CHAPTER XX

THE CONSTRUCTION OF EQUATIONS

486. The matters which have been established in the above chapter are accustomed to be applied chiefly to the construction of equations of higher orders. Indeed since we may have come upon an equation from two proposed curves, the roots may show the position of the intersections, thus in turn the intersections of two curves are able to attend to the roots of the indicated equations. And this method may provide the greatest use, if the roots of some equation must be expressed by lines; and with each curve described to be adapted towards this end, the intersections will be noted easily, from which, if they may be sent to the applied axis, the abscissas will provide the true roots of the equation. But if an inconvenience mentioned above may have a place, then indeed all the abscissas thus found will provide roots, but it will arise, that the proposed equation may include more roots, than may be found by such a construction.

487. Therefore since an algebraic equation will have been proposed involving the unknown x, the roots of which it may be required to designate, two curved lines are required to be sought or two equations between the variables x and y, which shall be prepared thus, so that the applied line y may be eliminated from these, and the proposed equation itself may result. With what done these two curves may be described above a common axis at the same starting points of the abscissas and the points may be noted in which they will intersect mutually. Then from these points of intersections the applied lines may be sent normal to the axis, which will show on the axis the abscissas of the of the individual equations proposed with equal roots. And thus in this manner the true values of the individual roots sought may be assigned, unless perhaps it may eventuate, that the equation may contain more roots, than the intersections are to be taken from that.

488. But before I shall relate the manner, by which these two curves are able to be found serving for the construction of the given equation, from the following we may consider these equations, the solution of which is resolved from two given curves. And indeed in the first place both the resolving right lines EM and FM themselves shall intersect at the point M. The right line EF may be taken for the axis and on that some point A for the start of the abscissas, from which the normal ABC drawn may cut the

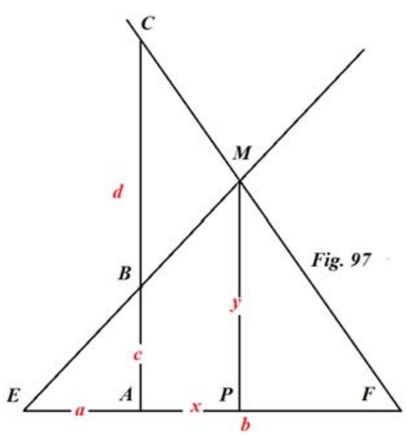


Fig. 97

first line at *B*, the latter at *C*. There shall be $AE = a,\ AF = b,\ AB = c,\ AC = d$; then truly the abscissa may be considered $AP = x$, the applied line $PM = y$, and for the first line *EM* there will be $a : c = a + x : y$ or $ay = c(a + x)$ and for the other $b : d = b - x : y$ or $by = d(b - x)$. From these equations if y may be eliminated, there will be produced

$$bc(a + x) = ad(b - x)$$

or

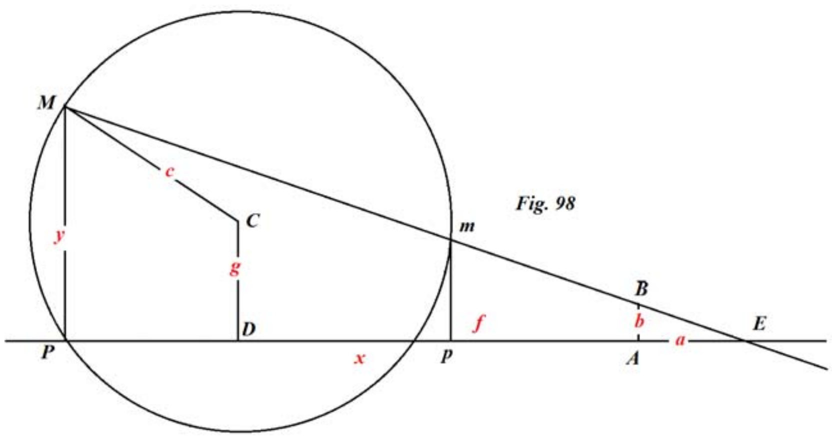
$$x = \frac{abd - abc}{bc + ad} = \frac{ab(d - c)}{bc + ad}.$$

Therefore the simple equation

$$x = \frac{ab(d - c)}{bc + ad};$$

will be able to be constructed through the intersection of the two right lines, to which form all simple equations are able to be recalled generally.

489. The circle follows right lines in the account of ease of being described, and on this account we may consider how equations of this kind are able to be constructed by the intersection of a straight line and a circle



Therefore the right line *EM* may be described (Fig. 98), on taking *AP* for the axis and *A* for the start of the abscissas, and by putting $AE = a,\ AB = b$ and with the coordinates *AP*= *x*, *PM*= *y*, there will be $a : b = a + x : y$ and thus $ay = b(a + x)$, which is the equation for a right line. Then the radius of the circle shall be $CM = c$, and with the perpendicular *CD* sent from its centre *C* to the axis calling $AD = f,\ CD = g$ there will be $DP = x - f$ and $PM - CD = y - g$. Now, since from the nature of the circle there shall be

$$CM^2 = DP^2 + (PM - \mathrm{CD})^2,$$

the equation for the circle will be

$$cc = xx - 2fx + ff + yy - 2gy + gg = (x-f)^2 + (y-g)^2 .$$

But the equation for the right line gives $y = \frac{ab+bx}{a}$, from which there becomes

$$y - g = \frac{a(b-g)+bx}{a} = b - g + \frac{bx}{a}$$

from which with the value of y placed in the other equation there emerges

$$cc = xx - 2fx + ff + (b-g)^2 + \frac{2b(b-g)x}{a} + \frac{bbxx}{aa}$$

or

$$\begin{matrix} +aaxx & +2ab(b-g)x & +aa(b-g)^2 = 0, \\ +bb & -2aaf & +aaff \\ & & -aacc \end{matrix}$$

therefore the roots of this equation may be found by the intersection of a right line and a circle, thus so that with the perpendiculars *MP*, *mp* sent from the intersections *M* and *m* to the axis, the values of y become *AP* and *Ap*.

490. Because all square equations are held in this equation, hence the construction of general square equations will be able to be established. Evidently this quadratic equation shall be proposed

$$Axx + Bx + C = 0 ,$$

which thus may be reduced to the first form above, so that the first terms may agree ; on multiplying by $\frac{aa+bb}{A}$:

$$(aa+bb)xx + \frac{B(aa+bb)x}{A} + \frac{C(aa+bb)}{A} = 0.$$

Now the equality of the remaining terms will give

$$2Aab(b-g) - 2Aaaf = B(aa+bb)$$

and thus there becomes

$$af = b(b-g) - \frac{B(aa+bb)}{2Aa}.$$

From which, since there shall be

$$aa(b-g)^2 + aaff - aacc = \frac{C(aa+bb)}{A},$$

there will be

$$(aa+bb)(b-g)^2 - \frac{Bb\left(b-g\right)(aa+bb)}{Aa} + \frac{BB(aa+bb)^2}{4AAaa} - aacc = \frac{C(aa+bb)}{A}$$

and thus

$$(b-g)^2 = \frac{Bb\left(b-g\right)}{Aa} - \frac{BB(aa+bb)}{4AAaa} + \frac{aacc}{aa+bb} + \frac{C}{A},$$

therefore

$$b-g = \frac{Bb}{2Aa} \pm \sqrt{\left(\frac{aacc}{aa+bb} + \frac{C}{A} - \frac{BB}{4AA}\right)}.$$

Therefore the three quantities *a*, *b* and *c* remain indeterminate at this stage, but which thus it is necessary to be taken, so that

$$\frac{aacc}{aa+bb} + \frac{C}{A} - \frac{BB}{4AA}$$

becomes a positive quantity, because otherwise $b-g = AA-CD$ and hence *CD* will become an imaginary quantity.

