A TEXT-BOOK

OF

EUCLID'S ELEMENTS.

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EUCLID'S ELEMENTS

FOR THE USE OF SCHOOLS

BOOKS I.—VI. AND XI.

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London
MACMILLAN AND CO., LIMITED
NEW YORK: THE MACMILLAN COMPANY
1898

RICHARD CLAY AND SONS, LIMITED,
LONDON AND BUNGAY

First Edition, 1888. *Second Edition (Book XI. added)*, 1889.
Reprinted 1890, 1891, 1892, 1893, 1894, 1895 (*Twice*), 1896, 1897, 1898 (*Twice*).

PREFACE TO THE FIRST EDITION.

THIS volume contains the first Six Books of Euclid's Elements, together with Appendices giving the most important elementary developments of Euclidean Geometry.

The text has been carefully revised, and special attention given to those points which experience has shewn to present difficulties to beginners.

In the course of this revision the Enunciations have been altered as little as possible; and, except in Book V., very few departures have been made from Euclid's proofs: in each case changes have been adopted only where the old text has been generally found a cause of difficulty; and such changes are for the most part in favour of well-recognised alternatives.

For example, the ambiguity has been removed from the Enunciations of Propositions 18 and 19 of Book I.: the fact that Propositions 8 and 26 establish the complete identical equality of the two triangles considered has been strongly urged; and thus the redundant step has been removed from Proposition 34. In Book II. Simson's arrangement of Proposition 13 has been abandoned for a well-known alternative proof. In Book III. Proposition 25 is not given at length, and its place is taken by a simple equivalent. Propositions 35 and 36 have been treated generally, and it has not been thought necessary to do more than call attention in a note to the special cases. Finally, in Book VI. we have adopted an alternative proof of Proposition 7, a theorem which has been too much neglected, owing to the cumbrous form in which it has been usually given.

These are the chief deviations from the ordinary text as regards method and arrangement of proof: they are points familiar as difficulties to most teachers, and to name them indicates sufficiently, without further enumeration, the general principles which have guided our revision.

A few alternative proofs of difficult propositions are given for the convenience of those teachers who care to use them.

With regard to Book V. we have established the principal propositions, both from the algebraical and geometrical definitions of ratio and proportion, and we have endeavoured to bring out clearly the distinction between these two modes of treatment.

In compiling the geometrical section of Book V. we have followed the system first advocated by the late Professor De Morgan; and here we derived very material assistance from the exposition of the subject given in the text-book of the Association for the Improvement of Geometrical Teaching. To this source we are indebted for the improved and more precise wording of definitions (as given on pages 286, 288 to 291), as well as for the order and substance of most of the propositions which appear between pages 297 and 306. But as we have not (except in the points above mentioned) adhered verbally to the text of the Association, we are anxious, while expressing in the fullest manner our obligation to their work, to exempt the

Association from all responsibility for our treatment of the subject.

One purpose of the book is to gradually familiarise the student with the use of legitimate symbols and abbreviations; for a geometrical argument may thus be thrown into a form which is not only more readily seized by an advanced reader, but is useful as a guide to the way in which Euclid's propositions may be handled in written work. On the other hand, we think it very desirable to defer the introduction of symbols until the beginner has learnt that they can only be properly used in Pure Geometry as abbreviations for verbal argument: and we hope thus to prevent the slovenly and inaccurate habits which are very apt to arise from their employment before this principle is fully recognised.

Accordingly in Book I. we have used no contractions or symbols of any kind, though we have introduced verbal alterations into the text wherever it appeared that conciseness or clearness would be gained.

In Book II. abbreviated forms of constantly recurring words are used, and the phrases *therefore* and *is equal to* are replaced by the usual symbols.

In the Third and following Books, and in additional matter throughout the whole, we have employed all such signs and abbreviations as we believe to add to the clearness of the reasoning, care being taken that the symbols chosen are compatible with a rigorous geometrical method, and are recognised by the majority of teachers.

It must be understood that our use of symbols, and the removal of unnecessary verbiage and repetition, by no means implies a desire to secure brevity at all hazards. On the contrary, nothing appears to us more mischievous than an abridgement which is attained by omitting steps, or condensing two or more steps into one. Such uses spring from the pressure of examinations; but an examination is not, or ought not to be, a mere race; and while we wish to indicate generally in the later books how a geometrical argument may be abbreviated for the purposes of written work, we have not thought well to reduce the propositions to the bare skeleton so often presented to an Examiner. Indeed it does not follow that the form most suitable for the page of a text-book is also best adapted to examination purposes; for the object to be attained in each case is entirely different. The text-book should present the argument in the clearest possible manner to the mind of a reader to whom it is new: the written proposition need only convey to the Examiner the assurance that the proposition has been thoroughly grasped and remembered by the pupil.

From first to last we have kept in mind the undoubted fact that a very small proportion of those who study Elementary Geometry, and study it with profit, are destined to become mathematicians in any real sense; and that to a large majority of students, Euclid is intended to serve not so much as a first lesson in *mathematical* reasoning, as the first, and sometimes the only, model of formal and rigid argument presented in an elementary education.

This consideration has determined not only the full treatment of the earlier Books, but the retention of the formal, if somewhat cumbrous, methods of Euclid in many places where proofs of greater brevity and mathematical elegance are available.

We hope that the additional matter introduced into the book will provide sufficient exercise for pupils whose study of Euclid is preliminary to a mathematical education.

The questions distributed through the text follow very easily from the propositions to which they are attached, and we think that teachers are likely to find in them all that is needed for an average pupil reading the subject for the first time.

The Theorems and Examples at the end of each Book contain questions of a slightly more difficult type: they have been very carefully classified and arranged, and brought into close connection with typical examples worked out either partially or in full; and it is hoped that this section of the book, on which much thought has been expended, will do something towards removing that extreme want of freedom in solving deductions that is so commonly found even among students who have a good knowledge of the text of Euclid.

In the course of our work we have made ourselves acquainted with most modern English books on Euclidean Geometry: among these we have already expressed our special indebtedness to the text-book recently published by the Association for the Improvement of Geometrical Teaching; and we must also mention the Edition of Euclid's Elements prepared by Dr. J. S. Mackay, whose historical notes and frequent references to original authorities have been of the utmost service to us.

Our treatment of Maxima and Minima on page 239 is based upon suggestions derived from a discussion of the subject which took place at the annual meeting of the Geometrical Association in January 1887.

Of the Riders and Deductions some are original; but the greater part have been drawn from that large store of floating material which has furnished Examination Papers for the last 30 years, and must necessarily form the basis of any elementary collection. Proofs which have been found in two or more books without acknowledgement have been regarded as common property.

As regards figures, in accordance with a usage not uncommon in recent editions of Euclid, we have made a distinction between given lines and lines of construction.

Throughout the book we have italicised those deductions on which we desired to lay special stress as being in themselves important geometrical results: this arrangement we think will be useful to teachers who have little time to devote to riders, or who wish to sketch out a suitable course for revision.

We have in conclusion to tender our thanks to many of our friends for the valuable criticism and advice which we received from them as the book was passing through the press, and especially to the Rev. H. C. Watson, of Clifton College, who added to these services much kind assistance in the revision of proof-sheets.

H. S. HALL,
F. H. STEVENS.

July, 1888.

PREFACE TO THE SECOND EDITION.

In the Second Edition the text of Books I—VI. has been revised; and at the request of many teachers we have added the first twenty-one Propositions of Book XI. together with a collection of Theorems and Examples illustrating the elements of Solid Geometry.

September, 1889.

CONTENTS.

BOOK I.

Theorems and Examples on Book I.

BOOK II.

BOOK III.

Theorems and Examples on Book III.

BOOK IV.

Theorems and Examples on Book IV

BOOK V.

BOOK VI.

Theorems and Examples on Book VI.

BOOK XI.

EUCLID'S ELEMENTS.

BOOK I.

DEFINITIONS.

1. A **point** is that which has position, but no magnitude.

2. A **line** is that which has length without breadth.

The extremities of a line are points, and the intersection of two lines is a point.

3. A **straight line** is that which lies evenly between its extreme points.

Any portion cut off from a straight line is called a **segment** of it.

4. A **surface** is that which has length and breadth, but no thickness.

The boundaries of a surface are lines.

5. A **plane surface** is one in which any two points being taken, the straight line between them lies wholly in that surface.

A plane surface is frequently referred to simply as a plane.

NOTE. Euclid regards a point merely as a *mark of position*, and he therefore attaches to it no idea of size and shape.

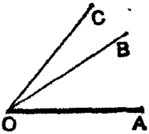
Similarly he considers that the properties of a line arise only from its *length* and *position*, without reference to that minute breadth which every line must really have *if actually drawn*, even though the most perfect instruments are used.

The definition of a surface is to be understood in a similar way.

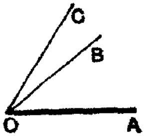
6. **A plane angle** is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.

The point at which the straight lines meet is called the **vertex** of the angle, and the straight lines themselves the **arms** of the angle.

When several angles are at one point O, any one of them is expressed by three letters, of which the letter that refers to the vertex is put between the other two. Thus if the straight lines OA, OB, OC meet at the point O, the angle contained by the straight lines OA, OB is named the angle AOB or BOA; and the angle contained by OA, OC is named the angle AOC or COA. Similarly the angle contained by OB, OC is referred to as the angle BOC or COB. But if there be only one angle at a point, it may be expressed by a single letter, as *the angle at* O.



Of the two straight lines OB, OC shewn in the adjoining figure, we recognize that OC is *more inclined* than OB to the straight line OA: this we express by saying that the angle AOC is greater than the angle AOB. Thus an angle must be regarded as having *magnitude*.

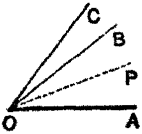


It should be observed that the angle AOC is the sum of the angles AOB and BOC; and that AOB is the difference of the angles AOC and BOC.

The beginner is cautioned against supposing that the size of an angle is altered either by increasing or diminishing the length of its arms.

[Another view of an angle is recognized in many branches of mathematics; and though not employed by Euclid, it is here given because it furnishes more clearly than any other a conception of what is meant by the *magnitude* of an angle.

Suppose that the straight line OP in the figure is capable of revolution about the point O, like the hand of a watch, but in the opposite direction; and suppose that in this way it has passed successively from the position OA to the positions occupied by OB and OC.

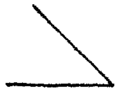


Such a line must have undergone *more turning* in passing from OA to OC, than in passing from OA to OB; and consequently the angle AOC is said to be greater than the angle AOB.]

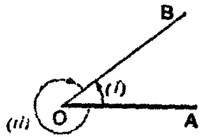
7. When a straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is called a **right angle**; and the straight line which stands on the other is called a **perpendicular** to it.

8. An **obtuse angle** is an angle which is greater than one right angle, but less than two right angles.

9. An **acute angle** is an angle which is less than a right angle.



[In the adjoining figure the straight line OB may be supposed to have arrived at its present position, from the position occupied by OA, by revolution about the point O in *either* of the two directions indicated by the arrows: thus two straight lines drawn from a point may be considered as forming *two* angles, (marked (i) and (ii) in the figure) of which the greater (ii) is said to be **reflex**.



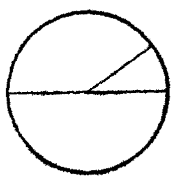
If the arms OA, OB are in the same straight line, the angle formed by them on either side is called a **straight angle**.]

10. Any portion of a plane surface bounded by one or more lines, straight or curved, is called a **plane figure**.

The sum of the bounding lines is called the **perimeter** of the figure.

Two figures are said to be equal in **area**, when they enclose equal portions of a plane surface.

11. A **circle** is a plane figure contained by one line, which is called the **circumference**, and is such that all straight lines drawn from a certain point within the figure to the circumference are equal to one another: this point is called the **centre** of the circle.



A **radius** of a circle is a straight line drawn from the centre to the circumference.

12. A **diameter** of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

13. A **semicircle** is the figure bounded by a diameter of a circle and the part of the circumference cut off by the diameter.

14. A **segment of a circle** is the figure bounded by a straight line and the part of the circumference which it cuts off.

15. **Rectilineal figures** are those which are bounded by straight lines.

16. A **triangle** is a plane figure bounded by *three* straight lines.

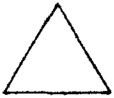
Any one of the angular points of a triangle may be regarded as its **vertex**; and the opposite side is then called the **base**.

17. A **quadrilateral** is a plane figure bounded by *four* straight lines.

The straight line which joins opposite angular points in a quadrilateral is called a **diagonal**.

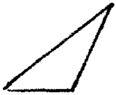
18. A **polygon** is a plane figure bounded by more than four straight lines.

19. An **equilateral triangle** is a triangle whose three sides are equal.

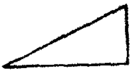


20. An **isosceles triangle** is a triangle two of whose sides are equal.

21. A **scalene triangle** is a triangle which has three unequal sides.

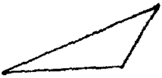


22. A **right-angled triangle** is a triangle which has a right angle.



The side opposite to the right angle in a right-angled triangle is called the **hypotenuse**.

23. An **obtuse-angled triangle** is a triangle which has an obtuse angle.



24. An **acute-angled triangle** is a triangle which has *three* acute angles.

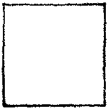
[It will be seen hereafter (Book I. Proposition 17) *that every triangle must have at least two acute angles.*]

25. **Parallel straight lines** are such as, being in the same plane, do not meet, however far they are produced in either direction.

26. A **Parallelogram** is a four-sided figure which has its opposite sides parallel.

27. A **rectangle** is a parallelogram which has one of its angles a right angle.

28. A **square** is a four-sided figure which has all its sides equal and all its angles right angles.



[It may easily be shewn that if a quadrilateral has all its sides equal and *one* angle a right angle, then *all* its angles will be right angles.]

29. A **rhombus** is a four-sided figure which has all its sides equal, but its angles are not right angles.

30. A **trapezium** is a four-sided figure which has *two* of its sides parallel.

ON THE POSTULATES.

In order to effect the *constructions* necessary to the study of geometry, it must be supposed that certain instruments are available; but it has always been held that such instruments should be as few in number, and as simple in character as possible.

For the purposes of the first Six Books *a straight ruler* and a pair of compasses are all that are needed; and in the following **Postulates**, or requests, Euclid demands the use of such instruments, and assumes that they suffice, theoretically as well as practically, to carry out the processes mentioned below.

POSTULATES.

Let it be granted,

1. That a straight line may be drawn from any one point to any other point.

When we draw a straight line from the point A to the point B, we are said to *join* AB.

2. That a *finite*, that is to say, a terminated straight line may be produced to any length in that straight line.

3. That a circle may be described from any centre, at any distance from that centre, that is, with a radius equal to any finite straight line drawn from the centre.

It is important to notice that the Postulates include no means of *direct measurement:* hence the straight ruler is not supposed to be *graduated;* and the compasses, in accordance with Euclid's use, are not to be employed for *transferring distances* from one part of a figure to another.

ON THE AXIOMS.

The science of Geometry is based upon certain simple statements, the truth of which is assumed at the outset to be self-evident.

These self-evident truths, called by Euclid *Common Notions*, are now known as the **Axioms**.

The necessary characteristics of an Axiom are

(i) That it should be *self-evident;* that is, that its truth should be immediately accepted without proof.

(ii) That it should be *fundamental;* that is, that its truth should not be derivable from any other truth more simple than itself.

(iii) That it should supply a basis for the establishment of further truths.

These characteristics may be summed up in the following definition.

DEFINITION. An **Axiom** is a self-evident truth, which neither requires nor is capable of proof, but which serves as a foundation for future reasoning.

Axioms are of two kinds, *general* and *geometrical.*

General Axioms apply to *magnitudes of all kinds.* Geometrical Axioms refer exclusively to *geometrical magnitudes,* such as have been already indicated in the definitions.

GENERAL AXIOMS.

1. Things which are equal to the same thing are equal to one another.

2. If equals be added to equals, the wholes are equal.

3. If equals be taken from equals, the remainders are equal.

4. If equals be added to unequals, the wholes are unequal, the greater sum being that which includes the greater of the unequals.

5. If equals be taken from unequals, the remainders are unequal, the greater remainder being that which is left from the greater of the unequals.

6. Things which are double of the same thing, or of equal things, are equal to one another.

7. Things which are halves of the same thing, or of equal things, are equal to one another.

9.* The whole is greater than its part.

* To preserve the classification of general and geometrical axioms, we have placed Euclid's *ninth* axiom before the *eighth.*

Geometrical Axioms.

8. Magnitudes which can be made to coincide with one another, are equal.

This axiom affords the ultimate test of the equality of two geometrical magnitudes. It implies that any line, angle, or figure, may be supposed to be taken up from its position, and without change in size or form, laid down upon a second line, angle, or figure, for the purpose of comparison.

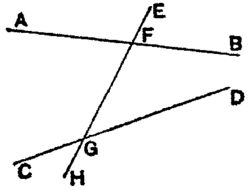
This process is called **superposition**, and the first magnitude is said to be **applied** to the other.

10. Two straight lines cannot enclose a space.

11. All right angles are equal.

[The statement that all right angles are equal, admits of proof, and is therefore perhaps out of place as an Axiom.]

12. If a straight line meet two straight lines so as to make the interior angles on one side of it together less than two right angles, these straight lines will meet if continually produced on the side on which are the angles which are together less than two right angles.



That is to say, if the two straight lines AB and CD are met by the straight line EH at F and G, in such a way that the angles BFG, DGF are together less than two right angles, it is asserted that AB and CD will meet if continually produced in the direction of B and D.

[Axiom 12 has been objected to on the double ground that it cannot be considered self-evident, and that its truth may be deduced from simpler principles. It is employed for the first time in the 29th Proposition of Book I., where a short discussion of the difficulty will be found.

The converse of this Axiom is proved in Book I. Proposition 17.]

INTRODUCTORY.

Plane Geometry deals with the properties of all lines and figures that may be drawn upon a plane surface.

Euclid in his first Six Books confines himself to the properties of straight lines, rectilineal figures, and circles.

The *Definitions* indicate the subject-matter of these books: the *Postulates and Axioms* lay down the fundamental principles which regulate all investigation and argument relating to this subject-matter.

Euclid's method of exposition divides the subject into a number of separate discussions, called **propositions**; each proposition, though in one sense complete in itself, is derived from results previously obtained, and itself leads up to subsequent propositions.

Propositions are of two kinds, **Problems** and **Theorems**.

A **Problem** proposes to effect some geometrical construction, such as to draw some particular line, or to construct some required figure.

A **Theorem** proposes to demonstrate some geometrical truth.

A Proposition consists of the following parts:

The General Enunciation, the Particular Enunciation, the Construction, and the Demonstration or Proof.

(1) The **General Enunciation** is a preliminary statement, describing in general terms the purpose of the proposition.

In a *problem* the Enunciation states the construction which it is proposed to effect: it therefore names first the **Data**, or things given, secondly the **Quæsita**, or things required.

In a *theorem* the Enunciation states the property which it is proposed to demonstrate: it names first, the **Hypothesis**, or the conditions assumed; secondly, the **Conclusion**, or the assertion to be proved.

(ii) The **Particular Enunciation** repeats in special terms the statement already made, and refers it to a diagram, which enables the reader to follow the reasoning more easily.

(iii) The **Construction** then directs the drawing of such straight lines and circles as may be required to effect the purpose of a problem, or to prove the truth of a theorem.

(iv) Lastly, the **Demonstration** proves that the object proposed in a problem has been accomplished, or that the property stated in a theorem is true.

Euclid's reasoning is said to be **Deductive**, because by a connected chain of argument it **deduces** new truths from truths already proved or admitted.

The initial letters Q.E.F., placed at the end of a problem, stand for **Quod erat Faciendum**, *which was to be done.*

The letters Q.E.D. are appended to a theorem, and stand for **Quod erat Demonstrandum**, *which was to be proved.*

A Corollary is a statement the truth of which follows readily from an established proposition; it is therefore appended to the proposition as an inference or deduction, which usually requires no further proof.

The following symbols and abbreviations may be employed in writing out the propositions of Book I., though their use is not recommended to beginners.

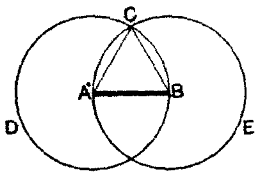
$\therefore$	*for*	therefore,	par^{l} (or $\parallel$)	*for*	parallel,
$=$	„	is, or are, equal to,	par^{m}	„	parallelogram,
$\angle$	„	angle,	sq.	„	square,
rt. $\llcorner$	„	right angle,	rectil.	„	rectilineal,
$\triangle$	„	triangle,	st. line	„	straight line,
perp.	„	perpendicular,	pt.	„	point;

and all obvious contractions of words, such as opp., adj., diag., &c., for opposite, adjacent, diagonal, &c.

SECTION I.

PROPOSITION 1. PROBLEM.

To describe an equilateral triangle on a given finite straight line.



Let AB be the given straight line.

It is required to describe an equilateral triangle on AB.

Construction. From centre A, with radius AB, describe the circle BCD. *Post.* 3.

From centre B, with radius BA, describe the circle ACE. *Post.* 3.

From the point C at which the circles cut one another, draw the straight lines CA and CB to the points A and B. *Post.* 1.

Then shall ABC be an equilateral triangle.

Proof. Because A is the centre of the circle BCD,
therefore AC is equal to AB. *Def.* 11.

And because B is the centre of the circle ACE,
therefore BC is equal to BA. *Def.* 11.

But it has been shewn that AC is equal to AB;
therefore AC and BC are each equal to AB.

But things which are equal to the same thing are equal to one another. *Ax.* 1.

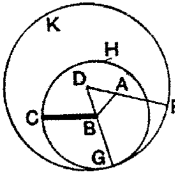
Therefore AC is equal to BC.

Therefore CA, AB, BC are equal to one another.

Therefore the triangle ABC is equilateral;
and it is described on the given straight line AB. Q.E.F.

PROPOSITION 2. PROBLEM.

From a given point to draw a straight line equal to a given straight line.



Let A be the given point, and BC the given straight line.

It is required to draw from the point A a straight line equal to BC.

Construction. Join AB; *Post.* 1.

and on AB describe an equilateral triangle DAB. I. 1.

From centre B, with radius BC, describe the circle CGH. *Post.* 3.

Produce DB to meet the circle CGH at G. *Post.* 2.

From centre D, with radius DG, describe the circle GKF.

Produce DA to meet the circle GKF at F. *Post.* 2.

Then AF shall be equal to BC.

Proof. Because B is the centre of the circle CGH,

therefore BC is equal to BG. *Def.* 11.

And because D is the centre of the circle GKF,

therefore DF is equal to DG; *Def.* 11.

and DA, DB, parts of them are equal; *Def.* 19.

therefore the remainder AF is equal to the remainder BG. *Ax.* 3.

And it has been shewn that BC is equal to BG;

therefore AF and BC are each equal to BG.

But things which are equal to the same thing are equal to one another. *Ax.* 1.

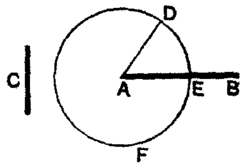
Therefore AF is equal to BC;

and it has been drawn from the given point A. Q.E.F.

[This Proposition is rendered necessary by the restriction, tacitly imposed by Euclid, that compasses shall not be used to *transfer distances*.]

Proposition 3. Problem.

From the greater of two given straight lines to cut off a part equal to the less.



Let AB and C be the two given straight lines, of which B is the greater.

It is required to cut off from AB a part equal to C.

Construction. From the point A draw the straight line AD equal to C; I. 2.
and from centre A, with radius AD, describe the circle DEF, meeting AB at E. *Post.* 3.

Then AE shall be equal to C.

Proof. Because A is the centre of the circle DEF,
therefore AE is equal to AD. *Def.* 11.
But C is equal to AD. *Constr.*
Therefore AE and C are each equal to AD.
Therefore AE is equal to C;
and it has been cut off from the given straight line AB.

Q.E.F

EXERCISES.

1. On a given straight line describe an isosceles triangle having each of the equal sides equal to a given straight line.

2. On a given base describe an isosceles triangle having each of the equal sides double of the base.

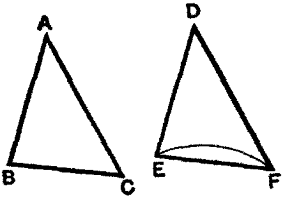
3. In the figure of I. 2, if AB is equal to BC, shew that D, the vertex of the equilateral triangle, will fall on the circumference of the circle CGH.

Obs. Every triangle has six **parts**, namely its three sides and three angles.

Two triangles are said to be **equal in all respects**, when they can be made to coincide with one another by *superposition* (see note on Axiom 8), and in this case each part of the one is equal to a corresponding part of the other.

PROPOSITION 4. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal; then shall their bases or third sides be equal, and the triangles shall be equal in area, and their remaining angles shall be equal, each to each, namely those to which the equal sides are opposite: that is to say, the triangles shall be equal in all respects.



Let ABC, DEF be two triangles, which have the side AB equal to the side DE, the side AC equal to the side DF, and the contained angle BAC equal to the contained angle EDF.

Then shall the base BC be equal to the base EF, and the triangle ABC shall be equal to the triangle DEF in area; and the remaining angles shall be equal, each to each, to which the equal sides are opposite,
namely the angle ABC to the angle DEF,
and the angle ACB to the angle DFE.

For if the triangle ABC be applied to the triangle DEF,
so that the point A may be on the point D,
and the straight line AB along the straight line DE,
then because AB is equal to DE, *Hyp.*
therefore the point B must coincide with the point E.

And because AB falls along DE,
and the angle BAC is equal to the angle EDF, *Hyp.*
therefore AC must fall along DF.
And because AC is equal to DF, *Hyp.*
therefore the point C must coincide with the point F.
Then B coinciding with E, and C with F,
the base BC must coincide with the base EF;
for if not, two straight lines would enclose a space; which is impossible. *Ax.* 10.

Thus the base BC coincides with the base EF, and is therefore equal to it. *Ax.* 8.

And the triangle ABC coincides with the triangle DEF, and is therefore equal to it in area. *Ax.* 8.

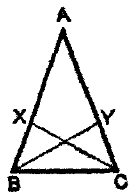
And the remaining angles of the one coincide with the remaining angles of the other, and are therefore equal to them,
namely, the angle ABC to the angle DEF,
and the angle ACB to the angle DFE.

That is, the triangles are equal in all respects. Q. E. D.

NOTE. It follows that two triangles which are equal in their several parts are equal also in *area;* but it should be observed that equality of area in two triangles does not necessarily imply equality in their several parts: that is to say, triangles may be equal in *area,* without being of the same *shape.*

Two triangles which are equal in all respects have *identity of form and magnitude,* and are therefore said to be **identically equal**, or **congruent.**

The following application of Proposition 4 anticipates the chief difficulty of Proposition 5.



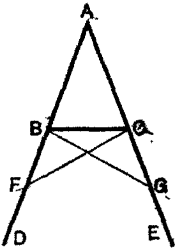
In the equal sides AB, AC *of* an isosceles triangle ABC, the points X and Y are taken, so that AX is equal to AY; and BY and CX are joined.

Shew that BY is equal to CX.

In the two triangles XAC, YAB,
XA is equal to YA, and AC is equal to AB; *Hyp.*
that is, the two sides XA, AC are equal to the two sides YA, AB, each to each;
and the angle at A, which is contained by these sides, is common to both triangles:
therefore the triangles are equal in all respects; I. 4.
so that XC is equal to YB. Q.E.D.

PROPOSITION 5. THEOREM.

The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced, the angles on the other side of the base shall also be equal to one another.



Let ABC be an isosceles triangle, having the side AB equal to the side AC, and let the straight lines AB, AC be produced to D and E:

then shall the angle ABC be equal to the angle ACB, and the angle CBD to the angle BCE.

Construction. In BD take any point F;
and from AE the greater cut off AG equal to AF the less, I. 3.
Join FC, GB.

Proof. Then in the triangles FAC, GAB,

Because { FA is equal to GA, *Constr.*
and AC is equal to AB, *Hyp.*
also the contained angle at A is common to the two triangles;

therefore the triangle FAC is equal to the triangle GAB in all respects; I. 4.

that is, the base FC is equal to the base GB,
and the angle ACF is equal to the angle ABG,
also the angle AFC is equal to the angle AGB.

Again, because the whole AF is equal to the whole AG,
of which the parts AB, AC are equal, *Hyp.*
therefore the remainder BF is equal to the remainder CG.

Then in the two triangles BFC, CGB,

Because { BF is equal to CG, *Proved.*
and FC is equal to GB, *Proved.*
also the contained angle BFC is equal to the contained angle CGB, *Proved.* }

therefore the triangles BFC, CGB are equal in all respects;
so that the angle FBC is equal to the angle GCB,
and the angle BCF to the angle CBG. I. 4.

Now it has been shewn that the whole angle ABG is equal to the whole angle ACF,

and that parts of these, namely the angles CBG, BCF, are also equal;

therefore the remaining angle ABC is equal to the remaining angle ACB;

and these are the angles at the base of the triangle ABC.

Also it has been shewn that the angle FBC is equal to the angle GCB;

and these are the angles on the other side of the base. Q.E.D.

COROLLARY. *Hence if a triangle is equilateral it is also equiangular.*

EXERCISES.

1. AB is a given straight line and C a given point outside it: shew how to find any points in AB such that their distance from C shall be equal to a given length L. Can such points always be found?

2. If the vertex C and one extremity A of the base of an isosceles triangle are given, find the other extremity B, supposing it to lie on a given straight line PQ.

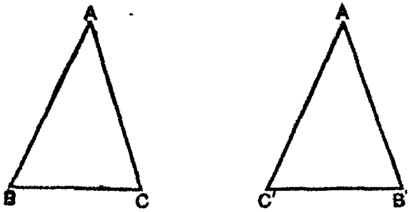
3. Describe a rhombus having given two opposite angular points A and C, and the length of each side.

4. AMNB is a straight line; on AB describe a triangle ABC such that the side AC shall be equal to AN and the side BC to MB.

5. In Prop. 2 the point A may be joined to *either* extremity of BC. Draw the figure and prove the proposition in the case when A is joined to C.

The following proof is sometimes given as a substitute for the first part of Proposition 5:

PROPOSITION 5. ALTERNATIVE PROOF.



Let ABC be an isosceles triangle, having AB equal to AC:
then shall the angle ABC be equal to the angle ACB.

Suppose the triangle ABC to be taken up, turned over and laid down again in the position A'B'C', where A'B', A'C', B'C' represent the new positions of AB, AC, BC.

Then A'B' is equal to A'C'; and A'B' is AB in its new position,
therefore AB is equal to A'C';
in the same way AC is equal to A'B';

and the included angle BAC is equal to the included angle C'A'B', for they are the same angle in different positions;

therefore the triangle ABC is equal to the triangle A'C'B' in all respects:
so that the angle ABC is equal to the angle A'C'B'. I. 4.

But the angle A'C'B' is the angle ACB in its new position;
therefore the angle ABC is equal to the angle ACB.

Q.E.D.

EXERCISES.

CHIEFLY ON PROPOSITIONS 4 AND 5.

1. Two circles have the same centre O; OAD and OBE are straight lines drawn to cut the smaller circle in A and B and the larger circle in D and E: prove that

(i) AD = BE. (ii) DB = EA.

(iii) The angle DAB is equal to the angle EBA.

(iv) The angle ODB is equal to the angle OEA.

2. ABCD is a square, and L, M, and N are the middle points of AB, BC, and CD: prove that

(i) LM = MN. (ii) AM = DM.

(iii) AN = AM. (iv) BN = DM.

[Draw a separate figure in each case.]

3. O is the centre of a circle and OA, OB are radii; OM divides the angle AOB into two equal parts and cuts the line AB in M: prove that AM = BM.

4. ABC, DBC are two isosceles triangles described on the same base BC but on opposite sides of it: prove that the angle ABD is equal to the angle ACD.

5. ABC, DBC are two isosceles triangles described on the same base BC, but on opposite sides of it: prove that if AD be joined, each of the angles BAC, BDC will be divided into two equal parts.

6. PQR, SQR are two isosceles triangles described on the same base QR, and on the same side of it: shew that the angle PQS is equal to the angle PRS, and that the line PS divides the angle QPR into two equal parts.

7. If in the figure of Exercise 5 the line AD meets BC in E, prove that BE = EC.

8. ABCD is a rhombus and AC is joined: prove that the angle DAB is equal to the angle DCB.

9. ABCD is a quadrilateral having the opposite sides BC, AD equal, and also the angle BCD equal to the angle ADC: prove that BD is equal to AC.

10. AB, AC are the equal sides of an isosceles triangle; L, M, N are the middle points of AB, BC, and CA respectively: prove that LM = MN.

Prove also that the angle ALM is equal to the angle ANM.

Definition. Each of two Theorems is said to be the **Converse** of the other, when the hypothesis of each is the conclusion of the other.

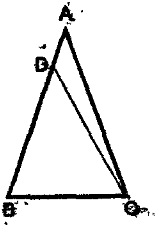
It will be seen, on comparing the hypotheses and conclusions of Props. 5 and 6, that each proposition is the converse of the other.

Note. Proposition 6 furnishes the first instance of an *indirect method of proof*, frequently used by Euclid. It consists in shewing that an absurdity must result from supposing the theorem to be otherwise than true. This form of demonstration is known as the **Reductio ad Absurdum**, and is most commonly employed in establishing the converse of some foregoing theorem.

It must not be supposed that the converse of a true theorem is itself necessarily true: for instance, it will be seen from Prop. 8, Cor. that if two triangles have their sides equal, each to each, then their angles will also be equal, each to each; but it may easily be shewn by means of a figure that the converse of this theorem is not necessarily true.

PROPOSITION 6. THEOREM.

If two angles of a triangle be equal to one another, then the sides also which subtend, or are opposite to, the equal angles, shall be equal to one another.



Let ABC be a triangle, having the angle ABC equal to the angle ACB:

then shall the side AC be equal to the side AB.

Construction. For if AC be not equal to AB,
one of them must be greater than the other.
If possible, let AB be the greater;
and from it cut off BD equal to AC. I. 3.
Join DC.

Proof. Then in the triangles DBC, ACB,

Because { DB is equal to AC, *Constr.*
and BC is common to both,
also the contained angle DBC is equal to the contained angle ACB; *Hyp.* }

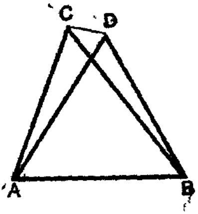
therefore the triangle DBC is equal in area to the triangle ACB, I. 4.
the part equal to the whole; which is absurd. *Ax.* 9.

Therefore AB is not unequal to AC;
that is, AB is equal to AC. Q.E.D.

COROLLARY. *Hence if a triangle is equiangular it is also equilateral.*

PROPOSITION 7. THEOREM.

On the same base, and on the same side of it, there cannot be two triangles having their sides which are terminated at one extremity of the base equal to one another, and likewise those which are terminated at the other extremity equal to one another.



If it be possible, on the same base AB, and on the same side of it, let there be two triangles ACB, ADB, having their sides AC, AD, which are terminated at A, equal to one another, and likewise their sides BC, BD, which are terminated at B, equal to one another.

CASE I. When the vertex of each triangle is without the other triangle.

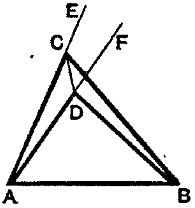
Construction. Join CD. *Post.* 1.

Proof. Then in the triangle ACD,
because AC is equal to AD, *Hyp.*
therefore the angle ACD is equal to the angle ADC. I. 5.

But the whole angle ACD is greater than its part, the angle BCD,
therefore also the angle ADC is greater than the angle BCD;
still more then is the angle BDC greater than the angle BCD.

Again, in the triangle BCD,
because BC is equal to BD, *Hyp.*
therefore the angle BDC is equal to the angle BCD: I. 5
but it was shewn to be greater; which is impossible.

CASE II. When one of the vertices, as D, is within the other triangle ACB.



Construction. As before, join CD; *Post.* 1.
and produce AC, AD to E and F. *Post.* 2.

Then in the triangle ACD, because AC is equal to AD, *Hyp.*
therefore the angles ECD, FDC, on the other side of the base, are equal to one another. I. 5.

But the angle ECD is greater than its part, the angle BCD;
therefore the angle FDC is also greater than the angle BCD:
still more then is the angle BDC greater than the angle BCD.

Again, in the triangle BCD,
because BC is equal to BD, *Hyp.*
therefore the angle BDC is equal to the angle BCD: I. 5.
but it has been shewn to be greater; which is impossible.

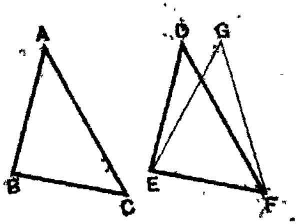
The case in which the vertex of one triangle is on a side of the other needs no demonstration.

Therefore AC cannot be equal to AD, and *at the same time*, BC equal to BD. Q.E.D.

NOTE. The sides AC, AD are called **conterminous** sides; similarly the sides BC, BD are conterminous.

PROPOSITION 8. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, then the angle which is contained by the two sides of the one shall be equal to the angle which is contained by the two sides of the other.



Let ABC, DEF be two triangles, having the two sides BA, AC equal to the two sides ED, DF, each to each, namely BA to ED, and AC to DF, and also the base BC equal to the base EF:

then shall the angle BAC be equal to the angle EDF.

Proof. For if the triangle ABC be applied to the triangle DEF, so that the point B may be on E, and the straight line BC along EF;

then because BC is equal to EF, *Hyp.*

therefore the point C must coincide with the point F.

Then, BC coinciding with EF,

it follows that BA and AC must coincide with ED and DF:

for if not, they would have a different situation, as EG, GF:

then, on the same base and on the same side of it there would be two triangles having their *conterminous* sides equal.

But this is impossible. I. 7.

Therefore the sides BA, AC coincide with the sides ED, DF.

That is, the angle BAC coincides with the angle EDF, and is therefore equal to it. *Ax.* 8.

Q. E. D.

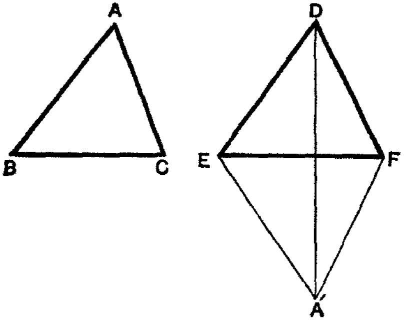
NOTE. In this Proposition the three sides of one triangle are given equal respectively to the three sides of the other; and from this it is shewn that the two triangles may be made *to coincide with one another*.

Hence we are led to the following important Corollary.

COROLLARY. *If in two triangles the three sides of the one are equal to the three sides of the other, each to each, then the triangles are equal in all respects.*

The following proof of Prop. 8 is worthy of attention as it is independent of Prop. 7, which frequently presents difficulty to a beginner.

PROPOSITION 8. ALTERNATIVE PROOF.



Let ABC and DEF be two triangles, which have the sides BA, AC equal respectively to the sides ED, DF, and the base BC equal to the base EF:

then shall the angle BAC be equal to the angle EDF.

For apply the triangle ABC to the triangle DEF, so that B may fall on E, and BC along EF, and so that the point A may be on the side of EF remote from D,

then C must fall on F, since BC is equal to EF.

Let A′EF be the new position of the triangle ABC.

If neither DF, FA′ nor DE, EA′ are in one straight line,

join DA′.

CASE I. When DA′ intersects EF.

Then because ED is equal to EA′,

therefore the angle EDA′ is equal to the angle EA′D. I. 5.

Again because FD is equal to FA′,

therefore the angle FDA′ is equal to the angle FA′D. I. 5.

Hence the whole angle EDF is equal to the whole angle EA′F;

that is, the angle EDF is equal to the angle BAC.

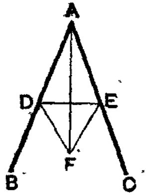
Two cases remain which may be dealt with in a similar manner: namely,

CASE II. When DA′ meets EF produced.

CASE III. When one pair of sides, as DF, FA′, are in one straight line.

PROPOSITION 9. PROBLEM.

To bisect a given angle, that is, to divide it into two equal parts.



Let BAC be the given angle:
it is required to bisect it.

Construction. In AB take any point D;
and from AC cut off AE equal to AD. I. 3.
Join DE;
and on DE, on the side remote from A, describe an equilateral triangle DEF. I. 1.
Join AF.
Then shall the straight line AF bisect the angle BAC.

Proof. For in the two triangles DAF, EAF,

Because { DA is equal to EA, *Constr.*
and AF is common to both;
and the third side DF is equal to the third side EF; *Def.* 19.

therefore the angle DAF is equal to the angle EAF. I. 8.
Therefore the given angle BAC is bisected by the straight line AF. Q.E.F.

EXERCISES.

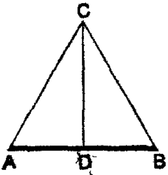
1. If in the above figure the equilateral triangle DFE were described on the same side of DE as A, what different cases would arise? And under what circumstances would the construction fail?

2. In the same figure, shew that AF also bisects the angle DFE.

3. Divide an angle into four equal parts.

PROPOSITION 10. PROBLEM.

To bisect a given finite straight line, that is, to divide it into two equal parts.



Let AB be the given straight line:
it is required to divide it into two equal parts.

Constr. On AB describe an equilateral triangle ABC, I. 1.
and bisect the angle ACB by the straight line CD, meeting AB at D. I. 9.

Then shall AB be bisected at the point D.

Proof. For in the triangles ACD, BCD,

Because { AC is equal to BC, *Def.* 19.
and CD is common to both;
also the contained angle ACD is equal to the contained angle BCD; *Constr.*

Therefore the triangles are equal in all respects:
so that the base AD is equal to the base BD. I. 4.
Therefore the straight line AB is bisected at the point D.

Q. E. F.

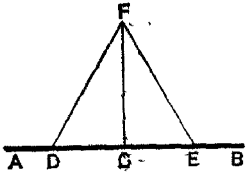
EXERCISES.

1. Shew that the straight line which bisects the vertical angle of an isosceles triangle, also bisects the base.

2. On a given base describe an isosceles triangle such that the sum of its equal sides may be equal to a given straight line.

PROPOSITION 11. PROBLEM.

To draw a straight line at right angles to a given straight line, from a given point in the same.



Let AB be the given straight line, and C the given point in it.

It is required to draw from the point C a straight line at right angles to AB.

Construction. In AC take any point D,
and from CB cut off CE equal to CD. I. 3.
On DE describe the equilateral triangle DFE. I. 1.
Join CF.

Then shall the straight line CF be at right angles to AB.

Proof. For in the triangles DCF, ECF,

Because { DC is equal to EC, *Constr.*
and CF is common to both;
and the third side DF is equal to the third side EF: *Def.* 19.

Therefore the angle DCF is equal to the angle ECF: I. 8.
and these are adjacent angles.

But when a straight line, standing on another straight line, makes the adjacent angles equal to one another, each of these angles is called a right angle; *Def.* 7.
therefore each of the angles DCF, ECF is a right angle.

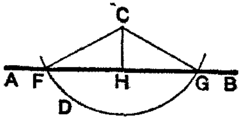
Therefore CF is at right angles to AB,
and has been drawn from a point C in it. Q.E.F.

EXERCISE.

In the figure of the above proposition, shew that *any* point in FC, or FC produced, is equidistant from D and E.

PROPOSITION 12. PROBLEM.

To draw a straight line perpendicular to a given straight line of unlimited length, from a given point without it.



Let AB be the given straight line, which may be produced in either direction, and let C be the given point without it.

It is required to draw from the point C a straight line perpendicular to AB.

Construction. On the side of AB remote from C take any point D;

and from centre C, with radius CD, describe the circle FDG, meeting AB at F and G. *Post.* 3.

Bisect FG at H; I. 10.

and join CH.

Then shall the straight line CH be perpendicular to AB.

Join CF and CG.

Proof. Then in the triangles FHC, GHC,

Because { FH is equal to GH, *Constr.*
and HC is common to both;
and the third side CF is equal to the third side CG, being radii of the circle FDG; *Def.* 11.

therefore the angle CHF is equal to the angle CHG; I. 8.

and these are adjacent angles.

But when a straight line, standing on another straight line, makes the adjacent angles equal to one another, each of these angles is called a right angle, and the straight line which stands on the other is called a perpendicular to it.

Therefore CH is a perpendicular drawn to the given straight line AB from the given point C without it. Q.E.F.

NOTE. The given straight line AB must be of unlimited length, that is, it must be capable of production to an indefinite length in either direction, to ensure its being intersected in two points by the circle FDG.

EXERCISES ON PROPOSITIONS 1 TO 12.

1. Shew that the straight line which joins the vertex of an isosceles triangle to the middle point of the base is perpendicular to the base.

2. Shew that the straight lines which join the extremities of the base of an isosceles triangle to the middle points of the opposite sides, are equal to one another.

3. Two given points in the base of an isosceles triangle are equidistant from the extremities of the base: shew that they are also equidistant from the vertex.

4. If the opposite sides of a quadrilateral are equal, shew that the opposite angles are also equal.

5. Any two isosceles triangles XAB, YAB stand on the same base AB: shew that the angle XAY is equal to the angle XBY; and that the angle AXY is equal to the angle BXY.

6. Shew that the opposite angles of a rhombus are bisected by the diagonal which joins them.

7. Shew that the straight lines which bisect the base angles of an isosceles triangle form with the base a triangle which is also isosceles.

8. ABC is an isosceles triangle having AB equal to AC; and the angles at B and C are bisected by straight lines which meet at O: shew that OA bisects the angle BAC.

9. Shew that the triangle formed by joining the middle points of the sides of an equilateral triangle is also equilateral.

10. The equal sides BA, CA of an isosceles triangle BAC are produced beyond the vertex A to the points E and F, so that AE is equal to AF; and FB, EC are joined: shew that FB is equal to EC.

