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Distribution and expectation bounds on order statistics from possibly dependent variates

Nickos Papadatos

Section of Statistics and O.R., Department of Mathematics, University of Athens, Panepistemiopolis, 157 84 Athens, Greece

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Abstract

Let $X_1, X_2, \dots, X_n$ be n random variables with an arbitrary n-variate distribution. We say that the X's are maximally (resp. minimally) stable of order j ($j \in \{1,2,\dots,n\}$), if the distribution $F_{(j)}$ of $\max\{X_{k_1},\dots,X_{k_j}\}$ (resp. $G_{(j)}$ of $\min\{X_{k_1},\dots,X_{k_j}\}$) is the same, for any j-subset $\{k_1,\dots,k_j\}$ of $\{1,2,\dots,n\}$. Under the assumption of maximal (resp. minimal) stability of order j, sharp upper (resp. lower) bounds are given for the distribution $F_{k:n}$ of the kth order statistic $X_{k:n}$, in terms of $F_{(j)}$ (resp. $G_{(j)}$), and the corresponding expectation bounds are derived. Moreover, some expectation bounds in the case of j-independent-F samples (i.e., when each j-tuple $X_{k_1},\dots,X_{k_j}$ is independent with a common marginal distribution F) are given. © 2001 Elsevier Science B.V. All rights reserved

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1. Introduction

Let $X_1, X_2, \dots, X_n$ be a sample of arbitrary, possibly dependent, random variables, with possibly different marginal distributions, and let us denote by $X_{1:n} \leqslant X_{2:n} \leqslant \cdots \leqslant X_{n:n}$ the corresponding order statistics. Assume that for some $j \in \{1,2,\dots,n\}$, the distribution of $\max\{X_{k_1},\dots,X_{k_j}\}$ is the same for all $1 \leqslant k_1 < \cdots < k_j \leqslant n$, denoted by $F_{(j)}$ (this condition is satisfied, for example, when the X's are exchangeable). In this case, we say that the X's are *maximally stable of order j*. Observe that this condition always holds for $j = n$. On the other hand, for $j = 1$, the condition is equivalent to the fact that the X's are identically distributed with common marginals $F_{(1)}$.

E-mail address: npapadat@cc.uoa.gr (N. Papadatos).

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In this note it is proved that, if the X's are maximally stable of order j, then for any $k \geqslant j$, the distribution $F_{k:n}$ of $X_{k:n}$ satisfies the inequality

$$F_{k:n}(x) \leqslant \min\left\{1, \frac{(n)_j}{(k)_j} F_{(j)}(x)\right\}, \tag{1}$$

where $(s)_j = s(s-1)\cdots(s-j+1)$. Furthermore, it is proved that (1) is attainable for any given $F_{(j)}$. Similarly, if the distribution $G_{(j)}$ of $\min\{X_{k_1},\dots,X_{k_j}\}$ is the same for all $1 \leqslant k_1 < \cdots < k_j \leqslant n$ (in which case we say that the X's are *minimally stable of order* j), then the inequality

$$F_{k:n}(x) \geqslant \max\left\{0, 1 - \frac{(n)_j}{(n+1-k)_j}(1 - G_{(j)}(x))\right\} \tag{2}$$

holds for any $k \leqslant n+1-j$, and the bound (2) is attainable for any given $G_{(j)}$.

It should be noted that for $j=1$, bounds (1) and (2) are known; in this case, $G_{(1)} = F_{(1)}$ is the common marginal of X's, and the corresponding inequalities were established by Rychlik (1992) and Caraux and Gascuel (1992).

In Section 3, we derive the relative expectation bounds, and compare them with the corresponding results for $j=1$, given by Rychlik (1992) and Gascuel and Caraux (1992). Finally, in Section 4 we derive some moment inequalities for order statistics in the case of j-independent-F samples (i.e., when each j-tuple $X_{k_1},\dots,X_{k_j}$ is independent with a common marginal distribution F), and compare our results with the corresponding ones obtained by Kemperman (1997).

2. Distribution bounds

The upper bound is given in the following theorem.

Theorem 2.1. *If the X's are maximally stable of order j and*

$$F_{(j)}(x) = \mathbb{P}[X_1 \leqslant x,\dots,X_j \leqslant x], \tag{3}$$

then (1) *holds for all* $k \geqslant j$. *Conversely, for any distribution function* $F_{(j)}$ *and any* $k \geqslant j$, *there are exchangeable random variables* $X_1, X_2, \dots, X_n$ *for which* (3) *holds and the equality is attained in* (1).