491. Therefore nothing hinders that we may not put $b=0$, and there will be

$$g = \sqrt{\left(cc + \frac{-BB+4AC}{4AA}\right)} \quad \text{and} \quad f = -\frac{B}{2A}.$$

Then truly, since the proposed equation $Axx + Bx + C = 0$ shall have no real roots, unless *BB* shall be greater than 4*AC*, in this case $\frac{BB-4AC}{4AA}$ will be a positive quantity, for which if it may be put equal to *cc*, so that there shall be

$$c = \frac{\sqrt{(BB-4AC)}}{2A}$$

also makes $g = 0$ and a in short departs from the calculation. Therefore the right line *EM* falls on the axis itself *AP* and the centre of the circle *C* must be located at the point D with

$$AD = -\frac{B}{2A},$$

from which centre, if a circle be described with the radius

$$c = \frac{\sqrt{(BB-4AC)}}{2A},$$

its intersections with the axis itself will show the roots of the proposed equation. But so that according to this there shall be no need for an irrational formula, there may be put

$g = c - \frac{k}{2A}$, so that there shall be

$$cc - \frac{2ck}{2A} + \frac{kk}{4AA} = cc + \frac{-BB+4AC}{4AA},$$

there will be

$$c = \frac{kk+BB-4AC}{4kA} \quad \text{and} \quad g = \frac{BB-4AC-kk}{4kA}.$$

Therefore in our arbitrary determination the quantity *k* remains, which takes any value, because the right line *CM* falls on the axis itself, the circle must be described in the following manner. On taking $AD = -\frac{B}{2A}$, the perpendicular may be taken

$$CD = \frac{BB-4AC-kk}{4Ak}.$$

and with center *C* a circle may be described, whose radius

$$= \frac{BB-4AC+kk}{4Ak};$$

and its intersections with the axis will show the roots of the proposed equation.

If therefore there may be put $k=-B$, on taking $AD=-\frac{B}{2A}$ it is found that $CD=\frac{C}{B}$, and the radius of the circle required to be described with centre C will be

$$\frac{-BB+2AC}{2AB}=-\frac{B}{2A}+\frac{C}{B},$$

from which the radius of the circle will be $=AD+CD$; which construction may be considered for the most convenient use.

492. Now we will consider two circles (Fig. 99) intersecting each other and for the first there shall be $AD=a$, $CD=b$, and its radius $CM=c$; and there will be on putting $AP=x$ and $PM=y$: $DP=a-x$, $CD-PM=b-y$; and thus from the nature of the circle there will be had

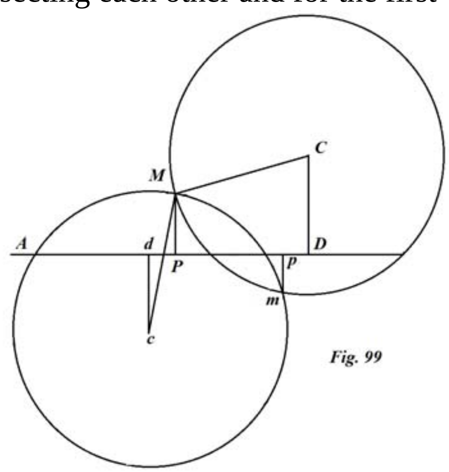


Fig. 99

$$xx-2ax+aa+yy-2by+bb=cc.$$

In a similar manner for the other circle there shall be $Ad=f$, $dc=g$ and its radius $cM=h$, and there will be

$$xx-2fx+ff+yy+2gy+gg=hh,$$

with which equations subtracted from each other there will remain

$$2(f-a)x+aa-ff-2(b+g)y+bb-gg=cc-hh,$$

therefore

$$y=\frac{aa+bb-ff-gg-cc-hh+2(a-f)x}{2(b+g)}$$

and hence

$$b-y=\frac{bb+2gbh-aa+ff+gg+cc-hh+2(a-f)x}{2(b+g)}$$

and

$$a-x=\frac{2a(b+g)-2(b+g)x}{2(b+g)}.$$

Therefore since there shall be $(a-x)^2+(b-y)^2=cc$, with the substitution made there shall be

$$\begin{array}{lll} +4(a-f)^2xx & -4(a-f)(b+g)^2x + & (b+g)^4=0. \\ & & +2(aa-cc)(b+g)^2 \\ +4(b+g)^2 & -4(a-f)(aa-ff) & +2(ff-hh)(b+g)^2 \\ & +4(a-f)(cc-hh) & +(aa-cc-ff+hh)^2 \end{array}$$

With the aid of this equation therefore, the equation

$$Axx+Bx+C=0$$

will be able to be constructed in an infinite number of ways ; likewise truly it is understood the a higher quadratic equation cannot be constructed by the intersection of two circles, because two circles therefore are unable to intersect in more than two points. Therefore since the same quadratic equation may be constructed by the intersection of a straight line and a circle, this construction for that, which requires two circles, deservedly is preferred, unless perhaps in some singular cases the determination of the lines *a, b, f, g, c* and *h* may be produced easier and at once.

493. Now a circle may intersect with a parabola (Fig. 100) ; evidently with the *CD* perpendicular sent from the centre of the circle *C* to the axis *AP*, there shall be $AD=a,\ CD=b$ and the radius of the circle $CM=c$, the equation between the orthogonal coordinates $AP=x,\ PM=y$ for the circle will be $(x-a)^2+(y-b)^2=cc.$ Truly the axis *FB* of the parabola here may be put in place assumed normal to this axis *AP* and there shall be $AE=f,\ EF=g$ and the parameter of the parabola $=2h$*;* from the nature of the parabola there shall be $EP^2=2h(EF+PM)$ or in symbols $(x-f)^2=2h(g+y)$, so that there shall be

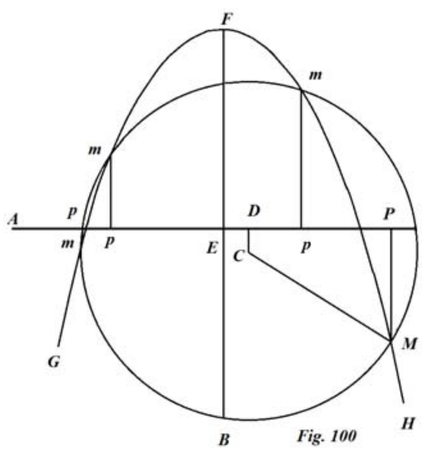


Fig. 100

$$y=\frac{(x-f)^2}{2h}-g \text{ and } y-b=\frac{(x-f)^2}{2h}-(b+g).$$

Which value if it may be substituted into the first equation, and *y* will be eliminated, will be

$$\frac{(x-f)^4}{4hh}-\frac{(b+g)(x-f)^2}{h}+(b+g)^2+(x-a)^2=cc$$

or

$$\begin{aligned}x^4-4fx^3+6ff+\ &xx-4f^3+x+\ f^4 &&=0\ ,\\ &-4h(b+g)+4fh(b+g)-4ffh(b+g)\\ &+4hh \qquad -8ahh \qquad +4hh(b+g)^2\\ &\qquad\qquad\qquad\qquad +4aahh\\ &\qquad\qquad\qquad\qquad -4cckh\end{aligned}$$

the roots of which equation will be the abscissas *AP*, *Ap*, *Ap*, *Ap*, so that the applied lines will pass through the points of intersection *M, m, m, m*.