11. Shew that the diagonals of a rhombus bisect one another at right angles.

12. In the equal sides AB, AC of an isosceles triangle ABC two points X and Y are taken, so that AX is equal to AY; and CX and BY are drawn intersecting in O: shew that

(i) the triangle BOC is isosceles;
(ii) AO bisects the vertical angle BAC;
(iii) AO, if produced, bisects BC at right angles.

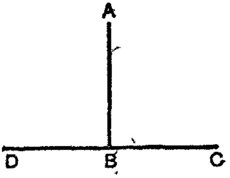
13. Describe an isosceles triangle, having given the base and the length of the perpendicular drawn from the vertex to the base.

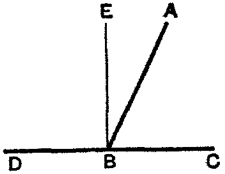
14. In a given straight line find a point that is equidistant from two given points.

In what case is this impossible?

PROPOSITION 13. THEOREM.

If one straight line stand upon another straight line, then the adjacent angles shall be either two right angles, or together equal to two right angles.





Let the straight line AB stand upon the straight line DC: then the adjacent angles DBA, ABC shall be either two right angles, or together equal to two right angles.

CASE I. For if the angle DBA is equal to the angle ABC, each of them is a right angle. *Def.* 7.

CASE II. But if the angle DBA is not equal to the angle ABC,

from B draw BE at right angles to CD. I. 11.

Proof. Now the angle DBA is made up of the two angles DBE, EBA;

to each of these equals add the angle ABC;

then the two angles DBA, ABC are together equal to the three angles DBE, EBA, ABC. *Ax.* 2.

Again, the angle EBC is made up of the two angles EBA, ABC;

to each of these equals add the angle DBE.

Then the two angles DBE, EBC are together equal to the three angles DBE, EBA, ABC. *Ax.* 2.

But the two angles DBA, ABC have been shewn to be equal to the same three angles;

therefore the angles DBA, ABC are together equal to the angles DBE, EBC. *Ax.* 1.

But the angles DBE, EBC are two right angles; *Constr.*

therefore the angles DBA, ABC are together equal to two right angles. Q.E.D.

DEFINITIONS.

(i) The **complement** of an acute angle is its *defect from* a right angle, that is, the angle by which it falls short of a right angle.

Thus two angles are **complementary**, when their sum is a right angle.

(ii) The **supplement** of an angle is its *defect from* two right angles, that is, the angle by which it falls short of two right angles.

Thus two angles are **supplementary**, when their sum is two right angles.

COROLLARY. *Angles which are complementary or supplementary to the same angle are equal to one another.*

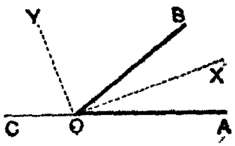
EXERCISES.

1. If the two exterior angles formed by producing a side of a triangle both ways are equal, shew that the triangle is isosceles.

2. *The bisectors of the adjacent angles which one straight line makes with another contain a right angle.*

NOTE. In the adjoining figure AOB is a given angle; and one of its arms AO is produced to C: the adjacent angles AOB, BOC are bisected by OX, OY.

Then OX and OY are called respectively the **internal** and **external bisectors** of the angle AOB.



Hence Exercise 2 may be thus enunciated:

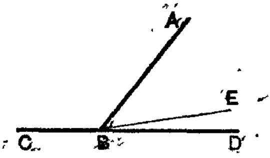
The internal and external bisectors of an angle are at right angles to one another.

3. Shew that the angles AOX and COY are complementary.

4. Shew that the angles BOX and COX are supplementary; and also that the angles AOY and BOY are supplementary.

PROPOSITION 14. THEOREM.

If, at a point in a straight line, two other straight lines, on opposite sides of it, make the adjacent angles together equal to two right angles, then these two straight lines shall be in one and the same straight line.



At the point B in the straight line AB, let the two straight lines BC, BD, on the opposite sides of AB, make the adjacent angles ABC, ABD together equal to two right angles:

then BD shall be in the same straight line with BC.

Proof. For if BD be not in the same straight line with BC, if possible, let BE be in the same straight line with BC.

Then because AB meets the straight line CBE,

therefore the adjacent angles CBA, ABE are together equal to two right angles. I. 13.

But the angles CBA, ABD are also together equal to two right angles. *Hyp.*

Therefore the angles CBA, ABE are together equal to the angles CBA, ABD. *Ax.* 11.

From each of these equals take the common angle CBA; then the remaining angle ABE is equal to the remaining angle ABD; the part equal to the whole; which is impossible.

Therefore BE is not in the same straight line with BC.

And in the same way it may be shewn that no other line but BD can be in the same straight line with BC.

Therefore BD is in the same straight line with BC. Q.E.D.

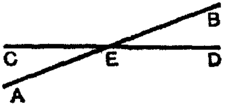
EXERCISE.

ABCD is a rhombus; and the diagonal AC is bisected at O. If O is joined to the angular points B and D; shew that OB and OD are in one straight line.

Obs. When two straight lines intersect at a point, four angles are formed; and any two of these angles *which are not adjacent*, are said to be **vertically opposite** to one another.

PROPOSITION 15. THEOREM.

If two straight lines intersect one another, then the vertically opposite angles shall be equal.



Let the two straight lines AB, CD cut one another at the point E:

then shall the angle AEC be equal to the angle DEB,

and the angle CEB to the angle AED.

Proof. Because AE makes with CD the adjacent angles CEA, AED,

therefore these angles are together equal to two right angles. I. 13.

Again, because DE makes with AB the adjacent angles AED, DEB,

therefore these also are together equal to two right angles.

Therefore the angles CEA, AED are together equal to the angles AED, DEB.

From each of these equals take the common angle AED;

then the remaining angle CEA is equal to the remaining angle DEB. *Ax.* 3.

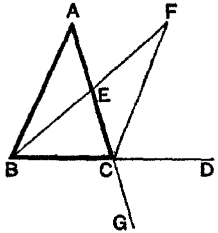
In a similar way it may be shewn that the angle CEB is equal to the angle AED. Q.E.D.

COROLLARY 1. *From this it is manifest that, if two straight lines cut one another, the angles which they make at the point where they cut, are together equal to four right angles.*

COROLLARY 2. *Consequently, when any number of straight lines meet at a point, the sum of the angles made by consecutive lines is equal to four right angles.*

PROPOSITION 16. THEOREM.

If one side of a triangle be produced, then the exterior angle shall be greater than either of the interior opposite angles.



Let ABC be a triangle, and let one side BC be produced to D: then shall the exterior angle ACD be greater than either of the interior opposite angles CBA, BAC.

Construction. Bisect AC at E: I. 10.
Join BE; and produce it to F, making EF equal to BE. I. 3.
Join FC.

Proof. Then in the triangles AEB, CEF,

Because { AE is equal to CE, *Constr.*
and EB to EF; *Constr.*
also the angle AEB is equal to the vertically opposite angle CEF; I. 15.

therefore the triangle AEB is equal to the triangle CEF in all respects: I. 4.

so that the angle BAE is equal to the angle ECF.

But the angle ECD is greater than its part, the angle ECF; therefore the angle ECD is greater than the angle BAE; that is, the angle ACD is greater than the angle BAC.

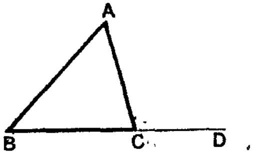
In a similar way, if BC be bisected, and the side AC produced to G, it may be shewn that the angle BCG is greater than the angle ABC.

But the angle BCG is equal to the angle ACD: I. 15.
therefore also the angle ACD is greater than the angle ABC.

Q. E. D.

Proposition 17. Theorem.

Any two angles of a triangle are together less than two right angles.



Let ABC be a triangle: then shall any two of its angles, as ABC, ACB, be together less than two right angles.

Construction. Produce the side BC to D.

Proof. Then because ACD is an exterior angle of the triangle ABC;

therefore it is greater than the interior opposite angle ABC. I. 16.

To each of these add the angle ACB:

then the angles ACD, ACB are together greater than the angles ABC, ACB. *Ax.* 4.

But the adjacent angles ACD, ACB are together equal to two right angles. I. 13.

Therefore the angles ABC, ACB are together less than two right angles.

Similarly it may be shewn that the angles BAC, ACB, as also the angles CAB, ABC, are together less than two right angles. Q. E. D.

Note. It follows from this Proposition that *every triangle must have at least two acute angles:* for if one angle is obtuse, or a right angle, each of the other angles must be less than a right angle.

EXERCISES.

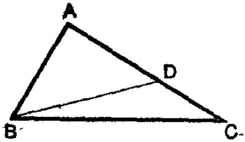
1. Enunciate this Proposition so as to shew that it is the converse of Axiom 12.

2. If any side of a triangle is produced both ways, the exterior angles so formed are together greater than two right angles.

3. Shew how a proof of Proposition 17 may be obtained by joining each vertex in turn to any point in the opposite side.

PROPOSITION 18. THEOREM.

If one side of a triangle be greater than another, then the angle opposite to the greater side shall be greater than the angle opposite to the less.



Let ABC be a triangle, in which the side AC is greater than the side AB :

then shall the angle ABC be greater than the angle ACB.

Construction. From AC, the greater, cut off a part AD equal to AB. I. 3.

Join BD.

Proof. Then in the triangle ABD,

because AB is equal to AD,

therefore the angle ABD is equal to the angle ADB. I. 5.

But the exterior angle ADB of the triangle BDC is greater than the interior opposite angle DCB, that is, greater than the angle ACB. I. 16.

Therefore also the angle ABD is greater than the angle ACB; still more then is the angle ABC greater than the angle ACB. Q.E.D.

Euclid enunciated Proposition 18 as follows:

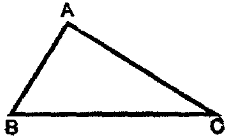
The greater side of every triangle has the greater angle opposite to it.

[This form of enunciation is found to be a common source of difficulty with beginners, who fail to distinguish what is *assumed* in it and what is *to be proved*.]

[For Exercises see page 38.]

Proposition 19. Theorem.

If one angle of a triangle be greater than another, then the side opposite to the greater angle shall be greater than the side opposite to the less.



Let ABC be a triangle in which the angle ABC is greater than the angle ACB:

then shall the side AC be greater than the side AB.

Proof. For if AC be not greater than AB,

it must be either equal to, or less than AB.

But AC is not equal to AB,

for then the angle ABC would be equal to the angle ACB; I. 5.

but it is not. *Hyp.*

Neither is AC less than AB;

for then the angle ABC would be less than the angle ACB; I. 18.

but it is not: *Hyp.*

Therefore AC is neither equal to, nor less than AB.

That is, AC is greater than AB. Q.E.D.

Note. The mode of demonstration used in this Proposition is known as the **Proof by Exhaustion.** It is applicable to cases in which one of certain mutually exclusive suppositions must necessarily be true; and it consists in shewing the falsity of each of these suppositions in turn *with one exception:* hence the truth of the remaining supposition is inferred.

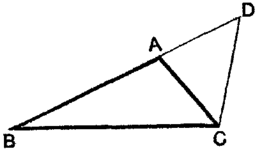
Euclid enunciated Proposition 19 as follows:

The greater angle of every triangle is subtended by the greater side, or, has the greater side opposite to it.

[For Exercises see page 38.]

PROPOSITION 20. THEOREM.

Any two sides of a triangle are together greater than the third side.



Let ABC be a triangle:
then shall any two of its sides be together greater than the third side:
namely, BA, AC, shall be greater than CB;
AC, CB greater than BA;
and CB, BA greater than AC.

Construction. Produce BA to the point D, making AD equal to AC. I. 3.
Join DC.

Proof. Then in the triangle ADC,
because AD is equal to AC, *Constr.*
therefore the angle ACD is equal to the angle ADC. I. 5.
But the angle BCD is greater than the angle ACD; *Ax.* 9.
therefore also the angle BCD is greater than the angle ADC,
that is, than the angle BDC.

And in the triangle BCD,
because the angle BCD is greater than the angle BDC, *Pr.*
therefore the side BD is greater than the side CB. I. 19.

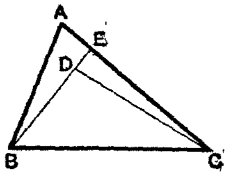
But BA and AC are together equal to BD;
therefore BA and AC are together greater than CB.

Similarly it may be shewn
that AC, CB are together greater than BA;
and CB, BA are together greater than AC. Q. E. D.

[For Exercises see page 38.]

PROPOSITION 21. THEOREM.

If from the ends of a side of a triangle, there be drawn two straight lines to a point within the triangle, then these straight lines shall be less than the other two sides of the triangle, but shall contain a greater angle.



Let ABC be a triangle, and from B, C, the ends of the side BC, let the two straight lines BD, CD be drawn to a point D within the triangle:

then (i) BD and DC shall be together less than BA and AC;
(ii) the angle BDC shall be greater than the angle BAC.

Construction. Produce BD to meet AC in E.

Proof. (i) In the triangle BAE, the two sides BA, AE are together greater than the third side BE: I. 20.

to each of these add EC;

then BA, AC are together greater than BE, EC. *Ax.* 4.

Again, in the triangle DEC, the two sides DE, EC are together greater than DC: I. 20.

to each of these add BD;

then BE, EC are together greater than BD, DC.

But it has been shewn that BA, AC are together greater than BE, EC:

still more then are BA, AC greater than BD, DC.

(ii) Again, the exterior angle BDC of the triangle DEC is greater than the interior opposite angle DEC; I. 16.

and the exterior angle DEC of the triangle BAE is greater than the interior opposite angle BAE, that is, than the angle BAC; I. 16.

still more then is the angle BDC greater than the angle BAC.

Q.E.D.

EXERCISES

ON PROPOSITIONS 18 AND 19.

1. The hypotenuse is the greatest side of a right-angled triangle.

2. If two angles of a triangle are equal to one another, the sides also, which subtend the equal angles, are equal to one another. Prop. 6. Prove this indirectly by using the result of Prop. 18.

3. BC, the base of an isosceles triangle ABC, is produced to any point D; shew that AD is greater than either of the equal sides.

4. If in a quadrilateral the greatest and least sides are opposite to one another, then each of the angles adjacent to the least side is greater than its opposite angle.

5. In a triangle ABC, if AC is not greater than AB, shew that any straight line drawn through the vertex A and terminated by the base BC, is less than AB.

6. ABC is a triangle, in which OB, OC bisect the angles ABC, ACB respectively: shew that, if AB is greater than AC, then OB is greater than OC.

ON PROPOSITION 20.

7. The difference of any two sides of a triangle is less than the third side.

8. In a quadrilateral, if two opposite sides which are not parallel are produced to meet one another; shew that the perimeter of the greater of the two triangles so formed is greater than the perimeter of the quadrilateral.

9. The sum of the distances of any point from the three angular points of a triangle is greater than half its perimeter.

10. The perimeter of a quadrilateral is greater than the sum of its diagonals.

11. Obtain a proof of Proposition 20 by bisecting an angle by a straight line which meets the opposite side.

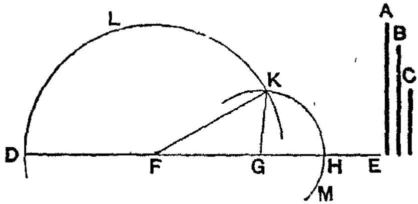
ON PROPOSITION 21.

12. In Proposition 21 shew that the angle BDC is greater than the angle BAC by joining AD, and producing it towards the base.

13. The sum of the distances of any point within a triangle from its angular points is less than the perimeter of the triangle.

PROPOSITION 22. PROBLEM.

To describe a triangle having its sides equal to three given straight lines, any two of which are together greater than the third.



Let A, B, C be the three given straight lines, of which any two are together greater than the third.

It is required to describe a triangle of which the sides shall be equal to A, B, C.

Construction. Take a straight line DE terminated at the point D, but unlimited towards E.

Make DF equal to A, FG equal to B, and GH equal to C. I. 3.

From centre F, with radius FD, describe the circle DLK.

From centre G with radius GH, describe the circle MHK, cutting the former circle at K.

Join FK, GK.

Then shall the triangle KFG have its sides equal to the three straight lines A, B, C.

Proof. Because F is the centre of the circle DLK,

therefore FK is equal to FD: *Def.* 11.

but FD is equal to A; *Constr.*

therefore also FK is equal to A. *Ax.* 1.

Again, because G is the centre of the circle MHK,

therefore GK is equal to GH: *Def.* 11.

but GH is equal to C; *Constr.*

therefore also GK is equal to C. *Ax.* 1.

And FG is equal to B. *Constr.*

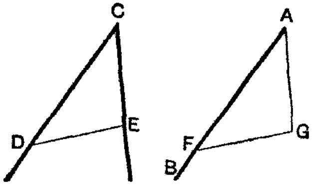
Therefore the triangle KFG has its sides KF, FG, GK equal respectively to the three given lines A, B, C. Q.E.F.

EXERCISE.

On a given base describe a triangle, whose remaining sides shall be equal to two given straight lines. Point out how the construction fails, if any one of the three given lines is greater than the sum of the other two.

Proposition 23. Problem.

At a given point in a given straight line, to make an angle equal to a given angle.



Let AB be the given straight line, and A the given point in it; and let DCE be the given angle.

It is required to draw from A a straight line making with AB an angle equal to the given angle DCE.

Construction. In CD, CE take any points D and E; and join DE.

From AB cut off AF equal to CD. I. 3.

On AF describe the triangle FAG, having the remaining sides AG, GF equal respectively to CE, ED. I. 22.

Then shall the angle FAG be equal to the angle DCE.

Proof. For in the triangles FAG, DCE,

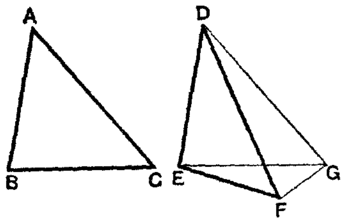
Because { FA is equal to DC, *Constr.*
and AG is equal to CE; *Constr.*
and the base FG is equal to the base DE: *Constr.* }

therefore the angle FAG is equal to the angle DCE. I. 8.

That is, AG makes with AB, at the given point A, an angle equal to the given angle DCE. Q.E.F.

Proposition 24.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one greater than the angle contained by the corresponding sides of the other; then the base of that which has the greater angle shall be greater than the base of the other.



Let ABC, DEF be two triangles, in which the two sides BA, AC are equal to the two sides ED, DF, each to each, but the angle BAC greater than the angle EDF:

then shall the base BC be greater than the base EF.

* Of the two sides DE, DF, let DE be that which is not greater than the other.

Construction. At the point D, in the straight line ED, and on the same side of it as DF, make the angle EDG equal to the angle BAC. I. 23.

Make DG equal to DF or AC; I. 3.

and join EG, GF.

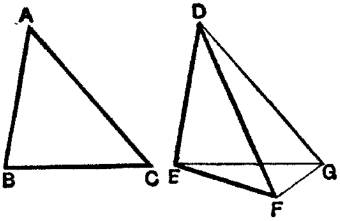
Proof. Then in the triangles BAC, EDG,

Because { BA is equal to ED, *Hyp.*
and AC is equal to DG, *Constr.*
also the contained angle BAC is equal to the contained angle EDG; *Constr.* }

Therefore the triangle BAC is equal to the triangle EDG in all respects: I. 4.

so that the base BC is equal to the base EG.

* See note on the next page.



Again, in the triangle FDG,
because DG is equal to DF,
therefore the angle DFG is equal to the angle DGF, I. 5.
but the angle DGF is greater than the angle EGF;
therefore also the angle DFG is greater than the angle EGF;
still more then is the angle EFG greater than the angle EGF.

And in the triangle EFG,
because the angle EFG is greater than the angle EGF,
therefore the side EG is greater than the side EF; I. 19.
but EG was shewn to be equal to BC;
therefore BC is greater than EF. Q.E.D.

* This condition was inserted by Simson to ensure that, in the complete construction, the point F should fall *below* EG. Without this condition it would be necessary to consider three cases: for F might fall *above*, or *upon*, or *below* EG; and each figure would require separate proof.

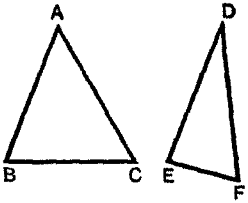
We are however scarcely at liberty to employ Simson's condition without *proving* that it fulfils the object for which it was introduced.

This may be done as follows:

Let EG, DF, produced if necessary, intersect at K.
Then, since DE is not greater than DF,
that is, since DE is not greater than DG,
therefore the angle DGE is not greater than the angle DEG. I. 18.
But the exterior angle DKG is greater than the angle DEK: I. 16.
therefore the angle DKG is greater than the angle DGK.
Hence DG is greater than DK. I. 19.
But DG is equal to DF;
therefore DF is greater than DK.
So that the point F must fall *below* EG.

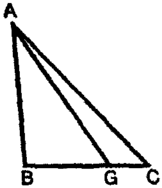
Or the following method may be adopted.

PROPOSITION 24. [ALTERNATIVE PROOF.]

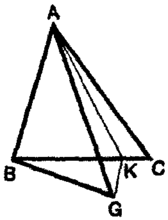


In the triangles ABC, DEF,
let BA be equal to ED,
and AC equal to DF,
but let the angle BAC be greater than the angle EDF:
then shall the base BC be greater than the base EF.

For apply the triangle DEF to the triangle ABC, so that D may fall on A, and DE along AB:
then because DE is equal to AB,
therefore E must fall on B.
And because the angle EDF is less than the angle BAC, *Hyp.*
therefore DF must fall between AB and AC.
Let DF occupy the position AG.



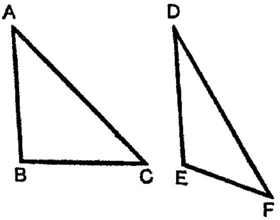
CASE I. If G falls on BC:
Then G must be between B and C:
therefore BC is greater than BG.
But BG is equal to EF:
therefore BC is greater than EF.



CASE II. If G does not fall on BC.
Bisect the angle CAG by the straight line AK which meets BC in K. I. 9.
Join GK.
Then in the triangles GAK, CAK,
Because { GA is equal to CA, *Hyp.*
and AK is common to both;
and the angle GAK is equal to the angle CAK; *Constr.* }
therefore GK is equal to CK. I. 4.
But in the triangle BKG,
the two sides BK, KG are together greater than the third side BG, I. 20.
that is, BK, KC are together greater than BG;
therefore BC is greater than BG, or EF. Q.E.D.

PROPOSITION 25. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of one greater than the base of the other; then the angle contained by the sides of that which has the greater base, shall be greater than the angle contained by the corresponding sides of the other.



Let ABC, DEF be two triangles which have the two sides BA, AC equal to the two sides ED, DF, each to each, but the base BC greater than the base EF:
then shall the angle BAC be greater than the angle EDF.

Proof. For if the angle BAC be not greater than the angle EDF, it must be either equal to, or less than the angle EDF.

But the angle BAC is not equal to the angle EDF,
for then the base BC would be equal to the base EF; I. 4.
but it is not. *Hyp.*
Neither is the angle BAC less than the angle EDF,
for then the base BC would be less than the base EF; I. 24.
but it is not. *Hyp.*
Therefore the angle BAC is neither equal to, nor less than the angle EDF;
that is, the angle BAC is greater than the angle EDF. Q.E.D.

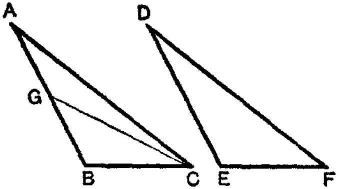
EXERCISE.

In a triangle ABC, the vertex A is joined to X, the middle point of the base BC; shew that the angle AXB is obtuse or acute, according as AB is greater or less than AC.

PROPOSITION 26. THEOREM.

If two triangles have two angles of the one equal to two angles of the other, each to each, and a side of one equal to a side of the other, these sides being either adjacent to the equal angles, or opposite to equal angles in each; then shall the triangles be equal in all respects.

CASE I. When the equal sides are *adjacent* to the equal angles in the two triangles.



Let ABC, DEF be two triangles, which have the angles ABC, ACB, equal to the two angles DEF, DFE, each to each; and the side BC equal to the side EF:
then shall the triangle ABC be equal to the triangle DEF in all respects;
that is, AB shall be equal to DE, and AC to DF, and the angle BAC shall be equal to the angle EDF.

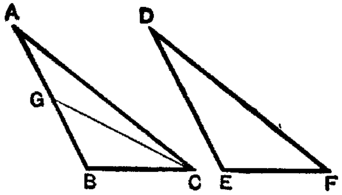
For if AB be not equal to DE, one must be greater than the other. If possible, let AB be greater than DE.

Construction. From BA cut off BG equal to ED, I. 3.
and join GC.

Proof. Then in the two triangles GBC, DEF,

Because { GB is equal to DE, *Constr.*
and BC to EF, *Hyp.*
also the contained angle GBC is equal to the contained angle DEF; *Hyp.* }

therefore the triangles are equal in all respects; I. 4.
so that the angle GCB is equal to the angle DFE.
But the angle ACB is equal to the angle DFE; *Hyp.*
therefore also the angle GCB is equal to the angle ACB; *Ax.* I.
the part equal to the whole, which is impossible.



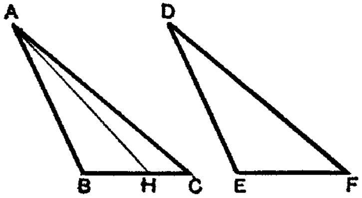
Therefore AB is not unequal to DE,
that is, AB is equal to DE.

Hence in the triangles ABC, DEF,

Because { AB is equal to DE, *Proved.*
and BC is equal to EF; *Hyp.*
also the contained angle ABC is equal to the contained angle DEF; *Hyp.* }

therefore the triangles are equal in all respects: I. 4.
so that the side AC is equal to the side DF;
and the angle BAC to the angle EDF. Q.E.D.

CASE II. When the equal sides are *opposite* to equal angles in the two triangles.



Let ABC, DEF be two triangles which have the angles ABC, ACB equal to the angles DEF, DFE, each to each, and the side AB equal to the side DE:
then shall the triangles ABC, DEF be equal in all respects;
that is, BC shall be equal to EF, and AC to DF,
and the angle BAC shall be equal to the angle EDF.

For if BC be not equal to EF, one must be greater than the other. If possible, let BC be greater than EF.

Construction. From BC cut off BH equal to EF, I. 3.
and join AH.

Proof. Then in the triangles ABH, DEF,

Because { AB is equal to DE, *Hyp.*
and BH to EF, *Constr.*
also the contained angle ABH is equal to the contained angle DEF; *Hyp.* }

therefore the triangles are equal in all respects, I. 4.
so that the angle AHB is equal to the angle DFE.

But the angle DFE is equal to the angle ACB; *Hyp.*
therefore the angle AHB is equal to the angle ACB; *Ax.* 1.
that is, an exterior angle of the triangle ACH is equal to an interior opposite angle; which is impossible. I. 16.
Therefore BC is not unequal to EF,
that is, BC is equal to EF.

Hence in the triangles ABC, DEF,

Because { AB is equal to DE, *Hyp.*
and BC is equal to EF; *Proved.*
also the contained angle ABC is equal to the contained angle DEF; *Hyp.* }

therefore the triangles are equal in all respects; I. 4.
so that the side AC is equal to the side DF,
and the angle BAC to the angle EDF.

Q.E.D.

COROLLARY. *In both cases of this Proposition it is seen that the triangles may be made* to coincide with one another; *and they are therefore equal* in area.

ON THE IDENTICAL EQUALITY OF TRIANGLES.

At the close of the first section of Book I., it is worth while to call special attention to those Propositions (viz. Props. 4, 8, 26) which deal with the *identical equality* of two triangles.

The results of these Propositions may be summarized thus:

Two triangles are equal to one another in all respects, when the following parts in each are equal, each to each.

1. Two sides, and the included angle. *Prop.* 4.
2. The three sides. *Prop.* 8, *Cor.*
3. (*a*) Two angles, and the adjacent side. (*b*) Two angles, and the side opposite one of them. } *Prop.* 26.

From this the beginner will perhaps surmise that two triangles may be shewn to be equal in all respects, when they have *three parts* equal, each to each; but to this statement two obvious exceptions must be made.

(i) When in two triangles the *three angles* of one are equal to the *three angles* of the other, each to each, it does *not* necessarily follow that the triangles are equal in all respects.

(ii) When in two triangles two sides of the one are equal to two sides of the other, each to each, and one angle equal to one angle, these not being the angles included by the equal sides; the triangles are *not* necessarily equal in all respects.

In these cases a further condition must be added to the hypothesis, before we can assert the identical equality of the two triangles.

[See Theorems and Exercises on Book I., Ex. 13, Page 92.]

We observe that in each of the three cases already proved of identical equality in two triangles, namely in Propositions 4, 8, 26, it is shewn that the triangles may be made to *coincide with one another*; so that they are equal in *area*, as in all other respects. Euclid however restricted himself to the use of Prop. 4, when he required to deduce the equality in *area* of two triangles from the equality of certain of their parts.

This restriction has been abandoned in the present text-book. [See note to Prop. 34.]

EXERCISES ON PROPOSITIONS 12—26.

1. If BX and CY, the bisectors of the angles at the base BC of an isosceles triangle ABC, meet the opposite sides in X and Y; shew that the triangles YBC, XCB are equal in all respects.

2. Shew that the perpendiculars drawn from the extremities of the base of an isosceles triangle to the opposite sides are equal.

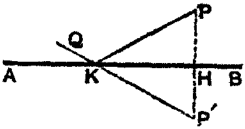
3. *Any point on the bisector of an angle is equidistant from the arms of the angle.*

4. Through O, the middle point of a straight line AB, any straight line is drawn, and perpendiculars AX and BY are dropped upon it from A and B: shew that AX is equal to BY.

5. If the bisector of the vertical angle of a triangle is at right angles to the base, the triangle is isosceles.

6. *The perpendicular is the shortest straight line that can be drawn from a given point to a given straight line; and of others, that which is nearer to the perpendicular is less than the more remote; and two, and only two equal straight lines can be drawn from the given point to the given straight line, one on each side of the perpendicular.*

7. *From two given points on the same side of a given straight line, draw two straight lines, which shall meet in the given straight line and make equal angles with it.*



Let AB be the given straight line, and P, Q the given points.

It is required to draw from P and Q to a point in AB, two straight lines that shall be equally inclined to AB.

Construction. From P draw PH perpendicular to AB: produce PH to P′, making HP′ equal to PH. Draw QP′, meeting AB in K. Join PK.

Then PK, QK shall be the required lines. [Supply the proof.]

8. In a given straight line find a point which is equidistant from two given intersecting straight lines. In what case is this impossible?

9. Through a given point draw a straight line such that the perpendiculars drawn to it from two given points may be equal.

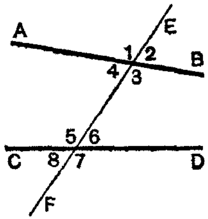
In what case is this impossible?

SECTION II.

PARALLEL STRAIGHT LINES AND PARALLELOGRAMS.

DEFINITION. Parallel straight lines are such as, being in the same plane, do not meet however far they are produced in both directions.

When two straight lines AB, CD are met by a third straight line EF, *eight* angles are formed, to which for the sake of distinction particular names are given.



Thus in the adjoining figure,

1, 2, 7, 8 are called **exterior** angles,

3, 4, 5, 6 are called **interior** angles,

4 and 6 are said to be **alternate** angles; so also the angles 3 and 5 are alternate to one another.

Of the angles 2 and 6, 2 is referred to as the exterior angle, and 6 as the **interior opposite** angle on the same side of EF.

2 and 6 are sometimes called **corresponding** angles.

So also, 1 and 5, 7 and 3, 8 and 4 are corresponding angles.

Euclid's treatment of parallel straight lines is based upon his twelfth Axiom, which we here repeat.

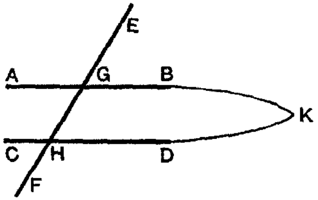
AXIOM 12. If a straight line cut two straight lines so as to make the two interior angles on the same side of it together less than two right angles, these straight lines, being continually produced, will at length meet on that side on which are the angles which are together less than two right angles.

Thus in the figure given above, if the two angles 3 and 6 are together less than two right angles, it is asserted that AB and CD will meet towards B and D.

This Axiom is used to establish I. 29: some remarks upon it will be found in a note on that Proposition.

PROPOSITION 27. THEOREM.

If a straight line, falling on two other straight lines, make the alternate angles equal to one another, then the straight lines shall be parallel.



Let the straight line EF cut the two straight lines AB, CD at G and H, so as to make the alternate angles AGH, GHD equal to one another:

then shall AB and CD be parallel.

Proof. For if AB and CD be not parallel,

they will meet, if produced, either towards B and D, or towards A and C.

If possible, let AB and CD, when produced, meet towards B and D, at the point K.

Then KGH is a triangle, of which one side KG is produced to A:

therefore the exterior angle AGH is greater than the interior opposite angle GHK. I. 16.

But the angle AGH is equal to the angle GHK: *Hyp.*

hence the angles AGH and GHK are both equal and unequal;

which is impossible.

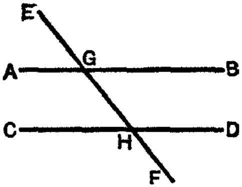
Therefore AB and CD cannot meet when produced towards B and D.

Similarly it may be shewn that they cannot meet towards A and C:

therefore they are parallel. Q.E.D.

PROPOSITION 28. THEOREM.

If a straight line, falling on two other straight lines, make an exterior angle equal to the interior opposite angle on the same side of the line; or if it make the interior angles on the same side together equal to two right angles, then the two straight lines shall be parallel.



Let the straight line EF cut the two straight lines AB, CD in G and H: and

First, let the exterior angle EGB be equal to the interior opposite angle GHD:

then shall AB and CD be parallel.

Proof. Because the angle EGB is equal to the angle GHD; and because the angle EGB is also equal to the vertically opposite angle AGH; I. 15.

therefore the angle AGH is equal to the angle GHD;

but these are alternate angles;

therefore AB and CD are parallel. I. 27.

Q. E. D.

Secondly, let the two interior angles BGH, GHD be together equal to two right angles:

then shall AB and CD be parallel.

Proof. Because the angles BGH, GHD are together equal to two right angles; *Hyp.*

and because the adjacent angles BGH, AGH are also together equal to two right angles; I. 13.

therefore the angles BGH, AGH are together equal to the two angles BGH, GHD.

From these equals take the common angle BGH:

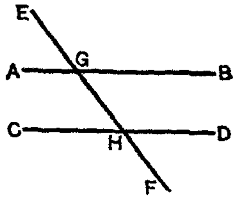
then the remaining angle AGH is equal to the remaining angle GHD: and these are alternate angles;

therefore AB and CD are parallel. I. 27.

Q. E. D.

Proposition 29. Theorem.

If a straight line fall on two parallel straight lines, then it shall make the alternate angles equal to one another, and the exterior angle equal to the interior opposite angle on the same side; and also the two interior angles on the same side equal to two right angles.



Let the straight line EF fall on the parallel straight lines AB, CD:

then (i) the alternate angles AGH, GHD shall be equal to one another;

(ii) the exterior angle EGB shall be equal to the interior opposite angle GHD;

(iii) the two interior angles BGH, GHD shall be together equal to two right angles.

Proof. (i) For if the angle AGH be not equal to the angle GHD, one of them must be greater than the other.

If possible, let the angle AGH be greater than the angle GHD;

add to each the angle BGH:

then the angles AGH, BGH are together greater than the angles BGH, GHD.

But the adjacent angles AGH, BGH are together equal to two right angles; I. 13.

therefore the angles BGH, GHD are together less than two right angles;

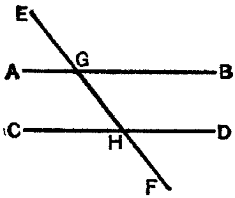
therefore AB and CD meet towards B and D. *Ax.* 12.

But they never meet, since they are parallel. *Hyp.*

Therefore the angle AGH is not unequal to the angle GHD:

that is, the alternate angles AGH, GHD are equal.

(*Over*)



(ii) Again, because the angle AGH is equal to the vertically opposite angle EGB; I. 15.
and because the angle AGH is equal to the angle GHD; *Proved.*
therefore the exterior angle EGB is equal to the interior opposite angle GHD

(iii) Lastly, the angle EGB is equal to the angle GHD; *Proved.*
add to each the angle BGH;
then the angles EGB, BGH are together equal to the angles BGH, GHD.
But the adjacent angles EGB, BGH are together equal to two right angles: I. 13.
therefore also the two interior angles BGH, GHD are together equal to two right angles. Q.E.D.

EXERCISES ON PROPOSITIONS 27, 28, 29.

1. Two straight lines AB, CD bisect one another at O: shew that the straight lines joining AC and BD are parallel. [I. 27.]

2. *Straight lines which are perpendicular to the same straight line are parallel to one another.* [I. 27 or I. 28.]

3. *If a straight line meet two or more parallel straight lines, and is perpendicular to one of them, it is also perpendicular to all the others.* [I. 29.]

4. *If two straight lines are parallel to two other straight lines, each to each, then the angles contained by the first pair are equal respectively to the angles contained by the second pair.* [I. 29.]

NOTE ON THE TWELFTH AXIOM.

It must be admitted that Euclid's twelfth Axiom is unsatisfactory as the basis of a theory of parallel straight lines. It cannot be regarded as either simple or self-evident, and it therefore falls short of the essential characteristics of an axiom: nor is the difficulty entirely removed by considering it as a corrollary to Proposition 17, of which it is the converse.

Many substitutes have been proposed; but we need only notice here the system which has met with most general approval.

This system rests on the following hypothesis, which is put forward as a fundamental Axiom.

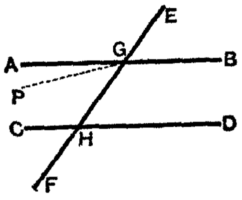
AXIOM. *Two intersecting straight lines cannot be both parallel to a third straight line.*

This statement is known as **Playfair's Axiom**; and though it is not altogether free from objection, it is recommended as both simpler and more fundamental than that employed by Euclid, and more readily admitted without proof.

Propositions 27 and 28 having been proved in the usual way, the first part of Proposition 29 is then given thus.

PROPOSITION 29. [ALTERNATIVE PROOF.]

If a straight line fall on two parallel straight lines, then it shall make the alternate angles equal.



Let the straight line EF meet the two parallel straight lines AB, CD, at G and H:

then shall the alternate angles AGH, GHD be equal.

For if the angle AGH is not equal to the angle GHD:

at G in the straight line HG make the angle HGP equal to the angle GHD, and alternate to it. I. 23.

Then PG and CD are parallel. I. 27.

But AB and CD are parallel: *Hyp.*

therefore the two intersecting straight lines AG, PG are both parallel to CD:

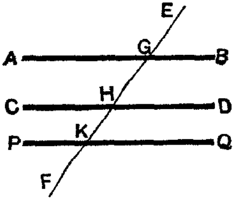
which is impossible. *Playfair's Axiom.*

Therefore the angle AGH is not unequal to the angle GHD, that is, the alternate angles AGH, GHD are equal. Q.E.D.

The second and third parts of the Proposition may then be deduced as in the text; and Euclid's Axiom 12 follows as a Corollary.

PROPOSITION 30. THEOREM.

Straight lines which are parallel to the same straight line are parallel to one another.



Let the straight lines AB, CD be each parallel to the straight line PQ:

then shall AB and CD be parallel to one another.

Construction. Draw any straight line EF cutting AB, CD, and PQ in the points G, H, and K.

Proof. Then because AB and PQ are parallel, and EF meets them,

therefore the angle AGK is equal to the alternate angle GKQ. I. 29.

And because CD and PQ are parallel, and EF meets them,

therefore the exterior angle GHD is equal to the interior opposite angle HKQ. I. 29.

Therefore the angle AGH is equal to the angle GHD;

and these are alternate angles;

therefore AB and CD are parallel. I. 27.

Q.E.D.

NOTE. If PQ lies between AB and CD, the Proposition may be established in a similar manner, though in this case it scarcely needs proof; for it is inconceivable that two straight lines, which do not meet an intermediate straight line, should meet one another.

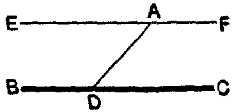
The truth of this Proposition may be readily deduced from Playfair's Axiom, of which it is the converse.

For if AB and CD were not parallel, they would meet when produced. Then there would be two intersecting straight lines both parallel to a third straight line: which is impossible.

Therefore AB and CD never meet; that is, they are parallel.

PROPOSITION 31. PROBLEM.

To draw a straight line through a given point parallel to a given straight line.



Let A be the given point, and BC the given straight line. It is required to draw through A a straight line parallel to BC.

Construction. In BC take any point D; and join AD. At the point A in DA, make the angle DAE equal to the angle ADC, and alternate to it. I. 23.

and produce EA to F.

Then shall EF be parallel to BC.

Proof. Because the straight line AD, meeting the two straight lines EF, BC, makes the alternate angles EAD, ADC equal; *Constr.*

therefore EF is parallel to BC; I. 27.

and it has been drawn through the given point A.

Q. E. F.

EXERCISES.

1. Any straight line drawn parallel to the base of an isosceles triangle makes equal angles with the sides.

2. If from any point in the bisector of an angle a straight line is drawn parallel to either arm of the angle, the triangle thus formed is isosceles.

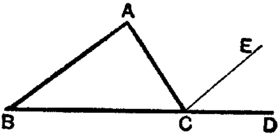
3. From a given point draw a straight line that shall make with a given straight line an angle equal to a given angle.

4. From X, a point in the base BC of an isosceles triangle ABC, a straight line is drawn at right angles to the base, cutting AB in Y, and CA produced in Z: shew the triangle AYZ is isosceles.

5. If the straight line which bisects an exterior angle of a triangle is parallel to the opposite side, shew that the triangle is isosceles.

PROPOSITION 32. THEOREM.

If a side of a triangle be produced, then the exterior angle shall be equal to the sum of the two interior opposite angles: also the three interior angles of a triangle are together equal to two right angles.



Let ABC be a triangle, and let one of its sides BC be produced to D:

then (i) the exterior angle ACD shall be equal to the sum of the two interior opposite angles CAB, ABC;

(ii) the three interior angles ABC, BCA, CAB shall be together equal to two right angles.

Construction. Through C draw CE parallel to BA. I. 31.

Proof. (i) Then because BA and CE are parallel, and AC meets them,

therefore the angle ACE is equal to the alternate angle CAB. I. 29.

Again, because BA and CE are parallel, and BD meets them,

therefore the exterior angle ECD is equal to the interior opposite angle ABC. I. 29.

Therefore the whole exterior angle ACD is equal to the sum of the two interior opposite angles CAB, ABC.

(ii) Again, since the angle ACD is equal to the sum of the angles CAB, ABC; *Proved.*

to each of these equals add the angle BCA:

then the angles BCA, ACD are together equal to the three angles BCA, CAB, ABC.

But the adjacent angles BCA, ACD are together equal to two right angles; I. 13.

therefore also the angles BCA, CAB, ABC are together equal to two right angles. Q. E. D.

From this Proposition we draw the following important inferences.

1. *If two triangles have two angles of the one equal to two angles of the other, each to each, then the third angle of the one is equal to the third angle of the other.*

2. *In any right-angled triangle the two acute angles are complementary.*

3. *In a right-angled isosceles triangle each of the equal angles is half a right angle.*

4. *If one angle of a triangle is equal to the sum of the other two, the triangle is right-angled.*

5. *The sum of the angles of any quadrilateral figure is equal to four right angles.*

6. *Each angle of an equilateral triangle is two-thirds of a right angle.*

EXERCISES ON PROPOSITION 32

1. Prove that the three angles of a triangle are together equal to two right angles,

   (i) by drawing through the vertex a straight line parallel to the base;

   (ii) by joining the vertex to any point in the base.

2. If the base of any triangle is produced both ways, shew that the sum of the two exterior angles diminished by the vertical angle is equal to two right angles.

3. *If two straight lines are perpendicular to two other straight lines, each to each, the acute angle between the first pair is equal to the acute angle between the second pair.*

4. *Every right-angled triangle is divided into two isosceles triangles by a straight line drawn from the right angle to the middle point of the hypotenuse.*

   Hence the joining line is equal to half the hypotenuse.

5. *Draw a straight line at right angles to a given finite straight line from one of its extremities, without producing the given straight line.*

   [Let AB be the given straight line. On AB describe any isosceles triangle ACB. Produce BC to D, making CD equal to BC. Join AD. Then shall AD be perpendicular to AB.]

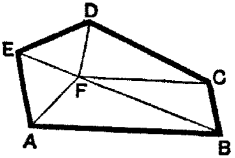
6. *Trisect a right angle.*

7. The angle contained by the bisectors of the angles at the base of an isosceles triangle is equal to an exterior angle formed by producing the base.

8. The angle contained by the bisectors of two adjacent angles of a quadrilateral is equal to half the sum of the remaining angles.

The following theorems were added as corollaries to Proposition 32 by Robert Simson.

COROLLARY 1. *All the interior angles of any rectilineal figure, with four right angles, are together equal to twice as many right angles as the figure has sides.*



Let ABCDE be any rectilineal figure.
Take F, any point within it,
and join F to each of the angular points of the figure.

Then the figure is divided into as many triangles as it has sides.

And the three angles of each triangle are together equal to two right angles. I. 32.

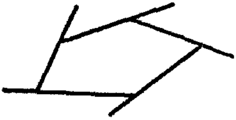
Hence all the angles of all the triangles are together equal to twice as many right angles as the figure has sides.