Proof. As in Caraux and Gascuel (1992), for any fixed x, consider the non-negative random variable

$$N_j(x) = \sum_{1 \leqslant k_1 < \cdots < k_j \leqslant n} I(X_{k_1} \leqslant x, \dots, X_{k_j} \leqslant x),$$

where $I((Y_1,\dots,Y_j) \in A)$ is the indicator function of the event $\{(Y_1,\dots,Y_j) \in A\}$, for any random vector $(Y_1,\dots,Y_j)$ and arbitrarily measurable A in $\mathbb{R}^j$. By (3) and maximal stability of order j, it follows that

$$\mathbb{E}N_j(x) = \binom{n}{j} F_{(j)}(x).$$

Therefore, for $k \geqslant j$, Markov's inequality yields

$$F_{k:n}(x) = \mathbb{P}\left[N_j(x) \geqslant \binom{k}{j}\right] \leqslant \binom{n}{j} F_{(j)}(x) \Big/ \binom{k}{j},$$

proving (1). We now show the existence of n exchangeable variables satisfying (3) and attaining equality in (1). Since for $k=n$ one can trivially take $X_1 = X_2 = \cdots = X_n$ with distribution $F_{(j)}$, we may assume that $1 \leqslant j \leqslant k < n$. For any fixed distribution $F_{(j)}$, let

$$t = \inf\left\{x\colon F_{(j)}(x) \geqslant \frac{(k)_j}{(n)_j}\right\} := F_{(j)}^{-1}\left(\frac{(k)_j}{(n)_j}\right)$$

and consider independent random variables Y, Z with distributions

$$F_Y(x) = \min\left\{1, \frac{(n)_j}{(k)_j} F_{(j)}(x)\right\} \quad \text{and} \quad F_Z(x) = \max\left\{0, \frac{(n)_j F_{(j)}(x) - (k)_j}{(n)_j - (k)_j}\right\}.$$

It follows that $Y \leqslant t \leqslant Z$ a.s. Therefore, the order statistic $Y_{k:n}$ of the random vector $(Y_1, \ldots, Y_n)$ with $Y_1 = \cdots = Y_k = Y$, $Y_{k+1} = \cdots = Y_n = Z$, satisfies $Y_{k:n} = Y$ a.s. We define

$$(X_1, X_2, \ldots, X_n) = (Y_{\pi(1)}, Y_{\pi(2)}, \ldots, Y_{\pi(n)}),$$

where $(\pi(1), \pi(2), \ldots, \pi(n))$ is stochastically independent of the X's and uniformly distributed over the $n!$ permutations of $(1, 2, \ldots, n)$. Then, it is easily verified that the X's are exchangeable (and thus, maximally stable of order j), and that they satisfy (3). Furthermore, since $X_{k:n} = Y_{k:n} = Y$ a.s., the distribution of $X_{k:n}$ attains the desired equality in (1) by definition. □

The lower bound can be derived from Theorem 2.1.

Corollary 2.1. *If the X's are minimally stable of order j and*

$$G_{(j)}(x) = 1 - \mathbb{P}[X_1 > x, \ldots, X_j > x], \tag{4}$$

then (2) *holds for all* $k \leqslant n + 1 - j$. *Conversely, for any distribution function* $G_{(j)}$ *and any* $k \leqslant n + 1 - j$, *there are exchangeable random variables* $X_1, X_2, \ldots, X_n$ *for which* (4) *holds and the equality is attained in* (2).

Proof. Let $Y_i = -X_i$, $i = 1, 2, \ldots, n$. Since the X's are minimally stable of order j, it follows immediately that the Y's are maximally stable of the same order, and $F_{(j)}(-x) = 1 - G_{(j)}(x_-)$. Then, for $n + 1 - k \geqslant j$, (1) yields

$$1 - F_{k:n}(x_-) = F_{Y_{n+1-k:n}}(-x) \leqslant \min\left\{1, \frac{(n)_j}{(n+1-k)_j} F_{(j)}(-x)\right\}$$
$$= \min\left\{1, \frac{(n)_j}{(n+1-k)_j}(1 - G_{(j)}(x_-))\right\},$$

that is,

$$F_{k:n}(x_-) \geqslant \max\left\{0, 1 - \frac{(n)_j}{(n+1-k)_j}(1 - G_{(j)}(x_-))\right\},$$

from which (2) follows on taking right limits. Conversely, from Theorem 2.1 it follows that for any fixed $k \leqslant n + 1 - j$ and any distribution function $F_{(j)}$, there are exchangeable random variables $Y_1, Y_2, \ldots, Y_n$ such that $F_{(j)}(x) = \mathbb{P}[Y_1 \leqslant x, \ldots, Y_j \leqslant x]$ and

$$F_{Y_{n+1-k:n}}(x) = \min\left\{1, \frac{(n)_j}{(n+1-k)_j} F_{(j)}(x)\right\}.$$

Therefore, taking $F_{(j)}(x) = 1 - G_{(j)}(-x_-)$ and $X_i = -Y_i$ for $i = 1, 2, \ldots, n$, we conclude that the X's are exchangeable and satisfy

$$G_{(j)}(x_-) = 1 - \mathbb{P}[X_1 \geqslant x, \ldots, X_j \geqslant x]$$

and

$$F_{k:n}(x_-) = \max\left\{0, 1 - \frac{(n)_j}{(n+1-k)_j}(1 - G_{(j)}(x_-))\right\}.$$

The desired result follows on taking right limits in the last two expressions. □

3. Expectation bounds

For any distribution function H, we denote its left-continuous inverse by $H^{-1}(u) = \inf\{x: H(x) \geqslant u\}$, $0 < u < 1$. Then, we have the following result.

