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AN EXACT TEST FOR THE SEQUENTIAL ANALYSIS OF VARIANCE.(U)
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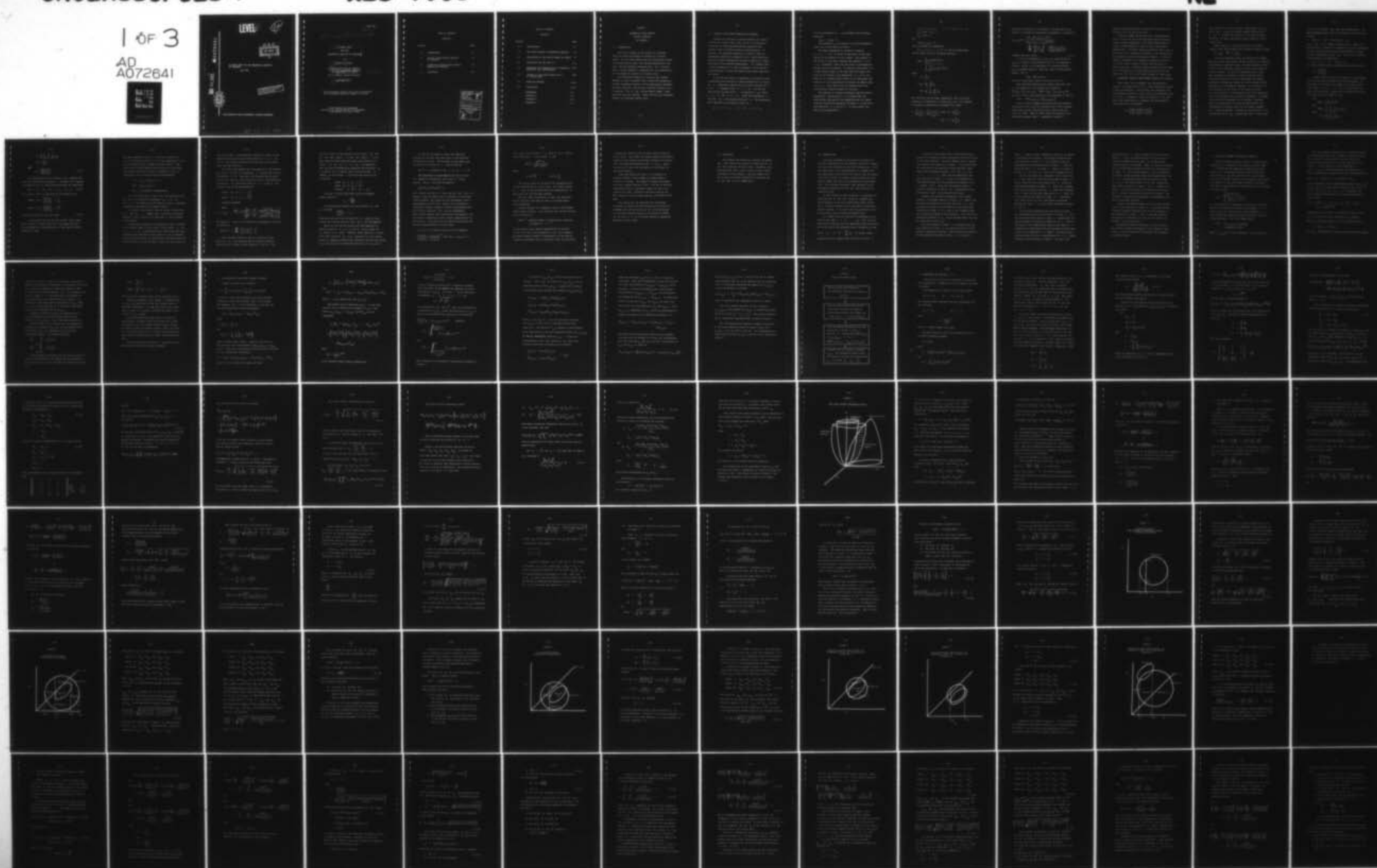

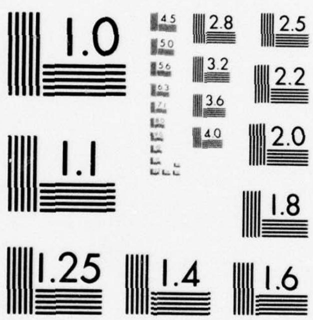

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AN EXACT TEST FOR THE SEQUENTIAL ANALYSIS OF VARIANCE

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AN EXACT TEST FOR THE SEQUENTIAL ANALYSIS OF VARIANCE*

by

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*This monograph contains the first two chapters of the doctoral thesis of Robert W. Miller

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TABLE OF CONTENTS

CHAPTER 1

TABLE OF CONTENTS

CHAPTER 2

CHAPTER 1

SEQUENTIAL FIXED EFFECTS ONE-WAY ANALYSIS OF VARIANCE

1.0 INTRODUCTION

This first chapter of the thesis will consider both the fixed and sequential analysis of variance tests. For the fixed sample test the discussions consist of the statistical model, the optimum properties of the test, and the operating characteristic (OC) function. Each of these concepts is important for the consideration of the sequential analysis of variance test.

The sequential analysis of variance test (termed SANOVA) is first discussed from a historical perspective. Further discussions consist of the experimental procedure, the test statistic, and the test statistic decision rule or regions. The OC and average sample number (ASN) functions are also defined. These functions are extremely helpful for designing SANOVA tests.

1.1 ONE-WAY FIXED EFFECTS ANALYSIS OF VARIANCE

Analysis of variance, a term introduced into statistics by R.A. Fisher (1918, 1925, 1935), is a statistical technique for analyzing measurements depending upon several kinds of effects operating simultaneously. In general, this technique consists of a body of tests of hypotheses, methods of estimation, etc., using statistics which are linear combinations of sums of squares of linear functions of the observed measurements. The simplest case in which analysis of variance is applied, is the one-way classification, in which the observations depend upon only one factor.

In the one-way layout, a population is stratified into m subpopulations according to some characteristic or factor and n_i independent observations are taken from each of k of the m subpopulations $(i = 1,\ldots,k)$. Let the jth observation from population i be denoted by x_{ij} where $i = 1,\ldots,k$ and $j = 1,\ldots,n_i$. Given that population i has mean $\mu+\sigma_i$ and standard deviation σ_i, the statistical model employed in the one-way layout is

$$x_{ij} = \mu + \sigma + e_{ij} \ , \quad i = 1,\cdots,k; \quad j = 1,\cdots,n_i$$

with the parameters $\delta_1,\ldots,\delta_k$ satisfying the following condition

$$n_1\delta_1+\ldots+n_k\delta_k = 0$$

The parameter δ_i is referred to as the differential effect due to the factor at level i.

The usual hypothesis of interest is whether $\delta_1 = \delta_2 = .. = \delta_k = 0$, which is equivalent to the hypothesis of the equality of the k means. The analysis of the effect of the factor depends upon whether $k < m$ or $k = m$. Eisenhardt (1947) was the first to differentiate between the two situations. He used the terms Model I or a fixed effects model as the case where the sample consists of all groups in the population, i.e., $k = m$, and Model II or a random effects model as the case where the interest is in the population from which the sample came, i.e., $k < m$. This thesis will be concerned with only fixed-effects one-way analysis of variance.

The analysis of variance technique requires several assumptions. Specifically, it is assumed that the observations from each of the subpopulations are random variables distributed normally with mean $\mu + \delta_i$ and standard deviation $\sigma = \sigma_i$ for all i. In other words, the model may be expressed as

$$x_{ij} = \mu + \delta_i + e_{ij} \qquad i = 1,\ldots,k;\ j = 1,\ldots,n$$

$$x_{ij} \sim N(\mu + \delta_i, \sigma)$$

$$e_{ij} \sim N(0, \sigma)$$

and

$$\mathrm{cov}(x_{ij}, x_{\ell m}) = 0.$$

With this model the hypotheses

$$H_0: \mu_1 = \mu_2 = \cdots \mu_k \text{ vs. } H_1: \text{not all means equal}$$

can be tested with the following statistic

$$F_{cal} = \frac{\sum_{i=1}^{k} n_i (\bar{x}_i - \bar{\bar{x}})^2/(k-1)}{\sum_{i=1}^{k}\sum_{j=1}^{n_i} (x_{ij} - x_i)^2/(N-k)}$$

where

$$N = \sum_{i=1}^{k} n_i$$

$$\bar{x}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} x_{ij}$$

$$\bar{\bar{x}} = \frac{1}{N}\sum_{i=1}^{k}\sum_{j=1}^{n_i} x_{ij}$$

This statistic can be shown (Kempthorne, 1952) to be distributed as a noncentral F variate with (k-1, N-k) degrees of freedom and noncentrality parameter $\bar{n}\lambda$, where

$$\lambda = \frac{\sum_{i=1}^{k} \delta_i^2 n_i}{\sigma^2} = \frac{\sum_{i=1}^{k} (\mu_i - \bar{\mu})^2 n_i}{\sigma^2} \quad \text{with} \quad \bar{\mu} = \frac{1}{k}\sum_{i=1}^{k} \mu_i$$

$$\text{and} \quad \bar{n} = \frac{1}{k}\sum_{j=1}^{k} n_i$$

The density function of a noncentral F variate with v_1, v_2 degrees of freedom and noncentrality parameter λ is given by:

$$f_{v_1,v_2,\lambda}(x) = \frac{e^{-\frac{1}{2}\lambda}\; v_1^{\frac{1}{2}v_1}\; v_2^{\frac{1}{2}v_2}\; x^{\frac{1}{2}v_1 - 1}}{B(\frac{1}{2}v_1, \frac{1}{2}v_2)\; (v_2 + v_1 x)^{\frac{1}{2}(v_1+v_2)}} \sum_{j=0}^{\infty} \left[\frac{\frac{1}{2}\lambda v_1 x}{v_2 + v_1 x}\right]^j \frac{\Gamma\{\frac{1}{2}(2j + v_1 + v_2)\}}{j!\,\Gamma(\frac{1}{2}v_2)\; \Gamma\frac{1}{2}(2j + v_1)}$$

(Johnson and Kotz, 1970). (1.1.1)

If the null hypothesis is true, the distribution of F_{cal} is a central F distribution with k-1, N-k degrees of freedom. Hence, if the hypothesis is rejected whenever F_{cal} is greater than the $100(1-\alpha)\%$ point of this distribution, that is

$$F_{cal} > F^*_{k-1,N-k,1-\alpha}$$

then the significance level of the test will be α.

The operating characteristic curve of the test, that is, the probability of accepting H_0 is given by $Pr\{F_{cal} \leq F^*_{k-1,N-k,1-\alpha}\}$. Since $F_{cal} \sim F_{k-1,N-k,\bar{n}_\xi}$ the OC of the test is characterized by the parameter $\xi = \bar{n}\lambda$, i.e.

$$OC(\lambda) = Pr\{F_{k-1,N-k,\xi} \leq F^*_{k-1,N-k,1-\alpha}\}$$

Several sets of tables and curves have been prepared from which the OC curve for selected tests can be obtained (Tang 1938, Pearson and Hartley 1951, Lehmer 1944, Fox 1956, Fix 1949). Most of these tables are entered with a different parameter than ξ. Appendix A contains a computer program which will calculate the OC curve (as a function of λ) for any given test.

Originally ANOVA was derived from a distributional point of view, but the F-test has been found to possess several optimum properties. Hsu (1941) showed that the F-test is UMP amongst all tests of size α whose power depends upon λ, and Wald (1942a) proved that the F-test is best when one is interested uniformly in all alternatives, as expressed by uniform weighting on spheres. As far as ANOVA is concerned it is immaterial whether the value of λ is built up by a number of small contributions or a single large one. Situations where instead the main emphasis is on detection of large deviations should not use ANOVA since the test is no longer optimum in these cases.

1.2 SEQUENTIAL ONE-WAY FIXED EFFECTS ANALYSIS OF VARIANCE

Wald (1947) first presented, and systematically studied, the sequential test of a simple hypothesis against a simple alternative. Let H_0 denote the hypothesis that the population density is $f_0(x)$, and H_1 the hypothesis that it is $f_1(x)$. Constants A and B are chosen (A > B), and after each observation in a sequence the corresponding likelihood ratio is computed:

$$\Lambda_n = \frac{f_1(x_1)\cdot f_1(x_2)\cdots f_1(x_n)}{f_0(x_1)\cdot f_0(x_2)\cdots f_0(x_n)}$$

The procedure is then as follows: reject H_0 if $\Lambda_n' \geq A$, accept H_0 if $\Lambda_n \leq B$, and obtain another observation if $B < \Lambda_n < A$. A and B are chosen so as to make the probabilities of Type-I and Type-II errors equal to α and β respectively.

Exact values of A and B are difficult to obtain. However, Wald (1947) proved that for small values of α and β

$$A \simeq \frac{1-\beta}{\alpha} \quad \text{and} \quad B \simeq \frac{\beta}{1-\alpha}$$

Since the hypothesis about the equality of K normal population means with common unknown variance is a composite multiparameter hypothesis with a nuisance parameter, Wald's theory of the sequential probability ratio test cannot be directly applied. To deal with problems such as these, Wald introduced the method of weight functions which, through the notion of a prior distribution for unknown parameters, essentially reduced the basic problem to test hypotheses in one parameter families. A difficulty with this procedure is the choice of the weight function. Cox (1952) devised a unified method under which sequential tests can be obtained for composite hypotheses. The basic idea behind Cox's procedure is to consider a sequence formed by transforming the original observations, the transformation chosen so that the new sequence depends upon a single parameter. Although the distribution of the transformed values $\{T_n\}$ depends upon only a single parameter Θ, the sequence $\{T_n\}$ may not be independent. Cox gave conditions under which the following factorization is possible

$$f(T_1,T_2,\cdots,T_n) = f(T_n|\Theta)f(T_2,\cdots,T_n)$$

where $f(T_2,\cdots,T_n)$ does not depend upon Θ. When this factorization is possible a sequential test can be developed to make a decision about this single parameter Θ, using only the transformed values $\{T_n\}$. The test for discriminating between the hypotheses

$$H_0: \ \Theta = \Theta_o \quad \text{vs.} \quad H_1: \Theta = \Theta_1$$

can now be constructed by considering the following ratio

$$\Lambda_n = \frac{f(T_n|\Theta_1)}{f(T_n|\Theta_0)} .$$

Johnson (1953) applied Cox's method to the following one-way fixed effects analysis of variance problem. An experiment is carried out in stages, and at each stage a fixed number r_i, for $i = 1,\cdots,k$, of observations are taken from each group. Denote the jth observation on the ith group at the nth stage by X_{ijn}.

Let

$$SSB_n = n\sum_{i=1}^{k} r_i(\bar{X}_i-\bar{\bar{X}})^2$$

and

$$SSW_n = \sum_{i=1}^{k}\sum_{j=1}^{r_i}\sum_{s=1}^{n}(X_{ijs}-\bar{X}_i)^2$$

with

$$\bar{X}_i = \frac{1}{nr_i}\sum_{j=1}^{n_i}\sum_{s=1}^{r} X_{ijs}$$

$$\bar{\bar{X}} = \frac{1}{N} \sum_{i=1}^{k} \sum_{j=1}^{n_i} \sum_{s=1}^{r} X_{ijs}$$

$$N = n \sum_{i=1}^{k} r_i$$

and

$$F_n = \frac{SSB_n/(k-1)}{SSW_n/(N-k)} \qquad (1.2.1)$$

The distribution of the sequence $\{F_n\}$ depends only upon the noncentrality parameter λ. Applying Cox's theorem, a sequential test for discriminating between the hypotheses

$$H_0: \ \lambda = \lambda_0 \quad \text{vs.} \quad H_1: \ \lambda = \lambda_1 \ , \ \lambda_1 > \lambda_0 \qquad (1.2.2)$$

for a given α and β is specified by the decision rule

$$\text{Accept } H_0 \text{ if } \frac{f(F_n|\lambda_1)}{f(F_n|\lambda_0)} \leq \frac{\beta}{1-\alpha}$$

$$\text{Reject } H_0 \text{ if } \frac{f(F_n|\lambda_1)}{f(F_n|\lambda_0)} \geq \frac{1-\beta}{\alpha}$$

otherwise continue to the next stage. (1.2.3)

An equivalent test was derived by Hoel (1955) using Wald's method of weight functions. The weight function Hoel employed was a generalization of that used for Wald's sequential t-test.

The same sequential test (i.e., the test statistic of (1.2.1) and decision rule (1.2.3) of the hypotheses (1.2.2) has also been by Hall, Wijsman and Ghosh (1965). Their derivation involved applying the principal of invariance. They showed that test statistic of equation (1.2.1) is unchanged by any of the following transformations:

(i) $X'_{ijn} = CX_{ijn} \quad C > 0$

(ii) $X'_{ijn} = X_{ijn} + C$

(iii) an orthogonal transformation

Also, they were able to prove that the sequential test was UMP for testing the hypotheses $H_0: \lambda \leq \lambda_0$ vs. $H_1: \lambda \geq \lambda_1$, by showing that the density $f(F_n|\lambda)$ possessed a monotone likelihood ratio (Lehman (1959)).

In addition, they proved that the vector of statistics $T_n = \{\bar{X}_1, \bar{X}_2, \ldots, SSW_n\}$ was a transitive sufficient sequence. This finding is of importance in later chapters of the thesis.

As previously explained, the sequential test is carried out in stages, where at each stage a fixed number r_i, for $i = 1, \ldots, k$, of observations are taken from each group. Throughout the remainder of this thesis it will be assumed that at the first stage two observations from each group will be taken (this is so the statistic SSB_1 will not be zero on the first stage). Each subsequent stage will result in one observation from each group being taken (i.e., $r_i = 1$ for all i). All future discussions will pertain to this particular testing situation.

As in the fixed sample test, the density of the statistic F_{nj} $(F_n|\lambda)$, is that of a noncentral F variate and is given in equation (1.1.1). Therefore, the decision rule of equation (1.2.3) requires calculating the ratio of two noncentral F densities. For specified values of α, β, λ_0 and λ_1 the decision rule can be reexpressed as:

accept H_0 if $\Lambda_n \leq \frac{\beta}{1-\alpha}$

reject H_0 if $\Lambda_n \geq \frac{1-\beta}{\alpha}$

continue otherwise

where

$$\Lambda = R(F_n) = \frac{e^{-\frac{n}{2}(\lambda_1-\lambda_0)} M\left[\frac{N-1}{2}, \frac{K-1}{2}, \frac{\lambda_0(K-1)F_n}{2(K(n-1)+(K-1)F_n)}\right]}{M\left[\frac{N-1}{2}, \frac{K-1}{2}, \frac{\lambda_1(K-1)F_n}{2(K(n-1)+(K-1)F_n)}\right]}$$

and $M(x,y,u)$, known as the confluent hypergeometric function is defined as

$$M(x,y,u) = \sum_{t=0}^{\infty} \frac{\Gamma(y)\ \Gamma(x+t)}{\Gamma(x)\ \Gamma(y+t)} \frac{u^t}{t!}$$

Since the above decision rule is a function of the statistic, F_n, the equations may be solved to obtain a decision rule in terms of that statistic. That is, two critical values of the statistic may be found; F_n^A and F_n^R such that $R(F_n^A) = \beta/(1-\alpha)$ and $R(F_n^R) = (1-\beta)/$. When these critical values have been calculated for all stages, F_n^A and F_n^R, $n = 2,\ldots,m_0$; the sequential test can then be conducted by comparing the statistic, F_n, of equation (1.2.1) against these critical values. In summary, at every stage n the following decision rule is applied:

$$\begin{array}{lll} \text{accept } H_0 & \text{if} & F_n \le F_n^A \\ \text{reject } H_0 & \text{if} & F_n \ge F_n^R \\ \text{continue} & \text{if} & F_n^A < F_n > F_n^R \end{array}$$

The test is usually performed using the somewhat simpler statistic

$$V_n = \frac{SSB_n}{SSW_n} .$$

The relationship between the two statistics F_n and V_n is simply

$$\frac{(N-K)V_n}{(K-1)} = F_n .$$

Conducting the test with the statistic V_n requires transforming the critical region as well $\left(\text{e.g., } V_n^A = (K-1)F_n^A/(N-K)\right)$.

Tables of the critical values have been prepared for selected values of α, β, K, λ_0 and λ_1 by Ray (1956) and B.K. Ghosh, et al. (1967). However, these tables are in terms of the test statistic $G_n = V_n/K$. Appendix B of this thesis contains a computer program which calculates the critical values of V_n; V_n^A and V_n^R, for specified values of α, β, K, λ_0, and λ.

As with all statistical tests, one important property of the test described above is the Operating Characteristic Curve. The OC curve for the above test is strictly a function of λ, and is given by

$$OC(\lambda^*) = Pr\{\text{accepting } H_0: \lambda = \lambda_0 \text{ if } \lambda = \lambda^*\}$$

Wald developed an approximation for the OC curve of a sequential probability ratio test of $f(X,\Theta_0)$ against $f(X,\Theta_1)$ provided the equation

$$E_\Theta\{[f(X,\Theta_1)/f(X,\Theta_0)]^h\} = 1$$

has a nonzero solution $h = h(\Theta)$, and the $\{X_1\}$ are i.i.d. However, since the above test is conducted on the transformed sequence $\{V_i\}$ which are not independent, Wald's approximation is not valid. Bhate (1955) developed a conjectural formula, similar to Wald's approximation for the OC curve, when the $\{X_i\}$ are not independent. Ghosh (1970) suggests that substituting the sequence $\{V_i\}$ into Bhate's formula may yield a useful approximation to the OC curve. The result of this substitution yields the following approximation to the OC curve.

If $h_i(\lambda)$ is a nonzero solution of the equation

$$\frac{f_i(V_i|V_1,\cdots,V_{i-1};\lambda_1)}{f_i(V_i|V_1,\cdots,V_{i-1};\lambda_0)}^{h_i} dF(V_i|V_1,\cdots,V_{i-1};\lambda) = 1$$

and $h_i(\lambda) \approx h(\lambda)$ for all $i \geq 1$, that is $h_i(\lambda)$ varies very little with i for a given λ, then

$$OC(\lambda) \simeq \frac{e^{Ah(\lambda)}-1}{e^{Ah(\lambda)}-e^{Bh(\lambda)}} .$$

Where

$$A \simeq \ell n \frac{1-\beta}{\alpha} \qquad B \simeq \ell n \frac{\beta}{1-\alpha}$$

The crucial point in the use of the conjecture lies in the verification of $h_i(\Theta) \simeq h(\Theta)$ for various values of i. Also it must be noted that this approximation is only valid for infinite Wald regions.

The only other alternative, to date, for obtaining the OC curve for this type of test is to employ Monte Carlo techniques.

Also of interest in a sequential test is the Average Sample Number function. For the above test the ASN function will be defined as:

$$ASN(\lambda^*) = \text{Expected number of stages until a decision is reached if } \lambda = \lambda^*.$$

As with the OC curve, Wald's approximation to the ASN, is not valid due to the dependence of the $\{V_i\}$ sequence. No general formula (exact or approximate) for the ASN for composite hypotheses exists, but Bhate (1955) has developed a conjectural formula along the same lines as that for the OC curve. Ray (1956) has applied Bhate's conjectural formula to the one-way fixed effect analysis of variance test, and obtained expressions for $\lambda = \lambda_0, \lambda_1$. Again, as with the OC curve this procedure is valid only for open Wald regions.

Since the regions are open, it is possible to progress through a large number of stages before a decision is reached. The number of stages will always be finite, however (Johnson, 1953). One way of assuming termination within a reasonable amount of time is to truncate the test. Truncation involves altering the Wald regions so that by some stage m_0 a decision can be made.

This thesis will be concerned with developing procedures to obtain the ASN function and OC curve for a SANOVA test with any given set of truncated regions. The following chapter contains a derivation of SANOVA for the case $k = 2$ by the Direct Method of Sequential Analysis (Aroian, 1968).

1.3 CONCLUSION

This chapter has served to introduce the SANOVA test. This thesis will pertain to obtaining the OC and ASN functions of such a test. Currently, only approximations exist, such as that of Bhate (1959), considered in this chapter. The next chapter will derive the first exact procedure for obtaining the OC and ASN of a k=2 SANOVA test.

2.0 INTRODUCTION

The major advantage to performing an analysis of variance sequentially is the possible reduction in sample size over that required for the fixed sample test. Since the sample size is not predetermined in a SANOVA test, the experimenter would like to be assured that the sequential test can offer an equally discriminating test with smaller sample size than the corresponding fixed sample test. As previously discussed, such assurance can be obtained by examining the OC and ASN curves of the sequential test.

In this chapter an exact procedure is developed for obtaining the OC and ASN curves of a SANOVA test. This procedure is the first which yields exact results and is versatile enough so as to be used for tests with general regions. It is hoped that the procedure will be an invaluable tool for designing SANOVA tests.

In a SANOVA test the decision of acceptance can be made at any stage i, $i = 2, \ldots, m_0$. Thus, the probability of accepting the null hypothesis must be calculated as the sum of the probabilities of accepting at each state, $P_A{}^i$; i.e. $OC = \sum_{i=2}^{m_0} P_A{}^i$. Of course, these probabilities will depend upon the state of nature λ.

Unlike the fixed sample test, these probabilities cannot be obtained by simply integrating the distribution of the test statistic. One must remember that in sequential analysis the statistic at stage i only exists when the statistics at all previous stages have had values within the continuation region, i.e., $F_A^j < F_j < F_R^j$, $j = 2, \ldots, i-1$. Thus, the distribution of the test statistic at stage i is not a true probability distribution since its total probability content is not 1 (rather P_C^{i-1}). Were this distribution known it could be integrated to obtain P_A^i. Unfortunately this distribution cannot be obtained analytically.

However, the procedure developed in this chapter obtains a "truncated" density at stage i-1. Rather than utilizing the density of the test statistic F_i, this procedure utilizes the joint density of the sufficient statistics at stage i (i.e. each of the K sample means and the pooled estimate of the variance). From this joint density the density of F_i can be obtained which then can be integrated to yield P_A^i.

The joint density at stage i is obtained from the joint density at stage i-1 by applying Aroian's direct method of sequential analysis. This consists of determining the mapping of points at stage i-1 to those at stage i (where a point represents a value of the vector of sufficient statistics). This mapping describes how the statistics at stage i-1 are changed by the new observations to yield statistics at stage i. Thus for any given point, A, at stage i, there is a region of points, P, at stage i-1 which can be mapped into it.

Due to the nature of a sequential test, some points in P may result in a decision being made at stage i-1. If so, the point can not be mapped into A, since the test would terminate at stage i-1. Thus, for a sequential test the region of points, H, which can be mapped into A must include only those points in P which lie in the continuation region at stage i-1. Ultimately, this restriction will yield the desired "truncated" density for stage i (i.e., the total probability content is P_C^{i-1}).

As previously mentioned the statistics at stage i are transformations of the statistics at stage i-1 and the new observations taken at stage i. Suppose that the required number of sufficient statistics is n, and that the number of new observations taken at any stage is K (assuming one observation from each population would imply a test for the equality of K means). The above transformation would then be a transformation of n + k random variables (the statistics at stage i and the K new observations) to n random variables (the statistics at stage i). Since the dimensionality of the two sets of random variables is not the same, K surplus random variables must be introduced. These K surplus variables will be judiciously selected functions of the statistics at stage $i-1$ and new observations. This introduction of surplus random variables makes the transformation from an $n + K$ dimensional space (the statistics at stage $i-1$ and K new observations) to an $n+K$ dimensional space (the statistics at stage i and K surplus variables). The joint density of the statistics at stage i and K surplus variables can be found by calculus. The procedure is essentially equivalent to transforming variables in multiple integrals.

Finally, the desired density (of the joint distribution of the statistics at stage i) is obtained by performing a multiple integration of the joint density of the statistics at stage i and K surplus variables. The region of integration will be over all points contained in the set of points H.

The above discussion has given a brief outline of the "exact" procedure developed in this chapter of the thesis. The following sections describe the procedure in greater detail.

2.1 THE DIRECT METHOD OF SEQUENTIAL ANALYSIS

Aroian developed a general theory for obtaining the properties of a sequential test exactly (Aroian, 1968).

To determine the properties (usually only the OC curve) for a fixed sample test one needs to know the distribution of the test statistic for a given sample size n for different values of the parameter being tested. For example, in the fixed sample analysis of variance test where n observations are taken from each of k groups, the test statistic

$$F_{cal} = \frac{\sum_{i=1}^{k} (\bar{X}_i - \bar{\bar{X}})^2/(k-1)}{\sum_{i=1}^{k} \sum_{j-1}^{n} (X_{ij} - X_i)^2/N-k)}$$

is distributed as a noncontrol F variate with [k-1,N-k] degrees of freedom and noncentrality parameter $\xi = n\lambda$. The OC curve for a given value of the parameter, ξ, is then obtained by integrating this distribution over the acceptance region. For fixed sample ANOVA

$$OC(\xi) = \int_{\text{Acceptance region}} f_{k-1,N-k,\xi}(F_{cal})\, dF_{cal}$$

where $f_{v_1,v_2,\xi}(X)$ is the noncentral F density function.

The direct method recognizes as its primary principle that observations are taken in stages in sequential testing, and that for this reason a way must be found to calculate the distribution of the test statistic T_n, at stage n. In most cases T_n is not independent of $T_1, T_2, \cdots, T_{n-1}$, so that the marginal distribution of T_n must be obtained by integrating the joint distribution, i.e.,

$$h_n(T_n) = \int\cdots\int_I h(T_1, T_2, \cdots, T_{n-1}) dT_1 + T_2 \cdots dT_{n-1}$$

Since a sequential test is terminated whenever any $T_m \leq T_m{}^A$ or $T_m \geq T_m{}^R$; it is not possible to have a value of T_n if any $T_i \leq T_i^A$ or $T_i \geq T_i{}^R$ $i=2,\cdots,n-1$. Therefore, the direct method considers only the truncated distribution $f_n(T_n)$, where

$$f_n(t) = \Pr\{T_2{}^A < T_2 < T_2{}^R,\ T_3{}^A < T_3 < T_3{}^R, \cdots T_{n-1}{}^A < T_{n-1} < T_{n-1}{}^R, T_n = t\}.$$

Mathematically f_n is not a true "density" function since $\int f_n \neq 1$, but will still be referred to as a density.

When T_n is dependent upon T_{n-1} in the following manner

$$T_n = g_1(T_{n-1}) + g_2(X_{(n)})$$

with $X_{(n)}$ representing the new observation at stage n, g_1 and g_2

arbitrary functions, $f_n(T_n)$ can be obtained from $f_n(T_{n-1})$. Bahadur generalized the dependence by introducing the notion of a transitive sequence of statistics (Bahadur, 1954). A transitive sufficient sequence $\{T_n\}$ is a sequence such that for every $n > 1$ the conditional distribution of T_{n+1}, given the set of observations up to stage n, is identical to the conditional distribution of T_{n+1}, given T_n. So, in general, whenever T_n is transitive sufficient, $f_n(T_n)$ can be obtained from $f_{n-1}(T_{n-1})$.

Instead of obtaining $f_n(T_n)$ via integration of a joint distribution, the direct method obtains $f_n(T_n)$ directly from $f_{n-1}(T_{n-1})$, due to the transitivity of T_n.

At each stage n, the direct method calculates the probability of accepting H_0, $P_A{}^n$, and the probability of rejecting H_0, $P_R{}^n$, by integrating $f_n(T_n)$ over the appropriate regions. In mathematical terms,

$$P_A{}^n = \int_{T_n \leq \underline{T}_n} f_n(T_n)\,dT_n$$

$$P_R{}^n = \int_{T_n \geq \overline{T}_n} f_n(T_n)\,dT_n.$$

These probabilities depend upon the state of nature Θ, since the distribution of T_n depends upon the parameter Θ. So for any given Θ, the OC and ASN curves may be calculated as:

$$OC(\theta) = \sum_{i=2}^{m_0} P_A^{\,i}$$

$$ASN(\theta) = \sum_{i=2}^{m_0} i(P_A^{\,i} + P_R^{\,i}) = 1 + \sum_{i=1}^{m} P_C^{\,i}$$

where m_0 is the truncation point of the sequential test.

Usually the density $f_n(T_n)$ cannot be obtained from $f_{n-1}(T_{n-1})$ analytically, so that the procedure must be performed numerically. In numerical terms $f_n(T_n)$ represents a "grid" of T_n values calculated for each n from a "grid" of T_{n-1} values.

The direct method has been used in a variety of applications, including tests for the mean of a normal distribution with the standard deviation known (Aroian and Robison, 1969) and unknown (Schmee, 1974), and tests of the standard deviation of a normal distribution with mean known and unknown (Aroian, Gorge, Goss and Robison, 1975).

The following section contains a discussion of the application of the direct method to SANOVA.

2.2 APPLICATION OF THE DIRECT METHOD TO SANOVA

SANOVA is based on the statistic

$$V_n = n \sum_{i=1}^{k} (\bar{X}_{i(n)} - \bar{\bar{X}}_{(n)})^2 / \sum_{i=1}^{k} \sum_{j=1}^{n} (X_{ij} - \bar{X}_{i(n)})^2 .$$

In order to solve this problem by the direct method a transitive, sufficient sequence $\{T_n\}$ must be used. The sequence $\{V_n\}$ is not transitive, so one must use the multidimensional transitive sequence

$$\{T_n\} = \{X_{1(n)}, X_{2(n)}, \cdots, X_{k(n)}, S^2_{(n)}\}$$

where

$$X_{i(n)} = \sum_{j=1}^{n} X_{ij}$$

and

$$S_{(n)}{}^2 = \sum_{i=1}^{k} \sum_{j=1}^{n} \left[X_{ij} - \frac{X_{i(n)}}{n} \right]^2$$

(Hall, Wijsman, Ghosh, 1965). Similarly, one must now work with the joint distribution $f_n(X_{1(n)}, X_{2(n)}, \cdots X_{k(n)}, S_{(n)}{}^2)$. From this distribution $P_A{}^n$ and $P_R{}^n$ can be obtained by a k+1 dimensional integration,

$$P^n = \iiint_A \cdots \int f_n(X_{1(n)}, X_{2(n)}, \cdots, S^2_{(n)}) dX_{1(n)} \cdots dS^2_{(n)}$$

where A is the region in k+1 space such that

$$V_n = \sum_{i=1}^{k} \left\{ X_{i(n)} - \left[\frac{X_{1(n)} + \cdots + X_{k(n)}}{k} \right] \right\} / S^2(n) \leq V_n^{L}$$

and

$$P_R^{\ n} = \int \cdots \int_R \cdots \int f_n(X_{1(n)}, \cdots, X_{k(n)}, S^2_{(n)}) dX_{1(n)} \cdots dS^2_{(n)}$$

where R is the region such that $V_n \geq V_n^{u}$.

The problem lies in obtaining $f_n(T_n)$. If the first stage at which a decision can be made is $n_1 \geq 2$, then since $X_{1(n_1)}, X_{2(n_1)}, \cdots, X_{k(n_1)}$, and $S^2_{(n_1)}$ are all independent

$$f_{n_1}(T_{n_1}) = f_{n_1}\left(X_{1_{(n_1)}}, X_{2_{(n_1)}}, \ldots, X_{K_{(n_1)}}, S_{(n_1)}^{\ 2}\right)$$

$$= \phi\left[\frac{X_{1(n_1)} - n_1\mu_1}{\sigma}\right] \cdot \phi\left[\frac{X_{2(n_1)} - n_1\mu_2}{\sigma}\right] \cdots \phi\left[\frac{X_{k(n_1)} - n_1\mu_2}{\sigma}\right]$$

$$\sigma^2 f_{\chi^2 k(n_1-1)} \left(S^2_{(n_1)}\right)$$

where

$$\phi(X) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}X^2}$$

is the standard normal density function and

$$f_{\chi^2_v}(X) = \frac{X^{v/2-1}e^{-X/2}}{2^{v/2}\Gamma(v/2)}, \quad X \geq 0$$

is the χ^2 density function with v degrees of freedom. Since the power of the SANOVA test depends only upon λ, the density $f_{n_1}(T_{n_1})$ for given λ can be calculated by assuming $\mu_1 = \mu_2 = \cdots = \mu_k$, $\sigma = 1$ and

$$\mu_k = \sqrt{\frac{\lambda^*}{k-1}} .$$

The probabilities $P_A^{n_1}$ and $P_R^{n_1}$ need not be obtained by integration of $f_{n_1}(T_{n_1})$ since the distribution of V_{n_1} is known to be related to the noncentral F-distribution;

$$\frac{k(n_1-1)}{(k-1)}V_{n_1} \sim F_{k-1,k(n_1-1)}(n_1\lambda^*) . \quad \text{Therefore}$$

$$P_A^{n_1} = \int_0^{\frac{k(n_1-1)}{(k-1)}V_{n_1}^A} f_{k-1,k(n_1-1),n,\lambda^*}(X)\, dX$$

and

$$P_R^{n_1} = \int_{\frac{k(n_1-1)V_{n_1}^R}{(k-1)}}^{\infty} f_{k-1,k(n_1-1),n_1\lambda^*}(X)\, dX$$

These integrals are evaluated by the methods discussed in Appendix A.

To determine $f_{n_1+1}(T_{n_1+1})$ the direct method will be applied. Since $\{T_n\}$ is transitive, $f_{n_1+1}(T_{n_1+1})$ can be obtained directly from $f_{n_1}(T_{n_1})$. Suppose the following relationships exist between the elements of T_{n_1+1} and T_{n_1};

$$\begin{aligned} X_{1(n_1+1)} &= g_{11}(X_{1(n_1)}) + g_{21}(X_{(n_1+1)}) \\ &\vdots \\ X_{k(n_1+1)} &= g_{1k}(X_{k(n_1)}) + g_{2k}(X_{(n_1+1)}) \end{aligned}$$

$$S^2_{(n_1+1)} = g_{1k+1}(S^2_{(n_1)}) + g_{2k+1}(X_{(n_1+1)})$$

where g_{1i} and g_{2i}, $i=1,\cdots,k+1$ are arbitrary functions, and $X_{(n_1+1)}$ is the vector of new observations from stage n_1+1. The statistic T_{n_1+1} defines a transformation which maps points in the 2k+1 dimensional space of $T_{n_1}, X_{(n_1+1)}$ to the K+1 dimensional space of T_{n_1+1}. To make the transformation from a 2k+1 space to a 2k+1 space, the following additional variables will be defined

$$\left.\begin{aligned} E_{1(n_1+1)} &= g_{2\ k+2}(X_{(n_1+1)}) \\ &\vdots \\ E_{k(n_1+1)} &= g_{2\ 2k+1}(X_{(n_1+1)}) \end{aligned}\right\} : E_{n_1+1}$$

where the functions g_{2i}, $i=k+2,\cdots,2k+1$ are arbitrary functions. Since the transformation is now 2k+1 to 2k+1, the joint distribution of $X_{1(n_1+1)},\cdots,E_{k(n_1+1)}$ can be obtained. From this distribution the joint marginal distribution of $X_{1(n_1+1)},\cdots,S^2_{(n_1+1)}$ will be obtained by integrating out $E_{1(n_1+1)},\cdots,E_{k(n_1+1)}$. To obtain the joint distribution of T_{n_1+1} and E_{n_1+1}, one must first obtain the joint distribution of T_{n_1} and $X_{(n_1+1)}$. Since $X_{(n_1+1)}$ is independent of T_{n_1}, the joint distribution is simply the product of the respective densities; i.e.

$$g\left(X_{1(n_1)},\cdots,S^2_{(n_1)},X_{(n_1+1)}\right) = f_{n_1}\left(X_{1(n_1)},\cdots,S^2_{(n_1)}\right) \cdot f\left(X_{(n_1+1)}\right)$$

Then under certain conditions (which for this general discussion will be assumed to be true, but are dependent upon the functions g_{1i} and g_{2i}) the joint distribution of T_{n_1+1} and E_{n_1+1} is given by

$$f\left(T_{n_1+1},E_{n_1+1}\right) = g\Big(u_1\left(T_{n_1},X_{(n_1+1)}\right),\cdots,u_{2k+1}\left(T_{n_1},X_{(n_1+1)}\right)\Big)\,|J|$$

where $u_i(T_{n_1}, X_{(n_1+1)})$, i=1,···,2k+1 is the set of inverse transformations and |J| is the Jacobian for the transformation. As previously mentioned the density of $f_{n_1}(T_{n_1+1})$ can now be obtained as follows

$$f_{n_1}(T_{n_{1+1}}) = \iint_{R^*}\cdots\int f(T_{n_1+1}, E_{n_1+1})\,dE_{1(n_1+1)}\cdots dE_{k(n_1+1)}\ .$$

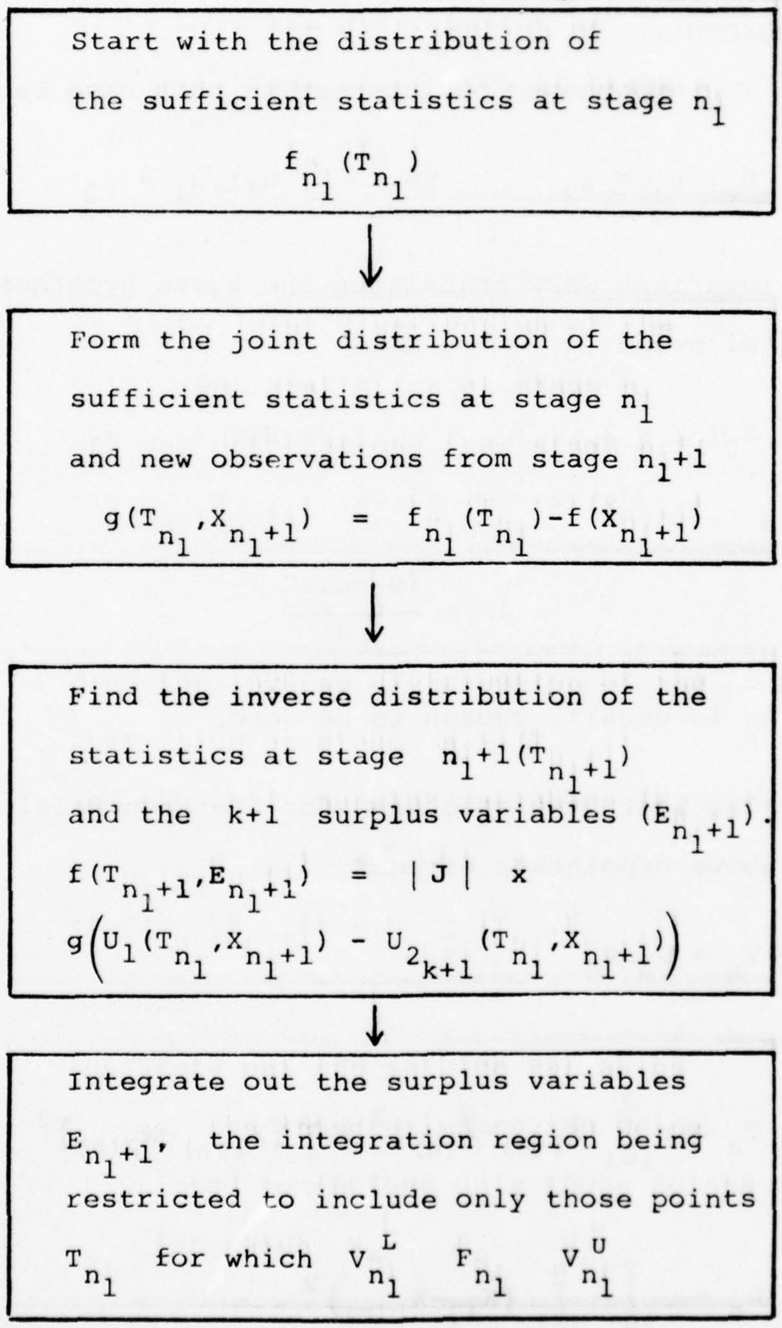
where R^* represents the integration region in k space.

The direct method restricts the set of points $T_{n_1}, X_{(n_1+1)}$ to be mapped into T_{n_1+1}, to include only those points for which $V_{n_1}^{L} < V_{n_1} < V_{n_1}^{u}$. This entire procedure can be represented diagramatically as shown in Figure 1.