But all the angles of all the triangles make up the interior angles of the figure, together with the angles at F;

and the angles at F are together equal to four right angles: I. 15, *Cor.*

Therefore all the interior angles of the figure, with four right angles, are together equal to twice as many right angles as the figure has sides. Q. E. D.

Corollary 2. *If the sides of a rectilineal figure, which has no re-entrant angle, are produced in order, then all the exterior angles so formed are together equal to four right angles.*



For at each angular point of the figure, the interior angle and the exterior angle are together equal to two right angles. I. 13.

Therefore all the interior angles, with all the exterior angles, are together equal to twice as many right angles as the figure has sides.

But all the interior angles, with four right angles, are together equal to twice as many right angles as the figure has sides. I. 32, *Cor.* 1.

Therefore all the interior angles, with all the exterior angles, are together equal to all the interior angles, with four right angles.

Therefore the exterior angles are together equal to four right angles. Q. E. D.

EXERCISES ON SIMSON'S COROLLARIES.

[A polygon is said to be **regular** when it has all its sides and all its angles equal.]

1. Express in terms of a right angle the magnitude of each angle of (i) a regular hexagon, (ii) a regular octagon.

2. If one side of a regular hexagon is produced, shew that the exterior angle is equal to the angle of an equilateral triangle.

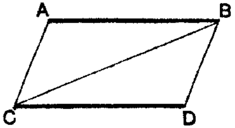
3. Prove Simson's first Corollary by joining one vertex of the rectilineal figure to each of the other vertices.

4. Find the magnitude of each angle of a regular polygon of n sides.

5. If the alternate sides of any polygon be produced to meet, the sum of the included angles, together with eight right angles, will be equal to twice as many right angles as the figure has sides.

PROPOSITION 33. THEOREM.

The straight lines which join the extremities of two equal and parallel straight lines towards the same parts are themselves equal and parallel.



Let AB and CD be equal and parallel straight lines; and let them be joined towards the same parts by the straight lines AC and BD:

then shall AC and BD be equal and parallel.

Construction. Join BC.

Proof. Then because AB and CD are parallel, and BC meets them,

therefore the alternate angles ABC, BCD are equal. I. 29.

Now in the triangles ABC, DCB,

Because { AB is equal to DC, *Hyp.*
and BC is common to both;
also the angle ABC is equal to the angle DCB; *Proved.* }

therefore the triangles are equal in all respects; I. 4.

so that the base AC is equal to the base DB,

and the angle ACB equal to the angle DBC;

but these are alternate angles;

therefore AC and BD are parallel: I. 27

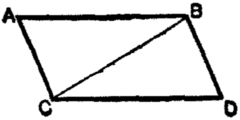
and it has been shewn that they are also equal.

Q. E. D.

DEFINITION. A **Parallelogram** is a four-sided figure whose opposite sides are parallel.

PROPOSITION 34. THEOREM.

The opposite sides and angles of a parallelogram are equal to one another, and each diagonal bisects the parallelogram.



Let ACDB be a parallelogram, of which BC is a diagonal: then shall the opposite sides and angles of the figure be equal to one another; and the diagonal BC shall bisect it.

Proof. Because AB and CD are parallel, and BC meets them,

therefore the alternate angles ABC, DCB are equal. I. 29.

Again, because AC and BD are parallel, and BC meets them,

therefore the alternate angles ACB, DBC are equal. I. 29.

Hence in the triangles ABC, DCB,

Because { the angle ABC is equal to the angle DCB,
and the angle ACB is equal to the angle DBC;
also the side BC, which is adjacent to the equal angles, is common to both,

therefore the two triangles ABC, DCB are equal in all respects; I. 26.

so that AB is equal to DC, and AC to DB;
and the angle BAC is equal to the angle CDB.

Also, because the angle ABC is equal to the angle DCB,
and the angle CBD equal to the angle BCA,
therefore the whole angle ABD is equal to the whole angle DCA.

And since it has been shewn that the triangles ABC, DCB are equal in all respects,
therefore the diagonal BC bisects the parallelogram ACDB.

Q.E.D.

[See note on next page.]

NOTE. To the proof which is here given Euclid added an application of Proposition 4, with a view to shewing that the triangles ABC, DCB are equal *in area*, and that therefore the diagonal BC bisects the parallelogram. This equality of area is however sufficiently established by the step which depends upon I. 26. [See page 48.]

EXERCISES.

1. *If one angle of a parallelogram is a right angle, all its angles are right angles.*

2. *If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*

3. *If the opposite angles of a quadrilateral are equal, the figure is a parallelogram.*

4. *If a quadrilateral has all its sides equal and one angle a right angle, all its angles are right angles.*

5. *The diagonals of a parallelogram bisect each other.*

6. *If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.*

7. If two opposite angles of a parallelogram are bisected by the diagonal which joins them, the figure is equilateral.

8. If the diagonals of a parallelogram are equal, all its angles are right angles.

9. In a parallelogram which is not rectangular the diagonals are unequal.

10. Any straight line drawn through the middle point of a diagonal of a parallelogram and terminated by a pair of opposite sides, is bisected at that point.

11. *If two parallelograms have two adjacent sides of one equal to two adjacent sides of the other, each to each, and one angle of one equal to one angle of the other, the parallelograms are equal in all respects.*

12. *Two rectangles are equal if two adjacent sides of one are equal to two adjacent sides of the other, each to each.*

13. In a parallelogram the perpendiculars drawn from one pair of opposite angles to the diagonal which joins the other pair are equal.

14. If ABCD is a parallelogram, and X, Y respectively the middle points of the sides AD, BC; shew that the figure AYCX is a parallelogram.

MISCELLANEOUS EXERCISES ON SECTIONS I. AND II.

1. Shew that the construction in Proposition 2 may generally be performed in eight different ways. Point out the exceptional case.

2. The bisectors of two vertically opposite angles are in the same straight line.

3. In the figure of Proposition 16, if AF is joined, shew

(i) that AF is equal to BC;

(ii) that the triangle ABC is equal to the triangle CFA in all respects.

4. ABC is a triangle right-angled at B, and BC is produced to D: shew that the angle ACD is obtuse.

5. Shew that in any regular polygon of n sides each angle contains $\frac{2(n-2)}{n}$ right angles.

6. The angle contained by the bisectors of the angles at the base of any triangle is equal to the vertical angle together with half the sum of the base angles.

7. The angle contained by the bisectors of two exterior angles of any triangle is equal to halt the sum of the two corresponding interior angles.

8. If perpendiculars are drawn to two intersecting straight lines from any point between them, shew that the bisector of the angle between the perpendiculars is parallel to (or coincident with) the bisector of the angle between the given straight lines.

9. If two points P, Q be taken in the equal sides of an isosceles triangle ABC, so that BP is equal to CQ, shew that PQ is parallel to BC.

10. ABC and DEF are two triangles, such that AB, BC are equal and parallel to DE, EF, each to each; shew that AC is equal and parallel to DF.

11. Prove the second Corollary to Prop. 32 by drawing through any angular point lines parallel to all the sides.

12. If two sides of a quadrilateral are parallel, and the remaining two sides equal but not parallel, shew that the opposite angles are supplementary; also that the diagonals are equal.

SECTION III.

THE AREAS OF PARALLELOGRAMS AND TRIANGLES.

Hitherto when two figures have been said to be *equal*, it has been implied that they are *identically* equal, that is, equal in all respects.

In Section III. of Euclid's first Book, we have to consider the equality in *area* of parallelograms and triangles which are not necessarily equal in all respects.

[The ultimate test of equality, as we have already seen, is afforded by Axiom 8, which asserts that magnitudes which *may be made to coincide with one another* are equal. Now figures which are not identically equal, cannot be made to coincide without first undergoing some change of form: hence the method of direct *superposition* is unsuited to the purposes of the present section.

We shall see however from Euclid's proof of Proposition 35, that two figures which are not identically equal, may nevertheless be so related to a third figure, that it is possible to infer the equality of their areas.]

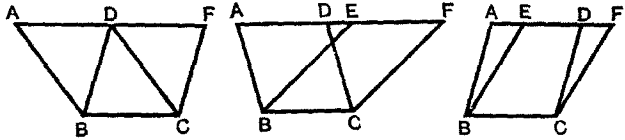
DEFINITIONS.

1. The **Altitude** of a parallelogram with reference to a given side as base, is the perpendicular distance between the base and the opposite side.

2. The **Altitude** of a triangle with reference to a given side as base, is the perpendicular distance of the opposite vertex from the base.

PROPOSITION 35. THEOREM.

Parallelograms on the same base, and between the same parallels, are equal in area.



Let the parallelograms ABCD, EBCF be on the same base BC, and between the same parallels BC, AF :

then shall the parallelogram ABCD be equal in area to the parallelogram EBCF.

CASE I. If the sides of the given parallelograms, opposite to the common base BC, are terminated at the same point D:

then because each of the parallelograms is double of the triangle BDC; I. 34.

therefore they are equal to one another. *Ax.* 6.

CASE II. But if the sides AD, EF, opposite to the base BC, are not terminated at the same point:

then because ABCD is a parallelogram,

therefore AD is equal to the opposite side BC; I. 34.

and for a similar reason, EF is equal to BC ;

therefore AD is equal to EF. *Ax.* 1.

Hence the whole, or remainder, EA is equal to the whole, or remainder, FD.

Then in the triangles FDC, EAB,

Because { FD is equal to EA, *Proved.*
and DC is equal to the opposite side AB, I. 34.
also the exterior angle FDC is equal to the interior opposite angle EAB, I. 29.

therefore the triangle FDC is equal to the triangle EAB. I. 4.

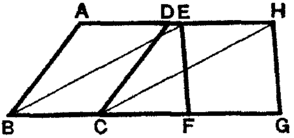
From the whole figure ABCF take the triangle FDC; and from the same figure take the equal triangle EAB ;

then the remainders are equal ; *Ax.* 3.

that is, the parallelogram ABCD is equal to the parallelogram EBCF. Q. E. D.

PROPOSITION 36. THEOREM.

Parallelograms on equal bases, and between the same parallels, are equal in area.



Let ABCD, EFGH be parallelograms on equal bases BC, FG, and between the same parallels AH, BG:
then shall the parallelogram ABCD be equal to the parallelogram EFGH.

Construction. Join BE, CH.

Proof. Then because BC is equal to FG; *Hyp.*
and FG is equal to the opposite side EH; I. 34.
therefore BC is equal to EH: *Ax.* 1.
and they are also parallel; *Hyp.*
therefore BE and CH, which join them towards the same parts, are also equal and parallel. I. 33.
Therefore EBCH is a parallelogram. *Def.* 26.
Now the parallelogram ABCD is equal to EBCH;
for they are on the same base BC, and between the same parallels BC, AH. I. 35.
Also the parallelogram EFGH is equal to EBCH;
for they are on the same base EH, and between the same parallels EH, BG. I. 35.
Therefore the parallelogram ABCD is equal to the parallelogram EFGH. *Ax.* 1.

Q. E. D.

From the last two Propositions we infer that:

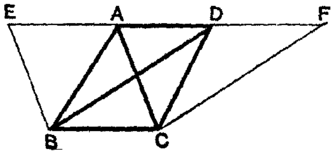
(i) *A parallelogram is equal in area to a rectangle of equal base and equal altitude.*

(ii) *Parallelograms on equal bases and of equal altitudes are equal in area.*

(iii) *Of two parallelograms of equal altitudes, that is the greater which has the greater base; and of two parallelograms on equal bases, that is the greater which has the greater altitude.*

PROPOSITION 37. THEOREM.

Triangles on the same base, and between the same parallels, are equal in area.



Let the triangles ABC, DBC be upon the same base BC, and between the same parallels BC, AD.

Then shall the triangle ABC be equal to the triangle DBC.

Construction. Through B draw BE parallel to CA, to meet DA produced in E; I. 31.
through C draw CF parallel to BD, to meet AD produced in F.

Proof. Then, by construction, each of the figures EBCA, DBCF is a parallelogram. *Def.* 26.

And EBCA is equal to DBCF;
for they are on the same base BC, and between the same parallels BC, EF. I. 35.

And the triangle ABC is half of the parallelogram EBCA, for the diagonal AB bisects it. I. 34.

Also the triangle DBC is half of the parallelogram DBCF, for the diagonal DC bisects it. I. 34.

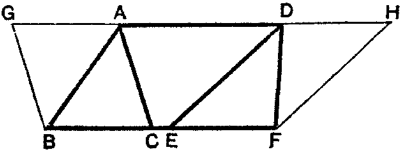
But the halves of equal things are equal; *Ax.* 7.
therefore the triangle ABC is equal to the triangle DBC.

Q.E.D.

[For Exercises see page 73.]

Proposition 38. Theorem.

Triangles on equal bases, and between the same parallels, are equal in area.



Let the triangles ABC, DEF be on equal bases BC, EF, and between the same parallels BF, AD:

then shall the triangle ABC be equal to the triangle DEF.

Construction. Through B draw BG parallel to CA, to meet DA produced in G; I. 31.
through F draw FH parallel to ED, to meet AD produced in H.

Proof. Then, by construction, each of the figures GBCA, DEFH is a parallelogram. *Def.* 26.

And GBCA is equal to DEFH;

for they are on equal bases BC, EF, and between the same parallels BF, GH. I. 36.

And the triangle ABC is half of the parallelogram GBCA, for the diagonal AB bisects it. I. 34.

Also the triangle DEF is half the parallelogram DEFH, for the diagonal DF bisects it. I. 34.

But the halves of equal things are equal: *Ax.* 7.
therefore the triangle ABC is equal to the triangle DEF.

Q.E.D.

From this Proposition we infer that:

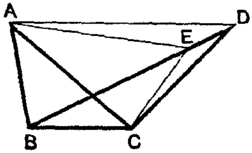
(i) *Triangles on equal bases and of equal altitude are equal in area.*

(ii) *Of two triangles of the same altitude, that is the greater which has the greater base: and of two triangles on the same base, or on equal bases, that is the greater which has the greater altitude.*

[For Exercises see page 73.]

PROPOSITION 39. THEOREM.

Equal triangles on the same base, and on the same side of it, are between the same parallels.



Let the triangles ABC, DBC which stand on the same base BC, and on the same side of it, be equal in area:

then shall they be between the same parallels;

that is, if AD be joined, AD shall be parallel to BC.

Construction. For if AD be not parallel to BC,

if possible, through A draw AE parallel to BC, I. 31.

meeting BD, or BD produced, in E.

Join EC.

Proof. Now the triangle ABC is equal to the triangle EBC, for they are on the same base BC, and between the same parallels BC, AE. I. 37.

But the triangle ABC is equal to the triangle DBC; *Hyp.*

therefore also the triangle DBC is equal to the triangle EBC;

the whole equal to the part; which is impossible.

Therefore AE is not parallel to BC.

Similarly it can be shewn that no other straight line through A, except AD, is parallel to BC.

Therefore AD is parallel to BC.

Q.E.D.

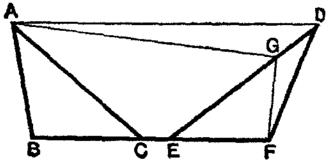
From this Proposition it follows that:

Equal triangles on the same base have equal altitudes.

[For Exercises see page 73.]

PROPOSITION 40. THEOREM.

Equal triangles, on equal bases in the same straight line, and on the same side of it, are between the same parallels.



Let the triangles ABC, DEF which stand on equal bases BC, EF, in the same straight line BF, and on the same side of it, be equal in area:

then shall they be between the same parallels;

that is, if AD be joined, AD shall be parallel to BF.

Construction. For if AD be not parallel to BF,

if possible, through A draw AG parallel to BF, I. 31.

meeting ED, or ED produced, in G.

Join GF.

Proof. Now the triangle ABC is equal to the triangle GEF, for they are on equal bases BC, EF, and between the same parallels BF AG. I. 38.

But the triangle ABC is equal to the triangle DEF: *Hyp.*

therefore also the triangle DEF is equal to the triangle GEF:

the whole equal to the part; which is impossible.

Therefore AG is not parallel to BF.

Similarly it can be shewn that no other straight line through A, except AD, is parallel to BF.

Therefore AD is parallel to BF.

Q.E.D.

From this Proposition it follows that:

(i) *Equal triangles on equal bases have equal altitudes.*

(ii) *Equal triangles of equal altitudes have equal bases.*

EXERCISES ON PROPOSITIONS 37—40.

DEFINITION. Each of the three straight lines which join the angular points of a triangle to the middle points of the opposite sides is called a **Median** of the triangle.

ON PROP. 37.

1. If, in the figure of Prop. 37, AC and BD intersect in K, shew that
   (i) the triangles AKB, DKC are equal in area.
   (ii) the quadrilaterals EBKA, FCKD are equal.

2. In the figure of I. 16, shew that the triangles ABC, FBC are equal in area.

3. On the base of a given triangle construct a second triangle, equal in area to the first, and having its vertex in a given straight line.

4. Describe an isosceles triangle equal in area to a given triangle and standing on the same base.

ON PROP. 38.

5. *A triangle is divided by each of its medians into two parts of equal area.*

6. A parallelogram is divided by its diagonals into four triangles of equal area.

7. ABC is a triangle, and its base BC is bisected at X; if Y be any point in the median AX, shew that the triangles ABY, ACY are equal in area.

8. In AC, a diagonal of the parallelogram ABCD, any point X is taken, and XB, XD are drawn: shew that the triangle BAX is equal to the triangle DAX.

9. If two triangles have two sides of one respectively equal to two sides of the other, and the angles contained by those sides *supplementary*, the triangles are equal in area.

ON PROP. 39.

10. *The straight line which joins the middle points of two sides of a triangle is parallel to the third side.*

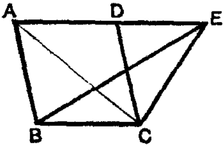
11. *If two straight lines* AB, CD *intersect in* O, *so that the triangle* AOC *is equal to the triangle* DOB, *shew that* AD *and* CB *are parallel.*

ON PROP. 40.

12. Deduce Prop. 40 from Prop. 39 by joining AE, AF in the figure of page 72.

Proposition 41. Theorem.

If a parallelogram and a triangle be on the same base and between the same parallels, the parallelogram shall be double of the triangle.



Let the parallelogram ABCD, and the triangle EBC be upon the same base BC, and between the same parallels BC, AE:

then shall the parallelogram ABCD be double of the triangle EBC.

Construction. Join AC.

Proof. Then the triangle ABC is equal to the triangle EBC, for they are on the same base BC, and between the same parallels BC, AE. I. 37.

But the parallelogram ABCD is double of the triangle ABC, for the diagonal AC bisects the parallelogram. I. 34.

Therefore the parallelogram ABCD is also double of the triangle EBC. Q.E.D.

EXERCISES.

1. ABCD is a parallelogram, and X, Y are the middle points of the sides AD, BC; if Z is any point in XY, or XY produced, shew that the triangle AZB is one quarter of the parallelogram ABCD.

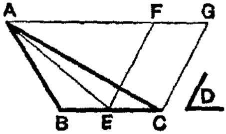
2. Describe a right-angled isosceles triangle equal to a given square.

3. If ABCD is a parallelogram, and XY any points in DC and AD respectively: shew that the triangles AXB, BYC are equal in area.

4. ABCD is a parallelogram, and P is any point within it; shew that the sum of the triangles PAB, PCD is equal to half the parallelogram.

Proposition 42. Problem.

To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given angle.



Let ABC be the given triangle, and D the given angle. It is required to describe a parallelogram equal to ABC, and having one of its angles equal to D.

Construction. Bisect BC at E. I. 10.

At E in CE, make the angle CEF equal to D; I. 23.

through A draw AFG parallel to EC; I. 31.

and through C draw CG parallel to EF.

Then FECG shall be the parallelogram required.

Join AE.

Proof. Now the triangles ABE, AEC are equal, for they are on equal bases BE, EC, and between the same parallels; I. 38.

therefore the triangle ABC is double of the triangle AEC.

But FECG is a parallelogram by construction; *Def.* 26.

and it is double of the triangle AEC,

for they are on the same base EC, and between the same parallels EC and AG. I. 41.

Therefore the parallelogram FECG is equal to the triangle ABC;

and it has one of its angles CEF equal to the given angle D.

Q. E. F.

EXERCISES.

1. Describe a parallelogram equal to a given square standing on the same base, and having an angle equal to half a right angle.

2. Describe a rhombus equal to a given parallelogram and standing on the same base. When does the construction fail?

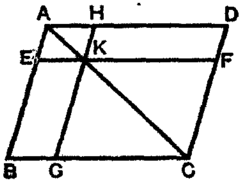
DEFINITION. If in the diagonal of a parallelogram any point is taken, and straight lines are drawn through it parallel to the sides of the parallelogram; then of the four parallelograms into which the whole figure is divided, the two through which the diagonal passes are called **Parallelograms about that diagonal**, and the other two, which with these make up the whole figure, are called the **complements** of the parallelograms about the diagonal.

Thus in the figure given below, AEKH, KGCF are parallelograms about the diagonal AC; and HKFD, EBGK are the complements of those parallelograms.

NOTE. A parallelogram is often named by *two* letters only, these being placed at opposite angular points.

PROPOSITION 43. THEOREM.

The complements of the parallelograms about the diagonal of any parallelogram, are equal to one another.



Let ABCD be a parallelogram, and KD, KB the complements of the parallelograms EH, GF about the diagonal AC: then shall the complement BK be equal to the complement KD.

Proof. Because EH is a parallelogram, and AK its diagonal, therefore the triangle AEK is equal to the triangle AHK. I. 34.

For a similar reason the triangle KGC is equal to the triangle KFC.

Hence the triangles AEK, KGC are together equal to the triangles AHK, KFC.

But the whole triangle ABC is equal to the whole triangle ADC, for AC bisects the parallelogram ABCD; I. 34.
therefore the remainder, the complement BK, is equal to the remainder, the complement KD. Q.E.D.

EXERCISES.

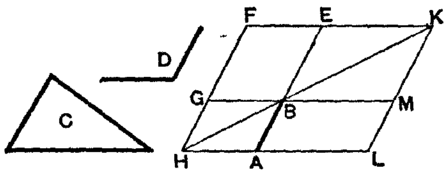
In the figure of Prop. 43, prove that

(i) The parallelogram ED is equal to the parallelogram BH.

(ii) If KB, KD are joined, the triangle AKB is equal to the triangle AKD.

Proposition 44. Problem.

To a given straight line to apply a parallelogram which shall be equal to a given triangle, and have one of its angles equal to a given angle.



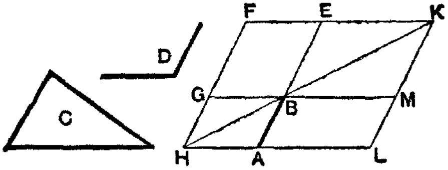
Let AB be the given straight line, C the given triangle, and D the given angle.

It is required to apply to the straight line AB a parallelogram equal to the triangle C, and having an angle equal to the angle D.

Construction. On AB produced describe a parallelogram BEFG equal to the triangle C, and having the angle EBG equal to the angle D; I. 22 *and* I. 42*.
through A draw AH parallel to BG or EF, to meet FG produced in H. I. 31.

Join HB.

* This step of the construction is effected by first describing on AB produced a triangle whose sides are respectively equal to those of the triangle C (I. 22); and by then making a parallelogram equal to the triangle so drawn, and having an angle equal to D (I. 42).



Then because AH and EF are parallel, and HF meets them, therefore the angles AHF, HFE are together equal to two right angles: I. 29.
hence the angles BHF, HFE are together less than two right angles;
therefore HB and FE will meet if produced towards B and E. *Ax.* 12.

Produce them to meet at K.

Through K draw KL parallel to EA or FH; I. 31.
and produce HA, GB to meet KL in the points L and M.

Then shall BL be the parallelogram required.

Proof. Now FHLK is a parallelogram, *Constr.*
and LB, BF are the complements of the parallelograms about the diagonal HK:

therefore LB is equal to BF. I. 43.

But the triangle C is equal to BF; *Constr.*

therefore LB is equal to the triangle C.

And because the angle GBE is equal to the vertically opposite angle ABM, I. 15.

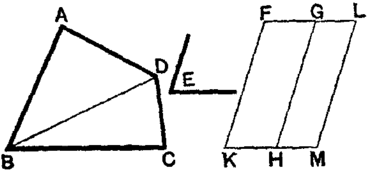
and is likewise equal to the angle D; *Constr.*

therefore the angle ABM is equal to the angle D.

Therefore the parallelogram LB, which is applied to the straight line AB, is equal to the triangle C, and has the angle ABM equal to the angle D. Q.E.F.

Proposition 45. Problem.

To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given angle.



Let ABCD be the given rectilineal figure, and E the given angle.

It is required to describe a parallelogram equal to ABCD, and having an angle equal to E.

Suppose the given rectilineal figure to be a quadrilateral.

Construction. Join BD.

Describe the parallelogram FH equal to the triangle ABD, and having the angle FKH equal to the angle E. I. 42.

To GH apply the parallelogram GM, equal to the triangle DBC, and having the angle GHM equal to E. I. 44.

Then shall FKML be the parallelogram required.

Proof. Because each of the angles GHM, FKH is equal to E, therefore the angle FKH is equal to the angle GHM.

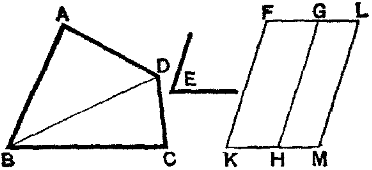
To each of these equals add the angle GHK;

then the angles FKH, GHK are together equal to the angles GHM, GHK.

But since FK, GH are parallel, and KH meets them, therefore the angles FKH, GHK are together equal to two right angles: I. 29.

therefore also the angles GHM, GHK are together equal to two right angles:

therefore KH, HM are in the same straight line. I. 14.



Again, because KM, FG are parallel, and HG meets them, therefore the alternate angles MHG, HGF are equal: I. 29

to each of these equals add the angle HGL;

then the angles MHG, HGL are together equal to the angles HGF, HGL.

But because HM, GL are parallel, and HG meets them, therefore the angles MHG, HGL are together equal to two right angles: I. 29.

therefore also the angles HGF, HGL are together equal to two right angles:

therefore FG, GL are in the same straight line. I. 14.

And because KF and ML are each parallel to HG, *Constr.*

therefore KF is parallel to ML; I. 30.

and KM, FL are parallel; *Constr.*

therefore FKML is a parallelogram. *Def.* 26.

And because the parallelogram FH is equal to the triangle ABD, *Constr.*

and the parallelogram GM to the triangle DBC; *Constr.*

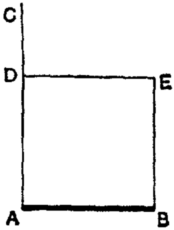
therefore the whole parallelogram FKML is equal to the whole figure ABCD;

and it has the angle FKM equal to the angle E.

By a series of similar steps, a parallelogram may be constructed equal to a rectilineal figure of more than four sides. Q.E.F.

PROPOSITION 46. PROBLEM.

To describe a square on a given straight line.



Let AB be the given straight line:
it is required to describe a square on AB.

Constr. From A draw AC at right angles to AB; I. 11.
and make AD equal to AB. I. 3.
Through D draw DE parallel to AB; I. 31.
and through B draw BE parallel to AD, meeting DE in E.
Then shall ADEB be a square.

Proof. For, by construction, ADEB is a parallelogram:
therefore AB is equal to DE, and AD to BE. I. 34.
But AD is equal to AB; *Constr.*
therefore the four straight lines AB, AD, DE, EB are equal to one another;
that is, the figure ADEB is equilateral.

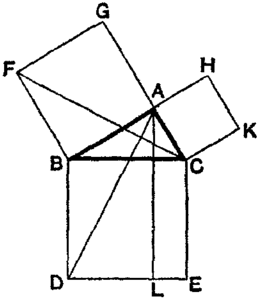
Again, since AB, DE are parallel, and AD meets them,
therefore the angles BAD, ADE are together equal to two right angles; I. 29.
but the angle BAD is a right angle; *Constr.*
therefore also the angle ADE is a right angle.
And the opposite angles of a parallelogram are equal; I. 34.
therefore each of the angles DEB, EBA is a right angle:
that is the figure ADEB is rectangular.
Hence it is a square, and it is described on AB.

Q.E.F.

COROLLARY. *If one angle of a parallelogram is a right angle, all its angles are right angles.*

PROPOSITION 47. THEOREM.

In a right-angled triangle the square described on the hypotenuse is equal to the sum of the squares described on the other two sides.



Let ABC be a right-angled triangle, having the angle BAC a right angle:
then shall the square described on the hypotenuse BC be equal to the sum of the squares described on BA, AC.

Construction. On BC describe the square BDEC; I. 46.
and on BA, AC describe the squares BAGF, ACKH.
Through A draw AL parallel to BD or CE; I. 31.
and join AD, FC.

Proof. Then because each of the angles BAC, BAG is a right angle,
therefore CA and AG are in the same straight line. I. 14.

Now the angle CBD is equal to the angle FBA,
for each of them is a right angle.
Add to each the angle ABC:
then the whole angle ABD is equal to the whole angle FBC.

Then in the triangles ABD, FBC,

Because { AB is equal to FB,
and BD is equal to BC,
also the angle ABD is equal to the angle FBC ;

therefore the triangle ABD is equal to the triangle FBC. I. 4.

Now the parallelogram BL is double of the triangle ABD, for they are on the same base BD, and between the same parallels BD, AL. I. 41.

And the square GB is double of the triangle FBC, for they are on the same base FB, and between the same parallels FB, GC. I. 41.

But doubles of equals are equal: *Ax.* 6.

therefore the parallelogram BL is equal to the square GB.

In a similar way, by joining AE, BK, it can be shewn that the parallelogram CL is equal to the square CH.

Therefore the whole square BE is equal to the sum of the squares GB, HC :

that is, the square described on the hypotenuse BC is equal to the sum of the squares described on the two sides BA, AC. Q.E.D.

NOTE. It is not necessary to the proof of this Proposition that the three squares should be described *external* to the triangle ABC; and since *each* square may be drawn either *towards* or *away from* the triangle, it may be shewn that there are $2 \times 2 \times 2$, or *eight*, possible constructions.

EXERCISES.

1. In the figure of this Proposition, shew that

    (i) If BG, CH are joined, these straight lines are parallel;

    (ii) The points F, A, K are in one straight line;

    (iii) FC and AD are at right angles to one another;

    (iv) If GH, KE, FD are joined, the triangle GAH is equal to the given triangle in all respects; and the triangles FBD, KCE are each equal in area to the triangle ABC. [See Ex. 9, p. 73.]

2. On the sides AB, AC of *any* triangle ABC, squares ABFG, ACKH are described both toward the triangle, or both on the side remote from it: shew that the straight lines BH and CG are equal.

3. On the sides of any triangle ABC, equilateral triangles BCX, CAY, ABZ are described, all externally, or all towards the triangle: shew that AX, BY, CZ are all equal.

4. *The square described on the diagonal of a given square, is double of the given square.*

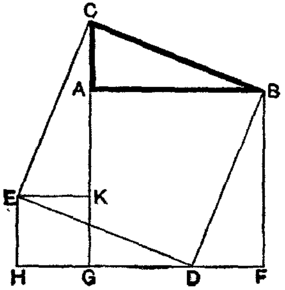
5. ABC *is an equilateral triangle, and* AX *is the perpendicular drawn from* A *to* BC: *shew that the square on* AX *is three times the square on* BX.

6. Describe a square equal to the sum of two given squares.

7. From the vertex A of a triangle ABC, AX is drawn perpendicular to the base: shew that the difference of the squares on the sides AB and AC, is equal to the difference of the squares on BX and CX, the segments of the base.

8. If from any point O within a triangle ABC, perpendiculars OX, OY, OZ are drawn to the sides BC, CA, AB respectively; shew that the sum of the squares on the segments AZ, BX, CY is equal to the sum of the squares on the segments AY, CX, BZ.

PROPOSITION 47. ALTERNATIVE PROOF.



Let CAB be a right-angled triangle, having the angle at A a right angle:

then shall the square on the hypotenuse BC be equal to the sum of the squares on BA, AC.

On AB describe the square ABFG. I. 46.
From FG and GA cut off respectively FD and GK, each equal to AC. I. 3.
On GK describe the square GKEH: I. 46.
then HG and GF are in the same straight line. I. 14.
Join CE, ED, DB.

It will first be shewn that the figure CEDB is the square on CB.

Now CA is equal to KG; add to each AK:
therefore CK is equal to AG.
Similarly DH is equal to GF:
hence the four lines BA, CK, DH, BF are all equal.

Then in the triangles BAC, CKE,

Because { BA is equal to CK, *Proved.*
and AC is equal to KE; *Constr.*
also the contained angle BAC is equal to the contained angle CKE, being right angles;

therefore the triangles BAC, CKE are equal in all respects. I. 4.
Similarly the four triangles BAC, CKE, DHE, BFD may be shewn to be equal in all respects.
Therefore the four straight lines BC, CE, ED, DB are all equal;
that is, the figure CEDB is equilateral.

Again the angle CBA is equal to the angle DBF; *Proved.*
add to each the angle ABD:
then the angle CBD is equal to the angle ABF:
therefore the angle CBD is a right angle.
Hence the figure CEDB is the square on BC. *Def.* 28.

And EHGK is equal to the square on AC. *Constr.*

Now the square on CEDB is made up of the two triangles BAC, CKE, and the rectilineal figure AKEDB;
therefore the square CEDB is equal to the triangles EHD, DFB together with the same rectilineal figure;
but these make up the squares EHGK, AGFB:
hence the square CEDB is equal to the sum of the squares EHGK, AGFB:
that is, the square on the hypotenuse BC is equal to the sum of the squares on the two sides CA, AB. Q. E. D.

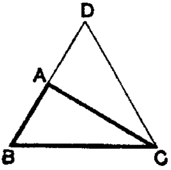
Obs. The following properties of a square, though not formally enunciated by Euclid, are employed in subsequent proofs. [See I. 48.]

(i) *The squares on equal straight lines are equal.*

(ii) *Equal squares stand upon equal straight lines.*

PROPOSITION 48. THEOREM.

If the square described on one side of a triangle be equal to the sum of the squares described on the other two sides, then the angle contained by these two sides shall be a right angle.



Let ABC be a triangle; and let the square described on BC be equal to the sum of the squares described on BA, AC: then shall the angle BAC be a right angle.

Construction. From A draw AD at right angles to AC; I. 11.
and make AD equal to AB. I. 3.
Join DC.

Proof. Then, because AD is equal to AB, *Constr.*
therefore the square on AD is equal to the square on AB.
To each of these add the square on CA;
then the sum of the squares on CA, AD is equal to the sum of the squares on CA, AB.
But, because the angle DAC is a right angle, *Constr.*
therefore the square on DC is equal to the sum of the squares on CA, AD. I. 47.
And, by hypothesis, the square on BC is equal to the sum of the squares on CA, AB;
therefore the square on DC is equal to the square on BC:
therefore also the side DC is equal to the side BC.
Then in the triangles DAC, BAC,

Because { DA is equal to BA, *Constr.*
and AC is common to both;
also the third side DC is equal to the third side BC; *Proved.*

therefore the angle DAC is equal to the angle BAC. I. 8.
But DAC is a right angle; *Constr.*
therefore also BAC is a right angle. Q. E. D.

THEOREMS AND EXAMPLES ON BOOK I.

INTRODUCTORY.

HINTS TOWARDS THE SOLUTION OF GEOMETRICAL EXERCISES. ANALYSIS. SYNTHESIS.

It is commonly found that exercises in Pure Geometry present to a beginner far more difficulty than examples in any other branch of Elementary Mathematics. This seems to be due to the following causes.

(i) The main Propositions in the text of Euclid must be not merely understood, but thoroughly digested, before the exercises depending upon them can be successfully attempted.

(ii) The variety of such exercises is practically unlimited; and it is impossible to lay down for their treatment any definite methods, such as the student has been accustomed to find in the rules of Elementary Arithmetic and Algebra.

(iii) The arrangement of Euclid's Propositions, though perhaps the most *convincing* of all forms of argument, affords in most cases little clue as to the way in which the proof or construction *was discovered*.

Euclid's propositions are arranged **synthetically**: that is to say, they start from the hypothesis or data; they next proceed to a construction in accordance with postulates, and problems already solved; then by successive steps based on known theorems, they finally establish the result indicated by the enunciation.

Thus Geometrical Synthesis is a *building up* of *known* results, in order to obtain a *new* result.

But as this is not the way in which constructions or proofs are usually discovered, we draw the attention of the student to the following hints.

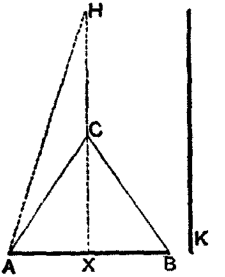
Begin by *assuming* the result it is desired to establish; then by working backwards, trace the consequences of the assumption, and try to ascertain its dependence on some simpler theorem which is already known to be true, or on some condition which suggests the necessary construction. If this attempt is successful, the steps of the argument may in general be re-arranged in reverse order, and the construction and proof presented in a synthetic form.

This unravelling of the conditions of a proposition in order to trace it back to some earlier principle on which it depends, is called **geometrical analysis**: it is the natural way of attacking most exercises of a more difficult type, and it is especially adapted to the solution of *problems*.

These directions are so general that they cannot be said to amount to a *method:* all that can be claimed for Geometrical Analysis is that it furnishes a mode of *searching for a suggestion*, and its success will necessarily depend on the skill and ingenuity with which it is employed: these may be expected to come with experience, but a thorough grasp of the chief Propositions of Euclid is essential to attaining them.

The practical application of these hints is illustrated by the following examples.

1. *Construct an isosceles triangle having given the base, and the sum of one of the equal sides and the perpendicular drawn from the vertex to the base.*



Let AB be the given base, and K the sum of one side and the perpendicular drawn from the vertex to the base.

ANALYSIS. *Suppose* ABC *to be the required triangle.*

From C draw CX perpendicular to AB:
then AB is bisected at X. I. 26.
Now if we produce XC to H, making XH equal to K,
it follows that CH = CA;
and if AH is joined,
we notice that the angle CAH = the angle CHA. I. 5.

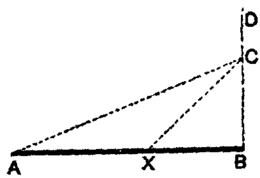
Now the straight lines XH and AH can be drawn *before the position of* C *is known;*

Hence we have the following construction, which we arrange synthetically.

SYNTHESIS. Bisect AB at X:
from X draw XH perpendicular to AB, making XH equal to K.
Join AH.
At the point A in HA, *make the angle* HAC *equal to the angle* AHX; and join CB.
Then ACB shall be the triangle required.

First the triangle is isosceles, for AC = BC. I. 4.
Again, since the angle HAC = the angle AHC, *Constr.*
∴ HC = AC. I. 6.
To each add CX;
then the sum of AC, CX = the sum of HC, CX
= HX.
That is, the sum of AC, CX = K. Q.E.F.

2. *To divide a given straight line so that the square on one part may be double of the square on the other.*



Let AB be the given straight line.

ANALYSIS. Suppose AB to be divided as required at X: that is, suppose the square on AX to be double of the square on XB.

Now we remember that in an isosceles right-angled triangle, the square on the hypotenuse is double of the square on either of the equal sides.

This suggests to us to draw BC perpendicular to AB, and to make BC equal to BX.

Join XC.

Then the square on XC is double of the square on XB, I. 47.
∴ XC = AX.
And when we join AC, we notice that
the angle XAC = the angle XCA. I. 5.
Hence the exterior angle CXB is double of the angle XAC. I. 32.
But the angle CXB is half of a right angle: I. 32.
∴ *the angle* XAC *is one-fourth of a right angle.*

This supplies the clue to the following construction:—

SYNTHESIS. From B draw BD perpendicular to AB;
and from A draw AC, *making* BAC *one-fourth of a right angle.*
From C, the intersection of AC and BD, draw CX, making the angle ACX equal to the angle BAC. I. 23.

Then AB shall be divided as required at X.

For since the angle XCA = the angle XAC,
∴ XA = XC. I. 6.
And because the angle BXC = the sum of the angles BAC, ACX, I. 32.
∴ the angle BXC is half a right angle;
and the angle at B is a right angle;
therefore the angle BCX is half a right angle; I. 32.
therefore the angle BXC = the angle BCX;
∴ BX = BC.
Hence the square on XC is double of the square on XB: I. 47.
that is, the square on AX is double of the square on XB. Q.E.F.

I. ON THE IDENTICAL EQUALITY OF TRIANGLES.

See Propositions 4, 8, 26.

1. If in a triangle the perpendicular from the vertex on the base bisects the base, then the triangle is isosceles.

2. If the bisector of the vertical angle of a triangle is also perpendicular to the base, the triangle is isosceles.

3. If the bisector of the vertical angle of a triangle also bisects the base, the triangle is isosceles.

[Produce the bisector, and complete the construction after the manner of I. 16.]

4. If in a triangle a pair of straight lines drawn from the extremities of the base, making equal angles with the sides, are equal, the triangle is isosceles.

5. If in a triangle the perpendiculars drawn from the extremities of the base to the opposite sides are equal, the triangle is isosceles.

6. Two triangles ABC, ABD on the same base AB, and on opposite sides of it, are such that AC is equal to AD, and BC is equal to BD: shew that the line joining the points C and D is perpendicular to AB.

7. If from the extremities of the base of an isosceles triangle perpendiculars are drawn to the opposite sides, shew that the straight line joining the vertex to the intersection of these perpendiculars bisects the vertical angle.

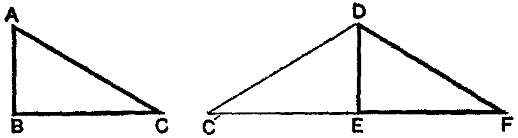
8. ABC is a triangle in which the vertical angle BAC is bisected by the straight line AX: from B draw BD perpendicular to AX, and produce it to meet AC, or AC produced, in E; then shew that BD is equal to DE.

9. In a quadrilateral ABCD, AB is equal to AD, and BC is equal to DC: shew that the diagonal AC bisects each of the angles which it joins.

10. In a quadrilateral ABCD the opposite sides AD, BC are equal, and also the diagonals AC, BD are equal: if AC and BD intersect at K, shew that each of the triangles AKB, DKC is isosceles.

11. If one angle of a triangle be equal to the sum of the other two, the greatest side is double of the distance of its middle point from the opposite angle.

12. *Two right-angled triangles which have their hypotenuses equal, and one side of one equal to one side of the other, are identically equal.*



Let ABC, DEF be two $\triangle^s$ right-angled at B and E, having AC equal to DF, and AB equal to DE:

then shall the $\triangle^s$ be identically equal.

For apply the $\triangle$ ABC to the $\triangle$ DEF, so that A may fall on D, and AB along DE; and so that C may fall on the side of DE remote from F.

Let C′ be the point on which C falls.

Then since AB = DE,

∴ B must fall on E;

so that DEC′ represents the $\triangle$ ABC in its new position.

Now each of the $\angle^s$ DEF, DEC′ is a rt. ∟; *Hyp.*

∴ EF and EC′ are in one st. line. I. 14.

Then in the $\triangle$ C′DF,

because DF = DC′,

∴ the $\angle$ DFC′ = the $\angle$ DC′F. I. 5.

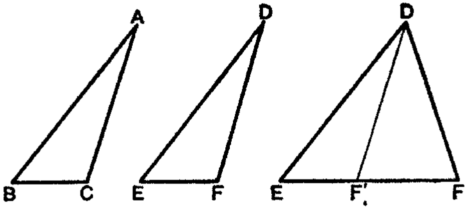
Hence in the two $\triangle^s$ DEF, DEC′,

Because { the $\angle$ DEF = the $\angle$ DEC′, being rt. ∟s; and the $\angle$ DFE = the $\angle$ DC′E; *Proved.* also the side DE is common to both;

∴ the $\triangle^s$ DEF, DEC′ are equal in all respects; I. 26.

that is, the $\triangle^s$ DEF, ABC are equal in all respects. Q.E.D.

18. *If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles opposite to one pair of equal sides equal, then the angles opposite to the other pair of equal sides are either equal or supplementary, and in the former case the triangles are equal in all respects.*



Let ABC, DEF be two triangles, having the side AB equal to the side DE, the side AC equal to the side DF, and the ∠ ABC equal to the ∠ DEF; then shall the ∠s ACB, DFE be either equal or supplementary, and in the former case the triangles shall be equal in all respects.

If the ∠ BAC = the ∠ EDF,
then the triangles are equal in all respects. I. 4.

But if the ∠ BAC be not equal to the ∠ EDF, one of them must be the greater.
Let the ∠ EDF be greater than the ∠ BAC.
At D in ED make the ∠ EDF' equal to the ∠ BAC.
Then the △s BAC, EDF' are equal in all respects. I. 26.
∴ AC = DF';
but AC = DF; *Hyp.*
∴ DF = DF',
∴ the ∠ DFF' = the ∠ DF'F. I. 5.
But the ∠s DF'F, DF'E are supplementary, I. 13.
∴ the ∠s DFF', DF'E are supplementary:
that is, the ∠s DFE, ACB are supplementary.

Q.E.D.

Three cases of this theorem deserve special attention.

It has been proved that if the angles ACB, DFE are not *equal*, they are *supplementary*;

And we know that of angles which are supplementary and unequal, one must be acute and the other obtuse.

COROLLARIES. Hence, in addition to the hypothesis of this theorem,

(i) If the angles ACB, DFE, opposite to the two equal sides AB, DE are both acute, both obtuse, or if one of them is a right angle,
it follows that these angles are equal,
and therefore that the triangles are equal in all respects.

(ii) If the two given angles are right angles or obtuse angles, it follows that the angles ACB, DFE must be both acute, and therefore equal, by (i):
so that the triangles are equal in all respects.

