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SOME MULTIPLICITY RESULTS FOR PERIODIC SOLUTIONS OF A RAYLEIGH DIFFERENTIAL EQUATION

P. Habets[1] and P.J. Torres[2]

[1]Institut de Mathématique Pure et Appliquée
Chemin du Cyclotron, 2, 1348 - Louvain la Neuve, Belgium
[2]Departamento de Matemática Aplicada
Universidad de Granada, 18071 Granada, Spain

Abstract. We study the existence and multiplicity of T-periodic solutions for a Rayleigh equation, with conditions on the nonlinearity that include some classical situations.
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1. Introduction

In this paper, we are concerned with the periodic problem for a Rayleigh equation

$$\begin{aligned} &x'' + f(x') + g(t,x,x') = \bar{p}, \\ &x(0) = x(T),\ x'(0) = x'(T). \end{aligned} \tag{1}$$

Our aim is to study the structure of the set of "mean values" $\bar{p}$ for which there exists at least one or at least two solutions of problem (1). Such sets turn out to be intervals which can be bounded or unbounded.

The literature contains a large variety of such results a example of which is the classical Ambrosetti-Prodi problem [1], see also [6], [4] and the references therein. Other related situations concern periodic nonlinearities (see for instance [10], [8]) or in case of a Dirichlet problem, nonlinearities depending only on the derivative [7]. Our main concern was to study similar situations for periodic solutions of the Rayleigh equation (1) which seems to be little studied. We were also concerned with extension to equations which satisfy Carathéodory conditions. This forced us to write in Section 2 a theorem which relates the existence of ordered strict upper and lower solutions with the degree of a suitable operator. Such a result in the framework of Carathéodory conditions does not seem to be classical.

In Section 3, we consider restoring forces $g(t,x,x')$ uniformly bounded by L^2-functions. This includes periodic nonlinearities such as in the pendulum equation. In this case, the set of admissible "mean values" $\bar{p}$ turns out to be an interval I, which we can estimate for the damped pendulum equation. If $g(t,x,x')$ is periodic in x, we prove existence of two solutions in the interior of I. The proof of this result is based on the existence of strict upper and lower solutions. This forced us to impose some uniform continuity condition on the function $g(t, x, x')$ (see condition (b) in Theorem 3). Some of our results are related to J. Mawhin [9].

The last section deals with restoring forces bounded from below. Here, we extend the Ambrosetti-Prodi type result in [6] to nonlinearities that satisfy Carathéodory conditions. In particular, we replace the usual condition $\lim_{|x|\to\infty} g(t,x,y) = +\infty$, uniformly in t and y, by the assumption that $g(t,x,y)$ can be broken into $\bar{g}(t,x,y) + h(t,x,y)$, where $\bar{g}(t,x,y)$ satisfies such a uniform limit for $|x| \to \infty$ and $h(t,x,y)$ is bounded by a L^2-function. These assumptions are modeled from a physical system with a restoring force $\bar{g}(t,x)$ and a forcing $h \in L^2$.

In the paper, we use the following notations. A function $g : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ is said to be a *Carathéodory function* if :
$g(\cdot, x, y)$ is measurable for all x, $y \in \mathbb{R}$ and
$g(t, \cdot, \cdot)$is continuous for almost all $t \in [0,T]$.
We call a function $g(t,x,y)$ a *L^1-Carathéodory function* if it is a Carathéodory function and for any $R > 0$ there exists a function $h \in L^1$ such that

$$\text{for almost all } t \in [0,T],\ \forall x \in [-R,R],\ \forall y \in [-R,R], \quad |g(t,x,y)| \le h(t).$$

We shall consider such nonlinearities throughout the paper, which implies solutions are in $W^{2,1}(0,T)$.

We write $\|\cdot\|_p$ for the usual norm in $L^p(0,T)$. Also, we shall use the spaces

$$\tilde{X} := \{x \in H^2(0,T) \mid x(0) = x(T), x'(0) = x'(T), \textstyle\int_0^T x\,dt = 0\},$$
$$\tilde{Y} := \{x \in L^2(0,T) \mid \textstyle\int_0^T x\,dt = 0\},$$

and the projector

$$P : L^2(0,T) \to \mathbb{R}, x \mapsto Px = \frac{1}{T}\int_0^T x\,dt. \tag{2}$$

We shall often write $\bar{x} = Px$ and $\tilde{x} = (I - P)x$.

2. Strict $W^{2,1}$-lower and upper solutions

In this section, we consider the problem

$$\begin{gathered} x'' + F(t,x,x') = 0, \\ x(0) = x(T),\ x'(0) = x'(T). \end{gathered} \tag{3}$$

where the real function $F(t,x,y)$ is defined for $(t,x,y) \in [0,T] \times \mathbb{R}^2$. To state the following basic definitions we extend F by periodicity for $t \in \mathbb{R}$

Definitions. A function $\alpha \in C(0,T)$ is a *strict $W^{2,1}$-lower solution* of (3) if it is not a solution on $[0,T]$ and its periodic extension on $\mathbb{R}$, defined by

$\alpha(t) = \alpha(t+T)$, is such that, for any $t_0 \in \mathbb{R}$,
either $D_-\alpha(t_0) < D^+\alpha(t_0)$,
or there exist an open interval I_0 and $\epsilon_0 > 0$ such that $t_0 \in I_0$, $\alpha \in W^{2,1}(I_0)$ and, for almost every $t \in I_0$, for all u with $\alpha(t) \le u \le \alpha(t) + \epsilon_0$, and all v with $\alpha'(t) - \epsilon_0 \le v \le \alpha'(t) + \epsilon_0$, we have