The following section contains a complete derivation of the direct method procedure to obtain $f_n(T_n)$ from $f_{n-1}(T_{n-1})$ for the special case k=2. This discussion will specify the functions g_{1i}, g_{2i}, u_i and derive the integration region R^*.

FIGURE 1

The Direct Method Logic



2.3 DERIVATION FOR THE CASE $k = 2$

This section will derive a procedure for obtaining the properties of a SANOVA test for the special case when k=2, groups.

The hypotheses of interest in this case become:

$$H_0: \mu_1 = \mu_2 \qquad \text{vs.} \qquad H_1: \mu_1 \neq \mu_2$$

The invariant SPRT translates the above hypotheses into the following

$$H_0: \lambda \leq \lambda_0 \qquad\qquad H_1: \lambda \geq \lambda_1$$

where

$$\lambda = \frac{(\mu_1 - \mu_2)^2}{2\sigma^2}$$

and λ_0 is usually chosen to be zero.

The test statistic used for the sequential test of the above hypotheses becomes

$$V_n = T_n / D_n$$

where

$$T_n = n \sum_{i=1}^{2} (\bar{X}_{i(n)} - \bar{\bar{X}}_{(n)})^2 = \frac{n}{2} [X_{1(n)} - X_{2(n)}]^2$$

and

$$D_n = \sum_{i=1}^{2} \sum_{j=1}^{n} (X_{ij} - X_{i(n)})^2 .$$

To conduct such a test requires the specification of the following quantities: the null hypothesis, λ_0; the alternative hypothesis, λ_1; and a set of regions V_A^i, V_R^i, $i=1,\cdots,m_0$ (m_0 being the test truncation point). The regions may be any type (Wald or modified Wald) that specify: accept H_0 if at any i, $V_i \leq V_A^i$ and reject H_0 if $V_i \geq V_R^i$; otherwise continue sampling. The properties of such a test consist of the OC and ASN curves as functions of λ; i.e., OC (λ), and ASN (λ), $\lambda_0 \leq \lambda \leq \lambda_1$.

The direct method involves calculating for a given λ^*, $f_n(T_n)$ at each stage n, from which the probabilities P_A^n and P_R^A are obtained. Once the quantities P_A^i, P_R^i, $i=1,\cdots m_0$ have been calculated, the points $OC(\lambda^*)$ and $ASN(\lambda^*)$ may be obtained. The following discussion will pertain to obtaining P_A^i, P_R^i and thus the OC and ASN for a given state of nature $\lambda = \lambda^*$. Unfortunately, the statistic V_n is not transitive, and in order to conserve all the necessary information, one must resort to a transitive sufficient sequence, such as $\{T_n\} = \{W_n, Q_n, R_n\}$ where

$$W_n = \sum_{j=1}^{n} X_{1_j}$$

$$Q_n = \sum_{j=1}^{n} X_{2_j}$$

$$R_n = \sum_{i=1}^{n} \sum_{j=1}^{n} X_{1_j}^{2} .$$

The reduction from $T_n \to V_n$ is performed at each stage in the following manner

$$V_n = \frac{\left[W_n - Q_n\right]^2}{2\left[nR_n - W_n{}^2 - Q_n{}^2\right]} \quad . \quad (2.3.1)$$

The direct method involves calculating, for every stage n, the joint density $f_n(W_n, Q_n, R_n)$.

Suppose the first stage at which a decision can be made is $n_1 \geq 2$. The density of $f_{n_1}(W_{n_1}, Q_{n_1}, R_{n_1})$ is obtained as follows:

Let

$$W_{n_1} = n_1 X$$

$$Q_{n_1} = n_1 Y$$

$$R_{n_1} = D_{n_1} + n_1 X^2 + n_1 Y^2$$

where

$$X = \bar{X}_{1(n_1)}$$

$$Y = \bar{X}_{2(n_1)}$$

$$D = \sum_{j=1}^{2} \sum_{i=1}^{n_1} (X_{ij} - \bar{X}_{i(n_1)})^2 .$$

Since the quantities X, Y, D are all independent, their joint distribution is given by:

$$f(X,Y,D) = \left[\chi^2_{2(n_1-1)}(D_{n_1})\cdot\sigma^2\right]\phi\left[\left(\frac{\bar{X}_{1(n_1)}-\mu_1}{\sigma/\sqrt{n_1}}\right)\right]\cdot\phi\left[\left(\frac{\bar{X}_{2(n_1)}-\mu_2}{/\sqrt{n_1}}\right)\right].$$

Since this procedure is being used to find the properties of the test when $\lambda = \lambda^*$, and the test is invariant with respect to λ, we can let $\mu_1 = 0$, $\sigma = 1$, and

$$\mu_2 = \sqrt{\lambda^*} .$$

So this density may be expressed as

$$f(X,y,D_{n_1}) = \chi^2_{2(n_1-1)}(D_{n_1})\cdot\phi(\sqrt{n_1}\,\bar{X}_{1(n_1)})\cdot\phi\left(\sqrt{n_1}\,(X_{2(n_1)}-\sqrt{\lambda^*})\right).$$

From this density we can determine the joint distribution of $W_{n_1}, Q_{n_1}, R_{n_1}$. The set of inverse transformations is given by

$$X = \frac{1}{n_1} W_{n_1}$$

$$Y = \frac{1}{n_1} Q_{n_1}$$

$$D = R_{n_1} - \frac{1}{n_1} W_{n_1}^{\;2} - \frac{1}{n_1} Q_{n_1}^{\;2}$$

which has a Jacobian

$$J = \begin{vmatrix} \frac{1}{n_1} & 0 & 0 \\ 0 & \frac{1}{n_1} & 0 \\ -\frac{2}{n_1} W_{n_1} & -\frac{2}{n_1} Q_{n_1} & 1 \end{vmatrix} = \frac{1}{n_1^{\;2}}$$

Since this transformation is one-to-one

$$f_{n_1}(W_{n_1},Q_{n_1},R_{n_1}) = \left(\frac{1}{n_1^{\,2}}\right)\chi^2_{2(n_1-1)}\left[R_{n_1}-\frac{1}{n_1}W_{n_1}^{\,2}-\frac{1}{n_1}Q_{n_1}^{\,2}\right]$$

$$\cdot\ \phi\left(\sqrt{n_1}\ (\tfrac{1}{n_1}W_{n_1})\right)\cdot\phi\left(\sqrt{n_1}\ (\tfrac{1}{n_1}Q_{n_1}-\sqrt{\lambda^*}\ )\right).$$

From this density, $F_{n_2}(W_{n_2},Q_{n_2},R_{n_2})$ will be obtained, where $n_2 = n_1+1$.

First consider the following functional relationships between the statistics at stage n_1 and stage n_2:

$$W_{n_2} = W_{n_1} + X_{1n_2} \qquad (2.3.2)$$

$$Q_{n_2} = Q_{n_1} + X_{2n_2}$$

$$R_{n_2} = R_{n_1} + X^2_{1n_2} + X^2_{2n_2}$$

The statistics are changed from stage n_1 to stage n_2 by two new observations, X_{1n_2} from group 1 and X_{2n_2} from group 2. Since X_{1n_2} and X_{2n_2} are independent of W_{n_1}, Q_{n_1}, R_{n_1}, the joint distribution of X_{1n_2}, X_{2n_2}, W_{n_1}, Q_{n_1}, R_{n_2} is simply

$$f^P_{n_1}(W_{n_1},Q_{n_1},R_{n_1},X_{1n_2},X_{2n_2}) = f_{n_1}(W_{n_1},Q_{n_1},R_{n_1})\cdot\phi(X_{1n_2})\cdot\phi(X_{1n_2}-\sqrt{\lambda^*}\ ).$$

Equations (2.3.2) represent a transformation from the 5 dimensional space of W_{n_1}, Q_{n_1}, R_{n_1}, X_{1n_2}, X_{2n_2} to the 3 dimensional space of W_{n_2}, Q_{n_2}, R_{n_2}. A transformation from

5 dimensional space to 5 dimensional space can be achieved by introducing the surplus variables Z and U, yielding the following transformation, T:

$$
\begin{aligned}
W_{n_2} &= W_{n_1} + X_{1n_2} \\
Q_{n_2} &= Q_{n_1} + X_{2n_2} \\
R_{n_2} &= R_{n_1} + X^2_{1n_2} + X^2_{2n_2} \\
Z &= X_{1n_2} \\
U &= X_{2n_2}
\end{aligned}
\qquad (2.3.3)
$$

The set of inverse transformations, T^{-1}, is then given by

$$
\begin{aligned}
W_{n_1} &= W_{n_2} - Z \\
Q_{n_1} &= Q_{n_2} - U \\
R_{n_1} &= R_{n_2} - Z^2 - U^2 \\
X_{1n_2} &= Z \\
X_{2n_2} &= U
\end{aligned}
\qquad (2.3.4)
$$

This transformation has a Jacobian matrix of the following form

$$
J = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2Z & -2U & 1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ B_{21} & B_{22} \end{bmatrix}
$$

so that

$$|J| = |I|\ |B_{22}-B_{21}I\ 0| = |I|\ |B_{22}| = |B_{22}| = 1.$$

Thus, the joint distribution of W_{n_2}, Q_{n_2}, R_{n_2}, Z, U is given by

$$f_{n_2}(W_{n_2}, Q_{n_2}, R_{n_2}, Z, U) = f^P_{n_1}(W_{n_2}-Z, Q_{n_2}-U, R_{n_2}-Z^2-U^2, Z, U) \ . \tag{2.3.5}$$

The marginal joint distribution of W_{n_2}, Q_{n_2}, R_{n_2} is obtained by integrating (2.3.5) with respect to U and Z over the appropriate regions. Ordinarily this region consists of all possible values of U and Z, $-\infty < U < \infty$, $-\infty < Z < \infty$; so that the marginal is obtained by

$$f(W_{n_2}, Q_{n_2}, R_{n_2}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{n_1}{}^P[W_{n_2}-Z, Q_{n_2}-U, R_{n_2}-Z^2-U^2, Z, U)\,dZdU$$

By substitution this integration becomes

$$f\left(W_{n_2}, Q_{n_2}, R_{n_2}\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{ \frac{1}{n_1^2} \chi^2_{2(n_1-1)} \left[R_{n_2} - Z^2 - U^2 - \frac{1}{n_1}\left(W_{n_2}-Z\right)^2 - \frac{1}{n_1}\left(Q_{n_2}-U\right)^2\right]\right.$$

$$\cdot\ \phi\left[\sqrt{n_1}\ \frac{1}{n_1}\left(W_{n_2}-Z\right)\right]\ \cdot\ \phi\left[\sqrt{n}\ \frac{1}{n}\left(Q_{n_2}-U\right)\right] - \sqrt{\lambda^*}$$

$$\left.\cdot\ \phi\left[(Z)\cdot\phi\left(U-\sqrt{\lambda^*}\right)\right]\right\} dZdU$$

Since the chi-squared density function is only defined for positive values, the integration region of U and Z must be chosen so that

$$R_{n_2} - Z^2 - U^2 - \frac{1}{n_1}\left(W_{n_2}-Z\right)^2 - \frac{1}{n_1}\left(Q_{n_2}-U\right)^2 \geq 0$$

Therefore, for a given value of U, say U*, the range of allowable Z values is given by the following roots.

$$Z_{\text{limits}} = \frac{W_{n_2}}{n_1} \pm \sqrt{\frac{n_1}{n_1+1}\left[R_{n_2} - \frac{W_{n_2}^{\ 2}}{n_1+1} - \frac{Q_{n_2}^{\ 2}}{n_1+1}\right] - \left(U^* - \frac{Q_{n2}}{n_1+1}\right)^2} \qquad (2.3.6).$$

Let the smaller root (the lower limit of Z integration) be denoted Z_L and the larger (the upper limit of Z) by Z_U.

The limits of the U integration are given by:

$$U_{limits} = \frac{Q_{n_2}^2}{n_1} \pm \sqrt{\frac{n_1}{n_1+1}\left[R_{n_2} - \frac{W_{n_2}^2}{n_1+1} - \frac{Q_{n_2}^2}{n_1+1}\right]} \quad (2.3.7).$$

Let the smaller root (the lower limit of U integration) be denoted by U_L and the larger by U_U (the upper limit of U).

It should be noted that equations (2.3.6) and (2.3.7) have solutions only if $R_{n_2} - \frac{W_{n_2}^2}{n_1+1} - \frac{Q_{n_2}^2}{n_1+1} \geq 0$.

If this is not the case, all the above limits can be regarded as zero, so that $f\left(W_{n_2}, Q_{n_2}, R_{n_2}\right) = 0$.

For all points R_{n_2}, W_{n_2}, Q_{n_2}, such that $R_{n_2} - \frac{W_{n_2}^2}{n_1+1} - \frac{Q_{n_2}^2}{n_1+1} \geq 0$, the joint density is obtained by more

$$f\left(W_{n_2},Q_{n_2},R_{n_2}\right) = \int_{U_L}^{U_U}\int_{Z_L}^{Z_U} f_{n_1}{}^{P}\left(W_{n_2}-Z,Q_{n_2}-U,R_{n_2}-Z^2-U^2,Z,U\right)dZdU. \quad (2.3.8)$$

The result of this integration yields

$$f(W_{n_2}, Q_{n_2}, R_{n_2}) = \frac{1}{(n_1+1)^2} \chi^2{}_{2(n_1)}\left[R_{n_2} - \frac{1}{(n_1+1)} W_{n_2}{}^2 - \frac{1}{(n_1+1)} Q_{n_2}{}^2\right]$$

$$\phi\left[\sqrt{n_1+1}\left(\frac{1}{n_1+1} W_{n_2}\right)\right] \cdot \phi\left(\sqrt{n_1+1}\left(\left(\frac{1}{n_1+1}\left(Q_{n_2} - \sqrt{\lambda^*}\right)\right)\right)\right).$$

This is the density which results if the first step at which a decision can be made is $n_2 = n_1 + 1$.

However, the direct method restricts the set of points $(W_{n_1}, Q_{n_1}, R_{n_1}, X_{2n_2}, X_{2n_2})$ to consist of only those points such that $V_{n_1}{}^A < V_{n_1} < V_{n_1}{}^R$. The limits in equation (2.3.8) do not consider this restriction. The result of applying this restriction involves altering the U and Z limits of integration. The integration region consists of all point U,Z such that:

$$(1) \qquad R_{n_2} - Z^2 - U^2 - \frac{1}{n_1}\left(W_{n_2}-Z\right)^2 - \frac{1}{n_1}\left(Q_{n_2}-U\right)^2 \geq 0$$

$$(2) \qquad V_A^{\,n_1} < \frac{\left[W_{n_2}-Z-Q_{n_2}+U\right]^2}{2\left[n_1\left(R_{n_2}-Z^2-U^2\right)-\left(W_{n_2}-Z\right)^2-\left(Q_{n_2}-U\right)^2\right]} < V_R^{\,n_1}$$

From these constraints integration limits U_U, U_L and Z_U, Z_L can be obtained, such that

$$f_{n_2}\left(W_{n_2}, Q_{n_2}, R_{n_2}\right) = \int_{U_L}^{U_U}\int_{Z_L}^{Z_U} f_{n_1}^{\,P}\left(W_{n_1}-Z,\ Q_{n_2}-U,\ R_{n_2}-U^2-Z^2,\ Z,U\right)dZdU.$$

Explicit expressions for these limits can be best derived geometrically.

Let $V_A = V_A^{\,n_1}$ and $V_R = V_R^{\,n_1}$ such that at stage n_1 H_0 is accepted if

$$\frac{\left[W_{n_1}-Q_{n_1}\right]^2}{2\left[n_1 R_{n_1}-W_{n_1}^{\,2}-Q_{n_1}^{\,2}\right]} \leq V_A \qquad (2.3.9)$$

and H_0 is rejected if

$$\frac{\left[W_{n_1}-Q_{n_1}\right]^2}{2\left[n_1R_{n_1}-W_{n_1}{}^2-Q_{n_1}{}^2\right]} \geq V_R . \quad (2.3.10)$$

Solving the above expressions, when the equalities are satisfied, yields the following two surfaces:

$$B_A: \quad R_{n_1} = \frac{(2V_A+1)W_{n_1}{}^2+(2V_A+1)Q_{n_1}{}^2-2W_{n_1}Q_{n_1}}{2n_1V_A}$$

$$R_{n_1} = C_AW_{n_1}{}^2+C_AQ_{n_1}{}^2-2P_AW_{n_1}Q_{n_1}$$

and

$$B_R: \quad R_{n_1} = \frac{(2V_R+1)W_{n_1}{}^2+(2V_R+1)Q_{n_1}{}^2-2W_{n_1}Q_{n_1}}{2n_1V_R}$$

$$R_{n_1} = C_RW_{n_1}{}^2+C_RQ_{n_1}{}^2-2P_RW_{n_1}Q_{n_1}$$

where

$$C_A = \frac{2V_A+1}{2n_1V_A} \quad \text{and} \quad P_A = \frac{1}{2n_1V_A}$$

with similar expressions for C_R and P_R.

The surface B_A is an elliptic paraboloid since the discriminant, D

$$D = 4P_A{}^2-4C_A{}^2 = 4\left(1-(2V_A+1)^2\right)$$

will always be negative for $V_A > 0$.

Similarly the surface B_R is an elliptic paraboloid, usually containing the surface B_A. All points lying between these two surfaces constitute the continuation region, C_{n_1}.

Next consider the surface induced by the transformation T. This surface contains all points in T_{n_1} space, $(W_{n_1}, Q_{n_1}, R_{n_1})$, which can be mapped into some point in T_{n_2} space $(W_{n_2} = a,\ Q_{n_2} = b,\ R_{n_2} = c)$.

Since

$$a = W_{n_1} + X_{1n}$$

$$b = Q_{n_1} + X_{2n_1}$$

$$c = R_{n_1} + X_{1n_1}^{\ 2} + X_{2n_1}^{\ 2};$$

this surface is given by

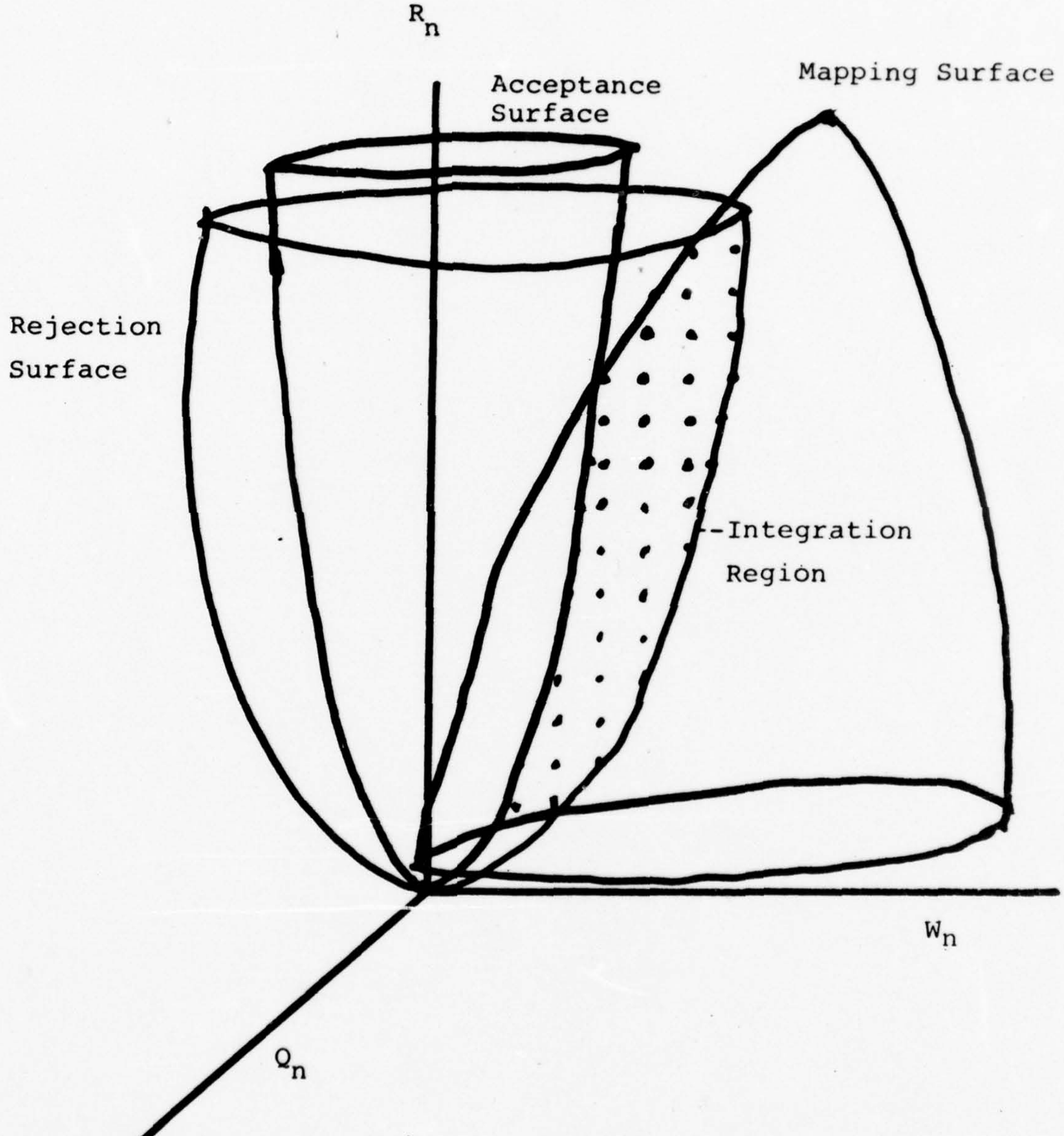
$$c = R_{n_1} + (a-W_{n_1})^2 + (b-Q_{n_1})^2 : \quad P$$

The surface P is an inverted elliptic paraboloid.

The intersection of the continuation region C_{n_1}, with the mapping surface P, determines the integration region for equation (2.3.8). This region is shown in Figure 2, and depends upon the point (a,b,c) as well as the regions V_A and V_R.

FIGURE 2

THE DIRECT METHOD INTEGRATION REGION



If this region is projected onto the W_{n_1}, Q_{n_1} axes one obtains the set of all W_{n_1}, Q_{n_1} points for which W_{n_1}, Q_{n_1}, R_{n_1} are contained on both the continuation surface and the mapping surface. Let this set be denoted by H;

$$H: \{W_{n_1}, Q_{n_1}\} \text{ s.t. } (W_{n_1}, Q_{n_1}, R_{n_1}) \in C_{n_1} \text{ and } P.$$

The integration region for U and Z, for a given point $W_{n_2} = a$, $Q_{n_2} = b$, $R_{n_2} = c$ will consist of all points U, Z such that the doublet $W = a - Z$, $Q = b - U$ is contained in the set H. Let this set be denoted by G,

$$G: \{Z, U\} \text{ such that } (a-Z, b-U) \in H.$$

Since there is a one-to-one relationship between the sets G and H, the limits U_U, U_L, Z_U, A_L can be found by inspection of the set H.

An analytic expression for H can be found by projecting $B_A \cap P$ and $B_R \cap P$ onto the W_{n_1}, Q_{n_1} axes. Since

$$B_R: \; R_{n_1} = C_R W_{n_1}^{\;2} + C_R Q_{n_1}^{\;2} - 2P_R W_{n_1} Q_n$$

$$P: \; R_{n_1} = C - (W_{n_1} - a)^2 - (Q_{n_1} - b)^2$$

the projection of $B_R \cap P$ onto the Q_{n_1}, W_{n_1} axes is obtained by substitution, yielding the curve RE:

$$(C_R+1)\, W_{n_1}^{\ 2} + (C_R+1)\, Q_{n_1}^{\ 2} - 2aW_{n_1} - 2bQ_{n_1} - 2P_R W_{n_1} Q_{n_1} = c - a^2 - b^2$$

Similarly the projection of $B_A \cap P$ onto the Q_{n_1}, W_{n_1} axes yields the curve AE:

$$(C_A+1)\, W_{n_1}^{\ 2} + (C_A+1)\, Q_{n_1}^{\ 2} - 2aW_{n_1} - 2bQ_{n_1} - 2P_A W_{n_1} Q_{n_1} = c - a^2 - b^2$$

Both RE and AE are equations of an ellipse; and since the coefficients of $W_{n_1}^{\ 2}$ and $Q_{n_1}^{\ 2}$ are equal the axes of the ellipse are rotated 45°. Thus, the set H consists of all points which are inside RE and outside AE. Figure **3** shows the integration region for a particular case. Many such integration regions can arise depending upon the values of a, b, c, V_A, V_R. However, the region will always be one of the following:*

I. A point not possible at step (n+1).

This consists of all points $(W_{n+1}, Q_{n+1}, R_{n+1})$ such that $R_{n+1} - \frac{W_{n+1}^{\ 2}}{n+1} - \frac{Q_{n+1}^{\ 2}}{n+1} \leq 0$, which means $f(W_{n+1}, Q_{n+1}, R_{n+1}) = 0$. All future discussions about integration regions will pertain to all points possible at step (n+1).

*For examining the types of integration regions that can arise, the following less cumbersome notation will be used: $n = n_1$.

$$RE': \quad \left(\frac{n+1}{n}\right)\left[Q_n - \frac{\sqrt{2}\,(a+b)\,n}{2(n+1)}\right]^2 + \left(\frac{V_R+nV_R+1}{nV_R}\right)\left[W_n' - \frac{\sqrt{2}\,(a-b)\,nV_R}{2(V_R+nV_R+2}\right]^2$$

$$= \left\{C-a^2-b^2 + \frac{n(a+b)^2}{2(n+1)} + \frac{nV_R(a-b)^2}{2(V_R+nV_R+1)}\right\}$$

Note that RE will only be defined if the following inequality is satisfied

$$C-a^2-b^2 + \frac{n(a+b)^2}{2(n+1)} + \frac{nV_R(a-b)^2}{2(V_R+nV_R+1)} > 0$$

or

$$C - \frac{a^2}{n+1} - \frac{b^2}{n+1} - \frac{(a-b)^2 n}{2(n+1)(V_R+nV_R+1)} > 0$$

Whenever this inequality is not satisfied, B_R never intersects P. This means that none of the points that can be mapped into a, b, c lie in the continuation region, resulting in f(a,b,c) = 0.

RE′ is an ellipse with center at

$$Q_n' = \frac{\sqrt{2}\,(a+b)\,n}{2(n+1)}$$

$$W_n' + \frac{\sqrt{2}\,(a-b)\,nV_R}{2(V_R+nV_R+1)}$$

II. Case when only a decision to reject H_0 is possible at stage n.

When V_A is a number less than zero it is not possible to accept H_0 at stage n, since the left hand side of equation (2.3.9) can never be less than zero. Assuming $V_R<\infty$, the only decision that can be made at stage n, is the decision to reject H_0.

If no decision could be made at stage n, $V_R=\infty$ and $V_A<0$, and as previously discussed, the set H consists of all W_n, Q_n inside the following circle, RE_∞:

$$\left(W_n-\frac{na}{n+1}\right)^2 + \left(Q_n-\frac{nb}{n+1}\right)^2 = \frac{n}{n+1}\left[C-\frac{a^2}{n+1} - \frac{b^2}{n+1}\right]$$

This is the set of all W_n, Q_n coordinates of all points W_n, Q_n, R_n which can be mapped into the point ($W_{n+1} = a$, $Q_{n+1} = b$, $R_{n+1} = c$) and which satisfy

$$0 < \frac{\left[W_n - Q_n\right]^2}{2\left[nR_n - W_n^2 - Q_n^2\right]} < \infty .$$

Once the W_n, Q_n limits of the set H are obtained, the U, Z limits are obtained by the following relationship between the sets H and G

$$U = b - Q_n$$

$$Z = a - W_n .$$

For the case when no decision can be made at stage n, these limits become those given by equations (2.3.6) and (2.3.7).

Whenever a decision to reject H_0 at stage n is possible, a set of points, W_n^*, Q_n^*, R_n^*, in W_n, Q_n, R_n space exist such that

$$V(W_n^*, Q_n^*, R_n^*) = \frac{\left[W_n^* - Q_n^*\right]^2}{2\left[nR_n^* - W_n^{*2} - Q_n^{*2}\right]} \geq V_R < \infty .$$

Since these points are not included in the set H, the set H now consists of all W_n, Q_n inside the following ellipse, RE:

$$(C_R+1)W_n^2 + (C_R+1)Q_n^2 - 2aW_n - 2bQ_n - 2P_R W_n Q_n = c - a^2 - b^2$$

To compare the two curves RE_∞ and RE consider the following rotated coordinate system:

$$Q_n' = \frac{\sqrt{2}}{2}\left[Q_n + W_n\right]$$

$$W_n' = \frac{\sqrt{2}}{2}\left[-Q_n + W_n\right]$$

The curves in this new coordinate system become

$$RE_\infty' : \left[Q_n' - \frac{\sqrt{2}\, n(a+b)}{2(n+1)}\right]^2 + \left[W_n' - \frac{\sqrt{2}\, n(a-b)}{2(n+1)}\right]^2 = \frac{n}{n+1}\left\{C - \frac{a^2}{n+1} - \frac{b^2}{n+1}\right\}$$

and

$$RE': \left(\frac{n+1}{n}\right)\left[Q_n' - \frac{2\ (a+b)n}{2(n+1)}\right]^2 + \left(\frac{V_R+nV_R+1}{nV_R}\right)\left[W_n' - \frac{2\ (a-b)nV_R}{2(V_R+nV_R+2)}\right]^2$$

$$= \left\{C-a^2-b^2 + \frac{n(a+b)^2}{2(n+1)} + \frac{nV_R(a-b)^2}{2(V_R+nV_R+1)}\right\}$$

Note that RE will only be defined if the following inequality is satisfied

$$C-a^2-b^2 + \frac{n(a+b)^2}{2(n+1)} + \frac{nV\ (a-b)^2}{2(V\ +nV\ +1} > 0$$

or

$$C - \frac{a^2}{n+1} - \frac{b^2}{n+1} - \frac{(a-b)^2 n}{2(n+1)(V_R+nV_R+1} > 0$$

Whenever this inequality is not satisfied, B_R never intersects P. This means that none of the points that can be mapped into a, b, c lie in the continuation region, resulting in f(a,b,c) = 0.

RE′ is an ellipse with center at

$$Q_n' = \frac{\sqrt{2}(a+b)n}{2(n+1)}$$

$$W_n' = \frac{\sqrt{2}(a-b)nV_R}{2(V_R+nV_R+1)}$$

and minor axis along the W_n' axis. The circle RE_∞' contains the ellipse RE''. This can be seen by substituting the ellipse end points into the equation of the circle. Consider first the end points given by

$$W_n' = \frac{\sqrt{2}(a-b)nV_R}{2(V_R+nV_R+1)}$$

$$Q_n' = \frac{\sqrt{2}(a+b)n}{n+1} \pm \sqrt{\frac{n}{n+1}\, C\left\{- \frac{a^2}{n+1} - \frac{b^2}{n+1} - \frac{(a-b)^2 n}{2(n+1)(V_R+nV_R+1)}\right\}}$$

Substituting these points into RE_∞' yields

$$\left[\frac{n}{n+1}\left\{C - \frac{a^2}{n+1} - \frac{b^2}{n+1} - \frac{(a-b)^2 n}{2(n+1)(V\ +nV\ +1)}\right\}\right] + \frac{n^2(a-b)^2}{2(n+1)^2(V\ +nV\ +1)^2}$$

$$= \frac{n}{n+1}\left\{C - \frac{a^2}{n+1} - \frac{b^2}{n+1}\right\}$$

which simplifies to

$$\frac{n^2(a-b)^2}{2(n+1)^2(V_R+nV_R+1)}\left[\frac{1}{(V_R+nV_R+1)} - 1\right]$$

Since this quantity will always be less than or equal to zero, this set of end points will be contained in RE_∞'.

Next consider the set of end points given by

$$W_n' = \frac{\sqrt{2}(a-b)nV_R}{2(V_R+nV_R+1)} \pm \sqrt{\frac{nV_R}{(V_R+nV_R+1}\left\{C - \frac{a^2}{n+1} - \frac{b^2}{n+1} - \frac{(a-b)^2 n}{2(n+1)(V_R+nV_R+1)}\right\}}$$

$$Q_n' = \frac{\sqrt{2}(a+b)n}{2(n+1)}$$

Substituting this into RE_∞' yields the following expression

$$\left\{-G - \frac{H(a-b)^2}{2} \mp \sqrt{2}(a-b)\sqrt{HG}\right\}\left[\frac{n}{(n+1)(V_R+nV_R+1}\right]$$

where

$$H = \frac{nV_R}{V_R+nV_R+1}$$

and

$$G = C - \frac{a^2}{n+1} - \frac{b^2}{n+1} - \frac{(a-b)^2 H}{2}$$

The above expression may be rearranged to yield

$$\left[\sqrt{G} \pm (a-b)\sqrt{H}\,\frac{\sqrt{2}}{2}\right]^2 \left[\frac{-n}{(n+1)(V+nV+1)}\right]$$

Since this quantity will always be zero or negative, this set of end points will also be contained in RE_∞'.

Since a decision to reject H_0 can be made at stage n, the integration region is reduced to the set of all points contained in ellipse RE, as shown in Figure 3. To determine the U integration limits on the integral (2.3.8) first requires finding the Q_n limits of RE.

Letting Q_{n_U} be the maximum value of Q_n and Q_{n_L} the minimum value of Q_n on this ellipse, the integration limits for U are given by

$$U_L = b - Q_{nu}$$

$$U_U = b - Q_{n_L} \quad . \tag{2.3.11}$$

Explicit expressions for Q_{n_U} and Q_{n_L} for given a, b, c, V_R may be obtained by noting that at both points

$$\frac{dW_n}{dQ_n} = \infty \quad .$$

Therefore an expression for $\frac{dW_n}{dQ_n}$ must be found and examined to see at what points it approaches infinity.

The derivative $\frac{dW_n}{dQ_n}$ is given by

$$\frac{dW_n}{dQ_n} = \frac{P_R}{C_R+1} \pm \frac{\frac{1}{2}\left[\frac{2(a+P_RQ_n)}{(C_R+1)^2} + \frac{2b-2(C+1)Q_n}{(C_R+1)}\right]}{\left[\frac{(a+P_RQ_n)^2}{(C_R+1)^2} - \left(\frac{a^2+b^2-C-2bQ_n+(C_R+1)Q_n^2}{(C_R+1)}\right)\right]^{\frac{1}{2}}}$$

In order for this derivative to approach infinity the denominator must be equal to zero. Equating the numerator to zero yields,

$$\left[\frac{(a+P_RQ_n)^2}{(C_R+1)^2} - \left(\frac{a^2+b^2-C-2bQ_n+(C_R+1)Q_n^2}{(C_R+1)}\right)\right]^{\frac{1}{2}} = 0$$

and solving for Q_n yields

$$Q_n = \frac{b(C+1)+aP_R}{(C_R+1)^2-P_R^2} \pm \sqrt{\left[\frac{b(C_R+1)+aP_R}{P_R^2-C_R+1^2}\right]^2 \pm \left[\frac{a^2-(C+1)(a^2+b^2-c)}{P_R^2-(C+1)^2}\right]} \qquad (2.3.12)$$

The larger root will be Q_{n_U} and the smaller will be Q_{n_L}.

The limits W_{n_L} and W_{n_U} depend upon the value of Q_n. For a given value $Q' = b - U'$; $Q_{n_L} \leq Q' \leq Q_{n_U}$ the limits for W_n can be found by solving the equation of the ellipse RE, yielding

$$W_n = \frac{(a+P_R Q')}{(C_R+1)} \pm \sqrt{\left[\frac{(a+P_R Q')^2}{(C_R+1)^2} - \frac{a^2+b^2-c-2bQ'+(C_R+1)Q'^2}{C_R+1}\right]} .$$

(2.3.13)

Letting W_{n_L} be the smaller root and W_{n_U} the larger, the Z integration limits become

$$Z_U = a - W_{n_L} \qquad (2.3.14)$$

$$Z_L = a - W_{n_U} \quad .$$

In summary, whenever $V_A < 0$ and $V_R < \infty$ the integration limits U_L, U_U for a point $W_n = a$, $Q_n = b$, $R_n = c$, can be obtained from equation (2.3.11), where Q_{n_L} and Q_{n_U} are values obtained from equation (2.3.12). The limits Z_L and Z_U depend upon the value of U; for a given value U' the limits are obtained from equation (2.3.14) where W_{n_L} and W_{n_U} are obtained from equation (2.3.13).

III. Case when only a decision to accept H_0 is possible at stage n.

When $V_R = \infty$ a decision to reject H_0 cannot be made at stage n. If $V_R = \infty$,

$$C_R = \frac{2V_R+1}{2nV_R} = \frac{1}{n}$$

and

$$P_R = \frac{1}{2nV_R} = 0$$

so the ellipse B_R becomes:

$$R_n = \left(1+\frac{1}{n}\right) W_n^2 + \left(1+\frac{1}{n}\right) Q_n^2.$$

The projection of $B_R \cap P$ onto the W_n, Q_n axes yields, RE:

$$\left(1+\frac{1}{n}\right) W_n^2 + \left(1+\frac{1}{n}\right) Q_n^2 - 2aW_n - 2bQ_n = c - a^2 - b^2,$$

which is now the equation of a circle with center at

$$W_n = \frac{a}{1+\frac{1}{n}} = \frac{na}{n+1}$$

$$Q_n = \frac{b}{1+\frac{1}{n}} = \frac{nb}{n+1}$$

and

$$\text{radius} = \sqrt{\frac{nc}{n+1} - \frac{na^2}{(n+1)^2} - \frac{nb^2}{(n+1)^2}}\ .$$

The equation for AE is still given by:

$$(C_A+1)\,W_n^{\;2} + (C_A+1)\,Q_n^{\;2} - 2aW_n - 2bQ_n - 2P_AW_nQ_n \;=\; c - a^2 - b^2$$

which is the equation of an ellipse with center at

$$Q_n \;=\; \frac{b(C_A+1)}{(C_A-P_A+1)(C_A+P_A+1)}$$

$$W_n \;=\; \frac{a(C_A+1)}{(C_A-P_A+1)(C_A+P_A+1)}$$

In this situation the set H consists of all W_n, Q_n which lie outside the curve AE yet inside RE.

By equating the left hand sides of RE and AE one obtains the following equation:

$$W_n^{\;2} + Q_n^{\;2} - 2W_nQ_n \;=\; 0$$

or

$$(W_n - Q_n)^2 \;=\; 0$$

This means that the two curves, AE and RE, will intersect only at the points where $W_n = Q_n$. Substituting this into RE yields

$$2\left(1+\frac{1}{n}\right)Q_n^{\;2} - 2(a+b)Q_n \;=\; c - a^2 - b^2$$

Solving for Q_n yields

$$\frac{(a+b) \pm \sqrt{(a+b)^2 - 2\left(1+\frac{1}{n}\right)\left(a^2+b^2-c\right)}}{2\left(1+\frac{1}{n}\right)}$$

At this point, it must be noted that there are a variety of ways in which the curves AE and RE can intersect. The above derivations have shown that when only a decision to accept H_0 is possible at stage n, the curves will intersect along the line $W_n = Q_n$. The specific points of intersection are given by the previous equation. This equation may yield zero, one, or two distinct intersection points, depending upon the value of the discriminant; i.e.

$$(a+b)^2 - 2\left(1+\frac{1}{n}\right)\left(a^2+b^2-c\right).$$

This equation reveals that the number of intersection points depends solely upon the point a, b, c.

Each of the intersection possibilities (i.e., zero, one, or two intersection points) indicates a different geometric relationship between AE and RE; which means that each results in a different $\mathcal{U}$, $\mathcal{Z}$ integration region. Thus, to obtain the entire density (i.e. the density at all points) requires deriving the integration regions of all the possible intersection situations. Each of these possibilities will now be considered.

Whenever the following condition occurs

$$(a+b)^2 - 2\left(1+\frac{1}{n}\right)(a^2+b^2-c) \leq 0$$

the two curves RE and AE will never intersect. This indicates one of the following geometric relationships must exist:

(1) the curve RE contains AE

(2) the curve AE contains RE

(3) the curves AE and RE contain no points in common, given they don't intersect.

Situation (3) will occur only if neither curve contains the other's center. This is equivalent to satisfying the following inequalities (if they don't intersect):

$$\left[\frac{n(a-b)^2}{2V_A(n+1)^2}\right] - \left[c - \frac{a^2}{n+1} - \frac{b^2}{n+1}\right] > 0 \qquad (2.3.15)$$

and

$$\left(\frac{(a^2+b^2)n}{n+1}\right)\left[\frac{1}{4[(n+1)V_A+1]^2} - 1\right] - \left(c - a^2 - b^2\right) > 0 \qquad (2.3.16)$$

When these inequalities are satisfied the set H consists of all W_n, Q_n contained inside RE. This is shown in Figure 4. The Q_n end points of this circle are given by:

$$\frac{nb}{n+1} \pm \sqrt{\frac{nc}{n+1} - \frac{na^2}{(n+1)^2} - \frac{nb^2}{(n+1)^2}} \qquad (2.3.17)$$

Let the smaller root be denoted by Q_{n_L} and the larger by Q_{n_U}; then the U integration limits are given by

$$U_U = b - Q_{n_L} \qquad (2.3.18)$$

$$U_L = b - Q_{n_U}$$

For a given value of U, say U^* the Z integration limits are given by

$$Z_U = a - W_{n_L} \qquad (2.3.19)$$

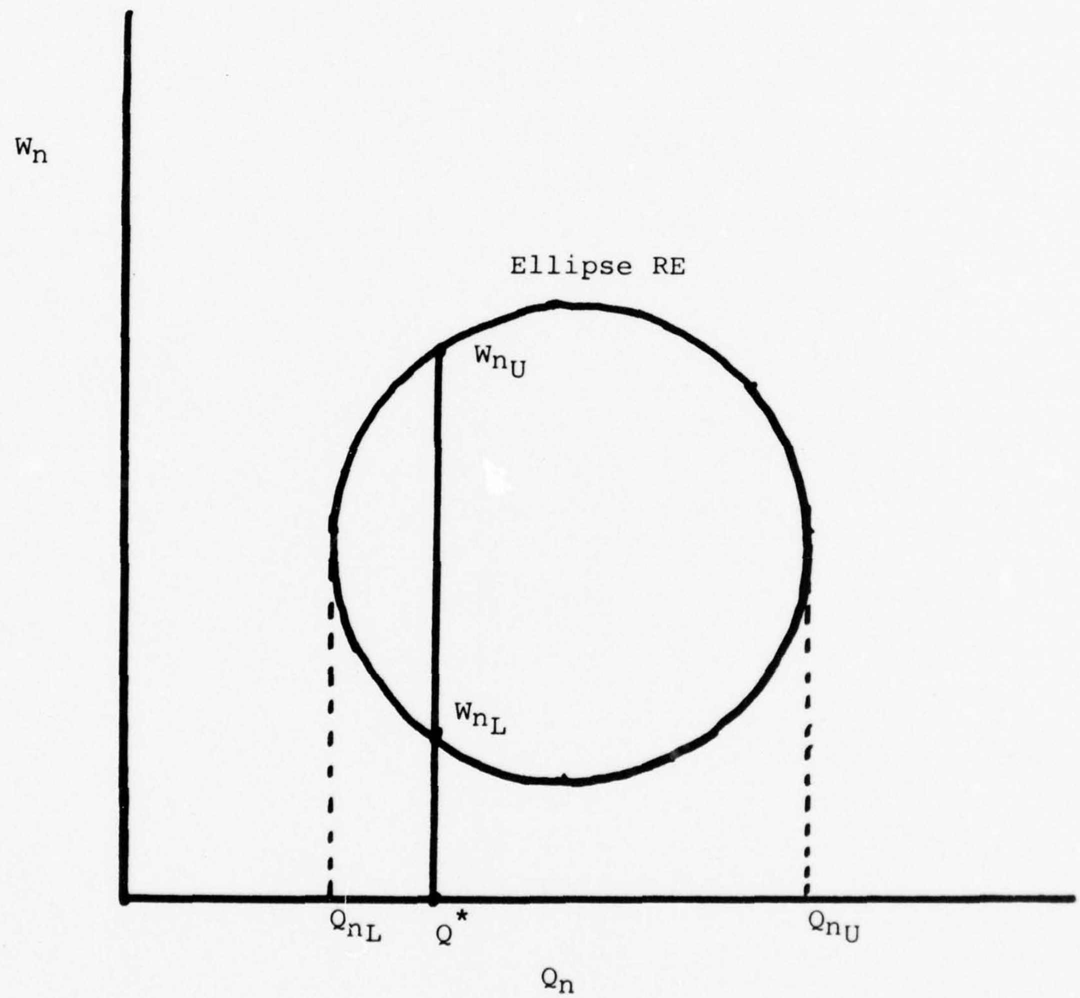
$$Z_L = a - W_{n_U}$$

Where W_{n_L} and W_{n_U} are the smallest and largest values of

$$\frac{na}{n+1} \pm \sqrt{\frac{n}{n+1}c - \frac{b^2}{n+1} - \frac{na^2}{(n+1)^2} - \frac{2bU^*}{n+1} - U^{*2}} \qquad (2.3.20)$$

FIGURE 4

Integration Region
When Neither a Decision to Accept
or Reject Can Be Made



These limits are identical to those in equations (2.3.6) and (2.3.8); which is to be expected, since situation (1) is a case where all points W_n, Q_n, R_n, which can be mapped into the point a, b, c, lie in the continuation region C_n.

