p,$$

or

$$\tau_1 = T_1 = 0{,}0375 \text{ sec.}$$

which determines the required compensating-signal level. Further, we have:

$$\Phi_{\text{ж}}(p) = \frac{K_0}{K_0 + K_0 T_1 p + p^2 + T_1 p^3} + \frac{K_0 T_1 p}{K_0 + K_0 T_1 p + p^2 + T_1 p^3} =$$
$$= \Phi_1(p) + \Phi_2(p).$$

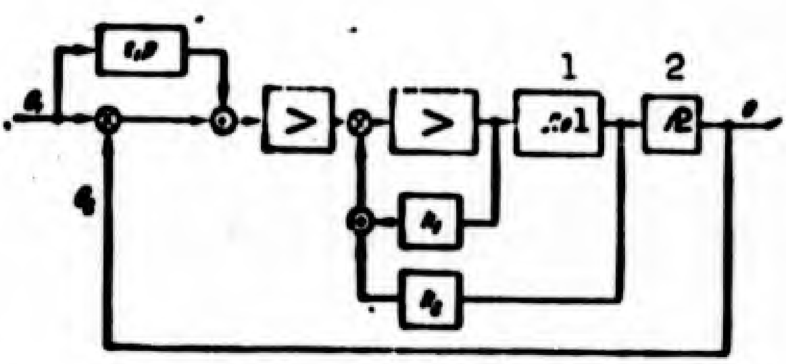


Fig. 221. Block diagram of system for Problem 303. 1) Motor; 2) reduction gear.

The desired transfer function of the initial servosystem is

$$W_{\text{ж}}(p) = \frac{\Phi_1(p)}{1 - \Phi_1(p)} = \frac{\frac{1}{T_1}}{p\left(1 + \frac{1}{K_0 T_1} p + \frac{T_1}{K_0 T_1} p^2\right)} =$$
$$= \frac{\frac{1}{T_1}}{p(1 + ap + bp^2)} = \frac{17{,}4}{p(1 + 0{,}0195p + 0{,}00022p^2)}.$$

The transfer function of the uncompensated system is

$$W(p) = \frac{K_0}{p(1 + T_{\text{д}} p)(1 + T_y p)} = \frac{K_0}{p[1 + (T_{\text{д}} + T_y)p + T_{\text{д}} T_y p^2]}.$$

Comparison of the last two expressions shows that if we are to have

$W_{zh}(p) = W(p)$, the following conditions must be satisfied:

$$K_\Omega = 17{,}4 \ 1/\text{sec},$$
$$T_a + T_y = a = 0{,}0193 \ \text{sec},$$
$$T_a T_y = b = 0{,}00022 \ \text{sec}^2.$$

No difficulty is presented by the first condition, since the velocity figure of merit K_Ω, which is the over-all transfer constant of the open-loop system may be arbitrary. Satisfaction of the second and third conditions requires the introduction of compensating elements decreasing the coefficients on p and p^2 in brackets in the expression for W(p) since without the compensating elements

$$T_a + T_y = 0{,}15 \ \text{sec and } T_a T_y = 0{,}005 \ \text{sec}^2.$$

This may be done by using proportional feedback around the amplifier and the amplifier together with the motor (Fig. 221). In this case, the open-loop transfer function of the circuit together with the feedback loops will be

$$W_{oc}(p) = \frac{K_\Omega}{p\left[1 + \left(\frac{T_a + T_y + k_1 T_a}{1 + k_1 + k_2}\right)p + \frac{T_a T_y}{1 + k_1 + k_2} p^2\right]}.$$

Comparing the last expression with the expression for $W_{zh}(p)$, we find that

$$\frac{T_a + T_y + k_1 T_a}{1 + k_1 + k_2} = a = 0{,}0193 \ \text{sec},$$
$$\frac{T_a T_y}{1 + k_1 + k_2} = b = 0{,}00022 \ \text{sec}^2,$$

from which we find the required gains for the first and second feedback loops (Fig. 221):

$$k_1 = 1{,}9, \quad k_2 = 22{,}5.$$

§34. CARRIER-FREQUENCY ELEMENTS

304. Select a circuit and parameters for an alternating-current element whose l.a.kh. corresponds in envelope to a differentiating ele-

ment (Fig. 222) with transfer function

$$W(p)=\frac{T_2}{T_1}\frac{1+T_1 p}{1+T_2 p},$$

where T_1 = 0.08 sec, T_2 = 0.01 sec. The carrier frequency ω_n equals 3140 1/sec.

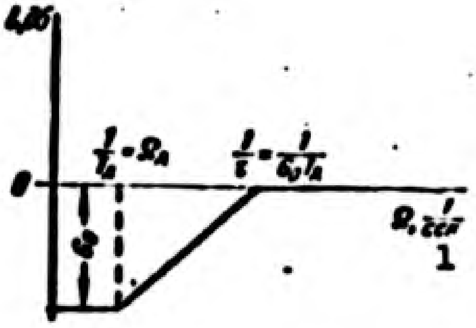


Fig. 222. Logarithmic amplitude characteristic of real differentiating element. 1) sec.

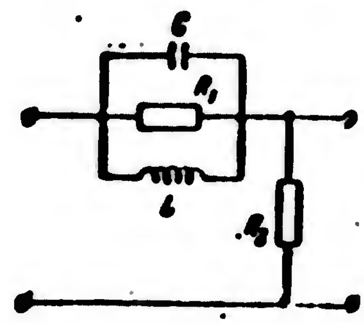


Fig. 223. Circuit of resonant RLC network.

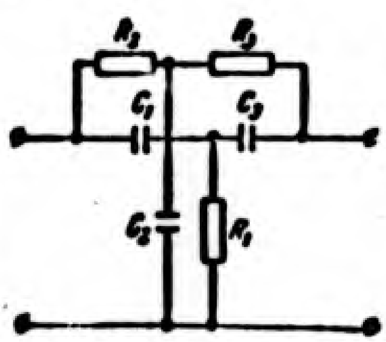


Fig. 224. Circuit of twin-T RC network.

Solution. The transfer function of a real alternating-current differentiating element with respect to the envelope frequency Ω may be written for a fairly narrow band of frequencies Ω in the form [3]

$$W(j\Omega)=G_0\frac{1+j\Omega T_d}{1+j\Omega T_d G_0}=G_0\frac{1+j\Omega T_1}{1+j\Omega T_2}.$$

By the problem hypothesis, $G_0 = T_2/T_1 =$ = 0.01/0.08 = 0.125, and $T_d = T_1$ = 0.08 sec.

At present, the following types of differentiating elements are commonly used: a) twin- or parallel-T RC network; b) bridge-T RC networks; c) bridge RC and LC networks; d) resonant RLC networks.

Let us consider the possibility of using a resonant RLC network (Fig. 223). The transfer function of such an element with respect to the envelope has the form (see Appendix 21)

$$W(j\Omega)=\frac{R_1}{R_1+R_0}\cdot\frac{1+j\Omega 2R_1C}{1+j\Omega 2R_1C\frac{R_1}{R_1+R_0}}.$$

For our case

$$2R_1C = T_d, \quad 2R_1C\frac{R_2}{R_1+R_2} = T_1G_0,$$
$$\frac{R_2}{R_1+R_2} = G_0 \text{ and } \omega_n = \frac{1}{\sqrt{LC}}.$$

We thus obtain four equations in four unknowns:

$$0{,}08 = 2R_1C, \quad 0{,}01 = 2R_1C\frac{R_2}{R_1+R_2},$$
$$0{,}125 = \frac{R_2}{R_1+R_2}, \quad 3140 = \frac{1}{\sqrt{LC}}.$$

The resistance R_2 is usually given (it equals the input resistance of the following element). Let R_2 = 100 kohm. We find R_1:

$$0{,}125R_1 + 0{,}125 \cdot 100 = 100,$$
$$R_1 = \frac{87{,}5}{0{,}125} = 700 \text{ kohm.}$$

We now find the capacitance C:

$$C = \frac{0{,}08}{2R_1} = \frac{0{,}08}{2 \cdot 0{,}7} = 0{,}057 \ \mu\text{f}.$$

To conclude we find the inductance L:

$$L = \frac{1}{\omega_n^2 C} = \frac{10^6}{3140^2 \cdot 0{,}057} = 1{,}8 \text{ henrys.}$$

305. Find the parameters of a twin T network (Fig. 324) working at the carrier frequency $\omega_n = 2\pi f_n = 314$ 1/sec. The remaining conditions are the same as for the preceding problem.

Solution. In order to determine the element parameters, we use the table given in Appendix 22.

By the hypothesis of the problem, the product $T_d\omega_n = 25$. By integrating the values, we can find the G_0 corresponding to the obtained product $T_d\omega_n$. The coefficient G_0 determined in this manner proves equal to 0.02. Thus G_0 may be made smaller than the given value of G_0. This in turn, provided the relationship $T_1 = T_d = 0.08$ sec is maintained, can lead to a reduction in the time constant T_2 to a value $T_2 = T_1G_0 = 0.08 \cdot 0.02 = 0.0016$ sec.

As a rule, a reduction in the time constant T_2 will not impair the dynamic properties of a compensating system.

Let us determine the parameters of the twin-T network. Let $C_1 = C_2 = C_3 = C = 0.5$ μf; then (see Appendix 22)

$$R_1 = \frac{a}{\omega_n C} = \frac{0{,}391 \cdot 10^6}{314 \cdot 0{,}5} \approx 2500 \text{ ohms},$$

$$R_2 = \frac{1}{2a\omega_n C} = \frac{10^6}{2 \cdot 314 \cdot 0{,}5 \cdot 0{,}391} \approx 8000 \text{ ohms},$$

$$R_3 = \frac{1}{\sqrt{2}\,\omega_n C} = \frac{10^6}{1{,}41 \cdot 314 \cdot 0{,}5} \approx 4500 \text{ ohms}.$$

306. Find the parameters of a twin-T network. The differentiating time constant $T_d = 0.0047$ sec. The carrier frequency $\omega_n = 2\pi f_n = 3140$ 1/sec. $C_1 = C_2 = C = 1$ μf.

Answer

$G_0 = 0.034$, $R_1 = 134$ ohms, $R_2 = 380$ ohms, $R_3 = 225$ ohms.

Manuscript Page No.	[Transliterated Symbols]
280	д = d = dvigatel' = motor
280	н = n = novyy = new
280	о.с = o.s = obratnaya svyaz' = feedback
280	у = u = usilitel' = amplifier
280	тг = tg = takhogenerator = tachometer generator
280	ЧЭ = ChE = chuvstvitel'nyy element = sensitive element
280	Р = R = reduktor = reduction gear
281	КЗ = KZ = korrektiruyushcheye zveno = compensating element
282	ж = zh = zhelayemyy = desired
282	р = r = regulyator = regulator
283	л.а.х. = l.a.kh. = logarifmicheskaya amplitudnaya kharakteristika = logarithmic amplitude characteristic

283 к = k = kontrol'nyy = control

284 п.з = p.z = posledovatel'nyy element = series element

284 с = s = sistema = system

287 ПД = PD = privodnoy dvigatel' = drive motor

287 ГУ = GU = gidravlicheskiy usilitel' = hydraulic booster

287 РМ = RM = rabochiy mekhanizm = working mechanism

287 УД = UD = upravlyayushchiy dvigatel' = control motor

287 гм = gm = gidromekhanicheskiy = hydromechanical

288 и = i = iskhodnyy = initial

290 пс = ps = parallel'naya svyaz' = parallel coupling <u>or</u> pryamaya svyaz' = feedforward

291 к = k = korrektirovanyy = compensated

296 э = e = ekvivalentnyy = equivalent

297 Дв = Dv = dvigatel' = motor

299 н = n = nesushiy = carrier

BLANK PAGE

Chapter 13

SPECIAL AUTOMATIC CONTROL SYSTEMS

§35. FORCE GYROSTABILIZATION SYSTEMS

307. Figure 225 shows the diagram of a single-axis force gyrostabilizer. Set up the differential equations if the base on which the stabilizer is mounted moves owing to rolling about the stabilization axis; find the system open-loop transfer function and construct the logarithmic amplitude characteristic [l.a.kh.].

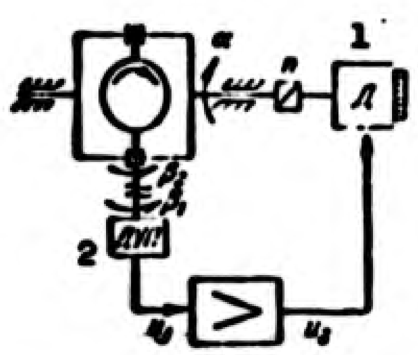


Fig. 225. Diagram of single-axis force gyrostabilizer. 1) Motor; 2) DUP.

The initial data are: 1) kinematic gyroscope moment $H = 12.9 \cdot 10^3$ g·cm·sec; 2) the moment of inertia of the parts involved in rolling, referred to the motor shaft is $J = 0.002$ g·cm·sec^2; 3) the moment of inertia of the gyroplatform about the stabilization axis is $A = 1000$ g·cm·sec^2; 4) the gyroscope moment of inertia about the precession axis is $B = 4$ g·cm·sec^2; 5) the load moment on the stabilization axis is $M_1 = 1000$ g·cm; 6) the load moment on the precession axis is $M_2 = 0$; 7) the amplifier time constant is $T_u = 0.1$ sec; 8) the gear ratio between motor and gyroplatform is $n = 200$; 9) the slope of the precession-angle pickoff characteristic is $k_1 = 1$ v/degree = 57.3 v/rad; 10) the motor develops a starting torque of $M_p = 20$ g·cm for a voltage across the control winding of $U_n = 110$ v; 11) the no-load speed is $\Omega_{kh.kh} = 1000$/sec for a voltage across the control winding of U_n =

= 110 v; 12) the amplifier voltage gain is $k_4 = 5$.

In the calculations we assume that the real frequency of free stabilizer oscillation is less than the computed frequency by a factor of 1.5 owing to elastic torsion and play.

Solution. We use the following definition: α is the stabilization angle (stabilization error), β is the precession angle, θ is the base roll angle, M_{dv} is the torque developed by the motor, u_1 and u_2 are the voltages across the amplifier input and output.

Then the initial differential equations will take the form

$$A\ddot{\alpha} - H\dot{\beta} - nM_{dv} = n^2 J\ddot{\theta} + M_t, \qquad (1)$$

$$B\ddot{\beta} + H\dot{\alpha} = M_t = 0, \qquad (2)$$

$$k_5 M_{dv} + k_6 n\dot{\alpha} - k_6 n\dot{\theta} = u_2, \qquad (3)$$

$$u_2 + T_y \dot{u}_2 = k_4 u_1, \qquad (4)$$

$$u_1 = k_1 \beta_1. \qquad (5)$$

In these formulas, we introduce the coefficients

$$k_5 = \frac{U_n}{M_n} = \frac{110}{20} = 5{,}5 \ \text{v/g}\cdot\text{cm}$$

$$k_6 = \frac{U_n}{\Omega_{x.x}} = \frac{110}{1000} = 0{,}11 \ \text{v}\cdot\text{sec/rad}.$$

In order to obtain the system open-loop transfer function, we cut the system as shown in Fig. 225. The angles β_1 and β_2 will determine the system open-loop transfer function

$$W(p) = \frac{\beta_2(p)}{\beta_1(p)}. \qquad (6)$$

The entire system will then be divided into two elements. The first element is the precession-angle pickoff. Its transfer function is determined from (5):

$$w_1(p) = \frac{u_1(p)}{\beta_1(p)} = k_1 = 57{,}3 \ \text{v/rad}. \qquad (7)$$

The second element is all the remaining system with transfer function

$$w_2(p) = \frac{\beta_2(p)}{u_1(p)}. \qquad (8)$$

In order to find the latter, in the system of equations (1)-(4), we go over to transforms and set the external disturbances equal to zero ($M_2 = 0$, $M_1 = 0$, and $\theta = 0$). Then, substituting β_2 for the angle β, we obtain

$$Ap^2\alpha - Hp\beta_2 - nM_{dv} = 0, \qquad (9)$$

$$Bp^2\beta_2 + Hp\alpha = 0, \qquad (10)$$

$$k_0 M_{dv} + k_3 n p\alpha = u_y, \qquad (11)$$

$$(T_y p + 1)u_y = k_1 u_1. \qquad (13)$$

We solve the resulting system of equations for the angle β_2, discarding the zero root (the gyrostabilizer is considered to be uncompensated):

$$\beta_2 = \frac{k_1 n u_1}{k_0 H p(1 + T_y p)\left(1 + T_{dv} p + \frac{p^2}{q^2}\right)} = w_2(p)u_1. \qquad (13)$$

The open-loop transfer function is

$$W(p) = w_1(p)w_2(p) = \frac{K}{p(1 + T_y p)\left(1 + T_{dv} p + \frac{p^2}{q^2}\right)}. \qquad (14)$$

The quantities entering into this formula are determined as follows. The over-all open-loop gain is

$$K = \frac{k_1 k_2 n}{k_0 H} = \frac{57.3\cdot 3\cdot 200}{3.3\cdot 12.9\cdot 10^3} = 0.8 \text{ 1/sec.}$$

The calculated free-oscillation frequency of the gyrostabilizer is

$$q_0 = \frac{H}{\sqrt{AB}} = \frac{12.9\cdot 10^3}{\sqrt{1000\cdot 4}} = 200 \text{ 1/sec.}$$

The real free-oscillation frequency is taken equal to

$$q = \frac{q_0}{1.5} = \frac{200}{1.5} = 133 \text{ 1/sec.}$$

The time constant of the motor together with the gyroscope is

$$T_{dv} = \frac{k_M n^2 R}{k_0 H^2} = \frac{0.11\cdot 200^2\cdot 1}{3.3\cdot 12.9^2\cdot 10^6} = 2.5\cdot 10^{-3} \text{ sec.}$$

Substitution of the numerical values into (14) yields

$$W(p) = \frac{0.8}{p(1 + 0.1p)\left(1 + 2.5\cdot 10^{-3}p + \frac{p^2}{133^2}\right)}. \qquad (15)$$

The l.a.kh. of the open-loop system has been plotted in Fig. 226

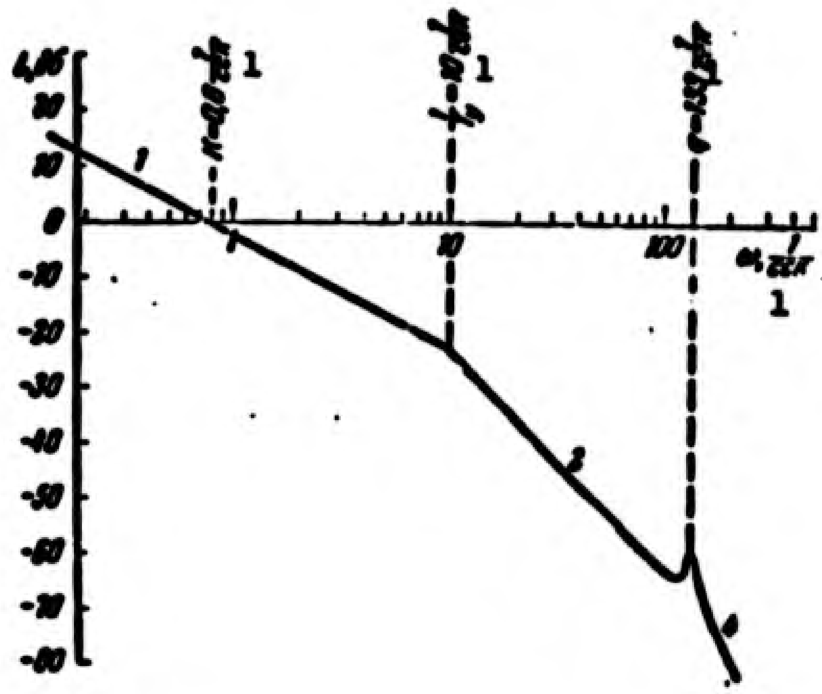


Fig. 226. Logarithmic amplitude characteristic for open-loop gyrostabilizer. 1) sec.

in accordance with (14) and (15).

308. Under the conditions of the preceding problem, determine the admissible over-all gain K and amplifier gain k_4 such that the magnitude ratio of the closed-loop system will not exceed M = 1.5.

Solution. From the condition that must be satisfied in order to assur permissible phase shift in the region of the cutoff frequency (Fig. 226) we have [3]

$$K < \frac{1}{T_y} \frac{M^2 + M\sqrt{M^2 - 1}}{2} = \frac{1}{0,1} \frac{1,5^2 + 1,5\sqrt{1,5^2 - 1}}{2} = 19,7 \text{ 1/sec.}$$

From the condition requiring that the resonance peak not enter the forbidden zone we have for the high-frequency part of the l.a.kh.

$$K < \frac{M}{M+1} q^2 T_{d.n} T_y = \frac{1,5}{1,5+1} \cdot 133^2 \cdot 2,5 \cdot 10^{-4} \cdot 0,1 = 35 \text{ 1/sec.}$$

Finally, we can let K $\leq$ 19.7 1/sec. The permissible amplifier gain is

$$k_4 = \frac{k_d H K}{k_y n} = \frac{5,5 \cdot 12,9 \cdot 10^4 \cdot 19,7}{57,3 \cdot 200} \approx 120.$$

309. Using the conditions of Problem 307, determine for a gyrostabilizer the amplitude of the stabilization error when K = 0.8 1/sec, if

the base rolls in accordance with a harmonic law at period T = 6 sec and amplitude a = 10°.

Answer. The amplitude of the moment component of the error is

$$\alpha_0 = \frac{2BM_1}{H^2} = \frac{2\cdot 4\cdot 1000}{12{,}9^2\cdot 10^6} = 4{,}8\cdot 10^{-5}\ \text{rad} = 0.165'.$$

The amplitude of the harmonic component is

$$\alpha_2 = \frac{\varepsilon_{max} T_{a.e}}{\sqrt{\omega^2 + k^2}} = \frac{11\cdot 2{,}5\cdot 10^{-4}}{\sqrt{1{,}05^2 + 0{,}5^2}} = 0{,}0021° = 0{,}126'.$$

where $\omega = 2\pi/T = 6.28/6 = 1.05$ 1/sec and $\varepsilon_{max} = \omega^2 a = 1.1\cdot 10 = 11$ degrees/sec^2. The resultant error is

$$\alpha_{max} = \alpha_0 + \alpha_2 = 0{,}165 + 0{,}126 \approx 0{,}29'.$$

310. In a force gyrostabilizer, find the over-all open-loop gain and the steady-state precession angle when a constant load torque M_1 = 1000 g·cm is applied to the stabilization axis. The gyrostabilizer has the following parameters: 1) kinetic moment of the gyroscope $H = 2\cdot 10^3$ g·cm·sec; 2) the gyroscope moment of inertia about the precession axis is B = 1.5 g·cm·sec^2; 3) the gear ratio of the reduction gear is n = = 150; 4) the slope of the stabilizing-motor torque characteristic is $1/k_2$ = 0.5 g·cm/v for k_2 = 2 v/g·cm; 5) the slope of the precession-angle pickoff curve is k_1 = 0.5 v/degree = 28.6 v/rad; 6) the amplifier voltage gain is k_4 = 10. The parameter symbols agree with those of Problem 307.

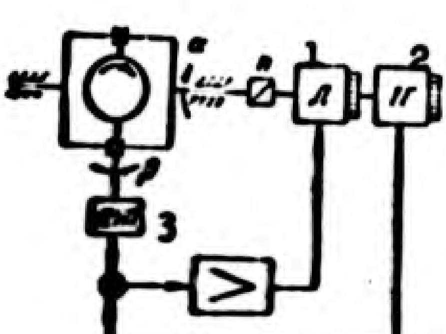


Fig. 227. Force stabilizer with tachometer feedback. 1) Motor; 2) tachometer generator; 3) DUP.

Answer. The over-all open-loop gain is

$$K = \frac{nk_1k_4}{k_2H} = \frac{150\cdot 28{,}6\cdot 10}{2\cdot 2\cdot 10^3} \approx 10{,}8\ \text{1/sec}.$$

The steady-state precession angle is

$$\beta_{ycT} = \frac{BM_1}{KH} = \frac{1{,}5\cdot 1000}{10{,}8\cdot 2\cdot 10^3} = 0{,}07\ \text{rad} = \text{–}°.$$

Another expression for the steady-state precession angle of the stabilizer is:

$$\beta_{ust} = \frac{Bk_3M_1}{ak_1k_2} = \frac{1.5\cdot 2\cdot 1000}{150\cdot 28.6\cdot 10} = 0.07 \text{ rad} = 4^{\circ}.$$

311. In a single-axis stabilizer (Fig. 227), perform the calculations for the voltage feedback circuit of a tachometer generator on one axis with a stabilizing motor and find the required amplifier gain such that the steady-state precession angle will not exceed $\beta_{ust} = 1^{\circ}$ when a constant load torque $M_1 = 500$ g·cm is applied to the stabilization axis. The initial values, in the notation of Problem 307 are: $N = 6.5\cdot 10^3$ g·cm·sec, $B = 5$ g·cm·sec^2, $A = 800$ g·cm·sec^2, $T_u = 0$, $M_2 = 0$, $k_1 = 1$ v/degree = 57.3 v/rad, $k_2 = 4$ v/g·cm, $k_3 = 0.2$ v·sec/rad, $n = 50$. The permissible magnitude ratio is $M = 1.3$.

<u>Solution</u>. The steady-state precession angle is determined from the expression

$$\beta_{ust} = \frac{BM_1}{KH}\ [\text{rad}] = \frac{57.3BM_1}{KH}\ [\text{degrees}]$$

From this we obtain the required over-all open-loop gain:

$$K = \frac{57.3BM_1}{\beta_{ust}H} = \frac{57.3\cdot 5\cdot 500}{1\cdot 6.5\cdot 10^3} = 22 \text{ 1/sec.}$$

The time constant for the motor together with the gyroscope is

$$T_{d.g} = \frac{k_3n^2B}{k_2H^2} = \frac{0.2\cdot 50^2\cdot 5}{4\cdot 6.5^2\cdot 10^6} = 1.47\cdot 10^{-5} \text{ sec.}$$

The square of the free-oscillation frequency of the gyrostabilizer is

$$q^2 = \frac{H^2}{AB} = \frac{6.5^2\cdot 10^6}{800\cdot 5} = 1.06\cdot 10^4 \text{ 1/sec}^2.$$

The permissible value of the over-all gain in the absence of tachometer feedback [3]

$$K < q^2T_{d.g}\frac{M}{M+1} = 1.06\cdot 10^4\cdot 1.47\cdot 10^{-5}\cdot\frac{1.3}{1.3+1} = 0.09 \text{ 1/sec.}$$

This gain is considerably below the required value K = 22 1/sec.