494. In this equation six constants are present *a*, *b*, *c*, *f*, *g* and *h*, of which truly the pair $b+g$ are to be considered as one, thus so that only five shall be agreed to be present on putting $b+g=k$. Evidently on putting $CD+EF=b+g=k$ the following equation will be had

$$\begin{aligned}x^4-4fx^3+6\,&ffxx-4f^3x+f^4 &&=0\ .\\ -4hk\ &+4fhk-4ffhk\\ +4hh\ &-8ahh+4hhkk\\ &\qquad\quad +4aahh\\ &\qquad\quad -4cchh\end{aligned}$$

Moreover the general biquadratic equation can be reduced to this form; for let this equation be proposed :

$$x^4-Ax^3+Bxx-Cx+D=0\,,$$

it will be prepared by putting in place

$$4f=A \quad \text{or} \quad f=\tfrac{1}{4}A\,,$$
$$6ff-4hk+4hh=B \quad \text{or} \quad \tfrac{3}{8}AA-4hk+4hh=B\,,$$

from which there becomes

$$k=\frac{3AA}{32h}+h-\frac{B}{4h}\,,$$

$$4f^3-4fhk+8ahh=C$$

or

$$\frac{1}{16}A^3 - \frac{3}{32}A^3 - Ahh + \tfrac{1}{4}AB + 8ahh = C$$

therefore

$$a = \frac{A^3}{256hh} + \frac{A}{8} - \frac{AB}{32hh} + \frac{C}{8hh}.$$

And then there is

$$(ff - 2hk)^2 + 4aahh - 4cckk = D.$$

But there is

$$ff - 2hk = \frac{B}{2} - 2hh - \frac{AA}{16}$$

and

$$2ah = \frac{A^3}{128h} + \frac{Ah}{4} - \frac{AB}{16h} + \frac{C}{4h}$$

with which values substituted an equation emerges involving c and h, which therefore thence it is required to be defined most appropriately, clearly thus so that each may retain a real value.

495. Truly because in any biquadratic equation the second term can be removed easily, now we may consider that itself to be removed, and thus this equation is to be constructed

$$x^4 * + Bxx - Cx + D = 0.$$

Therefore at first there will be $f = 0$, secondly $k = h - \frac{B}{4h}$, thirdly $a = \frac{C}{8hh}$ and on account of

$$2hk - ff = 2hh - \frac{B}{2} \quad \text{and} \quad 2ah = \frac{C}{4h}$$

in the fourth place

$$4h^4 - 2Bhh + \tfrac{1}{4}BB + \frac{CC}{16hh} - 4cchh = D,$$

so that there shall be

$$64cch^4 = CC + 4BBhh - 32Bh^4 + 64h^6 - 16Dhh$$

and thus

$$8chh = \sqrt{(4hh(B-4hh)^2 + CC - 16Dhh)}.$$

Truly because here especially it is to be effected, that c as well as h may retain real values, there is put $c = h - \frac{B+q}{4h}$ and there will be

$$CC - 16Dhh + 8Bhhq - 32h^4q - 4hhqq = 0.$$

So that therefore we may satisfy the equation, two cases are to be distinguished, the one in which D is a negative quantity, the other in which D is a positive quantity.

I.

Therefore let D be a positive quantity $= +EE$, thus so that this equation must be constructed

$$x^4 * + Bx^2 - Cx + EE = 0;$$

according to this there may be put $q = 0$, so that there shall be $c = \frac{4hh-B}{4h}$, and there will become

$$hh = \frac{CC}{16EE} \text{ and } h = \frac{C}{4E};$$

from which it becomes

$$c = \frac{CC-4BE}{4CE}$$

and again

$$k = c = \frac{CC-4BE}{4CE}, \quad a = \frac{2EE}{C} \text{ and } f = 0.$$

II.

But if D shall be a negative quantity, for example $D = -EE$, so that this equation must be constructed

$$x^4 * + Bx^2 - Cx - EE = 0;$$

there will become

$$64cch^4 = CC + 4hh(4hh-B)^2 + 16EEhh,$$

which equation provides a real value for c, whatever is assumed for h; for there will become

$$c = \frac{\sqrt{(CC + 4hh(4hh - B)^2 + 16EEhh)}}{8hh}$$

and *h* can be assumed as it pleases ; therefore in any case assumed thus, so that the construction of *c* thence may follow most easily. With which done there shall be, as before,

$$AE = f = 0, \quad CD + EF = k = \frac{4hh - B}{4h}$$

and

$$AD = a = \frac{C}{8hh}.$$

If there may be put $E = 0$, the construction of the cubic equation arises :

$$x^3 * + Bx - C = 0.$$

And Baker's rule depends on this construction commonly known well enough.

496. If any two lines of the second order or conic sections may be taken, of which the equations and likewise the start of the abscissas shall be related to a common axis :

$$ayy + bxy + cxx + dy + ex + f = 0$$

and

$$\alpha yy + \beta xy + \gamma xx + \delta y + \varepsilon x + \zeta = 0.$$

From which if *y* may be eliminated by the method treated above, so that it comes about in comparing these equations with these treated above in § 479, evidently

$$P + Qy + Ryy = 0$$

and

$$p + qy + ryy = 0,$$

P and *p* become functions of the second order of x, *Q* and *q* functions of the first order, *R* and *r* will be constants, from which the resulting equation is gathered to become a biquadratic. And thus by the intersections of some two conic sections equations cannot be constructed of higher degree than the biquadratic, but which we have seen able to be constructed by the intersection of a parabola with a circle. Truly likewise this can be understood from the nature of lines of the second order, which can be cut at two points by a straight line ; from which two right lines will be able to form four intersections, but two right lines considered jointly constitute an example of lines of the second order ; from which it is apparent two lines of the second order can intersect each other mutually in four points.

497. Two curves may be used to put into effect the intersections, truly the one curve of the second order and the other indeed of the third order, which may be expressed by these two equations

$$P + Qy + Ryy = 0$$

and

$$p + qy + ryy + sy^3 = 0.$$

Therefore *P* will be a function of two dimensions of *x*, *Q* a function of one dimension and *R* constant ; then truly *p* will be a function of three dimensions, *q* of two, *r* of one dimension and *s* constant. The reason for which may be had, if in the equation arising after the elimination of *y* (§480), the equation is seen to be of the sixth order ; whereby the intersection of a line of the third order with a conic section higher equations than of the sixth power will be unable to be constructed, which likewise is apparent from the nature of each order; for since lines of the third order will intersect with a right line at three points, the same as with two right lines, which taken jointly constitute an example of lines of the second order, which will be intersected in six points.

498. If we may transfer both the eliminations set out above as well as this reasoning demanded from the intersection of right lines to higher orders, it will be apparent by the intersections of two lines of the third order to be able to construct equations of the ninth power, but through the intersections of two lines of the fourth order equations cannot exceed the sixteenth power. And in general from the intersection of two curves, on which one shall be of order *m* with the other of order *n*, all the equations able to be constructed cannot exceed the power *mn*. Thus towards constructing an equation of the one hundredth power there will be a need for either two lines of the tenth order, or from two, of which the one shall be of the fifth order and the other of the twentieth order, and thus so on again, by resolving the number 100 into two factors. But if the maximum power may be required to be expressed by a prime number, or by not admitting other suitable factors, then in its place another number having suitable factors may be substituted ; for with these two curves equations can be constructed from greater powers, also equations can be constructed of each lesser grade. Thus for an equation of the thirty-ninth grade two curves are able to be used, the one of the sixth and the other of the seventh order, because with two curves of this kind an equation of the forty-second order can be constructed and this construction is agreed to be simpler, than if one curve of order three may be assumed, and the other of order thirteen.

499. Therefore from these it is evident any equation thus indeed can be constructed from two curves in innumerable ways by intersections, so that the real roots of this may be assigned. From which innumerable ways it will be convenient to select that chiefly,

which is resolved by the curved lines both most simply, as well as described most easily ; truly in the first case it will be required to apply oneself, so that all the real roots will be shown by intersections ; which may be the case, if we assume curves of this kind, which may be without imaginary intersections. But above we have seen no imaginary intersections to remain in place, of the equation for the other curve the applied line y may be equal to some uniform function of x ; for then, because this curve has no imaginary applied lines, it cannot happen, that imaginary intersections may arise, whatever the number of imaginary applied lines the other curve may also contain. Therefore in this business of the construction we may assume the other curve thus always, so that its equation may be contained in this form $P+Qy=0$, with P and Q denoting functions of x.