(iii) If in each triangle the side opposite the given angle is not less than the other given side; that is, if AC and DF are not less than AB and DE respectively, then
the angles ACB, DFE cannot be greater than the angles ABC, DEF respectively;
therefore the angles ACB, DFE, are both acute;
hence, as above, they are equal;
and the triangles ABC, DEF are equal in all respects.

II. ON INEQUALITIES.

See Propositions 16, 17, 18, 19, 20, 21, 24, 25.

1. In a triangle ABC, if AC is not greater than AB, shew that any straight line drawn through the vertex A, and terminated by the base BC, is less than AB.

2. ABC *is a triangle, and the vertical angle* BAC *is bisected by a straight line which meets the base* BC *in* X; *shew that* BA *is greater than* BX, *and* CA *greater than* CX. *Hence obtain a proof of* I. 20.

3. *The perpendicular is the shortest straight line that can be drawn from a given point to a given straight line; and of others, that which is nearer to the perpendicular is less than the more remote; and two, and only two equal straight lines can be drawn from the given point to the given straight line, one on each side of the perpendicular.*

4. The sum of the distances of any point from the three angular points of a triangle is greater than half its perimeter.

5. The sum of the distances of any point within a triangle from its angular points is less than the perimeter of the triangle.

6. The perimeter of a quadrilateral is greater than the sum of its diagonals.

7. The sum of the diagonals of a quadrilateral is less than the sum of the four straight lines drawn from the angular points to any given point. Prove this, and point out the exceptional case.

8. *In a triangle any two sides are together greater than twice the median which bisects the remaining side.* [See Def. p. 73.]

[Produce the median, and complete the construction after the manner of I. 16.]

9. *In any triangle the sum of the medians is less than the perimeter.*

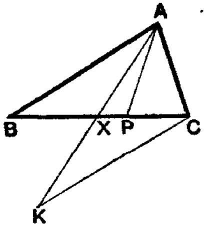
10. In a triangle an angle is acute, obtuse, or a right angle, according as the median drawn from it is greater than, less than, or equal to half the opposite side. [See Ex. 4, p. 59.]

11. The diagonals of a rhombus are unequal.

12. *If the vertical angle of a triangle is contained by unequal sides, and if from the vertex the median and the bisector of the angle are drawn, then the median lies within the angle contained by the bisector and the longer side.*

Let ABC be a △, in which AB is greater than AC; let AX be the median drawn from A, and AP the bisector of the vertical ∠ BAC:

then shall AX lie between AP and AB.



Produce AX to K, making XK equal to AX. Join KC.

Then the △ˢ BXA, CXK may be shewn to be equal in all respects; I. 4.

hence BA = CK, and the ∠ BAX = the ∠ CKX.

But since BA is greater than AC, *Hyp.*

∴ CK is greater than AC;

∴ the ∠ CAK is greater than the ∠ CKA: I. 18.

that is, the ∠ CAX is greater than the ∠ BAX:

∴ the ∠ CAX must be more than half the vert. ∠ BAC;

hence AX lies within the angle BAP. Q.E.D.

13. *If two sides of a triangle are unequal, and if from their point of intersection three straight lines are drawn, namely the bisector of the vertical angle, the median, and the perpendicular to the base, the first is intermediate in position and magnitude to the other two.*

III. ON PARALLELS.

See Propositions 27—31.

1. If a straight line meets two parallel straight lines, and the two interior angles on the same side are bisected; shew that the bisectors meet at right angles. [I. 29, I. 32.]

2. The straight lines drawn from any point in the bisector of an angle parallel to the arms of the angle, and terminated by them, are equal; and the resulting figure is a rhombus.

3. AB and CD are two straight lines intersecting at D, and the adjacent angles so formed are bisected: if through any point X in DC a straight line YXZ be drawn parallel to AB and meeting the bisectors in Y and Z, shew that XY is equal to XZ.

4. If two straight lines are parallel to two other straight lines, each to each; and if the angles contained by each pair are bisected; shew that the bisecting lines are parallel.

5. The middle point of any straight line which meets two parallel straight lines, and is terminated by them, is equidistant from the parallels.

6. A straight line drawn between two parallels and terminated by them, is bisected; shew that any other straight line passing through the middle point and terminated by the parallels, is also bisected at that point.

7. If through a point equidistant from two parallel straight lines, two straight lines are drawn cutting the parallels, the portions of the latter thus intercepted are equal.

PROBLEMS.

8. AB and CD are two given straight lines, and X is a given point in AB: find a point Y in AB such that YX may be equal to the perpendicular distance of Y from CD.

9. ABC is an isosceles triangle; required to draw a straight line DE parallel to the base BC, and meeting the equal sides in D and E, so that BD, DE, EC may be all equal.

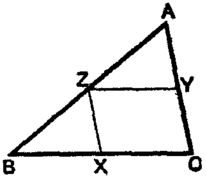
10. ABC is any triangle; required to draw a straight line DE parallel to the base BC, and meeting the other sides in D and E, so that DE may be equal to the sum of BD and CE.

11. ABC is any triangle; required to draw a straight line parallel to the base BC, and meeting the other sides in D and E, so that DE may be equal to the difference of BD and CE.

IV. ON PARALLELOGRAMS.

See Propositions 33, 34, and the deductions from these Props. given on page 64.

1. *The straight line drawn through the middle point of a side of a triangle parallel to the base, bisects the remaining side.*

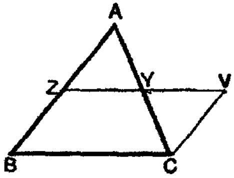


Let ABC be a $\triangle$, and Z the middle point of the side AB. Through Z, ZY is drawn par^{l} to BC; then shall Y be the middlepoint of AC.

Through Z draw ZX par^{l} to AC. I. 31.

Then in the $\triangle^{s}$ AZY, ZBX,
because ZY and BC are par^{l},
$\therefore$ the $\angle$ AZY = the $\angle$ ZBX; I. 29.
and because ZX and AC are par^{l},
$\therefore$ the $\angle$ ZAY = the $\angle$ BZX; I. 29.
also AZ = ZB; *Hyp.*
$\therefore$ AY = ZX. I. 26.
But ZXCY is a par^{m} by construction;
$\therefore$ ZX = YC. I. 34.
Hence AY = YC;
that is, AC is bisected at Y. Q.E.D.

2. *The straight line which joins the middle points of two sides of a triangle, is parallel to the third side.*



Let ABC be a $\triangle$, and Z, Y the middle points of the sides AB, AC:
then shall ZY be par^{l} to BC.
Produce ZY to V, making YV equal to ZY.
Join CV.
Then in the $\triangle^{s}$ AYZ, CYV,

Because { AY = CY, *Hyp.*
and YZ = YV, *Constr.*
and the $\angle$ AYZ = the vert. opp. $\angle$ CYV; I. 15.

$\therefore$ AZ = CV, I. 4.
and the $\angle$ ZAY = the $\angle$ VCY;
hence CV is par^{l} to AZ. I. 27.
But CV is equal to AZ, that is, to BZ: *Hyp.*
$\therefore$ CV is equal and par^{l} to BZ:
$\therefore$ ZV is equal and par^{l} to BC: I. 33.
that is, ZY is par^{l} to BC. Q.E.D.

[A second proof of this proposition may be derived from I. 38, 39.]

3. *The straight line which joins the middle points of two sides of a triangle is equal to half the third side.*

4. *Shew that the three straight lines which join the middle points of the sides of a triangle, divide it into four triangles which are identically equal.*

5. *Any straight line drawn from the vertex of a triangle to the base is bisected by the straight line which joins the middle points of the other sides of the triangle.*

6. Given the three middle points of the sides of a triangle, construct the triangle.

7. AB, AC are two given straight lines, and P is a given point between them; required to draw through P a straight line terminated by AB, AC, and bisected by P.

8. ABCD is a parallelogram, and X, Y are the middle points of the opposite sides AD, BC: shew that BX and DY trisect the diagonal AC.

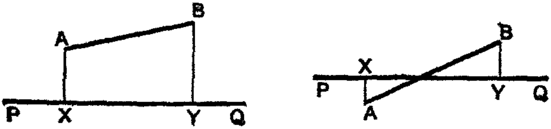
9. *If the middle points of adjacent sides of any quadrilateral be joined, the figure thus formed is a parallelogram.*

10. Shew that the straight lines which join the middle points of opposite sides of a quadrilateral, bisect one another.

11. The straight line which joins the middle points of the oblique sides of a trapezium, is parallel to the two parallel sides, and passes through the middle points of the diagonals.

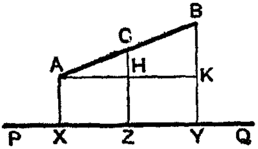
12. *The straight line which joins the middle points of the oblique sides of a trapezium is equal to half the* sum *of the parallel sides; and the portion intercepted between the diagonals is equal to half the* difference *of the parallel sides.*

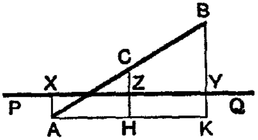
Definition. If from the extremities of one straight line perpendiculars are drawn to another, the portion of the latter intercepted between the perpendiculars is said to be the **Orthogonal Projection** of the first line upon the second.



Thus in the adjoining figures, if from the extremities of the straight line AB the perpendiculars AX, BY are drawn to PQ, then XY is the *orthogonal projection* of AB on PQ.

13. *A given straight line* AB *is bisected at* C; *shew that the projections of* AC, CB *on any other straight line are equal.*





Let XZ, ZY be the projections of AC, CB on any straight line PQ: then XZ and ZY shall be equal.

Through A draw a straight line parallel to PQ, meeting CZ, BY or these lines produced, in H, K. I. 31.

Now AX, CZ, BY are parallel, for they are perp. to PQ; I. 28.
∴ the figures XH, HY are par^{ms};
∴ AH = XZ, and HK = ZY. I. 34.

But through C, the middle point of AB, a side of the △ ABK, CH has been drawn parallel to the side BK;
∴ CH bisects AK; Ex. 1, p. 96.
that is, AH = HK;
∴ XZ = ZY. Q.E.D.

14. *If three parallel straight lines make equal intercepts on a fourth straight line which meets them, they will also make equal intercepts on any other straight line which meets them.*

15. *Equal and parallel straight lines have equal projections on any other straight line.*

16. AB is a given straight line bisected at O; and AX, BY are perpendiculars drawn from A and B on any other straight line: shew that OX is equal to OY.

17. AB *is a given straight line bisected at* O: *and* AX, BY *and* OZ *are perpendiculars drawn to any straight line* PQ, *which does not pass between* A *and* B: *shew that* OZ *is equal to half the sum of* AX, BY.

[OZ is said to be the **Arithmetic Mean** between AX and BY.]

18. AB is a given straight line bisected at O; and through A, B and O parallel straight lines are drawn to meet a given straight line PQ in X, Y, Z: shew that OZ is equal to half the *sum*, or half the *difference* of AX and BY, according as A and B lie on the *same* side or on *opposite* sides of PQ.

19. *To divide a given finite straight line into any number of equal parts.*

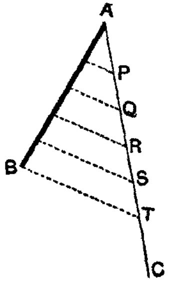
[For example, required to divide the straight line AB into *five* equal parts.

From A draw AC, a straight line of unlimited length, making any angle with AB.

In AC take *any* point P, and mark off successive parts PQ, QR, RS, ST each equal to AP.

Join BT; and through P, Q, R, S draw parallels to BT.

It may be shewn by Ex. 14, p. 98, that these parallels divide AB into five equal parts.]



20. *If through an angle of a parallelogram any straight line is drawn, the perpendicular drawn to it from the opposite angle is equal to the sum or difference of the perpendiculars drawn to it from the two remaining angles, according as the given straight line falls without the parallelogram, or intersects it.*

[Through the opposite angle draw a straight line parallel to the given straight line, so as to meet the perpendicular from one of the remaining angles, produced if necessary: then apply I. 34, I. 26. Or proceed as in the following example.]

21. From the angular points of a parallelogram perpendiculars are drawn to any straight line which is without the parallelogram: shew that the sum of the perpendiculars drawn from one pair of opposite angles is equal to the sum of those drawn from the other pair.

[Draw the diagonals, and from their point of intersection let fall a perpendicular upon the given straight line. See Ex. 17, p. 98.]

22. The sum of the perpendiculars drawn from any point in the base of an isosceles triangle to the equal sides is equal to the perpendicular drawn from either extremity of the base to the opposite side.

[It follows that the sum of the distances of *any* point in the base of an isosceles triangle from the equal sides is **constant**, that is, the same whatever point in the base is taken.]

23. In the base produced of an isosceles triangle any point is taken: shew that the difference of its distances from the equal sides is *constant*.

24. The sum of the perpendiculars drawn from any point within an equilateral triangle to the three sides is equal to the perpendicular drawn from any one of the angular points to the opposite side, and is therefore constant.

PROBLEMS.

[Problems marked (*) admit of more than one solution.]

*25. Draw a straight line through a given point, so that the part of it intercepted between two given parallel straight lines may be of given length.

26. Draw a straight line parallel to a given straight line, so that the part intercepted between two other given straight lines may be of given length.

27. Draw a straight line equally inclined to two given straight lines that meet, so that the part intercepted between them may be of given length.

28. AB, AC are two given straight lines, and P is a given point *without* the angle contained by them. It is required to draw through P a straight line to meet the given lines, so that the part intercepted between them may be equal to the part between P and the nearer line.

V. MISCELLANEOUS THEOREMS AND EXAMPLES.

Chiefly on I. 32.

1. *A is the vertex of an isosceles triangle* ABC, *and* BA *is produced to* D, *so that* AD *is equal to* BA; *if* DC *is drawn, shew that* BCD *is a right angle.*

2. *The straight line joining the middle point of the hypotenuse of a right-angled triangle to the right angle is equal to half the hypotenuse.*

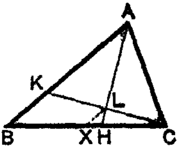
3. From the extremities of the base of a triangle perpendiculars are drawn to the opposite sides (produced if necessary); shew that the straight lines which join the middle point of the base to the feet of the perpendiculars are equal.

4. *In a triangle* ABC, AD *is drawn perpendicular to* BC; *and* X, Y, Z *are the middle points of the sides* BC, CA, AB *respectively: shew that each of the angles* ZXY, ZDY *is equal to the angle* BAC.

5. *In a right-angled triangle, if a perpendicular be drawn from the right angle to the hypotenuse, the two triangles thus formed are equiangular to one another.*

6. *In a right-angled triangle two straight lines are drawn from the right angle, one bisecting the hypotenuse, the other perpendicular to it: shew that they contain an angle equal to the difference of the two acute angles of the triangle.* [See above, Ex. 2 and Ex. 5.]

7. *In a triangle if a perpendicular be drawn from one extremity of the base to the bisector of the vertical angle,* (i) *it will make with either of the sides containing the vertical angle an angle equal to half the sum of the angles at the base;* (ii) *it will make with the base an angle equal to half the difference of the angles at the base.*



Let ABC be the given △, and AH the bisector of the vertical ∠ BAC.

Let CLK meet AH at right angles.

(i) Then shall each of the ∠s AKC, ACK be equal to half the sum of the ∠s ABC, ACB.

In the △s AKL, ACL,

Because { the ∠ KAL = the ∠ CAL, *Hyp.*
also the ∠ ALK = the ∠ ALC, being rt. ∟s;
and AL is common to both △s;

∴ the ∠ AKL = the ∠ ACL. I. 26.

Again, the ∠ AKC = the sum of the ∠s KBC, KCB; I. 32.

that is, the ∠ ACK = the sum of the ∠s KBC, KCB.

To each add the ∠ ACK,

then twice the ∠ ACK = the sum of the ∠s ABC, ACB,

∴ the ∠ ACK = half the sum of the ∠s ABC, ACB.

(ii) The ∠ KCB shall be equal to half the difference of the ∠s ACB, ABC.

As before, the ∠ ACK = the sum of the ∠s KBC, KCB.

To each of these add the ∠ KCB:

then the ∠ ACB = the ∠ KBC together with twice the ∠ KCB.

∴ twice the ∠ KCB = the difference of the ∠s ACB, KBC,

that is, the ∠ KCB = half the difference of the ∠s ACB, ABC.

COROLLARY. *If* X *be the middle point of the base, and* XL *be joined, it may be shewn by Ex.* 3, *p.* 97, *that* XL *is half* BK; *that is, that* XL *is half the difference of the sides* AB, AC.

8. *In any triangle the angle contained by the bisector of the vertical angle and the perpendicular from the vertex to the base is equal to half the difference of the angles at the base.* [See Ex. 3, p. 59.]

9. In a triangle ABC the side AC is produced to D, and the angles BAC, BCD are bisected by straight lines which meet at F; shew that they contain an angle equal to half the angle at B.

10. If in a right-angled triangle one of the acute angles is double of the other, shew that the hypotenuse is double of the shorter side.

11. If in a diagonal of a parallelogram any two points equidistant from its extremities be joined to the opposite angles, the figure thus formed will be also a parallelogram.

12. ABC is a given equilateral triangle, and in the sides BC, CA, AB the points X, Y, Z are taken respectively, so that BX, CY and AZ are all equal. AX, BY, CZ are now drawn, intersecting in P, Q, R: shew that the triangle PQR is equilateral.

13. If in the sides AB, BC, CD, DA of a parallelogram ABCD four points P, Q, R, S be taken in order, one in each side, so that AP, BQ, CR, DS are all equal; shew that the figure PQRS is a parallelogram.

14. In the figure of I. 1, if the circles intersect at F, and if CA and CB are produced to meet the circles in P and Q respectively; shew that the points P, F, Q are in the same straight line; and shew also that the triangle CPQ is equilateral.

[Problems marked (*) admit of more than one solution.]

15. Through two given points draw two straight lines forming with a straight line given in position, an equilateral triangle.

*16. From a given point it is required to draw to two parallel straight lines two equal straight lines at right angles to one another.

*17. Three given straight lines meet at a point; draw another straight line so that the two portions of it intercepted between the given lines may be equal to one another.

18. From a given point draw three straight lines of given lengths, so that their extremities may be in the same straight line, and intercept equal distances on that line. [See Fig. to I. 16.]

19. Use the properties of the equilateral triangle to trisect a given finite straight line.

20. In a given triangle inscribe a rhombus, having one of its angles coinciding with an angle of the triangle.

VI. ON THE CONCURRENCE OF STRAIGHT LINES IN A TRIANGLE.

DEFINITIONS. (i) Three or more straight lines are said to be **concurrent** when they meet in one point.

(ii) Three or more points are said to be **collinear** when they lie upon one straight line.

We here give some propositions relating to the concurrence of certain groups of straight lines drawn in a triangle: the importance of these theorems will be more fully appreciated when the student is familiar with Books III. and IV.

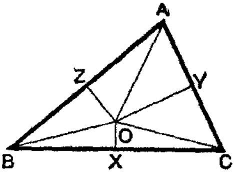
1. *The perpendiculars drawn to the sides of a triangle from their middle points are concurrent.*

Let ABC be a $\triangle$, and X, Y, Z the middle points of its sides:

then shall the perps drawn to the sides from X, Y, Z be concurrent.

From Z and Y draw perps to AB, AC; these perps, since they cannot be parallel, will meet at point O. *Ax.* 12.

Join OX.



It is required to prove that OX *is perp. to* BC.

Join OA, OB, OC.

In the $\triangle^s$ OYA, OYC,

Because { YA = YC, *Hyp.*
and OY is common to both;
also the $\angle$ OYA = the $\angle$ OYC, being rt. $\llcorner^s$. }

∴ OA = OC. I. 4.

Similarly, from the $\triangle^s$ OZA, OZB,
it may be proved that OA = OB.
Hence OA, OB, OC are all equal.

Again, in the $\triangle^s$ OXB, OXC

Because { BX = CX, *Hyp.*
and XO is common to both;
also OB = OC: *Proved.* }

∴ the $\angle$ OXB = the $\angle$ OXC, I. 8.

but these are adjacent $\angle^s$;

∴ they are rt. $\llcorner^s$; *Def.* 7.

that is, OX is perp. to BC.

Hence the three perps OX, OY, OZ meet in the point O.

Q. E. D.

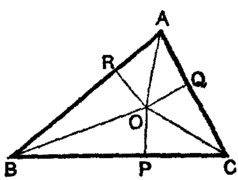
2. *The bisectors of the angles of a triangle are concurrent.*

Let ABC be a $\triangle$. Bisect the $\angle^s$ ABC, BCA, by straight lines which must meet at some point O. *Ax.* 12.

Join AO.

It is required to prove that AO *bisects the* $\angle$ BAC.

From O draw OP, OQ, OR perp. to the sides of the $\triangle$.

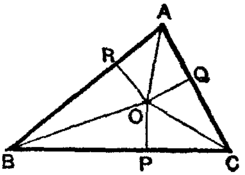


Then in the $\triangle^s$ OBP, OBR,

Because { the $\angle$ OBP = the $\angle$ OBR, *Constr.*
and the $\angle$ OPB = the $\angle$ ORB, being rt. $\llcorner^s$,
and OB is common; }

∴ OP = OR. I. 26.

Similarly from the △ˢ OCP, OCQ, it may be shewn that OP = OQ,
∴ OP, OQ, OR are all equal.



Again in the △ˢ ORA, OQA,

Because { the ∠ˢ ORA, OQA are rt. ∟ˢ, and the hypotenuse OA is common, also OR = OQ; *Proved.*

∴ the ∠ RAO = the ∠ QAO. Ex. 12, p. 91.

That is, AO is the bisector of the ∠ BAC.

Hence the bisectors of the three ∠ˢ meet at the point O.

Q. E. D.

3. *The bisectors of two exterior angles of a triangle and the bisector of the third angle are concurrent.*

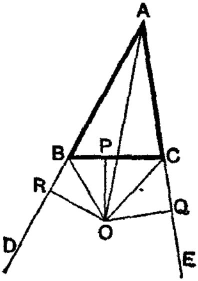
Let ABC be a △, of which the sides AB, AC are produced to any points D and E.

Bisect the ∠ˢ DBC, ECB by straight lines which must meet at some point O. *Ax.* 12.

Join AO.

It is required to prove that AO *bisects the angle* BAC.

From O draw OP, OQ, OR perp. to the sides of the △.



Then in the △ˢ OBP, OBR,

Because { the ∠ OBP = the ∠ OBR, *Constr.* also the ∠ OPB = the ∠ ORB, being rt. ∟ˢ, and OB is common;

∴ OP = OR. I. 26.

Similarly in the △ˢ OCP, OCQ, it may be shewn that OP = OQ:
∴ OP, OQ, OR are all equal.

Again in the △ˢ ORA, OQA,

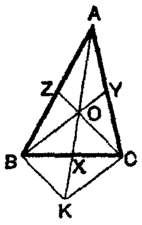
Because { the ∠ˢ ORA, OQA are rt. ∟ˢ, and the hypotenuse OA is common, also OR = OQ; *Proved.*

∴ the ∠ RAO = the ∠ QAO. Ex. 12, p. 91.

That is, AO is the bisector of the ∠ BAC.
∴ the bisectors of the two exterior ∠ˢ DBC, ECB, and of the interior ∠ BAC meet at the point O.

Q.E.D.

4. *The medians of a triangle are concurrent.*



Let ABC be a △. Let BY and CZ be two of its medians, and let them intersect at O.

Join AO,

and produce it to meet BC in X.

It is required to shew that AX *is the remaining median of the* △.

Through C draw CK parallel to BY:

produce AX to meet CK at K.

Join BK.

In the △ AKC,

because Y is the middle point of AC, and YO is parallel to CK,

∴ O is the middle point of AK. Ex. 1, p. 96.

Again in the △ ABK,

since Z and O are the middle points of AB, AK,

∴ ZO is parallel to BK, Ex. 2, p. 96.

that is, OC is parallel to BK:

∴ the figure BKCO is a par^m.

But the diagonals of a par^m bisect one another, Ex. 5, p. 64.

∴ X is the middle point of BC.

That is, AX is a median of the △.

Hence the three medians meet at the point O. Q.E.D.

COROLLARY. *The three medians of a triangle cut one another at a point of trisection, the greater segment in each being towards the angular point.*

For in the above figure it has been proved that

AO = OK,

also that OX is half of OK;

∴ OX is half of OA:

that is, OX is one third of AX.

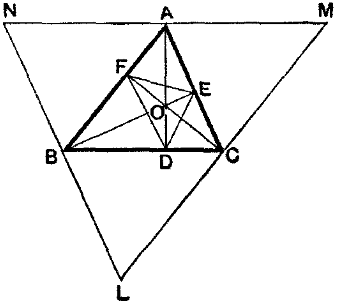
Similarly OY is one third of BY,

and OZ is one third of CZ. Q.E.D.

By means of this Corollary it may be shewn that in any triangle the shorter median bisects the greater side.

[The point of intersection of the three medians of a triangle is called the **centroid**. It is shewn in mechanics that a thin triangular plate will balance in any position about this point: therefore the centroid of a triangle is also its centre of gravity.]

5. *The perpendiculars drawn from the vertices of a triangle to the opposite sides are concurrent.*



Let ABC be a $\triangle$, and AD, BE, CF the three perp^s drawn from the vertices to the opposite sides:

then shall these perp^s be concurrent.

Through A, B, and C draw straight lines MN, NL, LM parallel to the opposite sides of the $\triangle$.

Then the figure BAMC is a par^m. *Def.* 26.

$\therefore$ AB = MC. I. 34.

Also the figure BACL is a par^m.

$\therefore$ AB = LC,

$\therefore$ LC = CM :

that is, C is the middle point of LM.

So also A and B are the middle points of MN and NL.

Hence AD, BE, CF are the perp^s to the sides of the $\triangle$ LMN from their middle points. Ex. 3, p. 54.

But these perp^s meet in a point: Ex. 1, p. 103.

that is, the perp^s drawn from the vertices of the $\triangle$ ABC to the opposite sides meet in a point. Q.E.D.

[For another proof see Theorems and Examples on Book III.]

DEFINITIONS.

(i) The intersection of the perpendiculars drawn from the vertices of a triangle to the opposite sides is called its **ortho-centre.**

(ii) The triangle formed by joining the feet of the perpendiculars is called the **pedal triangle.**

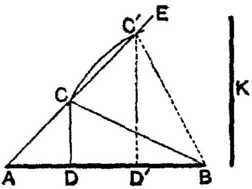
VII. ON THE CONSTRUCTION OF TRIANGLES WITH GIVEN PARTS.

No general rules can be laid down for the solution of problems in this section; but in a few typical cases we give constructions, which the student will find little difficulty in adapting to other questions of the same class.

1. *Construct a right-angled triangle, having given the hypotenuse and the sum of the remaining sides.*

It is required to construct a rt. angled △, having its hypotenuse equal to the given straight line K, and the sum of its remaining sides equal to AB.

From A draw AE making with BA an ∠ equal to half a rt. ∟. From centre B, with radius equal to K, describe a circle cutting AE in the points C, C′.



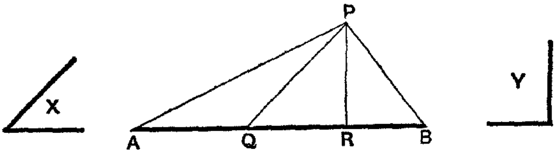
From C and C′ draw perp^s CD, C′D′ to AB; and join CB, C′B. Then either of the △^s CDB, C′D′B will satisfy the given conditions.

NOTE. If the given hypotenuse K be greater than the perpendicular drawn from B to AE, there will be *two* solutions. If the line K be equal to this perpendicular, there will be *one* solution; but if less, the problem is *impossible*.]

2. Construct a right-angled triangle, having given the hypotenuse and the difference of the remaining sides.

3. Construct an isosceles right-angled triangle, having given the sum of the hypotenuse and one side.

4. *Construct a triangle, having given the perimeter and the angles at the base.*



[Let AB be the perimeter of the required △, and X and Y the ∠^s at the base.

From A draw AP, making the ∠ BAP equal to half the ∠ X.
From B draw BP, making the ∠ ABP equal to half the ∠ Y.
From P draw PQ, making the ∠ APQ equal to the ∠ BAP.
From P draw PR, making the ∠ BPR equal to the ∠ ABP.
Then shall PQR be the required △.]

5. Construct a right-angled triangle, having given the perimeter and one acute angle.

6. Construct an isosceles triangle of given altitude, so that its base may be in a given straight line, and its two equal sides may pass through two fixed points. [See Ex. 7, p. 49.]

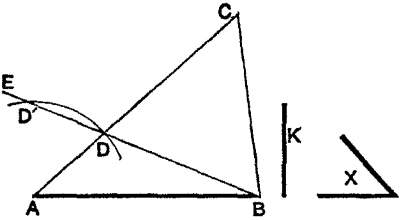
7. Construct an equilateral triangle, having given the length of the perpendicular drawn from one of the vertices to the opposite side.

8. Construct an isosceles triangle, having given the base, and the difference of one of the remaining sides and the perpendicular drawn from the vertex to the base. [See Ex. 1, p. 98.]

9. Construct a triangle, having given the base, one of the angles at the base, and the sum of the remaining sides.

10. Construct a triangle, having given the base, one of the angles at the base, and the difference of the remaining sides.

11. *Construct a triangle, having given the base, the difference of the angles at the base, and the difference of the remaining sides.*



[Let AB be the given base, X the difference of the ∠ˢ at the base, and K the difference of the remaining sides.

Draw BE, making the ∠ ABE equal to half the ∠ X.

From centre A, with radius equal to K, describe a circle cutting BE in D and D′. Let D be the point of intersection *nearer to* B.

Join AD and produce it to C.

Draw BC, making the ∠ DBC equal to the ∠ BDC.

Then shall CAB be the △ required. Ex. 7, p. 101.

NOTE. This problem is possible only when the given difference K is greater than the perpendicular drawn from A to BE.]

12. Construct a triangle, having given the base, the difference of the angles at the base, and the sum of the remaining sides.

13. Construct a triangle, having given the perpendicular from the vertex on the base, and the difference between each side and the adjacent segment of the base.

14. Construct a triangle, having given two sides and the median which bisects the remaining side. [See Ex. 18, p. 102.]

15. Construct a triangle, having given one side, and the medians which bisect the two remaining sides.

[See Fig. to Ex. 4, p. 105.
Let BC be the given side. Take two-thirds of each of the given medians; hence construct the triangle BOC. The rest of the construction follows easily.]

16. *Construct a triangle, having given its three medians.*

[See Fig. to Ex. 4, p. 105.
Take two-thirds of each of the given medians, and construct the triangle OKC. The rest of the construction follows easily.]

VIII. ON AREAS.

See Propositions 35—48.

It must be understood that throughout this section the word *equal* as applied to rectilineal figures will be used as denoting *equality of area* unless otherwise stated.

1. *Shew that a parallelogram is bisected by any straight line which passes through the middle point of one of its diagonals.* [I. 29, 26.]

2. Bisect a parallelogram by a straight line drawn through a given point.

3. Bisect a parallelogram by a straight line drawn perpendicular to one of its sides.

4. Bisect a parallelogram by a straight line drawn parallel to a given straight line.

5. ABCD *is a trapezium in which the side* AB *is parallel to* DC. *Shew that its area is equal to the area of a parallelogram formed by drawing through* X, *the middle point of* BC, *a straight line parallel to* AD. [I. 29, 26.]

6. A trapezium is equal to a parallelogram whose base is half the sum of the parallel sides of the given figure, and whose altitude is equal to the perpendicular distance between them.

7. ABCD is a trapezium in which the side AB is parallel to DC; shew that it is double of the triangle formed by joining the extremities of AD to X, the middle point of BC.

8. Shew that a trapezium is bisected by the straight line which joins the middle points of its parallel sides. [I. 38.]

In the following group of Exercises the proofs depend chiefly on Propositions 37 and 38, and the two converse theorems.

9. If two straight lines AB, CD intersect at X, and if the straight lines AC and BD, which join their extremities are parallel, shew that the triangle AXD is equal to the triangle BXC.

10. If two straight lines AB, CD intersect at X, so that the triangle AXD is equal to the triangle XCB, then AC and BD are parallel.

11. ABCD is a parallelogram, and X any point in the diagonal AC produced; shew that the triangles XBC, XDC are equal. [See Ex. 13, p. 64.]

12. ABC is a triangle, and R, Q the middle points of the sides AB, AC; shew that if BQ and CR intersect in X, the triangle BXC is equal to the quadrilateral AQXR. [See Ex. 5, p. 73.]

13. If the middle points of the sides of a quadrilateral be joined in order, the *parallelogram* so formed [see Ex. 9, p. 97] is equal to half the given figure.

14. Two triangles of equal area stand on the same base but on opposite sides of it: shew that the straight line joining their vertices is bisected by the base, or by the base produced.

15. The straight line which joins the middle points of the diagonals of a trapezium is parallel to each of the two parallel sides.

---

16. (i) *A triangle is equal to the sum or difference of two triangles on the same base (or on equal bases), if the altitude of the former is equal to the sum or difference of the altitudes of the latter.*

(ii) *A triangle is equal to the sum or difference of two triangles of the same altitude if the base of the former is equal to the sum or difference of the bases of the latter.*

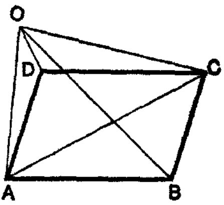
Similar statements hold good of parallelograms.

17. ABCD is a parallelogram, and O is any point outside it; shew that the sum or difference of the triangles OAB, OCD is equal to half the parallelogram. Distinguish between the two cases.

On the following proposition depends an important theorem in Mechanics: we give a proof of the first case, leaving the second case to be deduced by a similar method.

18. (i) ABCD *is a parallelogram, and* O *is any point* without the angle BAD and its opposite vertical angle; *shew that the triangle* OAC *is equal to the* sum *of the triangles* OAD, OAB.

(ii) *If* O *is* within the angle BAD or its opposite vertical angle, *the triangle* OAC *is equal to the difference of the triangles* OAD, OAB.



CASE I. If O is without the ∠ DAB and its opp. vert. ∠, then OA is without the parᵐ ABCD: therefore the perp. drawn from C to OA is equal to the sum of the perpˢ drawn from B and D to OA. [See Ex. 20, p. 99.]

Now the △ˢ OAC, OAD, OAB are upon the same base OA;

and the altitude of the △ OAC with respect to this base has been shewn to be equal to the sum of the altitudes of the △ˢ OAD, OAB.

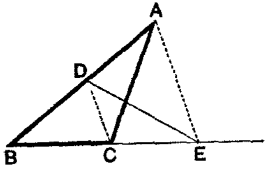
Therefore the △ OAC is equal to the sum of the △ˢ OAD, OAB. [See Ex. 16, p. 110.] Q.E.D.

19. ABCD is a parallelogram, and through O, any point within it, straight lines are drawn parallel to the sides of the parallelogram; shew that the difference of the parallelograms DO, BO is double of the triangle AOC. [See preceding theorem (ii).]

20. The area of a quadrilateral is equal to the area of a triangle having two of its sides equal to the diagonals of the given figure, and the included angle equal to either of the angles between the diagonals.

---

21. ABC *is a triangle, and* D *is any point in* AB: *it is required to draw through* D *a straight line* DE *to meet* BC *produced in* E, *so that the triangle* DBE *may be equal to the triangle* ABC.



[Join DC. Through A draw AE parallel to DC. I. 31.
Join DE.
The △ EBD shall be equal to the △ ABC. I. 37.]

22. On a base of given length describe a triangle equal to a given triangle and having an angle equal to an angle of the given triangle.

23. Construct a triangle equal in area to a given triangle, and having a given altitude.

24. On a base of given length construct a triangle equal to a given triangle, and having its vertex on a given straight line.

25. On a base of given length describe (i) an isosceles triangle; (ii) a right-angled triangle, equal to a given triangle.

26. Construct a triangle equal to the sum or difference of two given triangles. [See Ex. 16, p. 110.]

27. ABC is a given triangle, and X a given point: describe a triangle equal to ABC, having its vertex at X, and its base in the same straight line as BC.

28. ABCD *is a quadrilateral: on the base* AB *construct a triangle equal in area to* ABCD, *and having the angle at* A *common with the quadrilateral.*

[Join BD. Through C draw CX parallel to BD, meeting AD produced in X; join BX.]

29. *Construct a rectilineal figure equal to a given rectilineal figure, and having fewer sides by one than the given figure.*

Hence shew how to construct a triangle equal to a given rectilineal figure.

30. ABCD is a quadrilateral: it is required to construct a triangle equal in area to ABCD, having its vertex at a given point X in DC, and its base in the same straight line as AB.

31. Construct a rhombus equal to a given parallelogram.

32. Construct a parallelogram which shall have the same area and perimeter as a given triangle.

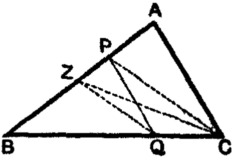
---

33. *Bisect a triangle by a straight line drawn through one of its angular points.*

34. Trisect a triangle by straight lines drawn through one of its angular points. [See Ex. 19, p. 102, and I. 38.]

35. Divide a triangle into any number of equal parts by straight lines drawn through one of its angular points.
[See Ex. 19, p. 99, and I. 38.]

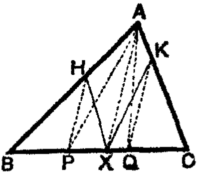
36. *Bisect a triangle by a straight line drawn through a given point in one of its sides.*



[Let ABC be the given △, and P the given point in the side AB.

Bisect AB at Z; and join CZ, CP.
Through Z draw ZQ parallel to CP.
Join PQ.
Then shall PQ bisect the △.
See Ex. 21, p. 111.]

37. *Trisect a triangle by straight lines drawn from a given point in one of its sides.*



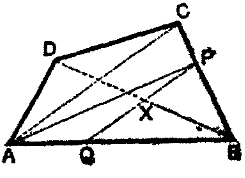
[Let ABC be the given △, and X the given point in the side BC.

Trisect BC at the points P, Q. Ex. 19, p. 99.
Join AX, and through P and Q draw PH and QK parallel to AX.
Join XH, XK.
These straight lines shall trisect the △; as may be shewn by joining AP, AQ.
See Ex. 21, p. 111.]

38. Cut off from a given triangle a fourth, fifth, sixth, or any part required by a straight line drawn from a given point in one of its sides. [See Ex. 19, p. 99, and Ex. 21, p. 111.]

39. *Bisect a quadrilateral by a straight line drawn through an angular point.*

[Two constructions may be given for this problem: the first will be suggested by Exercises 28 and 33, p. 112.



The second method proceeds thus.

Let ABCD be the given quadrilateral, and A the given angular point.

Join AC, BD, and bisect BD in X.
Through X draw PXQ parallel to AC, meeting BC in P; join AP.
Then shall AP bisect the quadrilateral.
Join AX, CX, and use I. 37, 38.]

40. Cut off from a given quadrilateral a third, a fourth, a fifth, or any part required, by a straight line drawn through a given angular point. [See Exercises 28 and 35, p. 112.]

[The following Theorems depend on I. 47.]

41. In the figure of I. 47, shew that

(i) the sum of the squares on AB and AE is equal to the sum of the squares on AC and AD.

(ii) the square on EK is equal to the square on AB with four times the square on AC.

(iii) the sum of the squares on EK and FD is equal to five times the square on BC.

42. If a straight line be divided into any two parts the square on the straight line is greater than the squares on the two parts.

43. If the square on one side of a triangle is less than the squares on the remaining sides, the angle contained by these sides is acute; if greater, obtuse.

44. ABC is a triangle, right-angled at A; the sides AB, AC are intersected by a straight line PQ, and BQ, PC are joined: shew that the sum of the squares on BQ, PC is equal to the sum of the squares on BC, PQ.

45. In a right-angled triangle four times the sum of the squares on the medians which bisect the sides containing the right angle is equal to five times the square on the hypotenuse.

46. Describe a square whose area shall be three times that of a given square.

47. Divide a straight line into two parts such that the sum of their squares shall be equal to a given square.

IX. ON LOCI.

It is frequently required in the course of Plane Geometry to find the position of a point which satisfies given conditions. Now all problems of this type hitherto considered have been found to be capable of definite determination, though some admit of more than one solution: this however will not be the case if *only one* condition is given. For example, if we are asked to find a point which shall be at a given distance from a given point, we observe at once that the problem is *indeterminate*, that is, that it admits of an indefinite number of solutions; for the condition stated is satisfied by any point on the circumference of the circle described from the given point as centre, with a radius equal to the given distance: moreover this condition is satisfied by no other point within or without the circle.

Again, suppose that it is required to find a point at a given distance from a given straight line.

Here, too, it is obvious that there are an infinite number of such points, and that they lie on the two parallel straight lines which may be drawn on either side of the given straight line at the given distance from it: further, no point that is not on one or other of these parallels satisfies the given condition.

Hence we see that when one condition is assigned it is not sufficient to determine the position of a point absolutely, but it may have the effect of restricting it to some definite line or lines, straight or curved. This leads us to the following definition.

DEFINITION. The **Locus** of a point satisfying an assigned condition consists of the line, lines, or part of a line, to which the point is thereby restricted; provided that the condition is satisfied by every point on such line or lines, and by no other.

A locus is sometimes defined as the path traced out by a point which moves in accordance with an assigned law.

Thus the locus of a point, which is always at a given distance from a given point, is a circle of which the given point is the centre: and the locus of a point, which is always at a given distance from a given straight line, is a pair of parallel straight lines.

We now see that in order to infer that a certain line, or system of lines, is the locus of a point under a given condition, it is necessary to prove

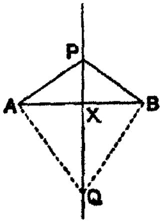
(i) that any point which fulfils the given condition is on the supposed locus;

(ii) that every point on the supposed locus satisfies the given condition.

1. *Find the locus of a point which is always equidistant from two given points.*

Let A, B be the two given points.

(a) Let P be any point equidistant from A and B, so that AP=BP.



Bisect AB at X, and join PX.

Then in the △ˢ AXP, BXP,

Because { AX=BX, *Constr.* and PX is common to both, also AP=BP, *Hyp.* }

∴ the ∠ PXA = the ∠ PXB; I. 8.

and they are adjacent ∠ˢ;

∴ PX is perp. to AB.

∴ **Any point which is equidistant from A and B is on the straight line which bisects AB at right angles.**

(β) Also every point in this line is equidistant from A and B.

For let Q be any point in this line.

Join AQ, BQ.

Then in the $\triangle^s$ AXQ, BXQ,

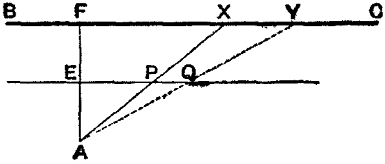
Because { AX = BX,
and XQ is common to both;
also the $\angle$ AXQ = the $\angle$ BXQ, being rt. $\llcorner^s$;

∴ AQ = BQ. I. 4.

That is, Q is equidistant from A and B.

Hence we conclude that the locus of the point equidistant from two given points A, B is the straight line which bisects AB at right angles.

2. *To find the locus of the middle point of a straight line drawn from a given point to meet a given straight line of unlimited length.*



Let A be the given point, and BC the given straight line of unlimited length.

(α) Let AX be any straight line drawn through A to meet BC, and let P be its middle point.

Draw AF perp. to BC, and bisect AF at E.

Join EP, and produce it indefinitely.

Since AFX is a $\triangle$, and E, P the middle points of the two sides AF, AX,

∴ EP is parallel to the remaining side FX. Ex. 2, p. 96.

∴ P is on the straight line which passes through the *fixed point* E, and is parallel to BC.

(β) Again, every point in EP, or EP produced, fulfils the required condition.

For, in this straight line take any point Q.

Join AQ, and produce it to meet BC in Y.

Then FAY is a $\triangle$, and through E, the middle point of the side AF, EQ is drawn parallel to the side FY,

∴ Q is the middle point of AY. Ex. 1, p. 96.

Hence the required locus is the straight line drawn parallel to BC, and passing through E, the middle point of the perp. from A to BC.

3. *Find the locus of a point equidistant from two given intersecting straight lines.* [See Ex. 3, p. 49.]

4. Find the locus of a point at a given radial distance from the circumference of a given circle.

5. Find the locus of a point which moves so that the sum of its distances from two given intersecting straight lines of unlimited length is constant.

6. Find the locus of a point when the differences of its distances from two given intersecting straight lines of unlimited length is constant.

7. A straight rod of given length slides between two straight rulers placed at right angles to one another: find the locus of its middle point. [See Ex. 2, p. 100.]

8. On a given base as hypotenuse right-angled triangles are described: find the locus of their vertices.

9. AB is a given straight line, and AX is the perpendicular drawn from A to any straight line passing through B: find the locus of the middle point of AX.

10. Find the locus of the vertex of a triangle, when the base and area are given.

11. Find the locus of the intersection of the diagonals of a parallelogram, of which the base and area are given.

12. Find the locus of the intersection of the medians of a triangle described on a given base and of given area.

X. ON THE INTERSECTION OF LOCI.

It appears from various problems which have already been considered, that we are often required to find a point, the position of which is subject to two given conditions. The method of loci is very useful in the solution of problems of this kind: for corresponding to each condition there will be a locus on which the required point must lie; hence all points which are common to these two loci, that is, all the points of intersection of the loci, will satisfy *both* the given conditions.

EXAMPLE 1. *To construct a triangle, having given the base, the altitude, and the length of the median which bisects the base.*

Let AB be the given base, and P and Q the lengths of the altitude and median respectively:

then the triangle is known if its *vertex* is known.

(i) Draw a straight line CD parallel to AB, and at a distance from it equal to P:

then the required vertex must lie on CD.

(ii) Again, from the middle point of AB as centre, with radius equal to Q, describe a circle:

then the required vertex must lie on this circle.