$$\alpha''(t) + F(t,u,v) \ge 0.$$

In the same way, a function $\beta \in C(0,T)$ is a *strict* $W^{2,1}$*-upper solution* of (3) if it is not a solution on $[0,T]$ and its periodic extension on $\mathbb{R}$, defined by $\beta(t) = \beta(t+T)$, is such that, for any $t_0 \in \mathbb{R}$,
either $D^-\beta(t_0) > D_+\beta(t_0)$,
or there exist an open interval I_0 and $\epsilon_0 > 0$ such that $t_0 \in I_0$, $\beta \in W^{2,1}(I_0)$ and, for almost every $t \in I_0$, for all u with $\beta(t) - \epsilon_0 \le u \le \beta(t)$, and all v with $\beta'(t) - \epsilon_0 \le v \le \beta'(t) + \epsilon_0$, we have

$$\beta''(t) + F(t,u,v) \le 0.$$

As a next step, we write (3) as a fixed point equation. To this aim, we define the operator

$$\mathcal{T} = K_1 N, \tag{4}$$

where

$$K_1 : L^1(0,T) \to C^1[0,T] \tag{5}$$

is the Green operator corresponding to $x'' - x + f = 0$, $x(0) = x(T)$, $x'(0) = x'(T)$, and $N : C^1[0,T] \to L^1(0,T)$ is defined from $Nx = F(\cdot, x, x') + x$. With these notations, the problem (3) is equivalent to

$$x = \mathcal{T}x.$$

Theorem 1 *Let $F : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ be a L^1-Carathéodory function. Assume*
(i) there exist strict $W^{2,1}$-lower and upper solutions α and $\beta \in W^{1,\infty}(a,b)$ of (3) such that $\alpha < \beta$ on $[0,T]$,
(ii) there exists $R > \max\{\|\alpha'\|_\infty, \|\beta'\|_\infty\}$ such that any solution x of $x'' + F(t,x,x') = 0$, $x'(t_0) = 0$, with $t_0 \in [0,T]$ and $\alpha < x < \beta$, verifies

$$\|x'\|_\infty < R. \tag{6}$$

Then,

$$\deg(I - \mathcal{T}, \Omega) = 1, \tag{7}$$

where $\mathcal{T}$ is defined in (4) and $\Omega := \{x \in C^1[0,T] \mid \alpha < x < \beta, |x'| < R\}$.

Proof. *A modified problem.* Let us consider the modified problem

$$\begin{gathered} x'' - x + \hat{F}(t, x, x') + \delta(\alpha(t), x, \beta(t)) = 0, \\ x(0) = x(T), \; x'(0) = x'(T), \end{gathered} \tag{8}$$

where $\hat{F}(t, x, y) := F(t, \delta(\alpha(t), x, \beta(t)), \delta(-R, y, R))$ and

$$\delta(A, u, B) := \begin{cases} A & \text{if } u \le A, \\ u & \text{if } A \le u < B, \\ B & \text{if } B \le u. \end{cases} \tag{9}$$

Claim 1 : Every solution x of (8) *satisfies $\alpha(t) < x(t) < \beta(t)$ on $[0, T]$.* Assume on the contrary that for some $t_0 \in [0, T]$,

$$\min_{t \in [0,T]} (x(t) - \alpha(t)) = x(t_0) - \alpha(t_0) \le 0.$$

Hence, $x'(t_0) - D_-\alpha(t_0) \le 0 \le x'(t_0) - D^+\alpha(t_0)$ and by definition of a strict $W^{2,1}$-lower solution $D_-\alpha(t_0) = D^+\alpha(t_0) = x'(t_0)$. Next, we choose from this definition an open interval I_0 and $\epsilon_0 > 0$ and take them small enough so that for any $t \in I_0$,

$$\begin{gathered} \alpha(t) \le \delta(\alpha(t), x(t), \beta(t)) \le \alpha(t) + \epsilon_0, \\ -R \le \alpha'(t) - \epsilon_0 \le x'(t) \le \alpha'(t) + \epsilon_0 \le R. \end{gathered}$$

Whence, for almost every $t \in]t_0, t_1[$

$$\alpha''(t) + \hat{F}(t, x(t), x'(t)) = \alpha''(t) + F(t, \delta(\alpha(t), x(t), \beta(t)), x'(t)) \ge 0.$$

Notice that as α is not a solution, t_0 could have been chosen such that for some $t_1 > t_0$, $t_1 \in I_0$, we have $x'(t_1) - \alpha'(t_1) > 0$, so that the following contradiction holds

$$\begin{aligned} 0 < x'(t_1) - \alpha'(t_1) &= \int_{t_0}^{t_1} (x''(s) - \alpha''(s))\, ds \\ &\le \int_{t_0}^{t_1} [-\hat{F}(s, x(s), x'(s)) + x - \delta(\alpha(t), x, \beta(t)) - \alpha''(s)]\, ds \le 0. \end{aligned}$$

Similarly, we prove that $\forall t \in [0, T]$, $x(t) < \beta(t)$, which proves the claim.

Claim 2 : Every solution x of (8) *satisfies $|x'(t)| < R$ on $[0, T]$.* If not, there exists a solution x of (8), $t_0 \in [0, T]$ and $t_1 \in [0, T]$, $t_1 > t_0$, such that $x'(t_0) = 0$, $|x'(t_1)| = R$ and for any $t \in [t_0, t_1]$, $|x'(t)| \le R$, $\alpha(t) < x(t) < \beta(t)$. Hence x satisfies $x'' + F(t, x, x') = 0$ on $[t_0, t_1]$ and contradicts assumption *(ii)*.

Claim 3 : $\deg(I - \hat{\mathcal{T}}, \Omega) = 1$. Define the operator $\hat{\mathcal{T}} = K_1 \hat{N}$, where

$$(\hat{N}x)(t) = \hat{F}(t, x(t), x'(t)) + \delta(\alpha(t), x, \beta(t)).$$

It is clear that $\hat{\mathcal{T}}$ is bounded and for $\hat{R} > 0$ large enough and any $\lambda \in [0,1]$,

$$\deg(I - \hat{\mathcal{T}}, B(0,\hat{R})) = \deg(I - \lambda\hat{\mathcal{T}}, B(0,\hat{R})) = \deg(I, B(0,\hat{R})) = 1.$$

Also if $\hat{R}$ is large enough, $\Omega \subset B(0,\hat{R})$ and from Claims 1 and 2 and the excision property of the degree

$$\deg(I - \hat{\mathcal{T}}, B(0,\hat{R})) = \deg(I - \hat{\mathcal{T}}, \Omega) = \deg(I - \mathcal{T}, \Omega) = 1,$$

which proves the theorem. ∎

3. Bounded restoring forces

In this section, we consider a nonlinearity bounded by some function $h \in L^2(0,T)$. A first result describes the structure of the set of $\bar{p}$ for which the periodic boundary value problem (1) has a solution.