500. Therefore with some proposed equation a certain suitable curve is selected with the equation $P+Qy=0$. And, because the equation for the other curve must be prepared thus, so that, in place of y the value $-\frac{P}{Q}$ may be substituted in that equation, the proposed equation itself may result, from that the equation proposed in turn will be able to be formed for the other curve by introducing y in place of $-\frac{P}{Q}$. So that if this equation were proposed

$$x^4+Ax^3+Bxx+Cx+D=0,$$

a parabola may be taken for the other curve held by the equation $ay=xx+bx$; from which since there shall be $xx=ay-bx$, this value may be substituted into the proposed equation as often as wished ; there will be

$$x^4=aayy-2abxy+bbxx,$$
$$Ax^3= \quad +Aaxy-Abxx$$

and therefore the equation of this kind of the second order will be obtained :

$$aayy+a(A-2b)xy+(B-Ab+bb)xx+Cx+D=0$$

the intersections of which thus with the curve $ay=xx+bx$ will indicate the roots of the proposed equation.

501. Just as both these curves can be varied in an indefinite number of ways for determining the arbitrary constants a and b, thus a much greater variation can be induced at this stage. For since from the first equation there shall be $xx-ay+bx=0$, also there will be $acxx-aacy+abcx=0$, which if it is added to the latter equation, a much broader equation for the line of the second order may arise, the intersections of which with the

former will indicate equally the roots of the proposed equation. Evidently both these curves serving to be constructed will be :

I.
$$ay = xx + bx\,,$$

II.
$$aayy + a(A-2b)xy + (B - Ab + bb + ac)xx - aacy + (C + abc)x + D = 0\,,$$

and this latter equation thus can be added to, so that it may include in itself some conic section ; evidently it is required to attend to this quantity

$$AA - 4B - 4ac\,,$$

which if positive, the curve will be a hyperbola, if it were $=0$, the curve will be a parabola, but if it shall be a negative quantity, the curve will be an ellipse.

[For if $b = \frac{1}{2}A$ then there is no cross-term, and the coefficient of x^2 becomes : $B - Ab + bb + ac = B - \frac{1}{2}A^2 + \frac{1}{4}A^2 + ac = B - \frac{1}{4}A^2 + ac = -\left(A^2 - 4B + 4ac\right)$, from which the result follows.]

Truly this other curve will be a circle, if there were

$$b = \tfrac{1}{2}A \quad\text{and}\quad aa = B - \tfrac{1}{4}AA + ac$$

or

$$c = a + \frac{AA}{4a} - \frac{B}{a}\,;$$

then indeed the equation for that curve will be

$$aayy + aaxx - \left(a^3 + \frac{AAa}{4} - Ba\right)y + \left(C + \frac{Aaa}{2} + \frac{A^3}{8} - \frac{AB}{2}\right)x + D = 0$$

or

$$\left(y - \frac{a}{2} - \frac{AA}{8a} + \frac{B}{2a}\right)^2 + \left(x + \frac{C}{2aa} + \frac{A}{4} + \frac{A^3}{16aa} - \frac{AB}{4aa}\right)^2$$
$$= \left(\frac{a}{2} + \frac{AA}{8a} - \frac{B}{2a}\right)^2 + \left(\frac{C}{2aa} + \frac{A}{4} + \frac{A^3}{16aa} - \frac{AB}{4aa}\right)^2 - \frac{D}{aa},$$

where this last term is the square of the radius of the circle.

502. Therefore thus from conic sections alone innumerable curves are had, which described with the parabola $ay = xx + bx$ will provide the roots of the proposed equation from their intersections. Therefore any one of these curves may be taken, the parabola will be intersected always at the same points and thus all these curves will cut each other mutually at the same points. On which account from these infinite curves it will be allowed to assume any two (with the parabola first assumed passed over), which if they may be described on a common axis, will always indicate the same roots by their intersections : Thus in this manner that equation will be able to be constructed either from a circle and a parabola, just as we have seen above now, by two parabolas, by a parabola and an ellipse or a hyperbola, two ellipses, two hyperbolas, or finally by an ellipse with hyperbola. Moreover a much greater variety of constructions will be multiplied together, if also it may be desired to use curves of a higher order to this end.

503. Equations of higher orders can be constructed in the same manner, by assuming for the other curve a line of the parabolic kind expressed by the equation $y = P$. Thus, if the equation shall be proposed to be constructed :

$$x^{12} - f^{10}xx + f^{9}gx - g^{12} = 0,$$

a parabolic equation of the fourth order may be taken : $x^4 = a^3 y$ *;* and since there shall be $x^{12} = a^9 y^3$, with this term put in place an equation will arise for a line of the third order :

$$a^9 y^3 - f^{10}xx + f^{9}gx - g^{12} = 0;$$

from which, it to that some multiple of the first equation $x^4 - a^3 y = 0$ may be added, innumerable lines of the fourth order will be formed, some two of which taken together will construct the proposed equation.

504. But if it may arise, that from the equation proposed to be constructed not a suitable enough method may be able to be derived from the preceding method, then the proposed equation may be multiplied by *x, xx* or x^3, or again by some higher power of *x*, thus so that to its roots some vanishing roots may be added extra, which may be indicated by their intersections made from the start of the abscissas and thus will be easily distinguished from the remaining true roots. Thus the equation proposed therefore will be of a higher order, yet this by not hindering on numerous occasions a more suitable construction to be obtained. Thus, if for example the cubic equation were proposed

$$x^3 + Axx + Bx + C = 0,$$

which on putting $xx = ay$, thus so that the one constructing curve shall become a parabola, the other will be always a hyperbola; for with *ay* substituted in place of *xx*, this equation will be produced

$$axy + Aay + Bx + C = 0\ ;$$

or by adding the first equation $cxx - acy = 0$ this more general equation arises

$$axy + cxx + a(A - c)y + Bx + C = 0\ ,$$

which also is a hyperbola always. Therefore if either a circle, ellipse or parabola may be considered more suitable to be used, then the proposed equation may be multiplied by *x*, so that this equation may be had :

$$x^4 + Ax^3 + Bxx + Cx = 0\ ,$$

which if it may be compared with the biquadratic equation constructed above, there will be $D = 0$ and this equation will be able to be constructed by a circle and parabola always.

505. Therefore because every equation of each order can be constructed by the intersections of two algebraic curves and that in an infinite number of ways, some line is allowed to be substituted in place of the other curve and hence the question has extricated itself, just as it may be able to be constructed with the aid of a given curve. But noting here in the first place the given curve must be from that kind, so that its applied line may be expressed by a uniform function of *x*, lest it be disturbed by imaginary intersections. Nor indeed may it suffice, that a curve or only a part of a proposed curve shall have abscissas equal to one root of the equation, which condition, if indeed only a single root of the proposed equation be desired, is accustomed to be added ; indeed it may happen, that this arc of the curve may experience no intersection, even if the abscissa corresponding to some point of its arc shall be a true root, because this root may be able to be indicated either an imaginary intersection or corresponding to some other branch through the same abscissa. On account of the cause of this question being more curious than useful I will not linger, since I may have made clear well enough the true fundamentals of this kind of construction.

CAPUT XX

DE CONSTRUCTIONE AEQUATIONUM

486. Quae in superiori capite de intersectione curvarum sunt exposita, potissimum ad constructiones aequationum altiorum graduum traduci solent. Cum enim duabus curvis propositis aequationem invenerimus, cuius radices intersectionum locos exhibeant, ita vicissim intersectiones duarum curvarum inservire possunt radicibus aequationum indicandis. Atque hic modus maximam affert utilitatem, si radices cuiuspiam aequationis per lineas exprimi debeant; descripta namque utraque curva ad hunc finem accommodata, intersectiones facile notabuntur, unde, si ad axem applicatae demittantur, abscissae praebebunt veras aequationis radices. Si autem incommodum supra memoratum locum habeat, tum quidem omnes abscissae sic inventae radices praebebunt, at fieri poterit, ut aequatio proposita plures complectatur radices, quam per talem constructionem reperiuntur.

487. Cum igitur proposita fuerit aequatio algebraica incognitam x involvens, cuius radices assignari oporteat, duae quaerendae sunt lineae curvae seu duae aequationes inter binas variabiles x et y, quae ita sint comparatae, ut, si ex iis applicata y eliminetur, ipsa aequatio proposita resultet. Quo facto istae duae curvae super communi axe atque ad idem abscissarum initium describantur punctaque, quibus se mutuo intersecabunt, notentur. Tum ex his intersectionum punctis ad axem applicatae normales demittantur, quae in axe exhibebunt abscissas singulis aequationis propositae radicibus aequales. Hoc itaque modo singularum radicum quaesitarum valores veri assignabuntur, nisi forte eveniat, ut aequatio plures contineat radices, quam intersectiones ad esse deprehendantur.