Theorem 3.1. (i) *If the random variables* $X_1, X_2, \ldots, X_n$ *are maximally stable of order* j *and* $\mathbb{E}\max\{0, X_1, \ldots, X_j\} < \infty$, *then for any* $k \geqslant j$,

$$\mathbb{E}X_{k:n} \geqslant \frac{1}{a}\int_0^a F_{(j)}^{-1}(u)\,\mathrm{d}u, \tag{5}$$

where $a = (k)_j/(n)_j$ *and* $F_{(j)}$ *is given by* (3). *Moreover, the bound* (5) *is attainable.*

(ii) *Assume that the random variables* $X_1, X_2, \ldots, X_n$ *are minimally stable of order* j, *and let* $b = (n+1-k)_j/(n)_j$. *If* $\mathbb{E}\min\{0, X_1, \ldots, X_j\} > -\infty$, *then for any* $k \leqslant n+1-j$,

$$\mathbb{E}X_{k:n} \leqslant \frac{1}{b}\int_{1-b}^1 G_{(j)}^{-1}(u)\,\mathrm{d}u, \tag{6}$$

where $G_{(j)}$ *is given by* (4). *Moreover, the bound* (6) *is attainable.*

Proof. We shall only prove (i) because (ii) is similar. If $1 \leqslant j \leqslant k = n$, (5) is trivial, and the equality is attained when $X_1 = X_2 = \cdots = X_n$ and the distribution of X_1 is $F_{(j)}$. Therefore, we may assume that $j \leqslant k < n$. In this case, $a \in (0,1)$. Consider a random variable Y with distribution given by the RHS of (1). By Theorem 2.1, Y is stochastically smaller than $X_{k:n}$ and thus, $\mathbb{E}X_{k:n} \geqslant \mathbb{E}Y$ (note that the condition $\mathbb{E}\max\{0, X_1, \ldots, X_j\} < \infty$ implies that $-\infty \leqslant \mathbb{E}X_{k:n} \leqslant \mathbb{E}X_{n:n} < \infty$). On the other hand, $Y \leqslant t$ a.s., where $t = F_{(j)}^{-1}((k)_j/(n)_j) = F_{(j)}^{-1}(a)$ (finite). Therefore,

$$\mathbb{E}[X_{k:n} - t] \geqslant \mathbb{E}[Y - t] = -\int_{-\infty}^0 \mathbb{P}[Y - t \leqslant x]\,\mathrm{d}x = -\int_{-\infty}^t \mathbb{P}[Y \leqslant x]\,\mathrm{d}x.$$

By using the identity (cf. Arnold, 1980, and Gascuel and Caraux, 1992)

$$F^{-1}(y) - \frac{1}{y}\int_{-\infty}^{F^{-1}(y)} F(x)\,\mathrm{d}x = \frac{1}{y}\int_0^y F^{-1}(u)\,\mathrm{d}u, \tag{7}$$

which holds for any distribution F and any $y \in (0,1)$ (in the sense that either both sides are $-\infty$ or they are finite and equal; for a proof of a more general result see Lemma 4.1, below), we conclude that

$$\mathbb{E}X_{k:n} \geqslant F_{(j)}^{-1}(a) - \frac{1}{a}\int_{-\infty}^{F_{(j)}^{-1}(a)} F_{(j)}(x)\,\mathrm{d}x = \frac{1}{a}\int_0^a F_{(j)}^{-1}(u)\,\mathrm{d}u,$$

which is (5). Finally, it is obvious that the equality is attained in (5) when $X_{k:n}$ is distributed like Y, and Theorem 2.1 shows that this can be achieved for any fixed $F_{(j)}$ and any fixed k ($k \geqslant j$). This completes the proof. □

The results are illustrated by two examples.

Example 1. Let $X_1, X_2, \ldots, X_n$ be jointly distributed so that the joint distribution of any two random variables (X_i, X_r) $(i \neq r)$ is uniform over $(0,1)^2$; in other words, the X's are 2-independent standard uniform variables. It follows that the X's are maximally and minimally stable of order 2, and $F_{(2)}(x) = x^2$, $G_{(2)}(x) = 1 - (1-x)^2$, $0 < x < 1$. In this case, Theorem 3.1 yields

$$\frac{2}{3}\sqrt{\frac{k(k-1)}{n(n-1)}} \leqslant \mathbb{E}X_{k:n} \leqslant 1 - \frac{2}{3}\sqrt{\frac{(n+1-k)(n-k)}{n(n-1)}},$$

for $2 \leqslant k \leqslant n-1$. Observe that for $3 \leqslant k \leqslant n-2$, these bounds dominate the corresponding ones for $j = 1$, given by Gascuel and Caraux (1992) and Rychlik (1992), namely

$$\frac{k}{2n} \leqslant \mathbb{E}X_{k:n} \leqslant \frac{1}{2} + \frac{k-1}{2n}.$$