Hence any points which are common to CD and the circle, satisfy both the given conditions: that is to say, if CD intersect the circle in E, F each of the points of intersection might be the vertex of the required triangle. This supposes the length of the median Q to be greater than the altitude.

EXAMPLE 2. *To find a point equidistant from three given points* A, B, C, *which are not in the same straight line.*

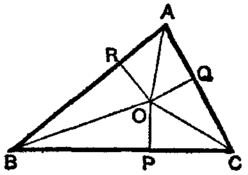
(i) The locus of points equidistant from A and B is the straight line PQ, which bisects AB at right angles. Ex. 1, p. 115.

(ii) Similarly the locus of points equidistant from B and C is the straight line RS which bisects BC at right angles.

Hence the point common to PQ and RS must satisfy both conditions: that is to say, the point of intersection of PQ and RS will be equidistant from A, B, and C.

These principles may also be used to prove the theorems relating to concurrency already given on page 103.

EXAMPLE. *To prove that the bisectors of the angles of a triangle are concurrent.*



Let ABC be a triangle.

Bisect the $\angle$s ABC, BCA by straight lines BO, CO: these must meet at some point O. *Ax.* 12.

Join OA.

Then shall OA bisect the $\angle$ BAC.

Now BO is the locus of points equidistant from BC, BA; Ex. 3, p. 49.

$\therefore$ OP=OR.

Similarly CO is the locus of points equidistant from BC, CA.

$\therefore$ OP=OQ; hence OR=OQ.

$\therefore$ O is on the locus of points equidistant from AB and AC: that is OA is the bisector of the $\angle$ BAC.

Hence the bisectors of the three $\angle$s meet at the point O.

It may happen that the data of the problem are so related to one another that the resulting loci do not intersect: in this case the problem is impossible.

For example, if in Ex. 1, page 118, the length of the given median *is less than* the given altitude, the straight line CD will not be intersected by the circle, and no triangle can fulfil the conditions of the problem. If the length of the median *is equal* to the given altitude, *one* point is common to the two loci; and consequently only one solution of the problem exists: and we have seen that there are two solutions, if the median is greater than the altitude.

In examples of this kind the student should make a point of investigating the relations which must exist among the data, in order that the problem may be possible; and he must observe that if under certain relations *two* solutions are possible, and under other relations no solution exists, there will always be some *intermediate* relation under which *one* and *only one* solution is possible.

EXAMPLES.

1. Find a point in a given straight line which is equidistant from two given points.

2. Find a point which is at given distances from each of two given straight lines. How many solutions are possible?

3. *On a given base construct a triangle, having given one angle at the base and the length of the opposite side. Examine the relations which must exist among the data in order that there may be two solutions, one solution, or that the problem may be impossible.*

4. On the base of a given triangle construct a second triangle equal in area to the first, and having its vertex in a given straight line.

5. Construct an isosceles triangle equal in area to a given triangle, and standing on the same base.

6. Find a point which is at a given distance from a given point, and is equidistant from two given parallel straight lines.

BOOK II.

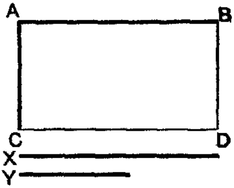
Book II. deals with the areas of rectangles and squares.

Definitions.

1. A **Rectangle** is a parallelogram which has one of its angles a right angle.

It should be remembered that if a parallelogram has *one* right angle, *all* its angles are right angles. [Ex. 1, p. 64.]

2. A rectangle is said to be **contained** by any two of its sides which form a right angle: for it is clear that both the form and magnitude of a rectangle are fully determined when the lengths of two such sides are given.

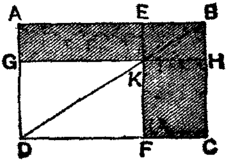


Thus the rectangle ACDB is said *to be contained* by AB, AC; or by CD, DB; and if X and Y are two straight lines equal respectively to AB and AC, then the rectangle contained by X and Y is equal to the rectangle contained by AB, AC.

[See Ex. 12, p. 64.]

After Proposition 3, we shall use the abbreviation *rect.* AB, AC to denote the *rectangle contained by* AB *and* AC.

3. In any parallelogram the figure formed by either of the parallelograms about a diagonal together with the two complements is called a **gnomon.**



Thus the shaded portion of the annexed figure, consisting of the parallelogram EH together with the complements AK, KC is the *gnomon* AHF.

The other gnomon in the figure is that which is made up of AK, GF and FH, namely the gnomon AFH.

INTRODUCTORY.

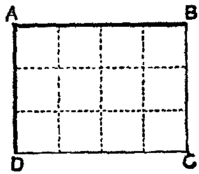
Pure Geometry makes no use of *number* to estimate the magnitude of the lines, angles, and figures with which it deals: hence it requires no *units of magnitude* such as the student is familiar with in Arithmetic.

For example, though Geometry is concerned with the relative lengths of straight lines, it does not seek to express those lengths in terms of *yards*, *feet*, or *inches:* similarly it does not ask how many *square yards* or *square feet* a given figure contains, nor how many *degrees* there are in a given angle.

This constitutes an essential difference between the method of Pure Geometry and that of Arithmetic and Algebra; at the same time a close connection exists between the results of these two methods.

In the case of Euclid's Book II., this connection rests upon the fact that *the number of units of area in a rectangular figure is found by multiplying together the numbers of units of length in two adjacent sides.*

For example, if the two sides AB, AD of the rectangle ABCD are respectively *four* and *three* inches long, and if through the points of division parallels are drawn as in the annexed figure, it is seen that the rectangle is divided into *three rows*, each containing *four* square inches, or into *four columns*, each containing *three* square inches.



Hence the whole rectangle contains 3×4, or 12, square inches.

Similarly if AB and AD contain m and n units of length respectively, it follows that the rectangle ABCD will contain mn units of area: further, if AB and AD are equal, each containing m units of length, the rectangle becomes a square, and contains m^2 units of area.

[It must be understood that this explanation implies that the lengths of the straight lines AB, AD are commensurable, that is, that they can be expressed *exactly* in terms of some common unit.

This however is not always the case: for example, it may be proved that the side and diagonal of a square are so related, that it is impossible to divide either of them into equal parts, *of which the other contains an exact number*. Such lines are said to be incommensurable. Hence if the adjacent sides of a rectangle are incommensurable, we cannot choose any linear unit in terms of which these sides may be *exactly* expressed; and thus it will be impossible to subdivide the rectangle into squares of unit area, as illustrated in the figure of the preceding page. We do not here propose to enter further into the subject of incommensurable quantities: it is sufficient to point out that further knowledge of them will convince the student that the area of a rectangle may be expressed *to any required degree of accuracy* by the product of the lengths of two adjacent sides, whether those lengths are commensurable or not.]

From the foregoing explanation we conclude that *the rectangle contained by two straight lines* in Geometry corresponds to *the product of two numbers* in Arithmetic or Algebra; and that *the square described on a straight line* corresponds to *the square of a number.* Accordingly it will be found in the course of Book II. that several theorems relating to the areas of rectangles and squares are analogous to well-known algebraical formulæ.

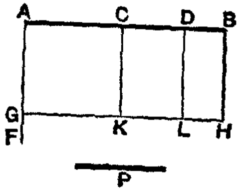
In view of these principles the rectangle contained by two straight lines AB, BC is sometimes expressed in the form of a product, as AB . BC, and the square described on AB as AB^2. This notation, together with the signs + and −, will be employed in the additional matter appended to this book; *but it is not admitted into Euclid's text* because it is desirable in the first instance to emphasize the distinction between geometrical magnitudes themselves and the numerical equivalents by which they may be expressed arithmetically.

Proposition 1. Theorem.

If there are two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the sum of the rectangles contained by the undivided straight line and the several parts of the divided line.

Let P and AB be two straight lines, and let AB be divided into any number of parts AC, CD, DB:

then shall the rectangle contained by P, AB be equal to the sum of the rectangles contained by P, AC, by P, CD, and by P, DB.



From A draw AF perp. to AB; I. 11.
and make AG equal to P. I. 3.
Through G draw GH par[l] to AB; I. 31.
and through C, D, B draw CK, DL, BH par[l] to AG.

Now the fig. AH is made up of the figs. AK, CL, DH: and of these,

the fig. AH is the rectangle contained by P, AB;
for the fig. AH is contained by AG, AB; and AG = P:
and the fig. AK is the rectangle contained by P, AC;
for the fig. AK is contained by AG, AC; and AG = P:
also the fig. CL is the rectangle contained by P, CD;
for the fig. CL is contained by CK, CD;
and CK = the opp. side AG, and AG = P: I. 34.
similarly the fig. DH is the rectangle contained by P, DB.

∴ the rectangle contained by P, AB is equal to the sum of the rectangles contained by P, AC, by P, CD, and by P, DB. Q.E.D.

CORRESPONDING ALGEBRAICAL FORMULA.

In accordance with the principles explained on page 122, the result of this proposition may be written thus:

$$P \cdot AB = P \cdot AC + P \cdot CD + P \cdot DB.$$

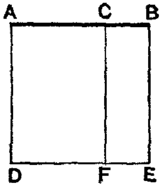
Now if the line P contains p units of length, and if AC, CD, DB contain a, b, c units respectively,

then $AB = a + b + c$,

and we have $p(a+b+c) = pa + pb + pc$.

Proposition 2. Theorem.

If a straight line is divided into any two parts, the square on the whole line is equal to the sum of the rectangles contained by the whole line and each of the parts.



Let the straight line AB be divided at C into the two parts AC, CB:

then shall the sq. on AB be equal to the sum of the rects. contained by AB, AC, and by AB, BC.

On AB describe the square ADEB. I. 46.
Through C draw CF par[l] to AD. I. 31.

Now the fig. AE is made up of the figs. AF, CE: and of these,
the fig. AE is the sq. on AB: *Constr.*
and the fig. AF is the rectangle contained by AB, AC;
for the fig. AF is contained by AD, AC; and AD = AB;
also the fig. CE is the rectangle contained by AB, BC;
for the fig. CE is contained by BE, BC; and BE = AB.
∴ the sq. on AB = the sum of the rects. contained by AB, AC, and by AB, BC. Q.E.D.

CORRESPONDING ALGEBRAICAL FORMULA.

The result of this proposition may be written

$$AB^2 = AB \cdot AC + AB \cdot BC.$$

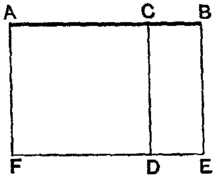
Let AC contain a units of length, and let CB contain b units,

then $AB = a + b$,

and we have $(a+b)^2 = (a+b)\,a + (a+b)\,b.$

PROPOSITION 3. THEOREM.

If a straight line is divided into any two parts, the rectangle contained by the whole and one of the parts is equal to the square on that part together with the rectangle contained by the two parts.



Let the straight line AB be divided at C into the two parts AC, CB:

then shall the rect. contained by AB, AC be equal to the sq. on AC together with the rect. contained by AC, CB.

On AC describe the square AFDC; I. 46.

and through B draw BE par[l] to AF, meeting FD produced in E. I. 31.

Now the fig. AE is made up of the figs. AD, CE;
and of these,
the fig. AE = the rect. contained by AB, AC;
for AF = AC;
and the fig. AD is the sq. on AC; *Constr.*
also the fig. CE is the rect. contained by AC, CB;
for CD = AC.

∴ the rect. contained by AB, AC is equal to the sq. on AC together with the rect. contained by AC, CB. Q.E.D.

CORRESPONDING ALGEBRAICAL FORMULA.

This result may be written $AB \cdot AC = AC^2 + AC \cdot CB$.

Let AC, CB contain a and b units of length respectively,

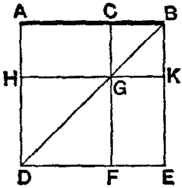
$$\text{then } AB = a + b,$$

and we have $$(a+b)\,a = a^2 + ab.$$

NOTE. It should be observed that Props. 2 and 3 are *special cases* of Prop. 1.

PROPOSITION 4. THEOREM.

If a straight line is divided into any two parts, the square on the whole line is equal to the sum of the squares on the two parts together with twice the rectangle contained by the two parts.



Let the straight line AB be divided at C into the two parts AC, CB:

then shall the sq. on AB be equal to the sum of the sqq. on AC, CB, together with twice the rect. AC, CB.

On AB describe the square ADEB; I. 46.
and join BD.
Through C draw CF par[l] to BE, meeting BD in G. I. 31.
Through G draw HGK par[l] to AB.

It is first required to shew that the fig. CK is the sq. on BC.

Because the straight line BGD meets the par[ls] CG, AD,
∴ the ext. angle CGB = the int. opp. angle ADB. I. 29.
But AB = AD, being sides of a square;
∴ the angle ADB = the angle ABD; I. 5.
∴ the angle CGB = the angle CBG.
∴ CB = CG. I. 6.
And the opp. sides of the par[m] CK are equal; I. 34.
∴ the fig. CK is equilateral;
and the angle CBK is a right angle; *Def.* 28.
∴ CK is a square, and it is described on BC. I. 46, *Cor.*

Similarly the fig. HF is the sq. on HG, that is, the sq. on AC,
for HG = the opp. side AC. I. 34.

Again, the complement AG = the complement GE. I. 43.

But the fig. AG = the rect. AC, CB; for CG = CB.

∴ the two figs. AG, GE = twice the rect. AC, CB.

*Now the sq. on AB = the fig. AE
= the figs. HF, CK, AG, GE
= the sqq. on AC, CB together with twice the rect. AC, CB.

∴ the sq. on AB = the sum of the sqq. on AC, CB with twice the rect. AC, CB. Q.E.D.

* For the purpose of oral work, this step of the proof may conveniently be arranged as follows:

Now the sq. on AB is equal to the fig. AE,
that is, to the figs. HF, CK, AG, GE;
that is, to the sqq. on AC, CB together with twice the rect. AC, CB.

COROLLARY. *Parallelograms about the diagonals of a square are themselves squares.*

CORRESPONDING ALGEBRAICAL FORMULA.

The result of this important Proposition may be written thus:

$$AB^2 = AC^2 + CB^2 + 2AC \cdot CB.$$

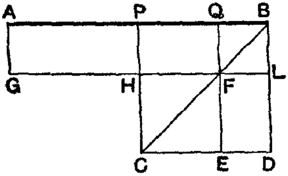
Let $AC = a$, and $CB = b$;

then $AB = a + b$,

and we have $(a+b)^2 = a^2 + b^2 + 2ab.$

Proposition 5. Theorem.

If a straight line is divided equally and also unequally, the rectangle contained by the unequal parts, and the square on the line between the points of section, are together equal to the square on half the line.



Let the straight line AB be divided equally at P, and unequally at Q:

then the rect. AQ, QB and the sq. on PQ shall be together equal to the sq. on PB.

On PB describe the square PCDB. I. 46.

Join BC.

Through Q draw QE par[l] to BD, cutting BC in F. I. 31.

Through F draw LFHG par[l] to AB.

Through A draw AG par[l] to BD.

Now the complement PF = the complement FD: I. 43.

to each add the fig. QL;

then the fig. PL = the fig. QD.

But the fig. PL = the fig. AH, for they are par[ms] on equal bases and between the same par[ls]. I. 36.

∴ the fig. AH = the fig. QD.

To each add the fig. PF;

then the fig. AF = the gnomon PLE.

Now the fig. AF = the rect. AQ, QB, for QB = QF;

∴ the rect. AQ, QB = the gnomon PLE.

To each add the sq. on PQ, that is, the fig. HE; II. 4.

then the rect. AQ, QB with the sq. on PQ

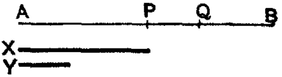
= the gnomon PLE with the fig. HE

= the whole fig. PD,

which is the sq. on PB.

That is, the rect. AQ, QB and the sq. on PQ are together equal to the sq. on PB. Q.E.D.

Corollary. From this Proposition it follows that *the difference of the squares on two straight lines is equal to the rectangle contained by their sum and difference.*



For let X and Y be the given st. lines, of which X is the greater.

Draw AP equal to X, and produce it to B, making PB equal to AP, that is to X.

From PB cut off PQ equal to Y.

Then AQ is equal to the sum of X and Y,

and QB is equal to the difference of X and Y.

Now because AB is divided equally at P and unequally at Q,

∴ the rect. AQ, QB with sq. on PQ = the sq. on PB; II. 5.

that is, the difference of the sqq. on PB, PQ = the rect. AQ, QB,

or, the difference of the sqq. on X and Y = the rect. contained by the sum and the difference of X and Y.

CORRESPONDING ALGEBRAICAL FORMULA.

This result may be written

$$AQ \cdot QB + PQ^2 = PB^2.$$

Let $AB = 2a$; and let $PQ = b$;

then AP and PB each $= a$.

Also $AQ = a + b$; and $QB = a - b$.

Hence we have

$$(a+b)(a-b) + b^2 = a^2,$$

or

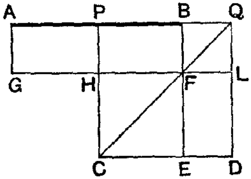
$$(a+b)(a-b) = a^2 - b^2.$$

EXERCISE.

In the above figure shew that AP *is half the sum of* AQ *and* QB; *and that* PQ *is half their difference.*

Proposition 6. Theorem.

If a straight line is bisected and produced to any point, the rectangle contained by the whole line thus produced, and the part of it produced, together with the square on half the line bisected, is equal to the square on the straight line made up of the half and the part produced.



Let the straight line AB be bisected at P, and produced to Q:

then the rect. AQ, QB and the sq. on PB shall be together equal to the sq. on PQ.

On PQ describe the square PCDQ. I. 46.

Join QC.

Through B draw BE par[l] to QD, meeting QC in F. I. 31.

Through F draw LFHG par[l] to AQ.

Through A draw AG par[l] to QD.

Now the complement PF = the complement FD. I. 43.

But the fig. PF = the fig. AH; for they are par[ms] on equal bases and between the same par[ls]. I. 36.

∴ the fig. AH = the fig. FD.

To each add the fig. PL,

then the fig. AL = the gnomon PLE.

Now the fig. AL = the rect. AQ, QB, for QB = QL;

∴ the rect. AQ, QB = the gnomon PLE.

To each add the sq. on PB, that is, the fig. HE;

then the rect. AQ, QB with the sq. on PB

= the gnomon PLE with the fig. HE

= the whole fig. PD,

which is the square on PQ.

That is, the rect. AQ, QB and the sq. on PB are together equal to the sq. on PQ. Q.E.D.

CORRESPONDING ALGEBRAICAL FORMULA.

This result may be written

$$AQ \cdot QB + PB^2 = PQ^2.$$

Let $AB = 2a$; and let $PQ = b$;

then AP and PB each $= a$.

Also $AQ = a + b$; and $QB = b - a$.

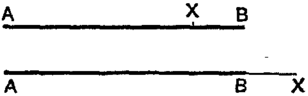
Hence we have

$$(a+b)(b-a) + a^2 = b^2,$$

or

$$(b+a)(b-a) = b^2 - a^2.$$

DEFINITION. If a point X is taken in a straight line AB, or in AB produced, the distances of the point of section from the extremities of AB are said to be the **segments** into which AB is divided at X.



In the former case AB is divided **internally**, in the latter case **externally**.

Thus in the annexed figures the segments into which AB is divided at X are the lines XA and XB.

This definition enables us to include Props. 5 and 6 in a single Enunciation.

If a straight line is bisected, and also divided (internally or externally) into two unequal segments, the rectangle contained by the unequal segments is equal to the difference of the squares on half the line, and on the line between the points of section.

EXERCISE.

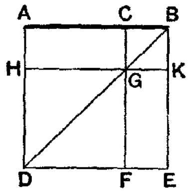
Shew that the Enunciations of Props. 5 and 6 may take the following form:

The rectangle contained by two straight lines is equal to the difference of the squares on half their sum and on half their difference.

[See Ex., p. 129.]

Proposition 7. Theorem.

If a straight line is divided into any two parts, the sum of the squares on the whole line and on one of the parts is equal to twice the rectangle contained by the whole and that part, together with the square on the other part.



Let the straight line AB be divided at C into the two parts AC, CB:

then shall the sum of the sqq. on AB, BC be equal to twice the rect. AB, BC together with the sq. on AC.

On AB describe the square ADEB. I. 46.

Join BD.

Through C draw CF par[1] to BE, meeting BD in G. I. 31.

Through G draw HGK par[1] to AB.

Now the complement AG = the complement GE; I. 43.

to each add the fig. CK:

then the fig. AK = the fig. CE.

But the fig. AK = the rect. AB, BC; for BK = BC.

∴ the two figs. AK, CE = twice the rect. AB, BC.

But the two figs. AK, CE make up the gnomon AKF and the fig. CK:

∴ the gnomon AKF with the fig. CK = twice the rect. AB, BC.

To each add the fig. HF, which is the sq. on AC:

then the gnomon AKF with the figs. CK, HF

= twice the rect. AB, BC with the sq. on AC.

Now the sqq. on AB, BC = the figs. AE, CK

= the gnomon AKF with the figs. CK, HF

= twice the rect. AB, BC with the sq. on AC.

CORRESPONDING ALGEBRAICAL FORMULA.

The result of this proposition may be written

$$AB^2 + BC^2 = 2\,AB \,.\, BC + AC^2.$$

Let $AB = a$, and $BC = b$; then $AC = a - b$.

Hence we have $$a^2 + b^2 = 2ab + (a-b)^2,$$

or $$(a-b)^2 = a^2 - 2ab + b^2.$$

Proposition 8. Theorem.

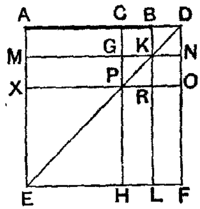
If a straight line be divided into any two parts, four times the rectangle contained by the whole line and one of the parts, together with the square on the other part, is equal to the square on the straight line which is made up of the whole and that part.

[As this proposition is of little importance we merely give the figure, and the leading points in Euclid's proof.]

Let AB be divided at C.

Produce AB to D, making BD equal to BC.

On AD describe the square AEFD; and complete the construction as indicated in the figure.



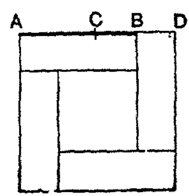
Euclid then proves (i) that the figs. CK, BN, GR, KO are all equal.

(ii) that the figs. AG, MP, PL, RF are all equal.

Hence the eight figures named above are four times the sum of the figs. AG, CK; that is, four times the fig. AK; that is, four times the rect. AB, BC.

But the whole fig. AF is made up of these eight figures, together with the fig. XH, which is the sq. on AC:

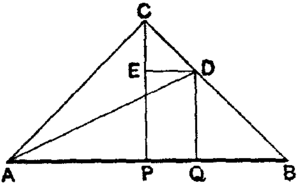
hence the sq. on AD = four times the rect. AB, BC, together with the sq. on AC. Q.E.D.



The accompanying figure will suggest a less cumbrous proof, which we leave as an Exercise to the student.

PROPOSITION 9. THEOREM. [EUCLID'S PROOF.]

If a straight line is divided equally and also unequally, the sum of the squares on the two unequal parts is twice the sum of the squares on half the line and on the line between the points of section.



Let the straight line AB be divided equally at P, and unequally at Q:

then shall the sum of the sqq. on AQ, QB be twice the sum of the sqq. on AP, PQ.

At P draw PC at rt. angles to AB; I. 11.
and make PC equal to AP or PB. I. 3.
Join AC, BC.
Through Q draw QD par[l] to PC; I. 31.
and through D draw DE par[l] to AB.
Join AD.

Then since PA = PC, *Constr.*
∴ the angle PAC = the angle PCA. I. 5.

And since, in the triangle APC, the angle APC is a rt. angle, *Constr.*
∴ the sum of the angles PAC, PCA is a rt. angle: I. 32.
hence each of the angles PAC, PCA is half a rt. angle.

So also, each of the angles PBC, PCB is half a rt. angle.
∴ the whole angle ACB is a rt. angle.

Again, the ext. angle CED = the int. opp. angle CPB, I. 29.
∴ the angle CED is a rt. angle:
and the angle ECD is half a rt. angle. *Proved.*
∴ also the angle EDC is half a rt. angle; I. 32.
∴ the angle ECD = the angle EDC;
∴ EC = ED. I. 6.

Again, the ext. angle DQB = the int. opp. angle CPB. I. 29.
∴ the angle DQB is a rt. angle.
And the angle QBD is half a rt. angle; *Proved.*
∴ also the angle QDB is half a rt. angle: I. 32.
∴ the angle QBD = the angle QDB;
∴ QD = QB. I. 6.

Now the sq. on AP = the sq. on PC; for AP = PC. *Constr.*
But the sq. on AC = the sum of the sqq. on AP, PC,
for the angle APC is a rt. angle. I. 47.
∴ the sq. on AC is twice the sq. on AP.

So also, the sq. on CD is twice the sq. on ED, that is, twice the sq. on the opp. side PQ. I. 34.
Now the sqq. on AQ, QB = the sqq. on AQ, QD
= the sq. on AD, for AQD is a rt. angle; I. 47.
= the sum of the sqq. on AC, CD, for ACD is a rt. angle; I. 47.
= twice the sq. on AP with twice the sq. on PQ. *Proved.*

That is,
the sum of the sqq. on AQ, QB = twice the sum of the sqq. on AP, PQ. Q.E.D.

CORRESPONDING ALGEBRAICAL FORMULA.

The result of this proposition may be written

$$AQ^2 + QB^2 = 2(AP^2 + PQ^2).$$

Let AB $= 2a$; and PQ $= b$;

then AP and PB each $= a$.

Also AQ $= a + b$; and QB $= a - b$.

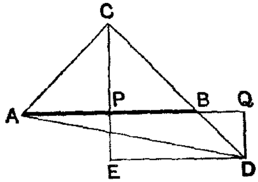
Hence we have

$$(a+b)^2 + (a-b)^2 = 2(a^2 + b^2).$$

[NOTE. For alternative proofs of this proposition, see page 137 A.]

PROPOSITION 10. THEOREM. [EUCLID'S PROOF.]

If a straight line is bisected and produced to any point, the sum of the squares on the whole line thus produced, and on the part produced, is twice the sum of the squares on half the line bisected and on the line made up of the half and the part produced.



Let the st. line AB be bisected at P, and produced to Q:

then shall the sum of the sqq. on AQ, QB be twice the sum of the sqq. on AP, PQ.

At P draw PC at right angles to AB; I. 11.

and make PC equal to PA or PB. I. 3.

Join AC, BC.

Through Q draw QD par[l] to PC, to meet CB produced in D; I. 31.

and through D draw DE par[l] to AB, to meet CP produced in E.

Join AD.

Then since PA = PC, *Constr.*

∴ the angle PAC = the angle PCA. I. 5.

And since in the triangle APC, the angle APC is a rt. angle,

∴ the sum of the angles PAC, PCA is a rt. angle. I. 32.

Hence each of the angles PAC, PCA is half a rt. angle.

So also, each of the angles PBC, PCB is half a rt. angle.

∴ the whole angle ACB is a rt. angle.

Again, the ext. angle CPB = the int. opp. angle CED: I. 29.

∴ the angle CED is a rt. angle:

and the angle ECD is half a rt. angle. *Proved.*

∴ the angle EDC is half a rt. angle. I. 32.

∴ the angle ECD = the angle EDC;

∴ EC = ED. I. 6.

Again, the angle DQB = the alt. angle CPB. I. 29.
∴ the angle DQB is a rt. angle.
Also the angle QBD = the vert. opp. angle CBP; I. 15.
that is, the angle QBD is half a rt. angle.
∴ the angle QDB is half a rt. angle: I. 32.
∴ the angle QBD = the angle QDB;
∴ QB = QD. I. 6.

Now the sq. on AP = the sq. on PC; for AP = PC. *Constr.*
But the sq. on AC = the sum of the sqq. on AP, PC,
for the angle APC is a rt. angle. I. 47.
∴ the sq. on AC is twice the sq. on AP.

So also, the sq. on CD is twice the sq. on ED, that is, twice the sq. on the opp. side PQ. I. 34.

Now the sqq. on AQ, QB = the sqq. on AQ, QD
= the sq. on AD, for AQD is a rt. angle; I. 47.
= the sum of the sqq. on AC, CD, for ACD is a rt. angle; I. 47.
= twice the sq. on AP with twice the sq. on PQ. *Proved.*

That is,
the sum of the sqq. on AQ, QB is twice the sum of the sqq. on AP, PQ. Q.E.D.

CORRESPONDING ALGEBRAICAL FORMULA.

The result of this proposition may be written

$$AQ^2 + BQ^2 = 2\,(AP^2 + PQ^2).$$

Let $AB = 2a$; and $PQ = b$;

then AP and PB each $= a$.

Also $AQ = a + b$; and $BQ = b - a$.

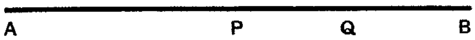
Hence we have

$$(a+b)^2 + (b-a)^2 = 2\,(a^2 + b^2).$$

[NOTE. For alternative proofs of this proposition, see page 137 B.]

Proposition 9. [Alternative Proof.]

If a straight line is divided equally and also unequally, the sum of the squares on the two unequal parts is twice the sum of the squares on half the line and on the line between the points of section.



Let the straight line AB be divided equally at P and unequally at Q:

then shall the sum of the sqq. on AQ, QB be twice the sum of the sqq. on AP, PQ.

For since AQ is divided at P,

∴ the sq. on AQ = the sum of the sqq. on AP, PQ with twice the rect. AP, PQ. II. 4.

And because PB is divided at Q,

∴ the sq. on QB with twice the rect. PB, PQ = the sum of the sqq. on PB, PQ. II. 7.

Adding together these pairs of equals,

the sqq. on AQ, QB with twice the rect. PB, PQ = the sum of the sqq. on AP, PQ, PB, PQ with twice the rect. AP, PQ.

But twice the rect. PB, PQ = twice the rect. AP, PQ.

Hence the sqq. on AQ, QB

= the sum of the sqq. on AP, PQ, PB, PQ

= twice the sum of the sqq. on AP, PQ.

A more concise proof of this proposition may be obtained from II. 4 and 5, as follows:

For $AQ \cdot QB = PB^2 - PQ^2$. II. 5.

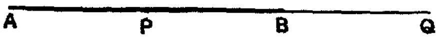
But $AQ^2 + QB^2 = AB^2 - 2AQ \cdot QB$ II. 4

$= 4PB^2 - 2(PB^2 - PQ^2)$

$= 2PB^2 + 2PQ^2$.

PROPOSITION 10. [ALTERNATIVE PROOF.]

If a straight line is bisected and produced to any point, the sum of the squares on the whole line thus produced and on the part produced, is twice the sum of the squares on half the line bisected and on the line made up of the half and the part produced.



Let the st. line AB be bisected at P and produced to Q:

then shall the sum of the sqq. on AQ, QB be twice the sum of the sqq. on AP, PQ.

For since AQ is divided at P,

∴ the sq. on AQ = the sum of the sqq. on AP, PQ with twice the rect. AP, PQ. II. 4.

And because PQ is divided at B,

∴ the sq. on QB with twice the rect. PQ, PB = the sum of the sqq. on PQ, PB. II. 7.

Adding together these pairs of equals,

the sqq. on AQ, QB with twice the rect. PQ, PB = the sum of the sqq. on AP, PQ, PQ, PB with twice the rect. AP, PQ.

But twice the rect. PQ, PB = twice the rect. AP, PQ;

∴ the sqq. on AQ, QB

= the sum of the sqq. on AP, PQ, PQ, PB

= twice the sum of the sqq. on AP, PQ.

A concise proof of this proposition may also be obtained from II. 6 and 7, as follows:

For $AQ \cdot QB = PQ^2 - PB^2$. II. 6.

But $AQ^2 + QB^2 = 2AQ \cdot QB + AB^2$ II. 7

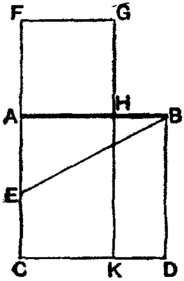
$= 2(PQ^2 - PB^2) + 4PB^2$

$= 2PB^2 + 2PQ^2$.

*

PROPOSITION 11. PROBLEM.

To divide a given straight line into two parts, so that the rectangle contained by the whole and one part may be equal to the square on the other part.



Let AB be the given straight line.

It is required to divide it into two parts, so that the rectangle contained by the whole and one part may be equal to the square on the other part.

On AB describe the square ACDB. I. 46.
Bisect AC at E. I. 10.
Join EB.
Produce CA to F, making EF equal to EB. I. 3.
On AF describe the square AFGH. I. 46.

Then shall AB be divided at H, so that the rect. AB, BH is equal to the sq. on AH.

Produce GH to meet CD in K.

Then because CA is bisected at E, and produced to F,
∴ the rect. CF, FA with the sq. on AE = the sq. on FE II. 6.
= the sq. on EB. *Constr.*
But the sq. on EB = the sum of the sqq. on AB, AE,
for the angle EAB is a rt. angle. I. 47.
∴ the rect. CF, FA with the sq. on AE = the sum of the sqq. on AB, AE.
From these take the sq. on AE:
then the rect. CF, FA = the sq. on AB.

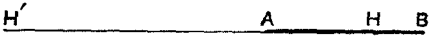
But the rect. CF, FA = the fig. FK; for FA = FG;
and the sq. on AB = the fig. AD. *Constr.*
∴ the fig. FK = the fig. AD.
From these take the common fig. AK,
then the remaining fig. FH = the remaining fig. HD.
But the fig. HD = the rect. AB, BH; for BD = AB;
and the fig. FH is the sq. on AH.
∴ the rect. AB, BH = the sq. on AH. Q.E.F.

DEFINITION. A straight line is said to be divided in **Medial Section** when the rectangle contained by the given line and one of its segments is equal to the square on the other segment.

The student should observe that this division may be *internal* or *external*.

Thus if the straight line AB is divided internally at H, and externally at H′, so that

(i) $AB . BH = AH^2$,
(ii) $AB . BH' = AH'^2$,



we shall in either case consider that AB is divided in medial section.

The case of *internal* section is alone given in Euclid II. 11; but a straight line may be divided *externally* in medial section by a similar process. See Ex. 21, p. 146.

ALGEBRAICAL ILLUSTRATION.

It is required to find a point H in AB, or AB produced, such that

$$AB . BH = AH^2.$$

Let AB contain a units of length, and let AH contain x units;

then $HB = a - x$:

and x must be such that $a(a - x) = x^2$,

or $$x^2 + ax - a^2 = 0.$$

Thus the construction for dividing a straight line in medial section corresponds to the algebraical solution of this quadratic equation.

EXERCISES.

In the figure of II. 11, shew that

(i) if CH is produced to meet BF at L, CL is at right angles to BF:

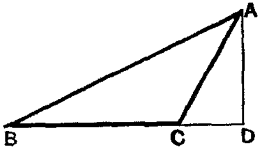
(ii) if BE and CH meet at O, AO is at right angles to CH:

(iii) the lines BG, DF, AK are parallel:

(iv) CF is divided in medial section at A.

PROPOSITION 12. THEOREM.

In an obtuse-angled triangle, if a perpendicular is drawn from either of the acute angles to the opposite side produced, the square on the side subtending the obtuse angle is greater than the squares on the sides containing the obtuse angle, by twice the rectangle contained by the side on which, when produced, the perpendicular falls, and the line intercepted without the triangle, between the perpendicular and the obtuse angle.



Let ABC be an obtuse-angled triangle, having the obtuse angle at C ; and let AD be drawn from A perp. to BC produced :

then shall the sq. on AB be greater than the sqq. on BC, CA, by twice the rect. BC, CD.

Because BD is divided into two parts at C,

∴ the sq. on BD = the sum of the sqq. on BC, CD, with twice the rect. BC, CD. II. 4.

To each add the sq. on DA.

Then the sqq. on BD, DA = the sum of the sqq. on BC, CD, DA, with twice the rect. BC, CD.

But the sum of the sqq. on BD, DA = the sq. on AB,

for the angle at D is a rt. angle. I. 47.

Similarly the sum of the sqq. on CD, DA = the sq. on CA.

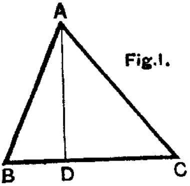
∴ the sq. on AB = the sum of the sqq. on BC, CA, with twice the rect. BC, CD.

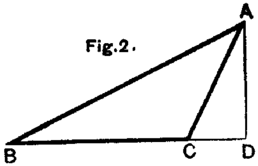
That is, the sq. on AB is greater than the sum of the sqq. on BC, CA by twice the rect. BC, CD. Q.E.D.

[For alternative Enunciations to Props. 12 and 13 and Exercises, see p. 142.]

PROPOSITION 13. THEOREM.

In every triangle the square on the side subtending an acute angle, is less than the squares on the sides containing that angle, by twice the rectangle contained by either of these sides, and the straight line intercepted between the perpendicular let fall on it from the opposite angle, and the acute angle.





Let ABC be any triangle having the angle at B an acute angle; and let AD be the perp. drawn from A to the opp. side BC :

then shall the sq. on AC be less than the sum of the sqq. on AB, BC, by twice the rect. CB, BD.

Now AD may fall within the triangle ABC, as in Fig. 1, or without it, as in Fig. 2.

Because {in Fig. 1. BC is divided into two parts at D,
{in Fig. 2. BD is divided into two parts at C,

∴ *in both cases,*

the sum of the sqq. on CB, BD = twice the rect. CB, BD with the sq. on CD. II. 7.

To each add the sq. on DA.

Then the sum of the sqq. on CB, BD, DA = twice the rect. CB, BD with the sum of the sqq. on CD, DA.

But the sum of the sqq. on BD, DA = the sq. on AB,

for the angle ADB is a rt. angle. I. 47.

Similarly the sum of the sqq. on CD, DA = the sq. on AC.

∴ the sum of the sqq. on AB, BC, = twice the rect. CB, BD, with the sq. on AC.

That is, the sq. on AC is less than the sqq. on AB, BC by twice the rect. CB, BD. Q.E.D.

Obs. If the perpendicular AD *coincides* with AC, that is, if ACB is a right angle, it may be shewn that the proposition merely repeats the result of I. 47.

NOTE. The result of Prop. 12 may be written

$$AB^2 = BC^2 + CA^2 + 2BC \cdot CD.$$

Remembering the definition of the **Projection** of a straight line given on page 97, the student will see that this proposition may be enunciated as follows:

In an obtuse-angled triangle the square on the side opposite the obtuse angle is greater than the sum of the squares on the sides containing the obtuse angle by twice the rectangle contained by either of those sides, and the projection of the other side upon it.

Prop. 13 may be written

$$AC^2 = AB^2 + BC^2 - 2CB \cdot BD,$$

and it may also be enunciated as follows:

In every triangle the square on the side subtending an acute angle, is less than the squares on the sides containing that angle, by twice the rectangle contained by either of these sides, and the projection of the other side upon it.

EXERCISES.

The following theorem should be noticed; it is proved by the help of II. 1.

1. *If four points* A, B, C, D *are taken in order on a straight line, then will*

$$AB \cdot CD + BC \cdot AD = AC \cdot BD.$$

ON II. 12 AND 13.

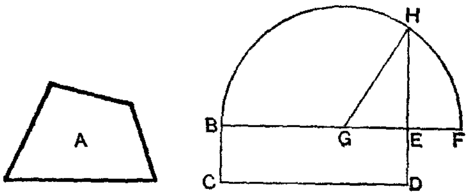
2. If from one of the base angles of an isosceles triangle a perpendicular is drawn to the opposite side, then twice the rectangle contained by that side and the segment adjacent to the base is equal to the square on the base.

3. If one angle of a triangle is one-third of two right angles, shew that the square on the opposite side is less than the sum of the squares on the sides forming that angle, by the rectangle contained by these two sides. [See Ex. 10, p. 101.]

4. If one angle of a triangle is two-thirds of two right angles, shew that the square on the opposite side is greater than the squares on the sides forming that angle, by the rectangle contained by these sides. [See Ex. 10, p. 101.]

PROPOSITION 14. PROBLEM.

To describe a square that shall be equal to a given rectilineal figure.



Let A be the given rectilineal figure.
It is required to describe a square equal to A.

Describe the par^{m} BCDE equal to the fig. A, and having the angle CBE a right angle. I. 45.

Then if BC = BE, the fig. BD is a square; and what was required is done.

But if not, produce BE to F, making EF equal to ED; I. 3.
and bisect BF at G. I. 10.

From centre G, with radius GF, describe the semicircle BHF:
produce DE to meet the semicircle at H.

Then shall the sq. on EH be equal to the given fig. A.

Join GH.

Then because BF is divided equally at G and unequally at E,

∴ the rect. BE, EF with the sq. on GE = the sq. on GF II. 5.
= the sq. on GH.

But the sq. on GH = the sum of the sqq. on GE, EH;
for the angle HEG is a rt. angle. I. 47.

∴ the rect. BE, EF with the sq. on GE = the sum of the sqq. on GE, EH.

From these take the sq. on GE:
then the rect. BE, EF = the sq. on HE.

But the rect. BE, EF = the fig. BD; for EF = ED; *Constr.*
and the fig. BD = the given fig. A. *Constr.*
∴ the sq. on EH = the given fig. A. Q.E.F.

THEOREMS AND EXAMPLES ON BOOK II.

ON II. 4 AND 7.

1. *Shew by* II. 4 *that the square on a straight line is four times the square on half the line.*

[This result is constantly used in solving examples on Book II, especially those which follow from II. 12 and 13.]

2. If a straight line is divided into any three parts, the square on the whole line is equal to the sum of the squares on the three parts together with twice the rectangles contained by each pair of these parts.

Shew that the algebraical formula corresponding to this theorem is

$$(a+b+c)^2=a^2+b^2+c^2+2bc+2ca+2ab.$$

3. *In a right-angled triangle, if a perpendicular is drawn from the right angle to the hypotenuse, the square on this perpendicular is equal to the rectangle contained by the segments of the hypotenuse.*

4. In an isosceles triangle, if a perpendicular be drawn from one of the angles at the base to the opposite side, shew that the square on the perpendicular is equal to twice the rectangle contained by the segments of that side together with the square on the segment adjacent to the base.

5. Any rectangle is half the rectangle contained by the diagonals of the squares described upon its two sides.

6. In any triangle if a perpendicular is drawn from the vertical angle to the base, the sum of the squares on the sides forming that angle, together with twice the rectangle contained by the segments of the base, is equal to the square on the base together with twice the square on the perpendicular.

ON II. 5 AND 6.

The student is reminded that these important propositions are both included in the following enunciation.

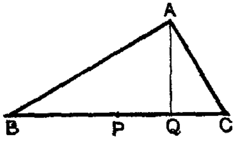
The difference of the squares on two straight lines is equal to the rectangle contained by their sum and difference.

7. In a right-angled triangle the square on one of the sides forming the right angle is equal to the rectangle contained by the sum and difference of the hypotenuse and the other side. [I. 47 and II. 5.]

8. *The difference of the squares on two sides of a triangle is equal to twice the rectangle contained by the base and the intercept between the middle point of the base and the foot of the perpendicular drawn from the vertical angle to the base.*

Let ABC be a triangle, and let P be the middle point of the base BC: let AQ be drawn perp. to BC.

Then shall $AB^2 - AC^2 = 2BC \cdot PQ$.



First, let AQ fall within the triangle.

Now $AB^2 = BQ^2 + QA^2$, I. 47.
also $AC^2 = QC^2 + QA^2$,

$\therefore\ AB^2 - AC^2 = BQ^2 - QC^2$ Ax. 3.
$= (BQ + QC)(BQ - QC)$ II. 5.
$= BC \cdot 2PQ$ Ex. 1, p. 129.
$= 2BC \cdot PQ$. Q.E.D.

The case in which AQ falls outside the triangle presents no difficulty.

9. *The square on any straight line drawn from the vertex of an isosceles triangle to the base is less than the square on one of the equal sides by the rectangle contained by the segments of the base.*

10. The square on any straight line drawn from the vertex of an isosceles triangle to the base produced, is greater than the square on one of the equal sides by the rectangle contained by the *segments into which the base is divided externally*.

11. If a straight line is drawn through one of the angles of an equilateral triangle to meet the opposite side produced, so that the rectangle contained *by the segments of the base* is equal to the square on the side of the triangle; shew that the square on the line so drawn is double of the square on a side of the triangle.

12. If XY be drawn parallel to the base BC of an isosceles triangle ABC, then the difference of the squares on BY and CY is equal to the rectangle contained by BC, XY. [See above, Ex. 8.]

13. In a right-angled triangle, if a perpendicular be drawn from the right angle to the hypotenuse, the square on either side forming the right angle is equal to the rectangle contained by the hypotenuse and the segment of it adjacent to that side.

ON II. 9 AND 10.

14. Deduce Prop. 9 from Props. 4 and 5, using also the theorem that the square on a straight line is four times the square on half the line.

15. Deduce Prop. 10 from Props. 7 and 6, using also the theorem mentioned in the preceding Exercise.

16. If a straight line is divided equally and also unequally, the squares on the two unequal segments are together equal to twice the rectangle contained by these segments together with four times the square on the line between the points of section.

ON II. 11.

17. *If a straight line is divided internally in medial section, and from the greater segment a part be taken equal to the less; shew that the greater segment is also divided in medial section.*

18. If a straight line is divided in medial section, the rectangle contained by the sum and difference of the segments is equal to the rectangle contained by the segments.

19. If AB is divided at H in medial section, and if X is the middle point of the greater segment AH, shew that a triangle whose sides are equal to AH, XH, BX respectively must be right-angled.

20. If a straight line AB is divided internally in medial section at H, prove that the sum of the squares on AB, BH is three times the square on AH.

21. *Divide a straight line externally in medial section.*

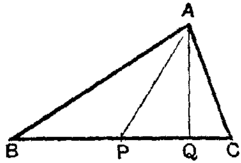
[Proceed as in II. 11, but instead of drawing EF, make EF′ equal to EB in the direction remote from A; and on AF′ describe the square AF′G′H′ on the side remote from AB. Then AB will be divided externally at H′ as required.]