488. Antequam autem modum tradam, quo binae illae curvae constructioni datae aequationis inservientes inveniri queant, a posteriori eas aequationes perpendamus, quarum resolutio ex datis duabus curvis absolvitur. Ac primo quidem sint ambae lineae resolventes rectae *EM, FM* sese in puncto *M* intersecantes. Sumatur recta *EF* pro axe in eoque punctum *A* pro initio abscissarum, unde educta normalis *ABC* rectam priorem in *B,* posteriorem in *C* secet. Sit $AE = a,\ AF = b,\ AB = c,\ AC = d$; tum vero ponatur abscissa $AP = x$, applicata $PM = y$, eritque pro priori recta *EM* $a : c = a + x : y$ seu $ay = c(a + x)$ et pro altera $b : d = b - x : y$ seu $by = d(b - x)$. Ex his aequationibus si eliminetur *y,* prodibit

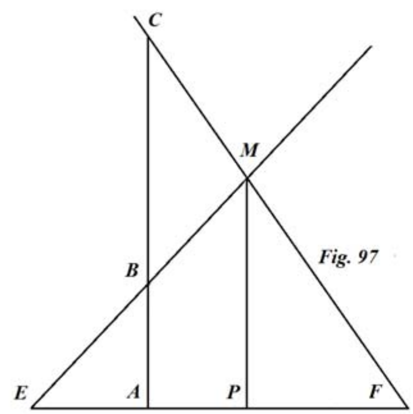


Fig. 97

$$bc(a + x) = ad(b - x)$$

seu

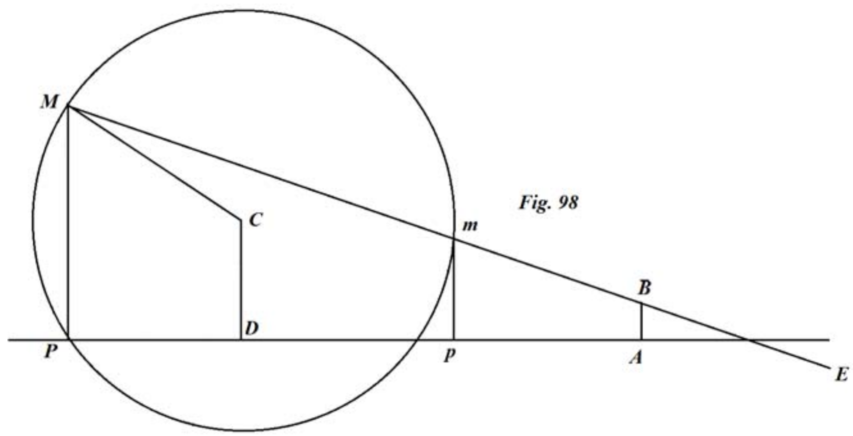
$$x = \frac{abd - abc}{bc + ad} = \frac{ab(d-c)}{bc+ad}.$$

Per intersectionem ergo duarum linearum rectarum construi poterit aequatio simplex

$$x = \frac{ab(d-c)}{bc+ad};$$

ad quam formam omnes omnino aequationes simplices revocari possunt.

489. Lineas rectas ratione facilitatis describendi excipit circulus et hanc ob rem videamus, cuiusmodi aequationes per intersectionem rectae et circuli



construi queant. Sit igitur (Fig. 98), sumta *AP* pro axe et *A* pro abscissarum initio, descripta linea recta *EM* positisque $AE = a$, $AB = b$ et coordinatis $AP = x$, $PM = y$, erit $a : b = a + x : y$ ideoque $ay = b(a+x)$, quae est aequatio pro linea recta. Deinde sit radius circuli $CM = c$, demissoque ex eius centro *C* in axem perpendiculo *CD* vocetur $AD = f$, $CD = g$ erit $DP = x - f$ et $PM - CD = y - g$. Iam,cum sit ex natura circuli

$$CM^2 = DP^2 + (PM - \mathrm{CD})^2,$$

erit aequatio pro circulo

$$cc = xx - 2fx + ff + yy - 2gy + gg = (x-f)^2 + (y-g)^2.$$

At aequatio pro recta dat $y = \dfrac{ab+bx}{a}$, unde fit

$$y - g = \frac{a(b-g)+bx}{a} = b - g + \frac{bx}{a}$$

quo ipsius *y* valore in altera aequatione substituto emerget

$$cc = xx - 2fx + ff + (b-g)^2 + \frac{2b(b-g)x}{a} + \frac{bbxx}{aa}$$

seu

$$\begin{array}{llll} +aaxx & +2ab(b-g)x & +aa(b-g)^2 & =0, \\ +bb & -2aaf & +aaff & \\ & & -aacc & \end{array}$$

cuius ergo aequationis radices invenientur per intersectiones rectae et circuli, ita ut demissis ex intersectionibus *M* et *m* in axem perpendiculis *MP, mp* valores ipsius *y* futuri sint *AP* et *Ap.*

490. Quoniam in hac aequatione omnes aequationes quadraticae continentur, hinc constructio generalis aequationum quadraticarum adornari poterit. Sit scilicet proposita haec aequatio quadratica

$$Axx + Bx + C = 0\,,$$

quae ad superiorem formam primum ita reducatur, ut primi termini conveniant ; multiplicando per $\frac{aa+bb}{A}$:

$$(aa+bb)xx + \frac{B(aa+bb)x}{A} + \frac{C(aa+bb)}{A} = 0.$$

Iam coaequatio reliquorum terminorum dabit

$$2Aab(b-g) - 2Aaaf = B(aa+bb)$$

ideoque fiet

$$af = b(b-g) - \frac{B(aa+bb)}{2Aa}\,.$$

Unde, cum sit

$$aa(b-g)^2 + aaff - aacc = \frac{C(aa+bb)}{A}\,,$$

erit

$$(aa+bb)(b-g)^2 - \frac{Bb(b-g)(aa+bb)}{Aa} + \frac{BB(aa+bb)^2}{4AAaa} - aacc = \frac{C(aa+bb)}{A}$$

ideoque

$$(b-g)^2=\frac{Bb(b-g)}{Aa}-\frac{BB(aa+bb)}{4AAaa}+\frac{aacc}{aa+bb}+\frac{C}{A},$$

ergo

$$b-g=\frac{Bb}{2Aa}\pm\sqrt{\left(\frac{aacc}{aa+bb}+\frac{C}{A}-\frac{BB}{4AA}\right)}.$$

Manent igitur tres quantitates *a*, *b* et *c* adhuc indeterminatae, quas autem ita accipi oportet, ut

$$\frac{aacc}{aa+bb}+\frac{C}{A}-\frac{BB}{4AA}$$

fiat quantitas affirmativa, quia alioquin $b-g=AA-CD$ hincque *CD* fieret quantitas imaginaria.