Theorem 2 *Let $\bar{p} \in \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and $g : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ be a Carathéodory function such that for some $h \in L^2(0,T)$ and all $(t,x,y) \in [0,T] \times \mathbb{R}^2$,*

$$|g(t,x,y)| \le h(t).$$

Then, there exists a nonempty interval $[a,b]$ such that
(i) if $\bar{p} \notin [a,b]$, problem (1) has no solution,
(ii) if $\bar{p} \in (a,b)$, problem (1) has at least one solution.

Proof. Let $\mathcal{M}$ be the set of all $\bar{p}$ such that (1) has a solution.

Claim 1 : Let $R > \sqrt{T}\,||h||_2$. Then any solution x of (1) is such that

$$||x'||_\infty < R. \tag{10}$$

Multiplying the equation (1) by x'' and integrating on $[0,T]$, we get

$$||x''||_2^2 \le -\int_0^T g(t,x,x')x''\,dt \le ||h||_2||x''||_2. \tag{11}$$

Hence,

$$||x''||_2 \le ||h||_2$$

and if we choose t_0 such that $x'(t_0) = 0$, we have

$$|x'(t)| = |\int_{t_0}^t x''(s)\,ds| \le \sqrt{T}\,||x''||_2 \le \sqrt{T}\,||h||_2 < R.$$

A modified problem. Consider the function

$$\hat{f}(y) := f(\delta(-R,y,R)),$$

with $\delta(A,y,B)$ defined in (9). Repeating the proof of Claim 1, it is clear that any solution x of

$$\begin{aligned} x'' + \hat{f}(x') + g(t,x,x') &= \bar{p}, \\ x(0) = x(T),\ x'(0) &= x'(T). \end{aligned} \tag{12}$$

satisfies (1). Hence x is a solution of (1) if and only if it is a solution of (12).

Claim 2 : $\mathcal{M}$ is non-empty. Let $\hat{\mathcal{T}} = K(I-P)\hat{N}$, where $K : \tilde{Y} \subset L^2(0,T) \to \tilde{X} \subset C^1[0,T]$ is the compact inverse of the operator $L : \tilde{X} \subset C^1[0,T] \to \tilde{Y} \subset L^2(0,T)$ with $Lx = -x''$, P is defined from (2) and

$$\hat{N} : C^1[0,T] \to L^2(0,T), x \mapsto \hat{N}x = \hat{f}(x') + g(\cdot, x, x').$$

The operator $\hat{\mathcal{T}}$ is completely continuous and bounded so that Schauder's fixed point Theorem provides a solution of the equation $x = \hat{\mathcal{T}}x$. This last equation is equivalent to

$$\begin{gathered} -x'' = \hat{f}(x') + g(t,x,x') - P(\hat{f}(x') + g(\cdot,x,x')) \\ x(0) = x(T),\ x'(0) = x'(T), \end{gathered}$$

which proves that $\bar{p} := P(\hat{f}(x') + g(.,x,x')) \in \mathcal{M}$.

Claim 3 : $\mathcal{M}$ is bounded. Direct integration of (12) shows that

$$|\bar{p}| \le ||\hat{f}||_\infty + \frac{||h||_2}{\sqrt{T}}.$$

Claim 4 : $\mathcal{M}$ is an interval. Consider $\bar{p}_1, \bar{p}_2 \in \mathcal{M}$ with $\bar{p}_1 < \bar{p}_2$ and let x_1, x_2 be the corresponding solutions of (1). For $p \in (p_1, p_2)$, the functions x_1 and x_2 are respectively upper and lower solutions of (12) as

$$x_1'' + \hat{f}(x_1') + g(t,x_1,x_1') - \bar{p} = \bar{p}_1 - \bar{p} < 0$$

and

$$x_2'' + \hat{f}(x_2') + g(t,x_2,x_2') - \bar{p} = \bar{p}_2 - \bar{p} > 0.$$

It follows now from an easy extension of Theorem 4.1 in [4] (see also [2]), that problem (12), and hence (1), has a solution. ■

Remark. Theorem 2 is best possible as follows from the following example

$$\begin{gathered} x'' + cx' + \arctan x = \bar{p}, \\ x(0) = x(T),\ x'(0) = x'(T), \end{gathered}$$

where $c \neq 0$. Multiplying the equation by x' and integrating, one obtains $||x'||_2^2 = 0$. This implies the solutions are constant and we compute $x(t) \equiv \tan\bar{p}$. Hence, this problem has exactly one solution for $\bar{p} \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and no solution for $\bar{p} \notin (-\frac{\pi}{2}, \frac{\pi}{2})$.

Remark. If g is continuous and uniformly bounded, Claims 2, 3 and 4 follow from Proposition 2.1 in [3].

In case the function $g(t,x,y)$ is periodic in x, Theorem 2 can be substantially improved, as it is shown by the next theorem.