ON II. 12 AND 13.

22. In a triangle ABC the angles at B and C are acute: if E and F are the feet of perpendiculars drawn from the opposite angles to the sides AC, AB, shew that the square on BC is equal to the sum of the rectangles AB, BF and AC, CE.

23. ABC is a triangle right-angled at C, and DE is drawn from a point D in AC perpendicular to AB: shew that the rectangle AB, AE is equal to the rectangle AC, AD.

24. *In any triangle the sum of the squares on two sides is equal to twice the square on half the third side together with twice the square on the median which bisects the third side.*



Let ABC be a triangle, and AP the median bisecting the side BC.

Then shall $AB^2 + AC^2 = 2\,BP^2 + 2\,AP^2$.

Draw AQ perp. to BC.

Consider the case in which AQ falls within the triangle, but does not coincide with AP.

Then of the angles APB, APC, one must be obtuse, and the other acute: let APB be obtuse.

Then in the $\triangle$ APB, $AB^2 = BP^2 + AP^2 + 2\,BP \,.\, PQ$. II. 12.

Also in the $\triangle$ APC, $AC^2 = CP^2 + AP^2 - 2\,CP \,.\, PQ$. II. 13.

But CP = BP,

$\therefore$ $CP^2 = BP^2$; and the rect. BP, PQ = the rect. CP, PQ.

Hence adding the above results

$AB^2 + AC^2 = 2 \,.\, BP^2 + 2 \,.\, AP^2$. Q.E.D.

The student will have no difficulty in adapting this proof to the cases in which AQ falls without the triangle, or coincides with AP.

25. *The sum of the squares on the sides of a parallelogram is equal to the sum of the squares on the diagonals.*

26. In any quadrilateral the squares on the diagonals are together equal to twice the sum of the squares on the straight lines joining the middle points of opposite sides. [See Ex. 9, p. 97.]

27. If from any point within a rectangle straight lines are drawn to the angular points, the sum of the squares on one pair of the lines drawn to opposite angles is equal to the sum of the squares on the other pair.

28. The sum of the squares on the sides of a quadrilateral is greater than the sum of the squares on its diagonals by four times the square on the straight line which joins the middle points of the diagonals.

29. O is the middle point of a given straight line AB, and from O as centre, any circle is described: if P be any point on its circumference, shew that the sum of the squares on AP, BP *is constant.*

30. Given the base of a triangle, and the sum of the squares on the sides forming the vertical angle; find the locus of the vertex.

31. ABC is an isosceles triangle in which AB and AC are equal. AB is produced beyond the base to D, so that BD is equal to AB. Shew that the square on CD is equal to the square on AB together with twice the square on BC.

32. In a right-angled triangle the sum of the squares on the straight lines drawn from the right angle to the points of trisection of the hypotenuse is equal to five times the square on the line between the points of trisection.

33. Three times the sum of the squares on the sides of a triangle is equal to four times the sum of the squares on the medians.

34. ABC is a triangle, and O the point of intersection of its medians: shew that

$$AB^2+BC^2+CA^2=3(OA^2+OB^2+OC^2).$$

35. ABCD is a quadrilateral, and X the middle point of the straight line joining the bisections of the diagonals; with X as centre *any* circle is described, and P is any point upon this circle: shew that $PA^2+PB^2+PC^2+PD^2$ is *constant*, being equal to

$$XA^2+XB^2+XC^2+XD^2+4XP^2.$$

36. The squares on the diagonals of a trapezium are together equal to the sum of the squares on its two oblique sides, with twice the rectangle contained by its parallel sides.

PROBLEMS.

37. Construct a rectangle equal to the difference of two squares.

38. Divide a given straight line into two parts so that the rectangle contained by them may be equal to the square described on a given straight line which is less than half the straight line to be divided.

39. Given a square and one side of a rectangle which is equal to the square, find the other side.

40. Produce a given straight line so that the rectangle contained by the whole line thus produced and the part produced, may be equal to the square on another given line.

41. Produce a given straight line so that the rectangle contained by the whole line thus produced and the given line shall be equal to the square on the part produced.

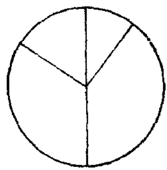
42. Divide a straight line AB into two parts at C, such that the rectangle contained by BC and another line X may be equal to the square on AC.

PART II.

BOOK III.

Book III. deals with the properties of Circles.

Definitions.



1. A **circle** is a plane figure bounded by one line, which is called the **circumference**, and is such that all straight lines drawn from a certain point within the figure to the circumference are equal to one another: this point is called the **centre** of the circle.

2. A **radius** of a circle is a straight line drawn from the centre to the circumference.

3. A **diameter** of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

4. A **semicircle** is the figure bounded by a diameter of a circle and the part of the circumference cut off by the diameter.

From these definitions we draw the following inferences:

(i) The distance of a point from the centre of a circle is less than the radius, if the point is within the circumference: and the distance of a point from the centre is greater than the radius, if the point is without the circumference.

(ii) A point is within a circle if its distance from the centre is less than the radius: and a point is without a circle if its distance from the centre is greater than the radius.

(iii) Circles of equal radius are equal in all respects; that is to say, their areas and circumferences are equal.

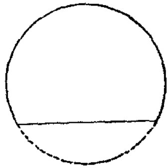
(iv) A circle is divided by any diameter into two parts which are equal in all respects.

5. Circles which have the same centre are said to be **concentric.**

6. An **arc** of a circle is any part of the circumference.

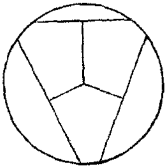
7. A **chord** of a circle is the straight line which joins any two points on the circumference.

From these definitions it may be seen that a chord of a circle, which does not pass through the centre, divides the circumference into two unequal arcs; of these, the greater is called the **major arc**, and the less the **minor arc.** Thus the major arc is *greater*, and the minor arc *less* than the semicircumference.

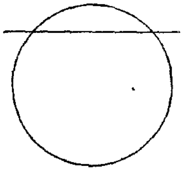


The major and minor arcs, into which a circumference is divided by a chord, are said to be **conjugate** to one another.

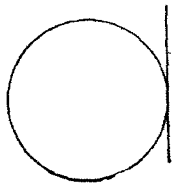
8. Chords of a circle are said to be **equidistant** from the centre, when the perpendiculars drawn to them from the centre are equal:
and one chord is said to be **further from the centre** than another, when the perpendicular drawn to it from the centre is greater than the perpendicular drawn to the other.



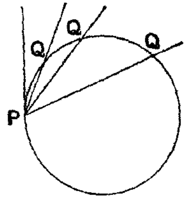
9. A **secant** of a circle is a straight line of indefinite length, which cuts the circumference in two points.



10. A **tangent** to a circle is a straight line which meets the circumference, but being produced, does not cut it. Such a line is said to **touch** the circle at a point; and the point is called the **point of contact.**

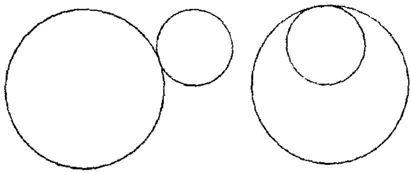


If a secant, which cuts a circle at the points P and Q, gradually changes its position in such a way that P remains fixed, the point Q will ultimately approach the fixed point P, until at length these points may be made to coincide. When the straight line PQ reaches this limiting position, it becomes the *tangent* to the circle at the point P.



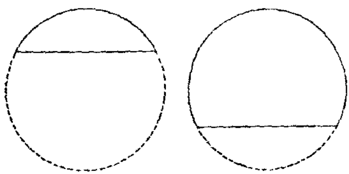
Hence a tangent may be defined as a straight line which passes through *two coincident points* on the circumference.

11. Circles are said to **touch one another** when they meet, but do not cut one another.



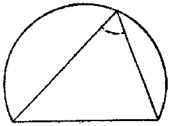
When each of the circles which meet is *outside the other*, they are said to touch one another **externally**, or to have **external contact**: when one of the circles is *within the other*, they are said to touch one another **internally**, or to have **internal contact**.

12. A **segment** of a circle is the figure bounded by a chord and one of the two arcs into which the chord divides the circumference.



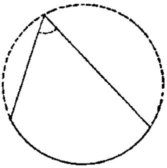
The chord of a segment is sometimes called its **base**.

13. An **angle in a segment** is one formed by two straight lines drawn from any point in the arc of the segment to the extremities of its chord.

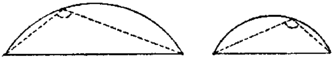


[It will be shewn in Proposition 21, that all angles in the same segment of a circle are equal.]

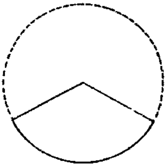
14. An **angle at the circumference** of a circle is one formed by straight lines drawn from a point on the circumference to the extremities of an arc: such an angle is said to **stand upon** the arc, which it subtends.



15. **Similar segments** of circles are those which contain equal angles.



16. A **sector** of a circle is a figure bounded by two radii and the arc intercepted between them.



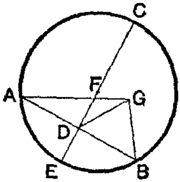
Symbols and Abbreviations.

In addition to the symbols and abbreviations given on page 10, we shall use the following.

$\odot$ *for* circle, $\bigcirc^{ce}$ *for* circumference.

PROPOSITION 1. PROBLEM.

To find the centre of a given circle.



Let ABC be a given circle:
it is required to find its centre.

In the given circle draw any chord AB,
and bisect AB at D. I. 10.

From D draw DC at right angles to AB; I. 11.
and produce DC to meet the $\bigcirc^{ce}$ at E and C.

Bisect EC at F. I. 10.

Then shall F be the centre of the $\odot$ ABC.

First, the centre of the circle must be in EC:
for if not, let the centre be at a point G without EC.

Join AG, DG, BG.

Then in the $\triangle^s$ GDA, GDB,

Because { DA = DB, *Constr.*
and GD is common;
and GA = GB, for by supposition they are radii;

$\therefore$ the $\angle$ GDA = the $\angle$ GDB; I. 8.

$\therefore$ these angles, being adjacent, are rt. angles.

But the $\angle$ CDB is a rt. angle; *Constr.*

$\therefore$ the $\angle$ GDB = the $\angle$ CDB, *Ax.* 11.

the part equal to the whole, which is impossible.

$\therefore$ G is not the centre.

So it may be shewn that no point outside EC is the centre;
$\therefore$ the centre lies in EC.

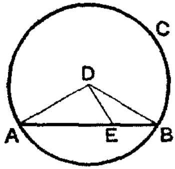
$\therefore$ F, the middle point of the diameter EC, must be the centre of the $\odot$ ABC. Q.E.F.

COROLLARY. *The straight line which bisects a chord of a circle at right angles passes through the centre.*

[For Exercises, see page 156.]

PROPOSITION 2. THEOREM.

If any two points are taken in the circumference of a circle, the chord which joins them falls within the circle.



Let ABC be a circle, and A and B any two points on its $\bigcirc^{ce}$:

then shall the chord AB fall within the circle.

Find D, the centre of the $\odot$ABC; III. 1.

and in AB take any point E.

Join DA, DE, DB.

In the $\triangle$DAB, because DA = DB, III. *Def.* 1.

$\therefore$ the $\angle$DAB = the $\angle$DBA. I. 5.

But the ext. $\angle$DEB is greater than the int. opp. $\angle$DAE; I. 16.

$\therefore$ also the $\angle$DEB is greater than the $\angle$DBE;

$\therefore$ in the $\triangle$DEB, the side DB, which is opposite the greater angle, is greater than DE which is opposite the less: I. 19.

that is to say, DE is less than a radius of the circle;

$\therefore$ E falls within the circle.

So also any other point between A and B may be shewn to fall within the circle.

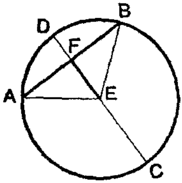
$\therefore$ AB falls within the circle. Q.E.D.

DEFINITION. A part of a curved line is said to be **concave** to a point when, *any* chord being taken in it, all straight lines drawn from the given point to the intercepted arc are cut by the chord: if, when any chord is taken, no straight line drawn from the given point to the intercepted arc is cut by the chord, the curve is said to be **convex** to that point.

Proposition 2 proves that the whole circumference of a circle is concave to its centre.

PROPOSITION 3. THEOREM.

If a straight line drawn through the centre of a circle bisects a chord which does not pass through the centre, it shall cut it at right angles:
and, conversely, if it cut it at right angles, it shall bisect it.



Let ABC be a circle; and let CD be a st. line drawn through the centre, and AB a chord which does not pass through the centre.

First. Let CD bisect AB at F:
then shall CD cut AB at rt. angles.

Find E, the centre of the circle; III. 1.
and join EA, EB.

Then in the $\triangle^s$ AFE, BFE,

Because { AF = BF, *Hyp.*
and FE is common;
and AE = BE, being radii of the circle;

$\therefore$ the $\angle$ AFE = the $\angle$ BFE; I. 8.

$\therefore$ these angles, being adjacent, are rt. angles,
that is, DC cuts AB at rt. angles. Q. E. D.

Conversely. Let CD cut AB at rt. angles:
then shall CD bisect AB at F.

As before, find E the centre; and join EA, EB.

In the $\triangle$ EAB, because EA = EB, III. *Def.* 1.
$\therefore$ the $\angle$ EAB = the $\angle$ EBA. I. 5.

Then in the $\triangle^s$ EFA, EFB,

Because { the $\angle$ EAF = the $\angle$ EBF, *Proved.*
and the $\angle$ EFA = the $\angle$ EFB, being rt. angles; *Hyp.*
and EF is common;

$\therefore$ AF = BF. I. 26.

Q. E. D.

[For Exercises, see page 156.]

EXERCISES.

ON PROPOSITION 1.

1. If two circles intersect at the points A, B, shew that the line which joins their centres bisects their common chord AB at right angles.

2. AB, AC are two equal chords of a circle; shew that the straight line which bisects the angle BAC passes through the centre.

3. *Two chords of a circle are given in position and magnitude: find the centre of the circle.*

4. *Describe a circle that shall pass through three given points, which are not in the same straight line.*

5. *Find the locus of the centres of circles which pass through two given points.*

6. Describe a circle that shall pass through two given points, and have a given radius.

ON PROPOSITION 2.

7. *A straight line cannot cut a circle in more than two points.*

ON PROPOSITION 3.

8. Through a given point within a circle draw a chord which shall be bisected at that point.

9. The parts of a straight line intercepted between the circumferences of two concentric circles are equal.

10. The line joining the middle points of two parallel chords of a circle passes through the centre.

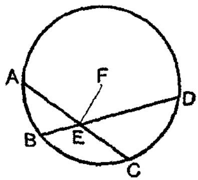
11. Find the locus of the middle points of a system of parallel chords drawn in a circle.

12. If two circles cut one another, any two parallel straight lines drawn through the points of intersection to cut the circles, are equal.

13. PQ and XY are two parallel chords in a circle: shew that the points of intersection of PX, QY, and of PY, QX, lie on the straight line which passes through the middle points of the given chords.

PROPOSITION 4. THEOREM.

If in a circle two chords cut one another, which do not both pass through the centre, they cannot both be bisected at their point of intersection.



Let ABCD be a circle, and AC, BD two chords which intersect at E, but do not both pass through the centre:

then AC and BD shall not be *both* bisected at E.

CASE I. If *one* chord passes through the centre, it is a diameter, and the centre is its middle point;

∴ it cannot be bisected by the other chord, which by hypothesis does not pass through the centre.

CASE II. If neither chord passes through the centre; then, if possible, let E be the middle point of *both*;

that is, let AE = EC; and BE = ED.

Find F, the centre of the circle: III. 1.

Join EF.

Then, because FE, which passes through the centre, bisects the chord AC, *Hyp.*

∴ the ∠ FEC is a rt. angle. III. 3.

And because FE, which passes through the centre, bisects the chord BD, *Hyp.*

∴ the ∠ FED is a rt. angle.

∴ the ∠ FEC = the ∠ FED,

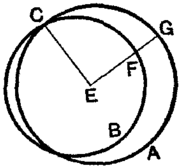
the whole equal to its part, which is impossible.

∴ AC and BD *are not both* bisected at E. Q. E. D.

[For Exercises, see page 158.]

PROPOSITION 5. THEOREM.

If two circles cut one another, they cannot have the same centre.



Let the two ⊙^{s} AGC, BFC cut one another at C:
then they shall not have the same centre.

For, if possible, let the two circles have the same centre; and let it be called E.

Join EC;

and from E draw any st. line to meet the ○^{ces} at F and G.

Then, because E is the centre of the ⊙ AGC, *Hyp.*

∴ EG = EC.

And because E is also the centre of the ⊙ BFC, *Hyp.*

∴ EF = EC.

∴ EG = EF,

the whole equal to its part, which is impossible.

∴ the two circles have not the same centre.

Q. E. D.

EXERCISES.

ON PROPOSITION 4.

1. If a parallelogram can be inscribed in a circle, the point of intersection of its diagonals must be at the centre of the circle.

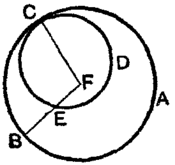
2. Rectangles are the only parallelograms that can be inscribed in a circle.

ON PROPOSITION 5.

3. Two circles, which intersect at one point, must also intersect at another.

Proposition 6. Theorem.

If two circles touch one another internally, they cannot have the same centre.



Let the two ⊙s ABC, DEC touch one another internally at C:

then they shall not have the same centre.

For, if possible, let the two circles have the same centre; and let it be called F.

Join FC;

and from F draw any st. line to meet the ○ces at E and B.

Then, because F is the centre of the ⊙ ABC, *Hyp.*

∴ FB = FC.

And because F is the centre of the ⊙ DEC, *Hyp.*

∴ FE = FC.

∴ FB = FE;

the whole equal to its part, which is impossible.

∴ the two circles have not the same centre. Q. E. D.

Note. From Propositions 5 and 6 it is seen that circles, whose circumferences have any point in common, cannot be concentric, unless they coincide entirely.

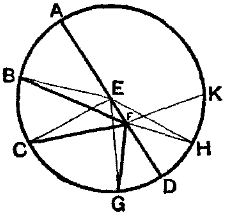
Conversely, the circumferences of concentric circles can have no point in common.

Proposition 7. Theorem.

If from any point within a circle which is not the centre, straight lines are drawn to the circumference, the greatest is that which passes through the centre; and the least is that which, when produced backwards, passes through the centre:

and of all other such lines, that which is nearer to the greatest is always greater than one more remote:

also two equal straight lines, and only two, can be drawn from the given point to the circumference, one on each side of the diameter.



Let ABCD be a circle, within which any point F is taken, which is not the centre: let FA, FB, FC, FG be drawn to the ○ce, of which FA passes through E the centre, and FB is nearer than FC to FA, and FC nearer than FG: and let FD be the line which, when produced backwards, passes through the centre: then of all these st. lines

(i) FA shall be the greatest;

(ii) FD shall be the least;

(iii) FB shall be greater than FC, and FC greater than FG;

(iv) also two, and only two, equal st. lines can be drawn from F to the ○ce.

Join EB, EC, EG.

(i) Then in the △ FEB, the two sides FE, EB are together greater than the third side FB. I. 20.

But EB = EA, being radii of the circle;

∴ FE, EA are together greater than FB;

that is, FA is greater than FB.

Similarly FA may be shewn to be greater than any other st. line drawn from F to the ○^{ce};

∴ FA is the greatest of all such lines.

(ii) In the △EFG, the two sides EF, FG are together greater than EG; I. 20.

and EG = ED, being radii of the circle;

∴ EF, FG are together greater than ED.

Take away the common part EF;

then FG is greater than FD.

Similarly any other st. line drawn from F to the ○^{ce} may be shewn to be greater than FD.

∴ FD is the least of all such lines.

(iii) In the △^{s} BEF, CEF,

Because { BE = CE, III. *Def.* 1.
and EF is common;
but the ∠ BEF is greater than the ∠ CEF;

∴ FB is greater than FC. I. 24.

Similarly it may be shewn that FC is greater than FG.

(iv) At E in FE make the ∠ FEH equal to the ∠ FEG. I. 23.

Join FH.

Then in the △^{s} GEF, HEF,

Because { GE = HE, III. *Def.* 1.
and EF is common;
also the ∠ GEF = the ∠ HEF; *Constr.*

∴ FG = FH. I. 4.

And besides FH no other straight line can be drawn from F to the ○^{ce} equal to FG.

For, if possible, let FK = FG.

Then, because FH = FG, *Proved.*

∴ FK = FH,

that is, a line nearer to FA, the greatest, is equal to a line which is more remote; which is impossible. *Proved.*

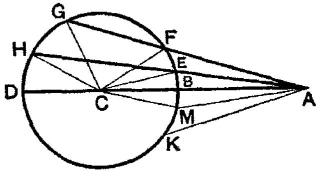
∴ two, and only two, equal st. lines can be drawn from F to the ○^{ce}. Q.E.D.

Proposition 8. Theorem.

If from any point without a circle straight lines are drawn to the circumference, of those which fall on the concave circumference, the greatest is that which passes through the centre; and of others, that which is nearer to the greatest is always greater than one more remote:

further, of those which fall on the convex circumference, the least is that which, when produced, passes through the centre; and of others that which is nearer to the least is always less than one more remote:

lastly, from the given point there can be drawn to the circumference two, and only two, equal straight lines, one on each side of the shortest line.



Let BGD be a circle of which C is the centre; and let A be any point outside the circle: let ABD, AEH, AFG, be st. lines drawn from A, of which AD passes through C, the centre, and AH is nearer than AG to AD:

then of st. lines drawn from A to the concave ○ce,

(i) AD shall be the greatest, and (ii) AH greater than AG:

and of st. lines drawn from A to the convex ○ce,

(iii) AB shall be the least, and (iv) AE less than AF.

(v) Also two, and only two, equal st. lines can be drawn from A to the ○ce.

Join CH, CG, CF, CE.

(i) Then in the △ ACH, the two sides AC, CH are together greater than AH: I. 20.

but CH = CD, being radii of the circle;

∴ AC, CD are together greater than AH:

that is, AD is greater than AH.

Similarly AD may be shewn to be greater than any other st. line drawn from A to the concave ○ce;

∴ AD is the greatest of all such lines.

(ii) In the $\triangle^s$ HCA, GCA,

Because $\begin{cases} \text{HC} = \text{GC}, & \text{III. } Def. \ 1. \\ \text{and CA is common;} \\ \text{but the } \angle \text{ HCA is greater than the } \angle \text{ GCA;} \end{cases}$

$\therefore$ AH is greater than AG. I. 24.

(iii) In the $\triangle$ AEC, the two sides AE, EC are together greater than AC: I. 20.

but EC = BC; III. *Def.* 1.

$\therefore$ the remainder AE is greater than the remainder AB.

Similarly any other st. line drawn from A to the convex $\bigcirc^{ce}$ may be shewn to be greater than AB;

$\therefore$ AB is the least of all such lines.

(iv) In the $\triangle$ AFC, because AE, EC are drawn from the extremities of the base to a point E within the triangle,

$\therefore$ AF, FC are together greater than AE, EC. I. 21.

But FC = EC, III. *Def.* 1.

$\therefore$ the remainder AF is greater than the remainder AE.

(v) At C, in AC, make the $\angle$ ACM equal to the $\angle$ ACE.

Join AM.

Then in the two $\triangle^s$ ECA, MCA,

Because $\begin{cases} \text{EC} = \text{MC}, & \text{III. } Def. \ 1. \\ \text{and CA is common;} \\ \text{also the } \angle \text{ ECA} = \text{the } \angle \text{ MCA;} & Constr. \end{cases}$

$\therefore$ AE = AM; I. 4.

and besides AM, no st. line can be drawn from A to the $\bigcirc^{ce}$, equal to AE.

For, if possible, let AK = AE:

then because AM = AE, *Proved.*

$\therefore$ AM = AK;

that is, a line nearer to the shortest line is equal to a line which is more remote; which is impossible. *Proved.*

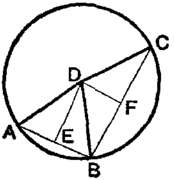
$\therefore$ two, and only two, equal st. lines can be drawn from A to the $\bigcirc^{ce}$. Q.E.D.

Where are the limits of that part of the circumference which is concave to the point A?

Obs. Of the following proposition Euclid gave two distinct proofs, the first of which has the advantage of being *direct*.

PROPOSITION 9. THEOREM. [FIRST PROOF.]

If from a point within a circle more than two equal straight lines can be drawn to the circumference, that point is the centre of the circle.



Let ABC be a circle, and D a point within it, from which more than two equal st. lines are drawn to the $\bigcirc^{ce}$, namely DA, DB, DC:

then D shall be the centre of the circle.

Join AB, BC:
and bisect AB, BC at E and F respectively. I. 10.
Join DE, DF.

Then in the $\triangle^s$ DEA, DEB,

Because { EA = EB, *Constr.*
and DE is common,
and DA = DB; *Hyp.*

∴ the ∠ DEA = the ∠ DEB; I. 8.

∴ these angles, being adjacent, are rt. angles.

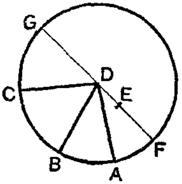
Hence ED, which bisects the chord AB at rt. angles, must pass through the centre. III. 1. *Cor.*

Similarly it may be shewn that FD passes through the centre.

∴ D, which is the only point common to ED and FD, must be the centre. Q.E.D.

PROPOSITION 9. THEOREM. [SECOND PROOF.]

If from a point within a circle more than two equal straight lines can be drawn to the circumference, that point is the centre of the circle.



Let ABC be a circle, and D a point within it, from which more than two equal st. lines are drawn to the $\bigcirc^{ce}$, namely DA, DB, DC:

then D shall be the centre of the circle.

For, if not, suppose E to be the centre.
Join DE, and produce it to meet the $\bigcirc^{ce}$ at F, G.

Then because D is a point within the circle, not the centre, and because DF passes through the centre E;

∴ DA, which is nearer to DF, is greater than DB, which is more remote: III. 7.

but this is impossible, since by hypothesis, DA, DB, are equal.

∴ E is not the centre of the circle.

*And wherever we suppose the centre E to be, otherwise than at D, two at least of the st. lines DA, DB, DC may be shewn to be unequal, which is contrary to hypothesis.

∴ D is the centre of the ⊙ ABC.

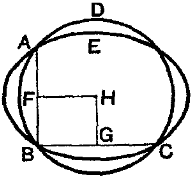
Q.E.D.

* NOTE. For example, if the centre E were supposed to be within the angle BDC, then DB would be greater than DA; if within the angle ADB, then DB would be greater than DC; if on one of the three straight lines, as DB, then DB would be greater than both DA and DC.

Obs. Two proofs of Proposition 10, both indirect, were given by Euclid.

PROPOSITION 10. THEOREM. [FIRST PROOF.]

One circle cannot cut another at more than two points.



If possible, let DABC, EABC be two circles, cutting one another at more than two points, namely at A, B, C.

Join AB, BC.

Draw FH, bisecting AB at rt. angles; I. 10, 11.
and draw GH bisecting BC at rt. angles.

Then because AB is a chord of *both* circles, and FH bisects it at rt. angles,

∴ the centre of both circles lies in FH. III. 1. *Cor.*

Again, because BC is a chord of both circles, and GH bisects it at right angles,

∴ the centre of both circles lies in GH. III. 1. *Cor.*

Hence H, the only point common to FH and GH, is the centre of both circles;

which is impossible, for circles which cut one another cannot have a common centre. III. 5.

∴ one circle cannot cut another at more than two points.

Q.E.D.

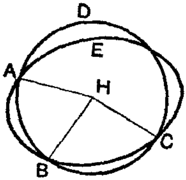
COROLLARIES. (i) *Two circles cannot meet in three points without coinciding entirely.*

(ii) *Two circles cannot have a common arc without coinciding entirely.*

(iii) *Only one circle can be described through three points, which are not in the same straight line.*

PROPOSITION 10. THEOREM. [SECOND PROOF.]

One circle cannot cut another at more than two points



If possible, let DABC, EABC be two circles, cutting one another at more than two points, namely at A, B, C.

Find H, the centre of the ⊙ DABC, III. 1.
and join HA, HB, HC.

Then since H is the centre of the ⊙ DABC,
∴ HA, HB, HC are all equal. III. *Def.* 1.

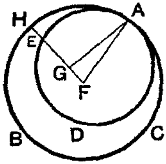
And because H is a point within the ⊙ EABC, from which more than two equal st. lines, namely HA, HB, HC are drawn to the ○ce,
∴ H is the centre of the ⊙ EABC : III. 9.
that is to say, the two circles have a common centre H;
but this is impossible, since they cut one another. III. 5.

Therefore one circle cannot cut another in more than two points. Q.E.D.

NOTE. This proof is imperfect, because it assumes that the centre of the circle DABC must fall within the circle EABC; whereas it may be conceived to fall either without the circle EABC, or on its circumference. Hence to make the proof complete, two additional cases are required.

PROPOSITION 11. THEOREM.

If two circles touch one another internally, the straight line which joins their centres, being produced, shall pass through the point of contact.



Let ABC and ADE be two circles which touch one another internally at A; let F be the centre of the ⊙ ABC, and G the centre of the ⊙ ADE:

then shall FG produced pass through A.

If not, let it pass otherwise, as FGEH.

Join FA, GA.

Then in the △ FGA, the two sides FG, GA are together greater than FA: I. 20.

but FA = FH, being radii of the ⊙ ABC: *Hyp.*

∴ FG, GA are together greater than FH.

Take away the common part FG;

then GA is greater than GH.

But GA = GE, being radii of the ⊙ ADE: *Hyp.*

∴ GE is greater than GH,

the part greater than the whole; which is impossible.

∴ FG, when produced, must pass through A.

Q.E.D.

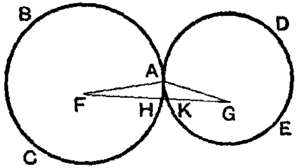
EXERCISES.

1. If the distance between the centres of two circles is equal to the difference of their radii, then the circles must meet in one point, but in no other; that is, they must touch one another.

2. *If two circles whose centres are* A *and* B *touch one another internally, and a straight line be drawn through their point of contact, cutting the circumferences at* P *and* Q; *shew that the radii* AP *and* BQ *are parallel.*

PROPOSITION 12. THEOREM.

If two circles touch one another externally, the straight line which joins their centres shall pass through the point of contact.



Let ABC and ADE be two circles which touch one another externally at A; let F be the centre of the ⊙ ABC, and G the centre of the ⊙ ADE:

then shall FG pass through A.

If not, let FG pass otherwise, as FHKG.

Join FA, GA.

Then in the △ FAG, the two sides FA, GA are together greater than FG: I. 20.

but FA = FH, being radii of the ⊙ ABC; *Hyp.*

and GA = GK, being radii of the ⊙ ADE; *Hyp.*

∴ FH and GK are together greater than FG;

which is impossible.

∴ FG must pass through A.

Q.E.D.

EXERCISES.

1. *Find the locus of the centres of all circles which touch a given circle at a given point.*

2. *Find the locus of the centres of all circles of given radius, which touch a given circle.*

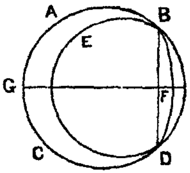
3. If the distance between the centres of two circles is equal to the sum of their radii, then the circles meet in one point, but in no other; that is, they touch one another.

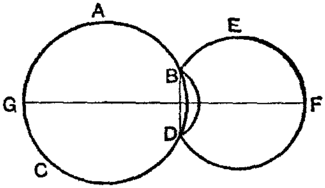
4. *If two circles whose centres are* A *and* B *touch one another externally, and a straight line be drawn through their point of contact cutting the circumferences at* P *and* Q; *shew that the radii* AP *and* BQ *are parallel.*

PROPOSITION 13. THEOREM.

Two circles cannot touch one another at more than one point, whether internally or externally.

Fig. 1 Fig. 2





If possible, let ABC, EDF be two circles which touch one another at more than one point, namely at B and D.

Join BD;
and draw GF, bisecting BD at rt. angles. I. 10, 11.

Then, whether the circles touch one another internally, as in Fig. 1, or externally as in Fig. 2,
because B and D are on the ○ces of both circles,
∴ BD is a chord of both circles:
∴ the centres of both circles lie in GF, which bisects BD at rt. angles. III. 1. *Cor.*
Hence GF which joins the centres must pass through a point of contact; III. 11, and 12.
which is impossible, since B and D are without GF.
∴ two circles cannot touch one another at more than one point.

Q.E.D.

NOTE. It must be observed that the proof here given applies, by virtue of Propositions 11 and 12, to *both* the above figures: we have therefore omitted the separate discussion of Fig. 2, which finds a place in most editions based on Simson's text.

EXERCISES ON PROPOSITIONS 1—13.

1. Describe a circle to pass through two given points and have its centre on a given straight line. When is this impossible?

2. All circles which pass through a fixed point, and have their centres on a given straight line, pass also through a second fixed point.

3. Describe a circle of given radius to touch a given circle at a given point. How many solutions will there be? When will there be only one solution?

4. From a given point as centre describe a circle to touch a given circle. How many solutions will there be?

5. Describe a circle to pass through a given point, and touch a given circle at a given point. [See Ex. 1, p. 169 and Ex. 5, p. 156.] When is this impossible?

6. Describe a circle of given radius to touch two given circles. [See Ex. 2, p. 169.] How many solutions will there be?

7. Two parallel chords of a circle are six inches and eight inches in length respectively, and the perpendicular distance between them is one inch: find the radius.

8. If two circles touch one another externally, the straight lines, which join the extremities of parallel diameters towards opposite parts, must pass through the point of contact.

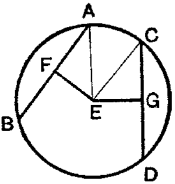
9. Find the greatest and least straight lines which have one extremity on each of two given circles, which do not intersect.

10. In any segment of a circle, of all straight lines drawn at right angles to the chord and intercepted between the chord and the arc, the greatest is that which passes through the middle point of the chord; and of others that which is nearer the greatest is greater than one more remote.

11. If from any point on the circumference of a circle straight lines be drawn to the circumference, the greatest is that which passes through the centre; and of others, that which is nearer to the greatest is greater than one more remote; and from this point there can be drawn to the circumference two, and only two, equal straight lines.

PROPOSITION 14. THEOREM.

Equal chords in a circle are equidistant from the centre: and, conversely, chords which are equidistant from the centre are equal.



Let ABC be a circle, and let AB and CD be chords, of which the perp. distances from the centre are EF and EG.

First, Let AB = CD :

then shall AB and CD be equidistant from the centre E.

Join EA, EC.

Then, because EF, which passes through the centre, is perp. to the chord AB; *Hyp.*

∴ EF bisects AB ; III. 3.

that is, AB is double of FA.

For a similar reason, CD is double of GC.

But AB = CD, *Hyp.*

∴ FA = GC.

Now EA = EC, being radii of the circle;

∴ the sq. on EA = the sq. on EC.

But the sq. on EA = the sqq. on EF, FA;

for the ∠ EFA is a rt. angle. I. 47.

And the sq. on EC = the sqq. on EG, GC ;

for the ∠ EGC is a rt. angle.

∴ the sqq. on EF, FA = the sqq. on EG, GC.

Now of these, the sq. on FA = the sq. on GC ; for FA = GC.

∴ the sq. on EF = the sq. on EG,

∴ EF = EG ;

that is, the chords AB, CD are equidistant from the centre.

Q.E.D.

Conversely. Let AB and CD be equidistant from the centre E;

that is, let EF = EG :
then shall AB = CD.

For, the same construction being made, it may be shewn as before that AB is double of FA, and CD double of GC;

and that the sqq. on EF, FA = the sqq. on EG, GC.
Now of these, the sq. on EF = the sq. on EG,
for EF = EG : *Hyp.*
∴ the sq. on FA = the square on GC;
∴ FA = GC;
and doubles of these equals are equal;
that is, AB = CD.

Q.E.D.

EXERCISES.

1. *Find the locus of the middle points of equal chords of a circle.*

2. If two chords of a circle cut one another, and make equal angles with the straight line which joins their point of intersection to the centre, they are equal.

3. *If two equal chords of a circle intersect, shew that the segments of the one are equal respectively to the segments of the other.*

4. In a given circle draw a chord which shall be equal to one given straight line (not greater than the diameter) and parallel to another.

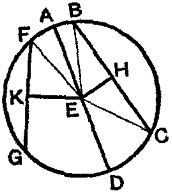
5. PQ is a fixed chord in a circle, and AB is any diameter: shew that the sum or difference of the perpendiculars let fall from A and B on PQ is constant, that is, the same for all positions of AB.

PROPOSITION 15. THEOREM.

The diameter is the greatest chord in a circle;

and of others, that which is nearer to the centre is greater than one more remote:

conversely, the greater chord is nearer to the centre than the less.



Let ABCD be a circle, of which AD is a diameter, and E the centre; and let BC and FG be any two chords, whose perp. distances from the centre are EH and EK:

then (i) AD shall be greater than BC:

(ii) if EH is less than EK, BC shall be greater than FG:

(iii) if BC is greater than FG, EH shall be less than EK.

(i) Join EB, EC.

Then in the △ BEC, the two sides BE, EC are together greater than BC; I. 20.

but BE = AE, III. *Def.* 1.

and EC = ED;

∴ AE and ED together are greater than BC;

that is, AD is greater than BC.

Similarly AD may be shewn to be greater than any other chord, not a diameter.

(ii) Let EH be less than EK;

then BC shall be greater than FG.

Join EF.

Since EH, passing through the centre, is perp. to the chord BC,

∴ EH bisects BC; III. 3.

that is, BC is double of HB.

For a similar reason FG is double of KF.

Now EB = EF, III. *Def.* 1.

∴ the sq. on EB = the sq. on EF.

But the sq. on EB = the sqq. on EH, HB;

for the ∠ EHB is a rt. angle; I. 47.

also the sq. on EF = the sqq. on EK, KF;

for the ∠ EKF is a rt. angle.

∴ the sqq. on EH, HB = the sqq. on EK, KF.

But the sq. on EH is less than the sq. on EK,

for EH is less than EK; *Hyp.*

∴ the sq. on HB is greater than the sq. on KF;

∴ HB is greater than KF:

hence BC is greater than FG.

(iii) Let BC be greater than FG;

then EH shall be less than EK.

For since BC is greater than FG, *Hyp.*

∴ HB is greater than KF:

∴ the sq. on HB is greater than the sq. on KF.

But the sqq. on EH, HB = the sqq. on EK, KF: *Proved.*

∴ the sq. on EH is less than the sq. on EK;

∴ EH is less than EK.

Q.E.D.

EXERCISES.

1. *Through a given point within a circle draw the least possible chord.*

2. AB is a fixed chord of a circle, and XY any other chord having its middle point Z on AB: what is the greatest, and what the least length that XY may have?

Shew that XY increases, as Z approaches the middle point of AB.

3. In a given circle draw a chord of given length, having its middle point on a given chord.

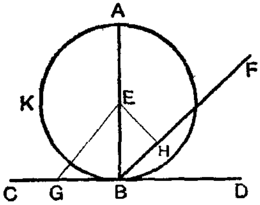
When is this problem impossible?

Obs. Of the following proofs of Proposition 16, the second (by reductio ad absurdum) is that given by Euclid; but the first is to be preferred, as it is *direct*, and not less simple than the other.

PROPOSITION 16. THEOREM. [ALTERNATIVE PROOF.]

The straight line drawn at right angles to a diameter of a circle at one of its extremities is a tangent to the circle:

and every other straight line drawn through this point cuts the circle.



Let AKB be a circle, of which E is the centre, and AB a diameter; and through B let the st. line CBD be drawn at rt. angles to AB:

then (i) CBD shall be a tangent to the circle;

(ii) any other st. line through B, as BF, shall cut the circle.

(i) In CD take any point G, and join EG.

Then, in the △EBG, the ∠EBG is a rt. angle, *Hyp.*

∴ the ∠EGB is less than a rt. angle; I. 17.

∴ the ∠EBG is greater than the ∠EGB;

∴ EG is greater than EB: I. 19.

that is, EG is greater than a radius of the circle;

∴ the point G is without the circle.

Similarly any other point in CD, except B, may be shewn to be outside the circle:

hence CD meets the circle at B, but being produced, does not cut it;

that is, CD is a tangent to the circle. III. *Def.* 10.

(ii) Draw EH perp. to BF. I. 12.

Then in the △ EHB, because the ∠ EHB is a rt. angle,

∴ the ∠ EBH is less than a rt. angle; I. 17.

∴ EB is greater than EH; I. 19.

that is, EH is less than a radius of the circle:

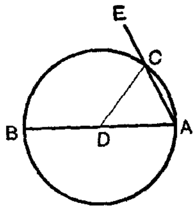
∴ H, a point in BF, is within the circle;

∴ BF must cut the circle. Q. E. D.

PROPOSITION 16. THEOREM. [EUCLID'S PROOF.]

The straight line drawn at right angles to a diameter of a circle at one of its extremities, is a tangent to the circle:

and no other straight line can be drawn through this point so as not to cut the circle.



Let ABC be a circle, of which D is the centre, and AB a diameter; let AE be drawn at rt. angles to BA, at its extremity A:

(i) then shall AE be a tangent to the circle.

For, if not, let AE cut the circle at C.

Join DC.

Then in the △ DAC, because DA = DC, III. *Def.* 1.

∴ the ∠ DAC = the ∠ DCA.

But the ∠ DAC is a rt. angle; *Hyp.*

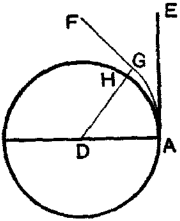
∴ the ∠ DCA is a rt. angle;

that is, two angles of the △ DAC are together equal to two rt. angles; which is impossible. I. 17.

Hence AE meets the circle at A, but being produced, does not cut it;

that is, AE is a tangent to the circle. III. *Def.* 10.

(ii) Also through A no other straight line but AE can be drawn so as not to cut the circle.



For, if possible, let AF be another st. line drawn through A so as not to cut the circle.

From D draw DG perp. to AF; I. 12.
and let DG meet the $\bigcirc^{ce}$ at H.

Then in the $\triangle$ DAG, because the $\angle$ DGA is a rt. angle,
$\therefore$ the $\angle$ DAG is less than a rt. angle; I. 17.
$\therefore$ DA is greater than DG. I. 19.
But DA = DH, III. *Def.* 1.
$\therefore$ DH is greater than DG,
the part greater than the whole, which is impossible.

$\therefore$ no st. line can be drawn from the point A, so as not to cut the circle, except AE.

COROLLARIES. (i) *A tangent touches a circle at one point only.*

(ii) *There can be but one tangent to a circle at a given point.*

PROPOSITION 17. PROBLEM.

To draw a tangent to a circle from a given point either on, or without the circumference.

Fig. 1

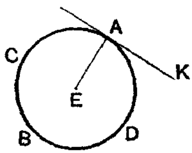
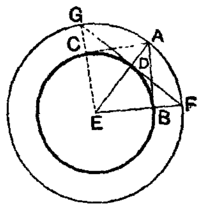


Fig. 2



Let BCD be the given circle, and A the given point: it is required to draw from A a tangent to the $\odot$ CDB.

CASE I. If the given point A is on the $\bigcirc^{ce}$.

Find E, the centre of the circle. III. 1.

Join EA.

At A draw AK at rt. angles to EA. I. 11.

Then AK being perp. to a diameter at one of its extremities, is a tangent to the circle. III. 16.

CASE II. If the given point A is without the $\bigcirc^{ce}$.

Find E, the centre of the circle; III. 1.

and join AE, cutting the $\odot$ BCD at D.

From centre E, with radius EA, describe the $\odot$ AFG.

At D, draw GDF at rt. angles to EA, cutting the $\odot$ AFG at F and G. I. 11.

Join EF, EG, cutting the $\odot$ BCD at B and C.

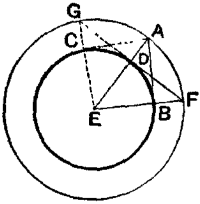
Join AB, AC.

Then both AB and AC shall be tangents to the $\odot$ CDB.

For in the $\triangle^s$ AEB, FED,

Because { AE = FE, being radii of the $\odot$ GAF;
and EB = ED, being radii of the $\odot$ BDC;
and the included angle AEF is common;

$\therefore$ the $\angle$ ABE = the $\angle$ FDE. I. 4.



But the ∠ FDE is a rt. angle, *Constr.*
∴ the ∠ ABE is a rt. angle;
hence AB, being drawn at rt. angles to a diameter at one of its extremities, is a tangent to the ⊙ BCD. III. 16.

Similarly it may be shewn that AC is a tangent. Q.E.F.

COROLLARY. *If two tangents are drawn to a circle from an external point, then* (i) *they are equal;* (ii) *they subtend equal angles at the centre;* (iii) *they make equal angles with the straight line which joins the given point to the centre.*

For, in the above figure,
Since ED is perp. to FG, a chord of the ⊙ FAG,
∴ DF = DG. III. 3.
Then in the △s DEF, DEG,
Because { DE is common to both,
and EF = EG; III. *Def.* 1.
and DF = DG; *Proved.*
∴ the ∠ DEF = the ∠ DEG. I. 8.
Again in the △s AEB, AEC,
Because { AE is common to both,
and EB = EC,
and the ∠ AEB = the ∠ AEC; *Proved.*
∴ AB = AC: I. 4.
and the ∠ EAB = the ∠ EAC. Q.E.D.

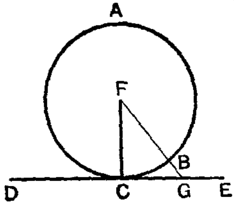
NOTE. If the given point A is within the circle, no solution is possible.