Situation (2) will occur whenever the center of the circle RE is a point inside the ellipse AE; and the following point on RE

$$W_{n_c} = \sqrt{\frac{na}{n+a} + \frac{nc}{n+1} - \frac{na^2}{(n+1)^2} - \frac{nb^2}{(n+1)^2}}$$

$$Q_{n_c} = \frac{nb}{n+1}\ , \qquad (2.3.21)$$

is inside the ellipse AE. This is equivalent to satisfying the following inequalities:

$$\frac{n(a-b)^2}{2V_n(n+1)^2} - \left[c - \frac{a^2}{n+1} - \frac{b^2}{n+1}\right] < 0 \qquad (2.3.22)$$

and

$$\frac{1}{2nV_A}\left[\frac{\sqrt{n}}{(n+1)}(a-b) + \frac{1}{\sqrt{n}}\sqrt{\frac{nc}{n+1} - \frac{na^2}{(n+1)^2} - \frac{nb^2}{(n+1)^2}}\right]^2 < 0 \qquad (2.3.23)$$

Since the second inequality can never be satisfied, situation (2) will never occur.

Situation (1) will occur whenever the center of the ellipse AE is a point inside RE; and the point W_{n_c}, Q_{n_c} defined in equation (2.3.21) is outside AE. The second constraint amounts to requiring the left hand side of equation (2.3.23) to be greater than zero; which will always be true. The first is equivalent to satisfying the following inequality:

$$\frac{n(a-b)^2}{2V_A(n+1)^2} - \left[c - \frac{a^2}{n+1} - \frac{b^2}{n+1}\right] < 0 \qquad (2.3.24)$$

Whenever this is true, the region of interest must be broken up into 4 subregions as shown in Figure 5. Thus the integral in equation (2.3.8) will be broken up into 4 separate integrals, so that

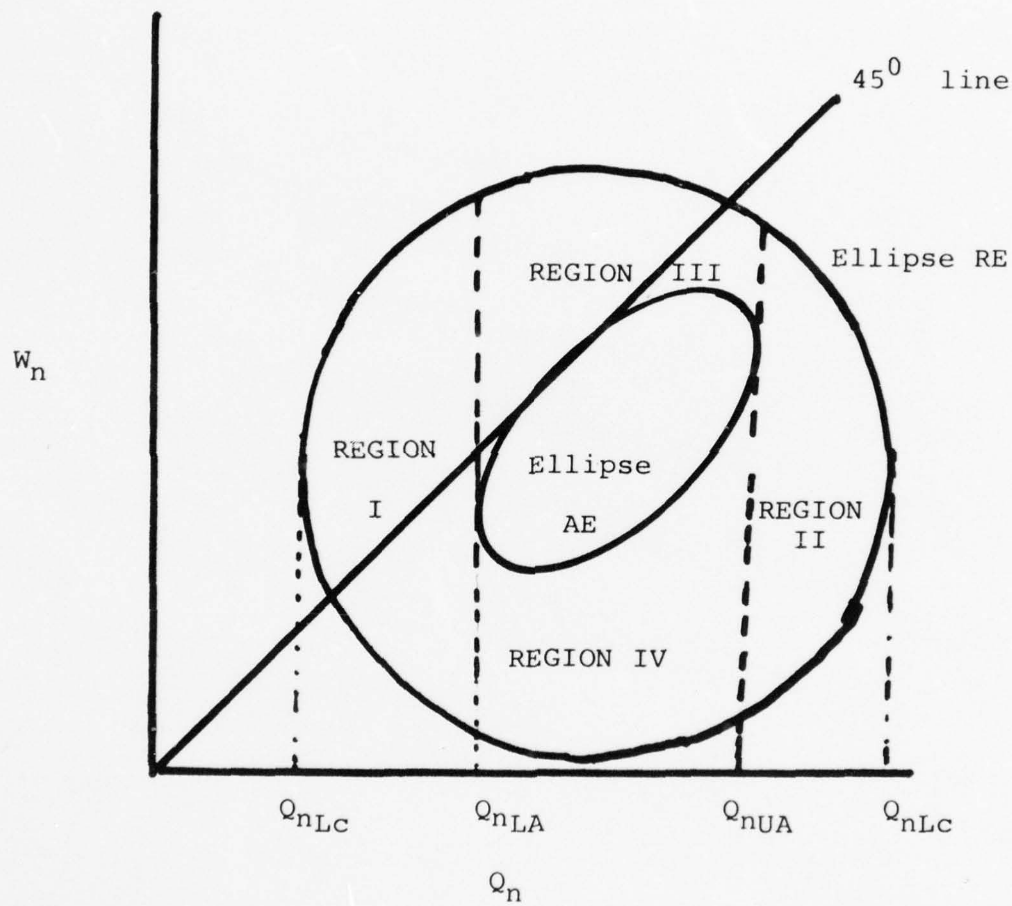
$$f_{n+1}(a,b,c) = \sum_{i=1}^{4} \left\{ \int_{U_{Li}}^{U_{Ui}} \int_{Z_{Li}}^{Z_{Ui}} f_n^P (a-Z,\ b-U,\ c-Z^2-U^2) dZdU \right\} \qquad (2.3.25)$$

The limits U_{U_i}, U_{L_i}, Z_{U_i}, Z_{L_i} will now be obtained for each region.

For each region a range of Q_n can be found; $Q_{n_{Li}} \leq Q_n \leq Q_{n_{Ui}}$ from which the U integration limits are obtained as: $U_{L_i} = b-Q_{n_{Ui}}$ and $U_{U_i} = b-Q_{n_{Li}}$.

FIGURE 5

An Integration Region
Consisting of Four Pieces



The range of Q_n for each of the subregions is as follows:

Region I: $Q_{n_{L1}} = Q_{n_{LC}} \leq Q_n \leq Q_{n_{LE}} = Q_{n_{U1}}$

Region II: $Q_{n_{L2}} = Q_{n_{UA}} \leq Q_n \leq Q_{n_{UC}} = Q_{n_{U2}}$

Region III: $Q_{n_{L3}} = Q_{n_{LA}} \leq Q_n \leq Q_{n_{UA}} = Q_{n_{U3}}$

Region IV: $Q_{n_{L4}} = Q_{n_{LA}} \leq Q_n \leq Q_{n_{UA}} = Q_{n_{U4}}$

Where $Q_{n_{LC}}$ and $Q_{n_{UC}}$ are the two Q_n end points of the circle RE, or the smallest and largest values of equation (2.3.17).

$Q_{n_{LA}}$ and $Q_{n_{UA}}$ represent the Q_n end points of the ellipse AE. These are obtained by the same methods employed for Case II (equation (2.3.12)); yielding $Q_{n_{LA}}$ and $Q_{n_{UA}}$ as the smallest and largest values of

$$\frac{b(c_A+1)+aP_A}{(c_A+1)^2-P_A^{\;2}} \pm \sqrt{\left[\frac{b(c_A+1)+aP_A}{P_A^{\;2}-(c_A+1)^2}\right]^2 - \left[\frac{a^2-(c_A+1)(a^2+b^2-c)}{P_A^{\;2}-(c_A+1)^2}\right]} \qquad (2.3.27)$$

Similarly, for each region a range of W_n values can be defined: $W_{n_{Li}} \leq W_n \leq W_{n_{Ui}}$ from which the Z limits are obtained as: $Z_{L_i} = a - W_{n_{Ui}}$ and $Z_{Ui} = a - W_{n_{Li}}$.

The range of W_n for each of the subregions is as follows:

Region I: $W_{n_{L1}} = W_{n_{LC}} \le W_n \le W_{n_{UC}} = W_{n_{U1}}$

Region II: $W_{n_{L2}} = W_{n_{LC}} \le W_n \le W_{n_{UC}} = W_{n_{U2}}$

Region III: $W_{n_{L3}} = W_{n_{UA}} \le W_n \le W_{n_{UC}} = W_{n_{U3}}$

Region IV: $W_{n_{L4}} = W_{n_{LC}} \le W_n \le W_{n_{LA}} = W_{n_{U4}}$

Where $W_{n_{LC}}$ and $W_{n_{UC}}$ are W_n points on the lower and upper portion of the circle RE; and $W_{n_{LA}}$ and $W_{n_{UA}}$ the analogous points on the ellipse AE. As in the previous cases, these values will depend upon the value of Q_n, $Q_{n_{Li}} \le Q_n \le Q_{n_{Ui}}$, or equivalently the value of U. For a given value of U, say U^*, $W_{n_{LC}}$ and $W_{n_{UC}}$ are the smallest and largest values of equation (2.3.20). The values $W_{n_{LA}}$ and $W_{n_{UA}}$ are the smallest and largest values of the following:

$$\frac{(a+P_A Q^*)}{(C_A+1)} \pm \sqrt{\frac{a+P_A Q^*}{C_A+1}^2 - \frac{a^2+b^2-c-2bQ^*+(C_A+1)Q^{*2}}{C_A+1}} \qquad (2.3.28)$$

where $Q^* = b - U^*$.

The case where the curves RE and AE intersect at only one point must also be considered. This will happen whenever

$$(a+b)^2 - 2(1+\frac{1}{n})(a^2+b^2-c) = 0.$$

If this is true the curves will intersect at the point

$$W_n = Q_n = \frac{n(a+b)}{2(n+1)} . \qquad (2.3.29)$$

Based on the previous discussion this can only occur in the following situations:

(1) the curve RE contains AE;

(2) the curves AE and RE contain no points in common, except for the point of intersection given in equation (2.3.29).

Situation (2) will occur whenever the inequalities given by equations (2.3.15) and (2.3.16) are satisfied. Since the point of intersection will be on the boundary of RE, the integration regions U_U, U_L, Z_U, and Z_L can still be obtained by equations (2.3.18) and (2.3.19).

Situation (1) will occur whenever the inequality given in equation (2.3.24) is satisfied. The integration region must now be broken up into three pieces as shown in Figure 6. This is simply a special case of Figure 5 and may be evaluated by the same methods used to evaluate equation (2.3.25).

The curves RE and AE may also intersect at two points. This will happen whenever

$$(a+b)^2 - 2(1+\frac{1}{n})(a^2+b^2-c) > 0.$$

This indicates that one of the following geometric relationships must exist:

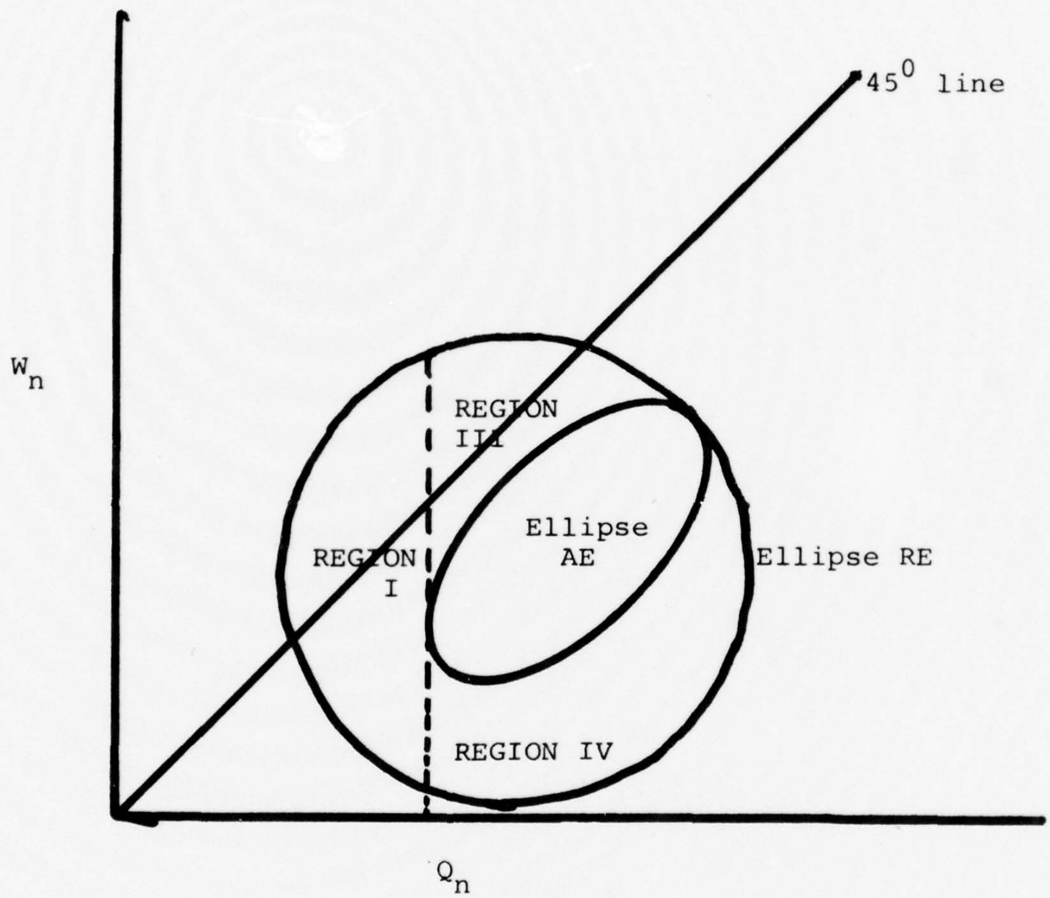
(1) The ellipse AE is contained inside the circle RE, with the Q_n end points of AE touching the circle.

(2) The intersection points fall below the axis of the ellipse AE, which is parallel to the line $W_n = Q_n$.

(3) The intersection points fall above the axis of the ellipse AE, which is parallel to the line $W_n = Q_n$.

FIGURE 6

An Integration Region
Consisting of Three Pieces



Consider the following set of rotated axes (45° rotation):

$$Q_n' = \frac{\sqrt{2}}{2}\left[Q_n + W_n\right] \tag{2.3.29}$$

$$W_n' = \frac{\sqrt{2}}{2}\left[-Q_n + W_n\right]$$

The ellipse AE in terms of this new coordinate system becomes:

$$(C_A + 1-P_A)\left[Q_n' - \frac{\sqrt{2}(a+b)}{2(C_A+1-P_A)}\right]^2 + (C_A+1+P_A)\left[W_n' - \frac{\sqrt{2}(a-b)}{2(C_A+1+P_A)}\right]^2$$

$$= c-a^2-b^2 + \frac{(a-b)^2}{2(C_A+1+P_A)} + \frac{(a+b)^2}{2(C_A+1-P_A)} \tag{2.3.30}$$

Also, the line $W_n = Q_n$ becomes:

$$W_n' = 0. \tag{2.3.31}$$

From these equations criteria for situations (1) - (3) can be established. Situation (1) will occur whenever a = b; situation (2) will occur whenever a > b; and situation (3) will occur whenever a < b.

Situation (1) is shown in Figure 7. The integration region must be divided into at most four subregions, as in Equation (2.3.25). The integration limits are the same as those obtained for equation (2.3.25), with the exception that one or two of the subregions may be empty.

Situation (2) is shown in Figure 8. The integration region must now be divided into three subregions. The range of Q_n for each of the subregions is as follows:

$$\text{Region I:} \quad Q_{n_{L1}} = Q_{n_{LC}} \leq Q_n \leq Q_{n_{LI}} = Q_{n_{U1}}$$

$$\text{Region II:} \quad Q_{n_{L2}} = Q_{n_{LI}} \leq Q_n \leq Q_{n_{UI}} = Q_{n_{U2}}$$

$$\text{Region III:} \quad Q_{n_{L3}} = Q_{n_{UI}} \leq Q_n \leq Q_{n_{UC}} = Q_{n_{U3}} \qquad (2.3.32)$$

The quantities $Q_{n_{LC}}$ and $Q_{n_{UC}}$ are again the two Q_n end points of the circle RE, or the smallest and largest values of equation (2.3.17). $Q_{n_{LI}}$ and $Q_{n_{UI}}$ are the intersection points of the ellipse AE with the circle RE, or the smallest and largest values of the following equation:

$$\frac{(a + b) \pm \sqrt{(a+b)^2 - 2\left(1+\frac{1}{n}\right)(a^2+b^2-c)}}{2\left(1 + \frac{1}{n}\right)} \qquad (2.3.33)$$

FIGURE 7

Integration Region When Ellipse AE
Intersects Circle RE at Two Points
Situation 1

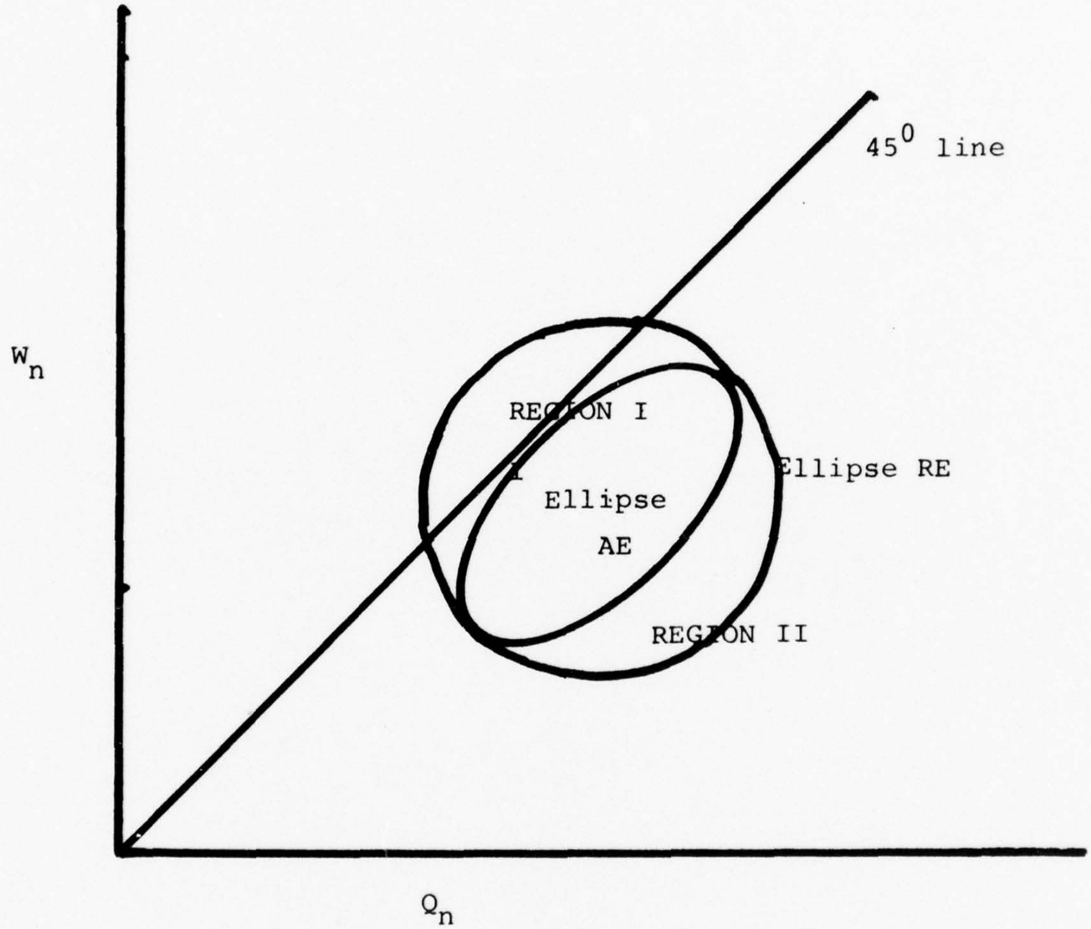
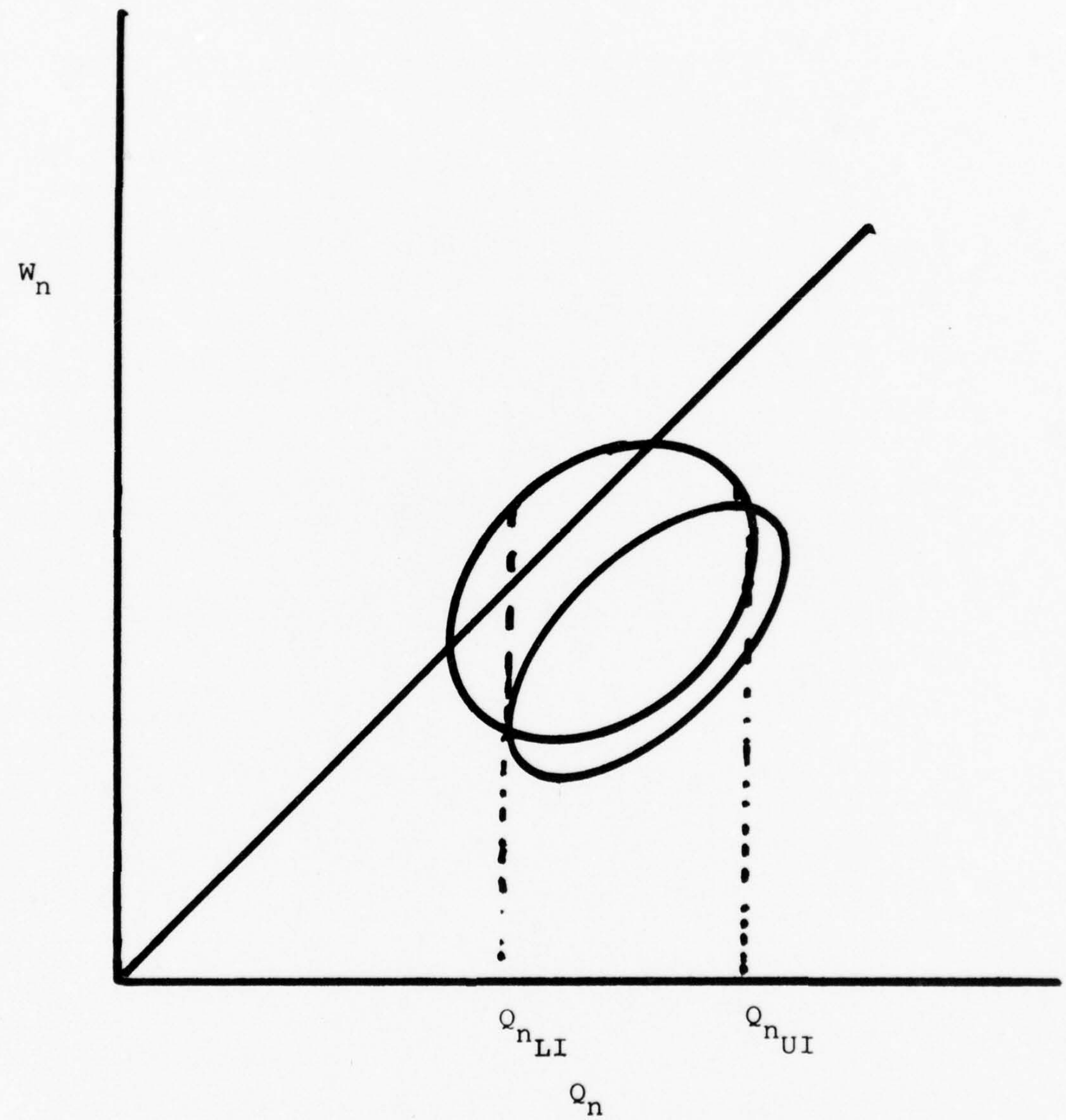


FIGURE 8

Integration Region When Ellipse AE Intersects Circle RE at Two Points
Situation 3



The U integration limits for each region are then given by:

$$U_{Li} = b - Q_{n_{Ui}}$$
$$U_{Ui} = b - Q_{n_{Li}}$$
$$i = 1,2,3 \qquad (2.3.34)$$

The range of W_n for each of the subregions is as follows:

Region I: $W_{n_{L1}} = W_{n_{LC}} \leq W_n \leq W_{n_{UC}} = W_{n_{U1}}$

Region II: $W_{n_{L2}} = W_{n_{LC}} \leq W_n \leq W_{n_{UE}} = W_{n_{U2}}$

Region III: $W_{n_{L3}} = W_{n_{LC}} \leq W_n \leq W_{n_{UC}} = W_{n_{U3}}$ (2.3.35)

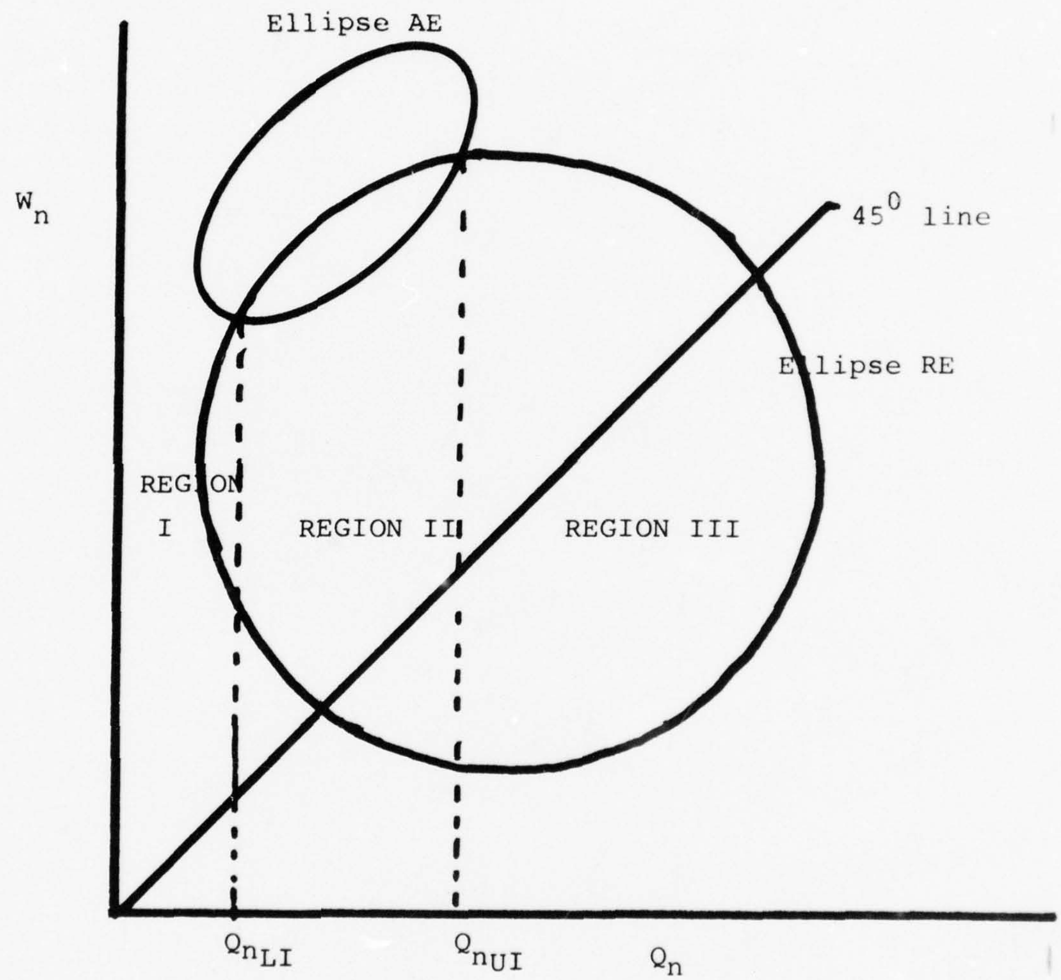
For a given value of U, say U^*, $W_{n_{LC}}$ and $W_{n_{UC}}$ are the smallest and largest values of equation (2.3.20) and $W_{n_{UC}}$ is the larger value of equation (2.3.28). Thus, the Z integration limits are obtained as:

$$Z_{Li} = a - W_{n_{Ui}}$$
$$Z_{u_i} = a - W_{n_{Li}}$$
$$i = 1,2,3 \qquad (2.3.36)$$

Situation (3) is shown in Figure 9. As in situation (2), the integration region must be broken up into three subregions. The range of Q_n for each of the subregions and the U integration limits are still given by equations (2.3.32) and

FIGURE 9

Integration Region When Ellipse AE Intersects Circle RE at Two Points
Situation 3



(2.3.34) respectively. However, the range of W_n for each of the subregions is now:

$$\begin{aligned}
&\text{Region I:} && W_{n_{L1}} = W_{n_{LC}} \le W_n \le W_{n_{UC}} = W_{n_{U1}} \\
&\text{Region II:} && W_{n_{L2}} = W_{n_{LC}} \le W_n \le W_{n_{LE}} = W_{n_{U2}} \\
&\text{Region III:} && W_{n_{L3}} = W_{n_{LC}} \le W_n \le W_{n_{UC}} = W_{n_{U3}}
\end{aligned} \qquad (2.3.36)$$

Where $W_{n_{LE}}$ is the smaller value of equation (2.3.28). Given these values the Z integration limits are given by equation (2.3.36).

It is also possible that the ellipse AE does not exist. This will occur whenever the surface B_A does not intersect the mapping function P, or whenever the following inequality is satisfied:

$$\frac{n(a-b)^2}{2(n+1)[(n+1)V_A+1]} - C - \left[\frac{a^2}{n+1} - \frac{b^2}{n+1}\right] \ge 0 \qquad (2.3.37)$$

This is not a special case, however, because whenever this inequality is satisfied, inequalities (2.3.15) and (2.3.16) are also satisfied. Thus the integration regions are obtained from equations (2.3.17) - (2.3.20).

In summary, this section has discussed the various types of integration regions that can result when only a decision to accept H_0 is possible at stage n, criteria for determining when each of these regions is appropriate, and formulas to calculate the required U, Z limits for each of these regions.

IV. Case when either a decision to accept or reject H_0 is possible at stage n.

Whenever $0 < V_A < V_R < \infty$, both acceptance and rejection are possible at stage n. In this case, both the curves AE and RE become equations of an ellipse. In order to determine the integration region, it is necessary to know the intersection points of the two ellipses.

The intersection points are most easily found by transforming AE and RE into a coordinate system rotated 45 degrees. The equation for RE in the rotated axes becomes, RE´:

$$(C_R-P_R+1)\ Q'^{\,2}_n + (C_R+1+P_R)W'^{\,2}_n - S(2a+2b)\ Q'_n - S(2a-2b)\ W'_n = C-a^2-b^2 \qquad (2.3.30)$$

and that of AE ,

AE´:

$$(C_A-P_A+1)\ Q'^{\,2}_n + (C_A+1+P_A)W'^{\,2}_n - S(2a+2b)\ Q'_n - S(2a-2b)\ W'_n = C-a^2-b^2 \qquad (2.3.31)$$

where S is given by

$$S = \text{Sin}(45°) = \frac{\sqrt{2}}{2}$$

These equations may be further simplified as:

RE´ :

$$(C_R-P_R+1)\left\{Q_n' - \frac{S(a+b)}{(C_R-P_R+1)}\right\}^2 + (C_R-P_R+1)\left\{W_n' - \frac{S(a-b)}{(C_R+P_R+1)}\right\}^2$$

$$= c-a^2-b^2 + \frac{(a+b)^2}{2(C_R-P_R+1)} + \frac{(a-b)^2}{2(C_R+P_R+1)}$$

and

AE´ :

$$(C_A-P_A+1)\left\{Q_n' - \frac{S(a+b)}{(C_A-P_A+1)}\right\}^2 + (C_A+P_A+1)\left\{W_n' - \frac{S(a-b)}{(C_A+P_A+1)}\right\}^2$$

$$= c-a^2-b^2 + \frac{(a+b)^2}{2(C_A-P_A+1)} + \frac{(a-b)^2}{2(C_A+P_A+1)}$$

Since

$$P_A = \frac{1}{2nV_A}$$

and

$$C_A = \frac{2V_A+1}{2nV_A} ,$$

(with similar expressions for P_R and C_R), these may be substituted into the above; which yields after combining similar terms, the following expressions:

RE´ :

$$(C_R-P_R+1)\left\{Q_n' - \frac{S(a+b)}{(C_R-P_R+1)}\right\}^2 + (C_R+P_R+1)\left\{W_n' - \frac{S(a-b)}{(C_R+P_R+1)}\right\}^2$$

$$= C - \frac{a^2}{n+1} - \frac{b^2}{n+1} - \frac{(a-b)^2 n}{2(n+1)(V_R+nV_R+1)}$$

and

AE´ :

$$(C_A-P_A+1)\left\{Q_n' - \frac{S(a+b)}{(C_A-P_A+1)}\right\}^2 + (C_A+P_A+1)\left\{W_n' - \frac{S(a-b)}{(C_A+P_A+1)}\right\}^2$$

$$= C - \frac{a^2}{n+1} - \frac{b^2}{n+1} - \frac{(a-b)^2 n}{2(n+1)(V_A+nV_A+1)}$$

Since

$$C_A-P_A+1 = C_R-P_R+1 ,$$

the above equations show that both curves will have identical Q_n' coordinates for their centers.

Solving for Q'_n in RE′ yields a solution of the following form:

$$Q'_n = K \pm D$$

where

$$K = \frac{S(a+b)}{(C_R - P_R + 1)}$$

$$D = \sqrt{\frac{.5(a+b)^2}{(C_R - P_R + 1)} - \left[\frac{a^2 + b^2 - c + (C_R + 1 + P_R)W'^2_n - S(2a+2b)W'_n}{C_R - P_R + 1}\right]}$$

Substituting this form in the equation for AE′ yields

$$(C_A - P_A + 1)(K^2 + D^2) + (C_A + 1 + P_A)W'^2_n \qquad (2.3.32)$$

$$- S(2a+2b)K - S(2a-2b)W'_n$$

$$+ \left(\pm 2KD(C_A - P_A + 1) - S(2a+2b)(\pm D)\right)$$

$$= C - a^2 - b^2$$

In general an equation describing the intersection of two ellipses will be a quartic. Equation (2.3.32) is a special case, however, and can be reduced to a quadratic by noting that the following term

$$\pm D\left(2K(C_n - P_A + 1) - S(2a+2b)\right)$$

$$= \pm D \left[\frac{S(2a+2b)(C_A - P_A + 1)}{(C_R - P_R + 1)} - S(2a+2b) \right]$$

is zero since

$$(C_A - P_A + 1) = (C_R - P_R + 1) = \frac{n+1}{n} .$$

Solving equation (2.3.32) for W_n' and substituting into RE' yields the following W_n', Q_n' intersection points:

$$W_n' = 0 \tag{2.3.33}$$

$$Q_n' = \frac{1}{n+1} \left[S(a+b)n \pm \sqrt{S\left[(a+b)n\right]^2 - n(n+1)(a^2+b^2-c)} \right]$$

In terms of the original W_n, Q_n axes the intersection points become:

$$W_n = Q_n = \frac{1}{2(n+1)} \left[n(a+b) \pm \sqrt{\left[(a+b)n\right]^2 - n(n+1)(a^2+b^2-c)} \right] \tag{2.3.34}$$

This means that the two ellipses, AE and RE, will intersect at zero, one, or two points. The sign of the discriminant of equation (2.3.34),

$$DIS = \left[(a+b)n\right]^2 - n(n+1)(a^2+b^2-c),$$

determines the number of intersection points. Whenever:

1. $DIS < 0$ (2.3.35)

AE and RE will not intersect.

2. DIS = 0 (2.3.36)

AE and RE will intersect at only one point, this point being

$$W_n = Q_n = \frac{n(a+b)}{2(n+1)}$$

3. DIS > 0 (2.3.37)

AE and RE will intersect at two points.

Consider first, the case when AE and RE do not intersect, or when equation (2.3.35) is satisfied. This indicates one of the following geometric relationships must exist:

(1) the curves AE and/or RE do not exist

(2) the curve RE contains AE

(3) the curve AE contains RE

(4) the curves AE and RE contain no points in common.

Situation (1) will occur if either of the ellipses has an imaginary radius, or whenever either of the following equations is satisfied:

$$C - \frac{a^2}{n+1} - \frac{b^2}{n+1} - \frac{(a-b)^2 n}{2(n+1)(V_R+nV_R+1)} \leq 0 \qquad (2.3.38)$$

$$C - \frac{a^2}{n+1} - \frac{b^2}{n+1} - \frac{(a-b)^2 n}{2(n+1)(V_A+nV_A+1)} \leq 0 . \qquad (2.3.39)$$

Since $V_A \leq V_R$, inequality (2.3.39) will be satisfied whenever (2.3.38) is satisfied. If (2.3.38) is satisfied, f(a,b,c) = 0, since none of the points that can be mapped into a,b,c lie in the continuation region.

If inequality (2.3.39) is satisfied and (2.3.38) is not, the point a,b,c is located such that the mapping function P intersects the rejection surface but never intersects the acceptance curve. This reduces to a case previously discussed, the case when only a decision to reject H_0 is possible, and the integration regions are given by equations (2.3.11) through (2.3.14).

Assuming neither inequality (2.3.38) or (2.3.39) is satisfied, situation (4) will occur when neither curve contains the other's center, or when the following inequalities are satisfied:

$$\left(\frac{V_R+nV_R+1}{nV_R}\right)\left(\frac{n^2(a-b)^2}{2}\right)\left[\frac{V_R}{(V_R+nV_R+1)}-\frac{V_A}{(V_A+nV_A+1)}\right]^2$$

$$-\left\{C-\frac{a^2}{n+1}-\frac{b^2}{n+1}-\frac{(a-b)^2n}{2(n+1)(V_R+nV_R+1)}\right\}>0$$

(2.3.40)

and

$$\left(\frac{V_A+nV_A+1}{nV_A}\right)\left(\frac{n^2(a-b)^2}{2}\right)\left[\frac{V_R}{(V_R+nV_R+1)}-\frac{V_A}{(V_A+nV_A+1)}\right]^2$$

$$-\left\{C-\frac{a^2}{n+1}-\frac{b^2}{n+1}-\frac{(a-b)^2n}{2(n+1)(V_A+nV_A+1)}\right\}>0$$

(2.3.41)

As it is necessary for both inequalities to hold, and inequality (2.3.41) implies inequality (2.3.40), it is only necessary to examine the former. In other words, situation (4) will occur whenever RE and AE do not intersect, and RE does not contain AE's center point.

The set H consists of all points W_n, Q_n contained inside the ellipse RE. The integration region in this case becomes identical to that required for the case when only rejection is possible and can be evaluated using equations (2.3.11) - (2.3.14).

Situation (3) will occur whenever the four end points of the ellipse RE´ are all points inside AE.´ First, consider the RE´ end points along the Q_n' axis. When this is substituted into AE´, the following inequality must hold for situation (3) to occur:

$$\left(\frac{n(a-b)^2}{2}\right)\left(\frac{V_R}{(V_R+nV_R+1)} - \frac{V_A}{(V_A+nV_A+1)}\right)$$

$$+\left(\frac{V_A+nV_A+1}{nV_A}\right)\left(\frac{(a-b)^2 n^2}{2}\right)\left(\frac{V_R}{(V_R+nV_R+1)} - \frac{V_A}{(V_R+nV_A+1)}\right)^2 < 0 .$$

Since $V_A \leq V_R$, this inequality can not be satisfied; and thus situation (3) can never occur.

Having shown that situation (3) cannot occur, situation (2) will occur whenever both the ellipses RE and AE exist (neither inequality (2.3.38) nor (2.3.31) is satisfied), and inequality (2.3.41) is not satisfied. In this case the integral given in equation (2.3.8) must be broken up into four separate pieces, similar to that given in equation (2.3.25). The limits U_{Ui}, U_{Li}, Z_{Ui} and Z_{Li}: $i = 1,\ldots,4$ must be determined for each region.

For each region a range of Q_n can be found; $Q_{n_{Li}} \leq Q_n \leq Q_{n_{Ui}}$ from which the U integration limits are obtained as:

$$U_{Li} = b - Q_{n_{Ui}}$$

$$U_{Ui} = b - Q_{n_{Li}} .$$

The range of Q_n for each of the pieces is as follows:

Region I: $Q_{n_{L1}} = Q_{n_{LR}} \leq Q_n \leq Q_{n_{LA}} = Q_{n_{U1}}$

Region II: $Q_{n_{L2}} = Q_{n_{UA}} \leq Q_n \leq Q_{n_{UR}} = Q_{n_{U2}}$

Region III: $Q_{n_{L3}} = Q_{n_{LA}} \leq Q_n \leq Q_{n_{UA}} = Q_{n_{U3}}$

Region IV: $Q_{n_{L4}} = Q_{n_{LA}} \leq Q_n \leq Q_{n_{UA}} = Q_{n_{U4}}$

(2.3.42) .

Where $Q_{n_{LR}}$ and $Q_{n_{UA}}$ are the minimum and maximum Q_n coordinatesson the ellipse RE; and $Q_{n_{LA}}$ and $Q_{n_{UA}}$ are the analogous quantities on the ellipse AE.

$Q_{n_{LR}}$ and $Q_{n_{UR}}$ have previously been derived as the smallest and largest values of equation (2.3.12). A similar expression derived for RE, yields $Q_{n_{LA}}$ and $Q_{n_{UA}}$ as the smallest and largest values of

$$\left[\frac{b(C_A+1) + aP_A}{(C_A+1)^2 - P_A^2}\right] \pm \sqrt{\left[\frac{b(C_A+1) + aP_A}{P_A^2 - (C_A+1)^2}\right]^2 - \left[\frac{a^2-(C_A+1)(a^2+b^2-c)}{P_A^2 - (C_A+1)^2}\right]}$$

(2.3.43) .

The range of W_n for each piece depends upon the value of Q_n (or equivalently U). For a given value of U_i, say U*, $U^* = b - Q_{n_i}^*$; $Q_{n_{Li}} \leq Q_{n_i}^* \leq Q_{n_{Ui}}$ a range of W_n values can be defined; $W_{n_{Li}} \leq W_n \leq W_{n_{ui}}$, from which the Z limits are obtained as:

$$Z_{L_i} = a - W_{n_{Ui}}$$

$$Z_{U_i} = a - W_{n_{Li}} \quad .$$

The range of W_n for each of the pieces is as follows:

$$\text{Region I:}\quad W_{n_{L1}} = W_{n_{LR}} \le W_n \le W_{n_{UR}} = W_{n_{U1}}$$

$$\text{Region II:}\quad W_{n_{L2}} = W_{n_{LR}} \le W_n \le W_{n_{UR}} = W_{n_{U2}}$$

$$\text{Region III:}\quad W_{n_{L3}} = W_{n_{UA}} \le W_n \le W_{n_{UR}} = W_{n_{U3}}$$

$$\text{Region IV:}\quad W_{n_{L4}} = W_{n_{LR}} \le W_n \le W_{n_{LA}} = W_{n_{U4}}$$

(2.3.44) .

$W_{n_{LR}}$ and $W_{n_{UR}}$ are the upper and lower points on the ellipse **RE**, for a given value of $U^* = b - Q_n^*$. These have been derived previously as the smallest and largest values of equation (2.3.13). $W_{n_{LA}}$ and $W_{n_{UA}}$ are the analogous points on the ellipse AE, and are obtained as the smallest and largest value of:

$$\left[\frac{(a+P_A Q^*)}{(C_A+1)}\right] \pm \sqrt{\left[\frac{a+P_A Q^*}{C_A+1}\right]^2 - \left[\frac{a^2+b^2-C-2bQ^*+(C_A+1)Q^{*2}}{(C_A+1)}\right]}$$

(2.3.45) .