When the tachometer feedback is introduced, the inequality

$$K < q^2 (T_{\text{д.р}} + T_{\text{о.с}}) \frac{M}{M+1}$$

should hold. From this we can determine the required feedback time constant:

$$T_{\text{о.с}} = \frac{K(M+1)}{q^2 M} - T_{\text{д.р}} = \frac{22(1.3+1)}{1.06 \cdot 10^4 \cdot 1.3} - 1.47 \cdot 10^{-3} = 0.37 \cdot 10^{-3} \text{ sec.}$$

On the other hand, the feedback time constant equals [3]

$$T_{\text{о.с}} = \frac{a^2 k_5 k_7 B}{k_4 H i^2},$$

where k_5 is the amplifier gain for the tachometer-generator signal, k_7 is the tachometer-generator sensitivity. From this last expression we can find the required value of the product:

$$k_5 k_7 = \frac{k_4 H i^2 T_{\text{о.с}}}{a^2 B} = \frac{4 \cdot 6.5^2 \cdot 10^6 \cdot 0.37 \cdot 10^{-3}}{30^2 \cdot 5} = 50 \cdot \text{v·sec/rad.}$$

Next, from the expression for the over-all open-loop gain

$$K = \frac{k_1 k_4 a}{k_6 H}$$

we can find the required amplifier gain for the main signal:

$$k_4 = \frac{k_6 H K}{k_1 a} = \frac{4 \cdot 6.5 \cdot 10^4 \cdot 22}{57.3 \cdot 30} \approx 200.$$

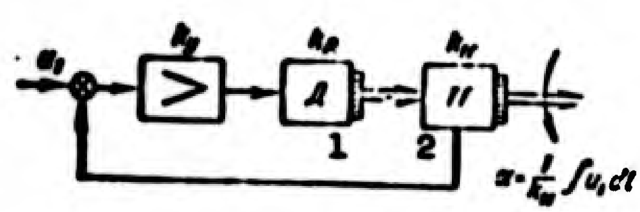


Fig. 228. Integrating-drive. 1) Motor; 2) tachometer generator.

When the feedback signal is added to the main signal as shown in Fig. 227, i.e., when $k_5 = k_4$, the required tachometer-generator sensitivity will be

$$k_7 = \frac{50}{200} = 0.25 \text{ v·sec/rad} = 0.026 \text{ v/rpm.}$$

§36. INTEGRATING (TACHOMETER) DRIVE

312. For the integrating drive mechanism illustrated in Fig. 228, determine the maximum permissible open-loop gain K such that the closed-loop system magnitude ratio will not exceed M = 1.3. Also find the amplifier gain k_u corresponding to the value found for K. The initial values are: 1) time constant of motor together with tachometer generator T_0 = 0.1 sec; 2) time constants of amplifier channel T_1 = 0.01 sec and T_2 = 0.005 sec; 3) tachometer-generator characteristic slope k_{tg} = $5\cdot 10^{-3}$ v/rpm; 4) motor-characteristic slope k_d = 10 rpm/v.

Solution. The permissible over-all open-loop gain for the static system may be found from the expression [3]

$$K<\frac{T_0}{\Sigma T}\,\frac{M^2+M\sqrt{M^2-1}}{2},$$

where T_0 is the largest time constant, ΣT is the sum of the small time constants. For the case under consideration, we have

$$K<\frac{T_0}{T_1+T_2}\,\frac{M^2+M\sqrt{M^2-1}}{2}=$$
$$=\frac{0,1}{0,015}\,\frac{1,3^2+1,3\sqrt{1,3^2-1}}{2}=9,2.$$

Since the over-all gain equals

$$K=k_y k_d k_{tg},$$

then the required amplifier gain will be

$$k_y=\frac{K}{k_d k_{tg}}=\frac{9,2}{10\cdot 5\cdot 10^{-3}}=184.$$

313. For the previous problem, determine the moment of inertia of the stabilizing flywheel on the motor shaft if it is necessary to bring the over-all gain to a value K = 1000, while the moment of inertia of the motor together with the tachometer generator is J_d = 0.05 $g\cdot cm\cdot sec^2$.

Answer

$$J_m=J_d\left(\frac{1000}{9,2}-1\right)=0,05\cdot 108=5,4\ g\cdot cm\cdot sec^2.$$

314. For the tachometer drive of Fig. 228, determine the required over-all open-loop gain K corresponding to a stable speed range D = = 10,000. The initial values are: 1) maximum motor speed n_{max} = 6000 rpm; 2) voltage across motor control winding U = 110 v, corresponding to a starting torque M_p = 100 g•cm and an ideal no-load speed $n_{kh.kh}$ = = 6000 rpm; 3) maximum motor shaft load M_n = 20 g•cm.

Solution [3]:

The minimum stable system speed is

$$n_{min} = \frac{n_{max}}{D} = \frac{6000}{10\,000} = 0,6 \text{ rpm}.$$

This speed is connected with the load torque by the relationship

$$n_{min} = \frac{M_n}{K_m}, \qquad (1)$$

where K_m is the system torque figure of merit. From (1) we find

$$K_m = \frac{M_n}{n_{min}} = \frac{20}{0,6} = 33,3 \text{ g•cm/rpm}.$$

The torque figure of merit is connected with the over-all gain by the expression

$$K_m = \frac{K}{\beta}, \qquad (2)$$

where the slope of the mechanical characteristic of the motor is

$$\beta = \frac{n_{x.x}}{M_n} = \frac{6000}{100} = 60 \text{ rpm/g•cm}.$$

From (2) we obtain the required over-all gain:

$$K = \beta K_m = 60 \cdot 33,3 = 2000.$$

315. Determine the operating range of the tachometer drive D_1 considered in the preceding problem if it is used in integration with a permissible relative reduced error δ = 1% for the same load torque of M_{n1} = 20 g•cm and for M_{n2} = 2 g•cm.

Answer

1) D_1 = δD = 0.01•10,000 = 100;

2) $D_1 = \delta D \, M_{n2}/M_{n1} = 0.01 \cdot 10{,}000 \cdot 20/2 = 1000.$

316. In order to determine the dynamic properties of two identical integrating drive mechanisms, they are connected into a test loop (Fig. 229). From the test it is found that when the initial conditions are introduced into the system, divergent oscillations appear at a frequency f = 1 cps. After n = = 10 complete oscillations, the increment in the deflection amplitude is 58% of the initial value. Find the equivalent time constant for each drive mechanism.

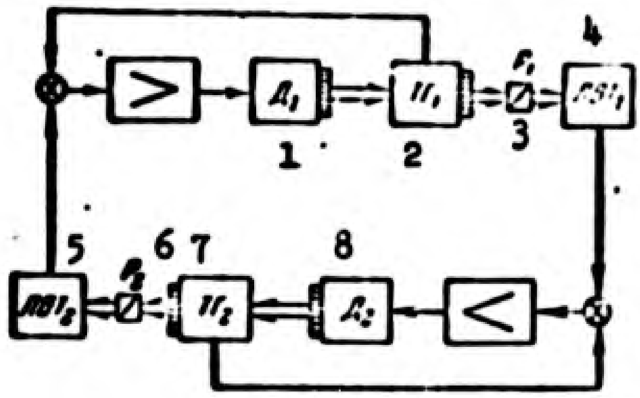


Fig. 229. Test circuit for integrating drives. 1) D_1; 2) TG_1; 3) R_1; 4) LVT_1; 5) LVT_2; 6) R_2; 7) TG_2; 8) D_2.

Solution [3]

$$T = \frac{2.3}{2\pi} \frac{1}{n\omega} \lg\left(1 + \frac{\Delta A}{A}\right) = \frac{2.3}{6.28} \frac{1}{10 \cdot 6.28} \lg 1.58 = 1.16 \cdot 10^{-3} \text{ sec.}$$

§37. SMOOTHING SYSTEMS

317. The useful signal at the input of a smoothing servosystem takes the form of a harmonic function with amplitude $\theta_{1max} = 15°$ and period $T_k = 20$ sec. The noise at the input is also harmonic with amplitude $\theta_{1p} = 1°$ and $T_p = 0.06$ sec. What should be the transfer function of a smoothing servosystem with first-order astatism if the error in reproduction of the useful signal is not to exceed $\theta_{max} = 0.1°$ and the noise-smoothing coefficient is to be at least $k_{sgl} = 10$ with acceptable dynamic properties of the system and the simplest possible structure?

Solution. The signal frequency is

$$\omega_k = \frac{2\pi}{T_k} = \frac{6.28}{20} = 0.314 \text{ 1/sec.}$$

At this frequency, the modulus of the frequency transfer function for the open-loop system should be no less than

$$|W(j\omega_k)| = \frac{\theta_{1max}}{\theta_{max}} = \frac{15}{0.1} = 150 = 43.5 \text{ db.}$$

Using these values, in Fig. 230 we have plotted a control point A_k. In order to provide the required accuracy it is necessary that the l.a.kh. of the open-loop servosystem remain above this point.

The noise frequency is

$$\omega_n = \frac{2\pi}{T_n} = \frac{6.28}{0.06} = 105 \quad 1/\text{sec}.$$

At this frequency, the modulus of the frequency transfer function for the open-loop system should not exceed the value

$$|W(j\omega_n)| = \frac{1}{k_{sm}} = \frac{1}{10} = -20 \text{ db}.$$

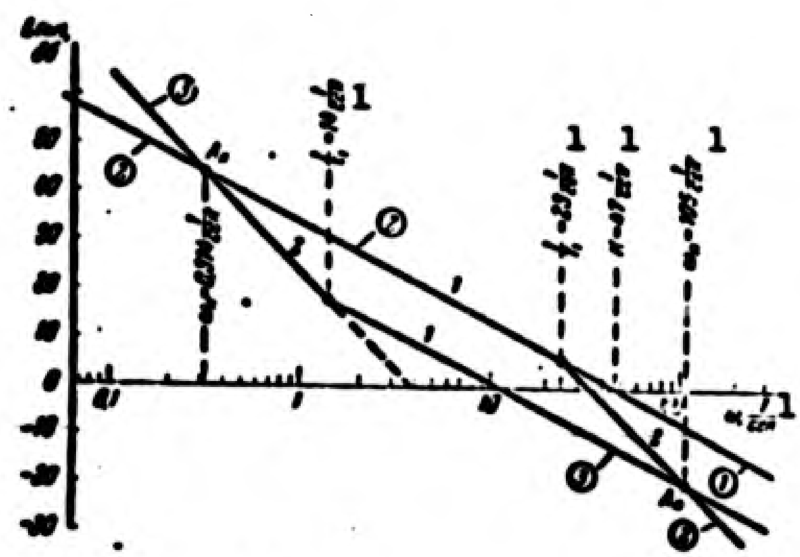


Fig. 230. Logarithmic amplifier characteristic of smoothing system. 1) sec.

The control point A_p has been plotted in Fig. 230 from this data. In order to provide the given smoothing coefficient it is necessary for the l.a.kh. of the open-loop system to remain below this point.

Let us see whether it is possible to use an elementary smoothing device [2] whose open-loop transfer function corresponds to an ideal integrating element (type 1):

$$W(j\omega) = \frac{K}{j\omega}.$$

If we so select the over-all gain that the l.a.kh. of the open-loop system passes through the point A_k (the l.a.kh. of Fig. 230), we then have

$$K = \omega_k |W(j\omega_k)| = 0.314 \cdot 150 = 47 \quad 1/\text{sec}.$$

At the noise frequency $\omega = \omega_p$ this gives a smoothing coefficient

$$k_{sg} \approx \frac{1}{|W(j\omega_p)|} = \frac{\omega_p}{K} = \frac{105}{47} = 2{,}24.$$

This is lower than the required value.

Let us see whether it is possible to use a smoothing device of the 1-2 type. The open-loop transfer function of such a device has the form

$$W(j\omega) = \frac{K}{j\omega(1 + j\omega T_1)}.$$

As above, from the condition stating the required signal-reproduction accuracy we obtain K = 47 1/sec. We next determine the smoothing coefficient at the noise frequency:

$$k_{sg} \approx \frac{1}{|W(j\omega_p)|} \approx \frac{\omega_p^2 T_1}{K}.$$

From this last expression we obtain the required time constant:

$$T_1 = \frac{k_{sg} K}{\omega_p^2} = \frac{10 \cdot 47}{105^2} = 0{,}043 \text{ sec.}$$

The smoothing-device open-loop transfer function will be

$$W(p) = \frac{K}{p(1 + T_1 p)} = \frac{47}{p(1 + 0{,}043p)}.$$

The smoothing-device l.a.kh. is shown in Fig. 230 (l.a.kh. 2).

Any stability-margin criterion may be used to check the dynamic properties. Thus, for example, let us find the system closed-loop magnitude ratio. From a well-known formula (see, for example, Problem 214), we have when the condition $KT_1 = 47 \cdot 0.043 = 2 > 0.5$:

$$M = \frac{2KT_1}{\sqrt{4KT_1 - 1}} = \frac{2 \cdot 47 \cdot 0{,}043}{\sqrt{4 \cdot 47 \cdot 0{,}043 - 1}} = 1{,}5,$$

which is acceptable.

318. Solve the preceding problem if the smoothing system must have second-order astatism.

<u>Answer</u>. The open-loop transfer function of a smoothing system of

2-1 type is

$$W(p) = \frac{K(1 + \tau_1 p)}{p^2} = \frac{15(1 + 0.7p)}{p^2}.$$

The l.a.kh. is shown in Fig. 230 (l.a.kh. 3).

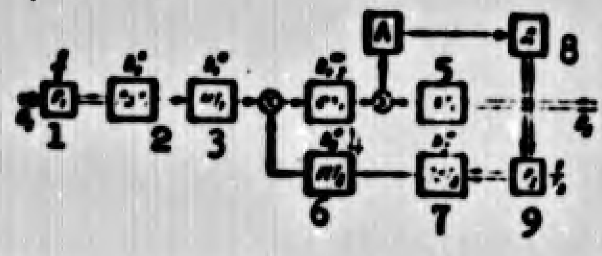


Fig. 231. Electromechanical diagram of type 1 smoothing system. 1) Reduction gear 1; 2) linear magslip 1; 3) scale transformer 1; 4) phase networks; 5) asynchronous tachometer generator 1; 6) scaling transformer 2; 7) linear magslip 2; 8) motor; 9) reduction gear 2.

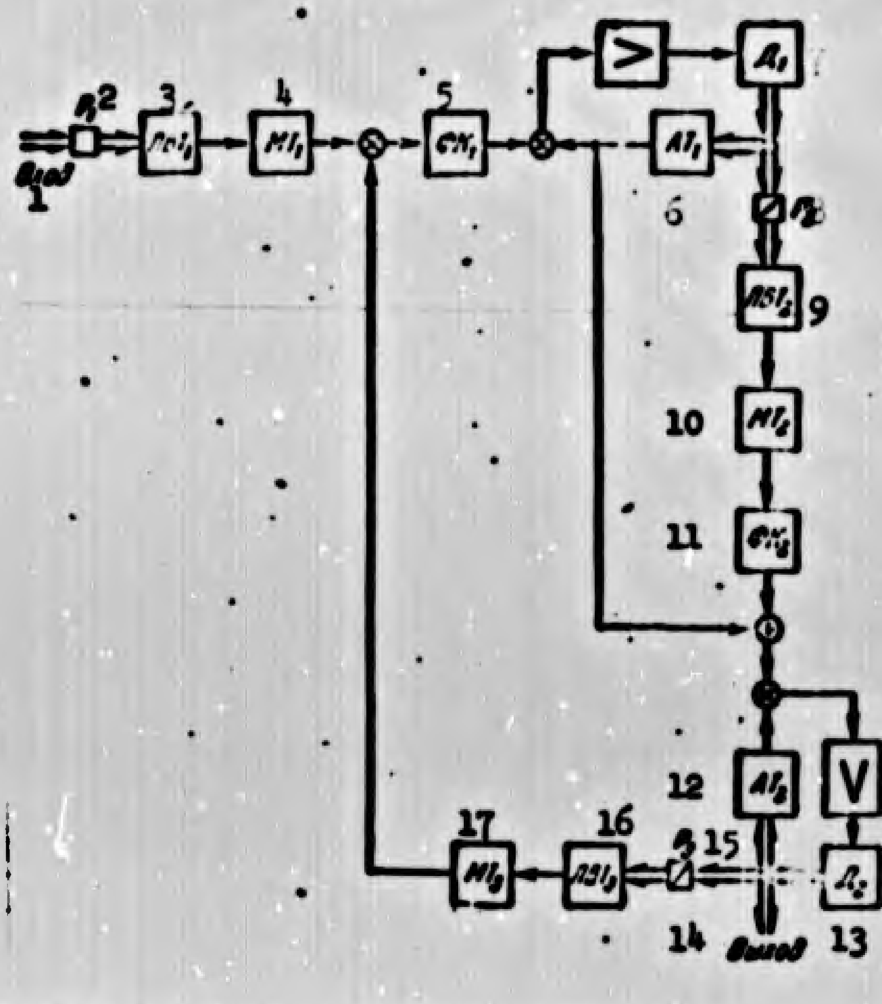


Fig. 232. Electromechanical diagram of type 2-1 smoothing system. 1) Input; 2) reduction gear 1; 3) linear magslip 1; 4) scaling transformer 1; 5) phasing network 1; 6) asynchronous tachometer generator 1; 7) motor 1; 8) reduction gear 2; 9) linear magslip 2; 10) scaling transformer 2; 11) phasing network 2; 12) asynchronous tachometer generator 2; 13) motor 2; 14) output; 15) reduction gear 3; 16) linear magslip 3; 17) scaling transformer 3.

319. Give the electromechanical diagrams for the smoothing systems of Problem 317 and 318 if they use induction elements: linear magslips

(LVT), asynchronous two-phase motors (AD), asynchronous tachometer generators (AT), scaling transformers (MT), and phasing networks (FK).

Answer. The diagrams are shown in Figs. 231 and 232.

320. For a type 1 smoothing device [2] with open-loop transfer function

$$W(p)=\frac{K}{p}$$

determine the relationship between the smoothing coefficient $k_{sgl} \gg 1$ for harmonic noise with period T_p and the observational time T_0 (settling time to an angle of Δ% of the initial value).

Answer. The oversational time is found from the system equation of motion (exponentials with time constant of 1/K):

$$T_0=\frac{1}{K}\ln\frac{1}{\Delta}. \qquad (1)$$

The smoothing coefficient

$$k_{sgl}\approx\frac{\omega_p}{K}=\frac{2\pi}{KT_p}. \qquad (2)$$

We have from (1) and (2)

$$T_0=\frac{k_{sgl}T_p}{2\pi}\ln\frac{1}{\Delta} \qquad (3)$$

or

$$k_{sgl}=\frac{2\pi T_0}{T_p\ln\frac{1}{\Delta}}. \qquad (4)$$

321. For a 1-2 type smoothing device [2] with an open-loop transfer function

$$W(p)=\frac{K}{p(1+T_1p)}$$

determine the relationship between the smoothing coefficient $k_{sgl} \gg 1$ for harmonic noise with a period T_p and an observational time T_0 for Δ = 1% for the case of multiple roots of the closed-loop characteristic equation.

Answer

$$k_{sgl}=0.0916\,\pi^2\frac{T_0^2}{T_p^2}.$$

§38. AMPLIFIERS WITH HEAVY FEEDBACK

322. For an operational amplifier with feedback (Fig. 233a), determine the exact and approximate values of the transfer function.

Answer

$$w(p)=-\frac{\frac{1}{z_1(p)}}{\frac{1}{K}\left[\frac{1}{R}+\frac{1}{z_1(p)}+\frac{1}{z_0(p)}\right]+\frac{1}{z_0(p)}}\approx-\frac{z_0(p)}{z_1(p)}.$$

323. Select the impedances $z_1(p)$ and $z_2(p)$ for the case in which

Fig. 233. Operational amplifiers.

the transfer function of an operational amplifier with feedback (Fig. 233a) must correspond to a floating-plus-proportional element (the minus sign is omitted):

$$w(p)=k_1+\frac{k_2}{p}=\frac{k_2(1+\tau_1 p)}{p},$$

where $k_1 = 0.8$, $k_2 = 2$ 1/sec, $\tau_1 = k_1/k_2 = 0.4$ sec.

Answer. The diagram is shown in Fig. 233b. The following conditions should be satisfied by the parameters:

$$\frac{R_2}{R_1}=k_1 \text{ and } \frac{1}{R_1C_2}=k_2.$$

The possible parameter values are:

$C_2 = 1$ µf; $R_2 = 400$ kohm, and $R_1 = 500$ kohm.

324. Solve the preceding problem for the case in which the operational amplifier with feedback should correspond to:

1) an integrating element with transfer function

$$w_1(p)=\frac{k_1}{p}, \text{ where } k_1 = 10 \text{ 1/sec};$$

2) a differentiating element with transfer function

$$w_2(p)=k_2p, \text{ where } k_2 = 0.2 \text{ sec};$$

3) an aperiodic element with transfer function

$$w_3(p)=\frac{1}{1+Tp}, \text{ where } T = 0.1 \text{ sec}.$$

Answer

1) $z_1(p)=R_1$ and $z_2(p)=\frac{1}{pC_2}$, $\frac{1}{R_1C_2}=k_1$

(occurring when $C_2 = 1$ µf, $R_1 = 100$ kohm);

2) $z_1(p)=\frac{1}{pC_1}$ and $z_2(p)=R_2$, $R_2C_1=k_2$

(occurring when $C_1 = 1$ µf, $R_2 = 200$ kohm);

3) $z_1(p)=R_1+\frac{1}{pC_1}$ and $z_2(p)=\frac{1}{pC_2}$, $C_1=C_2=C$,

$R_1C = T$ (occurring when $C = 1$ µf, $R_1 = 100$ kohm).

325. Calculate the required gain of an amplifier with heavy feedback (Fig. 234) under the following initial conditions: 1) gain instability of the amplifer without feedback $\Delta K/K = 50\%$; 2) required accur-

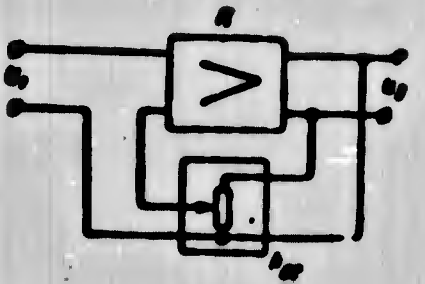


Fig. 234. Buffer amplifier with heavy feedback.

acy of gain stabilization in the presence of feedback $\delta = 0.01\%$; 3) resultant gain with feedback present $K_0 = 10$.

Solution. The required feedback-loop gain is

$$k_{o.s}K = \frac{\Delta K}{K}\frac{1}{\delta} = \frac{30}{0.01} = 3000.$$

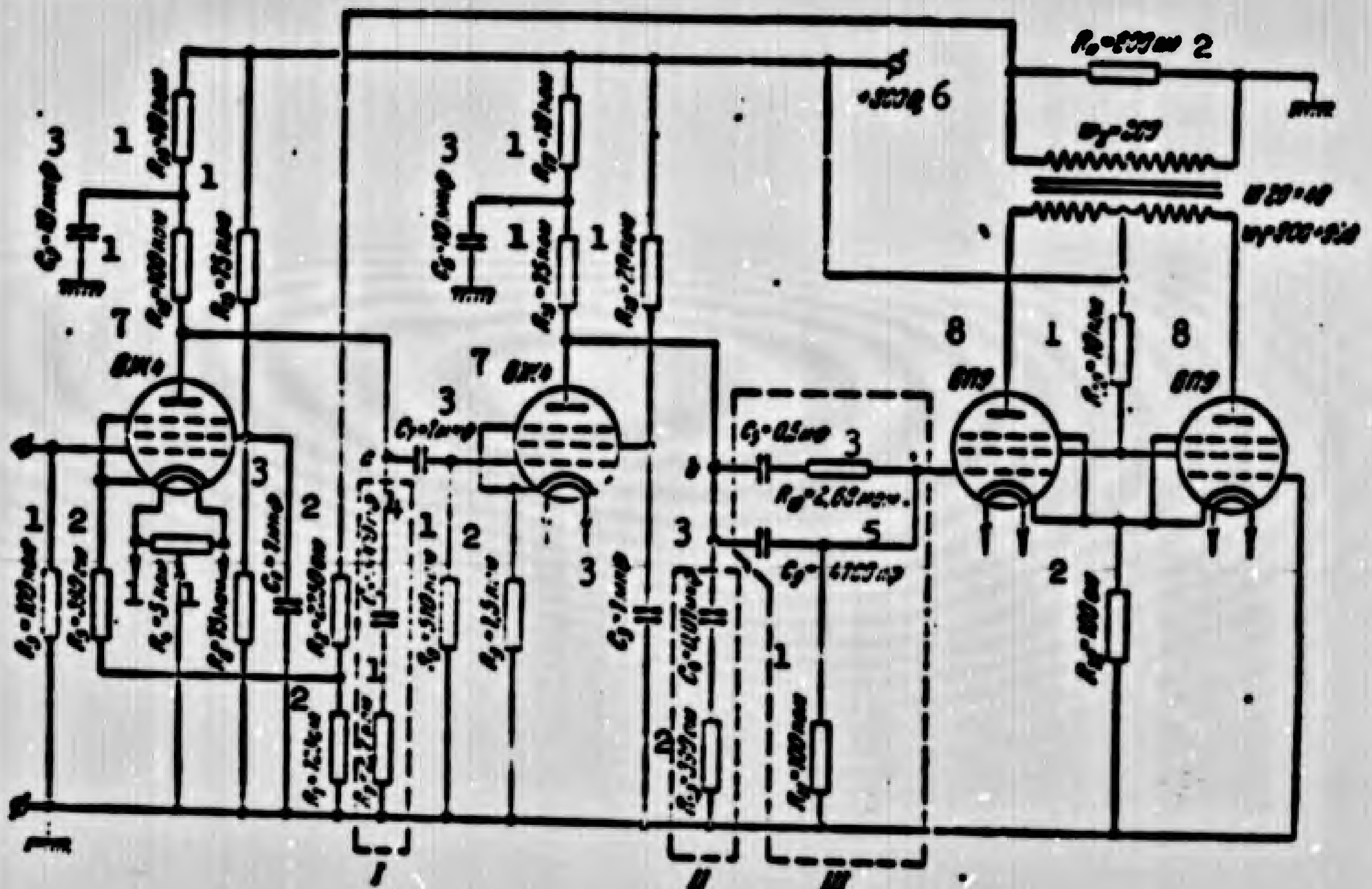


Fig. 235. Circuit of amplifier with heavy feedback. 1) kohm; 2) ohms; 3) μf; 4) pf; 5) megohm; 6) v; 7) 6Zh4; 8) 6P9.