Theorem 3 *Let $\bar{p} \in \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and $g : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ be a Carathéodory function such that*

(a) for almost all $t \in [0,T]$ *and all* $(x,y) \in \mathbb{R}^2$, $g(t,x,y) = g(t,x+2\pi,y)$,
(b) for all $t_0 \in [0,T]$, $(x_0,y_0) \in \mathbb{R}^2$ *and* $\epsilon > 0$, *there exists* $\delta > 0$ *such that*

$$|t-t_0| < \delta,\ |x-x_0| < \delta,\ |y-y_0| < \delta \quad \Rightarrow \quad |g(t,x,y) - g(t,x_0,y_0)| < \epsilon,$$

(c) for some $h \in L^2(0,T)$, *almost all* $t \in [0,T]$ *and all* $(x,y) \in \mathbb{R}^2$,

$$|g(t,x,y)| \leq h(t).$$

Then, there exists a nonempty interval $[a,b]$ *such that*
(i) if $\bar{p} \notin [a,b]$, *problem* (1) *has no solution,*
(ii) if $\bar{p} \in [a,b]$, *problem* (1) *has at least one solution.*
(iii) if $\bar{p} \in (a,b)$, *problem* (1) *has at least two solutions non-differing by a multiple of* 2π.

Remark. Note that assumption *(b)* is equivalent to

(b') given continuous functions μ, ν *in* $[0,T]$, $t_0 \in [0,T]$ *and* $\epsilon > 0$ *there exists* $\delta > 0$ *such that for any functions* u, v, *defined in* $[0,T]$, *satisfying* $||u-\mu||_\infty < \delta$, $||v-\nu||_\infty < \delta$, *the inequalities*

$$|g(t,u(t),v(t)) - g(t,\mu(t),\nu(t))| < \epsilon,$$

holds almost everywhere in $[t_0-\delta, t_0+\delta] \cap [0,T]$.
A simple example in which this may be checked is $g(t,x,y) = a(t)+b(t)\varphi(x,y)$ with $a \in L^2(0,T)$, $b \in L^\infty$ and $\varphi \in C(\mathbb{R}^2)$.

Proof. From Theorem 2, the set $\mathcal{M}$ of $\bar{p}$ such that (1) has at least one solution is a bounded interval, i.e. $\mathrm{cl}\mathcal{M} = [a,b]$.

A - Claim : $\mathcal{M}$ is closed. Let $(p_n)_n$ be a sequence in $\mathcal{M}$ converging to p and $(x_n)_n$ be the corresponding solutions of (1). From the periodicity, we can assume (adding a multiple of 2π to x_n if necessary) that $x_n(0) \in [0,2\pi]$. It follows now from (10) that

$$|x_n(t)| = |x_n(0) + \int_0^t x_n'(s)\,ds| \leq 2\pi + RT$$

which, using (11), gives that the sequence $(x_n)_n$ is bounded in $H^2(0,T)$. As $H^2(0,T)$ is compactly embedded in $C^1[0,T]$, a subsequence converges to some function $u \in C^1[0,T]$ and going to the limit in (1) (with $\bar{p} = p_n$), it follows that u is a solution of (1) with $\bar{p} = p$.

B - Existence of strict $W^{2,1}$-lower and upper solutions of (12) *for* $\bar{p} \in (a,b)$. Consider the modified problem (12), where $R > 0$ is defined from (10). Let x_a and x_b be the solutions of (1) with $\bar{p} = a$ and $\bar{p} = b$ respectively. As g is periodic, we can choose $k \in \mathbb{Z}$ such that $\alpha := x_b < \beta := x_a + 2k\pi$ and for some $t_* \in [0,T]$,

$$\alpha(t_*) + 2\pi \geq \beta(t_*). \tag{13}$$

Let us show that for $\bar{p} < b$ the function α is a strict $W^{2,1}$-lower solution. Given $t_0 \in [0,T]$, we can pick an open interval I_0 and $\epsilon_0 > 0$ small enough so that $t_0 \in I_0$ and for almost every $t \in I_0$, for all u with $\alpha(t) \leq u \leq \alpha(t) + \epsilon_0$, and all v with $\alpha'(t) - \epsilon_0 \leq v \leq \alpha'(t) + \epsilon_0$, we have

$$|\hat{f}(v) - \hat{f}(\alpha'(t))| \leq \frac{b - \bar{p}}{2}, \quad |g(t, u, v) - g(t, \alpha(t), \alpha'(t))| \leq \frac{b - \bar{p}}{2}.$$

It follows that for such t, u and v

$$\begin{aligned} &\alpha''(t) + \hat{f}(v) + g(t, u, v) \\ &\qquad = b + (\hat{f}(v) - \hat{f}(\alpha'(t))) + (g(t, u, v) - g(t, \alpha(t), \alpha'(t))) \\ &\qquad \geq b - (b - \bar{p}) = \bar{p}. \end{aligned}$$

Similarly, we can prove that $\alpha(t) + 2\pi$ is a strict $W^{2,1}$-lower solution of (12) if $\bar{p} < b$ and that $\beta(t)$ and $\beta(t) + 2\pi$ are strict $W^{2,1}$-upper solutions if $\bar{p} > a$.

C - Claim : If $\bar{p} \in (a, b)$, problem (1) *has at least two solutions non-differing by a multiple of* 2π. Define the sets

$$\Omega_1 := \{x \in C^1[0, T] \mid \forall t \in [0, T], \alpha(t) < x(t) < \beta(t), |x'(t)| < R\},$$

$$\Omega_2 := \{x \in C^1[0, T] \mid \forall t \in [0, T], \alpha(t) + 2\pi < x(t) < \beta(t) + 2\pi, |x'(t)| < R\},$$

and

$$\Omega_3 := \{x \in C^1[0, T] \mid \forall t \in [0, T], \alpha(t) < x(t) < \beta(t) + 2\pi, |x'(t)| < R\}.$$

Let $\mathcal{T}x := K_1 N x$, where K_1 is defined in (5) and $Nx = \bar{p} - \hat{f}(x') - g(\cdot, x, x') - x$. Theorem 1 applies,

$$\deg(I - \mathcal{T}, \Omega_1) = \deg(I - \mathcal{T}, \Omega_2) = \deg(I - \mathcal{T}, \Omega_3) = 1,$$

and by the excision property

$$\deg(I - \mathcal{T}, \Omega_3 \setminus (\overline{\Omega_1 \cup \Omega_2})) = -1.$$