Next consider the case when AE and RE intersect at only one point, which will occur whenever equation (2.3.36) is satisfied. Based on the previous discussion this can only occur in the following situations:

(1) either one or both of the curves AE and RE do not exist

(2) the curve RE contains AE

(3) the curves AE and RE contain no points in common, except for the point of intersection.

Situation (1) can never occur if equations(2.3.36) or (2.3.37) hold. This can be shown as follows:

If

$$(a+b)^2 - \left(2\frac{n+1}{n}\right)\left(a^2+b^2-c\right) \geq 0$$

then

$$C = a^2+b^2 - \frac{(a+b)^2}{2(n+1)} + \frac{S_1}{2(n+1)}$$

S_1 being a quantity greater than or equal to zero. Substituting this result into the equation of the radius of the ellipse AE and simplifying yields:

$$\text{Radius AE} = \frac{(a-b)^2}{2(C_A+1+P_A)} + \frac{S_1}{2(n+1)} .$$

Since this quantity will always be greater than equal to zero, the ellipse AE will always exist. The previous section also showed that a sufficient condition for RE to exist was the existence of AE. Hence, intersection of the ellipses AE and RE is a sufficient condition for their existence.

Situation (3) will occur whenever the inequality given in equation (2.3.11) is satisfied. Since the point of intersection will be on the boundary of RE, the integration regions U_L, U_L, Z_U, and Z_L can still be obtained by equations (2.3.11) and (2.3.14).

Similarly situation (2) occurs whenever inequality (2.3.41) is not satisfied, and requires the integration to be broken up into four pieces. The integration limits in each of these pieces may still be obtained by equations (2.3.42) through (2.3.45).

The curves AE and RE will intersect at two points whenever inequality (2.3.37) holds. In general, two ellipses can intersect at two points in many ways. However, consider the equations in the rotated axes coordinated system:

RE´:

$$\left(\frac{n+1}{n}\right)\left\{Q_n' - \left[\frac{\sqrt{2}(a+b)n}{2(n+1)}\right]\right\}^2 + \left(\frac{V_R+nV_R+1}{nV_R}\right)\left\{W_n' - \left[\frac{\sqrt{2}(a-b)nV_R}{2(V_R+nV_R+1)}\right]\right\}^2$$

$$= C - \frac{a^2}{n+1} - \frac{b^2}{n+1} - \frac{(a+b)^2 n}{2(n+1)(V_R+nV_R+1)}$$

and

AE´:

$$\left(\frac{n+1}{n}\right)\left\{Q_n' - \left[\frac{\sqrt{2}(a+b)n}{2(n+1)}\right]\right\}^2 + \left(\frac{V_A+nV_A+1}{nV_A}\right)\left\{W_n' - \left[\frac{\sqrt{2}(a-b)nV_A}{2(V_A+nV_A+1)}\right]\right\}^2$$

$$= C - \frac{a^2}{n+1} - \frac{b^2}{n+1} - \frac{(a-b)^2 n}{2(n+1)(V_A+nV_A+1)}$$

From these equations the following relationships may be noted:

(a) the curves RE´ and AE´ will have parallel major axes (i.e., parallel to the line $W_n' = 0$).

(b) the major axis of RE´ will be greater than or equal to the major axis of AE´.

(c) the major axes of RE´ and AE´ will always lie on the same side of the line $W_n' = 0$.

(d) the two curves will have the same center point and equal major axes whenever $a = b$.

(e) since

$$Q_n' = \frac{\sqrt{2}}{2}\left[Q_n + W_n\right]$$

$$W_n' = \frac{\sqrt{2}}{2}\left[-Q_n + W_n\right] ,$$

if the curves intersect, the intersection points will lie along the line $W_n = Q_n$ or $W_n' = 0$.

Given these relationships one can conclude that whenever the curves intersect at two points, one of the following geometric situations must exist:

(1) the ellipse RE´ circumscribes the ellipse AE´.

(2) the major axes of the ellipses lie above the line $W_n' = 0$.

(3) the major axes of the ellipses lie below the line $Wn' = 0$.

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AN EXACT TEST FOR THE SEQUENTIAL ANALYSIS OF VARIANCE.(U)

AUG 79 R W MILLER N00014-77-C-0438

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2 OF 3

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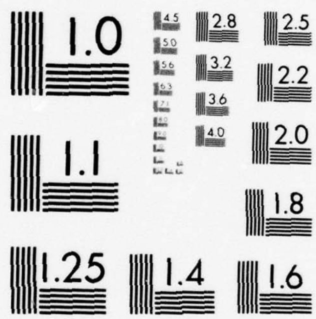

1.0
1.1
1.25
1.4
1.6
1.8
2.0
2.2
2.5
2.8
3.2
3.6
4.0
4.5
5.0
5.6
6.3

Situation (1) will occur whenever $a = b$, and requires the integral of equation (2.3.8) to be broken up into two pieces. These two pieces may be described by W_n, Q_n regions identical to those of Regions III and IV given in (2.3.42) and (2.3.44). Thus the integration limits U_{Li}, U_{Ui}, Z_{Li}, and Z_{Ui} may be obtained from equations (2.3.43) - (2.3.45).

Situation (2) will occur whenever $a > b$, and requires the integral of equation (2.3.8) to be broken up into five pieces, as shown in Figure 10. The range of Q_n for each of the subregions is as follows:

$$\text{Region I:} \quad Q_{n_{L1}} = Q_{n_{LR}} \leq Q_n \leq Q_{n_{LA}} = Q_{n_{U1}}$$

$$\text{Region II:} \quad Q_{n_{L2}} = Q_{n_{LA}} \leq Q_n \leq Q_{n_{UA}} = Q_{n_{U2}}$$

$$\text{Region III:} \quad Q_{n_{L3}} = Q_{n_{LA}} \leq Q_n \leq Q_{n_{LI}} = Q_{n_{U3}}$$

$$\text{Region IV:} \quad Q_{n_{L4}} = Q_{n_{UI}} \leq Q_n \leq Q_{n_{UA}} = Q_{n_{U4}}$$

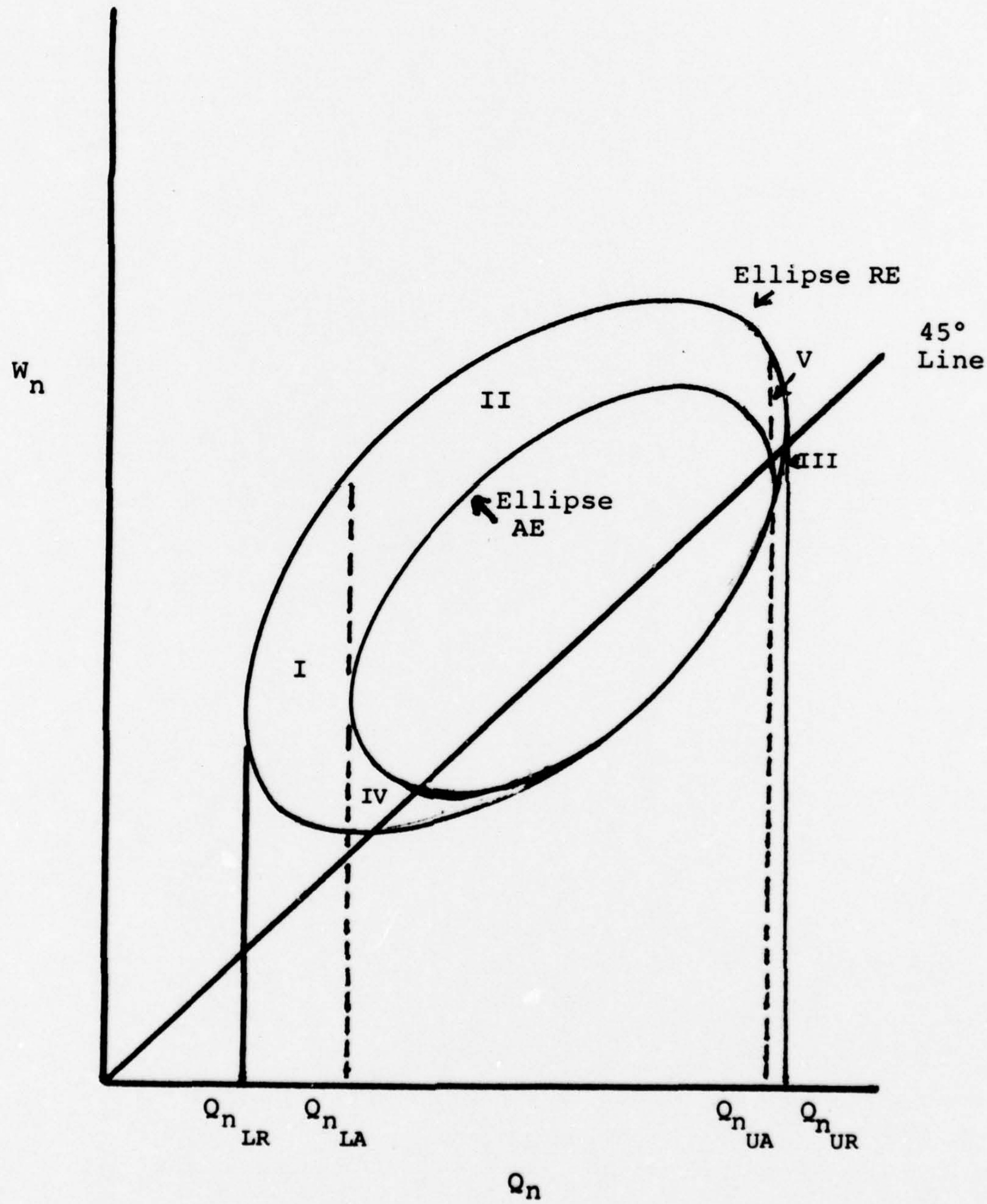
$$\text{Region V:} \quad Q_{n_{L5}} = Q_{n_{UA}} \leq Q_n \leq Q_{n_{UR}} = Q_{n_{U5}}$$

(2.3.46) .

The quantities $Q_{n_{LA}}$ and $Q_{n_{UA}}$ have previously been defined and may be obtained as the minimum and maximum of (2.3.43). Similarly $Q_{n_{LR}}$ and $Q_{n_{UR}}$ are the minimum and maximum of (2.3.12). $Q_{n_{LI}}$ and $Q_{n_{UI}}$ represent the two intersection points of AE and RE, and are defined as:

FIGURE 10

Integration Region When Both a Decision to Accept and Reject is Possible Situation 2



$$Q_{n_{LI}} = \min\left\{R_1, R_2\right\}$$

$$Q_{n_{UI}} = \max\left\{R_1, R_2\right\}$$

where

$$R_1 = \frac{(a+b)n + n\sqrt{(a+b)^2 - 2\left(\frac{n+1}{n}\right)\left(a^2+b^2-c\right)}}{2(n+1)}$$

$$R_2 = \frac{n(a+b) - n\sqrt{(a+b)^2 - 2\left(\frac{n+1}{n}\right)\left(a^2+b^2-c\right)}}{2(n+1)} \qquad (2.3.47).$$

The U integration limits for each piece are again obtained as:

$$U_{Ui} = b - Q_{n_{Li}}$$

$$U_{Li} = b - Q_{n_{Ui}}$$

Similarly for a given value of U, say $U^* = b - Q^*$, the range of W_n for each region is given by:

Region I: $W_{n_{L1}} = W_{n_{LR}} \leq W_n \leq W_{n_{UR}} = W_{n_{U1}}$

Region II: $W_{n_{L2}} = W_{n_{UA}} \leq W_n \leq W_{n_{UR}} = W_{n_{U2}}$

Region III: $W_{n_{L3}} = W_{n_{LR}} \leq W_n \leq W_{n_{LA}} = W_{n_{U3}}$

Region IV: $W_{n_{L4}} = W_{n_{LR}} \leq W_n \leq W_{n_{LA}} = W_{n_{U4}}$

Region V: $W_{n_{L5}} = W_{n_{LR}} \leq W_n \leq W_{n_{UR}} = W_{n_{U5}}$

$W_{n_{LR}}$ and $W_{n_{UR}}$ being the maximum and minimim of (2.3.45), and $W_{n_{LA}}$ and $W_{n_{UA}}$ the same for (2.3.13).

The Z limits are obtained for each region as:

$$Z_{Ui} = a - W_{n_{Li}}$$

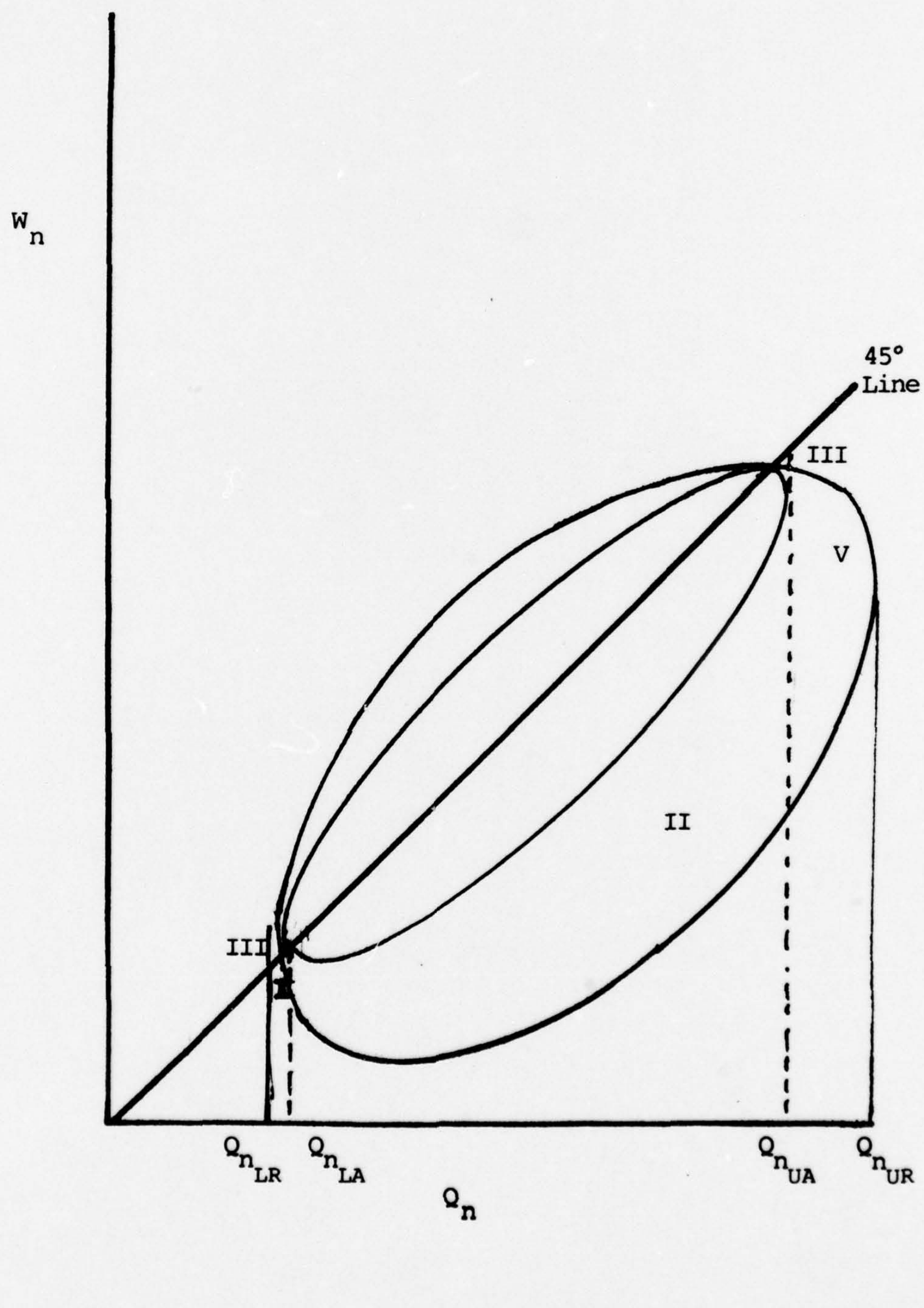
$$Z_{Li} = a - W_{n_{Ui}}$$

Situation (3) results whenever $a < b$, and again requires that equation (2.3.8) be split up into five separate integrals, as shown in Figure 11. In this case the range of Q_n for each of the subregions is as follows:

$$\begin{aligned}
&\text{Region I:} && Q_{n_{L1}} = Q_{n_{LR}} \le Q_n \le Q_{n_{LA}} = Q_{n_{U1}} \\
&\text{Region II:} && Q_{n_{L2}} = Q_{n_{LA}} \le Q_n \le Q_{n_{UA}} = Q_{n_{U2}} \\
&\text{Region III:} && Q_{n_{L3}} = Q_{n_{LA}} \le Q_n \le Q_{n_{LI}} = Q_{n_{U3}} \\
&\text{Region IV:} && Q_{n_{L4}} = Q_{n_{UI}} \le Q_n \le Q_{n_{UA}} = Q_{n_{U4}} \\
&\text{Region V:} && Q_{n_{L5}} = Q_{n_{UA}} \le Q_n \le Q_{n_{UR}} = Q_{n_{U5}}
\end{aligned} \qquad (2.3.48).$$

FIGURE 11

Integration Region When Both a Decision to Accept and Reject is Possible Situation 3



For a given value of $U^* = b - Q^*$ the W_n range for each region is:

$$\text{Region I:} \quad W_{n_{L1}} = W_{n_{LR}} \le W_n \le W_{n_{UR}} = W_{n_{U1}}$$

$$\text{Region II:} \quad W_{n_{L2}} = W_{n_{LR}} \le W_n \le W_{n_{LA}} = W_{n_{U2}}$$

$$\text{Region III:} \quad W_{n_{L3}} = W_{n_{UA}} \le W_n \le W_{n_{UR}} = W_{n_{U3}}$$

$$\text{Region IV:} \quad W_{n_{L4}} = W_{n_{UA}} \le W_n \le W_{n_{UR}} = W_{n_{U4}}$$

$$\text{Region V:} \quad W_{n_{L5}} = W_{n_{LR}} \le W_n \le W_{n_{UR}} = W_{n_{U5}} \tag{2.3.49}$$

2.4 OBTAINING THE PROBABILITIES OF ACCEPTANCE, REJECTION AND CONTINUATION

The previous section (2.3) has given methods for calculating the density $f_{n+1}(a,b,c)$, for a given point $W_{n+1} = a$, $Q_{n+1} = b$, $R_{n+1} = c$, from the density at stage n, $f_n(W_n,Q_n,R_n)$. Once this density has been obtained for all possible values of a, b, c, the probability of accepting $H_0 (P_A^{n+1})$, probability of rejecting H_0 as (P_R^{n+1}), and the probability of continuing (P_C^{n+1}) must be calculated. This requires integrating the three dimensional density $f_{n+1}(W_{n+1},Q_{n+1},R_{n+1})$ over all values of W_{n+1}, Q_{n+1}, R_{n+1} for which the statistic

$$V\left(W_{n+1},Q_{n+1},R_{n+1}\right) = \frac{\left[W_{n+1}-Q_{n+1}\right]^2}{2\left[(n+1)R_{n+1}-Q_{n+1}^{\;2}-W_{n+1}^{\;2}\right]}$$

is in the appropriate region (e.g., acceptance region, rejection region, or continuation region). Thus P_A^{n+1}, P_R^{n+1}, and P_C^{n+1} may be calculated as:

$$P_R^{n+1} = \iiint\limits_{0 \le V(W,Q,R) \le V_R^{n+1}} f_{n+1}(W,Q,R)\,dWdQdR$$

$$P_A^{n+1} = \iiint\limits_{V_R^{n+1} \le V(W,Q,R) \le \infty} f_{n+1}(W,Q,R)\, dW\, dQ\, dR$$

and

$$P_C^{n+1} \iiint\limits_{V_A^{n+1} \le V(W,Q,R) \le V_A^{n+1}} f_{n+1}(W,Q,R)\, dW\, dQ\, dR$$

These integrals amount to integrating $f_{n+1}(W,Q,R)$ over elliptic paraboloids, and may be reexpressed as the following iterated integrals:

$$P_A^{n+1} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{R_a}^{\infty} f_{n+1}(W,Q,R)\, dR\, dW\, dQ$$

$$P_R^{n+1} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{R_o}^{R_r} f_{n+1}(W,Q,R)\, dR\, dW\, dQ$$

$$P_C^{n+1} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{R_r}^{R_a} f_{n+1}(W,Q,R)\, dR\, dW\, dQ \qquad (2.4.1).$$

with

$$R_o = \frac{W^2}{n+1} + \frac{Q^2}{n+1}$$

$$R_a = \frac{[W-Q]^2}{2(n+1)V_A^{n+1}} + \frac{W^2}{n+1} + \frac{Q^2}{n+1}$$

$$R_r = \frac{[W-Q]^2}{q(n+1)V_R^{n+1}} + \frac{W^2}{n+1} + \frac{Q^2}{n+1}$$

In practice only two of the three integrals need be calculated due to the following identity:

$$P_C^{\,n} = P_A^{\,n+1} + P_R^{\,n+1} + P_C^{\,n+1} .$$

So if $P_A^{\,i}$ and $P_R^{\,i}$ are calculated at each stage i, $P_C^{\,i}$ may be obtained by subtraction,

$$P_C^{\,i} = P_C^{\,i-1} - P_A^{\,i} \; P_R^{\,i} .$$

2.5 SUMMARY OF THE DIRECT METHOD FOR A k=2 SANOVA TEST

The purpose of this section is to summarize the procedure for obtaining the OC and ASN curves for a k=2 SANOVA test.

First, a test of this type requires specification of the following quantities:

(1) The null hypothesis value, λ_0.

(2) The alternative hypothesis value, λ_1.

(3) A truncation point, m_0.

(4) A set of regions: V_A^i, V_R^i, $i = 2,\ldots, m_0$, such that at any stage N

These regions are to be compared with the statistic, V_n, of equation (2.3.1), such that at any stage n,

(a) H_0 is accepted if $V_n \leq V_A^n$

(b) H_1 is accepted if $V_n \geq V_R^n$.

(5) Values of α and β (needed only if the regions are to be modified Wald regions).

Second, the first step at which a decision can be made, say n_1, $2 \leq n_1 \leq m_0$, is determined.

Third, one must determine how many and which points on the OC and ASN curves will be calculated. Suppose L values are chosen, denoted by λ_ℓ^*, $\ell = 1,\ldots, L$, such that $\lambda_0 = \lambda_1^* < \lambda_2^* < \ldots < \lambda_L^* = \lambda_1$.

For a given λ_ℓ^*, the first stage density $f_{n_1}(W_{n_1}, Q_{n_1}, R_{n_1})$ may be calculated as follows:

$$f_{n_1}(W_{n_1}, Q_{n_1}, R_{n_1}) = \left(\frac{1}{n_1^{\,2}}\right) \chi^2_{2(n_1-1)}\left[R_{n_1} - \frac{W_{n_1}^{\;2}}{n_1} - \frac{Q_{n_1}^{\;2}}{n_1}\right]$$

$$\cdot \; \phi\left(\sqrt{n_1}\left(\frac{W_{n_1}}{n_1}\right)\right) \cdot \phi\left(\sqrt{n_1}\left(\frac{Q_{n_1}}{n_1} - \sqrt{\lambda^*_\ell}\right)\right) \qquad (2.5.1)$$

Note that this density is completely specified by χ^*_ℓ and n_1.

The probabilities of acceptance, rejection, and continuation at stage n_1 (the first stage at which a decision can be made); $P_A^{n}1$, $P_R^{n}1$, and $P_C^{n}1$, may be calculated using the noncentral F distribution (given in equation (1.1.1)) and is shown in appendix A.

To calculate the joint density at the next stage, $f_{n_1+1}(W_{n_1+1}, Q_{n_1+1}, R_{n_1+1})$, requires utilizing the procedures developed in section 2.3.

As shown in section 2.3, this consists of performing a bivariate integration of the following five dimensional joint density function.

$$f_{n_1}\left(^{P}(W_{n_1}, Q_{n_1}, R_{n_1}, X_{1n_2}, X_{2n_2}\right) = f_{n_1}\left(W_{n_1}, Q_{n_1}, R_{n_1}\right)$$

$$\cdot \; \phi\left(X_{1n_2}\right) \cdot \phi\left(X_{2n_2} - \sqrt{\lambda^*_\ell}\right) \qquad (2.5.2)$$

where $n_2 = n_1 + 1$.

This is the joint density of the statistics at stage n; W_{n_1}, Q_{n_1}, R_{n_1}; and the new observations taken at stage $n_2 = n_1 + 1$; X_{1n_2}, X_{2n_2}.

For any given point: $W_{n_1+1} = a$, $Q_{n_1+1} = b$, $R_{n_1+1} = C$; the joint density $f_{n_1+1}(a,b,c)$ is calculated by performing the following bivariate integration.

$$f_{n_1+1}(a,b,c) = \int_{U_L}^{U_U} \int_{Z_L}^{Z_U} f_{n_1}{}^{P}(a-Z,b-U,c-Z^2-U^2,Z,U)\, dZ\, dU \tag{2.5.3}$$

The limits U_L, U_U, Z_L, and Z_U are dependent upon the particular point (a,b,c) as well as the regions $V_A{}^{n_1}$ and $V_R{}^{n_1}$.

If no decision could be made at stage n_1, these limits are the limits for integrating around the following circle.

$$c - z^2 - u^2 - \frac{(a-z)^2}{n_1} - \frac{(b-u)^2}{n_1} = 0 \tag{2.5.4}$$

and are given in equations (2.3.6) and (2.3.7). Whenever a decision can be made at stage n, the integration region becomes a subset of the points contained inside this circle.

In some cases the integral given in equation (2.5.3) cannot be evaluated as one integral; rather it must be broken up into several pieces, with the overall integral being the sum of the individual integrals. Equation (2.3.25) is such an example. In such cases, the integration limits for each of the pieces must be determined.

The required integration region for equation (2.5.3) can be one of many. In section (2.3) every possible integration region has been explored; and for each case specific expressions for the U,Z integration limits have been given.

The U,Z integration determination may be best summarized in flowchart format, such as shown in Figure 12

This integration must be determined and performed for all points W_{n_1+1}, Q_{n_1+1}, R_{n_1+1}, thus obtaining the entire density $f_{n_1+1}(W_{n_1+1}, Q_{n_1+1}, R_{n_1+1})$. From this density the probabilities of acceptance $(P_A^{n_1+1})$, rejection $(P_R^{n_1+1})$, and continuation $(P_C^{n_1+1})$ must be calculated. Their calculation requires performing a trivariate integration of the density $f_{n_1+1}(W_{n_1+1}, Q_{n_1+1}, R_{n_1+1})$ over elliptic paraboloids. This is most easily performed as iterated integrals as shown in (2.4.1).

The entire process of obtaining the density, $f_i(W_i, Q_i, R_i)$, from the density, $f_{i-1}^P(W_{i-1}, Q_{i-1}, R_{i-1}, X_{1i}, X_{2i})$, and ultimately the probabilities, P_A^i, P_R^i, P_C^i, must be iterated for all stages, $i = n_1 + 2, \ldots m_0$.

The final result, for a given λ_ℓ^*, is the set of probabilities, P_A^i, P_R^i, P_C^i, $i = 2, \ldots m_0$. These probabilities will depend upon the value of λ_ℓ^*. This can easily be seen by noting that both the first step density of equation (2.5.1) as well as the five dimensional density of equation (2.5.2) are both functions of λ_ℓ^*. Therefore, the notation $P_A^i(\lambda_\ell^*)$, $P_R^i(\lambda_\ell^*)$, $P_C^i(\lambda_\ell^*)$, $i = 2, \ldots, m_0$, will be used to denote such a dependence. From these probabilities, the point on the OC and ASN curves for $\lambda = \lambda_\ell^*$ may be calculated. These quantities are calculated as follows:

$$OC(\lambda_\ell^*) = \sum_{L=Z}^{m_0} P_A^i(\lambda_\ell^*) \qquad (2.5.5)$$

and

$$ASN(\lambda_\ell^*) = \sum_{L=Z}^{m_0} P_R^i(\lambda_\ell^*) + P_A^i(\lambda_\ell^*) \cdot i = 1 + \sum_{L=Z}^{m_0} P_C^i(\lambda_\ell^*) \qquad (2.5.6)$$

Note that, by having all the probabilities $P_A^i(\lambda_\ell^*)$, $P_R^i(\lambda_\ell^*)$, $P_C^i(\lambda_\ell^*)$, other quantities of interest may also be calculated (e.g. variance of DSN, median of DSN, percentile of DSN, etc.).

This entire process has given a single point on the OC and ASN curves. To obtain the next point on the OC and ASN curves the density $f_{n_1}(W_{n_1}, Q_{n_1}, R_{n_1})$ must again be obtained from equation (2.5.1) with $\lambda = \lambda^*_{\ell+1}$. The process of obtaining the density, $f_i(W_i, Q_i, R_i)$, and the probabilities $P_A{}^i(\lambda^*_{\ell+1})$, $P_R{}^i(\lambda^*_{\ell+1})$, $P_C{}^i(\lambda^*_{\ell+1})$, must then be iterated for all stages $i = n_1+1, \ldots, m_0$.

The Direct Method for $K = 2$ SANOVA has been summarized in flowchart format as shown in Figure 13.

FIGURE 12

For any given stage, n: with regions V_A and V_R, the density of the point $(W_n = a, Q_n = b, R_n = c)$ is found by integrating the density of equation (2.3.5) as shown in equation (2.3.8). The integration limits U_U, U_L, Z_U, Z_L may be obtained from the following flowchart.

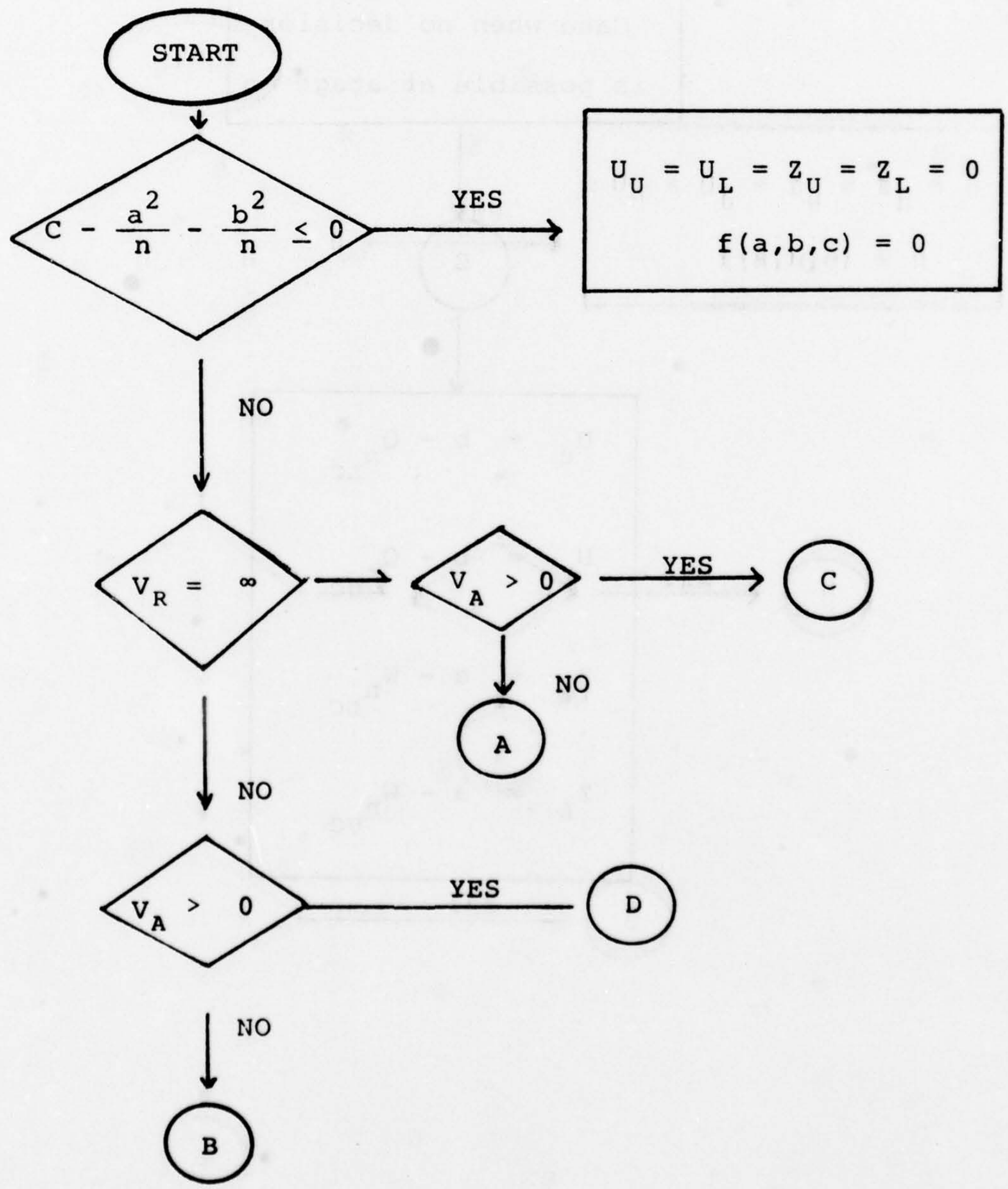


FIGURE 12 (continued)

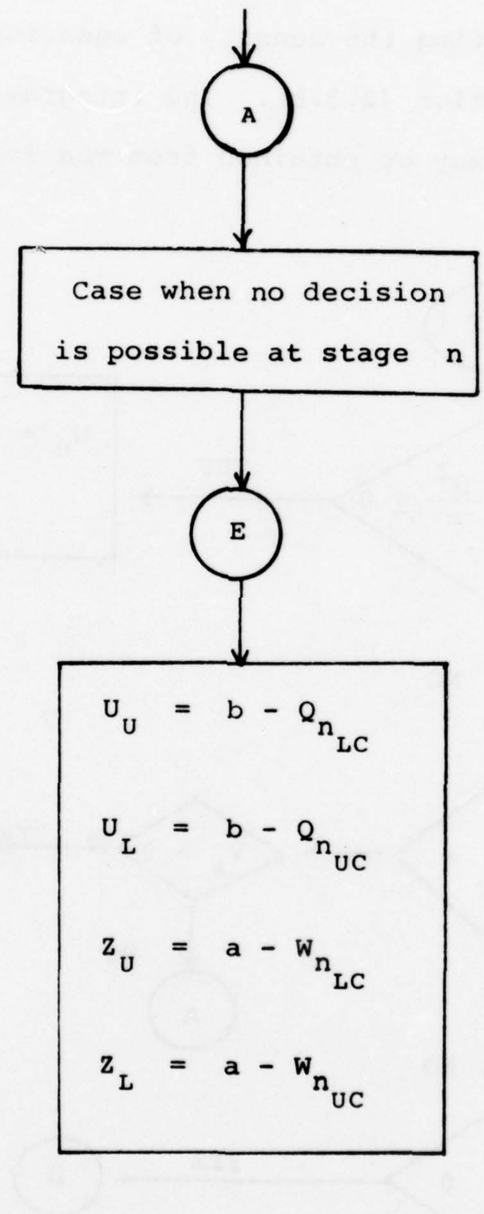


FIGURE 12 (continued)

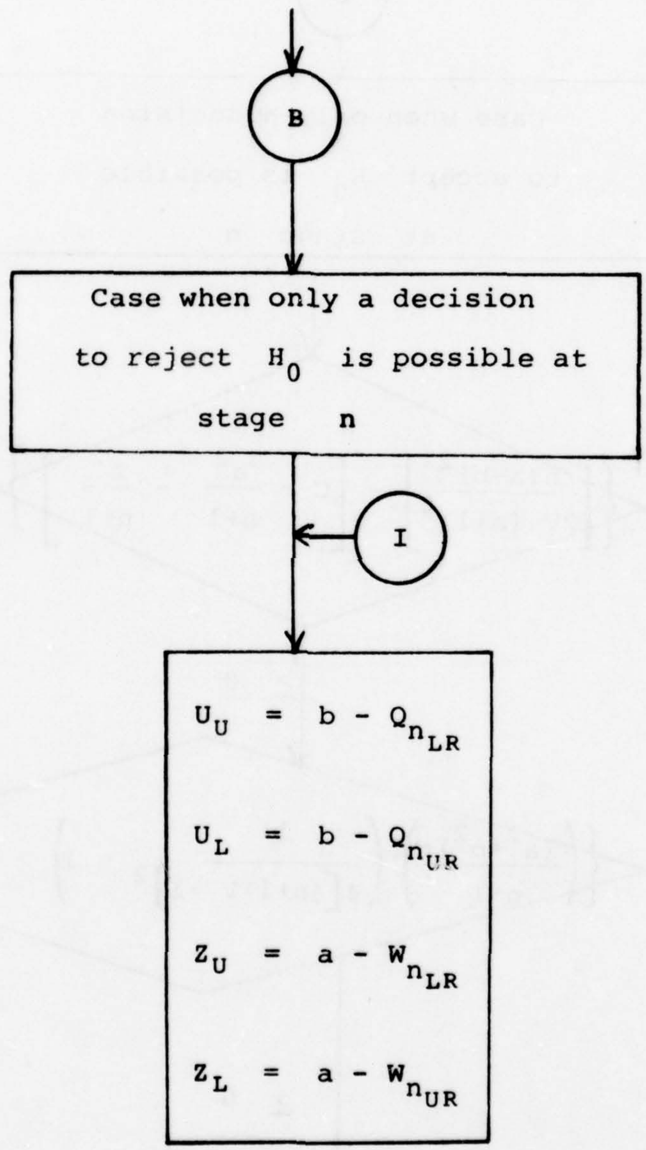


FIGURE 12 (Continued)

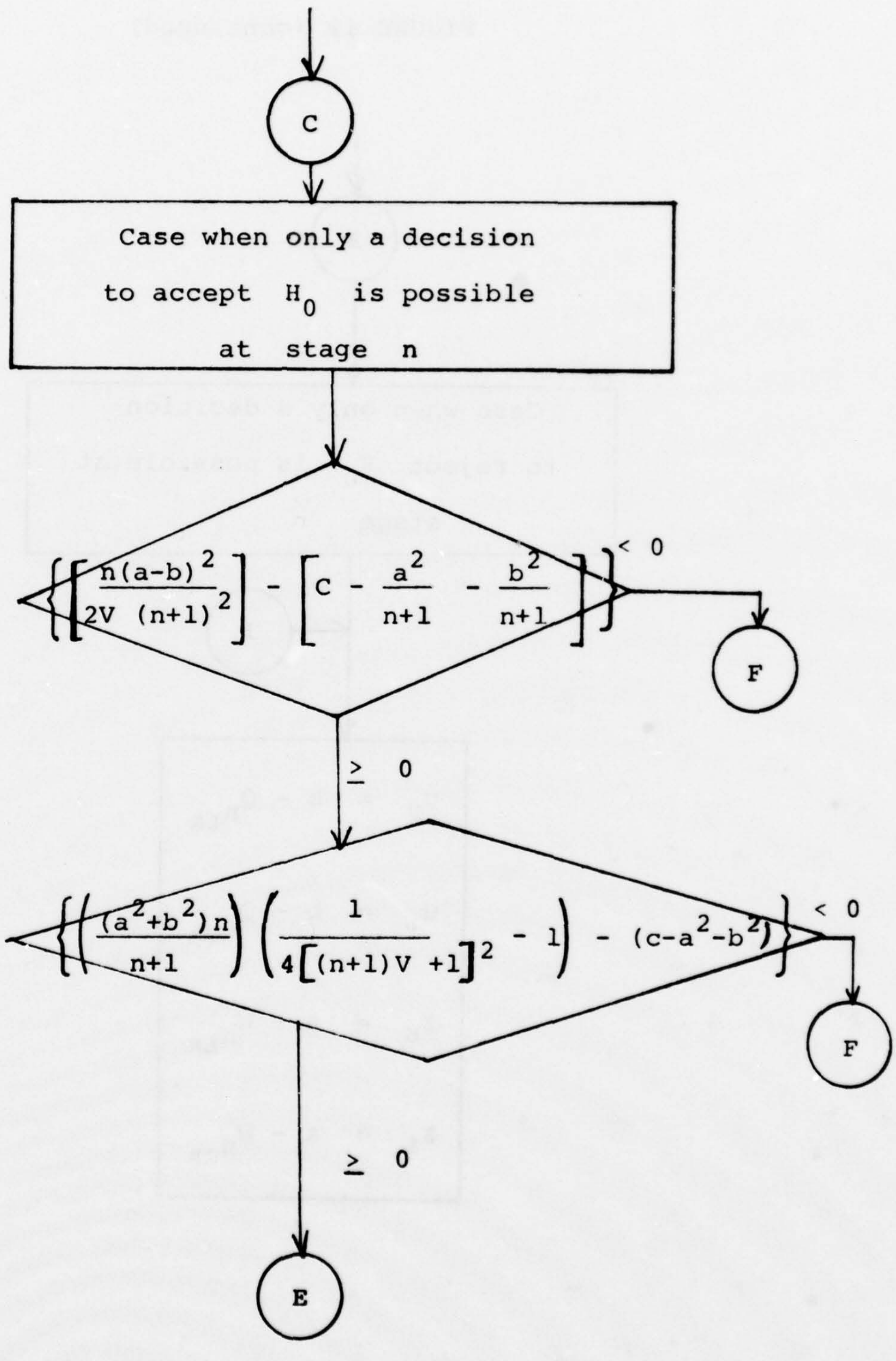


FIGURE 12 (Continued)

F

The integral must be broken up into at most 4 pieces as given in equation (2.3.25).