The required feedback factor is

$$k_{o.s} = \frac{1}{K_0} = \frac{1}{10} = 0.1.$$

The required amplifier gain is

$$K = \frac{k_{o.s}K}{k_{o.s}} = \frac{3000}{0.1} = 30\,000.$$

326. Design compensating networks for a buffer amplifier with heavy feedback having the block diagram shown in Fig. 234; the basic circuit is shown in Fig. 235. The initial values are: 1) load resist-

ance R_n = 600 ohms; 2) signal carrier frequency f_n = 500 cps; 3) closed-loop gain $K_0 = 4 \pm 0.05\%$; 4) amplifier open-loop gain with compensating networks inserted K = 4500; 5) transfer function of output transformer together with load resistance in low-frequency range; this may be represented in the form

$$W_{vykh}(p) = \frac{bp}{1 + T_{vykh}p},$$

where $T_{vykh} = 0.228 \cdot 10^{-3}$ sec; 6) the closed-loop magnitude ratio of the amplifier should not exceed M = 1.5.

Solution. The time constant corresponding to the carrier frequency is

$$T_n = \frac{1}{\omega_n} = \frac{1}{2\pi f_n} = \frac{1}{2\pi \cdot 500} = 0{,}32 \cdot 10^{-3} \text{ sec.}$$

The relationship between the time constants T_{vykh} and T_n is

$$a = \frac{T_n}{T_{vykh}} = \frac{0{,}32 \cdot 10^{-3}}{0{,}228 \cdot 10^{-3}} = 1{,}4.$$

The feedback factor is

$$k_{o.s} \approx \frac{1}{K_0} = \frac{1}{4} = 0{,}25.$$

The feedback-loop gain is

$$k_{o.s}K = 0{,}25 \cdot 4500 = 1125 = 61 \text{ db.}$$

The time constants corresponding to the conjugating frequencies of the desired l.a.kh. are found from expressions given in [3]:

$$T_1 = T_{vykh} = \frac{T_n}{a} = \frac{0{,}32 \cdot 10^{-3}}{1{,}4} = 0{,}228 \cdot 10^{-3} \text{ sec,}$$

$$T_2 = \frac{aT_n\tau_2}{(1 + a^2)\sqrt{k_{o.s}K}} = \frac{1{,}4 \cdot 0{,}32 \cdot 10^{-3} \cdot 1{,}73}{(1 + 1{,}4^2)\sqrt{1125}} = 7{,}8 \cdot 10^{-6} \text{ sec.}$$

The relative time constant entering here for the standard l.a.kh., τ_2, is set equal to

$$\tau_2 = \sqrt{\frac{M}{M - 1}} = \sqrt{\frac{1{,}5}{1{,}5 - 1}} = 1{,}73.$$

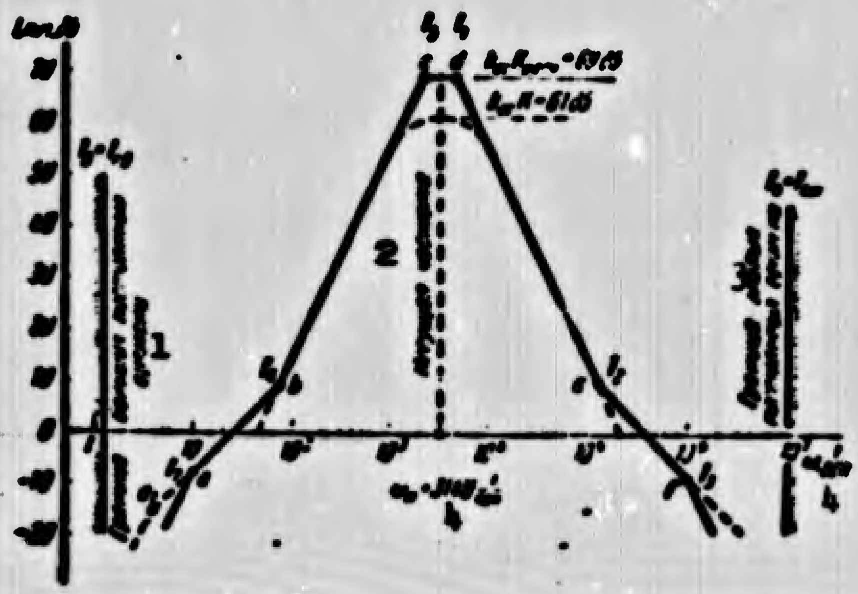


Fig. 236. Desired logarithmic amplitude characteristic for buffer amplifier. 1) Limit of large time constants; 2) carrier frequency; 3) limit of small time constants; 4) sec.

We next find

$$T_3 = \frac{aT_0\tau_0}{(1+a^2)\sqrt{k_{o.c}K}} = \frac{T_0}{h} = \frac{7.8\cdot10^{-6}}{7} = 1.1\cdot10^{-6}\ \text{sec.}$$

The relative time constant τ_3 is set equal to

$$\tau_3 = \frac{\sqrt{M(M-1)}}{M+1} - \sum\tau_m = \frac{\sqrt{1.5(1.5-1)}}{1.5+1} - 0.1 = 0.246,$$

where the sum of the relative small time constants $\Sigma\tau_m$ is taken equal to 0.1. The limiting value for the small time constants is

$$T_4 = T_{s.m} = \frac{aT_0\Sigma\tau_m}{4(1+a^2)\sqrt{k_{o.c}K}} = \frac{1.4\cdot0.32\cdot10^{-3}\cdot0.1}{4(1+1.4^2)\sqrt{1125}} = 0.112\cdot10^{-6}\ \text{sec.}$$

The time constants forming the low-frequency portion of the l.a.zh. are:

$$T_1 = aT_0 = 1.4\cdot0.32\cdot10^{-3} = 0.45\cdot10^{-3}\ \text{sec},$$
$$T_2 = \frac{(1+a^2)T_0\sqrt{k_{o.c}K}}{a\tau_0} = \frac{T_0^2}{T_3} = \frac{0.32^2\cdot10^{-6}}{7.8\cdot10^{-6}} = 13.2\cdot10^{-3}\ \text{sec},$$
$$T_3 = \frac{(1+a^2)T_0\sqrt{k_{o.c}K}}{a\tau_0} = hT_2 = 7\cdot13.2\cdot10^{-3} = 93\cdot10^{-3}\ \text{sec.}$$

The limiting value for the large time constants [3] is

$$T_5 = T_{s.l} = \frac{1000(1+a^2)k_{o.c}K}{a^2}T_{s.m} = \frac{1000(1+1.4^2)\cdot1125}{1.4^2}0.112\cdot10^{-6} = 0.91\ \text{sec.}$$

The theoretical gain (without the compensating networks) is

$$K_{theor} = \frac{(1+a^2)^2}{a^4} K = \frac{(1+1.4^2)^2}{1.4^4} 4300 = 10\,100.$$

The theoretical feedback-loop gain is

$$(k_{o.c} K)_{theor} = \frac{(1+a^2)^2}{a^4} k_{o.c} K = \frac{(1+1.4^2)^2}{1.4^4} 1125 = 2530 = 69 \text{ db}.$$

From the values obta.ned for the theoretical gain and time constants, we have plotted in Fig. 236 the asymptotic desired l.a.kh. The bends in the asymptotic l.a.kh. occur owing to the time constants introduced into the amplifier circuit by means of three compensating networks. The networks are shown in Fig. 235 by dashed lines.

Network I is a passive integrating element. Its transfer function is

$$W_a(p) = \frac{1+T'p}{1+T''p}.$$

The lower time constant $T'' = R_a C_2$ is taken equal to the second time constant $T_1 = 0.228 \cdot 10^{-3}$ sec (for a conjugating frequency $\omega_1 = 1/T_1$, the desired l.a.kh. should have two time constants T_1, since the bend at this point takes place at 40 db/decade; one of the time constants T_1 is introduced by means of the output transformer).

The upper time constant $T' = R_7 C_2$ is taken equal to $T_2 = 7.8 \times 10^{-6}$ sec.

The equivalent resistance R_a of point a in the circuit (see Fig. 235) with respec. to ground is calculated as the combination in parallel of the internal resistance of the 6Zh4 tube, R_1 = 750 kohm, and the two resistances R_{13} = 100 kohm and R_8 = 510 kohm:

$$R_a = \frac{750 \cdot 100 \cdot 510}{750 \cdot 100 + 750 \cdot 510 + 510 \cdot 100} = 75.3 \text{ kohm},$$

from which we obtain

$$C_2 = \frac{T_1}{R_a} = \frac{0.228 \cdot 10^{-3}}{75.3 \cdot 10^{3}} = 3020 \cdot 10^{-12} \text{ f}.$$

From a capacitance chart, we select C_2 = 3000 pf. Then

$$R_7 = \frac{T_2}{C_2} = \frac{7.8 \cdot 10^{-6}}{3 \cdot 10^{-9}} = 2.6 \cdot 10^3 \text{ ohms.}$$

From a resistance chart we take R_7 = 2.7 kohm.

Network II is also an integrating element. Its lower time constant is set equal to the first time constant $T_5 = 0.45 \cdot 10^{-3}$ sec (there should also be two time constants T_5).

The combination of the internal resitance of the 6Zh4 tube and resistances R_{16} and R_{11} in parallel gives the equivalent resistance of point b in the circuit with respect to ground:

$$R_b = \frac{150 \cdot 75 \cdot 100}{150 \cdot 75 + 150 \cdot 100 + 75 \cdot 100} = 40.5 \text{ kohm.}$$

From this we obtain the capacitance

$$C_4 = \frac{T_5}{R_b} = \frac{0.45 \cdot 10^{-3}}{40.5 \cdot 10^3} = 0.011 \cdot 10^{-6} \text{ f.}$$

From a capacitance chart we take C_4 = 0.01 μf.

The upper time constant for this network is used to compensate for the parasitic time constant at the input of the first tube. Taking the value of the input capacitance of the first tube, including the wiring capacitance, to equal C'_{vkh} = 40 pf, we obtain

$$T'_{vkh} = R_c C'_{vkh} = 100 \cdot 10^3 \cdot 40 \cdot 10^{-12} = 4 \cdot 10^{-6} \text{ sec,}$$

i.e., T'_{vkh} lies within the region of real time constants ($T_{g.m} < T'_{vkh} < T_{g.b}$) and it must be compensated. From this we determine the magnitude of R_{10}:

$$R_{10} = \frac{T'_{vkh}}{C_4} = \frac{4 \cdot 10^{-6}}{0.01 \cdot 10^{-6}} = 400 \text{ ohms.}$$

From a resistance chart, we welect R_{10} = 390 ohms.

Network III is a differentiating element. The transfer function of the given differentiating element has the form

$$W_3(p) = \frac{Tp(1 + T''p)}{(1 + T'''p)(1 + T^{IV}p)}.$$

where $T' = R_{11}C_5$, $T'' = R_{19}C_9$, $T''' = R_{19}C_5$, and $T^{IV} = R_{11}C_9$. The lower time constant of the network T''' is taken beyond the limit of large time constants:

$$T''' = R_{19}C_5 = 2{,}68 \cdot 10^6 \cdot 0{,}5 \cdot 10^{-6} = 1{,}34 \text{ sec} > T_{g.b} = 0{,}91 \text{ sec.}$$

The upper time constant of the network is

$$T'' = T_6 = 13{,}2 \cdot 10^{-3} \text{ sec.}$$

From this we obtain

$$C_9 = \frac{T_6}{R_{19}} = \frac{13{,}2 \cdot 10^{-3}}{2{,}68 \cdot 10^6} = 4900 \cdot 10^{-12} \text{ f.}$$

From a capacitance chart, we take C_9 = 4700 pf. The second low time constant of the network T^{IV} is taken equal to the second time constant $T_5 = 0.45 \cdot 10^{-3}$ sec. From this we obtain

$$R_{11} = \frac{T_5}{C_9} = \frac{0{,}45 \cdot 10^{-3}}{4{,}7 \cdot 10^{-9}} = 96 \cdot 10^3 \text{ ohms.}$$

From a resistance chart we take R_{11} = 100 kohm.

The parasitic time constant at the input of the second tube, for an input capacitance C''_{vkh} = 40 pf, will equal

$$T''_{vkh} = 2{,}7 \cdot 10^3 \cdot 40 \cdot 10^{-12} = 0{,}108 \cdot 10^{-6} \text{ sec} < T_{s.m} = 0{,}112 \cdot 10^{-6} \text{ sec.}$$

The parasitic time constant at the input of the third tube is

$$T'''_{vkh} = 390 \cdot 40 \cdot 10^{-12} = 0{,}0156 \cdot 10^{-6} \text{ sec} < 0{,}112 \cdot 10^{-6} \text{ sec.}$$

The time constant for the blocking capacitor C_7 is so chosen that it is close to $T_{g.b}$ = 0.91 sec. This time constant $C_7R_8 = 0.51 \cdot 1 = 0.51$ sec; as a result, the bend in the asymptotic l.a.kh. appearing at point a will be close to the limit of large time constants.

The bends in the asymptotic l.a.kh. at points a and f are moved away as far as possible toward the limiting values for the large and small time constants, as indicated in Fig. 236 by the dashed line. The increase in the length of both sections of the l.a.kh. having slopes of 20 db/decade makes it possible to increase the stability margin over the value specified.

Manuscript Page No.	[Transliterated Symbols]
302	л.а.х. = l.a.kh. = logarifmicheskaya amplitudnaya kharakteristika = logarithmic amplitude characteristic
302	ДУП = DUP = datchik ugla pretsessii = precession angle sensor
302	Д = D = dvigatel' = motor
302	п = p = pusk = starting
302	н = n = nominal'nyy = rated nominal
302	х.х = kh.kh = kholostoy khod = no load
302	у = u = usilitel' = amplifier
303	дв = dv = dvigatel' = motor
304	д = d = dvigatel' motor
304	г = g = giroskop = gyroscope
306	ТГ = TG = takhogenerator = tachometer generator
306	уст = ust = ustanovivshiysya = steady-state
308	о.с = o.s = obratnaya svyaz' = feedback
310	н = n = nagruzka = load
310	м = m = moment = torque
310	и = i = integrirovaniye = integration
311	Р = R = reduktor = reduction gear
311	ЛВТ = LVT = lineynyy vrashchayushchiy transformator = linear magslip
311	к = k [= kolebaniye = oscillation]
311	п = p = pomekha = noise
311	сгл = sgl = sglazhivaniye = smoothing
311	к = k = kontrol'nyy = control
314	МТ = MT = masshtabnyy transformator = scale transformer
314	АТ = AT = asinkhronnyy takhogenerator = asynchronous tachometer generator

314 ФК = FK = faziruyushchiy kontur = phase network

314 АД = AD = asinkhkronnyy dvigatel' = asynchronous motor

318 н = n = nesushchiy = carrier

318 вых = vykh = vykhodnoy = output

319 м = m = malyy = small

319 г = g = granichnyy = limiting

319 расч = rasch = raschetnyy = theoretical

319 б = b = bol'shoy = large

321 вх = vkh = vkhodnoy = input

321 д = d = differentsirovaniye = differentiation

Chapter 14

ADAPTIVE SYSTEMS

§39. INVESTIGATION OF STABILITY AND PERFORMANCE OF ADAPTIVE SYSTEMS

327. Figure 237 gives the basic circuit of a step-type optimum control system. Construct the block diagram of the system whose operating principle is as follows: the command pulse generator KG, producing pulses of varying length, controls the step switch ShR, which closes various electrical circuits. The first circuit through the object OR and wiper I is closed through the elements of the 6N8S tube (contact positions 1 and 3); the capacitors C_1 are connected in parallel to them. Depending on which spring touches the moving contact, the left or right side of the tube will be inserted. Thus two values of the optimization index are stored, each requiring one oscillator cycle. An intermediate

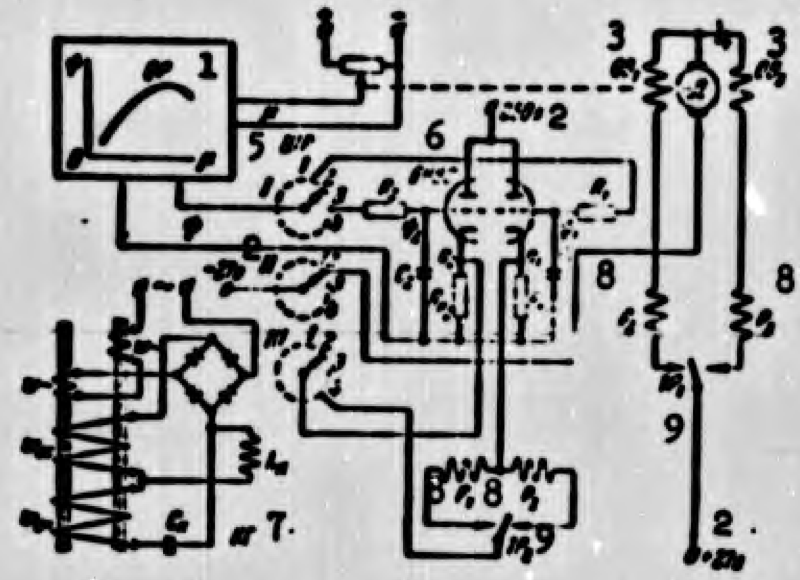


Fig. 237. Basic circuit of step-type optimum control system. 1) Object; 2) v; 3) field winding; 4) motor; 5) step switch; 6) 6N8S; 7) command pulse generator; 8) relay.

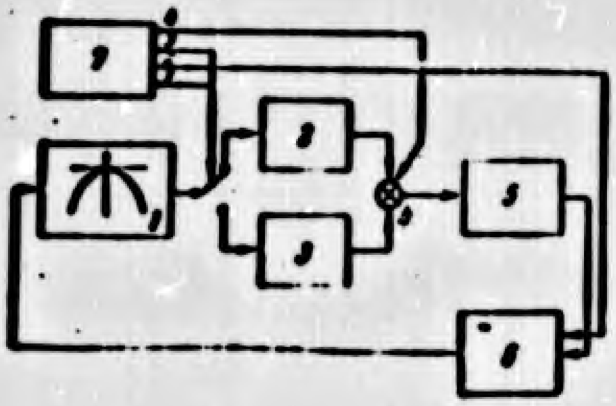


Fig. 238. Block diagram of step-type optimum system.

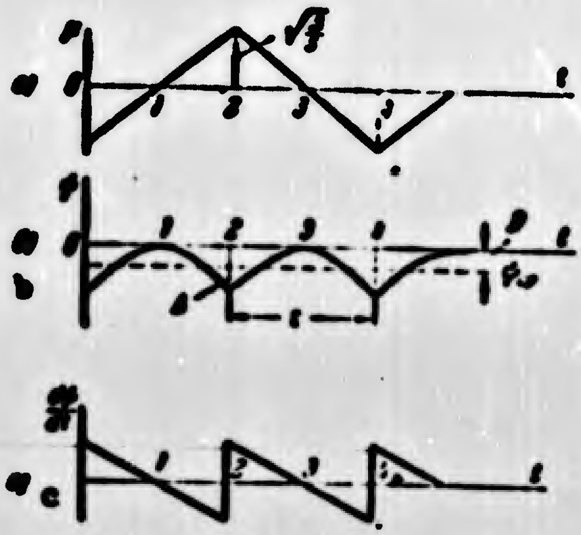


Fig. 239. Time responses of optimum system with continuous test motion. a) Measurement of manipulated variable; b) measurement of optimization index; c) measurement of rate of change of optimization index.

oscillator cycle coincides with closure of the second circuit which contains the actuating motor D (wiper II, position 2). This circuit is needed for test movement of the motor, which causes a change in the control factor μ. During the fourth cycle, the comparison is made and a regulator direction of motion selected with the aid of a logic element.

The logic element contains two relays R_1 and R_2. Their matched operation always ensures that the system will move stably toward the optimum value.

Let us assume that at time 1 the quantity φ_1 is recorded. While the moving contact is in the first position (I, 1), the capacitor is charged. The current through the tube will be proportional to the grid voltage, i.e., ψ_1. The potential of point a in the cathode circuit will be ψ_1. After position 1 has been passed, the grade circuit is opened, but the capacitor keeps a voltage ψ_1 on the grid until the next closure of circuit 1. In like manner ψ_2 is "stored" in the second half of the tube.

When the moving contact passes through position (III, 4), one of the relay windings of R_1 will be at a voltage $\psi_2 - \psi_1$. Depending on the sign of $\psi_2 - \psi_1$, the contacts of relay 1 R_1 will operate, and one of the motor windings will receive a voltage of +27 v. At the same time, the position of relay R_2 is determined; this in turn will determine the position of the contacts 1 R_1. For stable operation of the optimum regulator, the following conditions must be met:

$$\frac{d\psi}{d\mu} > 0 \text{ for } d\psi > 0 \text{ and } \frac{d\psi}{d\mu} < 0 \text{ for } d\psi < 0.$$

Answer. The block diagram is shown in Fig. 238, where 1 is the object possessing the optimum properties, 2 is the first storage device, 3 is the second storage device, 4 is the comparison unit, 5 is the logic unit performing sign inversion, 6 is the actuating element, 7 is the command oscillator; 1, 2, 3, 4 form the command sequence.

328. Figure 239 shows an oscillatory process in an optimizing system with continuous test motion [35, page 644]. Here μ is the manipulated variable, ψ is the optimization index. Find the relationship between the hunting amplitude Δ, search error D, hunting period τ, and slope of the optimization characteristic. Determine the critical reversal voltage. The magnitudes of D, τ, and Δ are indicated in Fig. 239b. The equation for the optimization characteristic is given in the form $\psi = -s\mu^2$.

Solution. Considering the variation in the optimization index ψ

and the manipulated variable μ (Fig. 239a,b), we can conclude that the system reverses at the points 0, 2, 4, 6, etc. We assume that the signal produced by the system logic element is proportional to the rate of change of the optimization index. As a consequence, $U = a\, d\psi/dt$, where a is a coefficient of proportionality. In this case, the voltage at which system reversal occurs (we call it the critical voltage) will equal

$$U_{kp} = a\left(\frac{d\psi}{dt}\right)_{kp}. \qquad (1)$$

On the other hand, differentiating the equation for the optimization characteristic $\psi = -s\mu^2$, we obtain

$$\frac{d\psi}{dt} = -2s\mu\frac{d\mu}{dt}. \qquad (2)$$

In view of the linear nature of the variation in the manipulated variable (Fig. 239a), we find $d\mu/dt$ as the tangent of the slope angle of the time response $\mu(t)$:

$$\frac{d\mu}{dt} = \frac{2\sqrt{\frac{A}{s}}}{\tau}. \qquad (3)$$

Substituting (3) into (2) and (1) with allowance for the fact that at the reversal points, $-s\mu = -\frac{A}{\sqrt{\frac{A}{s}}}$, we find

$$U_{kp} = -\frac{4aA}{\tau}. \qquad (4)$$

Averaging ψ over the period τ, we have

$$\psi_{cp} = \frac{1}{\tau}\int_0^\tau \psi\, dt =$$

$$= \frac{1}{\tau}\int_0^\tau -s\left(-\sqrt{\frac{A}{s}} + \frac{2\sqrt{\frac{A}{s}}}{\tau}t\right)^2 dt = -\frac{A}{3}. \qquad (5)$$

The search error is

$$D = |\psi_{max} - \psi_{cp}| = \frac{A}{3}, \qquad (6)$$

since $\psi_{max} = 0$.

329. The equations for the processes occurring in the elements of the optimizing system whose block diagram is shown in Fig. 240 are given in the form:

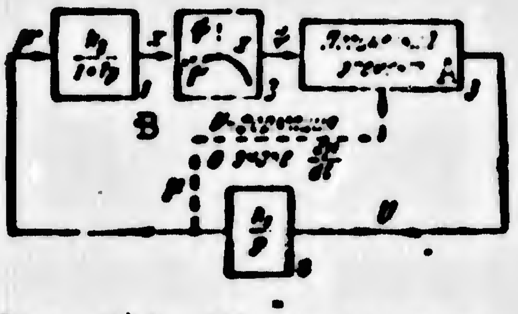


Fig. 240. Block diagram of optimizing system. 1) Object; 2) optimization characteristic; 3) logic element; 4) actuating element. A) logic element; B) information on sign of dμ/dt.

equation for the process in the actuating element

$$\frac{d\mu}{dt}=k_1U,$$

equation for the process in the linear section of the controlled object

$$T\frac{dx}{dt}+x=k_2\mu,$$

Equation of the object optimization characteristic

$$\psi=-sx^2,$$

Logic-element equation

$$U=\frac{d\psi}{dt}\,\mathrm{sign}\,\frac{d\mu}{dt}.$$

Here $\underline{x}$ is the angular velocity [1/sec], U is the voltage [$\underline{v}$], ψ is the optimization index [v·sec], μ is the displacement of the final control element [mm], k_1 is the transfer constant of the actuating element [mm/v·sec], k_2 is the transfer constant of the controlled object [1/ /sec·mm], $\underline{s}$ is the slope of the optimization characteristic [v·sec^3], T is the time constant of the controlled object [sec].

Determine the system transient response when the system has first been shifted away from optimum to the point M and has the parameters $T = 0.5$ sec, $s = 2\text{v}\cdot\text{sec}^3$, $k_1k_2 = 10\ 1/\text{v}\cdot\text{sec}^2$, $x_0 = -1$ 1/sec, $(dx/dt)_0 = 0$.