Hence, we obtain two solutions $x_1 \in \Omega_1$ and $x_2 \in \Omega_3 \setminus (\overline{\Omega_1 \cup \Omega_2})$ of (12). As this problem is equivalent to (1) on each of the sets Ω_i, x_1 and x_2 are also solutions of (1). Notice that $x_1 - 2n\pi \notin \Omega_3 \setminus (\Omega_1 \cup \Omega_2)$ for $n = 1, 2, \ldots$ since we have (using (13)) $x_1(t_*) - 2n\pi < \beta(t_*) - 2n\pi \leq \alpha(t_*) - 2(n-1)\pi \leq \alpha(t_*)$. Also $x_1 + 2n\pi$, with $n = 1, 2, \ldots$ cannot be in $\Omega_3 \setminus (\Omega_1 \cup \Omega_2)$ since $x_1 + 2n\pi > \alpha + 2\pi$. Hence $x_2 \in \Omega_3 \setminus (\Omega_1 \cup \Omega_2)$ cannot differ from x_1 by a multiple of 2π. ∎

A classical example of differential equation with periodic nonlinearity is the pendulum equation

$$x'' + f(x') + A \sin x = \bar{p} + \tilde{p}(t). \tag{14}$$

Following the ideas of [13], the next result gives some estimates on the set $\mathcal{M}$ of admissible $\bar{p}$.

Proposition 4 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, $A \in \mathbb{R}$ and $K \in (0, \pi)$. Then for all $\tilde{p} \in \tilde{Y}$ with $\|\tilde{p}\|_2 \leq K \frac{6\sqrt{5}}{T\sqrt{T}}$, there exists $\bar{p}_0 \in \mathbb{R}$ (independent of A) with*

$$|\bar{p}_0| \leq \max\{|f(y)| : |y| \leq \sqrt{\frac{T}{12}}\,\|\tilde{p}\|_2\} \tag{15}$$

such that (14) has at least a T-periodic solution for any $\bar{p}$ with

$$|\bar{p} - \bar{p}_0| \leq A \sin \frac{\pi - K}{2} \tag{16}$$

and two T-periodic solutions if the inequality is strict.

Proof. From Schauder's fixed point Theorem, there exists a solution $u_0 \in \tilde{X}$ of

$$u_0'' = \tilde{p} - f(u_0') + Pf(u_0'). \tag{17}$$

Multiplying by u_0'' and integrating we get $\|u_0''\|_2 \leq \|\tilde{p}\|_2$. Further, we obtain from classical estimates $\|u_0\|_\infty \leq \frac{T\sqrt{T}}{12\sqrt{5}}\|u_0''\|_2$ (see [12] p. 215) so that, for any t_1 and t_2,

$$|u_0(t_1) - u_0(t_2)| \leq |u_0(t_1)| + |u_0(t_2)| \leq \frac{T\sqrt{T}}{6\sqrt{5}}\|\tilde{p}\|_2 \leq K.$$

Let $\bar{p}_0 := Pf(u_0')$. Direct integration of (17) gives $T\bar{p}_0 = \int_0^T f(u_0')\,dt$ and (15) follows as

$$\|u_0'\|_\infty \leq \sqrt{\frac{T}{12}}\|u_0''\|_2 \leq \sqrt{\frac{T}{12}}\|\tilde{p}\|_2.$$

Let us now pick constants c_1 and c_2 so that $\alpha(t) := u_0(t) + c_1 \in [\frac{\pi}{2} - \frac{K}{2}, \frac{\pi}{2} + \frac{K}{2}]$ and $\beta(t) := u_0(t) + c_2 \in [\frac{3\pi}{2} - \frac{K}{2}, \frac{3\pi}{2} + \frac{K}{2}]$. We also fix $\bar{p}$ according to (16). It is then easy to check that α and β are lower and upper solutions for (14) so that the existence of a T-periodic solution is proved. Finally, the multiplicity result is a direct consequence of Theorem 3. ∎

If the friction force is linear, we obtain the following result.

Corollary 5 *Let $c \in \mathbb{R}$, $A \in \mathbb{R}$ and $K \in (0, \pi)$. Then, for all $\tilde{p} \in \tilde{Y}$ with $\|\tilde{p}\|_2 \leq K \frac{6\sqrt{5}}{T\sqrt{T}}$ and any $\bar{p}$ such that*

$$|\bar{p}| \leq A \sin \frac{\pi - K}{2},$$

the equation

$$x'' + cx' + A \sin x = \bar{p} + \tilde{p}(t)$$

has at least a T-periodic solution, and two if the inequality is strict.

Proof. One has to see that $\bar{p}_0 := P(u_0') = 0$. ∎

4. Restoring forces bounded from below

This section is concerned with the study of the number of T-periodic solutions of a Rayleigh equation with an unbounded nonlinearity.

Consider the periodic boundary value problem

$$\begin{gathered} x'' + f(x') + g(t,x,x') = \bar{p} + h(t,x,x'), \\ x(0) = x(T),\ x'(0) = x'(T). \end{gathered} \tag{18}$$

The first result of this section is the following theorem of Ambrosetti-Prodi type in the case of sublinear friction.

Theorem 6 *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $g, h : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ be Carathéodory functions such that for almost all $t_0 \in [0,T]$ and for all $(x_0, y_0) \in \mathbb{R} \times \mathbb{R}$ and $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\begin{aligned} &|t - t_0| < \delta,\ |x - x_0| < \delta,\ |y - y_0| < \delta \\ &\qquad \Rightarrow \quad |g(t,x,y) - g(t,x_0,y_0)| < \epsilon \text{ and } |h(t,x,y) - h(t,x_0,y_0)| < \epsilon. \end{aligned}$$

The following conditions are assumed:

(A) There exist $d \geq c > 0$ such that

$$c \leq \frac{f(y)}{y} \leq d$$

for all $y \in \mathbb{R}$.