If $n = 2m - 1$ and one uses the sample median $D_m = X_{m:2m-1}$ as an estimator of the true population median $\delta = \frac{1}{2}$, the bound for $j = 2$ shows that for any $\varepsilon > 0$, $\frac{1}{3} - \varepsilon < \mathbb{E}D_m < \frac{2}{3} + \varepsilon$ for all sufficiently large m. For $j = 1$ we can only conclude that asymptotically, $\frac{1}{4} - \varepsilon < \mathbb{E}D_m < \frac{3}{4} + \varepsilon$. More generally, if the X's are j-independent standard uniform variables for some fixed j, then it can be easily shown with the use of Theorem 3.1 that,

$$\frac{j}{2(j+1)} \leqslant \liminf_{m\to\infty} \mathbb{E}D_m \leqslant \limsup_{m\to\infty} \mathbb{E}D_m \leqslant \frac{j+2}{2(j+1)}$$

and thus, the asymptotic absolute bias of D_m is no greater than $(2(j+1))^{-1}$. However, this result is not optimal; see Example 3, below.

Example 2. If the X's are pairwise independent standard exponential, then they are minimally stable of order 2, and $G_{(2)}(x) = 1 - \exp(-2x)$ for $x > 0$. Thus, for any $k \leqslant n-1$, (6) yields

$$\mathbb{E}X_{k:n} \leqslant \frac{1}{2}\left(1 + \log\frac{n}{n+1-k} + \log\frac{n-1}{n-k}\right),$$

which is better than the upper bound obtained for $j = 1$, namely,

$$\mathbb{E}X_{k:n} \leqslant 1 + \log\frac{n}{n+1-k}.$$

4. The case of *j*-independent samples

A special case, where maximal and minimal stability of order j follows automatically, is the following (see Examples 1 and 2): Assume that there exists a distribution F and some fixed $j \in \{1, 2, \ldots, n\}$ such that for any j-subset $\{k_1, \ldots, k_j\}$ of $\{1, 2, \ldots, n\}$, the random variables $X_{k_1}, \ldots, X_{k_j}$ are independent with distribution F; i.e., the X's form a j-independent-F sample. In particular, the X's are maximally and minimally stable of any order $i \leqslant j$, with $F_{(i)} = F^i$ and $G_{(i)} = 1 - (1-F)^i$, $1 \leqslant i \leqslant j$. Obviously, for $j = n$, this condition is equivalent to the fact that the X's are i.i.d. with distribution F (and thus, $F_{k:n}$ is uniquely specified from F). The other extremal case $j = 1$ reduces to the fact that the X's are identically distributed with distribution F (and the known bounds for $j = 1$, given by Rychlik (1992) and Caraux and Gascuel (1992), are attainable). For $1 < j < n$, however, the bounds discussed in Section 1 are not attainable by j-independent-F samples, except in some trivial cases; this is an implication of Theorem 2.1, since from (1) we have that for all $k \geqslant j$,

$$F_{k:n}(x) \leqslant \min_{0\leqslant i\leqslant j}\left\{\frac{(n)_i}{(k)_i}F^i(x)\right\} = \frac{(n)_i}{(k)_i}F^i(x), \quad \text{if } a_{i+1} \leqslant F(x) \leqslant a_i,\ i = 0, 1, \ldots, j, \tag{8}$$

where $a_0 = 1$, $a_i = (k+1-i)/(n+1-i)$, $i = 1,\dots,j$, $a_{j+1} = 0$. Observe that (8) is usually strictly better than (1), which implies that (5) is not attainable in this case.

Kemperman (1997) established an effective method for obtaining sharp distribution bounds on order statistics from j-independent-F samples, exploiting some interesting relations between the joint factorial moments of a multinomial distribution and those of a multinomial random vector, defined by some events associated with the initial sample and a partition of the real line (for more details, see Theorem 1 in Kemperman (1997)). However, the resulting sharp bounds are quite complicated for $1 < j < n$ and, thus, difficult to work with; for instance, a careful reading of the pairwise independent case ($j = 2$; see pp. 302, 303 in Kemperman's paper) yields the bound (holding for all $1 \leqslant k \leqslant n$, $n \geqslant 2$):

$$\max_{\mathscr{F}(2;n;F)} F_{k:n}(x) = \begin{cases} \dfrac{nF(x)((n-1)F(x) - 2r) + r(r+1)}{(k-r)(k-r-1)} & \text{if } F(x) < \gamma, \\ F(x) + \frac{n-1}{k}F(x)(1-F(x)) & \text{if } \gamma \leqslant F(x) < \delta, \\ 1 & \text{if } F(x) \geqslant \delta, \end{cases} \tag{9}$$

where $\gamma = (k-1)/(n-1)$, $\delta = k/(n-1)$, $\mathscr{F}(j;n;F)$ denotes the space of j-independent-F samples of size n, and $r = r(k;n;F(x))$ is the greatest integer of $\{0,\dots,k-2\}$ less than or equal to $nF(x)(k-1-(n-1)F(x))/(k-nF(x))$ (provided that $k \geqslant 2$ and $F(x) < (k-1)/(n-1)$).