Solution. After eliminating the intervening variables we obtain

$$T\frac{d^2x}{dt^2}+\frac{dx}{dt}+2k_1k_2sx=0\ \left(\text{it is assumed } \frac{d\mu}{dt}>0\right).$$

Substituting in the numerical values, we write

$$0.5\frac{d^2x}{dt^2}+\frac{dx}{dt}+40x=0.$$

The roots of the characteristic equation will be

$$p_{1,2}=-1\pm j\sqrt{79}\ [1/\text{sec}].$$

They correspond to the general solution

$$x=C_1e^{(-1+j8.9)t}+C_2e^{(-1-j8.9)t}.$$

Using the initial conditions, we find

$$\begin{aligned}&1)\ C_1+C_2=-1,\\&2)\ C_1-C_2=0.\end{aligned}$$

From which we obtain

$$C_1=C_2=-\frac{1}{2}.$$

As a consequence,

$$x=-\frac{1}{2}e^{(-1+j8.9)t}-\frac{1}{2}e^{(-1-j8.9)t}.$$

Finally,

$$x=-e^{-t}\cos 8.9t.$$

It should be kept in mind that for the case given, we are solving the problem of finding the transient response when the system is displaced for a time at which it is on one of the sections of the optimization characteristic.

330. For the optimizing system with the block diagram shown in Fig. 241 and the logic circuit of Fig. 242, the equations for the processes occurring in the elements are given in the following forms:

the equation for the process in the actuating element

$$\frac{d\varphi}{dt}=k_1U,$$

the equation for the process in the linear portion of the controlled object

$$T\frac{dx}{dt}+x=k\alpha,$$

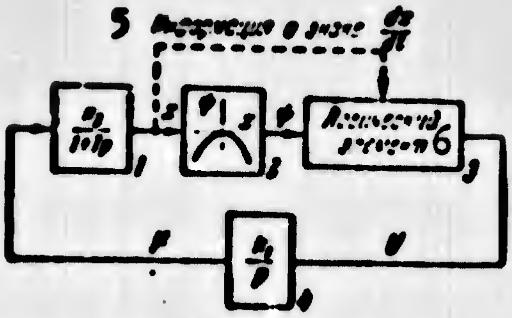


Fig. 241. Block diagram of optimum system. 1) Object; 2) optimization characteristic; 3) logic element; 4) actuating element; 5) information on sign of dx/dt; 6) logic element.

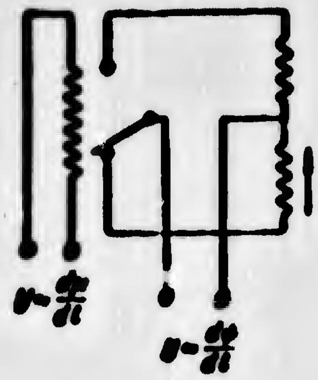


Fig. 242. Basic circuit of logic element.

The equation for the nonlinear (optimization) characteristic of the object

$$\varphi = -ax^2,$$

The logic-element equation

$$U = \frac{d\varphi}{dt}\,\mathrm{sign}\,\frac{dx}{dt}.$$

The quantities appearing and their dimensions are the same as in the preceding problem. Determine the phase path for the manipulated variable of the optimization system.

Solution. After eliminating the intervening variables, we obtain

$$T\frac{d^2x}{dt^2} + \frac{dx}{dt} + 2k_1k_2ax\frac{dx}{dt}\,\mathrm{sign}\,\frac{dx}{dt} = 0$$

We let dx/dt = v. Then

$$T\frac{dv}{dt} + v \pm 2k_1k_2axv = 0$$

or, after integrating this equation

$$v = v_0 - \frac{1}{T}(x - x_0) \mp \frac{k_1k_2a}{T}(x^2 - x_0^2).$$

In the expression for v, the upper sign corresponds to the case in which $v > 0$, and the lower sign to the case in which $v < 0$. x_0, v_0 are the initial values of the variable $\underline{x}$ and its derivative $\underline{v}$. The phase path of the system is shown in Fig. 243.

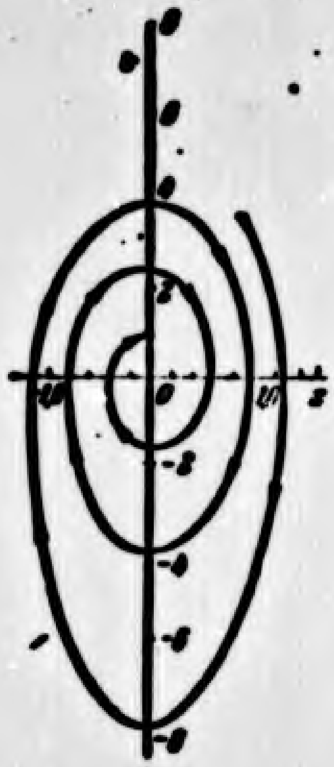


Fig. 243. Phase path for optimum system.

§40. SELECTING PARAMETERS FOR ELEMENTS OF ADAPTIVE SYSTEMS

331.* The block diagram of an optimizing system in shown in Fig. 244a. The controlled object 1 possesses an impulse response $w(t) = 100e^{-0.1t}$ [mm/sec], while its optimum properties 2 are described by the analytic formula $\psi = -2x^2$, where ψ is the optimization index and $\underline{x}$ the intervening output variable of the object.

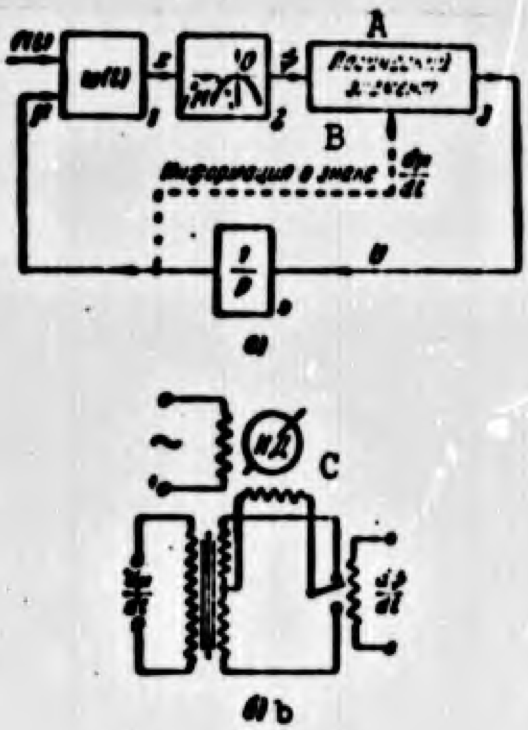


Fig. 244. a) Block diagram of optimum system; b) basic circuit of logic element. 1) Controlled object; 2) optimization characteristic; 3) logic element; 4) actuating element. A) logic element; B) information on sign of dμ/dt; C) actuating motor.

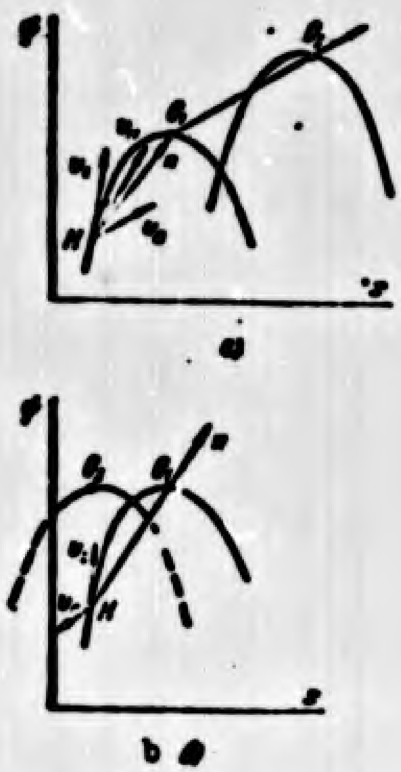


Fig. 245. Vector diagrams for: a) Stable optimization system: b) unstable optimization system.

An external disturbance written as

$$f(t)=2t,$$

acts on the object; as a result, the optimization characteristic drifts with a certain constant rate v_p (Fig. 245). Considering the displacement of point M on the optimization characteristic owing to the action of f(t) as a translational motion [38], and its motion along the optimization characteristic due to the effect of the actuating element as a relative motion, determine the actuating-element speed $dx/dt = v_{isp.el}$ required for the system to move stably toward the vertex of the optimization characteristic, i.e., for $v_M n > 0$, where v_M is the absolute speed of point M and n is the vector MO (Fig. 245).

Solution. We record the arbitrary system state. If in this case $v_p n > 0$ and $v_p v_\tau > 0$ simultaneously, then $v_M n > 0$ for any value of v_p and v_τ. Where $v_p n < 0$ we shall have $v_\tau v_p < 0$ and for stability it is necessary that $|v_\tau| > |v_p|$, since if this were not the case the condition for stable motion would not be satisfied (Fig. 245b). Let s be the length of the arc. Then

$$|v_\tau|=\left|\frac{ds}{dt}\right|=\frac{\left|\sqrt{(d\psi)^2+(dx)^2}\right|}{dt}=v_{isp.el}\sqrt{1+\left(\frac{d\psi}{dx}\right)^2},$$

where

$$v_{isp.el}=\frac{dx}{dt}.$$

We take the rate of translational motion as $v_p = d\psi/dt$. Since

$$\psi(t)=\int_0^\infty w(\tau)f(t-\tau)\,d\tau,$$

where $w(\tau)$ is the impulse response, then

$$\frac{d\psi}{dt}=\int_0^\infty w(\tau)\frac{df(t-\tau)}{dt}\,d\tau.$$

From this we obtain for $|v_\tau| > |v_p|$

$$v_{\text{вх.ос}}\sqrt{1+\left(\frac{dy}{dx}\right)^2} > \int_0^\infty w(\tau)\frac{df(t-\tau)}{dt}\,d\tau.$$

Substituting the initial values into the inequality, we find

$$v_{\text{вх.ос}}\sqrt{1+(4x)^2} > \int_0^\infty 100e^{-0.1\tau}\frac{d[2(t-\tau)]}{dt}\,d\tau$$

or

$$v_{\text{вх.ос}} > \frac{2000}{\sqrt{1+(4x)^2}}\ \frac{\text{mm}}{\text{sec}}$$

332. As one of the principles for adaptation, we may take self-adjustment on the basis of indirect indicators for dynamic properties.

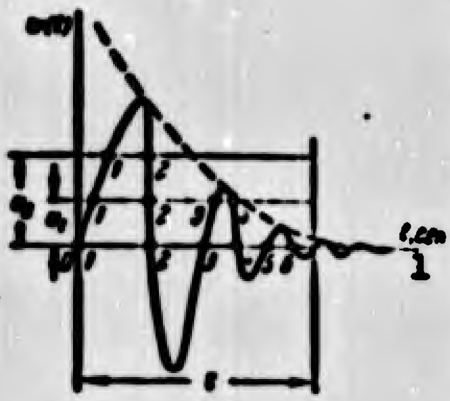


Fig. 246. Principle of indirect self-adjustment based on the number of times the impulse response passes through zero. 1) sec.

Thus, for example, if we represent the impulse response of a closed-loop system (Fig. 246), then by recording the number of times it passes through zero within a given time interval τ, we can obtain information as to the magnitude ratio of the characteristic curve. If we also introduce a shift in the axis of abscissas (for example, by supplying a constant displacement u_1, u_2, etc.), then by varying the number

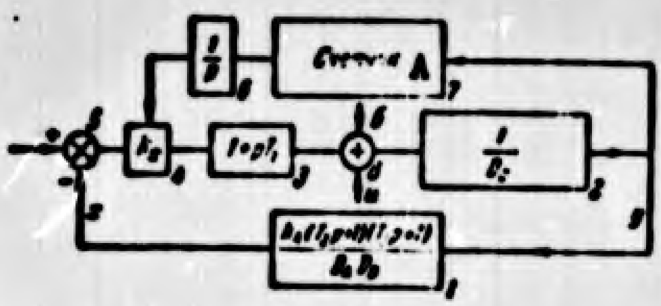


Fig. 247. Block diagram of adaptive system with register for counting the number of changes in sign of the impulse function. 1) Controlled object; 2) actuating element; 3) compensating element; 4) variable-coefficient element; 5) comparison element; 6) actuating element of self-adjustment circuit; 7) pulse counter; 8) adder. A) Register.

of times the impulse response passes through a given level it is also possible to determine the intensity of system process damping.

Using the principle discussed, consider an adaptive system with the block diagram of Fig. 247. We know that the optimum degree of damping $\zeta_{opt} = 0.35$ corresponds to three changes of sign.

The operators D_A, D_B, and D_C indicated on the block diagram have the following form:

$$D_A = T_A^2 p^2 + 2\zeta_A T_A p + 1,$$
$$D_B = T_B^2 p^2 + 2\zeta_B T_B p + 1,$$
$$D_C = T_C^2 p^2 + 2\zeta_C T_C p + 1.$$

We are to determine the range of variation in k_x and the number of digits in the register if for self-adjustment we have the condition that the system natural frequency $\omega_n = 1/T_{opt}$ remain unchanged. We are given the following system parameters: $T_{opt} = 0.0210$ sec, $T_1 = 0.033$ sec, $T_2 = 0.045$ sec, $T_3 = 0.2$ sec, $T_C = 0.0139$ sec, $\zeta_C = 0.7$, $T_A = 0.033$ sec, $\zeta_A = 0.72$, $T_B = 0.083$ sec, $\zeta_B = 0.72$, $k_A = 2 \cdot 10^3$.

<u>Solution</u>. We use the root-locus method [36] and construct the root locus, which is the locus of system transfer-function poles when the coefficient k_x varies (Fig. 248); p_A, p_B, p_C are the poles of $W(p)$. We see from an analysis of the locus that the system reaction is determined chiefly by the poles of the actuating element. By hypothesis, the frequency should remain unchanged, so that we determine the optimum damping ζ_{opt} and to do this we find the frequency.

We recall that the required frequency is found [24] as the magnitude of the root-locus vector drawn to a point on the optimum curve, while the damping is found as the cosine of the angle of this vector with respect to the negative real semiaxis. Using a radius of $1/T_{opt}$, we draw an arc until it intersects the root locus. This determines the range of variation in k_x and $\zeta_C = 0.22$-0.557.

Since the number of changes in sign φ of the impulse response and the damping factor ζ_C are inversely proportional, $\varphi = a/\zeta_C$, after substitution of ζ_C, we find the true range of variation in φ:

$$\varphi_{max} = \frac{a}{\zeta_{min}} = 3, \text{ i.e. } a = 1{,}03,$$

$$\varphi_1 = \frac{1{,}03}{0{,}557} = 1{,}8 \approx 2, \quad \varphi_2 = \frac{1{,}03}{0{,}22} = 4{,}8 \approx 5.$$

Thus, a three-digit binary counter is sufficient for determining the number of transitions. The system open-loop transfer function is

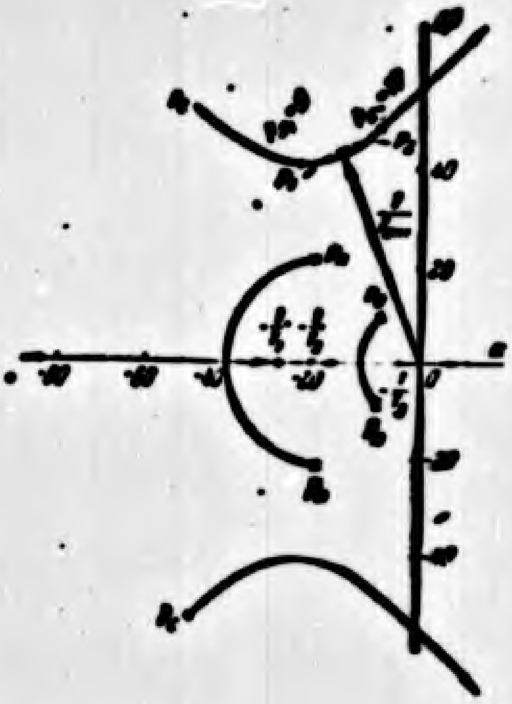


Fig. 248. System root locus.

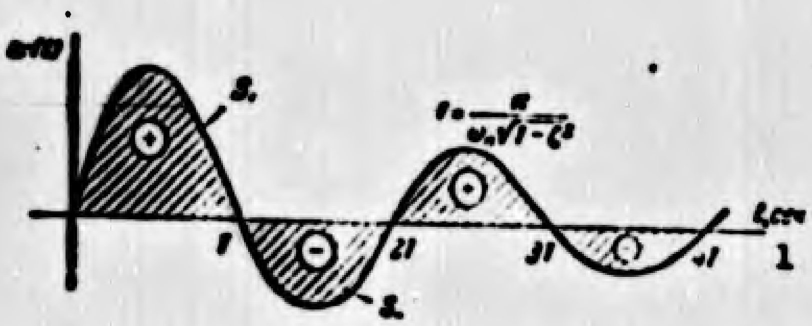


Fig. 249. Area criterion for self-adjustment. 1) sec.

$$W(p) = \frac{k_s k_A (1 + T_1 p)(1 + T_2 p)(1 + T_3 p)}{D_A D_B D_C}.$$

From the conditions under which the locus is drawn [36], the value of the transfer function at point p_1 of the locus is

$$W(p) = \frac{kABC}{DEFGHJ} \quad (k = k_s \cdot k_A).$$

Here A, B, C are the length of the vectors drawn from the zeros to point p_1, while D, E, F, G, H, J are the lengths of the vectors drawn from the poles to the same point. Taking into account the scale of the graph we find that $p_1 = p_2 = 48$; after this we calculate the transfer-function numerator:

$$(1+0{,}033\cdot 48)\cdot(1+0{,}045\cdot 48)\cdot(1+0{,}2\cdot 48)=86{,}6$$

and denominator:

$$D_A(p)=7{,}41,\quad D_B(p)=22{,}76,\quad D_C(p)=2{,}356.$$

As a consequence,

$$k=\frac{86{,}6}{7{,}41\cdot 22{,}76\cdot 2{,}356}\,\frac{27\cdot 96\cdot 20\cdot 62\cdot 36\cdot 53}{40\cdot 40\cdot 46}=1{,}79\cdot 10^{4}.$$

In like manner, for the second point ($p_2 = p_1$ by the conditions of the construction):

$$k=\frac{86{,}6}{7{,}41\cdot 22{,}76\cdot 2{,}356}\,\frac{41\cdot 108\cdot 28\cdot 70\cdot 39\cdot 57}{52\cdot 50\cdot 47}=3{,}42\cdot 10^{4}.$$

Since $k_A = 2\cdot 10^3$, $k_{x_1} = 8.95$, $k_{x_2} = 17.05$.

333.* As a self-adjustment criterion, we may use the intensity with which the impulse response is damped; this is evaluated by means of the function [11]

$$\varphi=S_{+}-\alpha S_{-},$$

where S_+ is the positive area enclosed by the pulse response, while S_- is the negative area (Fig. 249), α is the coefficient relating the areas. The coefficient α is so chosen that for the desired damping ζ, the value of φ will be zero. Where we select a system having an impulse response approximated by the analytic relationship

$$w(t)=\frac{\omega_n}{\sqrt{1-\zeta^2}}\,e^{-\zeta\omega_n t}\sin\left(\omega_n t\sqrt{1-\zeta^2}\right),$$

where ζ is the damping factor and ω_n the natural frequency, the relationship for φ is given by the formula

$$\varphi=\frac{1-\alpha e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}}{1-e^{-2\zeta\frac{\pi}{\sqrt{1-\zeta^2}}}}.$$

Find the value of α such that self-adjustment is accomplished when $\zeta = 0.5$.

<u>Answer</u>: $\alpha = 6.06$.

334. Frequency filters can also be used for self-adjustment on the basis of an indirect estimate of transient oscillation and transient damping intensity [14]. Thus, for example, if the over-all gain of a system (Fig. 250) kk_x varies owing to k, then by acting on k_x we can restore gain, thus restoring the dynamic properties of the system.

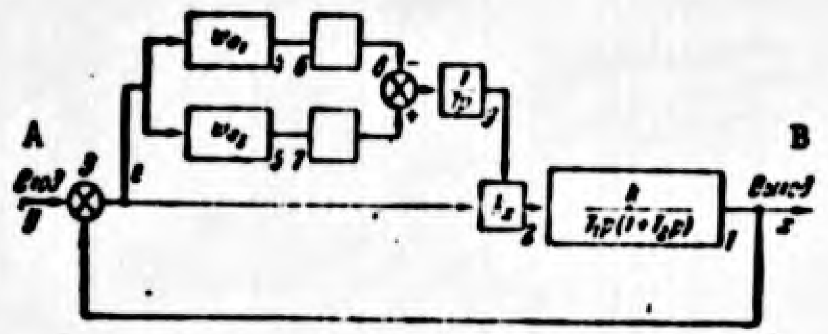


Fig. 250. Structural diagram of self-adjusting system with frequency filters. 1) Object; 2) controller; 3) companion element; 4) low-pass filter; 5) high-pass filter; 6,7) rectifiers; 8) companion element; 9) actuating element. A) Input; B) output.

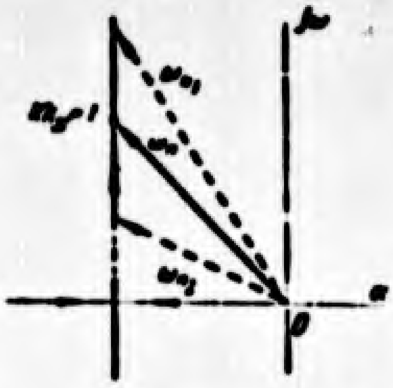


Fig. 251. Root-focus diagram of servosystem.

For many cases involving elementary systems where the controlled object has the transfer function

$$W(p) = \frac{k}{T_1 p(1 + T_2 p)},$$

the root locus has the form shown in Fig. 251. We shall henceforth assume that the optimum process with respect to natural frequency (ω_n) and damping factor (ζ_n) occurs at $kk_x = 1$ and $\omega_n = 1$ (Fig. 251). Then when kk_x changes so as to increase ($kk_x > 1$) we observe an increase in the natural frequency, while when kk_x decreases ($kk_x < 1$), conversely, there will be a decrease in the frequency. As a consequence, frequency variation may be used for self-adjustment. Here, however, it is necessary to ensure that the system frequency responses have a predetermined shape. Frequency filters are used to solve this problem.

Determine the frequency responses of filters such that a system with a given structure will perform self-adjustment over the range (0.2-9)k* where k* is the optimum value, where the input signal changes in steps by an amount r equal to 10 if $T_1 = 0.2$ sec, $T_2 = 5$ sec. In the calculations we assume that the minimum signal amplitude at the input of the actuating element A_1 is 3v.

<u>Solution</u>. From the given block diagram, we find the log-frequency responses for the system. The error transfer function with respect to the input signal equals

$$\frac{\varepsilon(p)}{y(p)}=\frac{T_1p(1+T_2p)}{T_1T_2p^2+T_1p+kk_2}=\frac{\frac{T_1}{kk_2}p(1+T_2p)}{\frac{T_1T_2}{kk_2}p^2+\frac{T_1}{kk_2}p+1}.$$

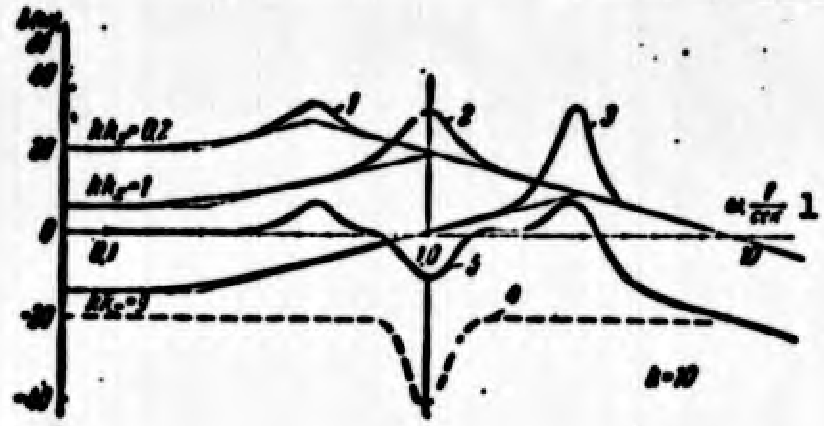


Fig. 252. Logarithmic amplitude-frequency responses of system for various gain values. 1) 1/sec.

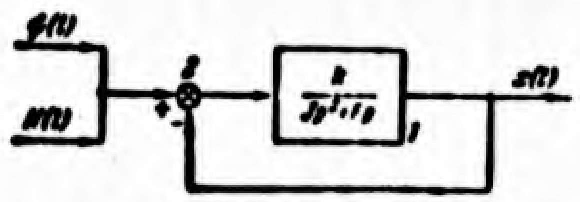


Fig. 253. Block diagram of servosystem. 1) Servo-system transfer function; 2) comparison element.

Since by hypothesis $y(p) = r/p$, we obtain finally

$$\varepsilon(p)=\frac{N(1+T_2p)}{1+2\xi Tp+T^2p^2},$$

where

$$N=\frac{rT_1}{kk_2},\quad \xi=\frac{1}{2}\sqrt{\frac{T_1}{T_2kk_2}},\quad T=\sqrt{\frac{T_1T_2}{kk_2}}.$$

When the numerical values are substituted, we find the logarithmic amplitude-frequency curves (Fig. 252) for three values of kk_x: $kk_x = 0.2$, $kk_x = 1$, and $kk_x = 9$. By hypothesis, for self-adjustment $kk_x = 1$ and the frequency $\omega_n = 1$ 1/sec.

We select the following filters:

$$w_{f_1}(p) = \frac{k_f T_f p}{T_{f_1} p + 1} \text{ and } w_{f_2}(p) = \frac{k_f}{T_{f_2} p + 1}.$$

Then

$$w_{f\,eq}(p) = \frac{k_f (T_f^2 p^2 + 2\zeta_{f_1} T_f p + 1)}{T_f^2 p^2 + 2\zeta_{f_2} T_f p + 1},$$

where

$$\zeta_{f_1} = \sqrt{\frac{T_{f_1}}{T_{f_2}}},\quad T_f^2 = T_{f_1} T_{f_2},\quad \zeta_{f_2} = \frac{1 + \zeta_{f_1}}{2\sqrt{\zeta_{f_1}}}.$$

The logarithmic amplitude characteristic for the self-adjusting unit has the form of curve 4, indicated in the figure by the dashed line. The filter constants should be so selected that they transform the input error signal in accordance with form 5 where at a frequency $\omega_n = 1$ 1/sec and in a small neighborhood of this frequency the output signal of the filter will not be reproduced by the actuating element while at higher and lower frequencies; on the other hand, the signal will appear with amplification sufficient to drive the actuating element of the self-adjusting unit.