491. Nihil ergo impedit, quominus ponamus $b=0$, eritque

$$g=\sqrt{\left(cc+\frac{-BB+4AC}{4AA}\right)} \quad \text{et} \quad f=-\frac{B}{2A}.$$

Deinde vero, cum aequatio proposita $Axx+Bx+C=0$ radices nullas habeat reales, nisi sit *BB* maior quam 4*AC*, erit hoc casu $\frac{BB-4AC}{4AA}$ quantitas affirmativa, cui si *cc* ponatur aequale, ut sit

$$c=\frac{\sqrt{(BB-4AC)}}{2A}$$

fiet quoque $g=0$ et *a* prorsus ex calculo excedit. Linea ergo recta *EM* in ipsum axem *AP* incidet et centrum circuli *C* collocari debebit in puncto *D* existente

$$AD=-\frac{B}{2A},$$

ex quo centro si circulus describatur radio

$$c=\frac{\sqrt{(BB-4AC)}}{2A},$$

huius intersectiones cum ipso axe ostendent aequationis propositae radices. Ne autem ad hoc constructione formulae irrationalis opus sit, ponatur

$g = c - \frac{k}{2A}$, ut sit

$$cc - \frac{2ck}{2A} + \frac{kk}{4AA} = cc + \frac{-BB + 4AC}{4AA},$$

erit

$$c = \frac{kk + BB - 4AC}{4kA} \quad \text{et} \quad g = \frac{BB - 4AC - kk}{4kA}.$$

In nostro ergo arbitrio determinatio quantitatis k relinquitur; qua utcunque assumta, quia recta CM in ipsum axem incidit, circulus sequenti modo describi debebit. Sumta $AD = -\frac{B}{2A}$, capiatur perpendiculum

$$CD = \frac{BB - 4AC - kk}{4Ak}.$$

et centro C describatur circulus, cuius radius

$$= \frac{BB - 4AC + kk}{4Ak};$$

huiusque intersectiones cum axe ostendent radices aequationis propositae.
Quodsi ergo statuatur $k = -B$, sumta $AD = -\frac{B}{2A}$ capiatur $CD = \frac{C}{B}$, et circuli centro C describendi radius erit

$$\frac{-BB + 2AC}{2AB} = -\frac{B}{2A} + \frac{C}{B},$$

ex quo radius circuli erit $= AD + CD$; quae constructio pro praxi commodissima videtur.

492. Consideremus iam (Fig. 99) duos circulos se intersecantes sitque pro primo $AD = a$, $CD = b$, et eius radius $CM = c$; eritque positis $AP = x$ et $PM = y$
$DP = a - x,\ \ CD - PM = b - y$; ideoque ex natura circuli habebitur

$$xx - 2ax + aa + yy - 2by + bb = cc.$$

Simili modo pro altero circulo sit $Ad = f$, $dc = g$ eiusque radius $cM = h$, eritque

$$xx - 2fx + ff + yy + 2gy + gg = hh,$$

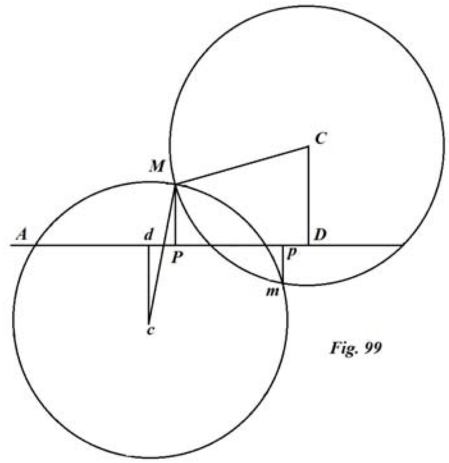


Fig. 99

quibus aequationibus a se invicem subtractis remanebit

$$2(f-a)x + aa - ff - 2(b+g)y + bb - gg = cc - hh,$$

ergo

$$y = \frac{aa + bb - ff - gg - cc - hh + 2(a-f)x}{2(b+g)}$$

hincque

$$b - y = \frac{bb + 2gbh - aa + ff + gg + cc - hh + 2(a-f)x}{2(b+g)}$$

et

$$a - x = \frac{2a(b+g) - 2(b+g)x}{2(b+g)}.$$

Cum igitur sit $(\mathrm{a}-x)^2 + (b-y)^2 = cc$, erit facta substitutione

$$\begin{aligned}&+4(a-f)^2xx - 4(a-f)(b+g)^2x + (b+g)^4 = 0.\\ &\qquad\qquad\qquad\qquad\qquad\qquad +2(aa-cc)(b+g)^2\\ &+4(b+g)^2 \quad -4(a-f)(aa-ff) + 2(ff-hh)(b+g)^2\\ &\qquad\qquad\qquad +4(a-f)(cc-hh) + (aa-cc-ff+hh)^2\end{aligned}$$

Huius ergo aequationis ope infinitis modis construi poterit aequatio

$$Axx + Bx + C = 0;$$

simul vero intelligitur aequationem quadraticam altiorem per intersectionem duorum circulorum construi non posse, propterea quod duo circuli se mutuo in pluribus quam duobus punctis intersecare nequeunt. Cum igitur eadem aequatio quadratica construi possit per intersectionem rectae et circuli, haec constructio illi, quae duos circulos requirit, merito praefertur, nisi forte in casibus quibusdam singularibus facilis linearum *a, b, f, g, c* et *h* determination sponte se prodat.

493. Intersecetur nunc (Fig. 100) circulus a parabola; sit scilicet, demisso ex centro circuli *C* in axem *AP* perpendiculo *CD*, $AD = a$, $CD = b$ et radius circuli $CM = c$, erit inter coordinatas orthogonales $AP = x$, $PM = y$ aequatio pro circulo $(x-a)^2 + (y-b)^2 = cc.$ Parabolae vero axis *FB* statuatur ad axem hic assumtum *AP* normalis sitque $AE = f$, $EF = g$ et parameter parabolae $= 2h$; erit ex natura parabolae $EP^2 = 2h(EF + PM)$ seu in symbolis $(x-f)^2 = 2h(g+y)$, unde erit

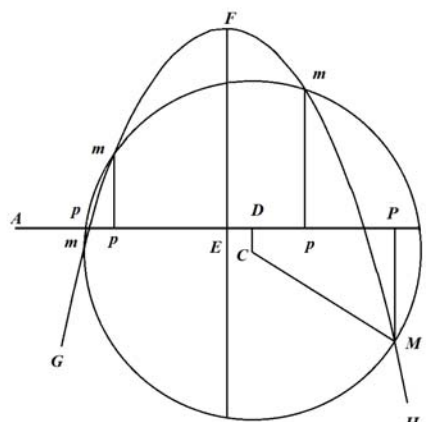


Fig. 100

$$y = \frac{(x-f)^2}{2h} - g \quad \text{et} \quad y - b = \frac{(x-f)^2}{2h} - (b+g).$$

Qui valor si in priori aequatione substituatur, eliminabitur *y* eritque

$$\frac{(x-f)^4}{4hh} - \frac{(b+g)(x-f)^2}{h} + (b+g)^2 + (x-a)^2 = cc$$

sive

$$\begin{array}{llll} x^4 - 4fx^3 + 6ff & + \; xx - 4f^3 & + \; x + \; f^4 & = 0\,, \\ \quad\quad - 4h(b+g) & + 4fh(b+g) & - 4ffh(b+g) & \\ \quad\quad + 4hh & - 8ahh & + 4hh(b+g)^2 & \\ & & + 4aahh & \\ & & - 4cckh & \end{array}$$

cuius aequationis radices erunt abscissae *AP, Ap, Ap, Ap,* unde applicatae per intersectionum puncta *M, m, m, m* transeunt.

494. In hac aequatione sex insunt constantes *a, b, c, f, g* et *h,* quarum vero binae $b+g$ pro una sunt reputandae, ita ut quinque solum ponendo $b+g=k$ inesse censendae sint. Posito scilicet $CD + EF = b + g = k$ sequens habebitur aequatio

$$\begin{aligned} x^4 - 4fx^3 + 6ffxx - 4f^3x + f^4 &= 0\,. \\ -4hk \quad +4fhk - 4ffhk & \\ +4hh \quad -8ahh + 4hhkk & \\ +4aahh & \\ -4cchh & \end{aligned}$$

Ad hanc autem formam omnis aequatio biquadratica revocari potest; sit enim proposita haec aequatio

$$x^4 - Ax^3 + Bxx - Cx + D = 0\,,$$

erit comparatione instituta

$$4f = A \quad \text{seu} \quad f = \tfrac{1}{4}A\,,$$

$$6ff - 4hk + 4hh = B \quad \text{seu} \quad \tfrac{3}{8}AA - 4hk + 4hh = B\,,$$

unde fit

$$k = \frac{3AA}{32h} + h - \frac{B}{4h}\,,$$

$$4f^3 - 4fhk + 8ahh = C$$

sive

$$\frac{1}{16}A^3 - \frac{3}{32}A^3 - Ahh + \tfrac{1}{4}AB + 8ahh = C$$

ergo

$$a = \frac{A^3}{256hh} + \frac{A}{8} - \frac{AB}{32hh} + \frac{C}{8hh}\,.$$

Denique est

$$(ff - 2hk)^2 + 4aahh - 4cckk = D\,.$$

At est

$$ff - 2hk = \frac{B}{2} - 2hh - \frac{AA}{16}$$

et

$$2ah = \frac{A^3}{128h} + \frac{Ah}{4} - \frac{AB}{16h} + \frac{C}{4h}$$

quibus valoribus substitutis emerget aequatio c et h involvens, quas propterea convenientissime inde definiri oportet, ita scilicet ut utraque valorem obtineat realem.