(B) There exist functions $k_1, k_2 \in L^2(0,T)$ such that for almost all $t \in [0,T]$ and all $(x,y) \in \mathbb{R}^2$

$$|h(t,x,y)| \leq k_1(t), \quad g(t,x,y) \geq k_2(t).$$

(C) For any $R \geq 0$, there exists a function $k \in L^\infty(0,T)$, such that for almost all $t \in [0,T]$ and all $(x,y) \in [-R,R] \times \mathbb{R}$,

$$|g(t,x,y)| \leq k(t).$$

(D) $\lim_{|x|\to\infty} g(t,x,y) = +\infty$, uniformly in t and y.

Then, there exists $a \in \mathbb{R}$ such that
(i) if $\bar{p} < a$, problem (18) has no solution.
(ii) if $\bar{p} = a$, problem (18) has at least one solution.
(iii) if $\bar{p} > a$, problem (18) has at least two solutions.

Proof. *Claim 1 : Given $\bar{p}_0$ there exist $R_0 > 0$ and $R_1 > 0$ so that any solution x of (18) with $\bar{p} \leq \bar{p}_0$ is such that*

$$||x||_\infty \leq R_0, \quad ||x'||_\infty \leq R_1. \tag{19}$$

Let $(x')_+(t) = \max\{x'(t), 0\}$. If we multiply equation (18) by $(x')_+$ and integrate, we obtain

$$\begin{aligned} c\int_0^T (x')_+^2\,dt &\le \int_0^T [x''(x')_+ + f(x')(x')_+ + (g(t,x,x') - k_2(t))(x')_+]\,dt \\ &= \int_0^T [\bar{p} + h(t,x,x') - k_2(t)](x')_+\,dt \\ &\le (|\bar{p}_0|T^{\frac{1}{2}} + ||k_1||_2 + ||k_2||_2)\,||(x')_+||_2. \end{aligned}$$

This gives a bound K_1 on $||(x')_+||_2$ and therefore $||(x')_+||_1 \le \sqrt{T}K_1$. Moreover, if we define $(x')_-(t) = \max\{-x'(t), 0\}$, then $0 = \int x'dt = ||(x')_-||_1 - ||(x')_-||_1$, so we have also $||x'||_1 = 2||(x')_+||_1 \le 2\sqrt{T}K_1$.

Now, take $K_2 := \frac{1}{T}(\bar{p}_0 T + ||k_1||_2\sqrt{T} + d2\sqrt{T}K_1)$. By assumption (D), there exists some $\nu > 0$ such that $g(t,x,y) > K_2$ for every $|x| > \nu$. If $x(t)$ is a solution of (18), direct integration gives

$$\begin{aligned} \int_0^T g(t,x,x')\,dt &= \bar{p}T + \int_0^T [h(t,x,x') - f(x')]\,dt \\ &\le \bar{p}_0 T + ||k_1||_2\sqrt{T} + d\int_0^T |x'|dt \\ &\le \bar{p}_0 T + ||k_1||_2\sqrt{T} + d2\sqrt{T}K_1 = K_2 T. \end{aligned}$$

Hence, for any solution x of (18) we can choose t_0 such that $|x(t_0)| \le \nu$. Thus,

$$|x(t)| \le |x(t_0)| + \int_0^T |x'(t)|\,dt \le \nu + 2\sqrt{T}K_1 =: R_0.$$

From hypothesis (C), there is a function $k \in L^\infty(0,T)$ such that for almost every $t \in [0,T]$ and all $(x,y) \in [-R_0, R_0] \times \mathbb{R}$,

$$|g(t,x,y)| \le k(t).$$

Now, if we multiply (18) by x'', integrate and take into account the bound on $x(t)$, we obtain

$$||x''||_2^2 = \int_0^T [h(t,x,x') - g(t,x,x')]x''\,dt \le (||k_1||_2 + ||k||_2)||x''||_2.$$

Hence, $||x''||_2$ is bounded by $||k_1||_2 + ||k||_2$ and as there exists t_0 such that $x'(t_0) = 0$ we deduce

$$|x'(t)| \le \int_{t_0}^t |x''|\,ds \le ||x''||_2\sqrt{T} \le (||k_1||_2 + ||k||_2)\sqrt{T} =: R_1.$$

Claim 2 : Equation (18) *has a solution for some* p_0 *large enough.* Let study the equation

$$x'' + f(x') + k_1(t) = \bar{p}_1. \tag{20}$$

It is known (see [11]) that there exists some $\bar{p}_1$ such that equation (20) has a family of T-periodic solutions $u + C$ with $C \in \mathbb{R}$. If u_0 is the element of this family such that $\bar{u}_0 = 0$, using the arguments of the proof of Proposition 4 we obtain that

$$||u_0||_\infty \le \frac{T\sqrt{T}}{12\sqrt{5}}||k_1||_2, \quad ||u_0'||_\infty \le \sqrt{\frac{T}{12}}||k_1||_2.$$

Fix $\bar{p} > \bar{p}_1 + \max\{g(t,x,y) : |x| \le \frac{T\sqrt{T}}{12\sqrt{5}}||k_1||_2, |y| \le \sqrt{\frac{T}{12}}||k_1||_2\}$. Then, we claim that u_0 is an upper solution of (18). Indeed,

$$\begin{aligned} &u_0'' + f(u_0') + g(t,u_0,u_0') - h(t,u_0,u_0') \\ &\qquad \le u_0'' + f(u_0') + g(t,u_0,u_0') + k_1(t) = \bar{p}_1 + g(t,u_0,u_0') < \bar{p}. \end{aligned}$$

On the other hand, it is possible to get an ordered lower solution by considering the equation

$$x'' + f(x') - k_1(t) = \bar{p}_2. \tag{21}$$

There exists some $\bar{p}_2$ such that equation (21) has a family of T-periodic solutions $u + C$, with $C \in \mathbb{R}$. Choose $u_1 := u - C$ with C large enough such that for every t, $u_1(t) < u_0(t)$ and $g(t,u_1,u_1') + \bar{p}_2 > \bar{p}$ (this is possible from (D)). Then,

$$\begin{aligned} &u_1'' + f(u_1') + g(t,u_1,u_1') - h(t,u_1,u_1') \\ &\qquad \ge u_1'' + f(u_1') + g(t,u_1,u_1') - k_1(t) = \bar{p}_2 + g(t,u_1,u_1') > \bar{p}. \end{aligned}$$

Claim 2 follows then from Claim 1 and the fact that u_1 and u_0 are ordered lower and upper solutions of (18).