The required values of ζ_f, ζ_{f_2}, and k_f are determined graphically, since the corrections to the damping factors ζ are given graphically (see Appendix 9). The system performance condition is:

$$20 \lg A \geq 20 \lg A_1 \text{ for } \omega = \omega_n,$$

where

$$20 \lg A = 20 \lg N + \delta_1(\zeta) - 20 \lg k_f - \delta_2(\zeta_{f_1}),$$

were $\delta_1(\zeta)$, $\delta_2(\zeta_{f_1})$ are, respectively, the corrections for the damping factors.

Since the given value $A_1 = 3$,

$$N = \frac{vT_0}{\Delta a} = \frac{10 \cdot 0.2}{1} = 2, \quad \zeta = \frac{1}{2}\sqrt{\frac{0.2}{5 \cdot 1}} = 0.1$$

and, as a consequence, $\delta_1(\zeta) = 14$ db; then from the equations

$$\zeta_{\phi_1} = \sqrt{\frac{T_{\phi_1}}{T_{\phi_2}}}, \quad \omega_c^2 = 1 = \frac{1}{T_{\phi_1} T_{\phi_2}}$$

and

$$20 \lg 2 + 14 - 20 \lg k_\phi - \delta_1(\zeta_{\phi_1}) = 20 \lg 3$$

we find for $k_f = 0.1$, $T_{f_1} = 0.2$ sec and $T_{f_2} = 5$ sec.

335. Figure 253 shows the block diagram of a servosystem; at its input there is a regular useful signal $\psi(t) = vt$ and random white noise $N(t)$ with a cross-correlation function of the form $R_N(\tau) = a^2\delta(\tau)$, where $a^2 = 600$ cm^2/sec^2. The system damping factor is $\zeta = 0.5$. The moment of inertia $J = 0.02$ g·cm·sec^2. System operation is ensured when the input velocity $v_1 \geq 600$ cm/sec, while the minimum permissible signal-to-noise ratio for a velocity gauge should be $k_1 = 0.5$ [14]. Owing to the fact that the velocity of the input signal varies, in order to obtain the minimum over-all square error it is necessary to adjust the dynamic parameters of the servosystem – the gain $\underline{k}$ and coefficient of viscous friction F. It is necessary to determine the time constant of the velocity gauge and the functional coupling relationships for adjustment of the coefficients $\underline{k}$ and F from the condition for the minimum mean-square error.

<u>Solution</u>. Taking into account the fact that the signal and noise are not tolerated, determine the mean-square total error [2]:

$$\overline{\varepsilon^2} = \varepsilon_\psi^2 + \overline{\varepsilon_N^2} = \frac{4\zeta^2}{\omega_0^2}v^2 + \frac{\omega_0}{4\zeta}a^2 = \frac{v^2}{\omega_0^2} + 300\omega_0.$$

Here ε_ψ^2 is the square of the following error, $\overline{\varepsilon_N^2}$ is the mean-square error due to noise, $\omega_0 = \sqrt{k/J}$ is the frequency of undamped system oscillations, ζ is the damping factor. From the condition $\partial\overline{\varepsilon^2}/d\omega_0 = 0$, determine the frequency ω_0^* for which the mean-square error will be a minimum:

$$\omega_0^* = 0{,}188\, v^{2/3}.$$

As a consequence, where the velocity v of the useful signal changes, the optimum frequency ω_0^* will also vary. The optimum coupling coefficients, which show how the gain and viscous-friction coefficient should change if we are to obtain $\bar{\varepsilon}^2_{min}$, equal, respectively:

$$k_{opt} = \omega_0^2 J = 0{,}00071\, v^{4/3},$$

$$F_{opt} = 2\zeta\omega_0 J = 0{,}00376\, v^{2/3}.$$

The time constant of the velocity gauge is determined for $v \geq 600$ m/sec from the formula [14]

$$T = \left[\frac{90 k_0^2 a^2}{v^2}\right]^{1/5}.$$

When v = 600 m/sec, T = 0.342 sec.

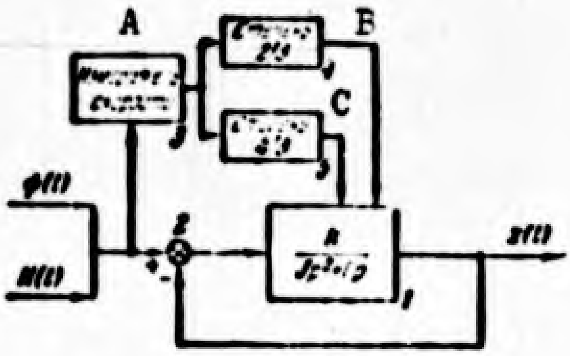


Fig. 254. Block diagram of adaptive servosystem. 1) Controlled object; 2) comparison element; 3) velocity gauge; 4,5) filter. A) velocity gauge; B) 2/3 power; C) 4/3 power.

The block diagram of an adaptive servosystem and the dependence on the input signal (variable speed v) are shown in Fig. 254.

336. To the input of the servosystem whose block diagram is shown in Fig. 255 we apply a stationary random signal $\psi(t)$ with spectral density $S_\psi(\omega) = k^2/\omega^4$ and stationary random white noise N(t) with spectral density $S_N(\omega) = a^2$. We assume that a and k can vary slowly. As a result, the servosystem, designed by the Weiner-Kolmogorov method for specific relationships of a* and k* from the condition for minimum mean-square error, requires adjustment

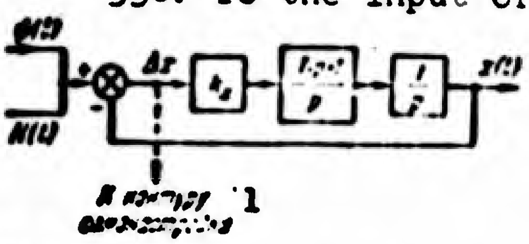


Fig. 255. Structural diagram of servosystem. 1) To self-adaptation circuit.

of the gain k_x. Show that it is possible to use frequency filters for self-adjustment and give the block diagram of the system.

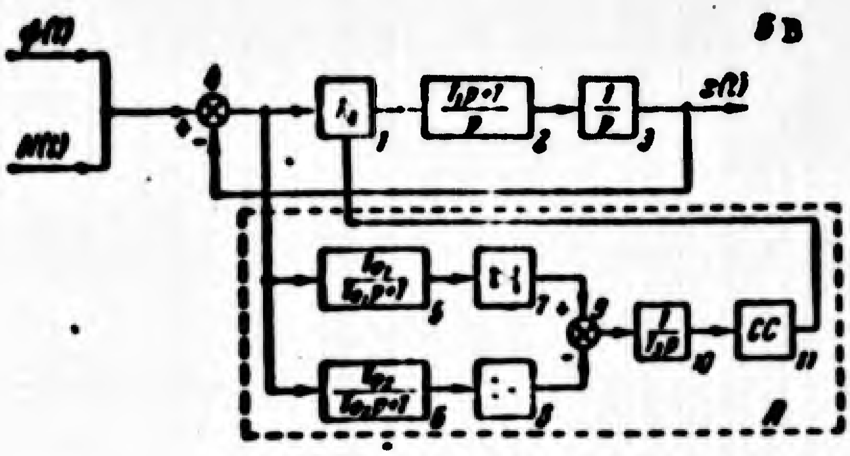


Fig. 256. Block diagram of adaptive servosystem. A) Self-adjustment unit; 3) main servosystem. 1) Variable-gain unit; 2) controlled object; 3) drive; 4) comparison element; 5,6) filter; 7,8) rectifiers; 9) comparison element; 10) drive; 11) servosystem.

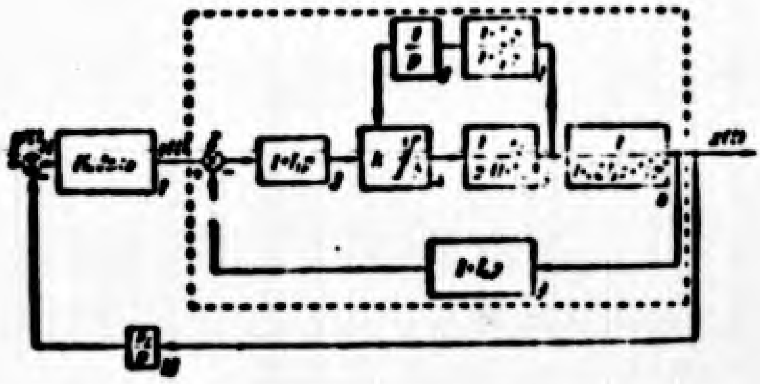


Fig. 257. Diagram for invariant adaptive system. 1) Model; 2) comparison element; 3) filter; 4) nonlinear element; 5,7,8) actuating elements of self-adjustment unit; 6) object; 9) compensating element; 10) integrating element; 11) model.

Solution. According to the Weiner-Kolmogorov method, for given S_ψ and S_N, the transfer function of the optimum servosystem ensuring minimum mean-square error will equal [14]

$$\Phi(p)=\frac{a(Tp+1)}{p^2+aTp+a},$$

where

$$a=\frac{k^2}{\sigma^2},\quad T=\sqrt{\frac{2}{a}}.$$

The value of the minimum mean-square error is computed from the formula [14]

$$a = 4\sqrt{k^{*}a^{*2}}.$$

For self-adjustment we use the resultant error Δx (Fig. 255), whose spectral density is calculated from the formula

$$S_{\Delta x}(\omega) = |1 - \Phi(j\omega)|^2 [S_v(\omega) + S_x(\omega)] = \frac{a^2\left(\omega^2 + \frac{k^2}{a^2}\right)}{\omega^2 + a^2}.$$

For optimum choice of k* and a* it is necessary that $S^{*}_{\Delta x}(\omega) = a^2$. As a consequence, the spectral density of the resultant system error in this case will be frequency-independent. When k and a vary, however, as indicated in the statement of the problem, $S_{\Delta x}(\omega)$ will vary monotonically as a function of frequency. This permits us to use frequency filters with various frequency responses in the self-adjustment unit. In particular, the frequency filters may be delay elements with transfer functions

$$w_1(p) = \frac{T_{0_1}}{T_{0_1}p + 1} \text{ and } w_2(p) = \frac{T_{0_2}}{T_{0_2}p + 1}.$$

Thus, for this case we may also use the adaptive system having the block diagram shown in Fig. 256.

337. Figure 257 shows the block diagram of an adaptive system. The system, including the self-adjustment unit, the controlled object, and the controller is enclosed by the dashed line. It is a so-called invariant element. The self-adjustment problem is to maintain the transfer function of the invariant element equal to unity. If the transfer function equals unity, then when combined with the model of a process that is optimum in terms of response time and overshoot, the system as a whole will provide ideal dynamic characteristics over a wide range. The self-adjustment system uses self-oscillation in an internal nonlinear circuit. The amplitude and frequency of these oscillations are such that they cannot be detected at the system output; their magnitude is sufficient, however, to permit detection of a change in controlled-pro-

cess parameters. A filter is needed in the self-adjustment unit to filter out the constant component. Here the role of the actuating element in the self-adjustment circuit is played by a high-frequency oscillator working into a relay element with the characteristic curves shown in the diagram. We are given: k_2 varies within the range 3 to 9, ζ_s = = 0.4, T_1 = 10 sec, T_3 = 0.33 sec, T_2 = 0.02 sec. We are to 1) calculate the parameters of the system simulation device, ensuring that ζ_n = = 0.7 and Ω_n = 3 rad/sec in the presence of a step disturbance; 2) select the range of the values for the nonlinear gain k at which system self-oscillation will appear with amplitude A, with self-adjustment taking place at A/b = 2; 3) determine the frequency of self-adjusting-unit self-oscillation.

Solution. We select a model transfer function of the form

$$w_m(p)=\frac{1}{1+T_m p}.$$

Then provided $|\Phi(j\omega)|\cong 1$ we shall have

$$\frac{X(p)}{Y(p)}=\frac{p}{T_m p^2+p+k_1}.$$

If Y(p) = r/p, where r is the step level, then

$$X(p)=\frac{r/k_1}{\frac{T_m}{k_1}p^2+\frac{1}{k_1}p+1}.$$

We let

$$\frac{r}{k_1}=k_0,\quad \frac{T_m}{k_1}=T_0^2,\quad \frac{1}{k_1}=2\zeta T_0.$$

Then

$$T_0=\sqrt{\frac{T_m}{k_1}},\quad \zeta=\frac{1}{2\sqrt{k_1 T_m}}$$

and, finally,

$$X(p)=\frac{k_0}{T_0^2 p^2+2\zeta_0 T_0 p+1}.$$

Since $T_k = 1/\omega_n$ and $1/2k_1T_k = \zeta_k$, then

$$T_k = 0{,}33 \text{ sec } k_1 = \frac{1}{2T_k\zeta_k} = 2{,}14 \left[\frac{1}{\text{sec}}\right],\ T_n = 0{,}238 \text{ sec.}$$

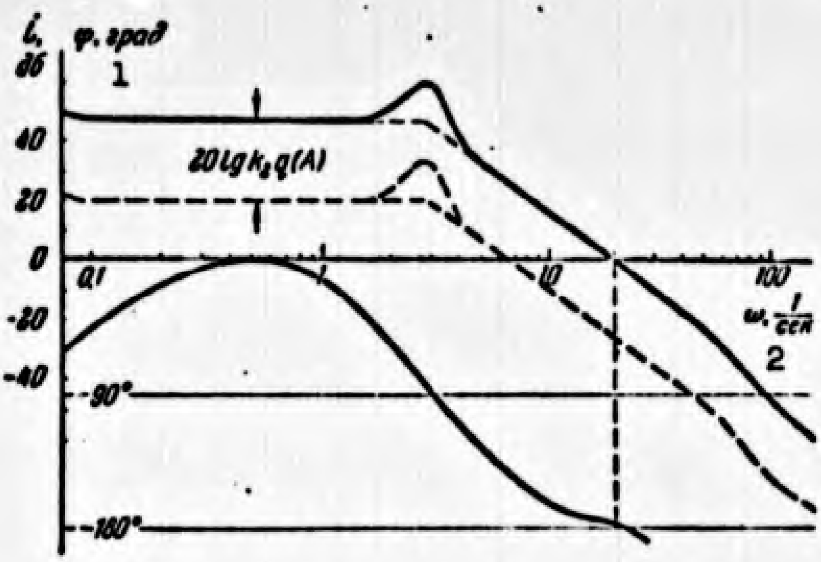


Fig. 258. Logarithmic characteristics for invariant system (the dashed line indicates characteristics with no allowance for nonlinear elements). 1) Degrees; 2) 1/sec.

In connection with the fact that the unit contains a nonlinear element, we use the describing function for the nonlinear element q(A) [20]. In this case

$$\Phi(j\omega) = \frac{\dfrac{(1+T_1\omega j)k_2q(A)}{j\omega(1+T_2j\omega)(1+2T_3\zeta\omega j-T_3^2\omega^2)}}{1+\dfrac{(1+T_4\omega j)(1+T_1\omega j)k_2q(A)}{j\omega(1+T_2\omega j)(1+2\zeta T_3\omega j-T_3^2\omega^2)}}.$$

If $|W(j\omega)| \ll 1$, then $|\Phi(j\omega)| = \sqrt{1+T_4^2\omega^2}$; thus $\Phi(j\omega) \approx 1$ if $T_4\omega_{kr} \ll 1$, where ω_{kr} is the frequency limit of the controlled-object passband ($\omega_{kr} \approx \omega_{sr}$). We set $T_4 = 0.01$ sec. We construct (Fig. 258) the logarithmic characteristics $W(j\omega)$ with no allowance for $k_2q(A)$ (in Fig. 258, this curve is shown by the dashed line). It is clear from Fig. 258 that $\omega_{sr} = 7$ rad/sec. As a consequence, the condition $T_4 \ll 1/\omega_{kr}$ is satisfied and $|\Phi(j\omega)| \approx 1$, i.e., the invariance condition is satisfied. The self-adjustment problem consists, however, not only in maintaining

$|\Phi(j\omega)| \approx 1$, but also in the fact that the unit contains high-frequency self oscillations which do not affect the system. The conditions for existence of these self oscillations are

$$|W(j\omega)k_2q(A)|=1, \quad \arg W(j\omega)k_2q(A)=\pi.$$

We thus select $k_2q(A)$ so that

$$20 \lg W(j\omega)+20 \lg k_2q(A)=0.$$

In the figure, the logarithmic amplitude characteristic is shown by the solid line. The oscillation frequency equals 22 rad/sec (three times the maximum system working frequency). We now have only to distribute the linear and nonlinear gain so that A/b will be obtained. For a nonlinearity of the given type we have (Appendix 20)

$$q(A)=\frac{2k}{\pi}\left(\arcsin\frac{b}{A}+\frac{b}{A}\sqrt{1-\frac{b^2}{A^2}}\right).$$

Setting A/b = 2 and 20 log $k_2q(A)$ = 30, we find

$$k_2q(A)=\frac{2k_2k}{\pi}(\arcsin 0{,}5+0{,}5\sqrt{0{,}75})=17{,}6.$$

As a consequence,

$$k_2k=5{,}7.$$

Since k_2 ranges from 3 to 9, then

$$k=1{,}9 \div 0{,}63.$$

Manuscript Page No.	[Transliterated Symbols]
323	КГ = KG = komandnyy generator = command generator
323	ШР = ShR = shagovoy raspredelitel' = step switch
323	ОР = OR = ob"yekt regulirovaniya = controlled system
323	ОВ = OV = obmotka vozbuzhdeniya = field winding
323	Д = D = dvigatel' = motor
323	Р = R = rele = relay

323 к = k

324 ср = sr = sredniy = average

326 кр = kr = kriticheskiy = critical

330 ИД = ID = ispolnitel'nyy dvigatel' = actuating motor

331 п = p = postoyannyy = constant

331 исп.эл = isp.el = ispolnitel'nyy element = actuating element

333 опт = opt = optimal'nyy = optimum

336 Ф(ф) = F(f) = fil'tr = filter

338 экв = ekv = ekvivalentnyy = equivalent

341 СС = SS = sledyashchaya sistema = servosystem

343 м = m = model' = model

344 ср = sr = srez = cutoff

BLANK PAGE

APPENDICES

1. Vyshnegradskiy diagram with lines of equal attenuation plotted in per cent per period

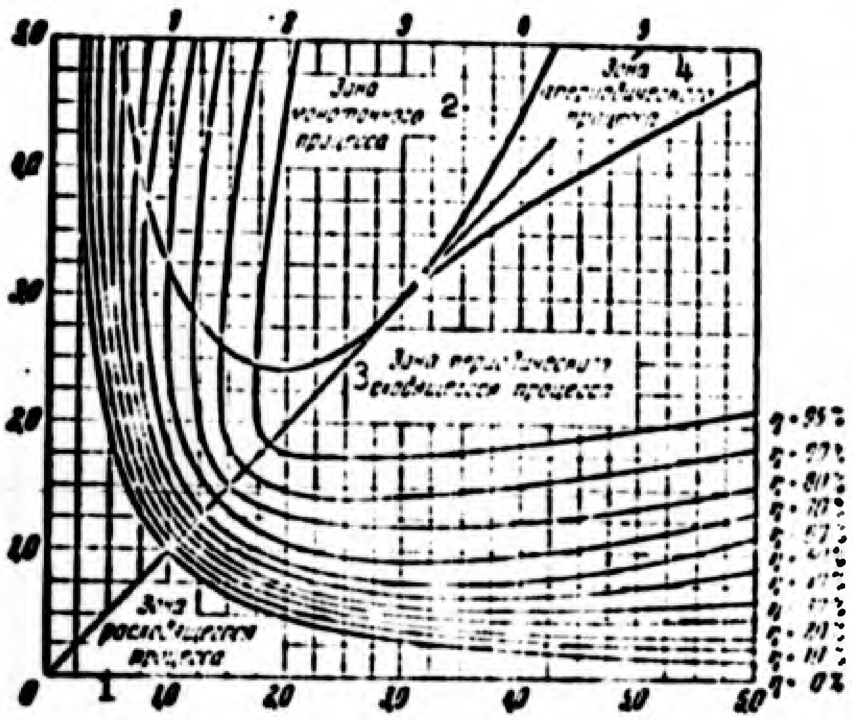


Fig. 259. 1) Divergent process zone; 2) monotonic process zone; 3) periodically-convergent process zone; 4) aperiodic process zone.

2. Vyshnegradskiy diagram with lines of equal stability

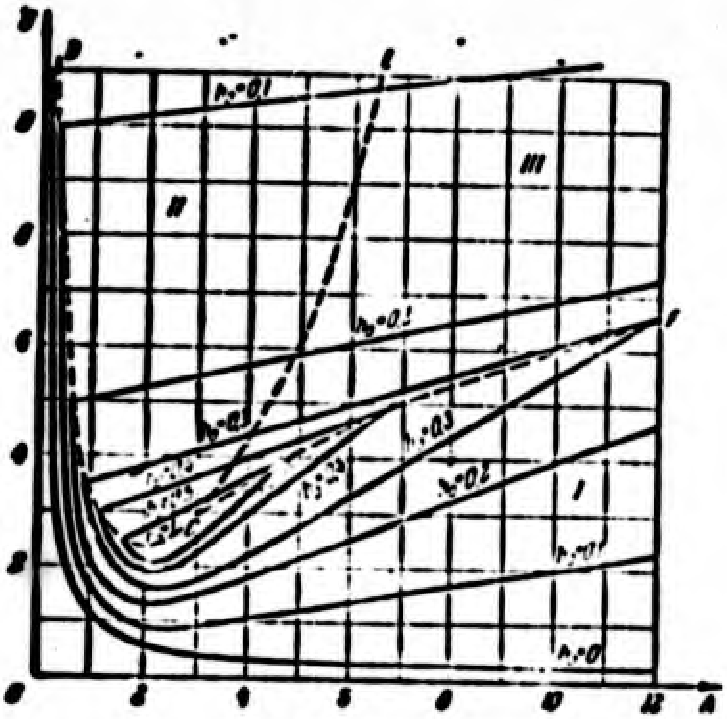


Fig. 260

3. Nomogram for constructing real frequency response of closed-loop system from open-loop gain-phase characteristic (real circle diagram)

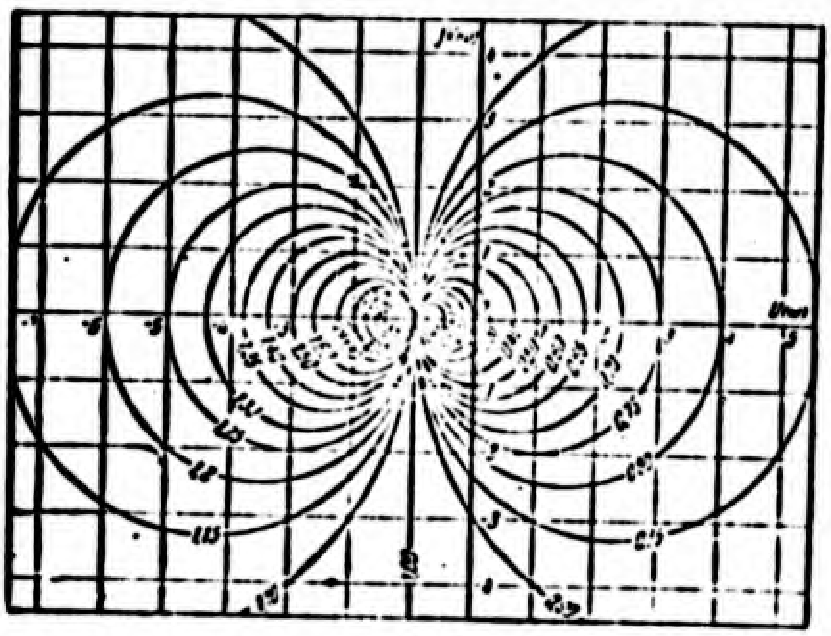


Fig. 261

4. Curves for determining transient time and overshoot from slope of trapezoidal real frequency curve

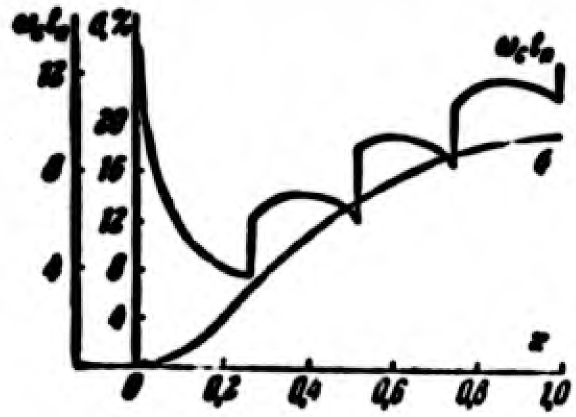


Fig. 262

5. Curves for determining transient time and overshoot for real frequency curve having a maximum

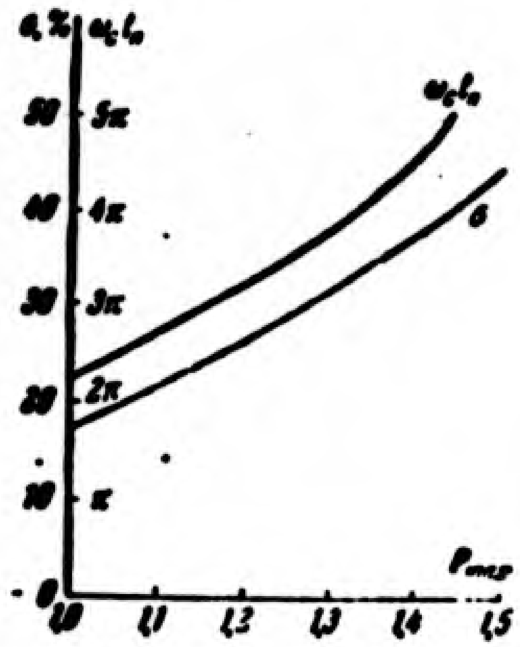


Fig. 263.