Bound (8) enables us to calculate some (non-attainable, in general) expectation bounds in terms of F^{-1}. For this reason, we first need to prove the following identities, perhaps of some independent interest.

Lemma 4.1. *Let F be a distribution function and F^{-1} its left-continuous inverse.*

(i) *For any $n = 1, 2, \dots$ and $y \in (0,1)$,*

$$\int_{-\infty}^{F^{-1}(y)} F^n(x)\,\mathrm{d}x = y^n F^{-1}(y) - n\int_0^y u^{n-1}F^{-1}(u)\,\mathrm{d}u. \tag{10}$$

(ii) *If, furthermore, for some $y \in (0,1)$*

$$\int_0^y F^{-1}(u)\,\mathrm{d}u > -\infty,$$

then for any $a \in (0,1)$, and $n = 2, 3, \dots$,

$$(n-1)\iint_{0<u<v<a} u^{n-2}(F^{-1}(v) - F^{-1}(u))\,\mathrm{d}v\,\mathrm{d}u = \int_0^a u^{n-2}(nu-(n-1)a)F^{-1}(u)\,\mathrm{d}u. \tag{11}$$

Proof. (i) Let $X_1,\dots,X_n$ be an i.i.d. sample from F and set

$$Y = \min\{X_{n:n}, F^{-1}(y)\} = X_{n:n}I(X_{n:n} \leqslant F^{-1}(y)) + F^{-1}(y)I(X_{n:n} > F^{-1}(y)),$$

where $X_{n:n} = \max\{X_1,\dots,X_n\}$. Since the distribution of Y is $F_Y(x) = F^n(x)I(x < F^{-1}(y)) + I(x \geqslant F^{-1}(y))$, it follows that

$$\mathbb{E}Y = F^{-1}(y) - \int_{-\infty}^{F^{-1}(y)} F^n(x)\,\mathrm{d}x$$

is either finite or $-\infty$. On the other hand, if $U_{n:n}$ is the maximum of n independent standard uniform variables, it follows that Y has the same distribution as

$$\begin{aligned} &F^{-1}(U_{n:n})I(F^{-1}(U_{n:n}) \leqslant F^{-1}(y)) + F^{-1}(y)I(F^{-1}(U_{n:n}) > F^{-1}(y)) \\ &\quad = F^{-1}(U_{n:n})I(U_{n:n} \leqslant F(F^{-1}(y))) + F^{-1}(y)I(U_{n:n} > F(F^{-1}(y))), \end{aligned}$$

where the equality holds because the sets $\{u \in (0,1)\colon F^{-1}(u) \leqslant t\}$ and $\{u \in (0,1)\colon u \leqslant F(t)\}$ are identical for each finite t. Therefore,

$$\mathbb{E}Y = n\int_0^{F(F^{-1}(y))} u^{n-1}F^{-1}(u)\,\mathrm{d}u + F^{-1}(y)(1 - F^n(F^{-1}(y))).$$

Equating the above two expressions for $\mathbb{E}Y$, we get

$$\int_{-\infty}^{F^{-1}(y)} F^n(x)\,\mathrm{d}x = n\int_0^{F(F^{-1}(y))} u^{n-1}(F^{-1}(y) - F^{-1}(u))\,\mathrm{d}u.$$

Since $y \leqslant F(F^{-1}(y))$ for each $y \in (0,1)$ and $F^{-1}(u) = F^{-1}(y)$ for all $u \in [y, F(F^{-1}(y))]$, (10) follows.

(ii) It suffices to observe that both sides of (11) present the same finite non-negative quantity

$$\frac{a^n}{n}\mathbb{E}[F^{-1}(V_{n:n}) - F^{-1}(V_{n-1:n})],$$

where $V_{1:n} < \cdots < V_{n:n}$ is the ordered sample corresponding to n independent uniform $(0,a)$ random variables. This completes the proof. □

It should be noted that several methods can be used for proving (10). For example, integrating, by parts, the LHS of (10) and setting $F^{-1}(u) = x$ yields the RHS of (10); another proof can be given by writing

$$F^n(x) = \int_{\mathbb{R}^n}\prod_{j=1}^n I(t_j \leqslant x)\,\mathrm{d}(F(t_1)\times\cdots\times F(t_n))$$

and using Tonelli's Theorem. However, we used the present 'non-parametric' approach, because the identity (10) follows as a natural property satisfied by an ordered sample. Also note that (10) with $n = 1$ yields (7).