Claim 3 : The set $\mathcal{M}$ of all the $\bar{p}$ such that (18) *has a solution is bounded below.* From Claim 2, $\mathcal{M}$ is not empty. Let $\bar{p}_0 \in \mathcal{M}$ and $x(t)$ be a solution of (18) with $\bar{p} \le \bar{p}_0$. From Claim 1, a direct integration of (18) leads to

$$\bar{p} \ge \bar{k}_2 - \bar{k}_1 - \sup\{|f(y)| : |y| < R_1\}.$$

Claim 4 : For every $\bar{p}_0 \in \mathcal{M}$, the set $\mathcal{M} \cap]-\infty, \bar{p}_0]$ is a compact interval. Define the functions

$$\hat{f}(y) := f(\delta(-R_1, y, R_1)),$$

and

$$\hat{g}(t,x,y) := g(t, \delta(-R_0, x, R_0), \delta(-R_1, y, R_1))$$

where $\delta(A,u,B)$ is defined in (9) and R_0 and R_1 are given by Claim 1. Repeating the proof of Claim 1 for the modified equation

$$\begin{aligned} x'' + \hat{f}(x') + \hat{g}(t,x,x') &= \bar{p} + h(t,x,x'), \\ x(0) = x(T),\ x'(0) &= x'(T), \end{aligned} \tag{22}$$

is clear that x is a solution of (18) with $\bar{p} \leq \bar{p}_0$, if and only if it is a solution of (22).

Now, repeating the proofs in Theorem 2 and 3, we can show that $\mathcal{M} \cap]-\infty, \bar{p}_0]$ is a closed interval. Finally, Claim 3 shows that $\mathcal{M} \cap]-\infty, \bar{p}_0] = [a, \bar{p}_0]$ for some a.

Claim 5 : If $p \in \mathrm{int}\mathcal{M}$, equation (18) *has two solutions.*

Let $\bar{p}_1$ and $\bar{p}_2 \in \mathcal{M}$ be such that $\bar{p}_1 < p < \bar{p}_2$. Then we prove as in Part B of the proof of Theorem 3 that the solution of (18) with $\bar{p} = \bar{p}_1$ is a strict $W^{2,1}$-upper solution, call it $\beta(t)$, and that an ordered strict $W^{2,1}$-lower solution $\alpha(t)$ can be obtained by the argument in Claim 2. Hence, writing problem (18) as a fixed point problem $x = \mathcal{T}x$, we have using Theorem 1

$$\deg(I - \mathcal{T}, \Omega) = 1,$$

where $\Omega = \{x \in C^1[0,T] \mid \alpha < x < \beta, |x'| < R_1\}$ and R_1 is obtained from Claim 1. Further, we can find $R > 0$ large enough so that $\Omega \subset B(0,R)$ and for all $\bar{p} \leq \bar{p}_2$ problem (18) has no solution on the boundary of $B(0,R) \subset C^1[0,T]$. It is easy to see that

$$\deg(I - \mathcal{T}, B(0,R)) = 0$$

and now the existence of a second solution in $B(0,R) \backslash \Omega$ follows from classical properties of the degree. ∎

Theorem 6 can be applied directly to the problem

$$\begin{aligned} x'' + f(x') + g(x) &= \bar{p} + h(t,x,x'), \\ x(0) = x(T), &\ x'(0) = x'(T). \end{aligned} \tag{23}$$

However, condition (A) on f is quite restrictive. From the physical point of wiew, it is very interesting to get results about superlinear damping. For instance, quadratic damping $f(y) = |y|y$ appear very often in applications (see [13] and the references herein). Hence, our purpose in the following result is to cover the model case

$$\begin{aligned} x'' + c|x'|^\rho x' + x^2 &= \bar{p} + h(t), \\ x(0) = x(T), &\ x'(0) = x'(T), \end{aligned}$$

with $h \in L^2$ and $\rho \geq 0$.

Theorem 7 *Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions, $h : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ be a Carathéodory function such that for almost all $t_0 \in [0,T]$ and for all $(x_0, y_0) \in \mathbb{R} \times \mathbb{R}$ and $\epsilon > 0$, there exists $\delta > 0$ such that*

$$|t - t_0| < \delta,\ |x - x_0| < \delta,\ |y - y_0| < \delta \ \Rightarrow\ |h(t,x,y) - h(t,x_0,y_0)| < \epsilon.$$

The following conditions are assumed:

(A) There exist $d \geq c > 0$ and $\rho \geq 0$ such that

$$c \leq \frac{f(y)}{|y|^\rho y} \leq d$$

for all $y \in \mathbb{R}$.

(B) There exists a function $k_1 \in L^2(0,T)$ such that

$$|h(t,x,y)| \leq k_1(t)$$

for almost all $t \in [0,T]$ and all $(x,y) \in \mathbb{R}^2$.

(C) $\lim_{|x|\to\infty} g(x) = +\infty$.

Then, there exists $a \in \mathbb{R}$ such that

(i) if $\bar{p} < a$, problem (23) has no solution,

(ii) if $\bar{p} = a$, problem (23) has at least one solution.

(iii) if $\bar{p} > a$, problem (23) has at least two solutions.

Proof. The proof of Theorem 6 is valid except for the a priori bounds on solutions of (23) obtained on Claim 1.