6. Nomogram for constructing real frequency response from given performance characteristics

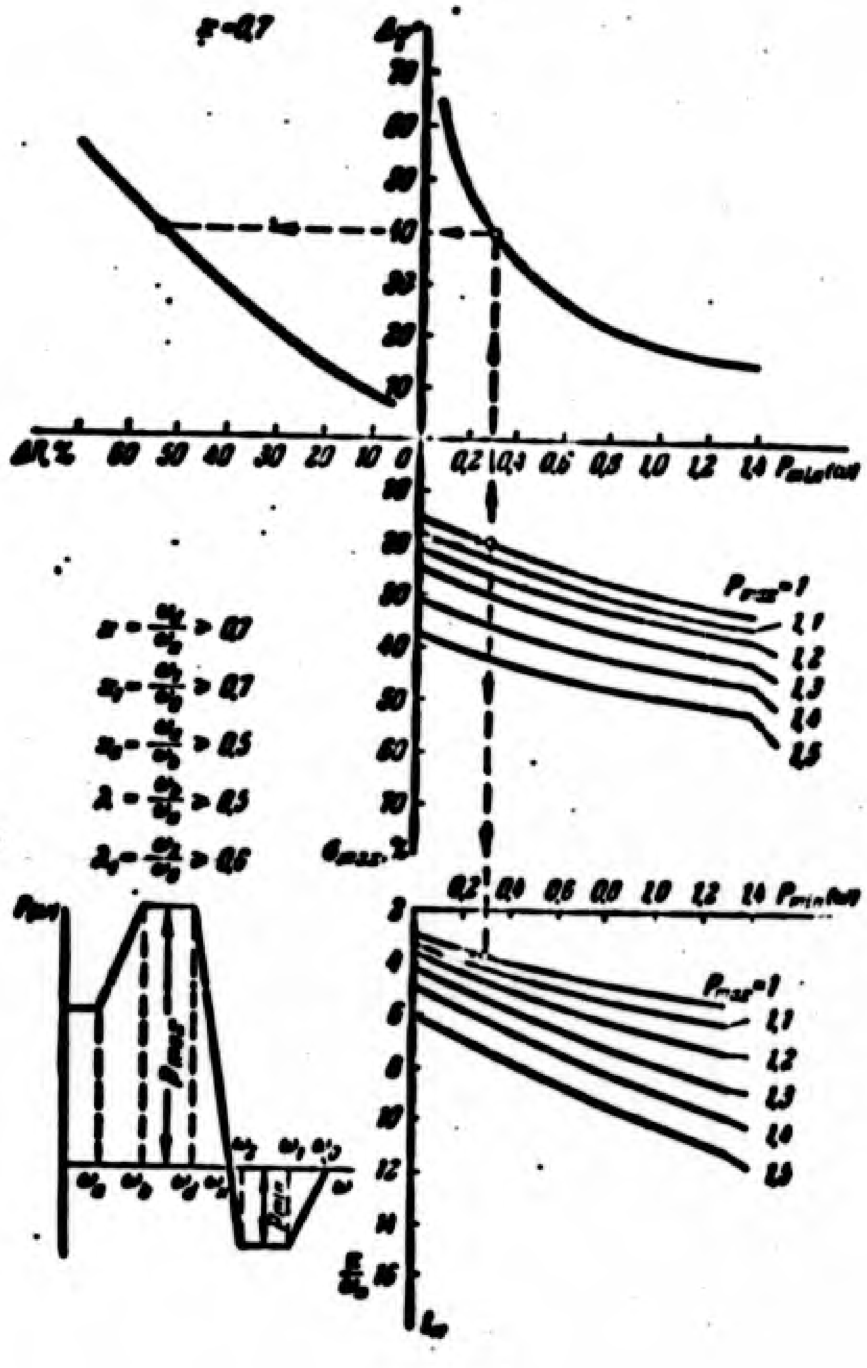


Fig. 264

7. Standardized logarithmic phase response of first-order aperiodic element

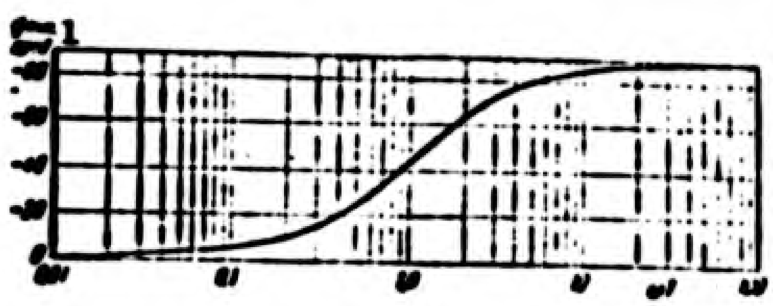


Fig. 265. 1) Degrees.

8. Standardized logarithmic amplitude and phase curves for oscillating element

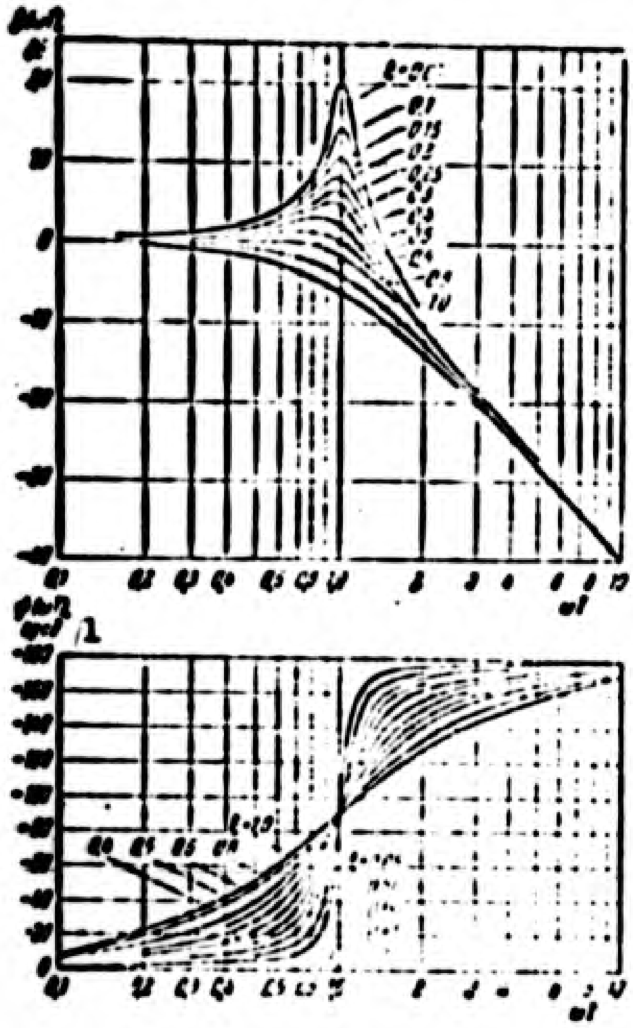


Fig. 266. 1) Degrees.

9. Deviation of asymptotic logarithmic amplitude characteristic for oscillating element from exact characteristic

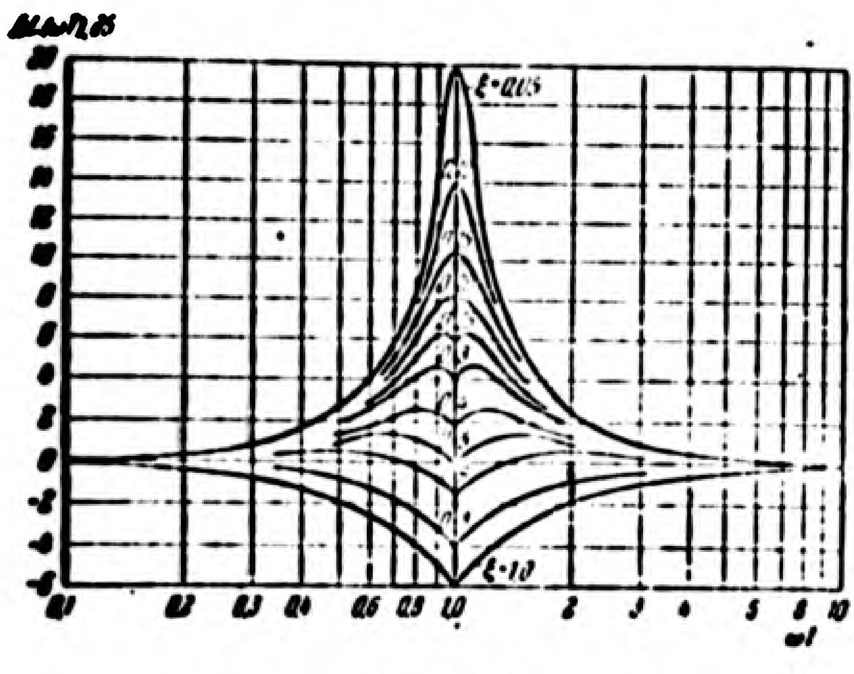


Fig. 267

10. Initial conditions in control system directly following application of unit step function

The system differential equation is

$$(a_0p^n + a_1p^{n-1} + \dots + a_n)x(t) = (b_0p^m + b_1p^{m-1} + \dots + b_m)f(t),$$

where x(t) is the output variable, f(t) is the disturbance variable.

$x_{-0}, x'_{-0}, \dots, x^{(x-1)}_{-0}$ are the initial conditions directly before application of the unit step function;

$x_{+0}, x'_{+0}, \dots, x^{(n-1)}_{+0}$ are the initial conditions directly following application of the unit step function.

$$x_{+0} = x_{-0},\ x'_{+0} = x'_{-0},\ \dots,\ x^{(n-m-1)}_{+0} = x^{(n-m-1)}_{-0},$$

$$x^{(n-m)}_{+0} = x^{(n-m)}_{-0} + \frac{b_0}{a_0}\cdot 1,$$

$$x^{(n-m+1)}_{+0} = x^{(n-m+1)}_{-0} + \frac{b_1}{a_0}\cdot 1 - \frac{a_1}{a_0}[x^{(n-m)}_{+0} - x^{(n-m)}_{-0}],$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$x^{(n-1)}_{+0} = x^{(n-1)}_{-0} + \frac{b_m}{a_0}\cdot 1 - \frac{a_{m-1}}{a_0}[x^{(n-m)}_{+0} - x^{(n-m)}_{-0}] - \dots$$

$$\dots - \frac{a_1}{a_0}[x^{(n-2)}_{+0} - x^{(n-2)}_{-0}].$$

11. Solutions for first-, second-, and third-order homogeneous differential equations

Порядок уравнения 1	2 Вещественные корни	3 Комплексные корни
1	$x = X_0 e^{-\alpha_1 t}$	—
2	$x = A_1 e^{-\alpha_1 t} + A_2 e^{-\alpha_2 t}$ $A_1 = \frac{\alpha_2 X_0 + V_0}{\alpha_2 - \alpha_1}$ $A_2 = \frac{\alpha_1 X_0 + V_0}{\alpha_1 - \alpha_2}$	$x = (B \cos \lambda t + C \sin \lambda t) e^{-\gamma t}$ $B = X_0$ $C = \frac{\gamma X_0 + V_0}{\lambda}$
3	$x = A_1 e^{-\alpha_1 t} + A_2 e^{-\alpha_2 t} + A_3 e^{-\alpha_3 t}$ $A_1 = \frac{\alpha_2 \alpha_3 X_0 + (\alpha_2 + \alpha_3) V_0 + W_0}{(\alpha_2 - \alpha_1)(\alpha_3 - \alpha_1)}$ $A_2 = \frac{\alpha_1 \alpha_3 X_0 + (\alpha_1 + \alpha_3) V_0 + W_0}{(\alpha_1 - \alpha_2)(\alpha_3 - \alpha_2)}$ $A_3 = \frac{\alpha_1 \alpha_2 X_0 + (\alpha_1 + \alpha_2) V_0 + W_0}{(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_3)}$	$x = A e^{-\alpha_1 t} + (B \cos \lambda t + C \sin \lambda t) e^{-\gamma t}$ $A = \frac{(\gamma^2 + \lambda^2) X_0 + 2\gamma V_0 + W_0}{(\gamma - \alpha_1)^2 + \lambda^2}$ $B = \frac{\alpha_1 (\alpha_1 - 2\gamma) X_0 - 2\gamma V_0 - W_0}{(\gamma - \alpha_1)^2 + \lambda^2}$ $C = \frac{\alpha_1 (\lambda^2 - \gamma^2 + \gamma \alpha_1) X_0 + (\alpha_1^2 - \gamma^2 + \lambda^2) V_0}{\lambda [(\gamma - \alpha_1)^2 + \lambda^2]} + \frac{(\alpha_1 - \gamma) W_0}{\lambda [(\gamma - \alpha_1)^2 + \lambda^2]}$

Note. Here α_1, α_2, α_3 are the absolute values of real nonmultiple roots, γ and λ are the absolute values of real and imaginary parts of complex roots, X_0 is the initial value of the function under investigation, $V_0 = x'(0)$ and $W_0 = x''(0)$ are the initial values of the rate of change and acceleration of the function under investigation.
1) Order of equation; 2) real roots; 3) complex roots.

12. Transforms of elementary functions

Наименование функции 1.	Оригинал 2	Изображение Лапласа 3	Изображение Карсона–Хевисайда 4
Единичная импульсная функция . . 5	$\delta(t)$	1	p
Единичная ступенчатая функция . 6	$1(t)$	$\frac{1}{p}$	1
Неединичная ступенчатая функция 7	$A1(t)$	$\frac{A}{p}$	A
Степенная функция . . . 8	$t^n 1(t)$	$\frac{n!}{p^{n+1}}$	$\frac{n!}{p^n}$
Экспонента . 9 . .	$e^{-at}1(t)$	$\frac{1}{p+a}$	$\frac{p}{p+a}$
Смещенная экспонента . . 10 . . .	$\frac{1}{a}(1-e^{-at})1(t)$	$\frac{1}{p(p+a)}$	$\frac{1}{p+a}$
Синусоида 11 . . .	$\sin \lambda t\, 1(t)$	$\frac{\lambda}{p^2+\lambda^2}$	$\frac{\lambda p}{p^2+\lambda^2}$
Косинусоида . 12 .	$\cos \lambda t\, 1(t)$	$\frac{p}{p^2+\lambda^2}$	$\frac{p^2}{p^2+\lambda^2}$
Затухающая синусоида 13	$e^{-\gamma t}\sin \lambda t\, 1(t)$	$\frac{\lambda}{(p+\gamma)^2+\lambda^2}$	$\frac{\lambda p}{(p+\gamma)^2+\lambda^2}$
Затухающая косинусоида 14	$e^{-\gamma t}\cos \lambda t\, 1(t)$	$\frac{p+\gamma}{(p+\gamma)^2+\lambda^2}$	$\frac{p(p+\gamma)}{(p+\gamma)^2+\lambda^2}$

1) Function; 2) original; 3) Laplace transform; 4) Karson-Heaviside transform; 5) unit impulse function; 6) unit step function; 7) nonunity step function; 8) step function; 9) exponential function; 10) mixed exponential function; 11) sinusoidal function; 12) cosinusoidal function; 13) damped sinusoidal function; 14) damped cosinusoidal function.

13. Table of z-transforms for elementary time functions

№	$f(t)$	$F(z)$
1	$1(t)$	$\frac{z}{z-1}$
2	t	$\frac{T_0 z}{(z-1)^2}$
3	$\frac{1}{2}t^2$	$\frac{T_0^2 z(z+1)}{2(z-1)^3}$
4	$e^{-\alpha t}$	$\frac{z}{z-d}, \quad d=e^{-\alpha T_0}$
5	$1-e^{-\alpha t}$	$\frac{(1-d)z}{(z-1)(z-d)}, \quad d=e^{-\alpha T_0}$
6	$\sin \beta t$	$\frac{z \sin \beta T_0}{z^2-2z\cos \beta T_0+1}$
7	$\cos \beta t$	$\frac{z^2-z\cos \beta T_0}{z^2-2z\cos \beta T_0+1}$
8	$e^{-\alpha t}\sin \beta t$	$\frac{zd \sin \beta T_0}{z^2-2zd\cos \beta T_0+d^2}$
9	$e^{-\alpha t}\cos \beta t$	$\frac{z^2-zd\cos \beta T_0}{z^2-2zd\cos \beta T_0+d^2}$

14. Finding the square integral estimate

$$I_0=\int_0^\infty x^2\,dt$$

The Laplace transform of the controlled variable takes the form

$$X(p)=\frac{b_0+b_1p+\ldots+b_mp^m}{a_0+a_1p+\ldots+a_np^n}\,\frac{1}{p} \qquad (n>m).$$

Then the integral estimate may be computed from the expression

$$I_0=\frac{1}{2a_0^2\Delta}\,(B_0\Delta_0+B_1\Delta_1+\ldots+B_m\Delta_m-2b_0b_1\Delta).$$

The determinant Δ is found as follows:

$$\Delta=\begin{vmatrix} a_0 & -a_2 & a_4 & -a_6 & \ldots & 0\\ 0 & a_1 & -a_3 & a_5 & \ldots & 0\\ 0 & -a_0 & a_2 & -a_4 & \ldots & 0\\ 0 & 0 & -a_1 & a_3 & \ldots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & 0 & \ldots & a_{n-1}\end{vmatrix}.$$

Δ_ν (ν = 0, 1, ..., m) is a determinant obtained from Δ by replacing the (ν + 1)-th column with the following column:

$$a_1,\ a_0,\ 0,\ \ldots,\ 0.$$

The coefficients $B_1, \ldots, B_m$ are calculated in the following manner:

$$
\begin{aligned}
B_0 &= b_0^2 \\
B_1 &= b_1^2 - 2b_0 b_2 \\
&\cdots\cdots\cdots\cdots \\
B_k &= b_k^2 - 2b_{k-1} b_{k+1} + \ldots + 2(-1)^k b_0 b_{2k} \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
B_m &= b_m^2.
\end{aligned}
$$

15. Formulas for integrating spectral density

The desired integral is represented in the form

$$I_n = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{G(j\omega)}{A(j\omega)A(-j\omega)} d\omega,$$

where

$$
\begin{aligned}
A(j\omega) &= a_0 (j\omega)^n + a_1 (j\omega)^{n-1} + \ldots + a_n \\
G(j\omega) &= b_0 (j\omega)^{2n-2} + b_1 (j\omega)^{2n-4} + \ldots + b_{n-1}.
\end{aligned}
$$

The polynomial $G(j\omega)$ contains only even powers of $j\omega$. The polynomial $A(j\omega)$ should have roots in the upper half plane, corresponding to a stable system.

For n = 1

$$I_1 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{b_0 d\omega}{|a_0 j\omega + a_1|^2} = \frac{b_0}{2a_0 a_1}.$$

For n = 2

$$I_2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{[b_0 (j\omega)^2 + b_1] d\omega}{|a_0 (j\omega)^2 + a_1 j\omega + a_2|^2} = \frac{-b_0 + \frac{a_0 b_1}{a_2}}{2a_0 a_1}.$$

For n = 3

$$I_3 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{[b_0 (j\omega)^4 + b_1 (j\omega)^2 + b_2] d\omega}{|a_0 (j\omega)^3 + a_1 (j\omega)^2 + a_2 j\omega + a_3|^2} =$$

$$= \frac{-a_2 b_0 + a_0 b_1 - \frac{a_0 a_1 b_2}{a_3}}{2a_0 (a_0 a_3 - a_1 a_2)}.$$

For n = 4

$$I_4 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{[b_0(j\omega)^6 + b_1(j\omega)^4 + b_2(j\omega)^2 + b_3]\,d\omega}{|a_0(j\omega)^4 + a_1(j\omega)^3 + a_2(j\omega)^2 + a_3 j\omega + a_4|^2} =$$

$$= \frac{b_0(-a_1a_4 + a_2a_3) - a_0a_3b_1 + a_0a_1b_2 + \frac{a_0b_3}{a_4}(a_0a_3 - a_1a_2)}{2a_0(a_0a_3^2 + a_1^2a_4 - a_1a_2a_3)}.$$

16. Standard normalized logarithmic amplitude characteristics

Every type of logarithmic amplitude characteristic is designated by a letter or a series of numbers indicating the slopes of the asymptotes (0 corresponds to a slope of 20 db/decade, 1 to a slope of 40 db/decade, 2 to a slope of 60 db/decade, etc.).

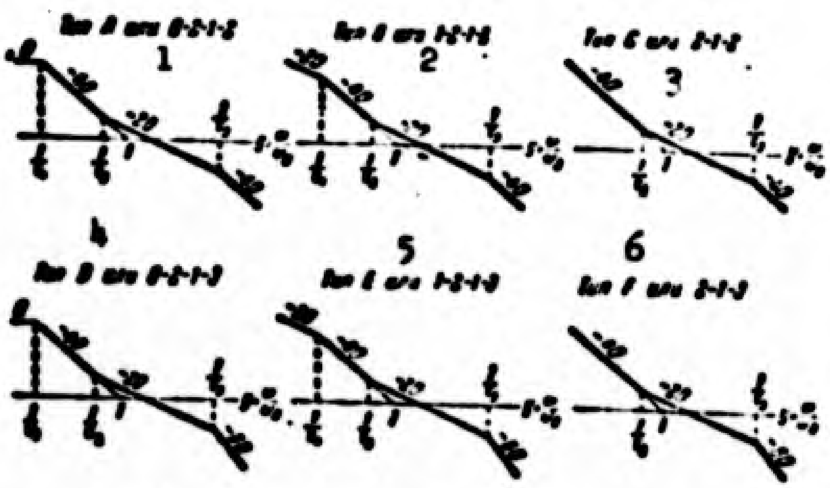


Fig. 268. 1) Type A or 0-2-1-2; 2) type B or 1-2-1-2; 3) type C or 2-1-2; 4) type D or 0-2-1-3; 5) type E or 1-2-1-3; 6) type F or 2-1-3.

The transfer functions corresponding to the standard logarithmic amplitude characteristics and the formulas for calculating the base frequency are given in Table 1.

TABLE 1

Степень астатизма 1	Тип л. а. х. 2	Передаточная функция разомкнутой системы 3	Нормированная передаточная функция 4	Базовая частота 5
0	A 0–2–1–2	$\frac{K(1+T_2p)}{(1+T_1p)^2(1+T_3p)}$	$\frac{\tau_1^2(1+\tau_2q)}{(1+\tau_1q)^2(1+\tau_3q)}$	$\frac{\sqrt{K}}{T_1}$
	D 0–2–1–3	$\frac{K(1+T_2p)}{(1+T_1p)^2(1+T_3p)^2}$	$\frac{\tau_1^2(1+\tau_2q)}{(1+\tau_1q)^2(1+\tau_3q)^2}$	
1	B 1–2–1–2	$\frac{K(1+T_2p)}{p(1+T_1p)(1+T_3p)}$	$\frac{\tau_1(1+\tau_2q)}{q(1+\tau_1q)(1+\tau_3q)}$	$\sqrt{\frac{K}{T_1}}$
	E 1–2–1–3	$\frac{K(1+T_2p)}{p(1+T_1p)(1+T_3p)^2}$	$\frac{\tau_1(1+\tau_2q)}{q(1+\tau_1q)(1+\tau_3q)^2}$	
2	C 2–1–2	$\frac{K(1+T_2p)}{p^2(1+T_3p)}$	$\frac{1+\tau_2q}{q^2(1+\tau_3q)}$	$\sqrt{K}$
	F 2–1–3	$\frac{K(1+T_2p)}{p^2(1+T_3p)^2}$	$\frac{1+\tau_2q}{q^2(1+\tau_3q)^2}$	

1) Degree of astatism; 2) type of logarithmic amplitude characteristic; 3) system open-loop transfer function; 4) normalized transfer function; 5) base frequency.

In the normalized transfer function we go to the new variable $q = q/\omega_0$, where ω_0 is the base frequency of the logarithmic amplitude characteristic.

The relative time constants are: $\tau_1 = \omega_0 T_1$, $\tau_2 = \omega_0 T_2$, and $\tau_3 = \omega_0 T_3$.

Where there are small time constants whose sum equals T_m, the relative sum of these small time constants $\tau_m = \omega_0 T_m$ is determined.

The parameters of normalized logarithmic amplitude characteristics are shown in Table 2 for types 2-1-2 and 2-1-3. The parameters for other types of logarithmic amplitude characteristic in the midfrequency range may be taken to be the same in first approximation.

TABLE 2

Тип л.а.х. 1	Показатель колебательности M 2	$t = \omega_0 T_2 = T_c = \frac{\omega_c}{\omega_0}$	Без учета малых постоянных времени 3		С учетом малых постоянных времени при $\tau_m = 0.1$ 4	
			$A = \frac{\omega_{[illegible]}}{\omega_0}$	$\tau_0 = \omega_0 T_0$	$A = \frac{\omega_{[illegible]}}{\omega_0}$	$\tau_0 = \omega_0 T_0$
C 2-1-2	1,1	3,32	21,0	0,158	57,2	0,059
	1,3	2,08	7,67	0,272	12,1	0,172
	1,5	1,73	5	0,316	7,04	0,246
	1,7	1,56	3,86	0,401	5,13	0,301
F 2-1-3	1,1	3,32	42,0	0,079	114	0,029
	1,3	2,08	15,3	0,136	21,2	0,086
	1,5	1,73	10	0,173	14,1	0,133
	1,7	1,56	7,72	0,202	10,3	0,152

1) Type of logarithmic amplitude characteristic; 2) magnitude ratio M; 3) neglecting small time constants; 4) taking into account small time constants for $\tau_m = 0.1$.

17. Normalized transient curves for standard logarithmic amplitude characteristics (after Appendix 16, Fig. 269)

Transient processes for systems having logarithmic amplitude characteristics of the 0-2-1-2 and 1-2-1-2 types are practically the same as for the 2-1-2 type.

The same is true for logarithmic amplitude characteristics of the 0-2-1-3, 1-2-1-3 and 2-1-3 types with respect to logarithmic amplitude characteristics of the 2-1-2 type.