The number of pieces will depend upon the sign of $(a+B)^2 - 2(1+\frac{1}{n})(a^2+b^2-c)$

The integration limits for each piece are given by:

Piece	U Limits	Z Limits
1	$U_U = b - Q_{n_{LC}}$	$Z_U = a - W_{n_{LC}}$
	$U_L = b - Q_{n_{LA}}$	$Z_L = a - W_{n_{UC}}$
2	$U_U = b - Q_{n_{UA}}$	$Z_U = a - W_{n_{LC}}$
	$U_L = b - Q_{n_{UC}}$	$Z_L = a - W_{n_{UC}}$
3	$U_U = b - Q_{n_{LA}}$	$Z_U = a - W_{n_{UA}}$
	$U_L = b - Q_{n_{UA}}$	$Z_L = a - W_{n_{UC}}$
4	$U_U = b - Q_{n_{LA}}$	$Z_U = a - W_{n_{LC}}$
	$U_L - b - Q_{n_{UA}}$	$Z_L = a - W_{n_{LA}}$

FIGURE 12 (Continued)

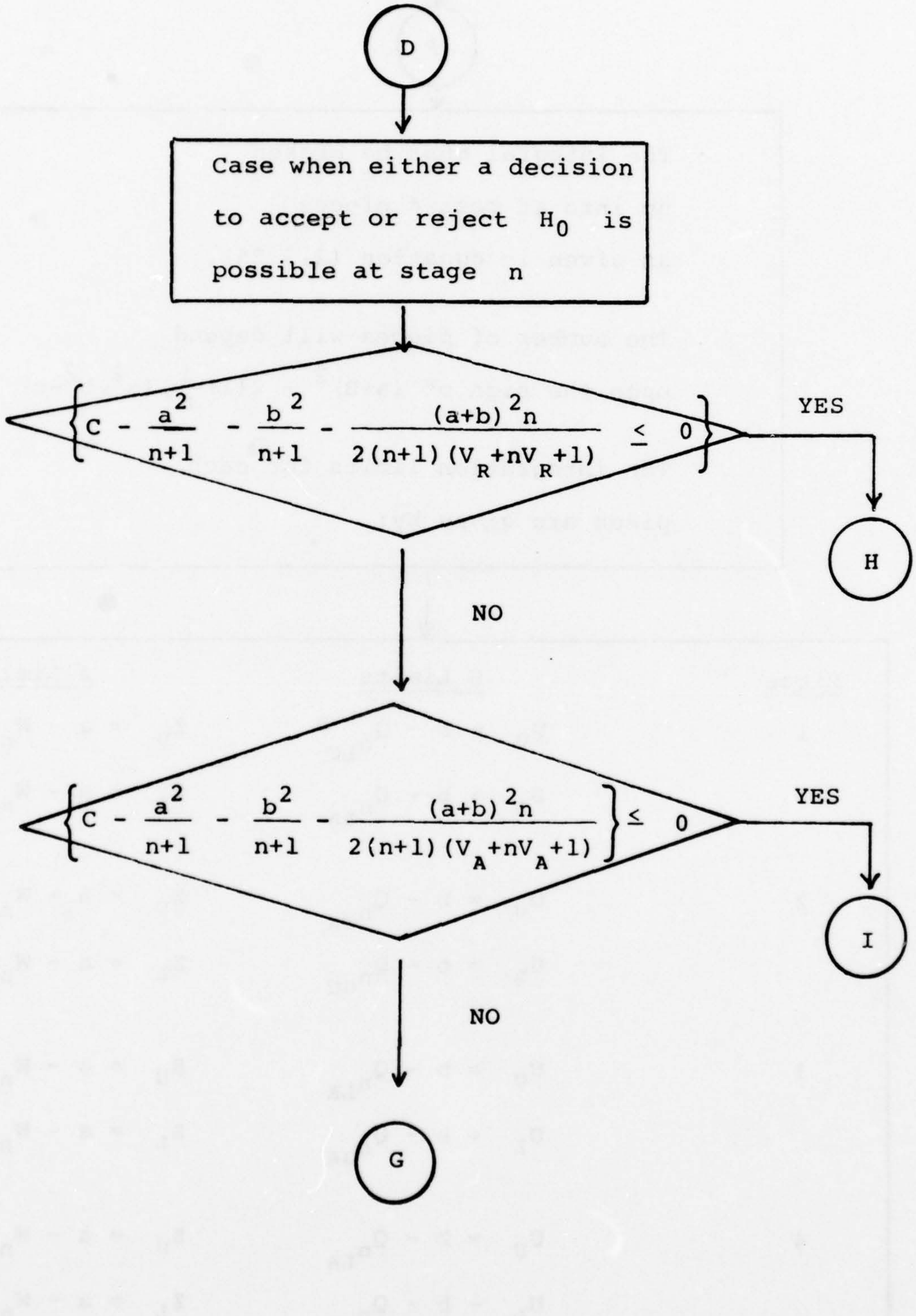


FIGURE 12 (Continued)

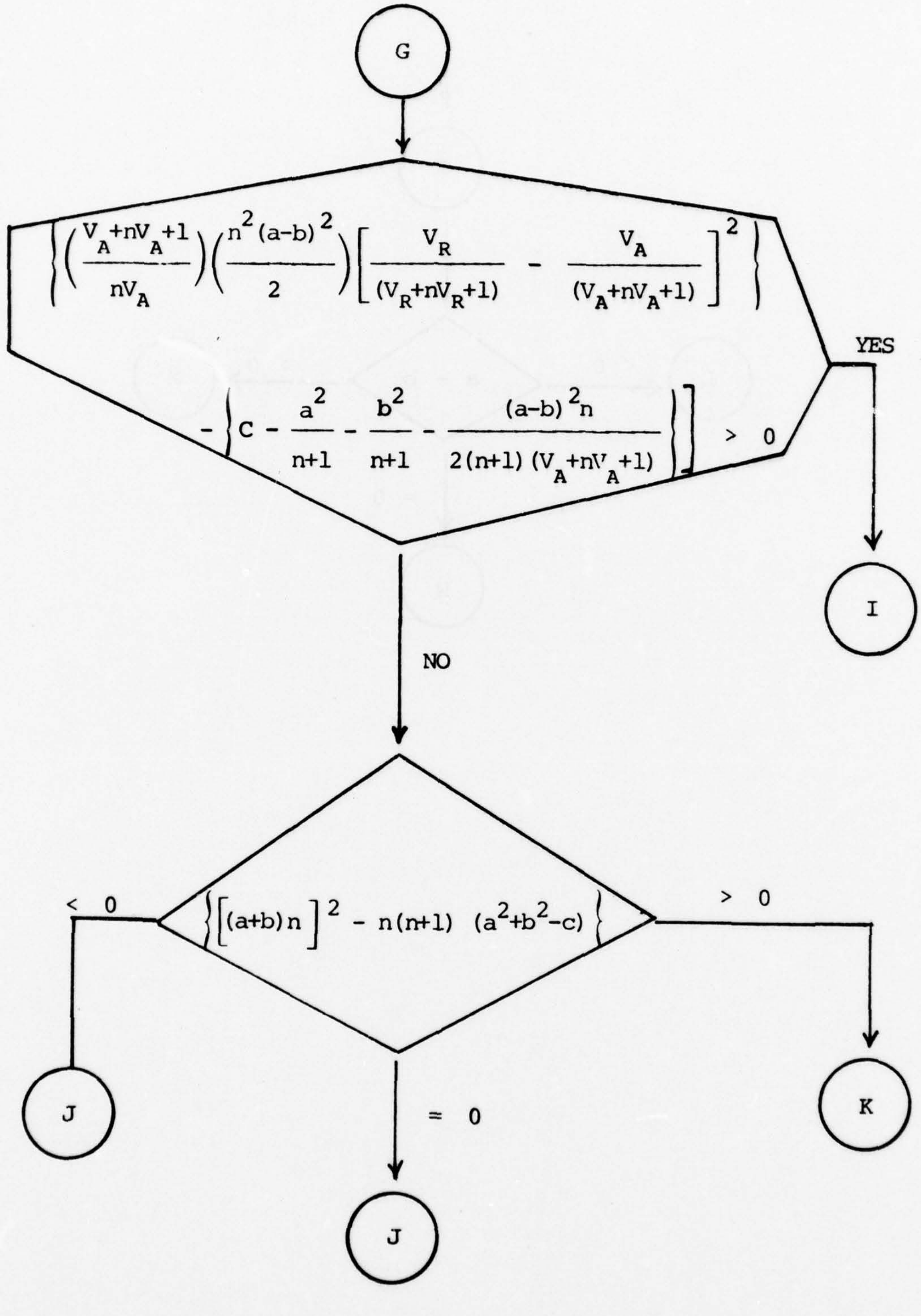


FIGURE 12 (Continued)

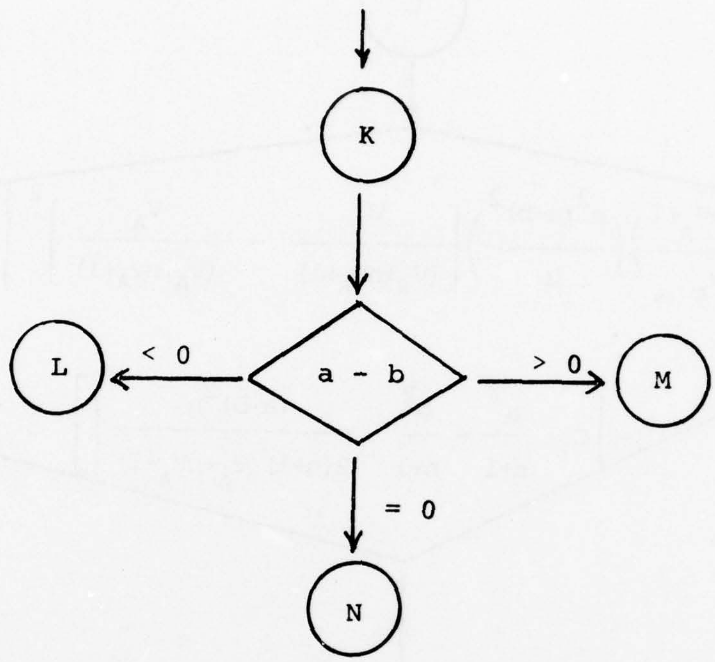


FIGURE 12 (Continued)

J

The integral must be broken up into at most four pieces.

One or two of the pieces may be null.

The integration regions for each piece are given as:

N

Piece	U Limits	Z Limits
1	$U_U = b - Q_{n_{LR}}$	$Z_U = a - W_{n_{LR}}$
	$U_L = b - Q_{n_{LA}}$	$Z_L = a - W_{n_{UR}}$
2	$U_U = b - Q_{n_{UA}}$	$Z_U = a - W_{n_{LR}}$
	$U_L = b - Q_{n_{UR}}$	$Z_L = a - W_{n_{UR}}$
3	$U_U = b - Q_{n_{LA}}$	$Z_U = a - W_{n_{UA}}$
	$U_L = b - Q_{n_{UA}}$	$Z_L = a - W_{n_{UR}}$
4	$U_U = b - Q_{n_{LA}}$	$Z_U = a - W_{n_{LR}}$
	$U_L = b - Q_{n_{UA}}$	$Z_L = a - W_{n_{LA}}$

FIGURE 12 (Continued)

L

Integral must be broken up into five pieces. The U,Z integration regions for each piece are as follows:

Piece #	U Limits	Z Limits
1	$U_U = b - Q_{n_{LR}}$	$Z_U = a - W_{n_{LR}}$
	$U_L = b - Q_{n_{LA}}$	$Z_L = a - W_{n_{UR}}$
2	$U_U = b - Q_{n_{LA}}$	$Z_U = a - W_{n_{LR}}$
	$U_L = b - Q_{n_{UA}}$	$Z_L = a - W_{n_{LA}}$
3	$U_U = b - Q_{n_{LA}}$	$Z_U = a - W_{n_{UA}}$
	$U_L = b - Q_{n_{LI}}$	$Z_L = a - W_{n_{UR}}$
4	$U_U = b - Q_{n_{UI}}$	$Z_U = a - W_{n_{UA}}$
	$U_L = b - Q_{n_{UA}}$	$Z_L = a - W_{n_{UR}}$
5	$U_U = b - Q_{n_{UA}}$	$Z_U = a - W_{n_{LR}}$
	$U_L = b - Q_{n_{UR}}$	$Z_L = a - W_{n_{UR}}$

FIGURE 12 (Continued)

M

Integral must be broken up into five pieces. The U,Z integration limits for each piece are given by:

Piece #	U Limits	Z Limits
1	$U_U = b - Q_{n_{LR}}$	$Z_U = a - W_{n_{LR}}$
	$U_L = b - Q_{n_{LA}}$	$Z_L = a - W_{n_{UR}}$
2	$U_U = b - Q_{n_{LA}}$	$Z_U = a - W_{n_{UA}}$
	$U_L = b - Q_{n_{UA}}$	$Z_L = a - W_{n_{UR}}$
3	$U_U = b - Q_{n_{LA}}$	$Z_U = a - W_{n_{LR}}$
	$U_L = b - Q_{n_{LI}}$	$Z_L = a - W_{n_{LA}}$
4	$U_U = b - Q_{n_{UI}}$	$Z_U = a - W_{n_{LR}}$
	$U_L = b - Q_{n_{UA}}$	$Z = a - W_{n_{LA}}$
5	$U_U = b - Q_{n_{UA}}$	$Z_U = a - W_{n_{LR}}$
	$U_L = b - Q_{n_{UR}}$	$Z_U = a - W_{n_{UR}}$

FIGURE 13

FLOWCHART

SUMMARY OF DIRECT METHOD

FOR

K=2 SANOVA

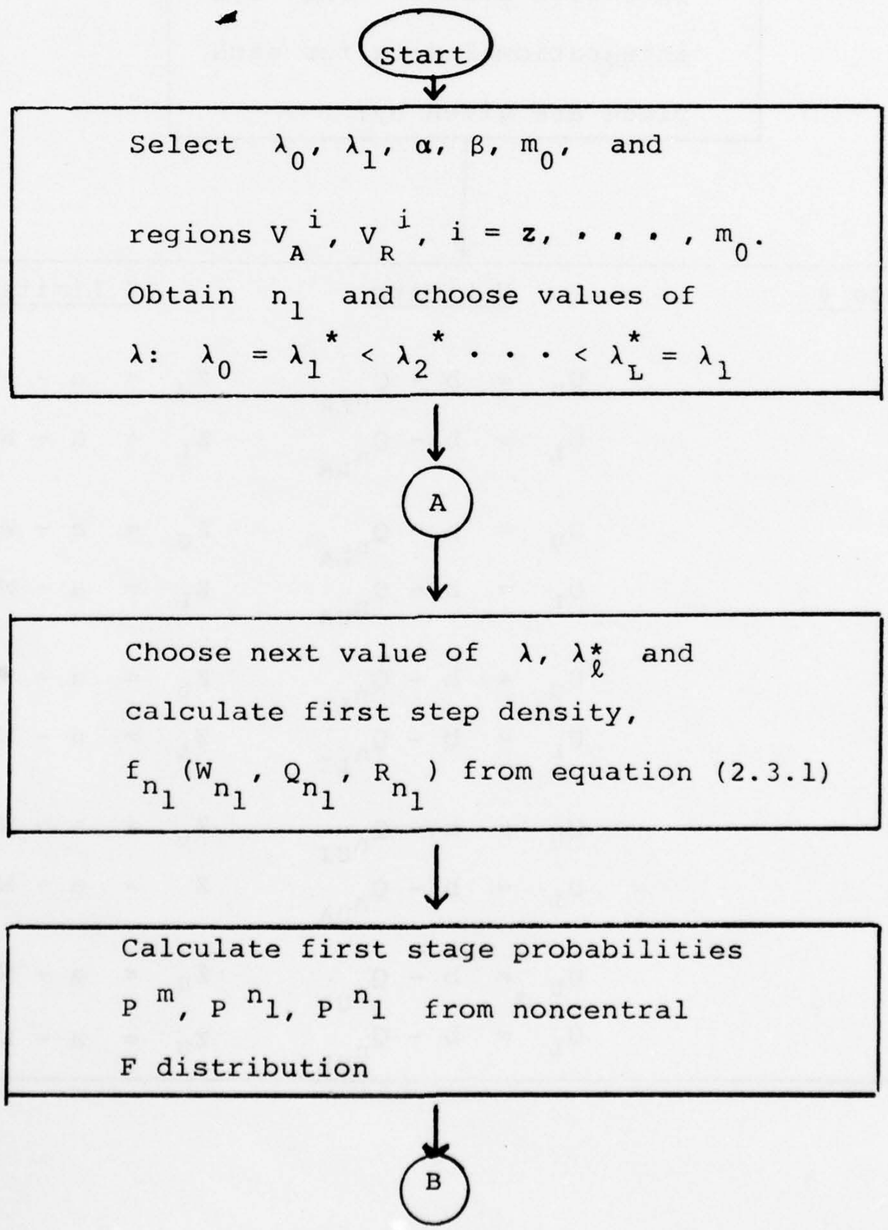


FIGURE 13 (Continued)

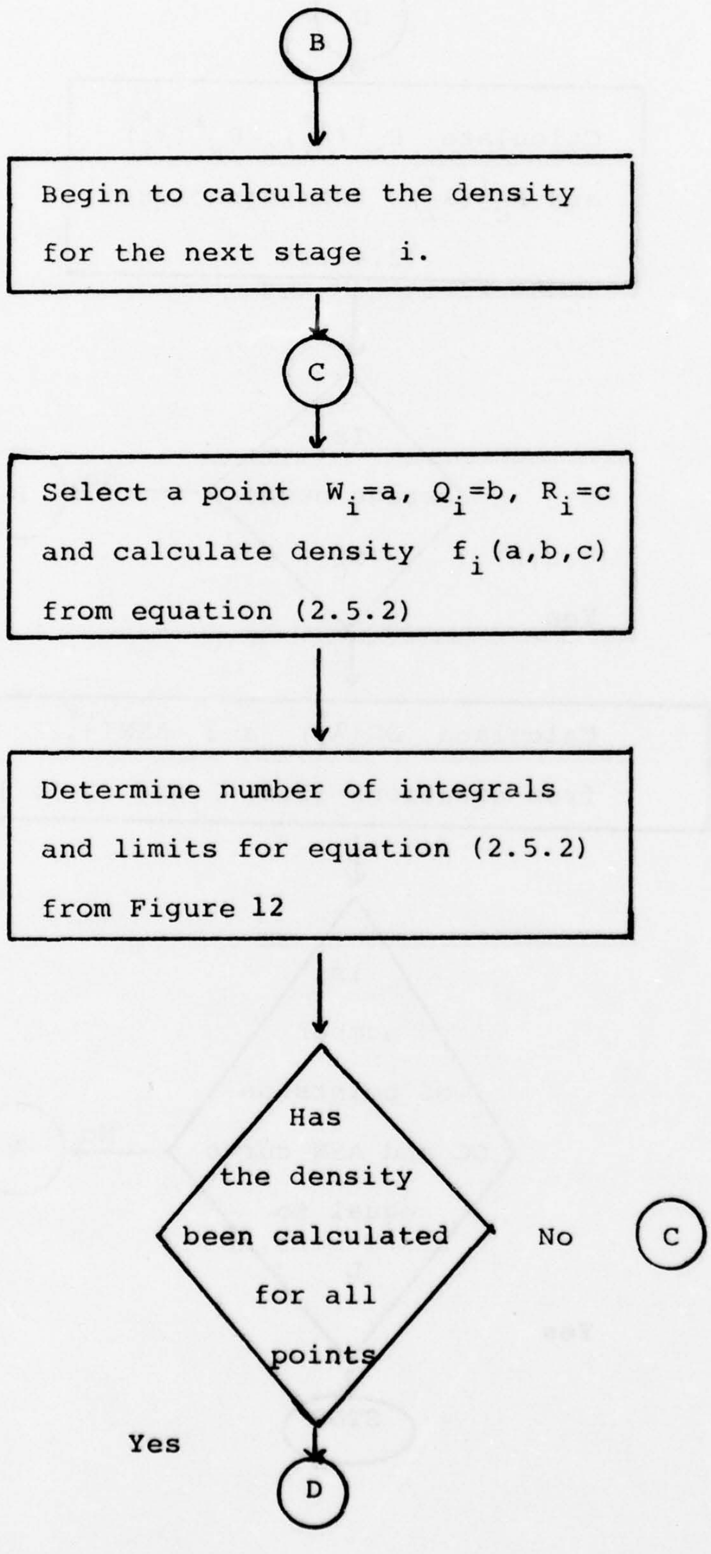
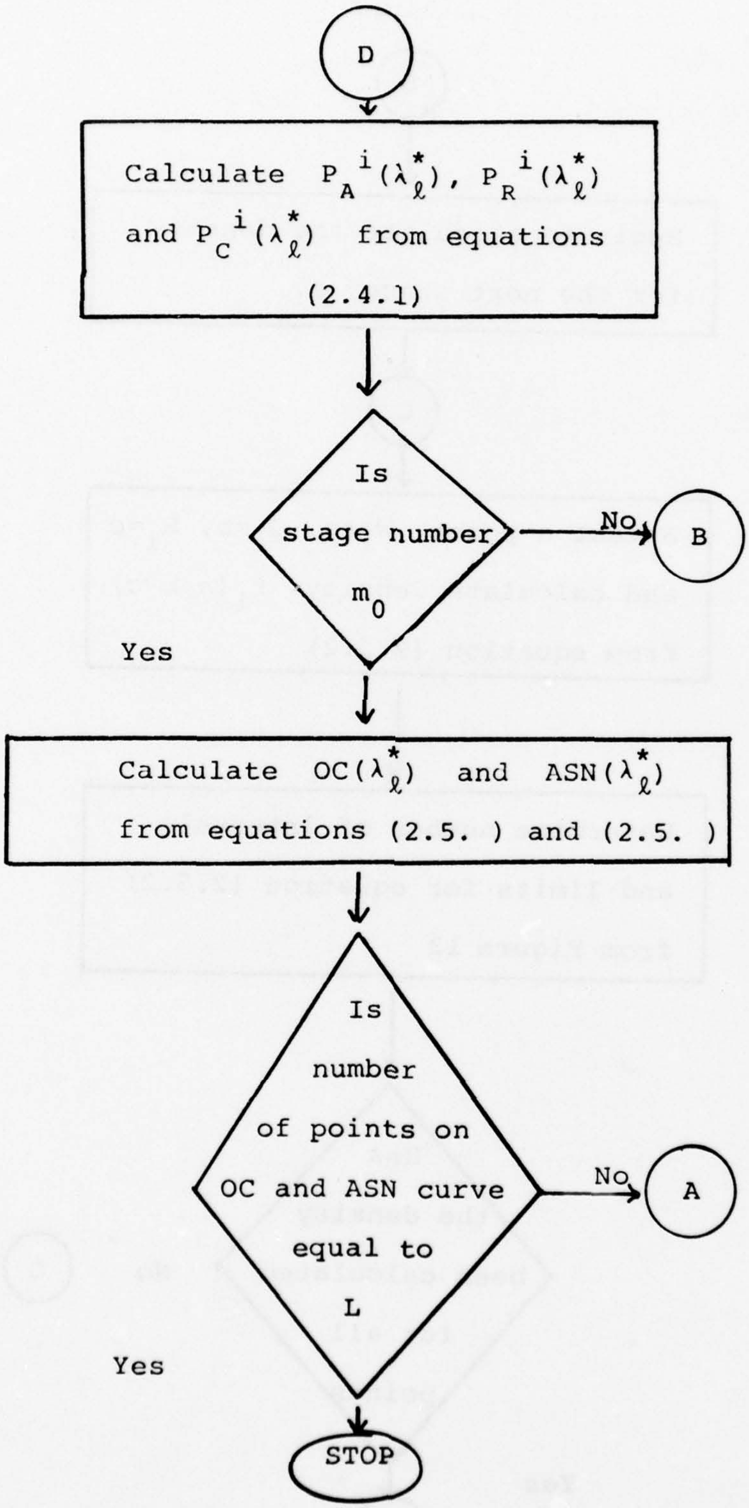


FIGURE 13 (Continued)



2.6 Numerical Methods

The previous sections have given a detailed description and derivation for obtaining the properties of a K=2 SANOVA test by the direct method.

In summary the procedure requires the following steps:

1. For a given value of $\lambda = \lambda^*$, determine the joint density at the first stage at which a decision can be made; $f_{n_1}(W_{n_1}, Q_{n_1}, R_{n_1})$.

2. Calculate the joint density $f_{n_{1+1}}(W_{n_{1+1}}, Q_{n_{1+1}}, R_{n_{1+1}})$, for all values of $W_{n_{1+1}}, Q_{n_{1+1}}, R_{n_{1+1}}$. This requires:

   a. Forming the five dimensional joint density $f_{n_1}{}^P(W_{n_1}, Q_{n_1}, R_{n_1}, X_{1n_2}, X_{2n_2})$ as given in equation (2.5.2).

   b. Performing the bivariate integration on this five dimensional joint density given in equation (2.5.3).

3. Performing a trivariate integration of the density $f_{n_1+1}(W_{n_1+1}, Q_{n_1+1}, R_{n_1+1})$ to obtain the probabilities of acceptance $(P_A{}^{n_1+1})$, rejection $(P_R{}^{n_1+1})$, and continuation $(P_C{}^{n_1+1})$.

4. Iterating steps 2 and 3 on the density $f_i(W_i, Q_i, R_i)$ for all $i = n_1+2,\ldots,M_0$.

5. Calculating the OC and ASN for $\lambda = \lambda^*$.

6. Repeating steps 1 through 5 for all values of λ^* of interest.

This section will discuss the practical evaluation of the integrals required in steps 2 and 3 of the above procedure.

These integrals will generally be very complicated expressions. For example, wherever no decision can be made, the U, Z region of integration required for step 2 consists of all U, Z contained inside the circle given in equation (2.5.4). The actual U, Z integration limits required are given in equations (2.3.6) and (2.3.7). This amounts to integrating a five dimensional joint density composed of the product of a χ^2 and four normal densities. This integration can be evaluated analytically, yielding the density given at the top of page 2-20. The cases which require integrating aroung ellipses (e.g., equations (2.3.12) - (2.3.14)) or those that require breaking the integral up into several pieces (e.g., equations (2.3.25) - (2.3.28)), generally can not be evaluated analytically.

One approach is to develop a numerical approximation to these bivariate integrals (e.g., series, partial fraction, or continued fraction expansions). Since the integration region required is dependent upon the point a, b, c (for a given set of regions), this approach would yield the following type of piecewise trivariate density for stage n_1+1:

$$f_{n_{1+1}}(W_{n_{1+1}}, Q_{n_{1+1}}, R_{n_{1+1}}) = \begin{cases} \text{Expression 1} & \text{all } W_{n_{1+1}}, Q_{n_{1+1}}, R_{n_{1+1}} \in \mathbf{R}_1 \\ \vdots & \\ \vdots & \\ \text{Expression K} & \text{all } W_{n_{1+1}}, Q_{n_{1+1}}, R_{n_{1+1}} \in \mathbf{R}_K \end{cases}$$

An analytic expression for the integral required in step 3 with this type of piecewise trivariate density function would probably not exist. Also, if one were to continue along these lines, the density at later stages; $f_i\ (W_i,\ Q_i,\ R_i)$, $i = n_1 + 2, \ldots, m_0$; would become a piecewise function with an infeasible number of pieces.

An alternative method for evaluating these integrals is via numerical integration. The analytic density $f_n\ (W_n,\ Q_n,\ R_n)$ may be represented by a discrete three dimensional grid of points; $f_n\ (W_{n_i},\ Q_{n_j},\ R_{n_k})$, $i = 1, \ldots, N_W$, $j = 1, \ldots, N_Q$, $K = 1, \ldots, N_R$; so that for a given point on this grid; $W_{n_{l+1}} = a = W_{n_{i*}}$, $Q_{n_{l+1}} = b = Q_{n_{j*}}$, $R_{n_{l+1}} = c = R_{n_{k*}}$; the joint density may be approximated by the following expression:

$$f_{n_{l+1}}(a,b,c) \simeq \sum_m \sum_\ell \omega_{1\ell}\, \omega_{2m}\, f^P_{n_{l+1}}(a-Z_\ell,\, b-U_m,\, c-Z_\ell^2-U_m^2,\, Z_\ell,\, U_m)$$

The quantities $\omega_{1\ell}$, ω_{zm} and Z_ℓ, U_m are the required weights and coordinates of the integration scheme employed and depend upon the U, Z region of integration.

Repeating this procedure for all a, b, c contained on this grid yields a new grid representing the density at stage n_{l+1}, $f_{n_{l+1}}(W_{n_i}, Q_{n_j}, R_{n_k})$. From this new grid the probabilities $P_A^{n_1+1}$, $P_R^{n_1+1}$, $P_C^{n_1+1}$ must be obtained. Obtaining these probabilities requires a trivariate integration which can also be done numerically. For example

$$P_A^{n_1+1} \simeq \sum_m \sum_\ell \sum_p \omega_{1m}\, \omega_{2\ell}\, \omega_{3p}\, f_{n_{l+1}}(W_m, Q_\ell, R_p)\ . \tag{2.6.2}$$

This new grid can again be manipulated to obtain the density at stage n_1+2 and ultimately the probabilities $P_A^{n_1+2}$, $P_R^{n_1+2}$ and $P_C^{n_1+2}$. Repeating the procedure to obtain a new grid $f(W_{n_i}, Q_{n_j}, R_{n_k})$ and P_A^n, P_R^n, P_C^n for all $n = n_1+2,...,m_0$, allows the calculation of a point on the ASN and OC curves.

In general, the density at any point not on the grid; say f_i (W_i^*, Q_i^*, R_i^*), $i = n_1+2,\ldots,M_0$; must be found by interpolation. Note that this could require the formidable task of interpolating in three dimensions. Thus, it would be desirable to use a grid scheme and integration rule that required a minimum amount of interpolation to evaluate equations (2.4.1) and (2.5.3).

First, consider the following grid scheme:

$$
\begin{aligned}
W_{n_i} &= \left[W_S + (i-1)\alpha_i\right] h_W & i &= 1,\ldots,N_W \\
Q_{n_j} &= \left[Q_S + (j-1)\beta_j\right] h_Q & j &= 1,\ldots,N_Q \\
R_{n_k} &= \left[R_S + (k-1)\gamma_k\right] h_R & k &= 1,\ldots,N_R
\end{aligned}
\qquad (2.6.3\ ).
$$

The quantities α_i, β_j, γ_k are all integers chosen such that:

$$
\begin{aligned}
W_{n_1} &< W_{n_2} < \ldots\ldots < W_{nn_W} \\
Q_{n_1} &< Q_{n_2} < \ldots\ldots < Q_{nn_Q} \\
R_{n_1} &< R_{n_2} < \ldots\ldots < R_{nn_R}
\end{aligned}
$$

The choice of the quantities W_S, Q_S, R_S, h_W, h_Q, and h_R will be discussed later.

Many integration rules are available (Davis and Rabinowitz (1967)); but to avoid excessive amounts of interpolation a rule should be chosen which allows the majority of the points to be located on the grid.

For the integration given in (2.6.1) this requires that not only $a - Z_\ell$ and $b - U_m$ be located on W_n, Q_n grid points, but also that $C - Z_\ell^2 - U_m^2$ be located on an R_n grid point. This can be guaranteed if the quantities h_W, h_Q, and h_R are chosen such that:

$$h_R = \Lambda_1 h_W^2 + \Lambda_2 h_Q^2$$

or

$$h_W^2 = \Lambda_3 h_R \quad \text{and} \quad h_Q^2 = \Lambda_4 h_R , \tag{2.6.4}$$

where

Λ_1, Λ_2, Λ_3, and Λ_4 are integers.

Using this type of grid and the trapezoid integration rule, equation (2.6.1) becomes:

$$f_i(a,b,c) \simeq \sum_{m=0}^{n_Q} \sum_{\ell=0}^{n_W} \omega_{1\ell}\, \omega_{2m}\, f_{i-1}^P \left(a-Z_\ell,\ b-U_m,\ c-Z_\ell^2-U_m^2,\ Z_\ell,\ U_m\right) \tag{2.6.5}$$

where the coordinates Z_ℓ and U_m are given by the following scheme:

$$Z_0 = Z_L \qquad\qquad U_0 = U_L$$

$$Z_S = \left[Z_L / h_W\right] \qquad\qquad U_S = \left[U_L / h_Q\right]$$

$$Z_F = \left[Z_U / h_W\right] \qquad\qquad U_F = \left[U_U / h_Q\right]$$

$$N_W - 1 = Z_F - Z_{SS} + 1 \qquad N_Q - 1 = U_F - U_S + 1$$

$$Z_\ell - 1 = Z_S + (\ell - 1)h$$
$$\ell = 1, \ldots, N_W - 2$$

$$Z_{n_{W-1}} = Z_F$$

$$Z_{n_W} = Z_U$$

$$U_m = U_S + (m-1)h_W$$
$$m = 1, \ldots, N -2$$

$$U_{n_{Q-1}} = U_F \qquad (2.6.6)$$

$$U_{n_Q} = U_U$$

with

$$[X] \equiv \text{sign}(X) \cdot \{\text{greatest integer in } |x|\}.$$

The weights $\omega_{1\ell}$ and ω_{2m} are given by:

$$\omega_{10} = \tfrac{1}{2}|(Z_L - Z_0)|$$

$$\omega_{11} = \tfrac{1}{2}|(Z_S - Z_0)| + \tfrac{1}{2}h_W$$

$$\omega_{1\ell} = \tfrac{1}{2}h_W \qquad \ell = 2, \ldots, N_W - 2$$

$$\omega_{1\ell} = \tfrac{1}{2}h_W + \tfrac{1}{2}|(Z_U - Z_F)| \qquad \ell = N_W - 1$$

$$\omega_{1\ell} = \tfrac{1}{2}|(Z_U - Z_F)| \qquad \ell = N_W$$

and

$$\omega_{20} = \tfrac{1}{2}|(U_L-U_S)|$$

$$\omega_{21} = \tfrac{1}{2}|(U_S-U_0)|+\tfrac{1}{2}h_W$$

$$\omega_{2m} = \tfrac{1}{2}h_Q \qquad m = 2,\ldots,N_Q-2$$

$$\omega_{2m} = \tfrac{1}{2}h_Q+\tfrac{1}{2}|(U_U-U_F)| \qquad m = N_W-1$$

$$\omega_{2m} = \tfrac{1}{2}|(U_U-U_F)| \qquad m = N_W$$

With this grid structure and integration scheme the density of some points may still need to be obtained by interpolation. Any values of Z^*, U^* that result in the point $(a-Z^*, b-U^*, c-Z^{*2}-U^{*2})$ not to be on a (W,Q,R) grid point will require that the five dimensional density $f_i^P(\quad)$ be obtained by interpolation. For example, there is no guarantee that the endpoints Z_L, Z_U, U_L and U_U will lie on a grid point. However, in such cases, the task of interpolation may be simplified by considering the form of the five dimensional density $f_i^P(\quad)$.

As shown in equation (2.5.2) the five dimensional joint density is given by:

$$f_i^P(a-Z,\ b-U,\ c-Z^2-U^2,\ Z,\ U)$$
$$= f_{i-1}(a-Z,\ b-U,\ c-z^2-U^2)\cdot\phi(Z)\cdot\phi(U-\sqrt{\lambda_\ell^*}).$$

Whenever interpolation is required to evaluate $f_i^P(\quad)$, it need only be performed in two or three dimensions on the density $f_{i-1}(\quad)$, since both ϕ's can be calculated exactly for any value of U and Z.

In other words, when interpolation is required

$$f_i^P(a-Z,\ b-U,\ c-Z^2-U^2,\ Z,\ U) \approx E^* \phi(Z) \cdot \phi\left(U-\sqrt{\lambda_\ell^*}\right)$$

where E^* is the interpolated value of the density $f_{i-1}(a-Z,\ b-U,\ c-Z^2-U^2)$.

For a given point a^*, b^*, c^* not on a (W,Q,R) grid point at stage i-1, the density $f_{i-1}(a^*,\ b^*,\ d^*)$ may be approximated by trivariate linear interpolation. This involves the following approximation:

$$f_{i-1}(a^*,b^*,c^*) = P^* \approx \sum_{\ell=1}^{2} \sum_{j=1}^{2} \sum_{k=1}^{2} f_{i-1}(a_\ell,\ b\ ,\ c\ )\alpha_\ell \beta_j \gamma_k \qquad (2.6.8)$$

where

$$a_1 = a^*/h_W+1 \ \{sign\ (a^*)$$

and

$$a_2 = a^*/h_W$$

and

$$\alpha_1 = \frac{a^*-a_2}{a_1-a_2} \quad , \quad \alpha_2 = \frac{a^*-a_1}{a_2-a_1}$$

and analagous expressions for the other quantities.

This should give fairly good approximations for small values of h_W, h_Q, and h_R. For large values of these quantities the result could be meaningless (i.e., $f_i(a^*, b^*, c^*) < 0$ or $f_i(a^*, b^*, c^*) > 1$), and should be modified in such cases. The modifications are of the following form:

$$f_i\ (a^*, b^*, c^*) = \begin{cases} P^* & \text{if} \quad 0 \leq P^* \leq 1 \\ 0 & \text{if} \quad P^* < 0 \\ \max(f_i(a_\ell, b_j, c_k) & \text{if} \quad P^* \geq 1 \end{cases}$$

By using the trapezoid rule and trivariate interpolation, the density $f_i(a, b, c)$ may be calculated. This must be repeated for all a, b, c contained on the grid. This will result in a new grid representing the density at stage i. From this new grid, the probabilities $P_h^{\ i}$, $P_R^{\ i}$, and $P_C^{\ i}$ must be calculated. These probabilities can also be calculated with a trapezoid rule integration scheme as given in (2.6.2).

In practice the following quantities must be specified:

(1) The grid sizes h_W, h_Q, h_R .

(2) The end points of the grid: W_S, W_F, Q_S, Q_F, R_S, R_F .

As in most numerical problems, the best choice of the grid sizes will depend upon the particular problem (i.e., $V_A{}^i$, $V_R{}^i$, and m_0). One approach to this problem is to start the procedure with a coarse grid and obtain answers; the procedure may then be redone using a finer grid and new answers obtained. This process is iterated until the results converge to answers accurate to the desired number of digits. One should note that the number of calculations required for each additional iteration increases exponentially. For example, suppose a grid is constructed, using grid sizes h_W, h_Q, and $h_R = h_W{}^2+h_Q{}^2$ and α_i, β_j, γ_k of (2.6.3) all equal to unity. Halving the h_W and h_Q grid sizes will result in an eight-fold increase in the total number of points on the grid. The density for each of these points must be calculated for each stage, which requires a bivariate integration for each point at each stage.

The grid end points must be chosen so as to exclude only a minute fraction of the density for all stages: $i = n_1,\ldots,m_0$. This amounts to choosing the quantities

W_S, W_F, Q_S, Q_F and R_S, R_F such that all points, W_n, Q_n. R_n, on the grid lie within these ranges, i.e.,

$$W_S \le W_b \le W_F$$

$$Q_S \le Q_n \le Q_F$$

$$R_S \le R_n \le R_F \, .$$

In most cases, the size of the required grid (i.e., W_S, W_F, Q_S, Q_F, R_S and R_F) is directly proportional to the value of m_0.

Since W_{n_1} (n_1 being the first stage at which a decision can be made) is distributed normally with mean zero and standard deviation $\sqrt{n_1}$, a W_n range of the following type:

$$W_S = -6\left[\sqrt{n_1}\,/h_W\right]*h_W$$

$$W_F = 6\left[\sqrt{n_1}\,/h_W\right]*h_W \, , \text{ where } [\;] \equiv \text{greatest integer}$$

should be sufficient for the grid at stage n_1. However, if the regions were such that no decision could be made until stage m_0, $W_{m_0} \sim N(0, \sqrt{m_0})$. Thus in order to insure that the grid is large enough, the following range should be used:

$$W_S = -6\left[\sqrt{m_0}\,/h_W\right]*h_W$$

$$W_F = +6\left[\sqrt{m_0}\,/h_W\right]*h_W$$

(2.6.9).

Employing similar logic to the Q dimension yields the following range:

$$Q_S = \min\left\{\left[(\sqrt{n_1\lambda} - 6\sqrt{n_1})/h_Q\right]*h_Q,\ \left[(\sqrt{m_0\lambda} - 6\sqrt{m_0})/h_Q\right]*h_Q\right\}$$

$$Q_F = \max\left\{\left[(\sqrt{n_1\lambda} + 6\sqrt{n_1})/h_Q\right]*h_Q,\ \left[(\sqrt{m_0\lambda} + 6\sqrt{m_0})/h_Q\right]*h_Q\right\}$$

where $[\ ] \equiv$ greatest integer. (2.6.10).

Since R must always be greater than the quantity $1/n(W^2 + Q^2)$, the R points for which

$$R_{m_0} < \frac{1}{m_0}\left(W_0^2 + Q_{m_0}^2\right),$$

need not be contained on the grid. Thus the range of R will depend upon the values of W and Q, and the overall grid structure becomes that of a cone as shown in Figure 14. An R range sufficient for the density $f_i\ (W_i, Q_i, R_i)$ for all i, is given by:

$$R_S = \left[\frac{1}{m_0}\ (W_{n_K}^2 + Q_{n_j}^2)/h_R\right]*h_R$$

$$R_F = R_S + \left[\chi^2_{99.9}\ (2m_0 - 2)/h_R\right]*h_R \qquad (2.6.11).$$

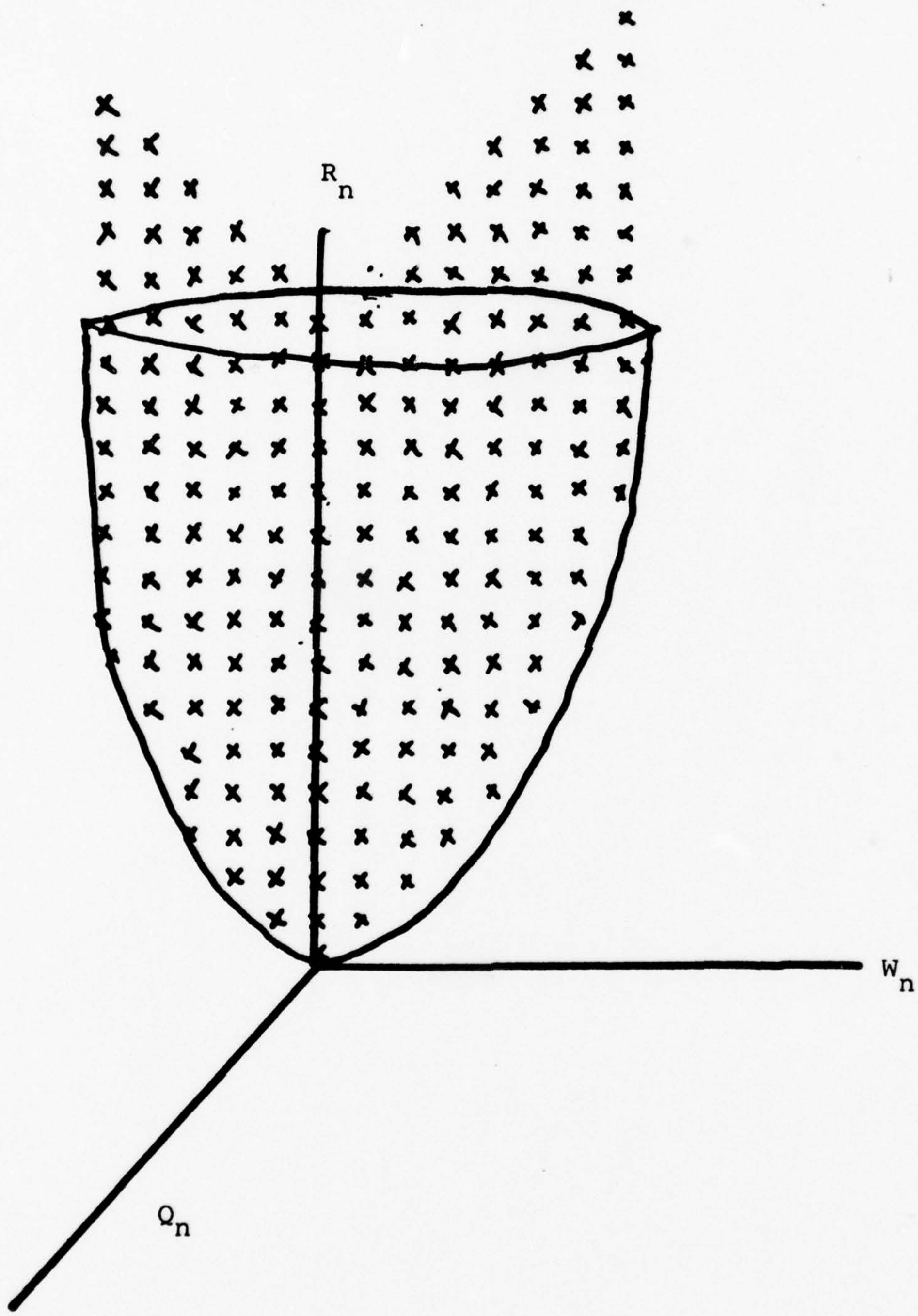
where $[\ ] \equiv$ greatest integer.

A grid of this form and size will allow for sufficient accuracy in the calculation of the OC and ASN curves.

In conclusion, this section has presented a procedure for implementing the theory of the previous sections.

FIGURE 14

STRUCTURE OF NUMERICAL GRID
FOR DIRECT METHOD IMPLEMENTATION



Since the density $f_i(W_i, Q_i, R_i)$ can not be expressed in a closed form for $i > n_1$, this section has discussed a numerical procedure which allows the implementation of the theory. The numerical procedure consists of:

1. Representing the density $f_i (W_i, Q_i, R_i)$ by a discrete 3-dimensional grid of points. The grid is shown in Figure 14 and is described mathematically by equations (2.6.3). The quantities R_S, R_F, Q_S, Q_F, W_S, W_F, h_R, h_W, h_a are given by equations (2.6.4) and (2.6.9) - (2.6.11).
2. "Carrying" this grid from stage to stage. The grid at stage i-1 is used to calculate a new grid for stage i, which represents the density $f_i(W_i, Q_i, R_i)$. To calculate the density of any point on this grid at stage i requires performing the bivariate integration of equation (2.5.3). However, the integration is now performed numerically. When the trapezoid integration rule is used, the calculation is given by equations (2.6.5) - (2.6.7).
3. After the density of all points at stage i has been calculated, the grid is then again numerically integrated to obtain P_A^i, P_R^i, P_C^i. This is calculated by the procedure shown in equation (2.6.2).

Since the density at stage i-1 is known only at the points on the grid, the density at points not on the grid must be obtained by interpolation. This can be done by three dimensional linear interpolation as given in equation (2.6.8).

The methods discussed in this section are only feasible if performed on an electronic computer. Appendix C discusses a program developed to calculate several points on the OC and ASN curves for any k=2 SANOVA test.