Hence we see that this problem admits of *two* solutions, *one* solution, or *no* solution, according as the given point A is *without*, *on*, or *within* the circumference of a circle.

For a simpler method of drawing a tangent to a circle from a given point, see page 202.

Proposition 18. Theorem.

The straight line drawn from the centre of a circle to the point of contact of a tangent is perpendicular to the tangent.



Let ABC be a circle, of which F is the centre;
and let the st. line DE touch the circle at C:
then shall FC be perp. to DE.

For, if not, suppose FG to be perp. to DE, I. 12.
and let it meet the ○ce at B.

Then in the △ FCG, because the ∠ FGC is a rt. angle, *Hyp.*
∴ the ∠ FCG is less than a rt. angle: I. 17.
∴ the ∠ FGC is greater than the ∠ FCG;
∴ FC is greater than FG: I. 19.
but FC = FB;
∴ FB is greater than FG,
the part greater than the whole, which is impossible.
∴ FC cannot be otherwise than perp. to DE:
that is, FC is perp. to DE. Q.E.D.

EXERCISES.

1. Draw a tangent to a circle (i) parallel to, (ii) at right angles to a given straight line.

2. *Tangents drawn to a circle from the extremities of a diameter are parallel.*

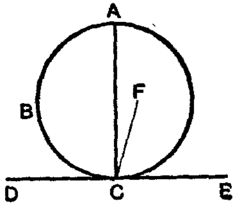
3. *Circles which touch one another internally or externally have a common tangent at their point of contact.*

4. *In two concentric circles any chord of the outer circle which touches the inner, is bisected at the point of contact.*

5. *In two concentric circles, all chords of the outer circle which touch the inner, are equal.*

PROPOSITION 19. THEOREM.

The straight line drawn perpendicular to a tangent to a circle from the point of contact passes through the centre.



Let ABC be a circle, and DE a tangent to it at the point C; and let CA be drawn perp. to DE:

then shall CA pass through the centre.

For if not, suppose the centre to be outside CA, as at F.

Join CF.

Then because DE is a tangent to the circle, and FC is drawn from the centre F to the point of contact,

∴ the ∠ FCE is a rt. angle. III. 18.

But the ∠ ACE is a rt. angle; *Hyp.*

∴ the ∠ FCE = the ∠ ACE;

the part equal to the whole, which is impossible.

∴ the centre cannot be otherwise than in CA;

that is, CA passes through the centre.

Q.E.D.

EXERCISES ON THE TANGENT.

PROPOSITIONS 16, 17, 18, 19.

1. *The centre of any circle which touches two intersecting straight lines must lie on the bisector of the angle between them.*

2. AB and AC are two tangents to a circle whose centre is O; shew that AO bisects the chord of contact BC at right angles.

3. If two circles are concentric all tangents drawn from points on the circumference of the outer to the inner circle are equal.

4. The diameter of a circle bisects all chords which are parallel to the tangent at either extremity.

5. *Find the locus of the centres of all circles which touch a given straight line at a given point.*

6. Find the locus of the centres of all circles which touch each of two parallel straight lines.

7. *Find the locus of the centres of all circles which touch each of two intersecting straight lines of unlimited length.*

8. Describe a circle of given radius to touch two given straight lines.

9. Through a given point, within or without a circle, draw a chord equal to a given straight line.

In order that the problem may be possible, between what limits must the given line lie, when the given point is (i) without the circle, (ii) within it?

10. Two parallel tangents to a circle intercept on any third tangent a segment which subtends a right angle at the centre.

11. *In any quadrilateral circumscribed about a circle, the sum of one pair of opposite sides is equal to the sum of the other pair.*

12. Any parallelogram which can be circumscribed about a circle, must be equilateral.

13. If a quadrilateral be described about a circle, the angles subtended at the centre by any two opposite sides are together equal to two right angles.

14. AB is any chord of a circle, AC the diameter through A, and AD the perpendicular on the tangent at B: shew that AB bisects the angle DAC.

15. Find the locus of the extremities of tangents of fixed length drawn to a given circle.

16. In the diameter of a circle produced, determine a point such that the tangent drawn from it shall be of given length.

17. In the diameter of a circle produced, determine a point such that the two tangents drawn from it may contain a given angle.

18. Describe a circle that shall pass through a given point, and touch a given straight line at a given point. [See page 183. Ex. 5.]

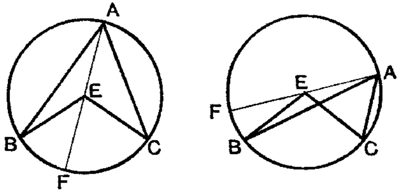
19. Describe a circle of given radius, having its centre on a given straight line, and touching another given straight line.

20. Describe a circle that shall have a given radius, and touch a given circle and a given straight line. How many such circles can be drawn?

PROPOSITION 20. THEOREM.

The angle at the centre of a circle is double of an angle at the circumference, standing on the same arc.

Fig. 1 Fig. 2



Let ABC be a circle, of which E is the centre; and let BEC be an angle at the centre, and BAC an angle at the ○ce, standing on the same arc BC:

then shall the ∠ BEC be double of the ∠ BAC.

Join AE, and produce it to F.

CASE I. When the centre E is within the angle BAC.

Then in the △ EAB, because EA = EB,

∴ the ∠ EAB = the ∠ EBA; I. 5.

∴ the sum of the ∠s EAB, EBA = twice the ∠ EAB.

But the ext. ∠ BEF = the sum of the ∠s EAB, EBA; I. 32.

∴ the ∠ BEF = twice the ∠ EAB.

Similarly the ∠ FEC = twice the ∠ EAC.

∴ the sum of the ∠s BEF, FEC = twice the sum of the ∠s EAB, EAC;

that is, the ∠ BEC = twice the ∠ BAC.

CASE II. When the centre E is without the ∠ BAC.

As before, it may be shewn that the ∠ FEB = twice the ∠ FAB;

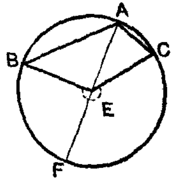
also the ∠ FEC = twice the ∠ FAC;

∴ the difference of the ∠s FEC, FEB = twice the difference of the ∠s FAC, FAB:

that is, the ∠ BEC = twice the ∠ BAC.

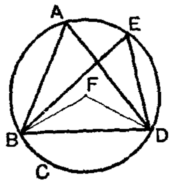
Q.E.D.

NOTE. If the arc BFC, on which the angles stand, is greater than a semi-circumference, it is clear that the angle BEC at the centre will be *reflex:* but it may still be shewn as, in Case I., that the reflex ∠ BEC is double of the ∠ BAC at the O^ce^, standing on the same arc BFC.



PROPOSITION 21. THEOREM.

Angles in the same segment of a circle are equal.



Let ABCD be a circle, and let BAD, BED be angles in the same segment BAED:

then shall the ∠ BAD = the ∠ BED.

Find F, the centre of the circle. III. 1.

CASE I. When the segment BAED is greater than a semicircle.

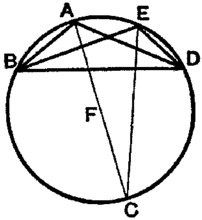
Join BF, DF.

Then the ∠ BFD at the centre = twice the ∠ BAD at the O^ce^, standing on the same arc BD: III. 20.

and similarly the ∠ BFD = twice the ∠ BED. III. 20.

∴ the ∠ BAD = the ∠ BED.

CASE II. When the segment BAED is not greater than a semicircle.



Join AF, and produce it to meet the $\bigcirc^{ce}$ at C.
Join EC.

Then since AEDC is a semicircle;
∴ the segment BAEC is greater than a semicircle:
∴ the ∠ BAC = the ∠ BEC, in this segment. *Case* 1.
Similarly the segment CAED is greater than a semicircle;
∴ the ∠ CAD = the ∠ CED, in this segment.
∴ the sum of the ∠s BAC, CAD = the sum of the ∠s BEC, CED:
that is, the ∠ BAD = the ∠ BED. Q. E. D.

EXERCISES.

1. P is any point on the arc of a segment of which AB is the chord. Shew that the sum of the angles PAB, PBA is constant.

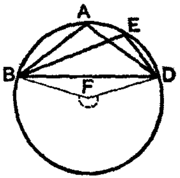
2. PQ and RS are two chords of a circle intersecting at X: prove that the triangles PXS, RXQ are equiangular.

3. Two circles intersect at A and B; and through A any straight line PAQ is drawn terminated by the circumferences: shew that PQ subtends a constant angle at B.

4. Two circles intersect at A and B; and through A any two straight lines PAQ, XAY are drawn terminated by the circumferences: shew that the arcs PX, QY subtend equal angles at B.

5. P is any point on the arc of a segment whose chord is AB: and the angles PAB, PBA are bisected by straight lines which intersect at O. Find the locus of the point O.

NOTE. If the extension of Proposition 20, given in the note on page 185, is adopted, a separate treatment of the second case of the present proposition is unnecessary.



For, as in Case I.,

the reflex ∠ BFD = twice the ∠ BAD; III. 20.

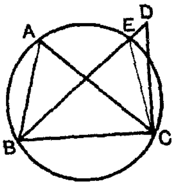
also the reflex ∠ BFD = twice the ∠ BED;

∴ the ∠ BAD = the ∠ BED.

The converse of Proposition 21 is very important. For the construction used in its proof, viz. *To describe a circle about a given triangle*, the student is referred to Book IV. Proposition 5. [Or see Theorems and Examples on Book I. Page 103, No. 1.]

CONVERSE OF PROPOSITION 21.

Equal angles standing on the same base, and on the same side of it, have their vertices on an arc of a circle, of which the given base is the chord.



Let BAC, BDC be two equal angles standing on the same base BC:

then shall the vertices A and D lie upon a segment of a circle having BC as its chord.

Describe a circle about the △ BAC: IV. 5.

then this circle shall pass through D.

For, if not, it must cut BD, or BD produced, at some other point E.

Join EC.

Then the ∠ BAC = the ∠ BEC, in the same segment: III. 21.

but the ∠ BAC = the ∠ BDC, by hypothesis;

∴ the ∠ BEC = the ∠ BDC;

that is, an ext. angle of a triangle = an int. opp. angle; which is impossible. I. 16.

∴ the circle which passes through B, A, C, cannot pass otherwise than through D.

That is, the vertices A and D are on an arc of a circle of which the chord is BC. Q. E. D.

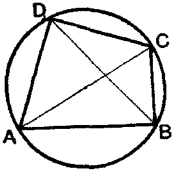
The following corollary is important.

All triangles drawn on the same base, and with equal vertical angles, have their vertices on an arc of a circle, of which the given base is the chord.

OR, *The locus of the vertices of triangles drawn on the same base with equal vertical angles is an arc of a circle.*

PROPOSITION 22. THEOREM.

The opposite angles of any quadrilateral inscribed in a circle are together equal to two right angles.



Let ABCD be a quadrilateral inscribed in the ⊙ABC;
then shall, (i) the ∠s ADC, ABC together = two rt. angles;
(ii) the ∠s BAD, BCD together = two rt. angles.

Join AC, BD.

Then the ∠ ADB = the ∠ ACB, in the segment ADCB; III. 21.
also the ∠ CDB = the ∠ CAB, in the segment CDAB.

∴ the ∠ ADC = the sum of the ∠s ACB, CAB.

To each of these equals add the ∠ ABC:

then the two ∠s ADC, ABC together = the three ∠s ACB, CAB, ABC.

But the ∠s ACB, CAB, ABC, being the angles of a triangle, together = two rt. angles. I. 32.

∴ the ∠s ADC, ABC together = two rt. angles.

Similarly it may be shewn that

the ∠s BAD, BCD together = two rt. angles.

Q. E. D.

EXERCISES.

1. If a circle can be described about a parallelogram, the parallelogram must be rectangular.

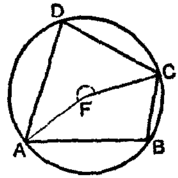
2. ABC is an isosceles triangle, and XY is drawn parallel to the base BC: shew that the four points B, C, X, Y lie on a circle.

3. *If one side of a quadrilateral inscribed in a circle is produced, the exterior angle is equal to the opposite interior angle of the quadrilateral.*

Proposition 22. [Alternative Proof.]

Let ABCD be a quadrilateral inscribed in the ⊙ ABC:
then shall the $\angle$s ADC, ABC together = two rt. angles.

Join FA, FC.



Then the $\angle$ AFC at the centre = twice the $\angle$ ADC at the $\bigcirc^{ce}$, standing on the same arc ABC. III. 20.

Also the reflex angle AFC at the centre = twice the $\angle$ ABC at the $\bigcirc^{ce}$, standing on the same arc ADC. III. 20.

Hence the $\angle$s ADC, ABC are together half the sum of the $\angle$ AFC and the reflex angle AFC;

but these make up four rt. angles: I. 15. *Cor.* 2.

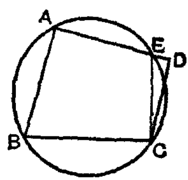
∴ the $\angle$s ADC, ABC together = two rt. angles. Q.E.D.

Definition. Four or more points through which a circle may be described are said to be **concyclic.**

Converse of Proposition 22.

If a pair of opposite angles of a quadrilateral are together equal to two right angles, its vertices are concyclic.

Let ABCD be a quadrilateral, in which the opposite angles at B and D together = two rt. angles;



then shall the four points A, B, C, D be concyclic.

Through the three points A, B, C describe a circle: IV. 5.

then this circle must pass through D.

For, if not, it will cut AD, or AD produced, at some other point E.

Join EC.

Then since the quadrilateral ABCE is inscribed in a circle,

∴ the $\angle$s ABC, AEC together = two rt. angles. III. 22.

But the $\angle$s ABC, ADC together = two rt. angles; *Hyp.*

hence the $\angle$s ABC, AEC = the $\angle$s ABC, ADC.

Take from these equals the $\angle$ ABC:

then the $\angle$ AEC = the $\angle$ ADC;

that is, an ext. angle of a triangle = an int. opp. angle; which is impossible. I. 16.

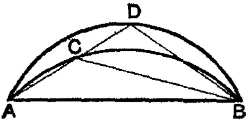
∴ the circle which passes through A, B, C cannot pass otherwise than through D:

that is the four vertices A, B, C, D are concyclic. Q.E.D.

DEFINITION. Similar segments of circles are those which contain equal angles.

PROPOSITION 23. THEOREM.

On the same chord and on the same side of it, there cannot be two similar segments of circles, not coinciding with one another.



If possible, on the same chord AB, and on the same side of it, let there be two similar segments of circles ACB, ADB, not coinciding with one another.

Then since the arcs ADB, ACB intersect at A and B,

∴ they cannot cut one another again; III. 10.

∴ one segment falls within the other.

In the outer arc take any point D;

join AD, cutting the inner arc at C:

join CB, DB.

Then because the segments are similar,

∴ the ∠ ACB = the ∠ ADB; III. *Def.*

that is, an ext. angle of a triangle = an int. opp. angle; which is impossible. I. 16.

Hence the two similar segments ACB, ADB, on the same chord AB and on the same side of it, must coincide.

Q.E.D.

EXERCISES ON PROPOSITION 22.

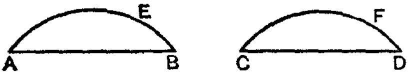
1. The straight lines which bisect any angle of a quadrilateral figure inscribed in a circle and the opposite exterior angle, meet on the circumference.

2. A triangle is inscribed in a circle: shew that the sum of the angles in the three segments exterior to the triangle is equal to four right angles.

3. Divide a circle into two segments, so that the angle contained by the one shall be double of the angle contained by the other.

PROPOSITION 24. THEOREM.

Similar segments of circles on equal chords are equal to one another.



Let AEB and CFD be similar segments on equal chords AB, CD:

then shall the segment ABE = the segment CDF.

For if the segment ABE be applied to the segment CDF, so that A falls on C, and AB falls along CD;

then since AB = CD,

∴ B must coincide with D.

∴ the segment AEB must coincide with the segment CFD; for if not, on the same chord and on the same side of it there would be two similar segments of circles, not coinciding with one another: which is impossible. III. 23.

∴ the segment AEB = the segment CFD. Q.E.D.

EXERCISES.

1. Of two segments standing on the same chord, the greater segment contains the smaller angle.

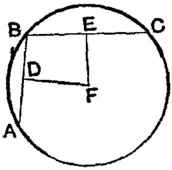
2. A segment of a circle stands on a chord AB, and P is any point on the same side of AB as the segment: shew that the angle APB is greater or less than the angle in the segment, according as P is within or without the segment.

3. *P, Q, R are the middle points of the sides of a triangle, and X is the foot of the perpendicular let fall from one vertex on the opposite side: shew that the four points P, Q, R, X are concyclic.*
[See page 96, Ex. 2: also page 100, Ex. 2.]

4. *Use the preceding exercise to shew that the middle points of the sides of a triangle and the feet of the perpendiculars let fall from the vertices on the opposite sides, are concyclic.*

PROPOSITION 25. PROBLEM*.

An arc of a circle being given, to describe the whole circumference of which the given arc is a part.



Let ABC be an arc of a circle:
it is required to describe the whole ○ce of which the arc ABC is a part.

In the given arc take any three points A, B, C.

Join AB, BC.

Draw DF bisecting AB at rt. angles, I. 10. 11.
and draw EF bisecting BC at rt. angles.

Then because DF bisects the chord AB at rt. angles,
∴ the centre of the circle lies in DF. III. 1. *Cor*

Again, because EF bisects the chord BC at rt. angles,
∴ the centre of the circle lies in EF. III. 1. *Cor.*

∴ the centre of the circle is F, the only point common to DF, EF.

Hence the ○ce of a circle described from centre F, with radius FA, is that of which the given arc is a part. Q. E. F.

* NOTE. Euclid gave this proposition a somewhat different form, as follows:

A segment of a circle being given, to describe the circle of which it is a segment.

Let ABC be the given segment standing on the chord AC.

Draw DB, bisecting AC at rt. angles. I. 10.

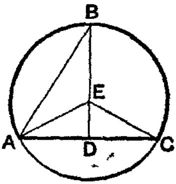
Join AB.

At A, in BA, make the ∠ BAE equal to the ∠ ABD. I. 23.

Let AE meet BD, or BD produced, at E.

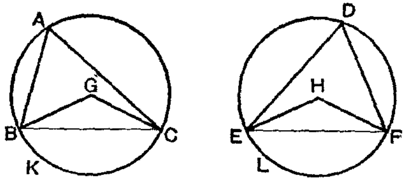
Then E shall be the centre of the required circle.

[Join EC; and prove (i) EA=EC; I. 4.
(ii) EA=EB. I. 6.]



Proposition 26. Theorem.

In equal circles the arcs which subtend equal angles, whether at the centres or at the circumferences, shall be equal.



Let ABC, DEF be equal circles and let the ∠ˢ BGC, EHF, at the centres be equal, and consequently the ∠ˢ BAC, EDF at the ○ᶜᵉˢ equal: III. 20.

then shall the arc BKC = the arc ELF.

Join BC, EF.

Then because the ⊙ˢ ABC, DEF are equal,

∴ their radii are equal.

Hence in the △ˢ BGC, EHF,

Because { BG = EH,
and GC = HF,
and the ∠ BGC = the ∠ EHF; } *Hyp.*

∴ BC = EF. I. 4.

Again, because the ∠ BAC = the ∠ EDF, *Hyp.*

∴ the segment BAC is similar to the segment EDF; III. *Def.* 15.

and they are on equal chords BC, EF;

∴ the segment BAC = the segment EDF. III. 24.

But the whole ⊙ ABC = the whole ⊙ DEF;

∴ the remaining segment BKC = the remaining segment ELF,

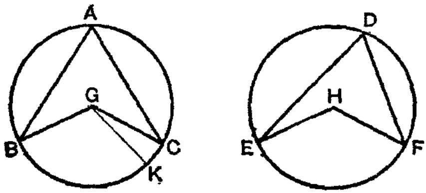
∴ the arc BKC = the arc ELF.

Q. E. D.

[For an Alternative Proof and Exercises see pp. 197, 198.]

Proposition 27. Theorem.

In equal circles the angles, whether at the centres or the circumferences, which stand on equal arcs, shall be equal.



Let ABC, DEF be equal circles,
and let the arc BC = the arc EF :
then shall the ∠ BGC = the ∠ EHF, at the centres;
and also the ∠ BAC = the ∠ EDF, at the ○ces.

If the ∠s BGC, EHF are not equal, one must be the greater.

If possible, let the ∠ BGC be the greater.
At G, in BG, make the ∠ BGK equal to the ∠ EHF. I. 23.

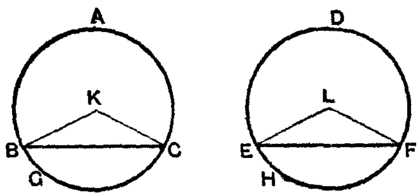
Then because in the equal ⊙s ABC, DEF,
the ∠ BGK = the ∠ EHF, at the centres; *Constr.*
∴ the arc BK = the arc EF. III. 26.
But the arc BC = the arc EF, *Hyp.*
∴ the arc BK = the arc BC,
a part equal to the whole, which is impossible.
∴ the ∠ BGC is not unequal to the ∠ EHF;
that is, the ∠ BGC = the ∠ EHF.

And since the ∠ BAC at the ○ce is half the ∠ BGC at the centre, III. 20.
and likewise the ∠ EDF is half the ∠ EHF,
∴ the ∠ BAC = the ∠ EDF. Q. E. D.

[For Exercises see pp. 197, 198.]

PROPOSITION 28. THEOREM.

In equal circles the arcs, which are cut off by equal chords, shall be equal, the major arc equal to the major arc, and the minor to the minor.



Let ABC, DEF be two equal circles,
and let the chord BC = the chord EF :
then shall the major arc BAC = the major arc EDF;
and the minor arc BGC = the minor arc EHF.

Find K and L the centres of the ⊙s ABC, DEF : III. 1.
and join BK, KC, EL, LF.

Then because the ⊙s ABC, DEF are equal,
∴ their radii are equal.

Hence in the △s BKC, ELF,

Because { BK = EL,
KC = LF,
and BC = EF; *Hyp.*

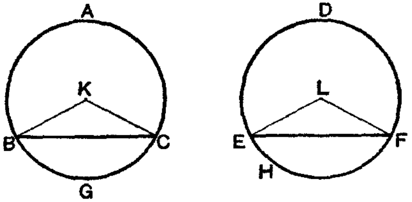
∴ the ∠ BKC = the ∠ ELF ; I. 8.
∴ the arc BGC = the arc EHF ; III. 26.
and these are the minor arcs.

But the whole ○ce ABGC = the whole ○ce DEHF; *Hyp.*
∴ the remaining arc BAC = the remaining arc EDF:
and these are the major arcs. Q.E.D.

[For Exercises see pp. 197, 198.]

PROPOSITION 29. THEOREM.

In equal circles the chords, which cut off equal arcs, shall be equal.



Let ABC, DEF be equal circles,
and let the arc BGC = the arc EHF:
then shall the chord BC = the chord EF.

Find K, L the centres of the circles. III. 1.
Join BK, KC, EL, LF.

Then in the equal ⊙s ABC, DEF,
because the arc BGC = the arc EHF,
∴ the ∠ BKC = the ∠ ELF. III. 27.

Hence in the △s BKC, ELF,

Because { BK = EL, being radii of equal circles;
KC = LF, for the same reason,
and the ∠ BKC = the ∠ ELF; *Proved.*
∴ BC = EF. I. 4.

Q. E. D.

EXERCISES

ON PROPOSITIONS 26, 27.

1. *If two chords of a circle are parallel, they intercept equal arcs.*

2. The straight lines, which join the extremities of two equal arcs of a circle towards the same parts, are parallel.

3. *In a circle, or in equal circles, sectors are equal if their angles at the centres are equal.*

4. If two chords of a circle intersect at right angles, the opposite arcs are together equal to a semicircumference.

5. *If two chords intersect within a circle, they form an angle equal to that subtended at the circumference by the sum of the arcs they cut off.*

6. *If two chords intersect without a circle, they form an angle equal to that subtended at the circumference by the difference of the arcs they cut off.*

7. *If* AB *is a fixed chord of a circle, and* P *any point on one of the arcs cut off by it, then the bisector of the angle* APB *cuts the conjugate arc in the same point, whatever be the position of* P.

8. Two circles intersect at A and B; and through these points straight lines are drawn from any point P on the circumference of one of the circles: shew that when produced they intercept on the other circumference an arc which is constant for all positions of P.

9. A triangle ABC is inscribed in a circle, and the bisectors of the angles meet the circumference at X, Y, Z. Find each angle of the triangle XYZ in terms of those of the original triangle.

ON PROPOSITIONS 28, 29.

10. The straight lines which join the extremities of parallel chords in a circle (i) towards the same parts, (ii) towards opposite parts, are equal.

11. Through A, a point of intersection of two equal circles two straight lines PAQ, XAY are drawn: shew that the chord PX is equal to the chord QY.

12. Through the points of intersection of two circles two parallel straight lines are drawn terminated by the circumferences: shew that the straight lines which join their extremities towards the same parts are equal.

13. Two equal circles intersect at A and B; and through A any straight line PAQ is drawn terminated by the circumferences: shew that BP = BQ.

14. ABC is an isosceles triangle inscribed in a circle, and the bisectors of the base angles meet the circumference at X and Y. Shew that the figure BXAYC must have four of its sides equal.

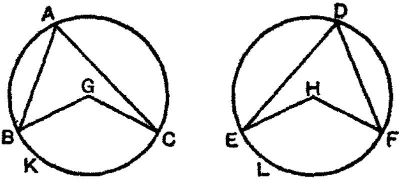
What relation must subsist among the angles of the triangle ABC, in order that the figure BXAYC may be equilateral?

Note. We have given Euclid's demonstrations of Propositions 26, 27, 28, 29; but it should be noticed that all these propositions also admit of direct proof by the method of *superposition*.

To illustrate this method we will apply it to Proposition 26.

Proposition 26. [Alternative Proof.]

In equal circles, the arcs which subtend equal angles, whether at the centres or circumferences, shall be equal.



Let ABC, DEF be equal circles, and let the ∠ˢ BGC, EHF at the centres be equal, and consequently the ∠ˢ BAC, EDF at the ⊙ᶜᵉˢ equal: III. 20.

then shall the arc BKC = the arc ELF.

For if the ⊙ ABC be applied to the ⊙ DEF, so that the centre G may fall on the centre H,

then because the circles are equal, *Hyp.*

∴ their ⊙ᶜᵉˢ must coincide;

hence by revolving the upper circle about its centre, the lower circle remaining fixed,

B may be made to coincide with E,

and consequently GB with HE.

And because the ∠ BGC = the ∠ EHF,

∴ GC must coincide with HF:

and since GC = HF, *Hyp.*

∴ C must fall on F.

Now B coinciding with E, and C with F,

and the ⊙ᶜᵉ of the ⊙ ABC with the ⊙ᶜᵉ of the ⊙ DEF,

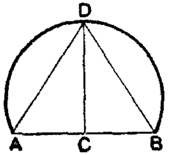
∴ the arc BKC must coincide with the arc ELF.

∴ the arc BKC = the arc ELF.

Q.E.D.

PROPOSITION 30. PROBLEM.

To bisect a given arc.



Let ADB be the given arc:
it is required to bisect it.

Join AB; and bisect it at C. I. 10.

At C draw CD at rt. angles to AB, meeting the given arc at D. I. 11.

Then shall the arc ADB be bisected at D.

Join AD, BD.

Then in the $\triangle^s$ ACD, BCD,

Because { AC = BC, *Constr.*
and CD is common;
and the $\angle$ ACD = the $\angle$ BCD, being rt. angles:

$\therefore$ AD = BD. I. 4.

And since in the $\odot$ ADB, the chords AD, BD are equal,
$\therefore$ the arcs cut off by them are equal, the minor arc equal to the minor, and the major arc to the major: III. 28.
and the arcs AD, BD are both minor arcs,
for each is less than a semi-circumference, since DC, bisecting the chord AB at rt. angles, must pass through the centre of the circle. III. 1. *Cor.*

$\therefore$ the arc AD = the arc BD:
that is, the arc ADB is bisected at D. Q. E. F.

EXERCISES.

1. If a tangent to a circle is parallel to a chord, the point of contact will bisect the arc cut off by the chord.

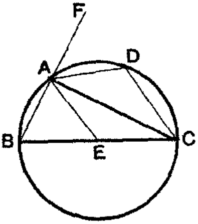
2. Trisect a quadrant, or the fourth part of the circumference, of a circle.

PROPOSITION 31. THEOREM.

The angle in a semicircle is a right angle:

the angle in a segment greater than a semicircle is less than a right angle:

and the angle in a segment less than a semicircle is greater than a right angle.



Let ABCD be a circle, of which BC is a diameter, and E the centre; and let AC be a chord dividing the circle into the segments ABC, ADC, of which the segment ABC is greater, and the segment is ADC less than a semicircle:

then (i) the angle in the semicircle BAC shall be a rt. angle;

(ii) the angle in the segment ABC shall be less than a rt. angle;

(iii) the angle in the segment ADC shall be greater than a rt. angle.

In the arc ADC take any point D;

Join BA, AD, DC, AE; and produce BA to F.

(i) Then because EA = EB, III. *Def.* 1.

∴ the ∠ EAB = the ∠ EBA. I. 5.

And because EA = EC,

∴ the ∠ EAC = the ∠ ECA.

∴ the whole ∠ BAC = the sum of the ∠s EBA, ECA:

but the ext. ∠ FAC = the sum of the two int. ∠s CBA, BCA;

∴ the ∠ BAC = the ∠ FAC;

∴ these angles, being adjacent, are rt. angles.

∴ the ∠ BAC, in the semicircle BAC, is a rt. angle.

(ii) In the $\triangle$ ABC, because the two $\angle$ s ABC, BAC are together less than two rt. angles; I. 17.

and of these, the $\angle$ BAC is a rt. angle; *Proved.*

$\therefore$ the $\angle$ ABC, which is the angle in the segment ABC, is less than a rt. angle.

(iii) Because ABCD is a quadrilateral inscribed in the $\odot$ ABC,

$\therefore$ the $\angle$ s ABC, ADC together = two rt. angles; III. 22.

and of these, the $\angle$ ABC is less than a rt. angle: *Proved.*

$\therefore$ the $\angle$ ADC, which is the angle in the segment ADC, is greater than a rt. angle. Q. E. D.

EXERCISES.

1. *A circle described on the hypotenuse of a right-angled triangle as diameter, passes through the opposite angular point.*

2. A system of right-angled triangles is described upon a given straight line as hypotenuse: find the locus of the opposite angular points.

3. A straight rod of given length slides between two straight rulers placed at right angles to one another: find the locus of its middle point.

4. Two circles intersect at A and B; and through A two diameters AP, AQ are drawn, one in each circle: shew that the points P, B, Q are collinear. [See Def. p. 102.]

5. A circle is described on one of the equal sides of an isosceles triangle as diameter. Shew that it passes through the middle point of the base.

6. Of two circles which have internal contact, the diameter of the inner is equal to the radius of the outer. Shew that any chord of the outer circle, drawn from the point of contact, is bisected by the circumference of the inner circle.

7. Circles described on any two sides of a triangle as diameters intersect on the third side, or the third side produced.

8. *Find the locus of the middle points of chords of a circle drawn through a fixed point.*

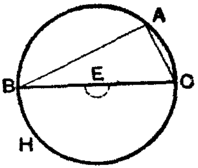
Distinguish between the cases when the given point is within, on, or without the circumference.

9. Describe a square equal to the difference of two given squares.

10. Through one of the points of intersection of two circles draw a chord of one circle which shall be bisected by the other.

11. On a given straight line as base a system of equilateral four-sided figures is described: find the locus of the intersection of their diagonals.

NOTE 1. The extension of Proposition 20 to *straight* and *reflex* angles furnishes a simple alternative proof of the first theorem contained in Proposition 31, viz.



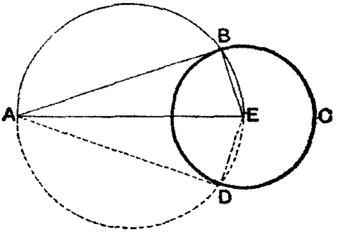
The angle in a semicircle is a right angle.

For, in the adjoining figure, the angle at the centre, standing on the arc BHC, is double the angle at the O^{ce}, standing on the same arc.

Now the angle at the centre is the *straight angle* BEC;
∴ the ∠ BAC is half of the *straight angle* BEC:
and a straight angle = two rt. angles;
∴ the ∠ BAC = one half of two rt. angles,
= one rt. angle. Q.E.D.

NOTE 2. From Proposition 31 we may derive a simple practical solution of Proposition 17, namely,

To draw a tangent to a circle from a given external point.



Let BCD be the given circle, and A the given external point:

it is required to draw from A a tangent to the ⊙ BCD.

Find E, the centre of the circle, and join AE.

On AE describe the semicircle ABE, to cut the given circle at B.

Join AB.

Then AB shall be a tangent to the ⊙ BCD.

For the ∠ ABE, being in a semicircle, is a rt. angle. III. 31.

∴ AB is drawn at rt. angles to the radius EB, from its extremity B;

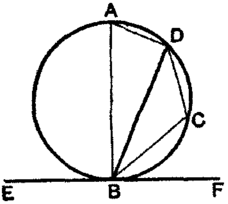
∴ AB is a tangent to the circle. III. 16.

Q.E.F.

Since the semicircle might be described on either side of AE, it is clear that there will be a second solution of the problem, as shewn by the dotted lines of the figure.

PROPOSITION 32. THEOREM.

If a straight line touch a circle, and from the point of contact a chord be drawn, the angles which this chord makes with the tangent shall be equal to the angles in the alternate segments of the circle.



Let EF touch the given ⊙ABC at B, and let BD be a chord drawn from B, the point of contact:

then shall (i) the ∠ DBF = the angle in the alternate segment BAD:

(ii) the ∠ DBE = the angle in the alternate segment BCD.

From B draw BA perp. to EF. I. 11.
Take any point C in the arc BD;
and join AD, DC, CB.

(i) Then because BA is drawn perp. to the tangent EF, at its point of contact B,

∴ BA passes through the centre of the circle: III. 19.

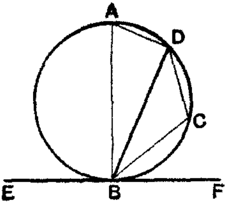
∴ the ∠ ADB, being in a semicircle, is a rt. angle: III. 31.

∴ in the △ABD, the other ∠ˢ ABD, BAD together = a rt. angle; I. 32.

that is, the ∠ˢ ABD, BAD together = the ∠ ABF.

From these equals take the common ∠ ABD;

∴ the ∠ DBF = the ∠ BAD, which is in the alternate segment.



(ii) Because ABCD is a quadrilateral inscribed in a circle,

∴ the $\angle^s$ BCD, BAD together = two rt. angles: III. 22.

but the $\angle^s$ DBE, DBF together = two rt. angles; I. 13.

∴ the $\angle^s$ DBE, DBF together = the $\angle^s$ BCD, BAD:

and of these the $\angle$ DBF = the $\angle$ BAD; *Proved.*

∴ the $\angle$ DBE = the $\angle$ DCB, which is in the alternate segment. Q. E. D.

EXERCISES.

1. State and prove the converse of this proposition.

2. Use this Proposition to shew that the tangents drawn to a circle from an external point are equal.

3. If two circles touch one another, any straight line drawn through the point of contact cuts off similar segments.
Prove this for (i) internal, (ii) external contact.

4. If two circles touch one another, and from A, the point of contact, two chords APQ, AXY are drawn: then PX and QY are parallel.
Prove this for (i) internal, (ii) external contact.

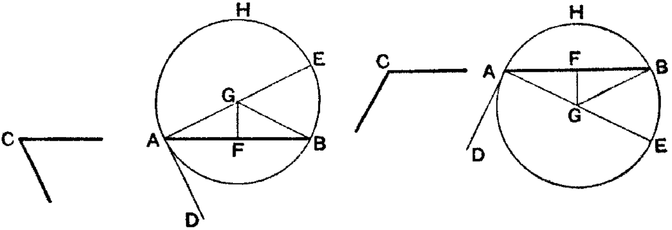
5. Two circles intersect at the points A, B: and one of them passes through O, the centre of the other: prove that OA bisects the angle between the common chord and the tangent to the first circle at A.

6. Two circles intersect at A and B; and through P, any point on the circumference of one of them, straight lines PAC, PBD are drawn to cut the other circle at C and D: shew that CD is parallel to the tangent at P.

7. If from the point of contact of a tangent to a circle, a chord be drawn, the perpendiculars dropped on the tangent and chord from the middle point of either arc cut off by the chord are equal.

PROPOSITION 33. PROBLEM.

On a given straight line to describe a segment of a circle which shall contain an angle equal to a given angle.



Let AB be the given st. line, and C the given angle: it is required to describe on AB a segment of a circle which shall contain an angle equal to C.

At A in BA, make the ∠ BAD equal to the ∠ C. I. 23.
From A draw AE at rt. angles to AD. I. 11.
Bisect AB at F; I. 10.
and from F draw FG at rt. angles to AB, cutting AE at G.
Join GB.

Then in the △s AFG, BFG.

Because { AF = BF, *Constr.*
and FG is common,
and the ∠ AFG = the ∠ BFG, being rt. angles;

∴ GA = GB : I. 4.

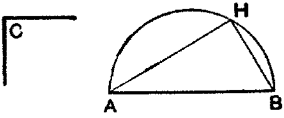
∴ the circle described from centre G, with radius GA, will pass through B.

Describe this circle, and call it ABH:
then the segment AHB shall contain an angle equal to C.

Because AD is drawn at rt. angles to the radius GA from its extremity A,

∴ AD is a tangent to the circle: III. 16.
and from A, its point of contact, a chord AB is drawn;
∴ the ∠ BAD = the angle in the alt. segment AHB. III. 32.
But the ∠ BAD = the ∠ C: *Constr*
∴ the angle in the segment AHB = the ∠ C.
∴ AHB is the segment required. Q.E.F.

NOTE. In the particular case when the given angle C is a rt. angle, the segment required will be the semicircle described on the given st. line AB; for the angle in a semicircle is a rt. angle. III. 31.



EXERCISES.

[The following exercises depend on the corollary to Proposition 21 given on page 187, namely

The locus of the vertices of triangles which stand on the same base and have a given vertical angle, is the arc of the segment standing on this base, and containing an angle equal to the given angle.

Exercises 1 and 2 afford good illustrations of the method of finding required points by the *Intersection of Loci*. See page 117.]

1. *Describe a triangle on a given base, having a given vertical angle, and having its vertex on a given straight line.*

2. *Construct a triangle, having given the base, the vertical angle and*
   (i) *one other side.*
   (ii) *the altitude.*
   (iii) *the length of the median which bisects the base.*
   (iv) *the point at which the perpendicular from the vertex meets the base.*

3. *Construct a triangle having given the base, the vertical angle, and the point at which the base is cut by the bisector of the vertical angle.*

[Let AB be the base, X the given point in it, and K the given angle. On AB describe a segment of a circle containing an angle equal to K; complete the $\odot^{ce}$ by drawing the arc APB. Bisect the arc APB at P: join PX, and produce it to meet the $\odot^{ce}$ at C. Then ABC shall be the required triangle.]

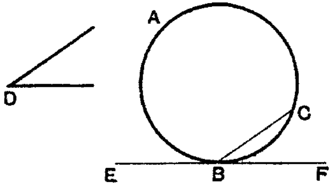
4. *Construct a triangle having given the base, the vertical angle, and the sum of the remaining sides.*

[Let AB be the given base, K the given angle, and H the given line equal to the sum of the sides. On AB describe a segment containing an angle equal to K, also another segment containing an angle equal to half the $\angle$ K. From centre A, with radius H, describe a circle cutting the last drawn segment at X and Y. Join AX (or AY) cutting the first segment at C. Then ABC shall be the required triangle.]

5. Construct a triangle having given the base, the vertical angle, and the difference of the remaining sides.

PROPOSITION 34. PROBLEM.

From a given circle to cut off a segment which shall contain an angle equal to a given angle.



Let ABC be the given circle, and D the given angle: it is required to cut off from the ⊙ ABC a segment which shall contain an angle equal to D.

Take any point B on the ○ce,
and at B draw the tangent EBF. III. 17.
At B, in FB, make the ∠ FBC equal to the ∠ D. I. 23.
Then the segment BAC shall contain an angle equal to D.

Because EF is a tangent to the circle, and from B, its point of contact, a chord BC is drawn,

∴ the ∠ FBC = the angle in the alternate segment BAC. III. 32.

But the ∠ FBC = the ∠ D; *Constr.*

∴ the angle in the segment BAC = the ∠ D.

Hence from the given ⊙ ABC a segment BAC has been cut off, containing an angle equal to D. Q.E.F.

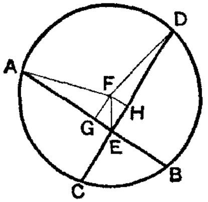
EXERCISES.

1. The chord of a given segment of a circle is produced to a fixed point: on this straight line so produced draw a segment of a circle similar to the given segment.

2. Through a given point without a circle draw a straight line that will cut off a segment capable of containing an angle equal to a given angle.

PROPOSITION 35. THEOREM.

If two chords of a circle cut one another, the rectangle contained by the segments of one shall be equal to the rectangle contained by the segments of the other.



Let AB, CD, two chords of the ⊙ACBD, cut one another at E:

then shall the rect. AE, EB = the rect. CE, ED.

Find F the centre of the ⊙ACB: III. 1.

From F draw FG, FH perp. respectively to AB, CD. I. 12.

Join FA, FE, FD.

Then because FG is drawn from the centre F perp. to AB,

∴ AB is bisected at G. III. 3.

For a similar reason CD is bisected at H.

Again, because AB is divided equally at G, and unequally at E,

∴ the rect. AE, EB with the sq. on EG = the sq. on AG. II. 5.

To each of these equals add the sq. on GF;

then the rect. AE, EB with the sqq. on EG, GF = the sum of the sqq. on AG, GF.

But the sqq. on EG, GF = the sq. on FE; I. 47.

and the sqq. on AG, GF = the sq. on AF;

for the angles at G are rt. angles.

∴ the rect. AE, EB with the sq. on FE = the sq. on AF.

Similarly it may be shewn that

the rect. CE, ED with the sq. on FE = the sq. on FD.

But the sq. on AF = the sq. on FD; for AF = FD.

∴ the rect. AE, EB with the sq. on FE = the rect. CE, ED with the sq. on FE.

From these equals take the sq. on FE:

then the rect. AE, EB = the rect. CE, ED. Q.E.D.

COROLLARY. *If through a fixed point within a circle any number of chords are drawn, the rectangles contained by their segments are all equal*

NOTE. The following special cases of this proposition deserve notice.

(i) when the given chords both pass through the centre:
(ii) when one chord passes through the centre, and cuts the other at right angles:
(iii) when one chord passes through the centre, and cuts the other obliquely.

In each of these cases the general proof requires some modification, which may be left as an exercise to the student.

EXERCISES.

1. *Two straight lines* AB, CD *intersect at* E, *so that the rectangle* AE, EB *is equal to the rectangle* CE, ED: *shew that the four points* A, B, C, D *are concyclic.*

2. The rectangle contained by the segments of any chord drawn through a given point within a circle is equal to the square on half the shortest chord which may be drawn through that point.

3. ABC is a triangle right-angled at C; and from C a perpendicular CD is drawn to the hypotenuse: shew that the square on CD is equal to the rectangle AD, DB.

4. ABC is a triangle, and AP, BQ the perpendiculars dropped from A and B on the opposite sides, intersect at O: shew that the rectangle AO, OP is equal to the rectangle BO, OQ.

5. Two circles intersect at A and B, and through any point in AB their common chord two chords are drawn, one in each circle; shew that their four extremities are concyclic.

6. A and B are two points within a circle such that the rectangle contained by the segments of any chord drawn through A is equal to the rectangle contained by the segments of any chord through B: shew that A and B are equidistant from the centre.

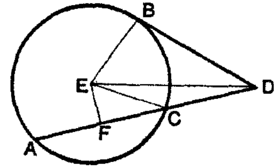
7. *If through* E, *a point without a circle, two secants* EAB, ECD *are drawn; shew that the rectangle* EA, EB *is equal to the rectangle* EC, ED.

[Proceed as in III. 35, using II. 6.]

8. Through A, a point of intersection of two circles, two straight lines CAE, DAF are drawn, each passing through a centre and terminated by the circumferences: shew that the rectangle CA, AE is equal to the rectangle DA, AF.

PROPOSITION 36. THEOREM.

If from any point without a circle a tangent and a secant be drawn, then the rectangle contained by the whole secant and the part of it without the circle shall be equal to the square on the tangent.



Let ABC be a circle; and from D a point without it, let there be drawn the secant DCA, and the tangent DB:
then the rect. DA, DC shall be equal to the sq. on DB.

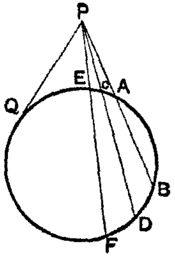
Find E, the centre of the ⊙ABC: III. 1.
and from E, draw EF perp. to AD. I. 12.
Join EB, EC, ED.

Then because EF, passing through the centre, is perp. to the chord AC,
∴ AC is bisected at F. III. 3.

And since AC is bisected at F and produced to D,
∴ the rect. DA, DC with the sq. on FC = the sq. on FD. II. 6.
To each of these equals add the sq. on EF:
then the rect. DA, DC with the sqq. on EF, FC = the sqq. on EF, FD.
But the sqq. on EF, FC = the sq. on EC; for EFC is a rt. angle;
= the sq. on EB.
And the sqq. on EF, FD = the sq. on ED; for EFD is a rt. angle;
= the sqq. on EB, BD; for EBD is a rt. angle. III. 18.
∴ the rect. DA, DC with the sq. on EB = the sqq. on EB, BD.
From these equals take the sq. on EB:
then the rect. DA, DC = the sq. on DB. Q.E.D.