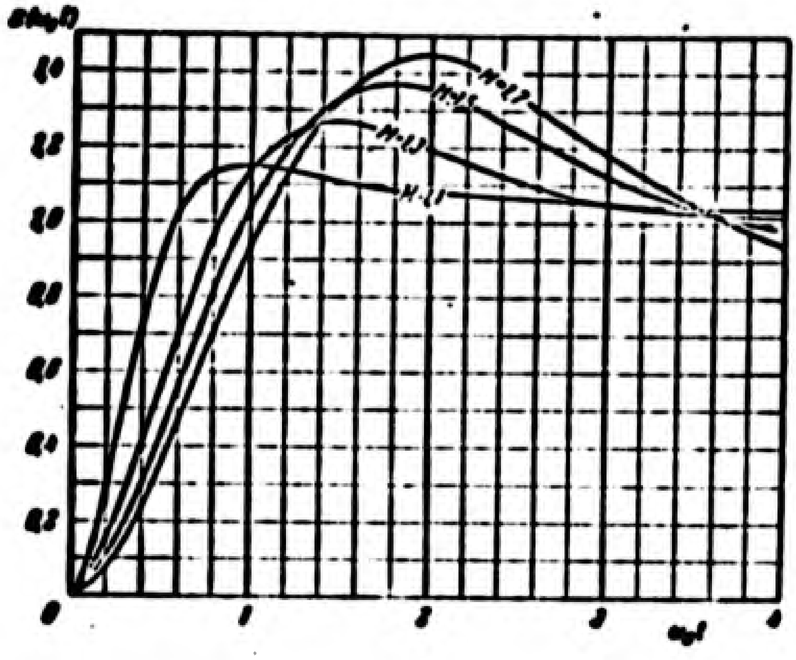


Fig. 269

18. Normalized system open-loop transfer functions

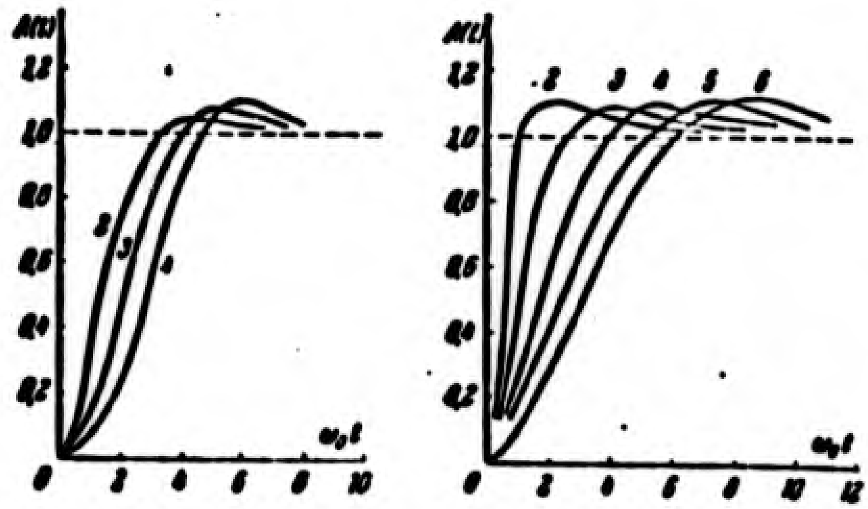


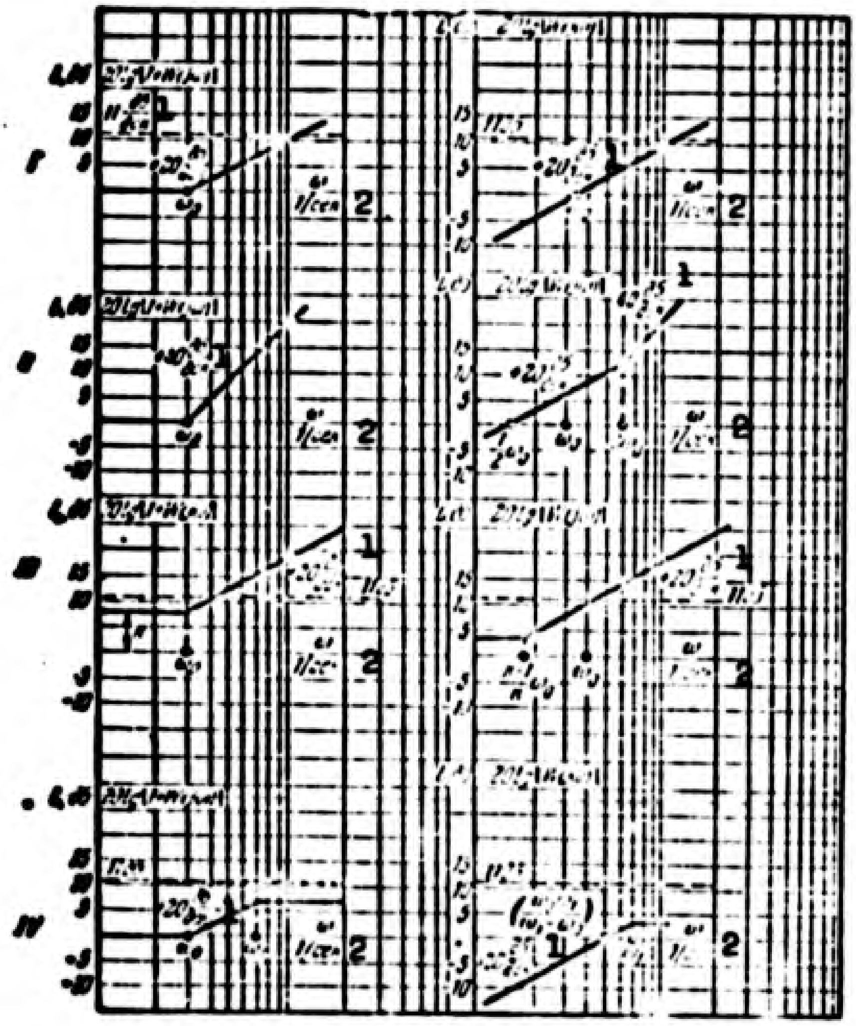
Fig. 270. Normalized transient curves standard transfer functions.

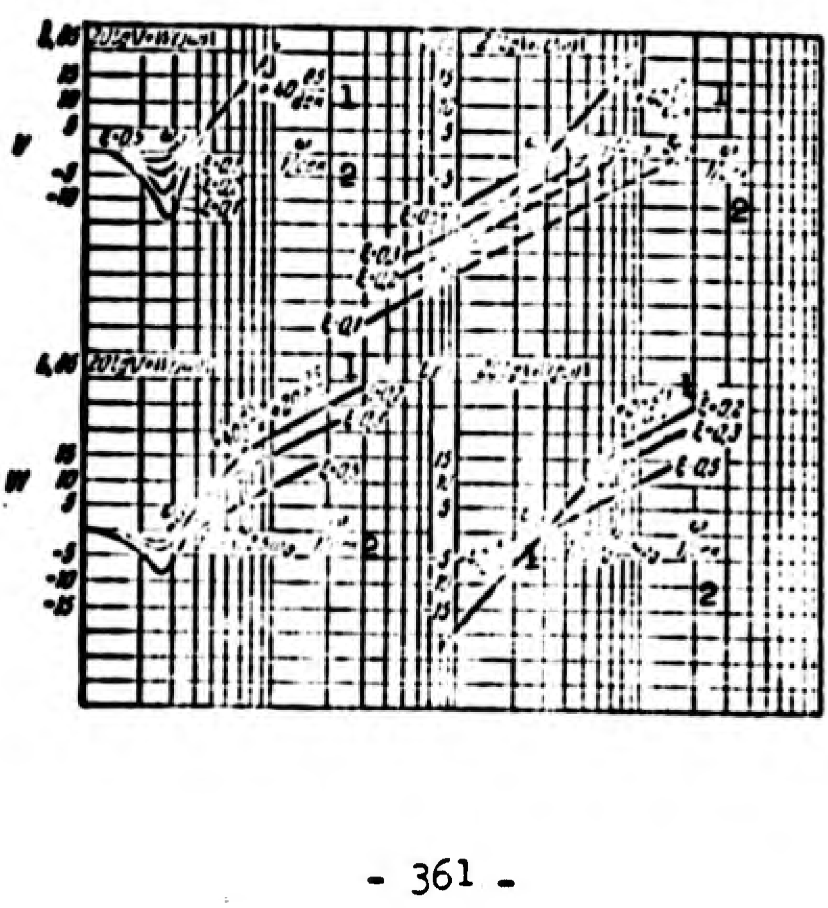
Transfer functions for various degrees n of a differential equation are shown in the Table. They contain the parameter ω_0 determining system response time. The transient curves corresponding to these transfer functions are shown in Fig. 270. The overshoot σ, % is given in the table.

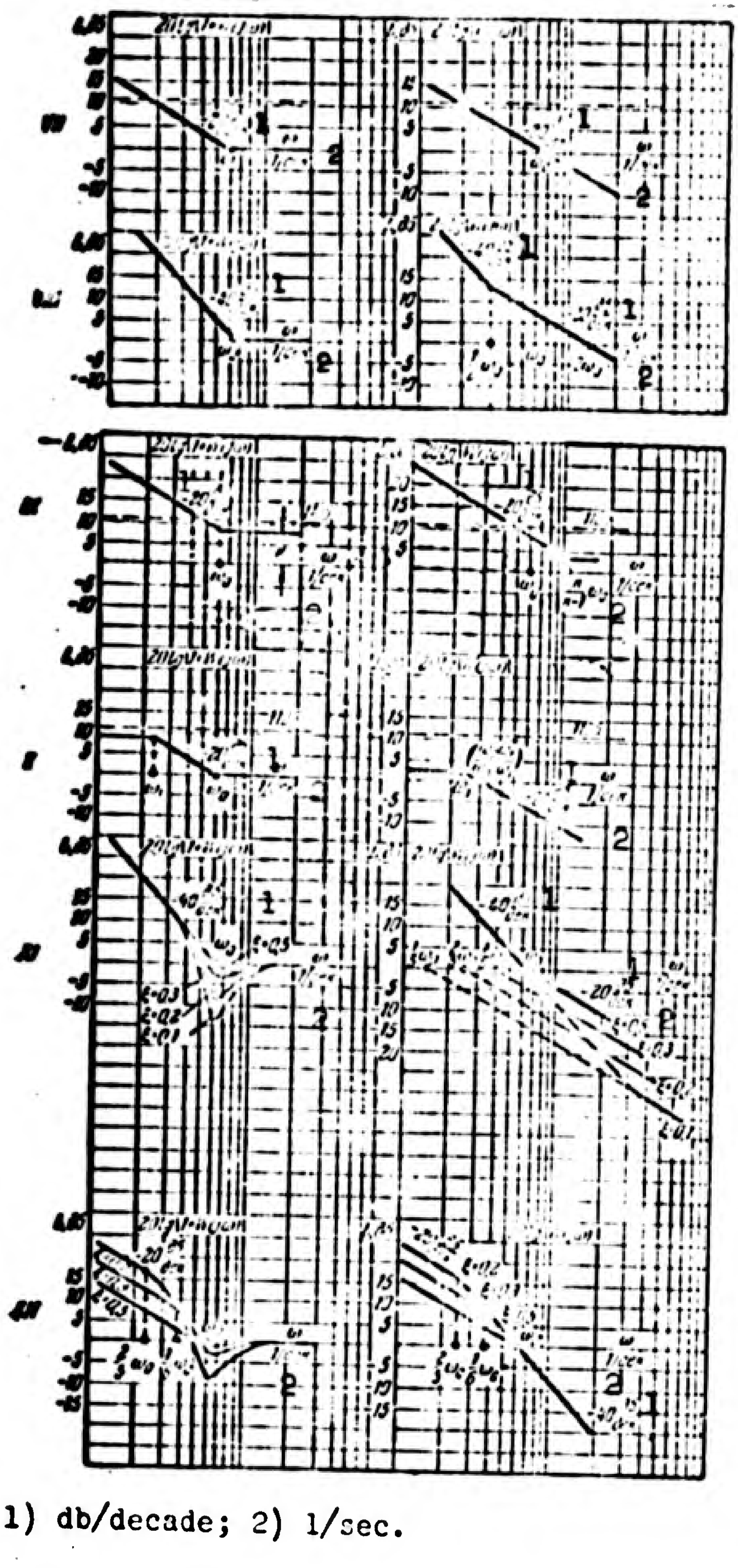
Степень астатизма 1)	n	σ, %	K	$W(p)$
1	2	5	$\frac{\omega_0}{1,4}$	$\frac{\omega_0^2}{p^2+1,4\omega_0 p}$
	3	8	$\frac{\omega_0}{2}$	$\frac{\omega_0^3}{p^3+2\omega_0 p^2+2\omega_0^2 p}$
	4	10	$\frac{\omega_0}{2,6}$	$\frac{\omega_0^4}{p^4+2,6\omega_0 p^3+3,4\omega_0^2 p^2+2,6\omega_0^3 p}$
2	2	10	ω_0^2	$\frac{2,5\omega_0 p+\omega_0^2}{p^2}$
	3	10	$\frac{\omega_0^2}{5,1}$	$\frac{6,3\omega_0^2 p+\omega_0^3}{p^3+5,1\omega_0 p^2}$
	4	10	$\frac{\omega_0^2}{16}$	$\frac{12\omega_0^3 p+\omega_0^4}{p^4+7,2\omega_0 p^3+16\omega_0^2 p^2}$
	5	10	$\frac{\omega_0^2}{38}$	$\frac{18\omega_0^4 p+\omega_0^5}{p^5+9\omega_0 p^4+29\omega_0^2 p^3+38\omega_0^3 p^2}$
	6	10	$\frac{\omega_0^2}{73}$	$\frac{25\omega_0^5 p+\omega_0^6}{p^6+11\omega_0 p^5+43\omega_0^2 p^4+83\omega_0^3 p^3+73\omega_0^4 p^2}$

1) Degree of astatism.

19. Table for converting from a logarithmic amplitude characteristic of the form 20 log |1 + W(jω)| to a logarithmic amplitude characteristic of the form 20 log |W(jω)|

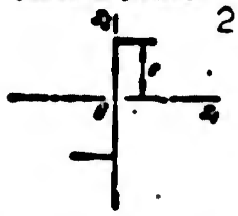
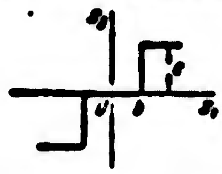
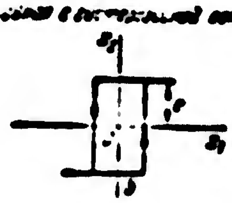




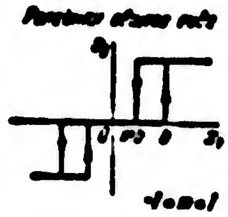
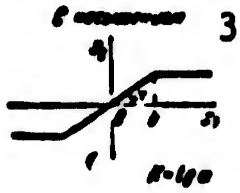


1) db/decade; 2) 1/sec.

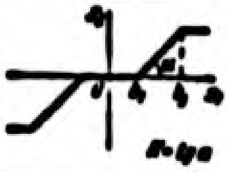
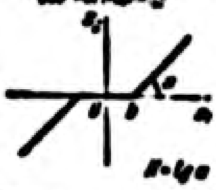
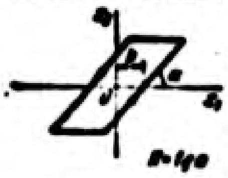
20. Coefficients of harmonic linearization for certain nonlinearities

Статические характеристики нелинейного звена 1	$q(A)$	$q'(A)$
Идеальное реле 2 	$\frac{4c}{\pi A}$	0
Реле с зоной нечувствительности 3 	$\frac{4c}{\pi A}\sqrt{1-\frac{b^2}{A^2}}$ при $A \geq b$ 5	0
Реле с гистерезисной петлей 4 	$\frac{4c}{\pi A}\sqrt{1-\frac{b^2}{A^2}}$ при $A \geq b$ 5	$-\frac{4cb}{\pi A^2}$ при $A \geq b$ 5

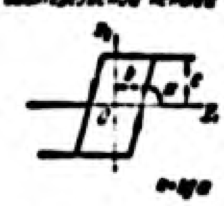
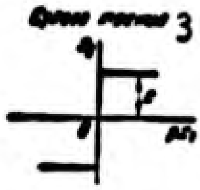
1) Static characteristic of nonlinear elements; 2) ideal relay; 3) relay with dead zone; 4) relay with hysteresis loop; 5) for.

Статические характеристики нелинейного звена 1	$q(A)$	$q'(A)$
Реле общего вида 2 	$\frac{2c}{\pi A}\left(\sqrt{1-\frac{b^2}{A^2}}+\sqrt{1-\frac{m^2b^2}{A^2}}\right)$ 4 при $A \geq b$	$-\frac{2cb}{\pi A^2}(1-m)$ при $A \geq b$ 4
С насыщением 3 	$\frac{2k}{\pi}\left(\arcsin\frac{b}{A}+\frac{b}{A}\sqrt{1-\frac{b^2}{A^2}}\right)$ 4 при $A \geq b$	0

1) Static characteristic of nonlinear element; 2) relay of general type; 3) with saturation; 4) for.

Статические характеристики нелинейного звена 1	$q(A)$	$q'(A)$
С зоной нечувствительности и насыщением 2 	$\frac{2k}{\pi}\left(\arcsin\frac{b_2}{A}-\arcsin\frac{b_1}{A}+\frac{b_2}{A}\sqrt{1-\frac{b_2^2}{A^2}}-\frac{b_1}{A}\sqrt{1-\frac{b_1^2}{A^2}}\right)$ 5 при $A\geq b_2$	0
С зоной нечувствительности без насыщения 3 	$k-\frac{2k}{\pi}\left(\arcsin\frac{b}{A}+\frac{b}{A}\sqrt{1-\frac{b^2}{A^2}}\right)$ 5 при $A\geq b$	0
Типа люфта или зазора 4 	$\frac{k}{\pi}\left[\frac{\pi}{2}+\arcsin\left(1-\frac{2b}{A}\right)+2\left(1-\frac{2b}{A}\right)\sqrt{\frac{b}{A}\left(1-\frac{b}{A}\right)}\right]$ 5 при $A\geq b$	$-\frac{4k}{\pi A}\left(1-\frac{b}{A}\right)$ при $A\geq b$ 5

1) Static characteristic of nonlinear element; 2) with dead zone and saturation; 3) with dead zone, no saturation; 4) play or gap type; 5) for.

Статические характеристики нелинейного звена 1	$q(A)$	$q'(A)$
С насыщением и гистерезисной петлей 2 	$\frac{k}{\pi}\left(\arcsin\frac{c+kb}{kA}+\arcsin\frac{c-kb}{kA}+\frac{c+kb}{kA}\sqrt{1-\frac{(c+kb)^2}{k^2A^2}}+\frac{c-kb}{kA}\sqrt{1-\frac{(c-kb)^2}{k^2A^2}}\right)$. при $A\geq\frac{c+kb}{k}$ 4	$-\frac{4bc}{\pi A^2}$ при $A\geq\frac{c+kb}{k}$ 4
Сухое трение 3 	$\frac{4c}{\pi A_{вх_1}}$	0

1) Static characteristic of nonlinear element; 2) with saturation and hysteresis loop; 3) dry friction; 4) for.

21. Parameters of twin-T network

$$C_1 = C_2 = C_3 = C;$$

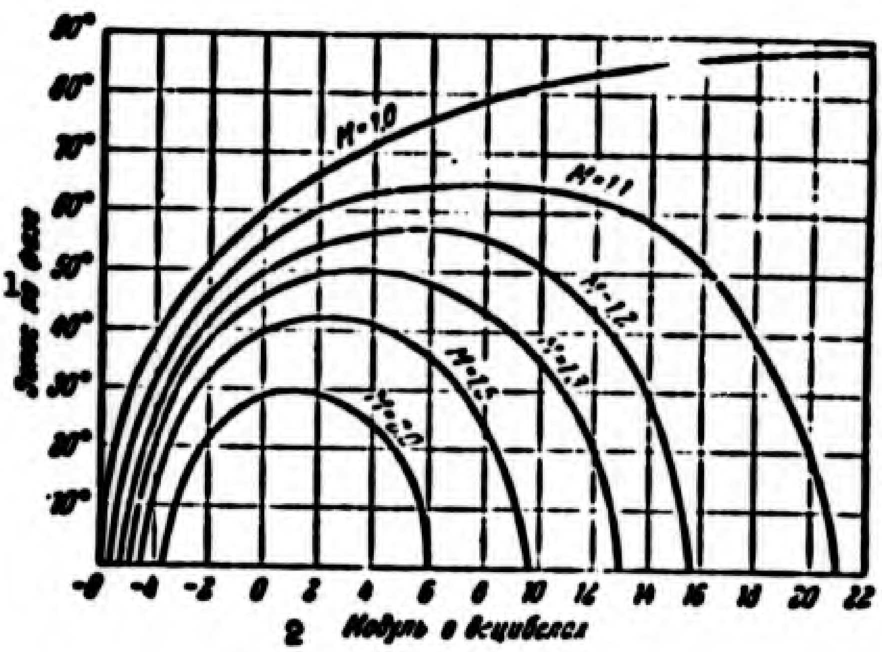
$$R_1 = \frac{a}{\omega_0 C}; \quad R_2 = \frac{1}{2\omega_0 C a}; \quad R_3 = \frac{1}{\sqrt{2}\,\omega_0 C};$$

G_0 is the carrier-frequency transfer constant.

$T_0\omega_0$	2.5	5.0	7.5	10	15	20	30	40	50	60	
1 Frequency bandwidth, cps	$\pm\frac{f_0}{2.5}$	$\pm\frac{f_0}{5.0}$	$\pm\frac{f_0}{7.5}$	$\pm\frac{f_0}{10}$	$\pm\frac{f_0}{15}$	$\pm\frac{f_0}{20}$	$\pm\frac{f_0}{30}$	$\pm\frac{f_0}{40}$	$\pm\frac{f_0}{50}$	$\pm\frac{f_0}{60}$	resonates 2
a	0.752	0.553	[illegible]	0.653	0.620	0.603	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
G_0	0.236	0.110	0.077	0.052	0.031	0.025	0.016	0.012	0.009	[illegible]	0

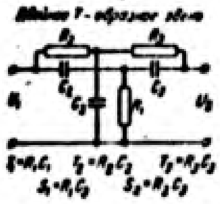
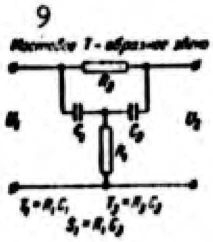
1) Frequency bandwidth, cps; 2) resonates.

22. Required phase margin as a function of modulus in decibels for various magnitude ratios M

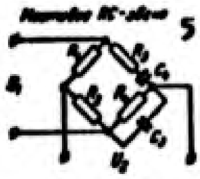
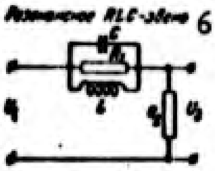


1) Phase margin; 2) modulus, decibels.

23. Alternating-current compensating networks

Схема 1	Передаточные функции звена 2	Связь между параметрическими постоянными времени и постоянными времени по огибающей 3	4 Условия осуществимости звена
5 Двойное T-образное звено 	Параметрическая 6 $\frac{U_2}{U_1}=\frac{A(p)}{B(p)}$, где 7 $A(p)=T_1T_2T_3p^3+T_1(s_2+T_3)p^2+$ $+(T_1+s_1)p+1$. $B(p)=T_1T_2T_3p^3+T_1(s_2+T_3)p^2+T_1(T_2+$ $+s_1+T_3)p^2+(T_1+s_1+T_2+T_3+s_2)p+1$. По огибающей 8 $\frac{U_2(j\Omega)}{U_1(j\Omega)}=Q_0\frac{1+T_д j\Omega}{1+\tau j\Omega}$	$T_1=Q_0\frac{\omega_0T_д\tau+2\tau+\omega_0T_д}{2\omega_0(1-Q_0)}$ $T_2=\frac{\tau^2}{T_1\omega_0}$; $T_3=\frac{1}{\tau\omega_0}$ $S_1=\frac{2}{\omega_0^2T_д}+\frac{\tau}{\omega_0}-T_1$ $S_2=\frac{2}{\omega_0^2\tau}-\frac{2}{\omega_0^2T_д}-\frac{1}{\omega_0\tau}-T_1$ $Q_0=\frac{\tau}{T_д}$	$\frac{1}{\tau\omega_0}-\frac{2}{T_д\omega_0}-\frac{1}{\tau\omega_0}\sqrt{1-\tau^2\omega_0^2}<$ $<\tau<\frac{1}{\tau\omega_0}-\frac{2}{T_д\omega_0}+\frac{1}{\tau\omega_0}\sqrt{1-\tau^2\omega_0^2}$ $\frac{2\tau\omega_0-\omega_0T_д(1+\sqrt{1-\tau^2\omega_0^2})}{4-\frac{4\tau}{T_д}-\tau T_д\omega_0^2}<\tau<$ $<\frac{2\tau\omega_0-\omega_0T_д(1-\sqrt{1-\tau^2\omega_0^2})}{4-\frac{4\tau}{T_д}-\tau T_д\omega_0^2}$ $\tau\leq\frac{1}{\omega_0}$; $Q_0\leq\frac{1}{T_д\omega_0}$; $\tau<\omega_0$
9 Мостовое T-образное звено 	Параметрическая 6 $\frac{U_2}{U_1}=\frac{T_1T_2p^2+(T_1+s_1)p+1}{T_1T_2p^2+(T_1+s_1+T_2)p+1}$ По огибающей 8 $\frac{U_2(j\Omega)}{U_1(j\Omega)}=Q_0\frac{1+T_д j\Omega}{1+\tau j\Omega}$	$T_2=\frac{2}{\omega_0^2}\left(\frac{1}{\tau}-\frac{1}{T_д}\right)$ $T_1=\frac{1}{2\left(\frac{1}{\tau}-\frac{1}{T_д}\right)}$. $S_1=\frac{2}{T_д\omega_0^2}-\frac{1}{2\left(\frac{1}{\tau}-\frac{1}{T_д}\right)}$ $Q_0=\frac{\tau}{T_д}$	$\tau<\omega_0$ $Q_0<\frac{4}{4+T_д^2\omega_0^2}$

1) Circuit; 2) network transfer function; 3) relationship between parametric time constants and envelope time constants; 4) conditions for network realization; 5) double T network; 6) parametric; 7) where; 8) envelope; 9) bridge T network.