2.7 CONCLUSION

This chapter of the thesis has derived a procedure for obtaining the OC and ASN curves of a k=2 SANOVA test. The procedure is the first to yield exact results.

Section (2.3) involved the theoretical derivation of the procedure, which has been summarized in Figures 12 and 15 of Section (2.5). Also, Section (2.6) contained a discussion of a numerical approach for implementing the procedure. Appendix C contains a computer program which calculates the OC and ASN curves via the methods discussed in this chapter.

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APPENDIX A

POWER CALCULATIONS FOR A FIXED SAMPLE ANOVA TEST

As shown in Section (1.1) of the thesis, the fixed sample test utilizes the statistic F_{cal}, where

$$F_{cal} = \frac{\sum_{i=1}^{K} n_i (\bar{X}_i - \bar{\bar{X}})^2/(K-1)}{\sum_{i=1}^{K} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2/(N-K)}$$

with

$$N = \sum_{i=1}^{K} n_i$$

$$\bar{X}_i = n_i^{-1} \sum_{j=1}^{n_i} X_{ij}$$

$$\bar{\bar{X}} = N^{-1} \sum_{i=1}^{K} \sum_{j=1}^{n_i} X_{ij}$$

For a test of K means with $n_i = n$ observations from each population

$$F_{cal} \sim F_{K-1,K(n-1)}(n\lambda)$$

where

$$\lambda = \frac{\sum_{i=1}^{K} (\mu_i - \bar{\mu})^2}{\sigma^2}$$

$$\mu = \sum_{i=1}^{K} \mu_i / K$$

and $F_{K-1,K(n-1)}$ $(n\lambda)$ is a noncentral F variate as defined in Section (1.1).

The ANOVA test is usually a test of the following hypotheses:

$$H_0: \lambda = 0 \quad \text{vs.} \quad H_1: \lambda \geq \lambda'$$

The decision criterion of the test is as follows:

(1) Accept H_o if $F_{CAL} < F^*_{K-1,K(n-1),\alpha}) = \alpha$.

The quantity α corresponds to the acceptable probability of a Type-I error.

The choice of any two of the three quantities (β (magnitude of the Type-II error), n, λ') completely determines the third.

The OC curve of the test is in terms of the parameter λ, and is defined as:

$$OC(\lambda^*) = Pr\ (\text{accepting } H_0 | \lambda = \lambda^*)$$

$$= Pr\ (F_{CAL} < F^*_{K-1,K(n-1),\alpha} | \lambda = \lambda^*)$$

$$= Pr\ (F_{CAL} < F^*_{K-1,K(n-1),\alpha} | F_{CAL}\ {}_{K-1,K(n-1)}(n\lambda^*))$$

$$= \int_0^{F^*_{K-1,K(n-1),\alpha}} f(F_{K-1,K(n-1)}(n\lambda^*)\ dF_{K-1,K(n-1)}(n\lambda^*)$$

where $f(F_{K-1,K(n-1)}(n\lambda^*))$ is the density of a noncentral F variate with $K-1, K(n-1)$ degrees of freedom and noncentral parameter $n\lambda^*$.

In order to calculate this integral the noncentral F distribution must be integrated. This integration can be expressed in terms of an infinite series of multiples of incomplete beta function ratios in the following manner:

$$OC(\lambda^*) = \sum_{j=0}^{\infty} \left(\frac{[\frac{1}{2}n\lambda^*]^j}{j!} e^{-\frac{1}{2}n\lambda^*} \right) I_g(\tfrac{1}{2}(K-1)+j,\ \tfrac{1}{2}K(n-1))$$

$$\text{where} \quad g = \frac{(K-1)F^*_{K-1,K(n-1),\alpha}}{\left[K(n-1) + (K-1)\ F^*_{K-1,K(n-1),\alpha}\right]} \qquad (A.1)$$

and

$$I_x(a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^x t^{a-1}(1-t)^{b-1}\,dt$$

the incomplete beta function.

Thus $OC(\lambda^*)$ may be calculated by summing terms in the series until the magnitude of a term is less than or equal to some ε.

The incomplete beta function cannot be evaluated analytically, so must be done numerically. One method is that of continued fractions. The incomplete beta function's continued fraction expansion was obtained by Aroian (1941) and is given in Abramowitz and Stegum (1969).

An approximation to the cumulative distribution function of the noncentral F distribution was given by Tiku (1966). His approximation consists of fitting the distribution of $F_{\nu_1,\nu_2}(\lambda)$ by that of $(b + cF_{\nu_1',\nu_2})$; choosing b, c, and ν_1' so as to make the first three moments agree. The values which do this are:

$$\nu_1' = \tfrac{1}{2}(\nu_2-2)\left[\sqrt{\frac{H^2}{H^2-4K^3}} - 1\right] \tag{A.2}$$

$$c = (\nu_1'/\nu_1)(2\nu_1+\nu_2-2)^{-1}(H/K)$$

$$b = -\nu_2(\nu_2-2)^{-1}(c-1-\lambda\ \nu_1^{-1})$$

where

$$H = 2(\nu_1+\lambda)^3 + 3(\nu_1+\lambda)(\nu_1+2\lambda)(\nu_2-2) + (\nu_1+3\lambda)(\nu_2-2)^3$$

and

$$K = (\nu_1+\lambda)^2 + (\nu_2-2)(\nu_1+2\lambda)$$

so that

$$\begin{aligned} Pr(F_{\nu_1,\nu_2}(\lambda) \le f_0) &\approx Pr(b+cF_{\nu_1,\nu_2} \le f_0) \\ &= Pr\left(F_{\nu_1,\nu_2} \le \frac{f_0-b}{c}\right) \end{aligned} \qquad (A.3)$$

This approximation simply requires a method for evaluating the cumulative distribution function of a central F with ν_1' and ν_2 degrees of freedom, which from above can be calculated as:

$$\Pr(F_{\nu_1',\nu_2} \leq X) = I_{\nu_1 X/(\nu_2+\nu_1' X)}(\tfrac{1}{2}\nu_1, \tfrac{1}{2}\nu_2') . \qquad (A.4)$$

The computer program contained at the end of Appendix B uses this approximation to calculate the OC curve of a fixed sample test with specified values of α, β and λ'.

APPENDIX B

OBTAINING WALD REGIONS FOR A SANOVA TEST

As discussed in the thesis, a SANOVA test is conducted using the test statistic F_n of equation (1.2.1) or the simpler statistic V_n, where

$$V_n = \frac{(K-1)}{(N-K)} F_n .$$

At each stage this statistic is calculated and compared with the quantities V_A^n and V_R^n; such that at any stage i:

(1) H_0 is accepted if $V_i \leq V_A^i$

(2) H_1 is accepted if $V_i \geq V_R^i$.

The regions V_A^i, V_R^i, are usually chosen so that the Type-I and Type-II errors are approximately equal to the risks acceptable to the experimenter (α and β). The regions developed by Wald are the most commonly used.

For a given set of quantities α, β, K, λ_0, and λ_1, Wald regions V_A^n and V_R^n are obtained as the solutions of the following equations:

$$\frac{\exp\left\{-\frac{n}{2}(\lambda_1-\lambda_0)\right\} M\left[\frac{Kn-1}{2}, \frac{K-1}{2}, \frac{m\lambda_1 V_A^{\,n}}{2(1+V_A^n)}\right]}{M\left[\frac{Kn-1}{2}, \frac{K-1}{2}, \frac{m\lambda_0 V_A^n}{2(1+V_A^n)}\right]} = \frac{\beta}{1-\alpha}$$

and

$$\frac{\exp\left\{-\frac{n}{2}(\lambda_1-\lambda_0)\right\} M\left[\frac{Kn-1}{2}, \frac{K-1}{2}, \frac{m\lambda_1 V_R^{\,n}}{2(1+V_R^{\,n})}\right]}{M\left[\frac{Kn-1}{2}, \frac{K-1}{2}, \frac{m\;{}_0 V_A^n}{2(1+V_R^n)}\right]} = \frac{1-\beta}{\alpha}$$

where M (X, Y, Z) is the confluent hypergeometric function given in Section (1.1) and discussed by Stater (1960).

These quantities are obtained by solving the above equations by a Newton-Raphson root solving technique (Carnahan, et al (1969)).

In some cases, i.e., for small values of n, a root does not exist for the equations above. In such cases it is not possible to make a decision at that stage.

The following pages contain a listing of a computer program which will calculate regions for any given values of α, β, K, λ_0 and λ_1.

Tables of such regions have been worked out by Ghosh and West (1967) for selected values of α, β, λ_0, and λ These regions however are only given for every fifth stage. Thus the following computer program also allows the Ghosh regions to be read in, and the missing region values calculated via Lagrangian interpolation (Ghosh and West (1967)).

```
FILE  5=CARD,UNIT=READER
FILE  6=PRINTER,UNIT=PRINTER
C *********************************************************************
C                    THIS PROGRAM ALLOWS
C                    DETERMINATION OF FIXED SIZE ANOVA TEST
C                    THIS PROGRAM WILL FIND A CRITICAL VALUE
C                    C20 AND THE SMALLEST INTEGER N FOR GIVEN
C                    VALUES OF ALPHA AND BETA
C                    IF V<C THEN HO IS ACCEPTED AND IF V>C HO IS REJECTED
C *********************************************************************
C
C
                                                           START OF SEGMEN'
      DIMENSION   BOUND(2),XLIN(2),REG(50,2)
      DOUBLE PRECISION VALO,HYP1,HYP2,HYP3,HYP4,VAL2,VALST,FX,XLN,AUX1
     1  ,AUX2 ,XNXT,FXNXT,XEVAL   ,FPX
      REAL LAMO,LAM1
      COMMON EPS
      TAU(Z)=((Z-0.5)*ALOG(Z))-Z+(0.5*ALOG(6.283185))+(1.0/(12.0*Z))
     1-(1.0/(360.0*(Z**3.0)))+(1.0/(1260.0*(Z**5.0)))-(1./(1680.*(Z**7.0
     2)))
      H1(GRN,SS,CP)=2.0*(((GRN-1.0)+(SS*CP))**3.0)
      H2(GRN,SS,CP)=3.0*((GRN-1.0)+(SS*CP))*((GRN-1.0)+(2.0*SS*CP))*(((G
     1RN*(SS-1.0))-2.0))
      H3(GRN,SS,CP)=((GRN-1.0)+(3.0*SS*CP))*(((GRN*(SS-1.0))-2.0)**2.0)
      CON2(GRN,SS,CP)=(((GRN-1.0)+(SS*CP))**2.0)+(((GRN*(SS-1.0))-2.0)
     2*((GRN-1.0)+(2.0*SS*CP)))
      CON3(GRN,SS,CP,H)=((GRN*(SS-1.0))/((GRN*(SS-1.0))-2.0))*(H-(((GRN-
     31.0)+(SS*CP))/(GRN-1.0)))
      EPS=1.0 E-8
      IREAD=5
      IRITE=6
      READ(IREAD,1) ALPHA,BETA,LAMO,LAM1,DEGF
1     FORMAT(5F10.4)
      WRITE(IRITE,701)
'01   FORMAT(1H1,20X,"FIXED SAMPLE ANOVA TEST")
      WRITE(IRITE,702)
702   FORMAT(/,20X,"*************************")
      WRITE(IRITE,703)  DEGF
703   FORMAT(///,24X,"K=",F3.1,2X,"GROUPS")
      READ(IREAD,713) SAM
13    FORMAT(F8.2)
      IF(LAM1=0.0   .AND. SAM   .GT. 0.0)  GO TO 21
      WRITE(IRITE,704)  LAMO,LAM1
04    FORMAT(////,13X,9HHO:LAMO =,F6.2,2X,"VS",2X,9HH1:LAM1 =,F6.2)
      WRITE(IRITE,705)  ALPHA,BETA
705   FORMAT(//,20X,"ALPHA =",F5.2,6X,"BETA =",F5.2)
      DO 10 N=3,1000
      SAM=FLOAT(N)
      IF(LAMO=0.0)  GO TO 5
      H=H1(DEGF,SAM,LAMO)+H2(DEGF,SAM,LAMO)+H3(DEGF,SAM,LAMO)
      CONK=CON2(DEGF,SAM,LAMO)
      E=(H**2.0)/(CONK**3.0)
      B=((DEGF*(SAM-1.0))-2.0)*(SQRT(E/(E-4.0))-1.0)
      B=B*0.5
      V=(B/(DEGF-1.0))*(H/CONK)*(1.0/((2.0*B)+(DEGF*(SAM-1.0))-2.0)  )
      C=CON3(DEGF,SAM,LAMO,V)
      T1=(DEGF*(SAM-1.0))/2.0
      T2=B/2.0
      GO TO 7
5     T1=0.5*DEGF*(SAM-1.0)
```

```
      T2=0.5*(DEGF-1.0)
      B=1.0
7     Y0=BETINC(1,T1,T2,ALPHA)
      F0=((V*DEGF*(SAM-1.0)*(1.0-(Y0*B)))/(Y0*B))-C
      IF(LAM0=0.0) F0=(DEGF*(SAM-1.0)*(1.0-Y0))/((DEGF-1.0)*Y0)
      T1=(DEGF*(SAM-1.0))/2.0
      HP=H1(DEGF,SAM,LAM1)+H2(DEGF,SAM,LAM1)+H3(DEGF,SAM,LAM1)
      CONKP=CON2(DEGF,SAM,LAM1)
      EP=(HP**2.0)/(CONKP**3.0)
      BP=((DEGF*(SAM-1.0))-2.0)*(SQRT(EP/(EP-4.0))-1.0)
      BP=BP*0.5
      VP=(BP/(DEGF-1.))*(HP/CONKP)*(1./((2.*BP)+(DEGF*(SAM-1.))-2.0))
      CP=CON3(DEGF,SAM,LAM1,VP)
      T3=BP/2.0
      Y1=1.0/(1.0+((BP/(DEGF*(SAM-1.0)))*((F0+ CP   )/VP))        )
      PROB=BETINC(0,T1,T3,Y1)
      A4=1.0-BETA
      IF(PROB .GE. A4)  GO TO 11
10    CONTINUE
11    WRITE(IRITE,706)  SAM
706   FORMAT(//,18X,"REQUIRED SAMPLE SIZE IS",F8.1)
      GO TO 13
C
C
C
C ******************************************************************
C
C
C                    THIS PART OF THE PROGRAM WILL CALCULATE A LAM1 FOR A
C                              GIVEN ALPHA,BETA, AND FIXED SAMPLE SIZE N
C
C
C ******************************************************************
C
C
C
21    T1=0.5*DEGF*(SAM-1.0)
      T2=0.5*(DEGF-1.0)
      Y0=BETINC(1,T1,T2,ALPHA)
      F0=(DEGF*(SAM-1.0)*(1.0-Y0))/((DEGF-1.0)*Y0)
      T1=(DEGF*(SAM-1.0))/2.0
      DO 625 LM=1,200
      ALTLAM=0.1*LM
      HP=H1(DEGF,SAM,ALTLAM)+H2(DEGF,SAM,ALTLAM)+H3(DEGF,SAM,ALTLAM)
      CONKP=CON2(DEGF,SAM,ALTLAM)
      EP=(HP**2.0)/(CONKP**3.0)
      BP=((DEGF*(SAM-1.0))-2.0)*(SQRT(EP/(EP-4.0))-1.0)
      BP=BP*0.5
      VP=(BP/(DEGF-1.))*(HP/CONKP)*(1./((2.*BP)+(DEGF*(SAM-1.))-2.0))
      CP=CON3(DEGF,SAM,ALTLAM,VP)
      T3=BP/2.0
      Y1=1.0/(1.0+((BP/(DEGF*(SAM-1.0)))*((F0+ CP   )/VP))        )
      PROB=BETINC(0,T1,T3,Y1)
      A4=1.0-BETA
      IF(PROB  .GE. A4)  GO TO 27
625   CONTINUE
27    LAM1=ALTLAM
      WRITE(IRITE,704)  LAM0,LAM1
      WRITE(IRITE,705)  ALPHA,BETA
      WRITE(IRITE,706)  SAM
13    CONTINUE
```

```
C **********************************************************************
C
C                     THIS PART OF THE PROGRAM CALCULATES THE OC FUNCTION
C                              FOR THE FIXED SIZE TEST
C
C **********************************************************************
C
C
      WRITE(IRITE,707)
707   FORMAT(///,20X,"OC FUNCTION FOR THE TEST")
      WRITE(IRITE,708)
708   FORMAT(///,14X,"LAMDA",10X,"PROB OF ACCEPTING HO")
      DO 401  IPOW=1,10
      APLAM=LAMO+(((LAM1-LAMO)/9.0)*FLOAT(IPOW-1))
      IF(APLAM>0.0)  GO TO 403
      ANOC=BETINC(0,T1,T2,Y0)
      ANOC=1.0-ANOC
      GO TO 402
403     HP=H1(DEGF,SAM,APLAM)+H2(DEGF,SAM,APLAM)+H3(DEGF,SAM,APLAM)
      CONKP=CON2(DEGF,SAM,APLAM)
      EP=(HP**2.0)/(CONKP**3.0)
      BP=((DEGF*(SAM-1.0))-2.0)*(SQRT(EP/(EP-4.0))-1.0)
      BP=BP*0.5
      VP=(BP/(DEGF-1.))*(HP/CONKP)*(1./((2.*BP)+(DEGF*(SAM-1.))-2.0))
      CP=CON3(DEGF,SAM,APLAM,VP)
      T3=BP/2.0
      Y1=1.0/(1.0+((BP/(DEGF*(SAM-1.0))))*((FO+ CP   )/VP))        )
      ANOC=BETINC(0,T1,T3,Y1)
      ANOC=1.0-ANOC
402   WRITE(IRITE,709)  APLAM,ANOC
709   FORMAT(/, 14X,F5.2,17X,F6.4)
401   CONTINUE
      WRITE(IRITE,710)  FO
710   FORMAT(///,16X,"CRITICAL VALUE OF F =",F7.2)
      CVV=(FO*(DEGF-1.0))/(DEGF*(SAM-1.0))
      WRITE(IRITE,711) CVV
711   FORMAT(//,16X,"CRITICAL VALUE OF V =",F10.5)
      READ(IREAD,101)  IREG
101   FORMAT(I2)
      IF(IREG=0)  GO TO 160
C **********************************************************************
C
C
C                     THIS PART OF THE PROGRAM WILL FIND WALD REGIONS FOR
C                     EVERY N≤NO (THE FIXED SIXE TEST) BY INTERPOLATION
C                     OF THE GHOSH AND WEST TABLES
C
C
C **********************************************************************
30    CONTINUE
      WRITE(IRITE,715)
715   FORMAT(1H1,20X,"SEQUENTIAL ANOVA TEST")
      WRITE(IRITE,702)
      WRITE(IRITE,703)  DEGF
      WRITE(IRITE,704)  LAMO,LAM1
      WRITE(IRITE,705)  ALPHA,BETA
      WRITE(IRITE,716)
716   FORMAT(///,20X,"THE WALD REGIONS ARE")
      WRITE(IRITE,717)
717   FORMAT(//,10X,"STEP",10X,"LOWER VN", 10X,"UPPER VN")
      NSAM=IFIX(SAM)
306   READ(IREAD,25)  I,AL1,AL2
25    FORMAT(I3,2F10.5)
```

```
      IF(I.LE. NSAM) GO TO 35
      ICOUNT=ICOUNT+1
      IF(ICOUNT>2)  GO TO 40
35    REG(I,1)=AL1
      REG(I,2)=AL2
      GO TO 306
40    DO 150  INDX=1,2
      N1SS=1
      N2SS=1
      N3SS=1
41    IF(REG(N1SS,INDX)>0)  GO TO 42
      N1SS=N1SS+1
      GO TO 41
42    N2SS=N2SS+1
43    IF(REG(N2SS,INDX)>0)  GO TO 44
      N2SS=N2SS+1
      GO TO 43
44    IF(N2SS>N1SS+1)  GO TO 46
      N1SS=N2SS
      GO TO 42
46    N3SS=N2SS+1
47    IF(REG(N3SS,INDX)>0)  GO TO 48
      N3SS=N3SS+1
      GO TO 47
48    L1=N1SS+1
      L2=N2SS-1
      IF(L2>NSAM)  L2=NSAM
      S1=FLOAT(N1SS)
      S2=FLOAT(N2SS)
      S3=FLOAT(N3SS)
      AN=S1*S2*S3
      DO 140  INBT=L1,L2
      SB=FLOAT(INBT)
      Z1=(((AN/SB)-(AN/S2))*((AN/SB)-(AN/S3)))/(((AN/S1)-(AN/S2))
     1*((AN/S1)-(AN/S3)))
      Z2=(((AN/SB)-(AN/S1))*((AN/SB)-(AN/S3)))/(((AN/S2)-(AN/S1))
     2 *((AN/S2)-(AN/S3)))
      Z3=(((AN/SB)-(AN/S1))*((AN/SB)-(AN/S2)))/(((AN/S3)-(AN/S1))
     3 *((AN/S3)-(AN/S2)))
      REG(INBT,INDX)=Z1*REG(N1SS,INDX)+Z2*REG(N2SS,INDX)
     4    +Z3*REG(N3SS,INDX)
140   CONTINUE
      IF( L2=NSAM)  GO TO 150
      N1SS=N2SS
      GO TO 42
150   CONTINUE
155   DO 156 JMF=1,NSAM
      IF(REG(JMF,1)=0.0)  REG(JMF,1)=99999.
      IF(REG(JMF,2)=0.0)  REG(JMF,2)=99999.
      WRITE(IRITE,301) JMF,REG(JMF,1),REG(JMF,2)
156   CONTINUE
      WRITE(IRITE,721)
721   FORMAT(//,80X,"INTERPOLATED")
      GO TO 445
C
C
C  ********************************************************************
C                   THIS PART OF THE PROGRAM WILL CALCULATE REGIONS FOR
C                      TESTS NOT CONTAINED IN THE GHOSH + WEST TABLES
C
C
C  ********************************************************************
```

```
160   W2=(DEGF-1.0)/2.0
      NSAM=IFIX(SAM)
      WRITE(IRITE,715)
      WRITE(IRITE,702)
      WRITE(IRITE,703)  DEGF
      WRITE(IRITE,704)  LAM0,LAM1
      WRITE(IRITE,705)  ALPHA,BETA
                                                                    SE
                                                              START OF
      WRITE(IRITE,716)
      WRITE(IRITE,717)
      BOUND(1)=    ALOG(BETA/(1.0-ALPHA))
      BOUND(2)=     ALOG((1.0-BETA)/ALPHA)
      XLIN(1)=  (2.0*BOUND(1)+ (LAM1-LAM0))/(-2.0*BOUND(1))
      XLIN(2)= (2.0*BOUND(2)+(LAM1-LAM0))/(-2.0*BOUND(2))
      DO 220  NSZ=2,NSAM
      ECON=-(FLOAT(NSZ)*(LAM1-LAM0))/2.0
      W1=((DEGF*FLOAT(NSZ))-1.0)/2.0
      ZCON1=(FLOAT(NSZ)*LAM1)/2.0
      ZCON0=(FLOAT(NSZ)*LAM0)/2.0
      DO 210 IB=1,2
      XEVAL=0.0
      IF( XLIN(IB)  .GT. 0.0  .AND. XLIN(IB)  .GT. 1.0)  GO TO 297
      IF(XLIN(IB)  .GT. 0.0)  GO TO 170
      VAL0=  EXP(ECON) - EXP(BOUND(IB))
      DO 296 ISR=1,9
      XSR=  -(ISR*0.1)
      SEAR=  ( XSR/(1.0+XSR))
      W3=ZCON1*SEAR
      W4=ZCON0*SEAR
      HYP1=  CONHYP(W1,W2,W3)
      HYP2=  CONHYP(W1,W2,W4)
      VAL2 =(  EXP(ECON)*HYP1) -(  EXP(BOUND(IB))*HYP2)
      IF( (VAL0*VAL2)  .LE.  0.0  .AND. XLIN(IB)  .LT. 0.0) GO TO 210
296   CONTINUE
      SEAR=0.99
      W3= ZCON1*SEAR
      W4=ZCON0*SEAR
      HYP1=    CONHYP(W1,W2,W3)
      HYP2=       CONHYP(W1,W2,W4)
      VAL2= (    EXP(ECON)*HYP1) -(  EXP( BOUND(IB))*HYP2)
      IF( (VAL0*VAL2)  .GT. 0.0)  GO TO 210
297   W3=ZCON1*0.5
      W4=ZCON0*0.5
      HYP1=  CONHYP(W1,W2,W3)
      HYP2=   CONHYP(W1,W2,W4)
      VALST=(  EXP(ECON)*HYP1) -(  EXP(BOUND(IB))*HYP2)
      IF( (VALST*VAL0) .GT. 0.0)  GO TO 197
      XLIN(IB)=1.0
      GO TO 170
197   DO 386 IFND=10,60 ,10
      SEARS=IFND/(1.0+IFND)
      W3=ZCON1*SEARS
      W4=ZCON0*SEARS
      HYP1=  CONHYP(W1,W2,W3)
      HYP2=   CONHYP(W1,W2,W4)
      VALS=  (  EXP(ECON)*HYP1) -( EXP(BOUND(IB))*HYP2)
      IF((VALST*VALS)) 387,386,386
387   XLIN(IB)=IFND-5.0
      GO TO 170
386   CONTINUE
      XLIN(IB)=IFND
```

```
170   CONTINUE
      IF(XLIN(IB)  .GT. 0.0)  GO TO  195
      XLIN(IB)= XLIN(IB)+XLN
      IF( XLIN(IB)  .EQ. -1.0)  XLIN(IB)=-0.5
      GO TO 170
195   W3= ZCON1*(XLIN(IB)/(1.0+XLIN(IB)))
      W4=ZCONO*(XLIN(IB)/(1.0+XLIN(IB))   )
      HYP1=CONHYP(W1,W2,W3)
      HYP2=CONHYP(W1,W2,W4)
      Y1=W1+1.0
      Y2=W2+1.0
      HYP3=CONHYP(Y1,Y2,W3)
      HYP4=CONHYP(Y1,Y2,W4)
      AUX1=(FLOAT(NSZ)*W1)/(2.0*W2*((1.0+XLIN(IB))**2.0))
      AUX2=((LAM1*HYP3)/HYP1)-((LAMO*HYP4)/HYP2)
      FPX=AUX1*AUX2
      HYP1=  DLOG(HYP1)
      HYP2=   DLOG(HYP2)
      FX=ECON+HYP1-HYP2-BOUND(IB)
      IF(  DABS(FPX)  .LE. 2.0)  GO TO 501
503   XLN=  XLIN(IB)-(FX/FPX)
      IF(  DABS(XLIN(IB)-XLN)  .LE. EPS   ) GO TO 210
      XINT=XLIN(IB)
      XLIN(IB)=XLN
      XLN=XINT
      XEVAL=FX
      GO TO 170
501   IF(  (XEVAL*FX)) 502,503,503
502   XNXT=  (  XLIN(IB)+XLN)*0.5
      W3=ZCON1*(XNXT/(1.0+XNXT))
      W4= ZCONO*(XNXT/(1.0+XNXT))
      HYP1=  CONHYP( W1,W2,W3)
      HYP2=   CONHYP(W1,W2,W4)
      FXNXT=  ECON+ DLOG(HYP1)- DLOG(HYP2)  -BOUND(IB)
      IF((XEVAL*FXNXT)) 505,210,506
505   XLIN(IB)=XNXT
      FX=FXNXT
      GO TO 507
506   XLN=XNXT
      XEVAL=FXNXT
507   IF(  DABS(XLIN(IB)-XLN)  .LE. EPS)  GO TO 210
      GO TO 502
210   CONTINUE
      WRITE(IRITE,301) NSZ,XLIN(1),XLIN(2)
301   FORMAT(11X,I2,10X,F8.4,10X,F8.4)

220   CONTINUE
      WRITE(IRITE,720)
720   FORMAT(//,80X,"CALCULATED")

445    CONTINUE
      STOP
      END
```

START OF SE

```
      FUNCTION ZI(X,A,B)
      FN=.7*(ALOG(15.+A+B))**2+AMAX1(X*(A+B)-A,0.0)
      N=INT(FN)
      C=1.-(A+B)*X/(A+2.*FN)
      ZI=2./((C+SQRT(C**2-4.*FN*(FN-B)*X/(A+2.*FN)**2))
      DO 60 J=1,N
      FN=N+1-J
      A2N=A+2.*FN
      ZI=(A2N-2.)*(A2N-1.-FN*(FN-B)*X*ZI/A2N)
      ZI=1./(1.-(A+FN-1.)*(A+FN-1.+B)*X/ZI)
60    CONTINUE
      RETURN
      END
```

SEG

```
      FUNCTION CGAM(A)
      AA=A
      CAC=0.0
      IF(A-2.)2,8,8
2     IF(A-1.)4,6,6
4     CAC=-2.+(A+.5)*ALOG(1.+1./A)+(A+1.5)*ALOG(1.+1./(A+1.))
      AA=A+2.
      GO TO 8
6     CAC=-1.+(A+.5)*ALOG(1.+1./A)
      AA=A+1.
8     CA=2.269489/AA
      CA=.52560647/(AA+1.0115231/(AA+1.5174737/(AA+CA)))
      CA=.0833333333/(AA+.03333333/(AA+.25238095/(AA+CA)))
      CGAM=CA+CAC
      RETURN
      END
```

START OF SE

```
      FUNCTION BETINC(IND,A,B,X)
C     INCOMPLETE BETA FUNCTION AND ITS INVERSE
C    MARK=1 FOR INVERSE (SEND DOWN PROB)
      CAB=CGAM(A+B)-CGAM(A)-CGAM(B)-.5*ALOG((A+B)*6.28318531)
      IF(IND)10,10,20
10    EP=CAB+A*ALOG(X*(1.+B/A))+B*ALOG((1.-X)*(1.+A/B))
       IF(X-A/(A+B))12,12,14
12    BETINC=ZI(X,A,B)*EXP(EP+.5*ALOG(B/A))
      RETURN
14    BETINC=1.-ZI(1.-X,B,A)*EXP(EP+.5*ALOG(A/B))
      RETURN
20    IF(X-.5)22,22,24
22    QZ=ALOG(X)
      IGO=1
      AA=A
      BB=B
      GO TO 26
24    QZ=ALOG(1.-X)
      IGO=2
      AA=B
      BB=A
26    XT=AA/(AA+BB)
      CABB=CAB+.5*ALOG(BB/AA)+AA*ALOG(1.+BB/AA)+BB*ALOG(1.+AA/BB)
      DO 40 NC=1,100
      ZZ=ZI(XT,AA,BB)
      QX=CABB+AA*ALOG(XT)+BB*ALOG(1.-XT)+ALOG(ZZ)
      XC=(QZ-QX)*(1.-XT)*ZZ/AA
      XC=AMAX1(XC,-.99)
      XC=AMIN1(XC,.5/XT-.5)
      XT=XT*(1.+XC)
      IF(ABS(XC)-1.E-6)42,40,40
40    CONTINUE
42    GO TO (44,46),IGO
44    BETINC=XT
      RETURN
46    BETINC=1.-XT
      RETURN
      END
```

START UP

```
      FUNCTION CONHYP(XF,YF,UF)
      COMMON  EPS
      DOUBLE PRECISION   TSUM
      TAU(AR)=ALGAMA(AR)
      X=XF
      Y=YF
      U=UF
      PMULT=1.0
      TSUM=1.0
      IF(X-Y)  101,100,101
100   CONHYP=  EXP(U)
      RETURN
101   IF(U)  103,102,104
102   CONHYP=  1.00
      RETURN
103   X=Y-X
      PMULT=  EXP(U)
      U= -U
104   IF(X)  105,102,106
105   ICHK=-IFIX(X)
      TEST=ICHK+X
      IF(TEST) 111,108 ,111
108   IF( ICHK-1)  111,107,109
107   CONHYP=(1.0-(U/Y))*PMULT
      RETURN
109   INDX=ICHK
      XSTAR=1.0
      DO 110  N=1,INDX
      XSTAR= XSTAR*( X+N-1.0)
      T1= Y+N
      T2=  FLOAT(N+1)
      T3=  FLOAT(N)
      YSTAR= (TAU(Y)-TAU(T1)-TAU(T2))+(T3*  ALOG(U))
      TSUM= TSUM+ (XSTAR*  EXP(YSTAR))
110   CONTINUE
      CONHYP=  PMULT*TSUM
      RETURN
111   DO 125  IT=1,50
      T=FLOAT(IT)
      T1=T+X
      T2=T+Y
      T3=T+1.0
      PS= (  GAMMA(Y)  / GAMMA(X))*(  GAMMA(T1) /GAMMA(T2))
      PF=  (T*  ALOG(U))- TAU(T3)
      PS=PS *  EXP(PF)
      TSUM=  TSUM+PS
      IF(  ABS(PS)  .LE. EPS)  GO TO 112
125   CONTINUE
112   CONHYP=  TSUM* PMULT
      RETURN
106   DO 115  IT=1,50
      T= FLOAT(IT)
      T1=T+X
      T2=T+Y
      T3=T+1.0
      PS=TAU(Y)+TAU(T1)-TAU(X)-TAU(T2)-TAU(T3)
      PS=  EXP(PS)*(U**T)
      TSUM=  TSUM+PS
      IF (  ABS(PS)  .LE.  EPS)  GO TO 120
115   CONTINUE
120   CONHYP=  TSUM* PMULT
      RETURN
```

FIXED SAMPLE ANOVA TEST

****

K=2.0 GROUPS

HO:LAMO = 0.00 VS H1:LAM1 = 1.00

ALPHA = 0.01 BETA = 0.01

REQUIRED SAMPLE SIZE IS 27.0

OC FUNCTION FOR THE TEST

LAMDA	PROB OF ACCEPTING HO
0.00	0.9900
0.11	0.8180
0.22	0.5793
0.33	0.3673
0.44	0.2154
0.56	0.1192
0.67	0.0631
0.78	0.0323
0.89	0.0160
1.00	0.0078

CRITICAL VALUE OF F = 7.15

CRITICAL VALUE OF V = 0.13748

SEQUENTIAL ANOVA TEST

****

K=2.0 GROUPS

H0:LAM0 = 0.00 VS H1:LAM1 = 1.00

ALPHA = 0.01 BETA = 0.01

THE WALD REGIONS ARE

STEP	LOWER VN	UPPER VN
2	-0.8912	-1.1088
3	-0.8912	-1.1088
4	-0.8912	-1.1088
5	-0.8912	42.7986
6	-0.8912	3.5556
7	-0.8912	1.8734
8	-0.8912	1.2850
9	-0.8912	0.9872
10	0.0049	0.8082
11	0.0104	0.6891
12	0.0156	0.6044
13	0.0204	0.5412
14	0.0250	0.4924
15	0.0293	0.4535
16	0.0333	0.4219
17	0.0371	0.3956
18	0.0406	0.3736
19	0.0438	0.3548
20	0.0468	0.3385
21	0.0497	0.3244
22	0.0523	0.3120
23	0.0548	0.3010
24	0.0571	0.2912
25	0.0593	0.2824
26	0.0614	0.2745
27	0.0633	0.2674

APPENDIX C

A COMPUTER PROGRAM FOR k=2 SANOVA

Chapter 2 of this thesis derived a procedure for obtaining the properties of a k=2 SANOVA test. The procedure has been summarized in Figures 12 and 13. As previously discussed, this procedure cannot feasibly be performed analytically. Section (2.6) considered an alternative, a numerical implementation of the theory. This appendix contains a computer program for obtaining the properties of a k=2 SANOVA test utilizing the approach discussed in Section (2.6).

The program is written in Fortran IV for use on a Burroughs or CDC computer. Its implementation on other machines may require modifications, specifically the statements involving a read or write from disc.

To use the program, the user must supply the following information:

1) λ_0, λ_1, k, m_0

   On one card in a (3F10.5,I3) format.

   Note k is the number of means and should always be input as 2; m_0 is the truncation point.

2) V_A^i, V_R^i, $i=1, \cdots, m_0$.

Two numbers per card (V_A^i being the acceptance region and V_R^i the rejection) in a (2F15.0) format. Any time acceptance is not possible at a given stage j, V_A^j should be input as a negative number. Similarly, any time rejection is not possible, V_R^j should be input as a number greater than 10^{10}. Note that the first region card should always be of the form $V_A^1 = -1$, $V_R^1 = 10^{10}$, since no decision can be made at this stage.

3) Gridsize in an (F10.0) format.

This represents the coarseness of the grid or the quantities h_Q and h_W of Section (2.6). The program assumes that $h_Q = h_W$. It is best to select this number as a power of 2, e.g., 0.5, 0.25, etc. In general, the smaller this number, the more accurate the results but the larger the amount of computation required. As discussed in Section (2.6), the most efficient approach is to perform several runs, using a finer grid size on each run.

The program uses two random access disc files (files 1 and 2). These files represent the density at stages i and $i+1 (i=n_1, \cdots, m_0-1)$. The first file is used to compute the second file as discussed in Section (2.6). The size of these files is dependent upon the choice of the gridsize parameter. However, 125,000 words per file should be sufficient for most problems.

The program output consists of the probability of acceptance, rejection and continuation $(P_A{}^i, P_R{}^i, P_C{}^i)$ for each stage for every value of λ. These quantities are then used to compute summary OC and ASN curves.

