Semantics and Domain theory

Exercises 9

1. Prove the following properties (by induction on τ). Here, M, M_1, M_2 range over closed terms, d_1, d_2 are domain elements.

   (a) If $d_2 \sqsubseteq d_1$ and $d_1 \lhd_\tau M_1$, then $d_2 \lhd_\tau M_1$.

   (b) If $d_1 \lhd_\tau M_1$ and $\forall V (M_1 \Downarrow_\tau V \Rightarrow M_2 \Downarrow_\tau V)$, then

   $$d_1 \lhd_\tau M_2$$

   These properties constitute Lemma 7.2.1 (iii).

2. Remember that $\lhd_\tau$ denotes the approximation relation (Slide 64).
   Show that, if $d \lhd_{\mathbf{nat}} M$, $e \lhd_{\mathbf{nat}} N$ and $b \lhd_{\mathbf{bool}} P$, then

   $$\text{if}(b, d, e) \lhd_{\mathbf{nat}} \mathbf{if}\ P\ \mathbf{then}\ M\ \mathbf{else}\ N$$

   (This is basically the "if" inductive case in the proof of the Fundamental Property, Slide 65)

3. Prove that $\mathbf{fn}\, x : \mathbf{nat}.\, \mathbf{succ}(\mathbf{pred}\, x) \leq_{\text{ctx}} \mathbf{fn}\, x : \mathbf{nat}.x$ in the following two ways:

   (a) By using the Proposition on Slide 68.

   (b) By using the Extensionality properties on Slide 69.

4. Consider the terms $M_1 := \mathbf{fix}(\mathbf{fn}\, f : \mathbf{nat} \to \mathbf{nat}.f)$ and
   $M_2 := \mathbf{fn}\, x : \mathbf{nat}.\mathbf{fix}(\mathbf{fn}\, x : \mathbf{nat}.x)$ of type $\mathbf{nat} \to \mathbf{nat}$. Use the Extensionality property of $\leq_{\text{ctx}}$ at function types (Slide 69) to show that $M_1 \cong_{\text{ctx}} M_2$.

5. Prove that for all $M_1, M_2 \in \text{PCF}_\tau$,
   $M_1 \leq_{\text{ctx}} M_2 : \tau$ if and only if
   $\forall M \in \text{PCF}_{\tau \to \mathbf{bool}} (M\, M_1 \Downarrow_{\mathbf{bool}} \mathbf{true} \Rightarrow M\, M_2 \Downarrow_{\mathbf{bool}} \mathbf{true})$.
   (Remember that PCF_τ are the *closed* PCF-terms of type τ.)

6. (Exercise 7.4.1.) For any PCF type τ and closed terms M_1, M_2 of type τ, show that

   $$(\forall V : \tau, (M_1 \Downarrow_\tau V \Leftrightarrow M_2 \Downarrow_\tau V)) \Rightarrow M_1 \cong_{\text{ctx}} M_2 : \tau. \qquad (**)$$

   [Hint: combine the Proposition on Slide 68 with Exercise 1 above (or Lemma 7.2.1(iii)).]

7. (Exercise 7.4.2.) For any PCF type τ and closed terms M_1, M_2 of type τ, we have

   $$(\forall V : \tau, (M_1 \Downarrow_\tau V \Leftrightarrow M_2 \Downarrow_\tau V)) \Rightarrow M_1 \cong_{\text{ctx}} M_2 : \tau. \qquad (**)$$

   Use (**) to show that β-conversion is valid up to contextual equivalence in PCF, in the sense that for all closed terms $\mathbf{fn}\, x : \tau_1.\, P : \tau_1 \to \tau_2$ and $Q : \tau_1$,

   $$(\mathbf{fn}\, x : \tau_1.\, P)\, Q \cong_{\text{ctx}} P[Q/x] : \tau_2.$$

8. (Exercise 7.4.3.) Show that the converse of (**) is not valid at all types by considering the terms M_1 and M_2 of Exercise 4