Схема 1	Передаточные функции звена 2	Связь между параметрическими постоянными времени и постоянными времени по огибающей 3	Условие осуществимости звена 4
Мостовое RC-звено 5 	Параметрическая 7 $\frac{U_2}{U_1} = \frac{R_2}{R_1+R_2}\frac{T_3T_4p^2+\left(T_3+T_4-\frac{R_2}{R_1}\tau_4\right)p+1}{T_3T_4p^2+(T_3+T_4+\tau_4)p+1}$ $T_3=C_3R_3;\ T_4=C_4R_4;\ \tau_4=C_4R_1$ По огибающей 8 $\frac{U_2(j\Omega)}{U_1(j\Omega)} = G_0\frac{1+T_Aj\Omega}{1+\tau j\Omega}$	$T_3=\frac{1}{\omega_n}\left[\frac{\rho}{n}+\frac{1-\rho}{m}\right]\pm$ $\pm\frac{1}{\omega_n}\left[\left(\frac{\rho}{n}+\frac{1-\rho}{m}\right)^2-1\right]^{1/2}$ $T_4=\frac{1}{\omega_n}\left[\frac{\rho}{n}+\frac{1-\rho}{m}\right]\mp$ $\mp\frac{1}{\omega_n}\left[\left(\frac{\rho}{n}+\frac{1-\rho}{m}\right)^2-1\right]^{1/2}$ $\tau_4=\frac{2\rho}{\omega_n}\left(\frac{1}{m}-\frac{1}{n}\right)$ $T_A=\frac{n}{\omega_n};\ G_0=\rho\frac{m}{n};\ \rho=\frac{R_2}{R_1+R_2}$ $\tau=\frac{m}{\omega_n}$	$\Omega\le\omega_n$ $T_A\omega_n\le\frac{\frac{R_2}{R_1+R_2}\omega_n}{\omega_n+\frac{R_2}{R_1+R_2}-1}$ $G_0\le\frac{1}{\sqrt{T_A\omega_n}}$
Резонансное RLC-звено 6 	$\frac{U_2}{U_1}=\frac{R_2}{R_1+R_2}\frac{1+j\Omega 2R_1C}{1+j\Omega\frac{2R_1CR_2}{R_1+R_2}}$	$T_A=2R_1C$ $\tau=2R_1CG_0$ $G_0=\frac{R_2}{R_1+R_2}$ $\omega_n=\frac{1}{\sqrt{LC}}$	$\Omega\le\omega_n$ 9 При малых ω_n необходимы большие индуктивности: $T_A>\tau$

1) Circuit; 2) network transfer function; 3) relationship between parametric time constants and envelope time constants; 4) conditions for network realization; 5) bridge RC network; 6) resonant RLC network; 7) parametric; 8) envelope; 9) for ω_n larger inductances are needed.

24. Table of functions $h(t_0)$

t_0	0,0	0,05	0,10	0,15	0,20	0,25	0,30	0,35	0,40	0,45
0,0	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
0,5	0,158	0,165	0,176	0,184	0,192	0,199	0,207	0,215	0,223	0,231
1,0	0,310	0,326	0,340	0,356	0,371	0,386	0,401	0,417	0,432	0,447
1,5	0,449	0,469	0,494	0,516	0,538	0,560	0,580	0,603	0,617	0,646
2,0	0,572	0,597	0,628	0,655	0,683	0,709	0,733	0,761	0,786	0,810
2,5	0,674	0,705	0,739	0,771	0,802	0,833	0,861	0,891	0,916	0,943
3,0	0,755	0,790	0,828	0,863	0,896	0,928	0,958	0,987	1,013	1,038
3,5	0,814	0,853	0,892	0,928	0,963	0,994	1,024	1,050	1,074	1,095
4,0	0,857	0,896	0,938	0,974	1,008	1,039	1,060	1,090	1,107	1,124
4,5	0,883	0,923	0,960	0,997	1,029	1,057	1,080	1,100	1,115	1,129
5,0	0,896	0,936	0,978	1,012	1,042	1,067	1,087	1,103	1,112	1,117
5,5	0,900	0,940	0,986	1,019	1,016	1,067	1,083	1,093	1,095	1,097
6,0	0,904	0,943	0,982	1,013	1,037	1,054	1,065	1,070	1,068	1,062
6,5	0,904	0,942	0,980	1,009	1,030	1,043	1,050	1,049	1,043	1,033
7,0	0,904	0,944	0,979	1,006	1,024	1,035	1,037	1,033	1,023	1,009
7,5	0,907	0,945	0,980	1,006	1,019	1,025	1,025	1,017	1,005	0,989
8,0	0,910	0,951	0,985	1,008	1,020	1,024	1,021	1,012	0,995	0,981
8,5	0,918	0,956	0,989	1,010	1,021	1,022	1,018	1,007	0,992	0,977
9,0	0,924	0,965	0,997	1,016	1,025	1,025	1,018	1,006	0,992	0,978
9,5	0,932	0,972	1,004	1,022	1,029	1,027	1,019	1,006	0,993	0,982
10,0	0,939	0,978	1,009	1,025	1,031	1,027	1,019	1,006	0,993	0,987
10,5	0,946	0,985	1,013	1,028	1,033	1,028	1,017	1,005	0,993	0,991
11,0	0,947	0,988	1,015	1,029	1,031	1,025	1,014	1,002	0,993	0,991
11,5	0,949	0,988	1,016	1,027	1,028	1,021	1,010	0,999	0,991	0,989
12,0	0,950	0,988	1,015	1,025	1,024	1,015	1,004	0,994	0,988	0,987
12,5	0,950	0,989	1,013	1,022	1,019	1,010	0,999	0,990	0,986	0,986
13,0	0,950	0,989	1,012	1,019	1,015	1,005	0,994	0,986	0,985	0,987
13,5	0,950	0,990	1,011	1,017	1,011	1,000	0,990	0,983	0,984	0,988

t_0	0,50	0,55	0,60	0,65	0,70	0,75	0,80	0,85	0,90	0,95	1,00
0,0	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
0,5	0,240	0,248	0,255	0,259	0,267	0,275	0,282	0,290	0,297	0,304	0,311
1,0	0,461	0,476	0,490	0,505	0,519	0,534	0,547	0,562	0,575	0,593	0,603
1,5	0,665	0,685	0,706	0,722	0,740	0,758	0,776	0,794	0,813	0,832	0,841
2,0	0,833	0,856	0,878	0,809	0,919	0,938	0,956	0,974	0,986	1,003	1,020
2,5	0,967	0,985	1,010	1,031	1,042	1,060	1,078	1,098	1,113	1,125	1,133
3,0	1,061	1,082	1,100	1,117	1,130	1,142	1,154	1,164	1,172	1,176	1,178
3,5	1,115	1,132	1,145	1,158	1,161	1,166	1,171	1,174	1,175	1,175	1,175
4,0	1,142	1,152	1,158	1,159	1,160	1,161	1,156	1,149	1,141	1,131	1,118
4,5	1,138	1,134	1,134	1,138	1,132	1,127	1,111	1,099	1,085	1,071	1,053
5,0	1,118	1,115	1,107	1,098	1,084	1,069	1,053	1,037	1,019	1,001	0,986
5,5	1,092	1,083	1,070	1,050	1,032	1,016	0,994	0,979	0,962	0,951	0,932
6,0	1,051	1,037	1,021	1,003	0,984	0,966	0,949	0,934	0,922	0,914	0,906
6,5	1,018	1,001	0,982	0,965	0,918	0,936	0,920	0,910	0,903	0,903	0,905
7,0	0,993	0,975	0,957	0,941	0,927	0,917	0,911	0,908	0,909	0,915	0,925
7,5	0,974	0,958	0,944	0,926	0,922	0,911	0,920	0,927	0,931	0,946	0,958
8,0	0,966	0,951	0,941	0,935	0,932	0,936	0,944	0,955	0,970	0,986	1,001
8,5	0,966	0,949	0,944	0,948	0,951	0,958	0,974	0,990	1,006	1,023	1,041
9,0	0,970	0,960	0,961	0,966	0,976	0,990	1,006	1,023	1,039	1,053	1,061
9,5	0,975	0,972	0,980	0,987	1,000	1,015	1,033	1,048	1,059	1,066	1,066
10,0	0,982	0,985	0,993	1,006	1,020	1,036	1,049	1,059	1,063	1,062	1,056
10,5	0,987	0,996	1,007	1,017	1,033	1,046	1,054	1,058	1,055	1,048	1,043
11,0	0,993	1,002	1,014	1,027	1,039	1,047	1,048	1,044	1,034	1,021	1,005
11,5	0,997	1,000	1,017	1,029	1,037	1,043	1,031	1,024	1,010	0,994	0,977
12,0	0,997	1,006	1,019	1,026	1,027	1,025	1,015	1,000	0,984	0,969	0,958
12,5	0,997	1,006	1,018	1,019	1,017	1,010	0,995	0,979	0,965	0,951	0,949
13,0	0,997	1,006	1,014	1,012	1,005	0,993	0,980	0,964	0,955	0,950	0,955
13,5	0,998	1,006	1,010	1,005	0,995	0,982	0,968	0,958	0,951	0,958	0,970

t°	0,0	0,05	0,10	0,15	0,20	0,25	0,30	0,35	0,40	0,45
14,0	0,952	0,989	1,011	1,016	1,009	0,997	0,988	0,983	0,985	0,991
14,5	0,954	0,990	1,012	1,015	1,008	0,996	0,987	0,985	0,988	0,996
15,0	0,956	0,993	1,012	1,014	1,007	0,995	0,988	0,987	0,991	1,000
15,5	0,959	0,995	1,014	1,014	1,006	0,995	0,989	0,988	0,996	1,001
16,0	0,961	0,997	1,015	1,014	1,006	0,995	0,991	0,992	0,998	1,007
16,5	0,964	0,999	1,016	1,014	1,005	0,995	0,993	0,995	1,002	1,009
17,0	0,965	1,001	1,016	1,013	1,005	0,995	0,991	0,997	1,005	1,010
17,5	0,966	1,002	1,015	1,012	1,003	0,995	0,991	0,998	1,006	1,010
18,0	0,966	1,002	1,015	1,011	1,002	0,995	0,995	1,001	1,008	1,010
18,5	0,966	1,001	1,015	1,009	1,001	0,994	0,995	1,001	1,007	1,009
19,0	0,967	1,000	1,015	1,008	0,998	0,992	0,995	1,001	1,006	1,006
19,5	0,967	1,000	1,014	1,005	0,996	0,991	0,995	1,001	1,005	1,001
20,0	0,967	1,000	1,013	1,005	0,995	0,991	0,995	1,001	1,005	1,002
20,5	0,968	1,002	1,012	1,001	0,991	0,991	0,996	1,002	1,004	1,001
21,0	0,968	1,002	1,011	1,003	0,991	0,992	0,997	1,003	1,004	1,001
21,5	0,969	1,002	1,011	1,003	0,995	0,992	0,999	1,001	1,001	1,001
22,0	0,971	1,002	1,011	1,002	0,995	0,993	1,000	1,005	1,004	1,000
22,5	0,973	1,002	1,011	1,002	0,996	0,995	1,002	1,006	1,001	0,999
23,0	0,974	1,005	1,011	1,002	0,996	0,997	1,001	1,007	1,003	0,999
23,5	0,975	1,005	1,010	1,002	0,996	0,998	1,004	1,008	1,003	0,998
24,0	0,975	1,005	1,010	1,001	0,996	0,999	1,005	1,007	1,002	0,997
24,5	0,975	1,005	1,009	1,000	0,995	0,999	1,005	1,006	1,001	0,997
25,0	0,975	1,005	1,008	1,000	0,995	0,999	1,005	1,001	1,000	0,996
25,5	0,975	1,005	1,008	0,999	0,995	0,999	1,004	1,003	0,998	0,996
26,0	0,975	1,005	1,007	0,999	0,995	0,999	1,004	1,002	0,997	0,996

t°	0,50	0,55	0,60	0,65	0,70	0,75	0,80	0,85	0,90	0,95	1,00
14,0	1,000	1,006	1,008	0,999	0,987	0,974	0,965	0,961	0,965	0,976	0,980
14,5	1,002	1,006	1,005	0,991	0,983	0,970	0,969	0,971	0,981	0,997	1,010
15,0	1,005	1,007	1,002	0,993	0,983	0,976	0,978	0,987	1,001	1,017	1,030
15,5	1,008	1,007	1,001	0,993	0,985	0,981	0,991	1,003	1,019	1,032	1,040
16,0	1,011	1,008	1,001	0,991	0,990	0,993	1,003	1,018	1,031	1,039	1,039
16,5	1,011	1,008	1,001	0,996	0,995	1,001	1,014	1,027	1,036	1,038	1,028
17,0	1,012	1,007	0,999	0,997	0,999	1,008	1,020	1,030	1,032	1,027	1,012
17,5	1,009	1,005	0,997	0,998	1,002	1,012	1,023	1,027	1,023	1,013	0,988
18,0	1,008	1,002	0,997	0,998	1,001	1,014	1,020	1,018	1,008	0,993	0,979
18,5	1,006	0,999	0,995	0,998	1,003	1,012	1,014	1,007	0,993	0,978	0,969
19,0	1,001	0,995	0,993	0,997	1,001	1,009	1,006	0,995	0,981	0,970	0,967
19,5	0,998	0,992	0,992	0,996	1,003	1,005	0,998	0,985	0,973	0,967	0,973
20,0	0,996	0,991	0,992	0,995	1,003	1,001	0,991	0,979	0,972	0,971	0,985
20,5	0,995	0,991	0,994	0,996	1,001	0,996	0,986	0,976	0,974	0,980	1,001
21,0	0,995	0,993	0,997	0,996	0,999	0,993	0,983	0,975	0,981	1,002	1,016
21,5	0,995	0,995	1,000	0,995	0,998	0,992	0,986	0,988	0,997	1,013	1,024
22,0	0,996	0,996	1,000	0,997	0,997	0,991	0,991	0,997	1,012	1,024	1,029
22,5	0,997	1,000	1,001	1,000	0,996	0,992	0,998	1,008	1,022	1,028	1,026
23,0	0,998	1,001	1,006	1,001	0,997	0,991	1,002	1,015	1,025	1,027	1,016
23,5	0,999	1,002	1,007	1,002	0,998	0,997	1,007	1,017	1,023	1,023	1,002
24,0	1,000	1,002	1,008	1,003	0,999	1,000	1,008	1,017	1,015	1,012	0,988
24,5	1,000	1,002	1,006	1,003	1,000	1,002	1,008	1,014	1,005	0,995	0,979
25,0	1,000	1,002	1,004	1,003	1,001	1,003	1,005	1,008	0,991	0,985	0,975
25,5	1,000	1,002	1,002	1,002	1,002	1,001	1,001	1,001	0,986	0,978	0,977
26,0	1,000	1,002	1,000	1,001	1,002	1,001	1,002	0,987	0,981	0,977	0,983

Manuscript Page No.	[Transliterated Symbols]
348	с = s = srez = cutoff
348	п = p = perekhodnyy = transient
358	м = m = malyy = small
365	н = n = nesushchiy = carrier
365	д = d [not identified]

BLANK PAGE

REFERENCES

1. Anisimov, V.I., Vavilov, A.A. and Fateyev, A.V. Sbornik primerov i zadach po lineynoy teorii avtomaticheskogo regulirovaniya [Collection of Examples and Problems in Linear Automatic Control Theory], Gosenergoizdat [State Power Engineering Press], 1959.

2. Besekerskiy, V.A., Vostokov, S.B. and Rzeytain, Ya.M. Elektromekhanicheskiye sglazhivayushchiye ustroystva na induktsionnykh elementakh [Electromechanical Smoothing Devices based on Inductive Elements], Sudpromgiz [State Publishing House for the Shipbuilding Industry], 1958.

3. Besekerskiy, V.A., Orlov, V.P., Polonskaya, L.V. and Fedorov, S.M. Proektirovaniye sledyashchikh sistem maloy moshchnosti [Design of Light-Duty Servosystems], Sudpromgiz, 1958.

4. Besekerskiy, V.A. and Fedorov, S.M. Raschet sledyatsikh sistem kombinirovannogo upravleniya metodom logarifmicheskikh chastotnykh kharakteristik [Design of Combined-Control Servosystems by the Method of Logarithmic Frequency Characteristics], in book entitled "Teoriya invariantnosti i yeye primeneniye" [Theory of invariance and its Applications], izd. AN USSR [Acad. Sci. UkrSSR Press], 1959.

5. Besekerskiy, V.A. and Fedorov, S.M. Primeneniye ekvivalentnoy peredatochnoy funktsii dlya rascheta sledyashchikh sistem s neyedinichnoy obratnoy svyaz'yu [Application of an Equivalent Transfer Function for the Design of Servosystems with Other Than Unity Feedback], Izvestiya AN SSSR, OTN, Energetika i avtomatika [Bull.

Acad. Sci. USSR, Div. Tech. Sci., Power Engineering and Automation], No. 2, 1960.

6. Besekerskiy, V.A. and Fedorov, S.M. Sintez sledyashchikh sistem s tsifrovymi vychislitel'nymi mashinami metodom logarifmicheskikh amplitudnykh kharakteristik [Synthesis of Servosystems with Digital Computers by the Method of Logarithmic Amplitude Characteristics], Izvestiya AN SSSR, OTN, Energetika i avtomatika, No. 3, 1961.

7. Blokh, Z.Sh. Dinamika lineynykh sistem avtomaticheskogo regulirovaniya [Dynamics of Linear Automatic Control Systems], Gostekhizdat [State Publishing House for Technical and Theoretical Literature], 1950.

8. Vasil'yev, D.V., Mitrofanov, B.A., Rabkin, G.L., Samokhvalov, G.H., Semenkovich, A.A., Fateyev, A.V. and Chicherin, N.I. Raschet sledyashchego privoda [Design of the Servosystem], Sudpromgiz, 1958.

9. Vasil'yev, D.V. and Chuich, V.G. Raschet sistem avtomaticheskogo upravleniya [Design of Automatic Control Systems], Mashgiz [State Scientific and Technical Publishing House for Literature on Machinery], 1959.

10. Voronov, A.A. Elementy teorii avtomaticheskogo regulirovaniya [Elements of Automatic Control Theory], Voenizdat [Military Press], 1954.

11. Gibson, Dzh. Sistemy avtomaticheskoy optimizatsii ili samonastraivayushchiyesya sistemy avtomaticheskogo regulirovaniya [Automatic Optimizing Systems or Self-Adaptive Automatic Control Systems], Report to the First International Conference of the IFAC, Moscow, 1960.

12. Gol'dfarb, L.S. O nekotorykh nelineynostyakh v sistemakh regulirovaniya [On Certain Nonlinearities in Control Systems], Avtomatika

i telemekhanika [Automation and Remote Control], No. 5, 1947.

13. Dzheyms, Kh., Nikol's, N. and Fillips, R. Teoriya sledyashchikh sistem [Theory of Servosystems], IL [Foreign Literature Press], 1953.

14. Ivakhnsnko, A.G. Tekhnicheskaya kiberistika [Engineering Cybernetics], Gostekhizdat Ukrainy [Ukrainian State Technical Publishing House], 1959.

15. Kazakov, I.Ye. Priblizhennyy veroyatnostnyy analiz tochnosti sushchestvenno nelineynykh avtomaticheskikh sistem [Approximate Probability Analysis of the Error of Essentially Nonlinear Automatic Systems], Avtomatika i telemakhanika [Automation and Remote Control], No. 5. 1956.

16. Krasovskiy, A.A. O dvukhkanal'nykh sistemakh avtomaticheskogo regulirovaniya s antisimmetrichenymi svyazyami [Two-Channel Automatic Control Systems with Antisymmetric Couplings], Avtomatika i telemekhanika [Automation and Remote Control], No. 2, 1957.

17. Kuntsevich, V.M. Issledovaniye sistemy ekstremal'nogo regulirovaniya pri nalichii sluchaynykh vozdeystviy [Investigation of Minimax Control Systems in the Presence of Random Disturbances], Izv. AN SSSR, OTN, Energetika i avtomatika, No. 1, 1960.

18. Lur'ye, A.I. Nekotoryye nelineynyye zadachi teorii avtomaticheskogo reguairovaniya [Certain Nonlinear Problems in Automatic Control Theory], Gostekhizdat, 1951.

19. Popov, Ye.P. Dinamika sistem avtomaticheskogo regulirovaniya [Dynamics of Automatic Control Systems], Gostekhizdat, 1954.

20. Popov, Ye.P. and Pal'tov, I.P. Priblizhennyye metody issledovaniya nelineynykh avtomaticheskikh sistem [Approximate Methods for Investigation of Nonlinear Automatic Systems], Fizmatgiz [State Publishing House of Physicomathematical Literature], 1960.

21. Presnukhin, L.N., Serebrovskiy, L.A. and Yudin, D.B. Ssnovy teorii i prosktirovaniya priborov upravleniya [Fundamentals of the Theory and Design of Control Devices], Oborongiz [State Publishing House for the Defense Industry], 1960.

22. Lugachev, V.S. Teoriya sluchaynykh funktsiy i yeye primeneniye k zadacham avtomaticheskogo upravleniya [Theory of Random Variables and its Application to Automatic Control Problems], Fizmatgiz, 1960.

23. Sokolov, T.N. Elektromekhanicheskiye sistemy avtomatichesgogo upravleniya [Electromechanical Automatic Control Systems], Energoizdat [Power Engineering Press], 1952.

24. Editor: Solodovnikov, V.V. Osnovy avtomaticheskogo regulirovaniya [Fundamentals of Automatic Control], Mashgiz, 1954.

25. Solodovnikov, V.V. Statisticheskaya dinamika lineynykh sistem avtomaticheskogo upravleniya [Statistical Dynamics of Linear Automatic Control Systems], Fizmatgiz, 1960.

26. Solodovnikov, V.V. Sintez korrektiruyushchikh ustroystv sledyashchikh sistem pri tipovykh vozdeystviyakh [Synthesis of Servosystem Compensating Devices for Standard Disturbances], Avtomatika i telemakhanika, No. 5, 1951.

27. Solodovnikov, V.V. Sintez korrektiruyushchikh ustroystv sledyashchikh sistem pri nomoshchi pomoshchi optimal'nykh i tinovykh logarifmicheskikh chastotnykh kharakteristik [Synthesis of Servosystem Compensating Devices with the Aid of Optimal and Standard Logarithmic Frequency Characteristics], Avtomatika i telemekhanika, No. 5, 1953.

28. Solodovnikov, V.V., Topcheyev, Yu.I. and Krutikova, G.V. Chastotnyy metod postroyeniya perekhodnykh protsessov s prilozheniyem tablits i nomogramm [Frequency Method for the Plotting of Tran-

sient Responses with Appended Tables and Alignment Charts], Gostekhizdat, Moscow, 1955.

29. Ulanov, G.M. Regulirovaniye po vozmushcheniyu [Disturbance Control], Energoizdat, 1960.

30. Fateyev, A.V. Osnovy lineynoy teorii avtomaticheskogo regulirovaniya [Fundamentals of Linear Automatic Control Theory], Gosenergoizdat [State Power Engineering Press], 1954.

31. Fedorov, S.M. K sintezu parallel'nykh korrektiruyushchikh ustroystv sledyashchikh sistem metodom pogarifmicheskikh chastotnykh kharakteristik [On the Synthesis of Parallel Servosystem Compensators by the Method of Logarithmic Frequency Characteristics], Avtomatika i telemekhanika, No. 9, 1956.

32. Tsypkin, Ya.Z. Teoriya impul'snykh sistem [Theory of Pulsed Systems], Fizmatgiz, 1959.

33. Tsyan', Syue-sen'. Tekhnicheskaya kibernetika [Engineering Cybernetics], IL, 1956.

34. Fel'dbaum, A.A. Electricheskiye sistemy avtomaticheskogo regulirovaniya [Electrical Automatic Control Systems], Oborongiz [State Publishing House for the Defense Industry], 1957.

35. Fel'dbaum, A.A. Vychislitel'nyye ustroystva v avtomaticheskikh sistemakh [Computing Devices in Automatic Systems], Fizmatgiz, 1959.

36. Chestnat, G. and Mayer, R. Proyektirovaniye i raschet sledyashchikh sistem i sistem regulirovaniya [Planning and Design of Servo and Control Systems], Goenergoizdat, 1959.

37. Chinayev, P.I. Nekotoryye voprosy teorii ekstremal'nykh sistem [Certain Problems in the Theory of Minimax Systems], Collection entitled "Avtomaticheskoye upravleniye i vychislitel'naya tekhnika [Automatic Control and Computer Engineering], No. 5, 1961.

38. Chinayev, P.I. Ekstremal'noye issledovaniye samonastraivayushchikhsya sistem impul'snogo deystviya [Minimax Investigation of Self-Adaptive Pulsed Systems], avtomatika, izd. AN USSR [Automation, Acad. Sci. UkrSSR Press], No. 3, 1959.

39. Whiteley, A.L. Teoriya sledyashchikh sistem osobenno v otnoshenii ikh stabilizatsin [Servosystem Theory, with Emphasis on Stabilization], Journ. IEE, 1946, August, Part II, Vol. 93.

DISTRIBUTION LIST

DEPARTMENT OF DEFENSE	Nr. Copies
HEADQUARTERS USAF	
ARL	1
OTHER AGENCIES	
AEC	2
AEC (Wash)	1
ARMY (FSTC)	3
NAVY	3
DIA	4
ATD	2
CIA	1
NASA (ATSS-T)	1
NSA	6
OAR	1
OTS	2
PWS	1
AFCRL (CRXLR)	1
FAA (SS-10)	1
RAND	1
SPECTRUM	1

MAJOR AIR COMMANDS	Nr. Copies
DDC	20
AFSC	
SCFTC	1
TDBDP-4 (Mrs. Webb)	1
TDBDP	2
TDBTL	2
TDGS	1
SSD (SSFAR)	2
TDEEC	1
AEDC (AEY)	1
AFETR (MTW)	1
AFWL (WLF)	1
APGC (PGF)	1
ASD (ASFS)	2
BSD (BSF)	1
ESD (ESY)	1
RADC (RAY)	1
TDBXT	1