Lemma 4.2. *Let $X_1,\ldots,X_n$ be i.i.d. with $\mathbb{E}|X_1| < \infty$. Then, for $s = 0,\ldots,n-1$,*

$$\mathbb{E}X_{n-s:n} = \sum_{j=s+1}^n \frac{s+1}{j}\mathbb{E}[X_{j-s:j} - X_{j-s-1:j}], \tag{12}$$

where $X_{1:j} \leqslant \cdots \leqslant X_{j:j}$ are the order statistics of $X_1,\ldots,X_j$, $j = s+1,\ldots,n$, and $X_{0:s+1} \equiv 0$.

Proof. Let F^{-1} be the inverse of the distribution of X_1. Writing the RHS of (12) as

$$(s+1)\int_0^1 F^{-1}(u)\sum_{j=s+1}^n \frac{1}{j}(g_{j-s:j}(u) - g_{j-s-1:j}(u))\,\mathrm{d}u,$$

where

$$g_{r:j}(u) = \frac{\Gamma(j+1)}{\Gamma(r)\Gamma(j+1-r)}u^{r-1}(1-u)^{j-r}, \quad 0 < u < 1,$$

for $j = s+1,\ldots,n$ and $r = j-s-1, j-s$ (with $g_{0:s+1} \equiv 0$), the desired result follows on verifying the simple algebraic identity

$$(s+1)\sum_{j=s+1}^n \frac{1}{j}(g_{j-s:j}(u) - g_{j-s-1:j}(u)) = g_{n-s:n}(u). \qquad \square$$

We are now in a position to prove the main result of this section.

Theorem 4.1. *Assume that $X_1,\dots,X_n$ is a j-independent-F sample $(1 \leqslant j \leqslant n)$ with $\mathbb{E}|X_1| < \infty$. Then, we have the following bounds.*

(i) *For $k \geqslant j$,*

$$\mathbb{E}X_{k:n} \geqslant \frac{n}{k}\int_0^{a_1} F^{-1}(u)\,\mathrm{d}u + \sum_{i=2}^{j} \frac{(i-1)(n)_i}{(k)_i} \iint_{0<u<v<a_i} u^{i-2}(F^{-1}(v)-F^{-1}(u))\,\mathrm{d}v\,\mathrm{d}u, \tag{13}$$

where $a_i = a_i(k,n) = (k+1-i)/(n+1-i),\ i=1,\dots,j$.

(ii) *For $k \leqslant n+1-j$,*

$$\mathbb{E}X_{k:n} \leqslant \frac{n}{n+1-k}\int_{1-b_1}^{1} F^{-1}(u)\,\mathrm{d}u$$

$$-\sum_{i=2}^{j} \frac{(i-1)(n)_i}{(n+1-k)_i} \iint_{1-b_i<u<v<1} (1-v)^{i-2}(F^{-1}(v)-F^{-1}(u))\,\mathrm{d}v\,\mathrm{d}u, \tag{14}$$

where $b_i = b_i(k,n) = (n-k+2-i)/(n+1-i),\ i=1,\dots,j$.

Proof. (i) If $j=1$, we simply ignore the sum of integrals in the RHS of (13), yielding the known bounds discussed before for the identically distributed case. Next, we assume that $2 \leqslant j \leqslant k \leqslant n$. If $k=n$ then $a_1 = \cdots = a_j = 1$, and (13) is equivalent to

$$\mathbb{E}X_{n:n} \geqslant \mathbb{E}[F^{-1}(U_1)] + \sum_{i=2}^{j} \frac{1}{i}\,\mathbb{E}[F^{-1}(U_{i:i}) - F^{-1}(U_{i-1:i})], \tag{15}$$

where $U_1,\dots,U_j$ are independent standard uniform variables and $U_{1:i} < \cdots < U_{i:i}$ are the order statistics corresponding to $U_1,\dots,U_i$, $i=2,\dots,j$. Since X_1 and $F^{-1}(U_1)$ have the same distribution and, by j-independence, the random vectors $(X_{i-1:i}, X_{i:i})$ and $(F^{-1}(U_{i-1:i}),\ F^{-1}(U_{i:i}))$ are identically distributed for each $i=2,\dots,j$, it follows that (15) is equivalent to the obvious inequality

$$\mathbb{E}X_{n:n} \geqslant \sum_{i=1}^{j} \frac{1}{i}\,\mathbb{E}[X_{i:i} - X_{i-1:i}] = \mathbb{E}X_{j:j}$$