Currently, the program is being implemented on a CDC computer by Mr. Kent Kaufmann of Western Illinois University. The program will be used to generate a brief set of OC and ASN curves for several $k=2$ SANOVA tests.

```
FILE  1=ST1/MIL,UNIT=DISK,SAVE=99,LOCK,RANDOM,BLOCKING=30,RECORD=1
FILE  2=ST2/MIL,UNIT=DISK,SAVE=99,LOCK,RANDOM,BLOCKING=30,RECORD=1
FILE  10=RES/MIL,UNIT=DISK,SAVE=99,LOCK,RANDOM,RECORD=32
FILE  11=CUR/MIL,UNIT=DISK,SAVE=99,LOCK,RANDOM,BLOCKING=5,RECORD=6
FILE  5=CARD,UNIT=READER
FILE  6=OUTPUT,UNIT=PRINTER
FILE  7=DEBUG,UNIT=PRINTER
FILE  8=TEMP,UNIT=PRINTER
C
C
C
C                   SEQUNTIAL ANALYSIS OF VARIANCE
C                         RW MILLER
C
C        THIS PROGRAM CALCULATES THE AVERAGE SAMPLE NUMBER.
C        MEDIAN SAMPLE SIZE,AND OPERATING CHARACTERISTIC FUNCTION FOR
C        A SEQUENTIAL TEST OF THE EQUALITY OF K MEANS
C        THE TEST IS CHARACTERIZED BY A LAMO,LAM1,AND REGIONS
C
C
C
                                                    START OF SEGMENT
      COMMON /CB10/CA,PA,CR,PR,A,B,C
      COMMON /CB1/GRIDW,GRIDQ,GRIDR
      COMMON /CB2/ JREC,TSTAT
      COMMON /CB3/ NCONW,NCONQ,NSTRT
      COMMON /CB5/ REG(30,2)
      COMMON /CB4/ DEGF,ALAM
      COMMON /CB6/ IMAXW,IMINW,IMAXQ,IMINQ
      COMMON /CB8/ JOINT,ICAL
      COMMON /CB11/ SINE45
      COMMON /CB12/ NUMW,NUMQ,NUMR
      COMMON / CB14/ XMEAN(2),XBR1,XBR2,VAR,DGF
      COMMON /CB7/ LSTP,ISUR
      COMMON /CB15/ GREGC(14,14)
      COMMON /CB20/ RECMAX
      COMMON/ERR/ IRP1,IRP2
      REAL LAMO,LAM1
C
C
C        THIS IS NECCESSARY FOR START-RESTART
C

      COMMON/RESTAR/OC(30,2),ASN(30),NTESTS
      COMMON /RESTAR/ KTEST,NOC,NSTP,I1,I2,I3,KREC,IRAC,NPFIAC,IRNR
      COMMON /RESTAR/ NPFINR,PROBAC,PROBNR,PRRAC,PRQAC,PRRNR,PRQNR
      COMMON /RESTAR/ RVALAC,RVALNR,RACBEG,SPROAC,RNBEG,SPRONR
      COMMON /RESTAR/ WN,QN,RN
      EQUIVALENCE (RNBEG,RNRBEG)
C
C
      ELIPFI(Q,W,R,N,VR)=(((Q*COS(45.)-W*SIN(45.))**2.)/(R*FLOAT(N)+RAD)
     1  )+(((Q*SIN(45.)+W*COS(45.))**2.)/(((R*FLOAT(N)*VR)/(2.*(FLOAT(N)
     2 **4.0)+VR))+RAD))-1.0
      PARAR(XC,XP,QC,WC)=(XC*((QC-(XP*WC/XC))**2.))+(XC*(WC**2.))
     1    -(((XP*WC)**2.)/XC)
      DATA IREAD,IRITE/5,6/
      EPS= 1.0E-6
      NPFI=6
      SINE45=SIN(0.78539816)
      READ(10=1) NTRYS,ITEST,NOC,MSTP
      NTRYS=NTRYS+1
      READ(IREAD,199) NTESTS
```

```
C
C         INPUT TO THIS PROGRAM CONSISTS OF LAM0,LAM1,K=NUMBER OF MEANS
C         AND THE TRUNCATION POINT   AS WELL AS THE REGIONS
C
      DO 9112 KTEST=1,NTESTS
      READ(IREAD,205)  LAM0,LAM1,DEGF,MTP
205   FORMAT(3F10.5,I3)
C
C         REG(I,1)=VALUE OF VI FOR WHICH ANY V≤VI RESULTS IN ACCEPTANCE
C         REG(I,2)=VALUE OF VI FOR WHICH ANY V≥VI RESULTS IN REJECTION
C
C
C         IF IT IS NOT POSSIBLE TO ACCEPT AT STEP I REG(I,1)=0.0
C         IF IT IS NOT POSSIBLE TO REJECT AT STEP I REG(I,2)=1.E10
C
      READ (IREAD,206) ((REG(J1,1),REG(J1,2)),J1=1,MTP)
206   FORMAT(2F15.0)
      NUM=IFIX(DEGF)-1
      DO 20  M=1,NUM
      XMEAN(M)=0.0
20    CONTINUE
      NUM=NUM+1
C
C
C         A SEARCH IS MADE TO DETERMINE THE FIRST STEP AT
C         WHICH A DECISON IS POSSIBLE
C
      DO 35 NSTP=1,MTP
      IF( REG(NSTP,1) .LE. 0.0 .AND. REG(NSTP,2) .GE. 1.E10) GO TO 25
      ISUR=NSTP
      NCAL=ISUR+1
      GO TO 40
25    CALL FSTEPPROB(OC(NSTP,1),OC(NSTP,2),ASN(NSTP),NSTP)
      IF( KTEST .GT. ITEST .OR. NOC .GT. KOC)
     1WRITE(11=(NSTP+((NOC-1)*NTESTS)+(KTEST-1)*NTESTS*10)) OC(NSTP,1),
     2 OC(NSTP,2),ASN(NSTP),NSTP,ALAM,KTEST
35    CONTINUE
40    READ(IREAD,207)  GRIDSZ
207   FORMAT(F10.0)
      IF(ITEST .GT. KTEST) GO TO 9112
      GRIDW=GRIDSZ
      GRIDQ=GRIDSZ
      GRIDR=GRIDW**2.0
      RAD=(3.0*(GRIDW**2.0+GRIDQ**2.0))**0.5
C
C         THE DIMMENSIONS OF THE GRID ARE CALCULATED
C
      XVAL= 10.0*(((0.4)**DEGF)**0.5)
      MAXVAL=XVAL*((2.0*DEGF*FLOAT(ISUR-1))**0.5)+(DEGF*(FLOAT(ISUR-1)))
      STORE= MAXVAL/GRIDR
      NUMR=IFIX(STORE)+1
      VAR= 4.0*(FLOAT(ISUR)**0.5)
      NUMW=((IFIX(VAR/GRIDW)+1)*2)+1
      NUMQ=((IFIX(VAR/GRIDQ)+1)*2)+1
      RECMAX=NUMQ*NUMR*NUMW
      VAR= (FLOAT(ISUR)**0.5)
      DGF=DEGF*FLOAT(ISUR-1)
      NSTRT=ISUR
      WRITE(IRITE,1401)
1401  FORMAT(1H1,10X,"SEQUENTIAL ANALYSIS OF VARIANCE")
      WRITE(IRITE,1402)
1402  FORMAT(//,20X,"THE TEST IS")
      WRITE(IRITE,1403) LAM0,LAM1
```

```
1404  FORMAT(//,15X,"WITH K =",F2.0)
      WRITE(IRITE,1405)
1405  FORMAT(//,15X,"AND THE FOLLOWING REGIONS")
      WRITE(IRITE,1406)
1406  FORMAT(12X,"STEP",5X,"VN ACCEPT",5X,"VN REJECT")
      DO 1409 IAM=1,MTP
      IF(REG(IAM,1) .LE. 0.0) REG(IAM,1)=-9999.99
      WRITE(IRITE,1407) IAM,REG(IAM,1),REG(IAM,2)
1407  FORMAT(13X,I2,6X,F8.4,6X,F8.4)
1409  CONTINUE
      WRITE(IRITE,1410) GRIDW,GRIDQ,GRIDR
1410  FORMAT(//,12X,"GRIDW=",F6.3,3X,"GRIDQ=",F6.3,3X,"GRIDR=",F6.3)
      WRITE(IRITE,1411) NUMW,NUMQ,NUMR
1411  FORMAT(12X,"SIZEW=",I4,5X,"SIZEQ=",I4,5X,"SIZER=",I4)
      WRITE(IRITE,1412) RECMAX
1412  FORMAT(/,12X,"TOTAL NUMBER OF GRID POINTS USED=",I9)
      CALL BOOK(1,1,1,0,WN,QN,RN)
      PRB1=PHI(WN,0,VAR)
      VALUE=AMAX1(GRIDR,POSPROB(WN,QN,RN,ISUR))
      PRB2=CHISQ(VALUE,DGF)
      CALL PCAL(1,1,NUMR,0,WN,QN,RN)
      PRB3=CHISQ((POSPROB(WN,QN,RN,ISUR)),DGF)
      WRITE(IRITE,1413) PRB1,PRB2,PRB3
1413  FORMAT(12X,"MIN W PROB =",E15.5,/,12X,"RANGE OF R PROB =",E15.5,
     1 5X,F15.5)
      WRITE(IRITE,202)
202   FORMAT(1H1,30X,"SEQUENTIAL ANOVA")
      WRITE(IRITE,203)
203   FORMAT(///,5X,"LAMBDA",10X,"MSN",10X,"ASN",10X,"OC",10X,"POW")
C
C        SEVERAL POINTS ON THE OC CURVE
C        ARE CALCULATED FOR A SEQUENTIAL TEST
C                WITH THESE REGIONS
C
                                                               SEGMENT
      DO 500 NOC=1,10,9
      IF( KOC .GT. NOC) GO TO 500
      ALAM=LAM0+ ((LAM1-LAM0)/9.0)*FLOAT(NOC-1)
      WRITE(7,204) ALAM
204   FORMAT(1H1,30X,"LAMBDA=  ",F6.3)

C
C        FOR A GIVEN LAMDA VALUES OF MEAN1 AND MEAN2 CAN BE
C        CAN BE CALCULATED. THESE ARE NEEDED FOR THE BOOKING
C        SUBROUTINE AND TO CALCULATE CERTAIN PROBABLITIES
C        SUCH AS THE F(W,Q,R) AT THE FIRST STEP
C        AND FOR PHI(Z) AND PHI(U)
C

      XMEAN(NUM)=  SQRT((DEGF*ALAM)/(DEGF-1.0))
      XBR1= XMEAN(1)*FLOAT(ISUR)
      XBR2=XMEAN(2)*FLOAT(ISUR)
      NCONW=IFIX((ISUR*XMEAN(1))/GRIDW)-IFIX(NUMW/2.)
      NCONQ=IFIX((ISUR*XMEAN(2))/GRIDQ)-IFIX(NUMQ/2.)
      JOINT=1
      ICAL=2
      CALL BOOK(1,1,1,0,TMINW,TMINQ,TMINR)
      CALL BOOK(NUMW,NUMQ,NUMR,0,TMAXW,TMAXQ,TMAXR)
      WRITE(7,972) XMEAN(1),XMEAN(2),VAR,XBR1,XBR2
972   FORMAT(///,5X,"MEAN1=",E15.7,2X,"MEAN2=",E15.7,2X,"VAR=",E15.7,2X,
     1"SMEAN1=",E15.7,2X,"SMEAN2=",E15.7)
      WRITE(7,801)  NUMR,NUMQ,NUMW
801   FORMAT(20X,3I7)
      WRITE(7,802) TMINQ,TMINW,TMAXQ,TMAXW,TMAXR
```

```
241   FORMAT(///,8X,"STEP",10X,"PROB. ACCEPT",13X,"PROB. REJECT",11X,
     1"PROB. CONTINUE")
      CALL FSTEPPROB( OC(ISUR,1),OC(ISUR,2),ASN(ISUR),ISUR)
      IF( KTEST .GT. ITEST .OR. NOC .GT. KOC)
     1WRITE(11=(ISUR+((NOC-1)*NTESTS)+(KTEST-1)*NTESTS*10))OC(ISUR,1),
     2 OC(ISUR,2),ASN(ISUR),ISUR,  ALAM,KTEST
      WRITE(7,208) ISUR,OC(ISUR,1),OC(ISUR,2),ASN(ISUR)
      LSTP=ISUR
      CR=1./ISUR
      PR=0.0
      CA=0.0
C
C         THIS PART OF THE PROGRAM CALCULATES
C         F(WN,QN,RN) FROM G( W(N-1), Q(N-1), R(N-1), Z, U)
C     WHERE G(W(N-1),Q(N-1),R(N-1),Z,U)=FT(W(N-1),Q(N-1),R(N-1))*P(Z)*P(U)

C         IN ORDER TO FIND THE QUANTITIES OF INTEREST IN
C         SEQUENTIAL ANALYSIS, NAMELY THE ASN AND OC CURVE
C         THE PROBABILITY DISTRIBUTION MUST BE FOUND
C         AT EVERY STEP N
C         THEN THIS DISTRIBUTION CAN BE INTEGRATED TO FIND
C         THE PROBABILITIES OF ACCEPTING ,REJECTING AND CONTINUING AT
C                                         EVERY STEP

      WRITE(8,777)
777   FORMAT(1H1)
      DO 400 NSTP=NCAL,MTP
      IF( MSTP .GT. NSTP) GO TO 399
      LSTP=NSTP-1
      IF( REG(LSTP,1) .LE. 0.0) GO TO 131
      CA=((2.0*REG(LSTP,1))+1.0)/(2.0*LSTP*REG(LSTP,1))
      PA=1.0/(2.0*LSTP*REG(LSTP,1))
131   IF( REG(LSTP,2) .GE. 1.E6) GO TO 132
      CR=((2.0*REG(LSTP,2))+1.0)/(2.0*LSTP*REG(LSTP,2))
      PR=1.0/(2.0*LSTP*REG(LSTP,2))
      GO TO 133
132   CR=1.0/FLOAT(LSTP)
      PR=0.0
133   PROBAC=0.0
      PROBNR=0.0
      PRRAC=0.0
      PRQAC=0.0
      PRRNR=0.0
      PRQNR=0.0
      DO 1130 I1=1,NUMW
      PRQAC=0.0
      PRQNR=0.0
      DO 1131 I2=1,NUMQ
      CALL BOOK(I1,I2,1,0,WN,QN,RN)
      IRAC=0
      IRNR=0
      RVALAC=1.E30
      IF( CA .LE. 0) GO TO 1121
      RVALAC=PARAR(CA,PA,QN,WN)
1121  RVALNR=PARAR(CR,PR,QN,WN)
      PRRAC=0.0
      PRRNR=0.0
      DO 5132 I3=1,NUMR
      IF( NTRYS .GT. 1 .AND. PTIMI .LE. 0.0)
     1                CALL RESUME( PTIM1,PTIM2,PTIM3,$5132)
      IF( I3 .LE. 1) GO TO 1122
      CALL RCAL(I1,I2,I3,0,WN,QN,RN)
1122  A=WN
```

```
      KREC=JREC
      IF( POSPROB(WN,QN,RN,NSTP) .LT. 0.0) GO TO 1132

C
C        THIS STATEMENT CALCULATES THE DENSITY AT POINT A,B,C
C        AT NSTP BY INTEGRATING OVER A TWO DIMENSIONAL REGION IN LSTP
C

      PROBTS=DENSTS(A,B,C,KREC)

C
C
      IF( C .LT. RVALAC) GO TO 1124
      IRAC=IRAC+1
      IF( IRAC .GT. 1) GO TO 1123
      NPFIAC=NUMR-I3+1
      RACBEG=C
      SPROAC=PROBTS
C
C
C        THIS PART OF THE PROGRAM CALCULATES
C        THE PROBABILITIES OF ACCEPTING,REJECTING,AND CONTINUING
C                              AT STEP NSTP
C        THESE ARE OBTAINED BY PERFORMING A THREEE DIMENSIONAL INTEGRATI

C        THIS THREE DIMENSIONAL INTEGRATION IS IS DONE NUMERICALLY
C        BY THREE SUCCESSIVE 1 DIMENSIONAL INTEGRALS
C        EACH 1 DIMENSIONAL INTEGRATION IS DONE VIA
C        A 14 POINT(IF POSSIBLE) NEWTON-GREGORY FORMULA


 1123 PRRAC=PRRAC+WEIGHT(NPFIAC,IRAC)*PROBTS
 1124  IF( C .LT. RVALNR) GO TO 1132
      IRNR=IRNR+1
      IF( IRNR .GT. 1) GO TO 1125
      NPFINR=NUMR-I3+1
      RNRBEG=C
      SPRONR=PROBTS
 1125 PRRNR=PRRNR+WEIGHT(NPFINR,IRNR)*PROBTS
 1132 TMEL1=PTIM1+TIME(2)/3600.0
      TMEL2=PTIM2+TIME(3)/3600.0
      TMEL3=PTIM3+TIME(4)/3600.0
      WRITE(10=1)  NTRYS,KTEST,NOC,NSTP,JOINT,ICAL,I1,I2,I3,KREC,IRAC,
     1 NPFIAC,IRNR,NPFINR,A,B,C,PROBAC,PROBNR,PRRAC,PRQAC,PRRNR,PRQNR,
     2  RVALAC,RVALNR,RACBEG,SPROAC,RNBEG,SPRONR, TMEL1,TMEL2,TMEL3
 5132 CONTINUE
      IF( IRAC .EQ. 0) GO TO 1126
      Y1=TERPU(A,B,RVALAC,ICAL)
      ADDA=ABS(RVALAC-RACBEG)*0.5*(SPROAC+Y1)
      PRRAC=GRIDR*PRRAC+ADDA
      PRQAC=PRQAC+WEIGHT(NUMQ,I2)*PRRAC
 1126 IF( IRNR .EQ. 0) GO TO 1131
      Y1=TERPU(A,B,RVALNR,ICAL)
      ADDR=ABS(RVALNR-RNRBEG)*0.5*(SPRONR+Y1)
      PRRNR=GRIDR*PRRNR+ADDR
      PRQNR=PRQNR+WEIGHT(NUMQ,I2)*PRRNR
 1131 CONTINUE
      PRQNR=PRQNR*GRIDQ
      PRQAC=PRQAC*GRIDQ
      PROBAC=PROBAC+WEIGHT(NUMW,I1)*PRQAC
      PROBNR=PROBNR+WEIGHT(NUMW,I1)*PRQNR
 1130 CONTINUE
      PROBNR=PROBNR*GRIDW
      PROBAC=PROBAC*GRIDW
```

```
      JOINT=ICAL
      ICAL=INTER
                                                                      S
                                                                 START OF
C        OC(NSTP,1)=PROBABILITY OF ACCEPTING AT STEP NSTP
C        OC(NSTP,2)=PROBABILITY OF REJECTING AT STEP NSTP
C        ASN(NSTP)=PROBABILITY OF CONTINUING AT STEP NSTP
      ASN(NSTP)=PROBNR-PROBAC
      OC(NSTP,1)=PROBAC
      OC(NSTP,2)=ASN(NSTP-1)-ASN(NSTP)-OC(NSTP,1)
      WRITE(11=(NSTP+((NOC-1)*NTESTS)+(KTEST-1)*NTESTS*10)) OC(NSTP,1),
     2 OC(NSTP,2),ASN(NSTP),NSTP,ALAM,KTEST
399   IF( NTRYS .GT. 1 .AND. PTIMI .LE. 0.0)
     1  READ(11=(NSTP+((NOC-1)*NTESTS)+(KTEST-1)*NTESTS*10))
     2  OC(NSTP,1),OC(NSTP,2),ASN(NSTP),NCARE,GLAM,LOSCAS
      WRITE(7,208) NSTP,OC(NSTP,1),OC(NSTP,2),ASN(NSTP)
208   FORMAT(5X,I5,5X,E20.10,5X,E20.10,5X,E20.10)
                                                                      S
400   CONTINUE
C
C        THIS PART OF THE PROGRAM CALCULATES
C        E(N]ALAM)=AVERAGE SAMPLE NUMBER WHEN LAMDA=ALAM
C                                  AND
C        M(N]ALAM)=MEDIAN SAMPLE NUMBER WHEN LAMDA=ALAM
C                  AS WELL AS
C        OC(ALAM)=PROB(REJECTING HO]LAMDA=ALAM)
C        B(ALAM)=1-OC(ALAM)
C
C
      OCF=0.0
      AVR=1.0
      POW=0.0
      TMED=0.0
      DO 490 IN=1,MIP
      AVR=AVR+ASN(IN)
      OCF=OCF+OC(IN,1)
      POW=POW+OC(IN,2)
      TES=IN-1.0-AVR
      IF(TES .LT. 0.5 .OR. ( TMED .GT. 0.0 .AND. TES .GT. 0.5))GO TO 490
      TMED=IN
      IF(TES .GT. 0.5) TMED=IN-0.5
490   CONTINUE
      WRITE(IRITE,209) ALAM,TMED,AVR,OCF,POW
209   FORMAT(5X,F6.4,9X,F6.2,8X,F6.4,8X,F6.4,8X,F6.4)
500   CONTINUE
9112  CONTINUE
9113  CONTINUE
      WRITE(8,9117) IRP1,IRP2
9117  FORMAT(1H1,20X,"MISTAKES IN THEORY",I8,5X,I8)
      STOP
      END

         THE SUBROUTINES CALLED FOLLOW
```

START OF SEGMENT

```
      SUBROUTINE FSTEPPROB(PACC,PREJ,PCON,N)
C
C         THIS SUBROUTINE CALCULATES
C         THE PROBABILITIES OF ACCEPTING,REJECTING,
C         AND COTINUING FOR STEPS
C         FOR STEPS LESS THAN AND EQUAL TO THE FIRST
C         STEP AT WHICH A DECISION CAN BE MADE
C         THIS IS ACCOMPLISHED BY MEANS OF AN INFINITE
C         SUM OF INCOMPLETE BETA FUNCTIONS
C
      COMMON /CB4/DEGF,ALAM
      COMMON /CB5/ REG(30,2)
      DO 50 IB=1,2
      TSUM=0.0
      IF( IB .EQ. 1 .AND. REG(N,1) .LE. 0.0) GO TO 30
      IF( IB .EQ. 2 .AND. REG(N,2) .GE. 1.E6) GO TO 30
      F0=((DEGF*(FLOAT(N)-1.))/(DEGF-1.))*REG(N,IB)
      U0=1.0/(1.0+(((DEGF-1.0)/(DEGF*(FLOAT(N)    -1.)))*F0))
        W1=(DEGF*(FLOAT(N)-1.))*0.5
      W2=(DEGF-1.)*0.5
      TSUM=BETINC(0,W1,W2,U0)
      IF( ALAM .LE. 0.0) GO TO 20
      DO 10 JR=1,101
      W2=((DEGF-1.)*0.5)+FLOAT(JR)
      TOT=(FLOAT(JR)*ALOG(0.5*FLOAT(   N )*ALAM))+ALOG(BETINC(0,W1,W2,U0)
     1 ) -ALGAMA(FLOAT(JR+1))
      TOT=EXP(TOT)
      TSUM=TSUM+TOT
      IF( TOT .LE. 1.E-06) GO TO 20
 10   CONTINUE
 20   TSUM=TSUM*EXP(-.5*FLOAT(N)*ALAM)
 30   IF( IB .GT. 1) GO TO 40
      PACC=1.-TSUM
      GO TO 50
 40   PREJ=TSUM
 50   CONTINUE
 51   PCON=1.-PACC-PREJ
      RETURN
      END
```

SEGMENT

START OF SEGMENT

```
      FUNCTION BETINC(IND,A,B,X)
C     INCOMPLETE BETA FUNCTION AND ITS INVERSE
C    MARK=1 FOR INVERSE (SEND DOWN PROB)
C
C         THIS SUBFUNCTION CALCULATES THE INCOMPLETE BETA FUNCTION
C         THIS IS NEEDED TO CALCULATE THE PA,PR,PC AT THE FIRST STEP
C         A DECISION CAN BE MADE
C
      CAB=CGAM(A+B)-CGAM(A)-CGAM(B)-.5*ALOG((A+B)*6.28318531)
      IF(IND)10,10,20
10    EP=CAB+A*ALOG(X*(1.+B/A))+B*ALOG((1.-X)*(1.+A/B))
       IF(X-A/(A+B))12,12,14
12    BETINC=ZI(X,A,B)*EXP(EP+.5*ALOG(B/A))
      RETURN
14    BETINC=1.-ZI(1.-X,B,A)*EXP(EP+.5*ALOG(A/B))
      RETURN
20    IF(X-.5)22,22,24
22    QZ=ALOG(X)
      IGO=1
      AA=A
      BB=B
      GO TO 26
24    QZ=ALOG(1.-X)
      IGO=2
      AA=B
      BB=A
26    XT=AA/(AA+BB)
      CABB=CAB+.5*ALOG(BB/AA)+AA*ALOG(1.+BB/AA)+BB*ALOG(1.+AA/BB)
      DO 40 NC=1,100
      ZZ=ZI(XT,AA,BB)
      QX=CABB+AA*ALOG(XT)+BB*ALOG(1.-XT)+ALOG(ZZ)
      XC=(QZ-QX)*(1.-XT)*ZZ/AA
      XC=AMAX1(XC,-.99)
      XC=AMIN1(XC,.5/XT-.5)
      XT=XT*(1.+XC)
      IF(ABS(XC)-1.E-6)42,40,40
40    CONTINUE
42    GO TO (44,46),IGO
44    BETINC=XT
      RETURN
46    BETINC=1.-XT
      RETURN
      END
```

SEGMENT

START OF SEGM

```
      SUBROUTINE BOOK (L1,L2,L3,LN,WN,QN,RN)
C
C
C         THIS SUBROUTINE IS A BOOK KEEPING ROUTINE
C
C         THIS ROUTINE CONVERTS A POINT IN THE GRID F(W,Q,R) TO
C         A POINT IN THE RANDOM ACCESS DISK FILE
C         THE POINT IN THAT FILE FOR A PARTICULAR POINT
C         IS TERMED JREC
      COMMON /CB2/JREC,TSTAT
      COMMON /CB1/GRIDW,GRIDQ,GRIDR
      COMMON /CB3/ NCONW,NCONQ,NSTRT
      COMMON /CB12/NUMW,NUMQ,NUMR
      WN=(L1-1+NCONW)*GRIDW
      QN=(L2-1+NCONQ)*GRIDQ
      ENTRY REAL(L1,L2,L3,LN,WN,QN,RN)
      RN=(IFIX((WN**2.+QN**2.)/(NSTRT*GRIDR))+L3)*GRIDR
      IF(  LN .LE. 0)   GO TO 10
      ENTRY CRITV( WN,QN,RN,LN)
      DBLCHK=((LN*RN)-(WN**2.)-(QN**2.))*2.0
      IF( DBLCHK   .LE. 0.0)   GO TO 5
      TSTAT=((WN-QN)**2.0)/DBLCHK
      RETURN
    5 TSTAT= 1.0 E20
      RETURN
      ENTRY IENT(L1,L2,L3,WN,QN,RN)
      L1=(WN/GRIDW)+1-NCONW
      L2=(QN/GRIDQ)+1-NCONQ
      L3=(RN/GRIDR)-IFIX((WN**2.+QN**2.)/(NSTRT*GRIDR))
      SZCHK=L1+((L2-1)*NUMW)+((L3-1)*NUMW*NUMQ)
      IF(ABS(SZCHK) .LT. 549755813886) GO TO 30
      JREC=541111111111
      RETURN
   30 JREC= IFIX(SZCHK)
      RETURN
   10 JREC=L1+((L2-1)*NUMW)+((L3-1)*NUMW*NUMQ)
      RETURN
      END
```

SEGMENT

START OF SEGMENT

```
      FUNCTION POSPROB(WV,QV,RV,N)
C
C        THIS IS A FUNCTION TO DETERMINE IF
C        A POINT IS ALLOWABLE AT STEP N
C
      POSPROB=RV-((WV**2.+QV**2.)/FLOAT(N))
      IF( ABS(POSPROB) .LE. 1.E-4) POSPROB=0.0
      RETURN
      END
```

SEGMENT

START OF SEGMENT

```
      FUNCTION PHI(Y,XBAR,SIG)
C
C        THIS SUBFUNCTION CALCULATES THE NORMAL DENSITY FUNCTION
C
      PHI=0.39894220*EXP(-.5*(((Y-XBAR)/SIG)**2.))*(1./SIG)
      RETURN
      END
```

SEGMENT

START OF SEGMENT

```
      FUNCTION CHISQ(Y,DOF)
C
C        THIS SUBFUNCTION CALCULATES THE CHISQUARE DENSITY FUNCTION
C
      CHISQ      =((Y**((DOF/2.)-1.))*EXP(-Y/2.))/((2.**(DOF/2.))
     1            *GAMMA(DOF/2.0))
      IF( Y .EQ. 0.0 .AND. DOF .EQ. 2.0) CHISQ=0.5
      RETURN
      END
```

SEGMENT

```
                                                                  START OF SEGMENT
      FUNCTION DENSIS(A,B,C,KREC)
C
C          THIS SUBROUTINE CALCULATES FN(A,B,C)
C          FROM G(A-Z,B-U,R-Z**2-U**2)*P(U)*P(Z)
C          BY INTEGRATING OVER THE APPROPRIATE REGIONS
      COMMON /CB1/ GRIDW,GRIDQ,GRIDR
      COMMON /CB6/ TMAXW,TMINW,TMAXQ,TMINQ
      COMMON /CB7/ LSTP,ISUR
      COMMON /CB8/ JOINT,ICAL
      COMMON /CB9/ NIP,RINT,CUR
      COMMON /CB5/ REG(30,2)
      DIMENSION POINT(4,4),FVAL(4),RINT(5,4)
      VOLUME=0.0
C
C          THIS NUMERICAL INTEGRATION INVOLVES SUMMING THE VOLUMES
C          OF    TRAPEZOIDSC
C          U=QDIMENSION   = MEAN 2
C          Z=   W DIMENSION = MEAN 1
C
      CALL UBOUND(A,B,C,REG(LSTP,1),REG(LSTP,2),NREG,RINT)
      IF( NREG .EQ. 0)  GO TO 230
      DO 229 NIP=1,NREG
      CALL ZRANGE(RINT(NIP,1),RINT(NIP,3),RINT(NIP,4),TMX,TMIN)
      USTRT= (IFIX(RINT(NIP,1)/GRIDW)+1)*GRIDW
      IF(RINT(NIP,1) .LT. 0.0 .AND. RINT(NIP,1) .NE. (USTRT-GRIDW))
     1         USTRT=IFIX(RINT(NIP,1)/GRIDW)*GRIDW
      UFIN=IFIX(RINT(NIP,2)/GRIDW)*GRIDW
      IF(RINT(NIP,2) .LT. 0.0) UFIN=(IFIX(RINT(NIP,2)/GRIDW)-1)*GRIDW
      IF( UFIN .EQ. RINT(NIP,2)) UFIN=UFIN-GRIDW
      IF( UFIN .LE. USTRT ) GO TO 220
      IF( LSTP .EQ. ISUR) GO TO 157
      IF((B-UFIN) .GT. TMAXQ .AND. (B-USTRT) .GT. TMAXQ) GO TO 220
      IF((B-UFIN) .LT. TMINQ .AND. (B-USTRT) .LT. TMINQ) GO TO 220
      IF((B-USTRT) .LT. TMINQ) USTRT=B-TMINQ
      IF((B-USTRT) .GT. TMAXQ) USTRT=B-TMAXQ
      IF((B-UFIN) .LT. TMINQ) UFIN=B-TMINQ
      IF((B-UFIN) .GT. TMAXQ) UFIN=B-TMAXQ
      IF( USTRT .GE. UFIN ) GO TO 220
157   U1=USTRT
      U2=USTRT+GRIDW
      CALL   ZRANGE(U1,RINT(NIP,3),RINT(NIP,4),ZMAX1,ZMIN1)
      POINT(1,1)=RINT(NIP,1)
      POINT(1,2)=TMX
      POINT(2,1)=U1
      POINT(2,2)=ZMAX1
      POINT(3,1)=RINT(NIP,1)
      POINT(3,2)=TMIN
      IF( TMX .EQ. TMIN) GO TO 158
      POINT(4,1)=U1
      POINT(4,2)=ZMIN1
      CUR=RINT(NIP,3)
      CALL RESVOL(A,B,C,VOLUME,4,4,POINT,FVAL)
      VOLUME=VOLUME+
     2(AREA(RINT(NIP,4),A-POINT(4,2),B-POINT(4,1),RINT(NIP,4),A-POINT(3,
     52),B-POINT(3,1))*(FVAL(3)+FVAL(4))*0.5)
      NSIDES=4
      GO TO 159
158   POINT(3,1)=U1
      POINT(3,2)=ZMIN1
      CALL RESVOL(A,B,C,VOLUME,3,3,POINT,FVAL)
      NSIDES=3
159   ZBEG1= IFIX(ZMIN1/GRIDQ)*GRIDQ
      ZFIN1=IFIX( ZMAX1/GRIDQ)*GRIDQ
```

```
160   CALL  ZRANGE(U2,RINT(NIP,3),RINT(NIP,4),ZMAX2,ZMIN2)
      ZBEG2=IFIX(ZMIN2/GRIDQ)*GRIDQ
      ZFIN2=IFIX(ZMAX2/GRIDQ)*GRIDQ
      IF(ZMIN2 .GT. 0.0) ZBEG2=(IFIX(ZMIN2/GRIDW)+1)*GRIDW
      IF(ZMAX2 .LT. 0.0) ZFIN2=(IFIX(ZMAX2/GRIDW)-1)*GRIDW
      ZBEG=  AMAX1(ZBEG1,ZBEG2)
      ZFIN=  AMIN1(ZFIN1,ZFIN2)
      IF( ZBEG .GE. ZFIN) GO TO 197
      IF( LSTP .EQ. ISUR) GO TO 168
      IF((A-ZBEG) .GT. TMAXW .AND. (A-ZFIN) .GT. TMAXW) GO TO 197
      IF((A-ZBEG) .LT. TMINW .AND. (A-ZFIN) .LT. TMINW) GO TO 197
      IF((A-ZBEG) .LT. TMINW) ZBEG=A-TMAXW
      IF((A-ZBEG) .LT. TMINW) ZBEG=A-TMINW
      IF((A-ZFIN) .GT. TMAXW) ZFIN=A-TMAXW
      IF((A-ZFIN) .LT. TMINW) ZFIN=A-TMINW
      IF(ZBEG .GE. ZFIN) GO TO 197
168   ZINT=ZBEG
170   Y1=TERPOS(U1,ZINT)
      Y2=TERPOS(U2,ZINT)
178   IF(ZINT .NE. ZBEG .AND. ZINT .NE. ZFIN) GO TO 180
      ZDET=ZMIN2
      NPFI=1
      IF( ZINT .NE. ZBEG .OR. U1 .NE. USTRT) GO TO 174
      UBFU=U1
      ZBFU=ZMAX1
      FBFU=FVAL(2)
      COR=RINT(NIP,4)
      IF( NSIDES .EQ. 3) GO TO 173
      POINT(2,1)=POINT(4,1)
      POINT(2,2)=POINT(4,2)
      FVAL(2)=FVAL(4)
      GO TO 176
173   POINT(2,1)=U1
      POINT(2,2)=ZMIN1
      FVAL(2)=FVAL(3)
      IF( POINT(3,1) .NE. U1 .OR. POINT(3,2) .NE. ZMIN1)
     1 FVAL(2)=TERPOS(U1,ZMIN1)
      IF( POINT(2,1) .NE. U1 .OR. POINT(2,2) .NE. ZMAX1)
     2 FBFU=TERPOS(U1,ZMAX1)
      GO TO 176
174   IF( ZINT .NE. ZBEG) GO TO 175
      POINT(2,1)=UBFL
      POINT(2,2)=ZBFL
      FVAL(2)=FBFL
      COR=RINT(NIP,4)
      GO TO 176
175   POINT(2,1)=UBFU
      POINT(2,2)=ZBFU
      FVAL(2)=FBFU
      ZDET=ZMAX2
      COR=RINT(NIP,3)
176   POINT(1,1)=U2
      POINT(1,2)=ZDET
      POINT(3,1)=U2
      POINT(3,2)=ZINT
      FVAL(3)=Y2
      POINT(4,1)=U1
      POINT(4,2)=ZINT
      FVAL(4)=Y1
      NSIDES=4
      CALL RESVOL(A,B,C,VOLUME,NSIDES,NPFI,POINT,FVAL)
      IF( ZINT .NE. ZBEG) GO TO 177
      UBFL=POINT(1,1)
```

```
      GO TO 178
177   UBFU=POINT(1,1)
      ZBFU=POINT(1,2)
      FBFU=FVAL(1)
178   VOLUME=VOLUME+GRIDW*GRIDQ*1.5*(1.0/6.0)*(Y1+Y2)
        IF(ZINT .EQ. ZFIN)  GO TO 200
      GO TO 190
180   VOLUME=VOLUME+(1.0/5.0)*GRIDQ*GRIDW*3.0*(Y1+Y2)
190   ZINT=ZINT+GRIDQ
      IF(ZINT .LE. ZFIN) GO TO 170
      GO TO 200
197   IF( (ZMAX2 .EQ. ZMIN2) .AND. U2 .EQ. UFIN) GO TO 201
      POINT(1,1)=U1
      POINT(1,2)=ZMAX1
      POINT(2,1)=U2
      POINT(2,2)=ZMAX2
      POINT(3,1)=U1
      POINT(3,2)=ZMIN1
      POINT(4,1)=U2
      POINT(4,2)=ZMIN2
      CUR=RINT(NIP,3)
      CALL RESVOL(A,B,C,VOLUME,4,4,POINT,FVAL)
      VOLUME=VOLUME+
     2(AREA(RINT(NIP,4),A-POINT(4,2),B-POINT(4,1),RINT(NIP,4),A-POINT(3,
     52),B-POINT(3,1))*(FVAL(3)+FVAL(4))*0.5)
      UBFU=U2
      ZBFU=ZMAX2
      FBFU=FVAL(2)
      UBFL=U2
      ZBFL=ZMIN2
      FBFL=FVAL(4)
200   U1=U2
      U2=U2+GRIDW
      ZBEG1=ZBEG2
      ZFIN1=ZFIN2
      ZMIN1=ZMIN2
      ZMAX1=ZMAX2
      IF( U2  .LE.  UFIN) GO TO 160
201   CALL ZRANGE(RINT(NIP,2),RINT(NIP,3),RINT(NIP,4),TMX,TMIN)
      POINT(1,1)=RINT(NIP,2)
      POINT(1,2)=TMX
      POINT(2,1)=U1
      POINT(2,2)=ZMAX1
      POINT(3,1)=RINT(NIP,2)
      POINT(3,2)=TMIN
      IF(TMX .EQ. TMIN) GO TO 219
      POINT(4,1)=U1
      POINT(4,2)=ZMIN1
      CUR=RINT(NIP,3)
      CALL RESVOL(A,B,C,VOLUME,4,4,POINT,FVAL)
      VOLUME=VOLUME+
     2(AREA(RINT(NIP,4),A-POINT(4,2),B-POINT(4,1),RINT(NIP,4),A-POINT(3,
     52),B-POINT(3,1))*(FVAL(3)+FVAL(4))*0.5)
      GO TO 229
219   POINT(3,1)=U1
      POINT(3,2)=ZMIN1
      CALL RESVOL(A,B,C,VOLUME,3,3,POINT,FVAL)
      GO TO 229
220   UINT=(RINT(NIP,2)-RINT(NIP,1))*.5+RINT(NIP,1)
      NTC=1
      CALL ZRANGE(UINT,RINT(NIP,3),RINT(NIP,4),ZX,ZM)
221   IF( TMX .EQ. TMIN) GO TO 222
      NSIDES=4
      POINT(1,1)=RINT(NIP,NTC)
```

```
      POINT(2,1)=UINT
      POINT(2,2)=ZX
      POINT(3,1)=RINT(NIP,NTC)
      POINT(3,2)=TMIN
      POINT(4,1)=UINT
      POINT(4,2)=ZM
      CUR=RINT(NIP,3)
      CALL RESVOL(A,B,C,VOLUME,NSIDES,4,POINT,FVAL)
      VOLUME=VOLUME+
     2(AREA(RINT(NIP,4),A-POINT(4,2),B-POINT(4,1),RINT(NIP,4),A-POINT(3,
     52),B-POINT(3,1))*(FVAL(3)+FVAL(4))*0.5)
      GO TO 223
  222 NSIDES=3
      POINT(1,1)=RINT(NIP,NTC)
      POINT(1,2)=TMX
      POINT(2,1)=UINT
      POINT(2,2)=ZX
      POINT(3,1)=UINT
      POINT(3,2)=ZM
      CALL RESVOL(A,B,C,VOLUME,NSIDES,3,POINT,FVAL)
  223 IF(NTC .GE. 2) GO TO 229
      CALL ZRANGE(RINT(NIP,2),RINT(NIP,3),RINT(NIP,4),TMX,TMIN)
      NTC=NTC+1
      GO TO 221
  229 CONTINUE
  230 DENSTS=VOLUME
      WRITE(ICAL=KREC) VOLUME
      RETURN
      END
                                                                 SEGMENT

                                                        START OF SEGMENT
      FUNCTION CGAM(A)
C
C        THIS SUBROUTINE IS NEEDED FOR THE INCOMPLETE BETA FUNCTION CALC
C
      AA=A
      CAC=0.0
      IF(A-2.)2,8,8
    2 IF(A-1.)4,6,6
    4 CAC=-2.+(A+.5)*ALOG(1.+1./A)+(A+1.5)*ALOG(1.+1./(A+1.))
      AA=A+2.
      GO TO 8
    6 CAC=-1.+(A+.5)*ALOG(1.+1./A)
      AA=A+1.
    8 CA=2.269489/AA
      CA=.52560647/(AA+1.0115231/(AA+1.5174737/(AA+CA)))
      CA=.0833333333/(AA+.03333333/(AA+.25238095/(AA+CA)))
      CGAM=CA+CAC
      RETURN
      END
                                                                 SEGMENT
```

```
      SUBROUTINE   UBOUND( A,B,C,VAN,VRN,NREG,RINT)
      COMMON /CB7/ NSTP
      COMMON /CB10/ CA,PA,CR,PR
      COMMON /CB11/ SINE45
      DIMENSION  RINT(5,5)
C        THIS SUBROUTINE CALCULATES THE INTEGRATION
C        LIMITS OF U, AND ALSO DETERMINES THE
C        NUMBER OF INTEGRATION REGIONS AND TYPE
C        SO AS TO ALLOW DETERMINATION OF THE Z RANGE
C        THESE ARE NEEDED TO OBTAIN THE DENSITY F(W,U,R) AT STEP
C        N FROM THE DENSITY AT STEP N-1
C        THE FOLLOWING CODE IS EMPLOYED
C        1=REJECTION ELLIPSE LOWER
C        2=REJECTION ELLIPSE UPPER
C        3=ACCEPTANCE ELLIPSE LOWER
C        4=ACCEPTANCE ELLIPSE UPPER
C        5=CIRCLE LOWER
C        6=CIRCLE UPPER
C        RINT(1,3)= UPPER Z CURVE ( LOWER W)
C        RINT(2,4)= LOWER Z CURVE   ( UPPER W)
      TERM2(XCR,XA,XB,XPR)= (XB*(XCR+1.)+XA*XPR)/(((XCR+1.0)**2.0)-(XPR
     1 **2.0))
      DISCR(XCR,XA,XB,XPR,XC)= SQRT((TERM2(XCR,XA,XB,XPR)**2.0)-(((XA**
     1  2.0)-((XCR+1.0)*(XA**2.0+XB**2.0-XC)))/((XPR**2.)-((XCR+1.)**2.)
     2 )))
      IF( VAN .LE. 0.0)  GO TO 5
      DA=C-A**2.-B**2.+((.5*((A+B)**2.)*NSTP)/(NSTP+1.))+((.5*((A-B)**2.
     5   ))/(CA+PA+1.))
 5    IF( VRN .LE. 0.0)  GO TO 7
      DR=C-A**2.-B**2.+((.5*((A+B)**2.)*NSTP)/(NSTP+1.))
     7     +((.5*((A-B)**2.0))/(CR+PR+1.))
 7    IF(VAN .LE. 0.0)  GO TO 60
      IF(VRN  .LE. 0.0)  GO TO 70
      IF(  DR .LE. 0.0 .AND. DA .LE. 0.0)  GO TO 15
      IF( DA .LE. 0.0 .AND. DR .GT. 0.0) GO TO 60
      DR1=SQRT((DR*NSTP)/(NSTP+1.0))
      DR2=SQRT(DR/(CR+PR+1.0))
      DA1=  SQRT((DA*NSTP)/(NSTP+1.0))
      DA2=SQRT(DA/(CA+PA+1.0))
      HA=( SINE45   *(A+B)*NSTP)/(NSTP+1.0)
      HR=(SINE45  *(A+B)*NSTP)/(NSTP+1.0)
      TKA=(SINE45   *(A-B))/(CA+PA+1.0)
      TKR=(SINE45   *(A-B))/(CR+PR+1.0)
      CHK=.5*(((A+B)*NSTP)**2.0)-NSTP*(NSTP+1.)*(A**2.+B**2.-C)
      IF(CHK)20,10,10
 10   TLOC=(((HR-HA)**2.)/(DR1**2.))+((TKR-TKA)**2.)/(DR2**2.) -1.
      IF( TLOC .GT. 0.0)  GO TO 60
      TLOC=(((HA-HR)**2.)/(DA1**2.))+((TKA-TKR)**2.)/(DA2**2.) -1.
      TSPEC= (((HA-HR)**2.)/(DA1**2.))+((TKR+DR2-TKA)**2.)/(DA2**2.) -1.
      IF( TLOC .LT. 0.0 .AND. TSPEC .GT. 0.0) GO TO 20
 15   NREG=0
      RETURN
 20   CON=TERM2(CR,A,B,PR)
      QUAD=DISCR(CR,A,B,PR,C)
      QNLR=CON-QUAD
      QNUR=CON+QUAD
      QUAD=DISCR(CA,A,B,PA,C)
      CON=TERM2(CA,A,B,PA)
      QNLA=CON-QUAD
      QNUA=CON+QUAD
      IF(CHK) 30,30,40
 30   NREG=4
      RINT(1,1)=B-QNLA
```

```
      RINT(1,3)=2.
      RINT(1,4)=1.
      RINT(2,1)=B-QNUA
      RINT(2,2)=B-QNLA
      RINT(2,3)=4.
      RINT(2,4)=2.
      RINT(3,1)=RINT(2,1)
      RINT(3,2)=RINT(2,2)
      RINT(3,3)=3.
      RINT(3,4)=1.
      RINT(4,1)=B-QNUR
      RINT(4,2)=B-QNUA
      RINT(4,3)=2.
      RINT(4,4)=1.
      RETURN
40    NREG=5
      RINT(1,1)=B-QNLA
      RINT(1,2)=B-QNLR
      RINT(1,3)=2.
      RINT(1,4)=1.
      RINT(2,1)=B-QNUA
      RINT(2,2)=B-QNLA
      RINT(2,3)=2.
      RINT(2,4)=4.
      RINT(5,1)=B-QNUR
      RINT(5,2)=B-QNUA
      RINT(5,3)=2.
      RINT(5,4)=1.
      CON= SINE45   *(A+B)*NSTP
      QUAD=SQRT(CHK)
      QN1=(1./(NSTP+1.))*(CON+QUAD)* SINE45
      QN2=(1./(NSTP+1.))*(CON-QUAD)*SINE45
      RINT(3,1)=B-QN1
      RINT(3,2)=B-QNLA
      RINT(4,1)=B-QNUA
      RINT(4,2)=B-QN2
      IF(IKR  .LT. IKA)  GO TO 50
      RINT(3,3)=3.
      RINT(3,4)=1.
      RINT(4,3)=3.
      RINT(4,4)=1.
      RETURN
50    RINT(3,3)=1.
      RINT(3,4)=3.
      RINT(4,3)=1.
      RINT(4,4)=3.
      RETURN
60    IF(  DR  .LE. 0.0)  GO TO 15
      NREG=1
      CON=TERM2(CR,A,B,PR)
      QUAD=DISCR(CR,A,B,PR,C)
      QNLR=CON-QUAD
      QNUR=CON+QUAD
      RINT(1,1)=B-QNUR
      RINT(1,2)=B-QNLR
      RINT(1,3)=2.
      RINT(1,4)=1.
      RETURN
70    IF(DA  .GT. 0.0)  GO TO 80
      NREG=1
      RINT(1,1)=- SQRT(C)
      RINT(1,2)=SQRT(C)
      RINT(1,3)=6.
      RINT(1,4)=6.
```

```
80    NREG=4
      CON=FERM2(CA,A,B,PA)
      QUAD=DISCR(CA,A,B,PA,C)
      QNLA=CON-QUAD
      QNUA=CON+QUAD
      RINT(1,1)=B-QNLA
      RINT(1,2)=SQRT(C)
      RINT(1,3)=6.
      RINT(1,4)=5.
      RINT(2,1)=B-QNUA
      RINT(2,2)=B-QNLA
      RINT(2,3)=6.
      RINT(2,4)=4.
      RINT(3,1)=B-QNUA
      RINT(3,2)=B-QNLA
      RINT(3,3)=3.
      RINT(3,4)=5.
      RINT(4,1)=-   SQRT(C)
      RINT(4,2)=B-QNUA
      RINT(4,3)=6.
      RINT(4,4)=5.
      RETURN
      END
                                                              SEGMENT
```

```
                                                     START OF SEGMENT
      FUNCTION ZI(X,A,B)
C
C        THIS SUBROUTINE IS NEEDED FOR THE INCOMPLETE BETA FUNCTION CALC
C
      FN=.7*(ALOG(15.+A+B))**2+AMAX1(X*(A+B)-A,0.0)
      N=INT(FN)
      C=1.-(A+B)*X/(A+2.*FN)
      ZI=2./(C+SQRT(C**2-4.*FN*(FN-B)*X/(A+2.*FN)**2))
      DO 60 J=1,N
      FN=N+1-J
      A2N=A+2.*FN
      ZI=(A2N-2.)*(A2N-1.-FN*(FN-B)*X*ZI/A2N)
      ZI=1./(1.-(A+FN-1.)*(A+FN-1.+B)*X/ZI)
60    CONTINUE
      RETURN
      END
                                                              SEGMENT
```

```
      SUBROUTINE RESVOL(A,B,C,VOLUME,NSIDES,NPNTBT,POINT,FVAL)
      COMMON /CB9/NIP,RINT,CUR
      COMMON /CB1/ GRIDW,GRIDQ,GRIDR
      DIMENSION POINT(4,4),FVAL(4),RINT(5,4) ,S(4)
      TLINE(X1,Y1,X2,Y2,X)=((Y2-Y1)*(X-X1)/(X2-X1))+Y1

      DO 10 IP=1,NPNTBT
      FVAL(IP)=TERP05(POINT(IP,1),POINT(IP,2))
10    CONTINUE
      GO TO (40,40,20,30),NSIDES
20    PIEC1=AREA(RINT(NIP,3),A-POINT(1,2),B-POINT(1,1),RINT(NIP,3),A-
     1     POINT(2,2),B-POINT(2,1)         )
      PIEC2=AREA(RINT(NIP,4),A-POINT(1,2),B-POINT(1,1),RINT(NIP,4),A-
     1     POINT(3,2),B-POINT(3,1))
      VOLUME=VOLUME+(PIEC1*(FVAL(1)+FVAL(2))+PIEC2*(FVAL(1)+FVAL(3)))*.5
      CURQ=IFIX(((POINT(2,2)-POINT(3,2))*0.5+POINT(3,2))/GRIDQ)*GRIDQ
      FIMP=TERP05(POINT(2,1),CURQ)
      DO 21 I=1,2
      DO 21 J=1,3
      IF(FVAL(I) .GE. FVAL(J)) GO TO 21
      AINT1=FVAL(I)
      FVAL(I)=FVAL(J)
      FVAL(J)=AINT1
      AINT1=POINT(I,1)
      AINT2=POINT(I,2)
      POINT(I,1)=POINT(J,1)
      POINT(I,2)=POINT(J,2)
      POINT(J,1)=AINT1
      POINT(J,2)=AINT2
21    CONTINUE
      S(1)=SQRT(((POINT(1,1)-POINT(2,1))**2.)+((POINT(1,2)-POINT(2,2))
     1   **2.0))
      S(2)=SQRT(((POINT(1,1)-POINT(3,1))**2.)+((POINT(1,2)-POINT(3,2))
     1   **2.0))
      S(3)=SQRT(((POINT(2,1)-POINT(3,1))**2.)+((POINT(2,2)-POINT(3,2))
     1   **2.0))
      SPER=0.5*(S(1)+S(2)+S(3))
      BSAREA=      (SPER*(SPER-S(1))*(SPER-S(2))*(SPER-S(3)))
      IF( BSAREA .LE. 0.0) BSAREA=0.0
      BSAREA=SQRT(BSAREA)

      IF( FVAL(1) .LE. 0.0) GO TO 25
      RVOL=((FVAL(1)-FVAL(2)+FVAL(1)-FVAL(3))*BSAREA)/3.0
      VOLUME=VOLUME+((BSAREA*FVAL(1))-RVOL)
25    EXTRA=(BSAREA+PIEC1+PIEC2)*(FIMP-AMIN1(FVAL(1),FVAL(2),FVAL(3)))
     1 *     0.5
      EXTRA=AMAX1(0.0,EXTRA)
      VOLUME=VOLUME+EXTRA
40    RETURN
30    H=ABS(POINT(1,1)-POINT(2,1))
      IF( POINT(1,1) .EQ. POINT(2,1) .OR. POINT(3,1) .EQ. POINT(4,1))
     1                RETURN
      B1=0.5*ABS(POINT(2,2)-POINT(4,2))*(FVAL(2)+FVAL(4))
      BP=0.5*ABS(POINT(1,2)-POINT(3,2))*(FVAL(1)+FVAL(3))
      X=0.5*(POINT(2,1)-POINT(1,1))+POINT(1,1)
      Y1=TLINE(POINT(1,1),POINT(1,2),POINT(2,1),POINT(2,2),X)
      Z1=TLINE(POINT(1,1),FVAL(1),POINT(2,1),FVAL(2),X)
      X=0.5*(POINT(4,1)-POINT(3,1))+POINT(3,1)
      Y2=TLINE(POINT(3,1),POINT(3,2),POINT(4,1),POINT(4,2),X)
      Z2=TLINE(POINT(3,1),FVAL(3),POINT(4,1),FVAL(4),X)
      BH=ABS(Y1-Y2)
      BMID=0.5*BH*(Z1+Z2)
```

```
      VOLUME=VOLUME+
     1(AREA(CUR,A-POINT(2,2),B-POINT(2,1),CUR,A-POINT(1,2),B-POINT(1,1))
     2        *(FVAL(1)+FVAL(2))*0.5)
      RETURN
      END
```