First, multiplying (23) by x' and integrating, we obtain

$$\begin{aligned} c\|x'\|_{\rho+2}^{\rho+2} &\leq \int_0^T f(x')x'\,dt = \int_0^T h(t,x,x')x'\,dt \\ &\leq \int_0^T |k_1 x'|\,dt \leq \|k_1\|_{\frac{\rho+2}{\rho+1}}\|x'\|_{\rho+2}, \end{aligned}$$

whence

$$\|x'\|_{\rho+2} \leq \left(\frac{1}{c}\|k_1\|_{\frac{\rho+2}{\rho+1}}\right)^{\frac{1}{\rho+1}} =: K_1.$$

Now, take $K_2 := \frac{1}{T}(\bar{p}_0 T + \|k_1\|_2\sqrt{T} + dK_1^{\rho+1}T^{\frac{1}{\rho+2}})$. By assumption (C), there exists some $\nu > 0$ such that $g(x) > K_2$ for every $|x| > \nu$. If $x(t)$ is a solution of (18), direct integration gives

$$\begin{aligned} \int_0^T g(x)\,dt &= \bar{p}T + \int_0^T [h(t,x,x') - f(x')]\,dt \\ &\leq \bar{p}_0 T + \|k_1\|_2\sqrt{T} + d\int_0^T |x'|^{\rho+1}\,dt \\ &\leq \bar{p}_0 T + \|k_1\|_2\sqrt{T} + dK_1^{\rho+1}T^{\frac{1}{\rho+2}} = K_2 T. \end{aligned}$$

Hence, for any solution x of (18) we can choose t_0 such that $|x(t_0)| \leq \nu$. Thus,

$$|x(t)| \leq |x(t_0)| + \int_0^T |x'(t)|\,dt \leq \nu + K_1 T^{\frac{\rho+1}{\rho+2}} =: R_0.$$

and repeating step by step the arguments of Theorem 6 the result is proved. ∎

Finally, we are going to enlarge the set of admissible f assuming a uniform bound on the derivative of g. More precisely, the following result can be proved.

Theorem 8 *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, $g : \mathbb{R} \to \mathbb{R}$ have a continuous derivative and $h : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ be a Carathéodory function such that for almost all $t_0 \in [0,T]$ and for all $(x_0, y_0) \in \mathbb{R} \times \mathbb{R}$ and $\epsilon > 0$, there exists $\delta > 0$ such that*

$$|t - t_0| < \delta,\ |x - x_0| < \delta,\ |y - y_0| < \delta \ \Rightarrow\ |h(t,x,y) - h(t,x_0,y_0)| < \epsilon.$$

The following conditions are assumed:

(A) There exist $c > 0$ and $\rho \geq 0$ such that for all $y \in \mathbb{R}$, $f(y)y \geq c|y|^{\rho+2}$.

(B) There exists a function $k_1 \in L^2(0,T)$ such that

$$|h(t,x,y)| \leq k_1(t)$$

for almost all $t \in [0,T]$ and all $(x,y) \in \mathbb{R}^2$.

(C) $G = \sup\{|g'(x)| : x \in \mathbb{R}\} < +\infty$.

(D) $\lim_{|x| \to \infty} g(x) = +\infty$.

Then, there exists $a \in \mathbb{R}$ such that

(i) if $\bar{p} < a$, problem (23) has no solution,

(ii) if $\bar{p} = a$, problem (23) has at least one solution.

(iii) if $\bar{p} > a$, problem (23) has at least two solutions.

Proof. As before, the key idea is how to compute a priori bounds on the derivative of the solutions of (23). Multiplying (23) by x' and integrating, we obtain

$$\begin{aligned} c\|x'\|_{\rho+2}^{\rho+2} &\leq \int_0^T f(x')x'\,dt = \int_0^T h(t,x,x')x'\,dt \\ &\leq \int_0^T |k_1 x'|\,dt \leq \|k_1\|_{\frac{\rho+2}{\rho+1}} \|x'\|_{\rho+2}, \end{aligned}$$

whence

$$\|x'\|_{\rho+2} \leq \left(\frac{1}{c}\|k_1\|_{\frac{\rho+2}{\rho+1}}\right)^{\frac{1}{\rho+1}} =: K_1.$$

Now, multiplying the equation by x'' and integrating over a period, we obtain

$$\|x''\|_2^2 + \int_0^T g(x)x''dt = \int_0^T h(t,x,x')x''dt;$$

but an integration by parts gives $\int_0^T g(x)x''dt = -\int_0^T g'(x)(x')^2dt$, so that

$$\begin{aligned} \|x''\|_2^2 &\leq \|k_1\|_2\|x''\|_2 + \left|\int_0^T g'(x)(x')^2dt\right| \\ &\leq \|k_1\|_2\|x''\|_2 + G\|x'\|_2 \leq \|k_1\|_2\|x''\|_2 + GK_1^2 T^{\frac{\rho}{\rho+2}}. \end{aligned}$$

Hence, there exists some $K_2 > 0$ such that $||x''||_2 \leq K_2$, and in conclusion $||x'||_\infty \leq \sqrt{T}K_2 =: K_3$.

With this bound, we can define the truncated function

$$\hat{f}(y) := f(\delta(-K_3, y, K_3)),$$

problem (23) is equivalent to the truncated problem

$$\begin{aligned} x'' + \hat{f}(x') + g(x) = \bar{p} + h(t, x, x'), \\ x(0) = x(T),\ x'(0) = x'(T) \end{aligned} \tag{24}$$

and we are in the "bounded" case studied above. ∎

Further remarks. In the case of Corollary 8, it is possible to add a damping term $f_0(y)$ with the only assumption that $f_0(y)y \geq 0$ for any y, which means simply that the "friction force" is opposed to the movement.

It is not hard to state "dual" versions of the results of Section 4 by considering restoring forces bounded from above.

On the other hand, it is possible to extend all the results of this paper to equations with a discontinuous friction force f. The physical model of this kind of nonlinearities is the dry or Coulomb friction, a phenomenon that is modeled by the sign of the derivative. Then, the differential equation must be considered as a differential inclusion (see for example [5]), and the same method of approximation by single-valued functions developed in [13] is available, since the a priori bounds obtained in the proofs above guarantee convergence of the sequence of solutions by Arzelà-Ascoli theorem.

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