(the last equality is a consequence of (12) with $n=j$ and $s=0$, since $X_1,\dots,X_j$ are i.i.d.). This proves (13) when $k=n$. Finally, we assume that $2 \leqslant j \leqslant k < n$. In this case, we consider the points γ_i, t_i, with $\gamma_0 = 0$, $\gamma_i = (k-j+i)/(n-j+i)$, $i=1,\dots,j$, and $t_i = F^{-1}(\gamma_i)$. Clearly, $0=\gamma_0<\gamma_1<\cdots<\gamma_j<1$ and thus, $-\infty = t_0 < t_1 \leqslant \cdots \leqslant t_j < \infty$ (note that $\gamma_i = a_{j+1-i}$ for $i=1,\dots,j$). Let Y be a random variable with distribution given by the RHS of (8). Obviously $Y \leqslant t_j$ a.s., and hence, proceeding as in the proof of Theorem 3.1, we have

$$\begin{aligned}\mathbb{E}X_{k:n} &\geqslant t_j - \int_{-\infty}^{t_j} \mathbb{P}[Y \leqslant x]\,\mathrm{d}x \\ &= t_j - \sum_{i=1}^{j} \frac{(n)_i}{(k)_i}\int_{t_{j-i}}^{t_{j+1-i}} F^i(x)\,\mathrm{d}x \\ &= t_j - \frac{n}{k}\int_{-\infty}^{t_j} F(x)\,\mathrm{d}x + \sum_{i=2}^{j}\left(\frac{(n)_{i-1}}{(k)_{i-1}}\int_{-\infty}^{t_{j+1-i}} F^{i-1}(x)\,\mathrm{d}x - \frac{(n)_i}{(k)_i}\int_{-\infty}^{t_{j+1-i}} F^i(x)\,\mathrm{d}x\right).\end{aligned}$$

Since $t_{j+1-i} = F^{-1}(\gamma_{j+1-i}) = F^{-1}(a_i)$ and

$$\frac{(n)_{i-1}}{(k)_{i-1}}\,a_i^{i-1} = \frac{(n)_i}{(k)_i}\,a_i^i,$$

applying identity (10) to each integral yields

$$\mathbb{E}X_{k:n} \geqslant \frac{n}{k}\int_0^{k/n} F^{-1}(u)\,\mathrm{d}u + \sum_{i=2}^{j}\frac{(n)_i}{(k)_i}\int_0^{a_i} u^{i-2}(iu-(i-1)a_i)F^{-1}(u)\,\mathrm{d}u \tag{16}$$

and one more application of identity (11) completes the proof. □

(ii) This follows easily by applying (13) to the j-independent-G sample $Y_i = -X_i$, $i=1,2,\dots,n$, noting that $X_{k:n} = -Y_{n+1-k:n}$, $G(x) = 1-F(-x_-)$, and $G^{-1}(u) = -F^{-1}((1-u)_+)$, which is equal to $-F^{-1}(1-u)$ a.e. in $(0,1)$.

Observe that (16) is a version of (13) containing only single integrals. Similarly, the upper bound (14) can be written as a sum of single integrals. Obviously, as j increases, the lower bound in (13) increases and the upper bound in (14) decreases (the a_i's and b_i's do not depend on j), yielding improved estimates. It should be noted, however, that the present approach is neither sharp nor complete for the general j-independent-F sample with $j \geqslant 2$ (e.g., (8) and (13) do not provide any bounds for $X_{k:n}$ when $k<j$). Also note that, by construction, Kemperman's (1997) distribution bounds are sharp, and thus

$$G_{k:n;j}(x) \leqslant H_{k:n;j}(x), \quad \text{for all } x,$$

where $H_{k:n;j}$ is the distribution in the RHS of (8) and $G_{k:n;j}$ is the corresponding one described implicitly by Kemperman (1997, pp. 301–302). Therefore, by using $H_{k:n;j}$ in place of $G_{k:n;j}$ in Theorem 4.1, one could derive considerably better expectation bounds; this procedure, however, seems to be intractable, since no closed form is available for $G_{k:n;j}$ (e.g., see the form of $G_{k:n;2}$ given in (9)), in contrast to the relatively simple form of $H_{k:n;j}$ discussed here.

Perhaps the most interesting case arises when the n-variate distribution consists of pairwise i.i.d. marginals; this is the case of a 2-independent-F sample. Taking $j=2$ in Theorem 4.1, we can immediately derive the corresponding expectation bounds as follows: for $k \geqslant 2$,

$$\mathbb{E}X_{k:n} \geqslant \frac{n}{k}\int_0^{k/n} F^{-1}(u)\,\mathrm{d}u + \frac{n(n-1)}{k(k-1)}\iint_{0<u<v<(k-1)/(n-1)} (F^{-1}(v)-F^{-1}(u))\,\mathrm{d}v\,\mathrm{d}u \tag{17}$$

and similarly, for $k \leqslant n-1$,

$$\begin{aligned}\mathbb{E}X_{k:n} \leqslant{}& \frac{n}{n+1-k}\int_{(k-1)/n} F^{-1}(u)\,\mathrm{d}u \\ &- \frac{n(n-1)}{(n+1-k)(n-k)}\iint_{(k-1)/(n-1)<u<v<1} (F^{-1}(v)-F^{-1}(u))\,\mathrm{d}v\,\mathrm{d}u.\end{aligned} \tag{18}$$

Bound (17) is obviously sharper than (5) for both $j=1$ and $j=2$, and the bounds coincide only in some very particular cases: for $j=1$ they coincide iff there exists some constant c such that $\mathbb{P}[X_1=c] \geqslant (k-1)/(n-1)$ and $\mathbb{P}[X_1<c]=0$; for $j=2$ the bounds are identical iff $k=n$ or $k<n$ and there exists some constant c such that $\mathbb{P}[X_1<c] \leqslant (k-1)/(n-1)$ and $\mathbb{P}[X_1 \leqslant c] \geqslant k/n$. The relation between all these expectation bounds is illustrated in the following example.