SEGMENT

START OF SEGMENT

```
      SUBROUTINE    ZRANGE(UVAL,TCL,TCU,ZMAX,ZMIN)
      COMMON /CB10/CA,PA,CR,PR,A,B,C
C
C
C        THIS SUBROUTINE CALCULATES THE Z INTEGRATION LIMITS
C        FOR A GIVEN U VALUE
C
C
      ELLIPS(XPR,XCR,QP,POM)=((A+XPR*QP)/(XCR+1.))+((-1.)**POM)*
     1SQRT(ECHECK(1,(((A+XPR*QP)/(XCR+1.))**2.-(A**2.+B**2.-C-2.*B*QP
     2 +((XCR+1.)*(QP**2.)))/(XCR+1.))))
      CIRCLE(QP,POM)=((-1.)**POM)*
     1 SQRT(ECHECK(2,(C-(QP**2.))))
      MTYP=IFIX(TCU)
      LTYP=IFIX(TCL)
      QNP=B-UVAL
      GO TO (10,10,20,20,30,30) ,MTYP
10    ZCAL= ELLIPS(PR,CR,QNP,TCU)
      GO TO 40
20    ZCAL=ELLIPS(PA,CA,QNP,TCU)
      GO TO 40
30    ZCAL= CIRCLE(UVAL,TCU)
40    ZMAX=A-ZCAL
      GO TO (50,50,60,60,70,70)  ,LTYP
50    ZCAL=ELLIPS(PR,CR,QNP,TCL)
      GO TO 80
60    ZCAL=ELLIPS(PA,CA,QNP,TCL)
      GO TO 80
70    ZCAL= CIRCLE(UVAL,TCL)
80    ZMIN=A-ZCAL
      RETURN
      END
```

SEGMENT

AD-A072 641 UNION COLL AND UNIV SCHENECTADY NY INST OF ADMINISTR--ETC F/G 12/1
AN EXACT TEST FOR THE SEQUENTIAL ANALYSIS OF VARIANCE.(U)
AUG 79 R W MILLER N00014-77-C-0438
UNCLASSIFIED AES-7906 NL

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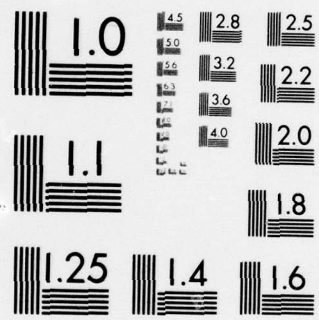

MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

```
      FUNCTION TERPO(W,Q,R,IREAD)
C         THIS ROUTINE ESTIMATES THE DENSITY F(W,Q,R)
C         FOR POINTS NOT LYING ON THE TRIVARIATE GRID
C         THIS SUBPROGRAM PERFORMS INTERPOLATION IN
C         ONE,TWO,OR THREE DIMENSIONS
      COMMON/CB1/GRIDW,GRIDQ,GRIDR
      COMMON /CB3/NCONW,NCONQ,NSTRT
      COMMON/CB14/XMEAN(2),XBR1,XBR2,VAR,DGF
      COMMON/CB7/ LSTP,ISUR
      COMMON /CB6/ IMAXW,IMINW,IMAXQ,IMINQ
      COMMON /CB12/ NUMW,NUMQ,NUMR
      COMMON /CB13/ QBEG,WBEG
      COMMON /CB2/ JREC,TSTAT
      COMMON /CB5/ REG(30,2)
      COMMON /CB20/ RECMAX
      DIMENSION COORD(8,4),XVAL(10),YVAL(10)
      DET2(A,B,C,D)=A*B-C*D
      TERPO=0.0
      IF( POSPROB(W,Q,R,LSTP+1) .LT. 0.0) RETURN
10    LB1=(W/GRIDW)+IFIX(NUMW/2.)+1-IFIX((ISUR*XMEAN(1))/GRIDW)
      WBEG=(LB1-1-IFIX(NUMW/2)+IFIX((ISUR*XMEAN(1))/GRIDW))*GRIDW
      LB2=(Q/GRIDQ)+IFIX(NUMQ/2)+1-IFIX((ISUR*XMEAN(2))/GRIDQ)
      QBEG=(LB2-1-IFIX(NUMQ/2)+IFIX((ISUR*XMEAN(2))/GRIDQ))*GRIDQ
      ENTRY TERPO1(W,Q,R,IREAD)
      JF=2
      JS=1
      JI=1
      NPSF=1
      NPNA=0
      IF(WBEG .EQ. W .AND. QBEG .EQ. Q) GO TO 15
      IF(WBEG .EQ. W) GO TO 45
      IF(QBEG .EQ. Q) GO TO 80
      GO TO 95
15    IF((W .GT. TMAXW .OR. W .LT. TMINW)
     1    .AND. (Q .GT. TMAXQ .OR. Q .LT. TMINQ)) GO TO 95
      IF(W .GT. TMAXW .OR. W .LT. TMINW) GO TO 80
      IF( Q .GT. TMAXQ .OR. Q .LT. TMINQ) GO TO 45
C
C         INTERP/LATION IN ONE DIMENSION(RN)
C                   VIA 4TH DEGRE LAGRANGE
C
      L4=(R/GRIDR)-IFIX(((W**2.)+(Q**2.))/(NSTRT*GRIDR))
      L5=L4+2
      POW=-1.0
      IF(L5 .GT. NUMR) L5=NUMR
      IF(L4) 16,16,17
16    L5=-1
      POW=1.0
17    WN=W
      QN=Q
      NP=1
      NTT=NUMR/2
      DENS=0.0
18    DO 20 I=1,NTT
      L6=L5+1+(POW*I)
      IF(L6 .LE. 0 .OR. L6 .GT. NUMR) GO TO 19
      CALL RCAL(0,0,L6,LSTP,WN,QN,RN)
      IF( TSTAT .GE. REG(LSTP,2)) GO TO 20
      CALL IENT(LJ1,LJ2,LJ3,WN,QN,RN)
      READ(IREAD=JREC) YVAL(NP)
      XVAL(NP)=RN
      IF(NP .GE. 4) GO TO 21
```

```
19    POW=(-1.0)*POW
      GO TO 16
21    DENS=POLY(XVAL,YVAL,R,NP)
40    IF( DENS .LT. 1.0E-35) DENS=0.0
      TERPO=DENS
      RETURN
45    IF(W .GT. TMAXW .OR. W .LT. TMINW) GO TO 95
      V1=W
C
C          TWO DIMENSIONAL INTERPOLATION (QN,RN)
C          USING A 4 POINT LATTICE FOR LAGRANGE
C          OR USING 3 POINT PLANAR IF ALL
C          POINTS ARE NOT AVAILABLE
C
      IF(Q .LT. 0.0) QBEG=QBEG-GRIDQ
      IF(QBEG .LT. TMINQ) QBEG=TMINQ+GRIDQ
      IF((QBEG+GRIDQ) .GT. TMAXQ) QBEG=TMAXQ-GRIDQ
      RBEG=IFIX(R/GRIDR)*GRIDR
47    DO 50 I1=1,JF,JS
      QN=QBEG+(I1-J1)*GRIDQ
      IF(QN .LT. TMINQ .OR. QN .GT. TMAXQ) GO TO 48
      DO 50 I2=1,JF,JS
      RN=RBEG+(I2-J1)*GRIDR
      L4=(RN/GRIDR)-IFIX(((W**2.)+(QN**2.))/(NSTRT*GRIDR))
      IF( L4 .LE. 0 .OR. L4 .GT. NUMR .OR. RN .LT. 0.0) GO TO 48
      CALL CRITV(WN,QN,RN,LSTP)
      IF(TSTAT .GE. REG(LSTP,2)) GO TO 48
      CALL IENT(JLW,JLQ,JLR,WN,QN,RN)
      READ(IREAD=JREC) COORD(NPSF,1)
      COORD(NPSF,2)=QN
      COORD(NPSF,3)=RN
      IF(NPSF .GE. 3 .AND. NPNA .GE. 1) GO TO 60
      NPSF=NPSF+1
      GO TO 50
48    NPNA=NPNA+1
50    CONTINUE
      IF( NPSF .EQ. 4) GO TO 70
      IF( JF .GT. 1) GO TO 55
      JF=4
      J1=2
      JS=2
      GO TO 47
55    RETURN
60    PLANE=DET2((COORD(2,3)-COORD(1,3)),(COORD(3,1)-COORD(1,1)),(COORD(
     13,3)-COORD(1,3)),(COORD(2,1)-COORD(1,1)))*(V1-COORD(1,2))
      PLANE=PLANE+(DET2((COORD(2,1)-COORD(1,1)),(COORD(3,2)-COORD(1,2))
     1 ,(COORD(3,1)-COORD(1,1)),(COORD(2,2)-COORD(1,2))))*(R-COORD(3,1))
      PMULT=DET2((COORD(2,2)-COORD(1,2)),(COORD(3,3)-COORD(1,3)),(COORD(
     1 3,2)-COORD(1,2)),(COORD(3,2)-COORD(1,2)))
      IF(PMULT .EQ. 0.0) GO TO 65
      DENS=COORD(1,1)-(PLANE/PMULT)
      GO TO 40
65    DENS=COORD(1,1)
      GO TO 40
70    DO 75 I=2,4
      IF( COORD(1,2) .NE. COORD(1,2)) X1=COORD(I,2)
      IF(COORD(I,3) .NE. COORD(1,3)) Y1=COORD(I,3)
      IF((COORD(I,2) .NE. COORD(1,2))
     1     .AND. (COORD(I,3) .EQ. COORD(1,3))) F1=COORD(I,1)
      IF((COORD(I,2) .NE. COORD(1,2))
     1  .AND. (COORD(I,3) .NE. COORD(1,3))) F3=COORD(I,1)
75    CONTINUE
      DENS=(1./((COORD(1,2)-X1)*(COORD(1,3)-Y1)))
```

```
     1 *((V1-X1)*(R-Y1)*COORD(1,1)-((V1-COORD(1,2))*(R-Y1)*F1)-( (V1-X1)
     2 *(R-COORD(1,3))*F2) +(V1-COORD(1,2))*(R-COORD(1,3))*F3)
      GO TO 40
80    IF( Q .GT. TMAXQ .OR. Q .LT. TMINQ) GO TO 95
      V1=W
C
C        TWO DIMENSIONAL INTERPOLATION      WN,QN
C
      IF( W .LT. 0.0) WBEG=WBEG-GRIDW
      IF( WBEG .LT. TMINW) WBEG=TMINW+GRIDW
      IF((WBEG+GRIDW) .GT.TMAXW) WBEG=TMAXW-GRIDW
      RBEG=IFIX(R/GRIDR)*GRIDR
83    DO 90 I1=1,JF,JS
      WN=WBEG+(I1-J1)*GRIDW
      IF( WN .LT. TMINW .OR. WN .GT. TMAXW) GO TO 85
      DO 90 I2=1,JF,JS
      L4=(RN/GRIDR)-IFIX(((Q**2.)+(WN**2.))/(NSTRT*GRIDR))
      IF( L4 .LE. 0 .OR. L4 .GT. NUMR .OR. RN .LT. 0.0) GO TO 85
      CALL CRITV(WN,QN,RN,LSTP)
      IF( TSTAT .GE. REG(LSTP,2)) GO TO 85
      CALL IENT(JLW,JLQ,JLR,WN,QN,RN)
      READ(IREAD=JREC) COORD(NPSF,1)
      COORD(NPSF,2)=WN
      COORD(NPSF,3)=RN
      IF(NPSF .GE. 3 .AND. NPNA .GE. I) GO TO 60
      NPSF=NPSF+1
      GO TO 90
85    NPNA=NPNA+1
90    CONTINUE
      IF( NPSF .EQ. 4) GO TO 70
      IF( JF .GT. 1) GO TO 55
      JF=4
      JI=2
      JS=2
      GO TO 83
C
C        THREE DIMENSIONAL INTERPOLATION
C        VIA AN , POINT LATTICE FOR LAGRANGE
C        OR HYPERPLANAR INTERPOLATION
C        IF POINTS ARENT AVAILABLE
C
95    IF(W .LT. 0.0) WBEG=WBEG-GRIDW
      IF(Q .LT. 0.0) QBEG=QBEG-GRIDQ
      IF(WBEG .LT. TMINW) WBEG=TMINW
      IF(QBEG .LT. TMINQ) QBEG=TMINQ
      IF((WBEG+GRIDW) .GT. TMAXW) WBEG=TMAXW-GRIDW
      IF((QBEG+GRIDQ) .GT. TMAXQ) QBEG=TMAXQ-GRIDQ
      RBEG=IFIX(R/GRIDR)*GRIDR
150   DO 120 I1=1,JF,JS
      WN=WBEG+(I1-JI)*GRIDW
      IF( WN .LT. TMINW .OR. WN .GT. TMAXW) GO TO 110
      DO 120 I2=1,JF,JS
      QN=QBEG+(I2-JI)*GRIDQ
      IF(QN .LT. TMINQ .OR. QN .GT. TMAXQ) GO TO 110
      DO 120 I3=1,JF,JS
      RN=RBEG+(I3-JI)*GRIDR
      IF(RN .LE. 0.0) GO TO 110
      L4=(RN/GRIDR)-IFIX(((WN**2.)+(QN**2.))/(NSTRT*GRIDR))
      IF( L4 .LE. 0 .OR. L4 .GT. NUMR) GO TO 110
      NPSF=NPSF+1
      CALL IENT(JLW,JLQ,JLR,WN,QN,RN)
      READ(IREAD=JREC) COORD(NPSF,1)
      COORD(NPSF,2)=WN
      COORD(NPSF,3)=RN
```

```
      GO TO 120
 110  NPNA=NPNA+1
 120  CONTINUE
      IF( NPSF .EQ. 8) GO TO 130
      IF( JF .GT. 1) GO TO 55
      JF=4
      JI=2
      JS=2
      GO TO 150
 130  DENS=0.0
      DO 190 I=1,8
      DO 181 J1=1,8
      IF( COORD(J1,2) .NE. COORD(I,2)) GO TO 182
 181  CONTINUE
 182  DO 183 J2=1,8
      IF( COORD(J2,3) .NE. COORD(I,3)) GO TO 184
 183  CONTINUE
 184  DO 185 J3=1,8
      IF( COORD(J3,4) .NE. COORD(I,3)) GO TO 186
 185  CONTINUE
 186  DENS=DENS+(((W-COORD(I,2))*(Q-COORD(I,3))*(R-COORD(I,4)))
     1 /((COORD(I,2)-COORD(J1,2))*(COORD(I,3)-COORD(J2,3))*(COORD(I,4)-
     2 COORD(J3,3))))*COORD(I,1)
 190  CONTINUE
      GO TO 40
 245  CONTINUE
      RETURN
      END
```

SEGMENT

```
      FUNCTION AREA(WHR1,CORW1,CORQ1,WHR2,CORW2,CORQ2)
      COMMON /CB10/ CA,PA,CR,PR,A,B,C
      Q1=CORQ1
      Q2=CORQ2
      W1=CORW1
      W2=CORW2
      AREA=0.0
      IF(           (WHR1 .EQ. WHR2)
     1                             .OR.
     2     ((AMOD(WHR1,2) .EQ. 0.0) .AND. (WHR2 .EQ. (WHR1+1.)))
     4                             .OR.
     5     ((AMOD(WHR1,2) .EQ. 1.) .AND.(WHR2 .EQ.(WHR1-1.))) ) GO TO 10
      RETURN
 10   IGO=IFIX(WHR2)
      GO TO (20,20,30,30,30,30) ,IGO
 20   XC=CR
      XP=PR
      GO TO 40
 30   XC=CA
      XP=PA
 40   IF((W1*W2) .LT. 0.0) GO TO 75
      EPART1=((A*(Q2-Q1)+0.5*XP*(Q2**2.-Q1**2.))/(XC+1.))
 45   C1=(XP**2.)-((XC+1.)**2.)
      C2=2.*(A*XP+B*(XC+1.))
      C3=A**2.-(XC+1.)*(A**2.+B**2.-C)
      TCURV=ABS(0.5*(Q2-Q1)*(W1+W2))
      TRM1=C1*(Q2**2.)+C2*Q2+C3
      IF(TRM1 .LT. 0.0 .AND. ABS(TRM1) .LT. 1.E-04) TRM1=0.0
      TRM1=(2.*C1*Q2+C2)*SQRT(TRM1)
      TRM2=C1*(Q1**2.)+C2*Q1+C3
      IF( TRM2 .LT. 0.0 .AND. ABS(TRM) .LT. 1.E-04) TRM2=0.0
      TRM2=(2.*C1*Q1+C2)*SQRT(TRM2)
      FINT1=(TRM1-TRM2)/(4.*C1)
      FINT2=(4.*C1*C3-C2**2.)/(8.*C1*SQRT(-C1))
      TRM3=SQRT(C2**2.-4.*C1*C3)
      ARG1=(2.*C1*Q1+C2)/TRM3
      IF( ABS(ARG1) .GT. 1.) ARG1=SIGN(1.,ARG1)
      ARG2=(2.*C1*Q2+C2)/TRM3
      IF( ABS(ARG2) .GT. 1.) ARG2=SIGN(1.,ARG2)
      FINT3=ARSIN(ARG1)-ARSIN(ARG2)
      EINTG=(FINT1+FINT2*FINT3)/(XC+1.)
      ELCURV=ABS(EPART1+((-1.)**(WHR2-1.))*EINTG)
      IF(      (    (W2 .GT. 0.0) .AND. (AMOD(WHR2,2) .EQ. 1.)          )
     1                                 .OR.
     2  (   (W2 .LT. 0.0) .AND. (AMOD(WHR2,2) .EQ. 0.0)        ) ) GO TO 60
      IF( W2 .NE. 0.0) GO TO 50
      IF(    (    (W1 .GT. 0.0) .AND. (AMOD(WHR1,2) .EQ. 1.))
     1                             .OR.
     2 (    (W1 .LT. 0.0) .AND.(AMOD(WHR1,2) .EQ.0.0))) GO TO 60
 50   AREA=TCURV-ELCURV
 55   AREA=AMAX1(0.0,AREA)
      RETURN
 60   AREA=ELCURV-TCURV
      GO TO 55
 75   IF( CORW2 .LT. 0.0) GO TO 76
      G=0.5+CORW2
      W2=-0.5
      W1=CORW1-G
      GO TO 77
 76   G=0.5+CORW1
      W1=-0.5
      W2=CORW2-G
```

```
30    CONTINUE
      RETURN
      END
                                                        SEGMENT

                                               START OF SEGMENT
      FUNCTION WEIGHT( NPA,NAN)
      COMMON /CBB/ GREGC(14,14)
      IF( NPA .GE. 15) GO TO 10
      WEIGHT=GREGC(NPA,NAN)
      RETURN
10    IF( NAN .GT. 7) GO TO 20
      WEIGHT=GREGC(14,NAN)
      RETURN
20    IF(NAN .LE. (NPA-7)) GO TO 30
      INDX=14-NPA+NAN
      WEIGHT=GREGC(14,INDX)
      RETURN
30    WEIGHT=1.
      RETURN
      END
                                                        SEGMENT
```

```
      FUNCTION  TERPO5(UCOR,ZCOR)
      COMMON /CB7/ LSTP,ISUR
      COMMON /CB1/ GRIDW,GRIDQ,GRIDR
      COMMON /CB12/ NUMW,NUMQ,NUMR
      COMMON /CB6/ TMAXW,TMINW,TMAXQ,TMINQ
      COMMON /CB2/ JREC,TSTAT
      COMMON /CB13/ QBEG,WBEG
      COMMON /CB14/ XMEAN(2),XBR1,XBR2,VAR,DGF
      COMMON /CB10/EXCES1,EXCES2,EXCES3,EXCES4,A,B,C
      COMMON /CB3/ NCONW,NCONQ,NSTRT
      COMMON/CB8/JOINT
      COMMON /CB20/ RECMAX
      DIMENSION XVAL(10),YVAL(10)
      TERPO5=0.0
      INZER=0
      IDID=0
      TERPO5=0.0
      W=A-ZCOR
      Q=B-UCOR
      R=C-(UCOR**2.+ZCOR**2.)
      VCH=POSPROB(W,Q,R,LSTP)
      IF( VCH .LT. 0.0) RETURN
      IF( LSTP .GT. ISUR) GO TO 5
      TERPO5=CHISQ(VCH,DGF)*PHI(W,XBR1,VAR)*PHI(Q,XBR2,VAR)*
     1     PHI(UCOR,XMEAN(2),1.)*PHI(ZCOR,XMEAN(1),1.)
      RETURN
5     LB1=(W/GRIDW)+IFIX(NUMW/2.)+1-IFIX((ISUR*XMEAN(1))/GRIDW)
      WBEG=(LB1-1-IFIX(NUMW/2)+IFIX((ISUR*XMEAN(1))/GRIDW))*GRIDW
      LB2=(Q/GRIDQ)+IFIX(NUMQ/2)+1-IFIX((ISUR*XMEAN(2))/GRIDQ)
      QBEG=(LB2-1-IFIX(NUMQ/2)+IFIX((ISUR*XMEAN(2))/GRIDQ))*GRIDQ
      IF(( W .GT. TMAXW .OR. W .LT. TMINW)     .AND.
     1  (Q .GT. TMAXQ .OR. Q .LT. TMINQ)) GO TO 55
      IF( WBEG .EQ. W .AND. QBEG .EQ. Q) GO TO 15
      IF( WBEG .EQ. W) GO TO 30
      IF( QBEG .EQ. Q) GO TO 40
      GO TO 50
15    IF( W .GT. TMAXW .OR. W .LT. TMINW) GO TO 40
      IF( Q .GT. TMAXQ .OR. Q .LT. TMINQ) GO TO 30
      WN=W
      QN=Q
20    L4=(R/GRIDR)-IFIX((W**2.+Q**2.)/(NSTRT*GRIDR))
      IF( L4 .LE. 0 .OR. L4 .GT. NUMR) GO TO 25
      CALL RCAL(LB1,LB2,L4,0,WN,QN,RN)
      IF( RN .NE. R) GO TO 25
      IF( JREC .LE. 0.0 .OR. JREC .GT. RECMAX) GO TO 25
      READ(JOINT=JREC) PROB3
      TERPO5=PROB3*PHI(UCOR,XMEAN(2),1.)*PHI(ZCOR,XMEAN(1),1.)
      RETURN
25    TERPO5=TERPO1(W,Q,R,JOINT)
     1 *PHI(UCOR,XMEAN(2),1.)*PHI(ZCOR,XMEAN(1),1.)
      RETURN
30    L8=LB2
      IF((L8-1) .LE. 0) L8=2
      IF((L8+1) .GT. NUMQ) L8=NUMQ-1
      L9=L8-1
      L10=L8+1
      QPFI=0
      ZPT=A-W
      DO 35 L11=L9,L10
      QH=(L11-1-IFIX(NUMQ/2.)+IFIX((ISUR*XMEAN(2))/GRIDQ))*GRIDQ
      UPT=B-QH
      IDID=IDID+1
```

```
      IF( YVAL(IDID) .GT. 0.0) INZER=INZER+1                            00135400
35    CONTINUE                                                          00135500
      CALL BESTINTERP(XVAL,YVAL,IDID,INZER,QPFI,TERP05)                 00135600
      IF( TERP05 .LT. 0.0) TERP05=0.0                                   00135700
      RETURN                                                            00135800
40    M8=LB1                                                            00135900
      IF((M8-1) .LE. 0) M8=2                                            00136000
      IF((M8+1) .GT. NUMW) M8=NUMW-1                                    00136100
      M9=M8-1                                                           00136200
      M10=M8+1                                                          00136300
      WPFI=W                                                            00136600
      QH=Q                                                              00136700
      QPT=B-QH                                                          00136800
      DO 45 M11=M9,M10                                                  00136900
      WH=(M11-1-IFIX(NUMW/2.)+IFIX((ISUR*XMEAN(1))/GRIDW))*GRIDW        00137000
      ZPT=A-WH                                                          00137100
      IDID=IDID+1                                                       00137200
      XVAL(IDID)=WH                                                     00137300
      YVAL(IDID)=TERP05(QPT,ZPT)                                        00137400
      IF( YVAL(IDID) .GT. 0.0) INZER=INZER+1                            00137500
45    CONTINUE                                                          00137600
      CALL BESTINTERP(XVAL,YVAL,IDID,INZER,WPFI,TERP05)                 00137700
      IF( TERP05 .LT. 0.0) TERP05=0.0                                   00137800
      RETURN                                                            00137900
50    QPT1=QBEG                                                         00138000
      WPT1=WBEG                                                         00138100
      QPT2=QPT1+GRIDQ                                                   00138200
      WPT2=WPT1+GRIDW                                                   00138300
      QI=Q                                                              00138400
      WI=W                                                              00138500
      CALL OVERFL(IND)                                                  00138800
      TLAGR1=(QI-QPT2)*(WI-WPT2)*TERP05(B-QPT1,A-WPT1)                  00138900
      TLAGR2=(QI-QPT2)*(WI-WPT1)*TERP05(B-QPT1,A-WPT2)                  00139000
      TLAGR3=(QI-QPT1)*(WI-WPT2)*TERP05(B-QPT2,A-WPT1)                  00139100
      TLAGR4=(QI-QPT1)*(WI-WPT1)*TERP05(B-QPT2,A-WPT2)                  00139200
      DVAL=TLAGR1+TLAGR2+TLAGR3+TLAGR4                                  00139300
      IF( DVAL .LT. 0.0 .OR. IND .EQ. 3) GO TO 51                       00139400
      TERP05=DVAL/(GRIDQ*GRIDW)                                         00139500
51    RETURN                                                            00139600
55    TERP05=TERP0I(W,Q,R,JOINT)                                        00139700
     1 *PHI(QCOR,XMEAN(2),1.)*PHI(ZCOR,XMEAN(1),1.)                     00139800
      RETURN                                                            00139900
      END                                                               00140000
                                                                       SEGMENT
```

START OF SEGMENT

```
      BLOCK DATA
      COMMON /CB8/GREGC(14,14)
C
C     THESE ARE THE WEIGHTS FOR THE NEWTON-GREGORY INTEGRATION FORMUL
C
      DATA GREGC(1,1)/0.0/
      DATA (GREGC(3,I),I=1,3)/.4166667,1.1166666,.4166667/
      DATA(GREGC(4,I),I=1,4)/.375,1.125,1.125,.375/
      DATA(GREGC(5,I),I=1,5)/.348611,1.2722222,.7583333,1.272222,
     1  .348611/
      DATA(GREGC(6,I),I=1,6)/.3298611,1.3020833,.8680555,.8680555,
     1 1.3020833,.3298611/
      DATA(GREGC(7,I),I=1,7)/.3298611,1.3020833,.7479167,1.2027777
     1,.7479167,1.3020833,.3298611/
      DATA(GREGC(8,I),I=1,8)/.3155919,1.3921792,.6382440,1.1539848,
     11.1539848,.6382440,1.3921792,.3155919/
      DATA(GREGC(9,I),I=1,9)/.3155919,1.3921792,.6239749,1.2583499,
     1.8198082,1.2583499,.6239749,1.3921792,.3155919/
      DATA(GREGC(10,I),I=1,10)/.3155919,1.3921792,.6239749,1.2440807,
     1.9241733,.9241733,1.2440807,.6239749,1.3921792,.3155919/
      DATA(GREGC(11,I),I=1,11)/.3155919,1.3921792,.6239749,1.2440807,
     1.9099041,1.0285384,.9099041,1.2440807,.6239749,1.3921792,.3155919/
      DATA(GREGC(12,I),I=1,12)/.3155919,1.3921792,.6239749,1.2440807,
     1.9099041,1.0142692,1.0142692,.9099041,1.2440807,.6239749,1.3921792
     2,.3155919 /
       DATA(GREGC(13,I),I=1,13)/.3042245,1.4603836,.4534640,1.4714286,
     1.7393932,1.0824735,.9772652,1.0824735,.7393932,1.4714286,.4534640,
     21.4603836,.3042245/
       DATA(GREGC(14,I),I=1,14)/.3042245,1.4603836,.4534640,1.4714286,
     1.7393932,1.0824735,.9886326,.9886326,1.0824735,.7393932,1.4714286,
     3,4534640,1.4603836,.3042245/
      END
```

SEGMENT

```
      SUBROUTINE BESTINTERP(X,Y,IDON,INOTZ,XINT,YINT)
      DIMENSION X(10),Y(10),EXT(10),WH(10)
      YINT=0.0
C
C
C         THIS SUBROUTINE DECIDES WHICH TYPE
C         OF INTERPOLATION IS MOST APPROPRIATE FOR A PARTICULAR POINT
C         SINCE THE DENSITY FUNCTION MAY BE TRUNCATED
C         LAGRAGIAN INTERPOLATION MAY NOT ALWAYS BE THE BEST
C
C
      IF( INOTZ .LE. 0) RETURN
      IF( INOTZ .NE. IDON) GO TO 7
      IF(XINT.LT. X(1).AND. XINT .LT. X(IDON) .AND. Y(1) .LE. 0) RETURN
      IF(XINT.GT. X(1).AND.XINT.GT.X(IDON).AND.Y(IDON).LE.0.0)RETURN
      IF(XINT .LT. X(1).AND.XINT.LT.X(IDON)) GO TO 3
      IF(XINT .GT. X(1) .AND. XINT .GT. X(IDON)) GO TO 3
      DO 2 IRUN=2,IDON
      IF((XINT.GT. X(IRUN-1).AND. XINT .LT. X(IRUN)) .OR.
     1(XINT.LT. X(IRUN-1) .AND. XINT .GT. X(IRUN))) INDEX=IRUN-1
      Y(IRUN)=ALOG(Y(IRUN))
2     CONTINUE
      Y(1)=ALOG(Y(1))
      YINT=SPLINEFIT(X,Y,IDON,INDEX,XINT)
      IF( YINT .LT. -30.) RETURN
      YINT=EXP(YINT)
      RETURN
3     DINT=POLY(X,Y,XINT,IDON)
      IF( DINT .LE. 0.0) RETURN
      YINT=DINT
      RETURN
7     DO 8 IRUN=2,IDON
      IF((XINT .GT. X(IRUN-1) .AND. XINT .LT. X(IRUN)) .OR.
     1(XINT .LT. X(IRUN-1) .AND. XINT .GT. X(IRUN))) GO TO 9
      GO TO 8
9     IF(Y(IRUN-1) .LE. 0.0 .AND. Y(IRUN) .LE. 0.0) RETURN
      INDEX=IRUN
      JCT=0
      IF(Y(IRUN-1) .LE. 0.0 .OR. Y(IRUN) .LE. 0.0) GO TO 25
      GO TO 10
8     CONTINUE
      RETURN
10    DO 15 JK=1,IDON
      IF(Y(JK) .LE. 0.0) GO TO 15
      JCT=JCT+1
      WH(JCT)=X(JK)
      EXT(JCT)=ALOG(Y(JK))
      IF(JK .EQ. INDEX) IDX=JK
15    CONTINUE
      IF(JCT .GE. 3) GO TO 20
      DINT=POLY(WH,EXT,XINT,JCT)
      IF( DINT .LE. -35.0) RETURN
      IF( DINT .GT. 0.0) GO TO 3
      YINT=EXP(DINT)
      RETURN
20    DINT=SPLINEFIT(WH,EXT,JCT,IDX,XINT)
      IF(DINT .LE. -35.) RETURN
      IF( DINT .GT. 0.0) GO TO 3
      YINT=EXP(DINT)
      RETURN
25    IF(Y(IRUN-1) .GT. 0.0)FAC=ABS(IFIX(ALOG10(Y(IRUN-1))))
      IF(Y(IRUN) .GT. 0.0) FAC=ABS(IFIX(ALOG10(Y(IRUN))))
      DO 50 JK=1,IDON
      IF( Y(JK) .GT. 0.0) GO TO 30
```

```
      IF( JK .NE. INDEX .OR. JK .NE. (INDEX-1)) GO TO 50
30    JCT=JCT+1
      WH(JCT)=X(JK)
      EXT(JCT)=ALOG(1.+(10.**FAC)*Y(JK))
      IF(JK .EQ. INDEX) IDX=JK
50    CONTINUE
      IF( JCT .GE. 3) GO TO 60
      DINT=POLY(WH,EXT,XINT,JCT)
      GO TO 61
60    DINT=SPLINEFIT(WH,EXT,JCT,IDX,XINT)
61    IF(DINT .LT. 0.0) RETURN
      YINT=(EXP(DINT)-1.)/(10.**FAC)
      RETURN
      END
                                                        SEGMENT

      FUNCTION POLY(X,Y,XINT,INUM)
      DIMENSION X(10),Y(10)
C
C         THIS ROUTINE PERFORMS INTERPOLATION VIA
C         GENERALIZED ONE DIMENSIONAL LAGRANGE
C
      DSUM=0.0
      DO 5 IK=1,INUM
      TTERM=1.
      BTERM=1.
      DO 4 JK=1,INUM
      IF( JK .EQ. IK) GO TO 4
      TTERM=(XINT-X(JK))*TTERM
      BTERM=(X(IK)-X(JK))*BTERM
4     CONTINUE
      DSUM=DSUM+(TTERM/BTERM)*Y(IK)
5     CONTINUE
      POLY=DSUM
      RETURN
      END
                                                        SEGMENT

                                               START OF SEGMENT
      FUNCTION ECHECK(INUM,VAL)
      COMMON/ERR/ IRP1,IRP2
      GO TO (10,20                                          ),INUM
C         THIS IS A CHECK ON ZRANGE- ELLIPSE
10    IF(VAL .LT. 0.0 .AND. ABS(VAL) .GT. 1.E-06) IRP1=IRP1+1
      VAL=AMAX1(0.0,VAL)
      RETURN
C         THIS IS A CHECK ON ZRANGE - CIRCL
20    IF( VAL .LE. 0.0 .AND. ABS(VAL) .GT. 1.E-06) IRP2=IRP2+1
      RETURN
      END
                                                        SEGMENT
```

START OF SEGMENT

```
      FUNCTION SPLINEFIT(X,Y,M,J,XINT)
      DIMENSION X(10),Y(10),D(10),P(10),E(10),C(4,10)
      DIMENSION A(10,3),B(10),Z(10)
C
C        SPLINEFIT PERFORMS INTERPOLATION BY FITTING
C        A CUBIC SPLINE FUNCTION TO THE POINTS
C
      MM=M-1
      DO 2 K=1,MM
      D(K)=X(K+1)-X(K)
      P(K)=D(K)/6.
2     E(K)=(Y(K+1)-Y(K))/D(K)
      DO 3 K=2,MM
3     B(K)=E(K)-E(K-1)
      A(1,2)=-1.-D(1)/D(2)
      A(1,3)=D(1)/D(2)
      A(2,3)=P(2)-P(1)*A(1,3)
      A(2,2)=2.*(P(1)+P(2))-P(1)*A(1,2)
      A(2,3)=A(2,3)/A(2,2)
      B(2)=B(2)/A(2,2)
      IF( M .EQ. 3) GO TO 5
      DO 4 K=3,MM
      A(K,2)=2.*(P(K-1)+P(K))-P(K-1)*A(K-1,3)
      B(K)=B(K)-P(K-1)*B(K-1)
      A(K,3)=P(K)/A(K,2)
4     B(K)=B(K)/A(K,2)
5     Q=D(M-2)/D(M-1)
      A(M,1)=1.+Q+A(M-2,3)
      A(M,2)=-Q-A(M,1)*B(M-1)
      Z(M)=B(M)/A(M,2)
      MN=M-2
      DO 6 I=1,MN
      K=M-I
6     Z(K)=B(K)-A(K,3)*Z(K+1)
      Z(1)=-A(1,2)*Z(2)-A(1,3)*Z(3)
      DO 7 K=1,MM
      Q=1./(6.*D(K))
      C(1,K)=Z(K)*Q
      C(2,K)=Z(K+1)*Q
      C(3,K)=Y(K)/D(K)-Z(K+1)*P(K)
7     C(4,K)=Y(K+1)/D(K)-Z(K+1)*P(K)
      TINT=(X(J+1)-XINT)*(C(1,J)*(X(K+1)-XINT)**2.+C(3,J))
      TINT=TINT+ (XINT-X(J))*(C(2,J)*(XINT-X(J))**2.+C(4,J))
      SPLINEFIT=TINT
      RETURN
      END
```

SEGMENT

```
      SUBROUTINE RESUME(X1,X2,X3,*)
C
C
C         THIS SUBROUTINE ALLOWS THE PROGRAM TO BE
C         RUN FOR A PERIOD OF TIME AND TERMINATED
C         BY THE COMPUTER OPERATOR. THIS SUB INITIALIZES
C         ALL THE IMPORTANT VARIABLES BACK TO WHAT THEY WERE WHEN THE
C         PROGRAM WAS RUNNING
C
C
      COMMON/CB7/ LSTP,ISUR
      COMMON /CB10/ CA,PA,CR,PR,A,B,C
      COMMON /CB8/ JOINT,ICAL
      COMMON/RESTAR/OC(30,2),ASN(30),NTESTS
      COMMON /RESTAR/ KTEST,NOC,NSTP,I1,I2,I3,KREC,IRAC,NPFIAC,IRNR
      COMMON /RESTAR/ NPFINR,PROBAC,PROBNR,PRRAC,PRQAC,PRRNR,PRQNR
      COMMON /RESTAR/ RVALAC,RVALNR,RACBEG,SPROAC,RNBEG,SPRONR
      COMMON /RESTAR/ WN,QN,RN
      DAY="  /  /"
      YEAR="  "
      DYM=TIME(5)
      DOFWK=TIME(6)
      TOD=TIME(1)/216000.0
      DAY=CONCAT(DAY,DYM,12,12,12)
      DAY=CONCAT(DAY,DYM,30,24,12)
      YEAR=CONCAT(YEAR,DYM,12,36,12)
      READ(10=1)    NTRYS,KTEST,NOC,NSTP,JOINT,ICAL,I1,I2,I3,KREC,IRAC,
     1 NPFIAC,IRNR,NPFINR,A,B,C,PROBAC,PROBNR,PRRAC,PRQAC,PRRNR,PRQNR,
     2  RVALAC,RVALNR,RACBEG,SPROAC,RNBEG,SPRONR,X1,X2,X3
      MTRYS=NTRYS+1
      WRITE(8,100) MTRYS,DOFWK,DAY,YEAR,TOD
100   FORMAT(1H1,20X,"RESUMING PROCESSING",20X,"TRY",I5,//,10X,A6,10X,
     1 2A6,25X,"AT",E15.7,"    HOURS")
      WRITE(8,101)
101   FORMAT(      40X,"SUMMARY FROM LAST RUN")
      WRITE(8,*/)  NTRYS,KTEST,NOC,NSTP,JOINT,ICAL,I1,I2,I3,KREC,IRAC,
     1 NPFIAC,IRNR,NPFINR,A,B,C,PROBAC,PROBNR,PPRAC,PRQAC,PRRNR,PRQNR,
     2  RVALAC,RVALNR,RACBEG,SPROAC,RNBEG,SPRONR,X1,X2,X3
                                                                 SEGMENT
      WRITE(8,102)
102   FORMAT(///,40X,"SUMMARY FROM ALL PREVIOUS RUNS")
      DO 10 J1=1,KTEST
      DO 10 J2=1,NOC,9
      DO 10 J3=1,NSTP-1
      READ(11=(J3+((J2-1)*NTESTS)+(J1-1)*NTESTS*10)) PACC,PREJ,PCON,
     1     NSTEP,ALAM,NCASE
      WRITE(8,*/) NCASE,ALAM,NSTEP,PACC,PREJ,PCON
      IF( NSTP .LE. (ISUR+1)) GO TO 10
      IF( J1 .LT. KTEST .OR. J2 .LT. NOC .OR. J3 .LE. ISUR) GO TO 10
      OC(J3,1)=PACC
      OC(J3,2)=PREJ
      ASN(J3)=PCON
10    CONTINUE
      WN=A
      QN=B
      RN=C
      RETURN 1
      END
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The Sequential Analysis of Variance test (SANOVA) is introduced in chapter one. The thesis will show how to obtain the OC and and ASN of such a test. Currently only an approximation exists due to Bhate (~~1959~~ 1959) discussed in chapter one. In chapter two the first exact procedure for obtaining the OC and ASN of a k=2 SANOVA test is derived. Section (2.3) provides the theoretical procedure, summarized in Figures 12 and 13 of section (2.5).

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20. (continued)

The numerical approach is discussed. ~~in section (2.6).~~
Appendix A gives the power calculation for a fixed sample ANOVA test; Appendix B shows how the Wald regions are found; Appendix C contains a computer program for the OC and ASN.