495. Quoniam vero in omni aequatione biquadratica secundus terminus facile tolli potest, ponamus ipsum iam esse sublatum ideoque construendam

esse hanc aequationem

$$x^4 * + Bxx - Cx + D = 0.$$

Erit ergo primum $f = 0$, secundo $k = h - \frac{B}{4h}$, tertio $a = \frac{C}{8hh}$ atque ob

$$2hk - ff = 2hh - \frac{B}{2} \text{ et } 2ah = \frac{C}{4h}$$

quarto

$$4h^4 - 2Bhh + \tfrac{1}{4}BB + \frac{CC}{16hh} - 4cchh = D,$$

unde fit

$$64cch^4 = CC + 4BBhh - 32Bh^4 + 64h^6 - 16Dhh$$

ideoque

$$8chh = \sqrt{(4hh(B - 4hh)^2 + CC - 16Dhh)}.$$

Quoniam vero hoc imprimis est efficiendum, ut tam c quam h obtineant valores reales, ponatur $c = h - \frac{B+q}{4h}$ eritque

$$CC - 16Dhh + 8Bhhq - 32h^4q - 4hhqq = 0.$$

Quo igitur quaesito satisfaciamus, duo casus sunt distinguendi, alter quo D est quantitas negativa, alter quo D est quantitas affirmativa. Sit igitur

I.

D quantitas affirmative $= +EE$, ita ut construi debeat haec aequatio

$$x^4 * + Bx^2 - Cx + EE = 0;$$

ponatur ad hoc $q = 0$, ut sit $c = \frac{4hh - B}{4h}$, fietque

$$hh = \frac{CC}{16EE} \text{ et } h = \frac{C}{4E};$$

unde fit

$$c = \frac{CC - 4BE}{4CE}$$

et porro

$$k = c = \frac{CC - 4BE}{4CE}, \quad a = \frac{2EE}{C} \text{ et } f = 0.$$

II.

Sit autem *D* quantitas negativa, puta $D = -EE$, ut construi debeat haec aequatio

$$x^4 * + Bx^2 - Cx - EE = 0;$$

fiet

$$64cch^4 = CC + 4hh(4hh - B)^2 + 16EEhh,$$

quae aequatio realem pro *c* valorem praebet, quicquid pro *h* assumatur; fiet enim

$$c = \frac{\sqrt{(CC + 4hh(4hh - B)^2 + 16EEhh)}}{8hh}$$

atque *h* pro lubitu assumi potest; quovis igitur casu ita assumatur, ut facillima ipsius *c* constructio inde consequatur. Quo facto erit, ut ante,

$$AE = f = 0, \quad CD + EF = k = \frac{4hh - B}{4h}$$

et

$$AD = a = \frac{C}{8hh}.$$

Si ponatur $E = 0$, orietur constructio aequationis cubicae

$$x^3 * + Bx - C = 0.$$

Hacque constructione nititur regula BACKERI vulgo satis nota.

496. Si sumantur duae quaecunque lineae secundi ordinis seu sectiones conicae, quarum aequationes ad communem axem idemque abscissarum initium relatae sint

$$ayy + bxy + cxx + dy + ex + f = 0$$

et

$$\alpha yy + \beta xy + \gamma xx + \delta y + \varepsilon x + \zeta = 0.$$

Ex quibus si methodo supra tradita *y* eliminetur, quod fiet istas aequationes comparando cum illis in § 479 tractatis, scilicet

$$P + Qy + Ryy = 0$$

et

$$p + qy + ryy = 0,$$

fient P et p functiones secundi ordinis ipsius x, Q et q functiones primi ordinis et R et r erunt constantes, unde colligitur aequatio resultans fore biquadratica. Atque adeo per intersectiones duarum quarumvis sectionum conicarum altioris gradus aequationes construi nequeunt quam biquadraticae, quas autem per circulum et parabolam construi posse vidimus. Hoc idem vero intelligere licet ex natura linearum secundi ordinis, quae a recta linea in duobus punctis secari possunt; unde duae rectae quatuor intersectiones formare poterunt, at duae lineae rectae iunctim consideratae speciem constituunt linearum secundi ordinis; unde patet duas lineas secundi ordinis se mutuo in quatuor punctis intersecare posse.

497. Adhibeantur ad intersectiones efficiendas duae lineae, altera secundi altero vero tertii ordinis, quae exprimantur his aequationibus

$$P + Qy + Ryy = 0$$

et

$$p + qy + ryy + sy^3 = 0.$$

Erit ergo P functio duarum dimensionum ipsius x, Q functio unius dimensionis et R constans; tum vero p functio trium dimensionum, q duarum, r unius dimensionis et s constans. Quarum ratio si in aequatione post eliminationem ipsius y orta (§ 480) habeatur, patebit eam fore ordinis sexti; quare per intersectiones lineae tertii ordinis cum sectione conica altiores aequationes quam sextae potestatis construi non poterunt, quod idem ex natura utriusque ordinis patet; cum enim lineae tertii ordinis a linea recta in tribus punctis intersecentur, eaedem a duabus rectis, quae iunctim sumptae speciem linearum secundi ordinis constituunt, in sex punctis intersecabuntur.

498. Si tam eliminationes supra expositas quam hoc ratiocinium ab intersectione rectarum petitum ad altiores ordines transferamus, patebit per intersectiones duarum linearum tertii ordinis construi posse aequationes nonae potestatis, per intersectiones duarum linearum quarti ordinis autem aequationes potestatem sextam decimam non superantes. Atque in genere per duarum linearum curvarum intersectiones, quarum altera sit ordinis m altera ordinis n, construi poterunt omnes aequationes potestatem mn non excedentes. Sic ad aequationem centesimae potestatis construendam opus erit vel duabus lineis decimi ordinis, vel duabus, quarum altera sit quinti altera vicesimi ordinis, et ita porro, resolvendo numerum 100 in duos factores. Quodsi autem aequationis construendae maxima potestas exponatur numero primo vel alio commodos factores non admittente, tum in eius locum alius numerus maior factores habens idoneos substituatur; quibus enim binis curvis aequationes maioris potestatis construi possunt, iisdem quoque aequationes inferioris cuiusque gradus construentur. Sic ad aequationem

gradus tricesimi noni adhiberi poterunt duae curvae, altera sexti altera septimi ordinis, quia duabus huiusmodi curvis aequatio quadragesimi secundi gradus construi potest haecque constructio simplicior est censenda, quam si altera curva ordinis tertii, altera decimi tertii assumeretur.

499. Ex his igitur perspicuum est unamquamque aequationem pluribus imo innumerabilibus modis per intersectiones duarum curvarum ita construi posse, ut eius radices reales assignentur. Ex quibus infinitis modis eum potissimum eligi conveniet, qui absolvitur lineis curvis cum simplicissimus tum descriptu facillimis; imprimis vero in id erit incumbendum, ut per intersectiones omnes radices reales exhibeantur; quod obtinetur, si eiusmodi curvae assumantur, quae intersectionibus imaginariis careant. Supra autem vidimus huiusmodi intersectionibus imaginariis nullum relinqui locum, si in aequatione pro altera curva applicata y aequetur functioni uniformi ipsius x ; tum enim, quia haec curva nullas habet applicatas imaginarias, fieri nequit, ut intersectiones imaginariae oriantur, quotcunque etiam applicatis imaginariis altera curva inquinetur. In hoc ergo constructionis negotio alteram curvam perpetuo ita assumamus, ut eius aequatio in hac forma $P+Qy=0$ continetur denotantibus P et Q functiones ipsius x.