Example 3. For pairwise independent standard uniform variables and $2 \leqslant k \leqslant n-1$, (17) and (18) yield

$$\frac{k}{2n} + \frac{n(k-1)^2}{6k(n-1)^2} \leqslant \mathbb{E}X_{k:n} \leqslant 1 - \frac{n+1-k}{2n} - \frac{n(n-k)^2}{6(n+1-k)(n-1)^2},$$

which is better than both bounds of Example 1. Using Kemperman's distribution bound (9), one finds that for $n \geqslant 3$,

$$\mathbb{E}X_{2:n} \geqslant \frac{13n-19}{12(n-1)^2} = \frac{13}{12}n^{-1} + \mathrm{o}(n^{-1})$$

and for $n \geqslant 4$,

$$\mathbb{E}X_{3:n} \geqslant \frac{31n-46}{18(n-1)^2} = \frac{31}{18}n^{-1} + \mathrm{o}(n^{-1}).$$

Hence, as $n \to \infty$, these bounds are of the same order as those presented in Theorem 4.1. The situation is completely different, however, if k is large. For instance, (17) yields the (almost trivial) lower bound

$$\mathbb{E}X_{n:n} \geqslant \tfrac{2}{3},$$

while Kemperman's bound (arising from (9)) is

$$\mathbb{E}X_{n:n} \geqslant 1 + \frac{5n-2}{3(n-1)^2} - \frac{n}{(n-1)^2}\sum_{i=1}^{n}\frac{1}{i} = 1 - n^{-1}\log n + \mathrm{O}(n^{-1}).$$

Kemperman's assertions (stated without proof) imply that, in general, if $k/n \to p$ as $n \to \infty$ with $0 < p < 1$, then (using the notation of (9))

$$\lim_{n\to\infty}\min_{\mathscr{F}(2;n;F)}\mathbb{E}X_{k:n} = \lim_{n\to\infty}\max_{\mathscr{F}(2;n;F)}\mathbb{E}X_{k:n} = p,$$

where F is the standard uniform distribution. This result is, clearly, much stronger than those presented in this example and in Example 1; this happens, however, because our results are based on the notion of maximal stability of order j rather than on j-independence, and the former condition is much weaker than the latter. Moreover, the present approach enables us to derive closed expressions for the bounds in the case of j-independent-uniform samples (as in Example 1). For instance:

$$\mathbb{E}X_{k:n} \geqslant \sum_{i=1}^{j}\frac{1}{i(i+1)}\frac{(n)_i}{(k)_i}\left(\frac{k+1-i}{n+1-i}\right)^{i+1},$$

for any $k \geqslant j$. If, however, $k/n \to p$ as $n \to \infty$, then the resulting bound approaches $pj/(j+1)$ (as in Example 1), which is again poor.

5. Concluding remarks

If the random variables $X_1, X_2, \ldots, X_n$ are non-negative, $R_{k:n}(x) = 1 - F_{k:n}(x)$ is the reliability of the k-out-of-n system with (possibly dependent) components $X_1, X_2, \ldots, X_n$, and $X_{k:n}$ is the corresponding failure time. Therefore, (1), (2) and (5), (6) provide sharp bounds for the reliability and the mean time to failure for systems of this kind, provided that the components of the system are maximally (minimally) stable of order j; this occurs if any j of the components creates the same parallel (serial) system (i.e., the reliability of any j-parallel (serial) system does not vary).

It should be noted that any sample $X_1, X_2, \ldots, X_n$ can be translated to an exchangeable sample $Y_1, Y_2, \ldots, Y_n$ with the same order statistics (for example, take $Y_i = X_{\pi(i)}$, $i = 1, 2, \ldots, n$, here $(\pi(1), \pi(2), \ldots, \pi(n))$ is a random permutation of $(1, 2, \ldots, n)$, as in the proof of Theorem 2.1). Then we can apply the results of Sections 2 and 3 (for all j), observing that

$$F_{(j)}(x) = \binom{n}{j}^{-1}\sum_{1\leqslant k_1<\cdots<k_j\leqslant n}\mathbb{P}[X_{k_1} \leqslant x, \ldots, X_{k_j} \leqslant x],$$

and, similarly,

$$G_{(j)}(x) = 1 - \binom{n}{j}^{-1}\sum_{1\leqslant k_1<\cdots<k_j\leqslant n}\mathbb{P}[X_{k_1} > x, \ldots, X_{k_j} > x].$$

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