500. Proposita ergo quacunque aequatione eligatur una quaedam conveniens curva in aequatione $P+Qy=0$. Et, quoniam aequatio pro altera curva ita debet esse comparata, ut, si in ea loco y substituatur valor $-\frac{P}{Q}$, ipsa aequatio proposita resultet, ex ipsa proposita vicissim efformari poterit aequatio pro altera curva introducendo y loco $-\frac{P}{Q}$.
Uti si proposita fuerit haec aequatio

$$x^4+Ax^3+Bxx+Cx+D=0,$$

sumatur parabola pro altera curva aequatione $ay=xx+bx$ contenta; ex qua cum sit $xx=ay-bx$, substituatur iste valor in aequatione proposita, quoties lubet; erit

$$\begin{aligned} x^4 &= aayy-2abxy+bbxx, \\ Ax^3 &= \quad +Aaxy-Abxx \end{aligned}$$

ideoque obtinebitur huiusmodi aequatio secundi ordinis

$$aayy+a(A-2b)xy+(B-Ab+bb)xx+Cx+D=0,$$

cuius adeo intersectiones cum curva $ay=xx+bx$ indicabunt radices aequationis propositae.

501. Quemadmodum hae curvae ambae determinandis pro arbitrio constantibus

a et *b* infinitis modis variari possunt, ita multo maior adhuc varietas induci potest. Cum enim ex aequatione priori sit $xx - ay + bx = 0$, erit quoque $acxx - aacy + abcx = 0$, quae si addatur ad posteriorem aequationem, multo latius patens orietur aequatio pro linea secundi ordinis, cuius intersectiones cum priori radices aequationis propositae aeque indicabunt. Ambae scilicet istae curvae constructioni inservientes erunt

I.
$$ay = xx + bx,$$

II.
$$aayy + a(A - 2b)xy + (B - Ab + bb + ac)xx - aacy + (C + abc)x + D = 0$$

haecque posterior aequatio ita adornari potest, ut quamvis sectionem conicam in se complectatur; attendendum scilicet est ad hanc quantitatem

$$AA - 4B - 4ac,$$

quae si fuerit affirmativa, curva erit hyperbola, si fuerit $= 0$, curva erit parabola, sin autem sit quantitas negativa, curva erit ellipsis. Circulus vero erit haec altera curva, si fuerit

$$b = \tfrac{1}{2}A \quad \text{et} \quad aa = B - \tfrac{1}{4}AA + ac$$

seu

$$c = a + \frac{AA}{4a} - \frac{B}{a};$$

tum enim aequatio pro eo erit

$$aayy + aaxx - \left(a^3 + \frac{AAa}{4} - Ba\right)y + \left(C + \frac{Aaa}{2} + \frac{A^3}{8} - \frac{AB}{2}\right)x + D = 0$$

seu

$$\left(y - \frac{a}{2} - \frac{AA}{8a} + \frac{B}{2a}\right)^2 + \left(x + \frac{C}{2aa} + \frac{A}{4} + \frac{A^3}{16aa} - \frac{AB}{4aa}\right)^2$$
$$= \left(\frac{a}{2} + \frac{AA}{8a} - \frac{B}{2a}\right)^2 + \left(\frac{C}{2aa} + \frac{A}{4} + \frac{A^3}{16aa} - \frac{AB}{4aa}\right)^2 - \frac{D}{aa},$$

ubi hoc membrum est quadratum radii circuli.

502. Sic igitur ex solis sectionibus conicis habentur innumerabiles curvae, quae cum parabola $ay = xx + bx$ descriptae intersectionibus suis radices aequationis propositae

praebebunt. Harum ergo curvarum quaecunque sumatur, parabola in iisdem semper punctis intersecabitur atque ideo illae curvae omnes se mutuo in iisdem punctis secabunt. Quocirca ex his curvis infinitis duas quascunque assumere licebit (praetermissa parabola primum assumta), quae si super communi axe describantur, per intersectiones suas radices aequationis propositae semper indicabunt: Hocque adeo modo ista aequatio construi poterit vel per circulum et parabolam, uti supra iam vidimus, vel per duas parabolas vel per parabolam et ellipsin hyperbolamve vel per duas ellipses vel per duas hyperbolas vel per ellipsin cum hyperbola. Multo magis autem varietas constructionum multiplicabitur, si etiam curvae altiorum ordinum in hunc finem adhiberi velint.

503. Simili modo construi poterunt aequationes altiorum graduum, assumendo pro altera curva lineam parabolici generis aequatione $y = P$ contentam. Sic, si proposita sit aequatio construenda

$$x^{12} - f^{10}xx + f^9gx - g^{12} = 0,$$

sumatur aequatio parabolica ordinis quarti $x^4 = a^3y$; et, cum sit $x^{12} = a^9y^3$, hoc termino substituto emerget aequatio pro linea tertii ordinis

$$a^9y^3 - f^{10}xx + f^9gx - g^{12} = 0;$$

ex qua, si ad eam addatur multiplum quodcunque prioris aequationis $x^4 - a^3y = 0$, innumerabiles formabuntur lineae quarti ordinis, quarum binae quaevis coniunctae aequationem propositam construent.

504. Quodsi eveniat, ut ex aequatione construenda proposita non satis idonea constructio praecedente methodo derivari queat, tum aequatio proposita multiplicetur per x vel xx vel x^3 vel altiorem quampiam potestatem ipsius x, ita ut ad eius radices aliquot insuper radices evanescentes addantur, quae per intersectiones in ipso abscissarum initio factas indicabuntur ideoque a reliquis radicibus veris aequationis propositae facile discernentur. Sic igitur aequatio proposita altoris fit gradus, hoc tamen non obstante saepenumero commodior constructio obtinebitur. Ita, si exempli gratia proposita fuerit aequatio cubica

$$x^3 + Axx + Bx + C = 0,$$

quae posito $xx = ay$, ita ut altera curva construens futura sit parabola, altera erit semper hyperbola; prodibit enim loco xx substituto ay haec aequatio

$$axy + Aay + Bx + C = 0;$$

vel addita aequatione priore $cxx - acy = 0$ nascetur haec latius patens

$$axy + cxx + a(A - c)y + Bx + C = 0,$$

quae quoque perpetuo est pro hyperbola. Quodsi ergo circulum vel ellipsin vel parabolam adhibere commodius videatur, tum aequatio proposita multiplicetur per x, ut habeatur haec aequatio

$$x^4 + Ax^3 + Bxx + Cx = 0,$$

quae si cum aequatione biquadratica supra constructa comparetur, erit $D = 0$ haecque aequatio semper per circulum et parabolam construi poterit.

505. Quoniam ergo omnis aequatio cuiusque gradus per intersectiones duarum curvarum algebraicarum construi potest idque infinitis modis, lineam quamcunque in locum alterius curvae substituere licebit hincque enata est quaestio, quemadmodum data aequatio ope datae curvae construi queat. Hic autem primum notandum est datam curvam ex eo genere esse debere, ut eius applicata exprimatur per functionem uniformem ipsius x, ne intersectiones imaginariae constructionem perturbent. Neque enim sufficeret, ut curva, vel tantum portio curvae proposita, habeat abscissas uni radici aequationis aequales, quae conditio, siquidem una tantum radix aequationis propositae desideretur, adiici est solita; fieri enim posset, ut iste arcus curvae nullam patiatur intersectionem, etiamsi abscissa cuipiam ipsius puncto respondens sit vera radix, quoniam haec radix vel per intersectionem imaginariam vel per alius rami eidem abscissae respondentia intersectionem indicari posset. Quam ob causam huic quaestioni curiosae magis quam utili non immoror, cum vera fundamenta omnium huiusmodi constructionum satis fuse ostenderim.