NOTE. This proof may easily be adapted to the case when the secant passes through the centre of the circle.

COROLLARY. *If from a given point without a circle any number of secants are drawn, the rectangles contained by the whole secants and the parts of them without the circle are all equal; for each of these rectangles is equal to the square on the tangent drawn from the given point to the circle.*



For instance, in the adjoining figure, each of the rectangles PB, PA and PD, PC and PF, PE is equal to the square on the tangent PQ:

∴ the rect. PB, PA

= the rect. PD, PC

= the rect. PF, PE.

NOTE. Remembering that the segments into which the chord AB is divided at P, are the lines PA, PB, (see Part I. page 131) we are enabled to include the corollaries of Propositions 35 and 36 in a single enunciation.

If any number of chords of a circle are drawn through a given point within or without a circle, the rectangles contained by the segments *of the chords are equal.*

EXERCISES.

1. Use this proposition to shew that tangents drawn to a circle from an external point are equal.

2. If two circles intersect, tangents drawn to them from any point in their common chord produced are equal.

3. If two circles intersect at A and B, and PQ is a tangent to both circles; shew that AB produced bisects PQ.

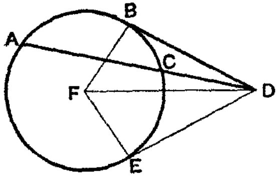
4. If P is any point on the straight line AB produced, shew that the tangents drawn from P to all circles which pass through A and B are equal.

5. ABC is a triangle right-angled at C, and from any point P in AC, a perpendicular PQ is drawn to the hypotenuse: shew that the rectangle AC, AP is equal to the rectangle AB, AQ.

6. ABC is a triangle right-angled at C, and from C a perpendicular CD is drawn to the hypotenuse: shew that the rect. AB, AD is equal to the square on AC.

Proposition 37. Theorem.

If from a point without a circle there be drawn two straight lines, one of which cuts the circle, and the other meets it, and if the rectangle contained by the whole line which cuts the circle and the part of it without the circle be equal to the square on the line which meets the circle, then the line which meets the circle shall be a tangent to it.



Let ABC be a circle; and from D, a point without it, let there be drawn two st lines DCA and DB, of which DCA cuts the circle at C and A, and DB meets it; and let the rect. DA, DC = the sq. on DB:

then shall DB be a tangent to the circle.

From D draw DE to touch the ⊙ABC: III. 17.
let E be the point of contact.
Find the centre F, and join FB, FD, FE. III. 1.

Then since DCA is a secant, and DE a tangent to the circle,
∴ the rect. DA, DC = the sq. on DE, III. 36.
But, by hypothesis, the rect. DA, DC = the sq. on DB;
∴ the sq. on DE = the sq. on DB,
∴ DE = DB.

Hence in the △s DBF, DEF.

Because { DB = DE, *Proved.*
and BF = EF; III. *Def.* 1.
and DF is common;
∴ the ∠ DBF = the ∠ DEF. I. 8.
But DEF is a rt. angle; III. 18.
∴ DBF is also a rt. angle;
and since BF is a radius,
∴ DB touches the ⊙ABC at the point B.

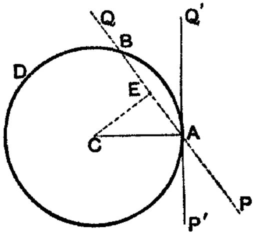
Q. E. D.

NOTE ON THE METHOD OF LIMITS AS APPLIED TO TANGENCY.

Euclid defines a tangent to a circle as *a straight line which meets the circumference, but being produced, does not cut it:* and from this definition he deduces the fundamental theorem that a tangent is perpendicular to the radius drawn to the point of contact. Prop. 16.

But this result may also be established by the Method of Limits, which regards the tangent as *the ultimate position of a secant when its two points of intersection with the circumference are brought into coincidence* [See Note on page 151]: and it may be shewn that every theorem relating to the tangent may be derived from some more general proposition relating to the secant, by considering the ultimate case when the two points of intersection coincide.

1. *To prove by the Method of Limits that a tangent to a circle is at right angles to the radius drawn to the point of contact.*



Let ABD be a circle, whose centre is C; and PABQ a secant cutting the ⊙^{ce} in A and B; and let P'AQ' be the limiting position of PQ when the point B is brought into coincidence with A: then shall CA be perp. to P'Q'.

Bisect AB at E and join CE:
then CE is perp. to PQ. III. 3.

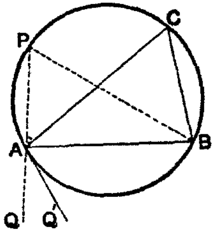
Now let the secant PABQ change its position in such a way that while the point A remains fixed, the point B continually approaches A, and ultimately coincides with it;

then, *however near* B *approaches to* A, the st. line CE is always perp. to PQ, since it joins the centre to the middle point of the chord AB.

But in the limiting position, when B coincides with A, and the secant PQ becomes the tangent P'Q', it is clear that the point E will also coincide with A; and the perpendicular CE becomes the radius CA. Hence CA is perp. to the tangent P'Q' at its point of contact A. Q. E. D.

NOTE. It follows from Proposition 2 that *a straight line cannot cut the circumference of a circle at more than two points.* Now when the two points in which a secant cuts a circle move towards coincidence, the secant ultimately becomes a tangent to the circle: we infer therefore that a tangent cannot meet a circle otherwise than at its point of contact. Thus Euclid's definition of a tangent may be deduced from that given by the Method of Limits.

2. *By this Method Proposition* 32 *may be derived as a special case from Proposition* 21.



For let A and B be two points on the ⊙^{ce} of the ⊙ ABC;
and let BCA, BPA be any two angles in the segment BCPA:
then the ∠ BPA = the ∠ BCA. III. 21.

Produce PA to Q.

Now let the point P continually approach the fixed point A, and ultimately coincide with it;
then, *however near* P *may approach to* A,
the ∠ BPQ = the ∠ BCA. III. 21.

But in the limiting position when P coincides with A,
and the secant PAQ becomes the tangent AQ′,
it is clear that BP will coincide with BA,
and the ∠ BPQ becomes the ∠ BAQ′.

Hence the ∠ BAQ′ = the ∠ BCA, in the alternate segment. Q. E. D.

The contact of circles may be treated in a similar manner by adopting the following definition.

Definition. If one or other of two intersecting circles alters its position in such a way that the two points of intersection continually approach one another, and ultimately coincide; in the limiting position they are said to **touch** one another, and the point in which the two points of intersection ultimately coincide is called the **point of contact.**

EXAMPLES ON LIMITS.

1. Deduce Proposition 19 from the Corollary of Proposition 1 and Proposition 3.

2. Deduce Propositions 11 and 12 from Ex. 1, page 156.

3. Deduce Proposition 6 from Proposition 5.

4. Deduce Proposition 13 from Proposition 10.

5. Shew that a straight line cuts a circle in two different points, two coincident points, or not at all, according as its distance from the centre is less than, equal to, or greater than a radius.

6. Deduce Proposition 32 from Ex. 3, page 188.

7. Deduce Proposition 36 from Ex. 7, page 209.

8. *The angle in a semi-circle is a right angle.*

To what Theorem is this statement reduced, when the vertex of the right angle is brought into coincidence with an extremity of the diameter?

9. From Ex. 1, page 190, deduce the corresponding property of a triangle inscribed in a circle.

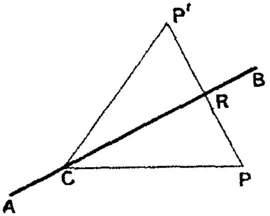
THEOREMS AND EXAMPLES ON BOOK III.

I. ON THE CENTRE AND CHORDS OF A CIRCLE.

See Propositions 1, 3, 14, 15, 25.

1. *All circles which pass through a fixed point, and have their centres on a given straight line, pass also through a second fixed point.*

Let AB be the given st. line, and P the given point.



From P draw PR perp. to AB;
and produce PR to P', making RP' equal to PR.

Then all circles which pass through P, and have their centres on AB, shall pass also through P'.

For let C be the centre of *any one* of these circles.
Join CP, CP'.
Then in the △ˢ CRP, CRP'

Because { CR is common,
and RP = RP', *Constr.*
and the ∠ CRP = the ∠ CRP', being rt. angles;
∴ CP = CP'; I. 4.

∴ the circle whose centre is C, and which passes through P, must pass also through P'.

But C is the centre of *any* circle of the system;

∴ all circles, which pass through P, and have their centres in AB, pass also through P'. Q. E. D.

2. *Describe a circle that shall pass through three given points not in the same straight line.*

3. Describe a circle that shall pass through two given points and have its centre in a given straight line. When is this impossible?

4. Describe a circle of given radius to pass through two given points. When is this impossible?

5. ABC is an isosceles triangle; and from the vertex A, as centre, a circle is described cutting the base, or the base produced, at X and Y. Shew that BX = CY.

6. If two circles which intersect are cut by a straight line parallel to the common chord, shew that the parts of it intercepted between the circumferences are equal.

7. If two circles cut one another, any two straight lines drawn through a point of section, making equal angles with the common chord, and terminated by the circumferences, are equal. [Ex. 12, p. 156.]

8. If two circles cut one another, of all straight lines drawn through a point of section and terminated by the circumferences, the greatest is that which is parallel to the line joining the centres.

9. Two circles, whose centres are C and D, intersect at A, B; and through A a straight line PAQ is drawn terminated by the circumferences: if PC, QD intersect at X, shew that the angle PXQ is equal to the angle CAD.

10. Through a point of section of two circles which cut one another draw a straight line terminated by the circumferences and bisected at the point of section.

11. AB is a fixed diameter of a circle, whose centre is C; and from P, any point on the circumference, PQ is drawn perpendicular to AB; shew that the bisector of the angle CPQ always intersects the circle in one or other of two fixed points.

12. Circles are described on the sides of a quadrilateral as diameters: shew that the common chord of any two consecutive circles is parallel to the common chord of the other two. [Ex. 9, p. 97.]

13. Two equal circles touch one another externally, and through the point of contact two chords are drawn, one in each circle, at right angles to each other: shew that the straight line joining their other extremities is equal to the diameter of either circle.

14. Straight lines are drawn from a given external point to the circumference of a circle: find the locus of their middle points. [Ex. 3, p. 97.]

15. Two equal segments of circles are described on opposite sides of the same chord AB; and through O, the middle point of AB, any straight line POQ is drawn, intersecting the arcs of the segments at P and Q: shew that OP = OQ.

II. ON THE TANGENT AND THE CONTACT OF CIRCLES.

See Propositions 11, 12, 16, 17, 18, 19.

1. All equal chords placed in a given circle touch a fixed concentric circle.

2. If from an external point two tangents are drawn to a circle, the angle contained by them is double the angle contained by the chord of contact and the diameter drawn through one of the points of contact.

3. Two circles touch one another externally, and through the point of contact a straight line is drawn terminated by the circumferences: shew that the tangents at its extremities are parallel.

4. Two circles intersect, and through one point of section any straight line is drawn terminated by the circumferences: shew that the angle between the tangents at its extremities is equal to the angle between the tangents at the point of section.

5. Shew that two parallel tangents to a circle intercept on any third tangent a segment which subtends a right angle at the centre.

6. Two tangents are drawn to a given circle from a fixed external point A, and any third tangent cuts them produced at P and Q: shew that PQ subtends a constant angle at the centre of the circle.

7. *In any quadrilateral circumscribed about a circle, the sum of one pair of opposite sides is equal to the sum of the other pair.*

8. *If the sum of one pair of opposite sides of a quadrilateral is equal to the sum of the other pair, shew that a circle may be inscribed in the figure.*

[Bisect two adjacent angles of the figure, and so describe a circle to touch three of its sides. Then prove indirectly by means of the last exercise that this circle must also touch the fourth side.]

9. Two circles touch one another internally, the centre of the outer being within the inner circle: shew that of all chords of the outer circle which touch the inner, the greatest is that which is perpendicular to the straight line joining the centres.

10. In any triangle, if a circle is described from the middle point of one side as centre and with a radius equal to half the sum of the other two sides, it will touch the circles described on these sides as diameters.

11. Through a given point, draw a straight line to cut a circle, so that the part intercepted by the circumference may be equal to a given straight line.

In order that the problem may be possible, between what limits must the given line lie, when the given point is (i) without the circle, (ii) within it?

12. A series of circles touch a given straight line at a given point: shew that the tangents to them at the points where they cut a given parallel straight line all touch a fixed circle, whose centre is the given point.

13. If two circles touch one another internally, and any third circle be described touching both; then the sum of the distances of the centre of this third circle from the centres of the two given circles is constant.

14. Find the locus of points such that the pairs of tangents drawn from them to a given circle contain a constant angle.

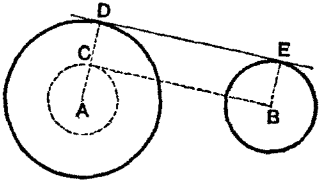
15. Find a point such that the tangents drawn from it to two given circles may be equal to two given straight lines. When is this impossible?

16. If three circles touch one another two and two; prove that the tangents drawn to them at the three points of contact are concurrent and equal.

The Common Tangents to Two Circles.

17. *To draw a common tangent to two circles.*

First, if the given circles are external to one another, or if they intersect.



Let A be the centre of the greater circle, and B the centre of the less.

From A, with radius equal to the diff^{ce} of the radii of the given circles, describe a circle: and from B draw BC to touch the last drawn circle. Join AC, and produce it to meet the greater of the given circles at D.

Through B draw the radius BE par^l to AD, and in the same direction.

Join DE:

then DE shall be a common tangent to the two given circles.

For since AC = the diff^{ce} between AD and BE, *Constr.*

∴ CD = BE:

and CD is par^l to BE; *Constr.*

∴ DE is equal and par^l to CB. I. 33.

But since BC is a tangent to the circle at C,

∴ the ∠ ACB is a rt. angle; III. 18.

hence each of the angles at D and E is a rt. angle: I. 29.

∴ DE is a tangent to both circles. Q.E.F.

It follows from hypothesis that the point B is outside the circle used in the construction:

∴ two tangents such as BC may always be drawn to it from B; hence *two* common tangents may always be drawn to the given circles by the above method. These are called the **direct common tangents.**

When the given circles are external to one another and do not intersect, two more common tangents may be drawn.

For, from centre A, with a radius equal to the *sum* of the radii of the given circles, describe a circle.

From B draw a tangent to this circle;

and proceed as before, but draw BE in the direction *opposite* to AD.

It follows from hypothesis that B is external to the circle used in the construction;

∴ two tangents may be drawn to it from B.

Hence *two* more common tangents may be drawn to the given circles: these will be found to pass between the given circles, and are called the **transverse common tangents.**

Thus, in general, *four* common tangents may be drawn to two given circles.

The student should investigate for himself the number of common tangents which may be drawn in the following special cases, noting in each case where the general construction fails, or is modified:—

(i) When the given circles intersect:

(ii) When the given circles have external contact:

(iii) When the given circles have internal contact:

(iv) When one of the given circles is wholly within the other.

18. *Draw the direct common tangents to two equal circles.*

19. If the two direct, or the two transverse, common tangents are drawn to two circles, the parts of the tangents intercepted between the points of contact are equal.

20. If four common tangents are drawn to two circles external to one another; shew that the two direct, and also the two transverse, tangents intersect on the straight line which joins the centres of the circles.

21. Two given circles have external contact at A, and a direct common tangent is drawn to touch them at P and Q: shew that PQ subtends a right angle at the point A.

22. Two circles have external contact at A, and a direct common tangent is drawn to touch them at P and Q: shew that a circle described on PQ as diameter is touched at A by the straight line which joins the centres of the circles.

23. Two circles whose centres are C and C′ have external contact at A, and a direct common tangent is drawn to touch them at P and Q: shew that the bisectors of the angles PCA, QC′A meet at right angles in PQ. And if R is the point of intersection of the bisectors, shew that RA is also a common tangent to the circles.

24. Two circles have external contact at A, and a direct common tangent is drawn to touch them at P and Q: shew that the square on PQ is equal to the rectangle contained by the diameters of the circles.

25. Draw a tangent to a given circle, so that the part of it intercepted by another given circle may be equal to a given straight line. When is this impossible?

26. Draw a secant to two given circles, so that the parts of it intercepted by the circumferences may be equal to two given straight lines.

Problems on Tangency.

The following exercises are solved by the Method of Intersection of Loci, explained on page 117.

The student should begin by making himself familiar with the following loci.

(i) *The locus of the centres of circles which pass through two given points.*

(ii) *The locus of the centres of circles which touch a given straight line at a given point.*

(iii) *The locus of the centres of circles which touch a given circle at a given point.*

(iv) *The locus of the centres of circles which touch a given straight line, and have a given radius.*

(v) *The locus of the centres of circles which touch a given circle, and have a given radius.*

(vi) *The locus of the centres of circles which touch two given straight lines.*

In each exercise the student should investigate the limits and relations among the data, in order that the problem may be possible.

27. Describe a circle to touch three given straight lines.

28. Describe a circle to pass through a given point and touch a given straight line at a given point.

29. Describe a circle to pass through a given point, and touch a given circle at a given point.

30. Describe a circle of given radius to pass through a given point, and touch a given straight line.

31. Describe a circle of given radius to touch two given circles.

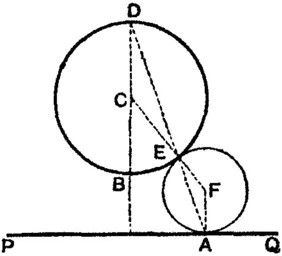
32. Describe a circle of given radius to touch two given straight lines.

33. Describe a circle of given radius to touch a given circle and a given straight line.

34. Describe two circles of given radii to touch one another and a given straight line, on the same side of it.

35. If a circle touches a given circle and a given straight line, shew that the points of contact and an extremity of the diameter of the given circle at right angles to the given line are collinear.

36. *To describe a circle to touch a given circle, and also to touch a given straight line at a given point.*



Let DEB be the given circle, PQ the given st. line, and A the given point in it:
it is required to describe a circle to touch the ⊙ DEB, and also to touch PQ at A.

At A draw AF perp. to PQ: I. 11.
then the centre of the required circle must lie in AF. III. 19.

Find C, the centre of the ⊙ DEB, III. 1.
and draw a diameter BD perp. to PQ:
join A to one extremity D, cutting the $\bigcirc^{ce}$ at E.

Join CE, and produce it to cut AF at F.

Then F is the centre, and FA the radius of the required circle.

[Supply the proof: and shew that a second solution is obtained by joining AB, and producing it to meet the $\bigcirc^{ce}$:
also distinguish between the nature of the contact of the circles, when PQ cuts, touches, or is without the given circle.]

37. Describe a circle to touch a given straight line, and to touch a given circle at a given point.

38. Describe a circle to touch a given circle, have its centre in a given straight line, and pass through a given point in that straight line.

[For other problems of the same class see page 235.]

Orthogonal Circles.

Definition. Circles which intersect at a point, so that the two tangents at that point are at right angles to one another, are said to be **orthogonal**, or **to cut one another orthogonally.**

39. In two intersecting circles the angle between the tangents at one point of intersection is equal to the angle between the tangents at the other.

40. *If two circles cut one another orthogonally, the tangent to each circle at a point of intersection will pass through the centre of the other circle.*

41. *If two circles cut one another orthogonally, the square on the distance between their centres is equal to the sum of the squares on their radii.*

42. Find the locus of the centres of all circles which cut a given circle orthogonally at a given point.

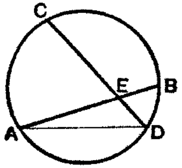
43. Describe a circle to pass through a given point and cut a given circle orthogonally at a given point.

III. On Angles in Segments, and Angles at the Centres and Circumferences of Circles.

See Propositions 20, 21, 22; 26, 27, 28, 29; 31, 32, 33, 34.

1. *If two chords intersect within a circle, they form an angle equal to that at the centre, subtended by half the sum of the arcs they cut off.*

Let AB and CD be two chords, intersecting at E within the given ⊙ADBC:
then shall the ∠AEC be equal to the angle at the centre, subtended by half the sum of the arcs AC, BD.



Join AD.

Then the ext. ∠AEC = the sum of the int. opp. ∠ˢ EDA, EAD;

that is, the sum of the ∠ˢ CDA, BAD.

But the ∠ˢ CDA, BAD are the angles at the Oᶜᵉ subtended by the arcs AC, BD;

∴ their sum = half the sum of the angles at the centre subtended by the same arcs;

or, the ∠AEC = the angle at the centre subtended by half the sum of the arcs AC, BD. Q. E. D.

2. *If two chords when produced intersect outside a circle, they form an angle equal to that at the centre subtended by half the difference of the arcs they cut off.*

3. The sum of the arcs cut off by two chords of a circle at right angles to one another is equal to the semi-circumference.

4. AB, AC are any two chords of a circle; and P, Q are the middle points of the minor arcs cut off by them: if PQ is joined, cutting AB and AC at X, Y, shew that AX = AY.

5. *If one side of a quadrilateral inscribed in a circle is produced, the exterior angle is equal to the opposite interior angle.*

6. If two circles intersect, and any straight lines are drawn, one through each point of section, terminated by the circumferences; shew that the chords which join their extremities towards the same parts are parallel.

7. ABCD is a quadrilateral inscribed in a circle; and the opposite sides AB, DC are produced to meet at P, and CB, DA to meet at Q: if the circles circumscribed about the triangles PBC, QAB intersect at R, shew that the points P, R, Q are collinear.

8. If a circle is described on one of the sides of a right-angled triangle, then the tangent drawn to it at the point where it cuts the hypotenuse bisects the other side.

9. Given three points not in the same straight line: shew how to find any number of points on the circle which passes through them, without finding the centre.

10. Through any one of three given points not in the same straight line, draw a tangent to the circle which passes through them, without finding the centre.

11. Of two circles which intersect at A and B, the circumference of one passes through the centre of the other: from A any straight line is drawn to cut the first at C, the second at D; shew that CB = CD.

12. Two tangents AP, AQ are drawn to a circle, and B is the middle point of the arc PQ, convex to A. Shew that PB bisects the angle APQ.

13. Two circles intersect at A and B; and at A tangents are drawn, one to each circle, to meet the circumferences at C and D: if CB, BD are joined, shew that the triangles ABC, DBA are equiangular to one another.

14. Two segments of circles are described on the same chord and on the same side of it; the extremities of the common chord are joined to any point on the arc of the exterior segment: shew that the arc intercepted on the interior segment is constant.

15. If a series of triangles are drawn standing on a fixed base, and having a given vertical angle, shew that the bisectors of the vertical angles all pass through a fixed point.

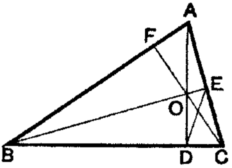
16. ABC is a triangle inscribed in a circle, and E the middle point of the arc subtended by BC on the side remote from A: if through E a diameter ED is drawn, shew that the angle DEA is half the difference of the angles at B and C. [See Ex. 7, p. 101.]

17. If two circles touch each other internally at a point A, any chord of the exterior circle which touches the interior is divided at its point of contact into segments which subtend equal angles at A.

18. If two circles touch one another internally, and a straight line is drawn to cut them, the segments of it intercepted between the circumferences subtend equal angles at the point of contact.

The Orthocentre of a Triangle.

19. *The perpendiculars drawn from the vertices of a triangle to the opposite sides are concurrent.*



In the △ABC, let AD, BE be the perp^s drawn from A and B to the opposite sides; and let them intersect at O. *Join* CO; and produce it to meet AB at F.

It is required to shew that CF *is perp. to* AB.

Join DE.

Then, because the ∠^s OEC, ODC are rt. angles, *Hyp.*

∴ the points O, E, C, D are concyclic:
∴ the ∠ DEC = the ∠ DOC, in the same segment;
= the vert. opp. ∠ FOA.

Again, because the ∠^s AEB, ADB are rt. angles, *Hyp.*
∴ the points A, E, D, B are concyclic:
∴ the ∠ DEB = the ∠ DAB, in the same segment.
∴ the sum of the ∠^s FOA, FAO = the sum of the ∠^s DEC, DEB
= a rt. angle: *Hyp.*
∴ the remaining ∠ AFO = a rt. angle: I. 32.
that is, CF is perp. to AB.

Hence the three perp^s AD, BE, CF meet at the point O. Q. E. D.

[For an Alternative Proof see page 106.]

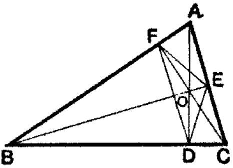
DEFINITIONS.

(i) The intersection of the perpendiculars drawn from the vertices of a triangle to the opposite sides is called its **ortho-centre.**

(ii) The triangle formed by joining the feet of the perpendiculars is called the **pedal** or **orthocentric triangle.**

20. *In an acute-angled triangle the perpendiculars drawn from the vertices to the opposite sides bisect the angles of the pedal triangle through which they pass.*

In the acute-angled △ ABC, let AD, BE, CF be the perp^s drawn from the vertices to the opposite sides, meeting at the orthocentre O; and let DEF be the pedal triangle:



then shall AD, BE, CF bisect respectively the ∠^s FDE, DEF, EFD.

For, as in the last theorem, it may be shewn that the points O, D, C, E are concyclic;

∴ the ∠ ODE = the ∠ OCE, in the same segment.

Similarly the points O, D, B, F are concyclic;
∴ the ∠ ODF = the ∠ OBF, in the same segment.

But the ∠ OCE = the ∠ OBF, each being the comp^t of the ∠ BAC.
∴ the ∠ ODE = the ∠ ODF.

Similarly it may be shewn that the ∠^s DEF, EFD are bisected by BE and CF. Q. E. D.

COROLLARY. (i) *Every two sides of the pedal triangle are equally inclined to that side of the original triangle in which they meet.*

For the ∠ EDC = the comp^t of the ∠ ODE
= the comp^t of the ∠ OCE
= the ∠ BAC.

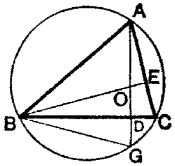
Similarly it may be shewn that the ∠ FDB = the ∠ BAC,
∴ the ∠ EDC = the ∠ FDB = the ∠ A.

In like manner it may be proved that
the ∠ DEC = the ∠ FEA = the ∠ B,
and the ∠ DFB = the ∠ EFA = the ∠ C.

COROLLARY. (ii) *The triangles* DEC, AEF, DBF *are equiangular to one another and to the triangle* ABC.

NOTE. If the angle BAC is *obtuse*, then the perpendiculars BE, CF bisect *externally* the corresponding angles of the pedal triangle.

21. *In any triangle, if the perpendiculars drawn from the vertices on the opposite sides are produced to meet the circumscribed circle, then each side bisects that portion of the line perpendicular to it which lies between the orthocentre and the circumference.*



Let ABC be a triangle in which the perpendiculars AD, BE are drawn, intersecting at O the orthocentre; and let AD be produced to meet the $\bigcirc^{ce}$ of the circumscribing circle at G:

then shall DO = DG.

Join BG.

Then in the two $\triangle^s$ OEA, ODB,

the ∠ OEA = the ∠ ODB, being rt. angles;

and the ∠ EOA = the vert. opp. ∠ DOB;

∴ the remaining ∠ EAO = the remaining ∠ DBO. I. 32.

But the ∠ CAG = the ∠ CBG, in the same segment;

∴ the ∠ DBO = the ∠ DBG.

Then in the $\triangle^s$ DBO, DBG,

Because { the ∠ DBO = the ∠ DBG, *Proved.*
the ∠ BDO = the ∠ BDG,
and BD is common;

∴ DO = DG. I. 26.

Q. E. D.

22. *In an acute-angled triangle the three sides are the external bisectors of the angles of the pedal triangle: and in an obtuse-angled triangle the sides containing the obtuse angle are the internal bisectors of the corresponding angles of the pedal triangle.*

23. *If* O *is the orthocentre of the triangle* ABC, *shew that the angles* BOC, BAC *are supplementary.*

24. *If* O *is the orthocentre of the triangle* ABC, *then any one of the four points* O, A, B, C *is the orthocentre of the triangle whose vertices are the other three.*

25. *The three circles which pass through two vertices of a triangle and its orthocentre are each equal to the circle circumscribed about the triangle.*

26. D, E are taken on the circumference of a semicircle described on a given straight line AB: the chords AD, BE and AE, BD intersect (produced if necessary) at F and G: shew that FG is perpendicular to AB.

27. ABCD is a parallelogram; AE and CE are drawn at right angles to AB, and CB respectively: shew that ED, if produced, will be perpendicular to AC.

28. ABC is a triangle, O is its orthocentre, and AK a diameter of the circumscribed circle: shew that BOCK is a parallelogram.

29. The orthocentre of a triangle is joined to the middle point of the base, and the joining line is produced to meet the circumscribed circle: prove that it will meet it at the same point as the diameter which passes through the vertex.

30. The perpendicular from the vertex of a triangle on the base, and the straight line joining the orthocentre to the middle point of the base, are produced to meet the circumscribed circle at P and Q: shew that PQ is parallel to the base.

31. *The distance of each vertex of a triangle from the orthocentre is double of the perpendicular drawn from the centre of the circumscribed circle on the opposite side.*

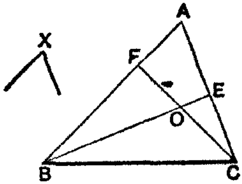
32. Three circles are described each passing through the orthocentre of a triangle and two of its vertices: shew that the triangle formed by joining their centres is equal in all respects to the original triangle.

33. ABC is a triangle inscribed in a circle, and the bisectors of its angles which intersect at O are produced to meet the circumference in PQR: shew that O is the orthocentre of the triangle PQR.

34. Construct a triangle, having given a vertex, the orthocentre, and the centre of the circumscribed circle.

Loci.

35. *Given the base and vertical angle of a triangle, find the locus of its orthocentre.*



Let BC be the given base, and X the given angle; and let BAC be any triangle on the base BC, having its vertical ∠ A equal to the ∠ X.

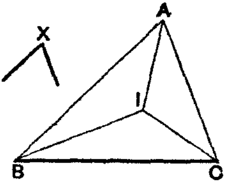
Draw the perp^s BE, CF, intersecting at the orthocentre O.

It is required to find the locus of O.

Since the ∠ ^s OFA, OEA are rt. angles,
∴ the points O, F, A, E are concyclic;
∴ the ∠ FOE is the supplement of the ∠ A: III. 22.
∴ the vert. opp. ∠ BOC is the supplement of the ∠ A.

But the ∠ A is constant, being always equal to the ∠ X;
∴ its supplement is constant;
that is, the △ BOC has a fixed base, and constant vertical angle; hence the locus of its vertex O is the arc of a segment of which BC is the chord. [See p. 187.]

36. *Given the base and vertical angle of a triangle, find the locus of the intersection of the bisectors of its angles.*



Let BAC be any triangle on the given base BC, having its vertical angle equal to the given ∠ X; and let AI, BI, CI be the bisectors of its angles: [see Ex. 2, p. 103.] it is required to find the locus of the point I.

Denote the angles of the △ ABC by A, B, C; and let the ∠ BIC be denoted by I.

Then from the △ BIC,

(i) $I + \frac{1}{2}B + \frac{1}{2}C =$ two rt. angles, I. 32.

and from the △ ABC,

$A + B + C =$ two rt. angles; I. 32.

(ii) so that $\frac{1}{2}A + \frac{1}{2}B + \frac{1}{2}C =$ one rt. angle,

∴, taking the differences of the equals in (i) and (ii),

$I - \frac{1}{2}A =$ one rt. angle:

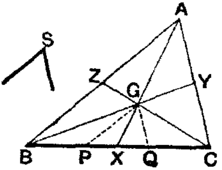
or, $I =$ one rt. angle $+ \frac{1}{2}A$.

But A is constant, being always equal to the ∠ X;

∴ I is constant:

∴, since the base BC is fixed, the locus of I is the arc of a segment of which BC is the chord.

37. *Given the base and vertical angle of a triangle, find the locus of the centroid, that is, the intersection of the medians.*



Let BAC be any triangle on the given base BC, having its vertical angle equal to the given angle S; let the medians AX, BY, CZ intersect at the centroid G [see Ex. 4, p. 105]: it is required to find the locus of the point G.

Through G draw GP, GQ par[l] to AB and AC respectively.

Then ZG is a third part of ZC; *Ex.* 4, p. 105.

and since GP is par[l] to ZB,

∴ BP is a third part of BC. *Ex.* 19, p. 99.

Similarly QC is a third part of BC;

∴ P and Q are fixed points.

Now since PG, GQ are par[l] respectively to BA, AC, *Constr.*

∴ the ∠ PGQ = the ∠ BAC, I. 29.

= the ∠ S,

that is, the ∠ PGQ is constant;

and since the base PQ is fixed,

∴ the locus of G is the arc of a segment of which PQ is the chord.

Obs. In this problem the points A and G move on the arcs of *similar segments.*

38. Given the base and the vertical angle of a triangle; find the locus of the intersection of the bisectors of the exterior base angles.

39. Through the extremities of a given straight line AB any two parallel straight lines AP, BQ are drawn; find the locus of the intersection of the bisectors of the angles PAB, QBA.

40. Find the locus of the middle points of chords of a circle drawn through a fixed point.

Distinguish between the cases when the given point is within, on, or without the circumference.

41. Find the locus of the points of contact of tangents drawn from a fixed point to a system of concentric circles.

42. Find the locus of the intersection of straight lines which pass through two fixed points on a circle and intercept on its circumference an arc of constant length.

43. A and B are two fixed points on the circumference of a circle, and PQ is any diameter: find the locus of the intersection of PA and QB.

44. BAC is any triangle described on the fixed base BC and having a constant vertical angle; and BA is produced to P, so that BP is equal to the sum of the sides containing the vertical angle: find the locus of P.

45. AB is a fixed chord of a circle, and AC is a moveable chord passing through A: if the parallelogram CB is completed, find the locus of the intersection of its diagonals.

46. A straight rod PQ slides between two rulers placed at right angles to one another, and from its extremities PX, QX are drawn perpendicular to the rulers: find the locus of X.

47. Two circles whose centres are C and D, intersect at A and B: through A, any straight line PAQ is drawn terminated by the circumferences; and PC, QD intersect at X: find the locus of X, and shew that it passes through B. [Ex. 9, p. 216.]

48. Two circles intersect at A and B, and through P, any point on the circumference of one of them, two straight lines PA, PB are drawn, and produced if necessary, to cut the other circle at X and Y: find the locus of the intersection of AY and BX.

49. Two circles intersect at A and B; HAK is a fixed straight line drawn through A and terminated by the circumferences, and PAQ is any other straight line similarly drawn: find the locus of the intersection of HP and QK.

50. Two segments of circles are on the same chord AB and on the same side of it; and P and Q are any points one on each arc: find the locus of the intersection of the bisectors of the angles PAQ, PBQ.

51. Two circles intersect at A and B; and through A any straight line PAQ is drawn terminated by the circumferences: find the locus of the middle point of PQ.

Miscellaneous Examples on Angles in a Circle.

52. ABC is a triangle, and circles are drawn through B, C, cutting the sides in P, Q, P′, Q′, ...: shew that PQ, P′Q′ . . are parallel to one another and to the tangent drawn at A to the circle circumscribed about the triangle.

53. Two circles intersect at B and C, and from any point A, on the circumference of one of them, AB, AC are drawn, and produced if necessary, to meet the other at D and E: shew that DE is parallel to the tangent at A.

54. A secant PAB and a tangent PT are drawn to a circle from an external point P; and the bisector of the angle ATB meets AB at C: shew that PC is equal to PT.

55. From a point A on the circumference of a circle two chords AB, AC are drawn, and also the diameter AF: if AB, AC are produced to meet the tangent at F in D and E, shew that the triangles ABC, AED are equiangular to one another.

56. O is any point within a triangle ABC, and OD, OE, OF are drawn perpendicular to BC, CA, AB respectively: shew that the angle BOC is equal to the sum of the angles BAC, EDF.

57. If two tangents are drawn to a circle from an external point, shew that they contain an angle equal to the difference of the angles in the segments cut off by the chord of contact.

58. Two circles intersect, and through a point of section a straight line is drawn bisecting the angle between the diameters through that point: shew that this straight line cuts off similar segments from the two circles.

59. Two equal circles intersect at A and B; and from centre A, with any radius less than AB a third circle is described cutting the given circles on the same side of AB at C and D: shew that the points B, C, D are collinear.

60. ABC and A′B′C′ are two triangles inscribed in a circle, so that AB, AC are respectively parallel to A′B′, A′C′: shew that BC′ is parallel to B′C.

61. Two circles intersect at A and B, and through A two straight lines HAK, PAQ are drawn terminated by the circumferences: if HP and KQ intersect at X, shew that the points H, B, K, X are concyclic.

62. Describe a circle touching a given straight line at a given point, so that tangents drawn to it from two fixed points in the given line may be parallel. [See Ex. 10, p. 183.]

63. C is the centre of a circle, and CA, CB two fixed radii: if from any point P on the arc AB perpendiculars PX, PY are drawn to CA and CB, shew that the distance XY is constant.

64. AB is a chord of a circle, and P any point in its circumference; PM is drawn perpendicular to AB, and AN is drawn perpendicular to the tangent at P: shew that MN is parallel to PB.

65. P is any point on the circumference of a circle of which AB is a fixed diameter, and PN is drawn perpendicular to AB; on AN and BN as diameters circles are described, which are cut by AP, BP at X and Y: shew that XY is a common tangent to these circles.

66. Upon the same chord and on the same side of it three segments of circles are described containing respectively a given angle, its supplement and a right angle: shew that the intercept made by the two former segments upon any straight line drawn through an extremity of the given chord is bisected by the latter segment.

67. Two straight lines of indefinite length touch a given circle, and any chord is drawn so as to be bisected by the chord of contact: if the former chord is produced, shew that the intercepts between the circumference and the tangents are equal.

68. Two circles intersect one another: through one of the points of contact draw a straight line of given length terminated by the circumferences.

69. On the three sides of any triangle equilateral triangles are described remote from the given triangle: shew that the circles described about them intersect at a point.

70. On BC, CA, AB the sides of a triangle ABC, any points P, Q, R are taken; shew that the circles described about the triangles AQR, BRP, CPQ meet in a point.

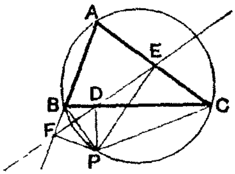
71. Find a point within a triangle at which the sides subtend equal angles.

72. Describe an equilateral triangle so that its sides may pass through three given points.

73. Describe a triangle equal in all respects to a given triangle, and having its sides passing through three given points.

Simson's Line.

74. *If from any point on the circumference of the circle circumscribed about a triangle, perpendiculars are drawn to the three sides, the feet of these perpendiculars are collinear.*



Let P be any point on the $\bigcirc^{ce}$ of the circle circumscribed about the $\triangle$ ABC; and let PD, PE, PF be the perp^s drawn from P to the three sides.

It is required to prove that the points D, E, F are collinear.

Join FD and DE:

then FD and DE shall be in the same st. line.

Join PB, PC.

Because the $\angle^s$ PDB, PFB are rt. angles, *Hyp.*

$\therefore$ the points P, D, B, F are concylic:

$\therefore$ the $\angle$ PDF = the $\angle$ PBF, in the same segment. III. 21.

But since BACP is a quad^l inscribed in a circle, having one of its sides AB produced to F,

$\therefore$ the ext. $\angle$ PBF = the opp. int. $\angle$ ACP. *Ex. 3, p. 188.*

$\therefore$ the $\angle$ PDF = the $\angle$ ACP.

To each add the $\angle$ PDE:

then the $\angle^s$ PDF, PDE = the $\angle^s$ ECP, PDE.

But since the $\angle^s$ PDC, PEC are rt. angles,

$\therefore$ the points P, D, E, C are concylic;

$\therefore$ the $\angle^s$ ECP, PDE together = two rt. angles:

$\therefore$ the $\angle^s$ PDF, PDE together = two rt. angles;

$\therefore$ FD and DE are in the same st. line; I. 14.

that is, the points D, E, F are collinear. Q.E.D.

[The line FDE is called the **Pedal** or **Simson's Line** of the triangle ABC for the point P; though the tradition attributing the theorem to Robert Simson has been recently shaken by the researches of Dr. J. S. Mackay.]

75. ABC is a triangle inscribed in a circle; and from any point P on the circumference PD, PF are drawn perpendicular to BC and AB: if FD, or FD produced, cuts AC at E, shew that PE is perpendicular to AC.

76. Find the locus of a point which moves so that if perpendiculars are drawn from it to the sides of a given triangle, their feet are collinear.

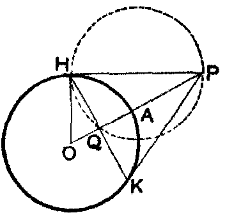
77. ABC and AB′C′ are two triangles having a common vertical angle, and the circles circumscribed about them meet again at P: shew that the feet of perpendiculars drawn from P to the four lines AB, AC, BC, B′C′ are collinear.

78. *A triangle is inscribed in a circle, and any point* P *on the circumference is joined to the orthocentre of the triangle: shew that this joining line is bisected by the pedal of the point* P.

IV. ON THE CIRCLE IN CONNECTION WITH RECTANGLES.

See Propositions 35, 36, 37.

1. *If from any external point* P *two tangents are drawn to a given circle whose centre is* O, *and if* OP *meets the chord of contact at* Q; *then the rectangle* OP, OQ *is equal to the square on the radius.*



Let PH, PK be tangents, drawn from the external point P to the ⊙ HAK, whose centre is O; and let OP meet HK the chord of contact at Q, and the ○^{ce} at A: then shall the rect. OP, OQ = the sq. on OA.

On HP as diameter describe a circle: this circle must pass through Q, since the ∠ HQP is a rt. angle. III. 31.

Join OH.

Then since PH is a tangent to the ⊙ HAK,

∴ the ∠ OHP is a rt. angle.

And since HP is a diameter of the ⊙ HQP,

∴ OH touches the ⊙ HQP at H. III. 16.

∴ the rect. OP, OQ = the sq. on OH, III. 36.

= the sq. on OA. Q. E. D.

2. ABC is a triangle, and AD, BE, CF the perpendiculars drawn from the vertices to the opposite sides, meeting in the orthocentre O: shew that the rect. AO, OD = the rect. BO, OE = the rect. CO, OF.

3. ABC is a triangle, and AD, BE the perpendiculars drawn from A and B on the opposite sides: shew that the rectangle CA, CE is equal to the rectangle CB, CD.

4. ABC is a triangle right-angled at C, and from D, any point in the hypotenuse AB, a straight line DE is drawn perpendicular to AB and meeting BC at E: shew that the square on DE is equal to the difference of the rectangles AD, DB and CE, EB.

5. From an external point P two tangents are drawn to a given circle whose centre is O, and OP meets the chord of contact at Q: shew that any circle which passes through the points P, Q will cut the given circle orthogonally. [See Def. p. 222.]

6. *A series of circles pass through two given points, and from a fixed point in the common chord produced tangents are drawn to all the circles: shew that the points of contact lie on a circle which cuts all the given circles orthogonally.*

7. *All circles which pass through a fixed point, and cut a given circle orthogonally, pass also through a second fixed point.*

8. Find the locus of the centres of all circles which pass through a given point and cut a given circle orthogonally.

9. Describe a circle to pass through two given points and cut a given circle orthogonally.

10. A, B, C, D are four points taken in order on a given straight line: find a point O between B and C such that the rectangle OA, OB may be equal to the rectangle OC, OD.

11. AB *is a fixed diameter of a circle, and* CD *a fixed straight line of indefinite length cutting* AB *or* AB *produced at right angles; any straight line is drawn through* A *to cut* CD *at* P *and the circle at* Q: *shew that the rectangle* AP, AQ *is constant.*

12. AB is a fixed diameter of a circle, and CD a fixed chord at right angles to AB; any straight line is drawn through A to cut CD at P and the circle at Q: shew that the rectangle AP, AQ is equal to the square on AC.

13. A *is a fixed point and* CD *a fixed straight line of indefinite length;* AP *is any straight line drawn through* A *to meet* CD *at* P; *and in* AP *a point* Q *is taken such that the rectangle* AP, AQ *is constant: find the locus of* Q.

14. Two circles intersect orthogonally, and tangents are drawn from any point on the circumference of one to touch the other: prove that the first circle passes through the middle point of the chord of contact of the tangents. [Ex. 1, p. 233.]

15. A semicircle is described on AB as diameter, and any two chords AC, BD are drawn intersecting at P: shew that

$$AB^2 = AC \cdot AP + BD \cdot BP.$$

16. Two circles intersect at B and C, and the two direct common tangents AE and DF are drawn: if the common chord is produced to meet the tangents at G and H, shew that $GH^2 = AE^2 + BC^2$.

17. If from a point P, without a circle, PM is drawn perpendicular to a diameter AB, and also a secant PCD, shew that

$$PM^2 = PC \cdot PD + AM \cdot MB.$$

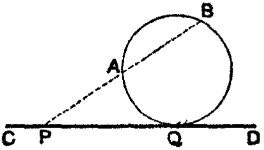
18. Three circles intersect at D, and their other points of intersection are A, B, C ; AD cuts the circle BDC at E, and EB, EC cut the circles ADB, ADC respectively at F and G : shew that the points F, A, G are collinear, and F, B, C, G concyclic.

19. A semicircle is described on a given diameter BC, and from B and C any two chords BE, CF are drawn intersecting within the semicircle at O; BF and CE are produced to meet at A: shew that the sum of the squares on AB, AC is equal to twice the square on the tangent from A together with the square on BC.

20. X and Y are two fixed points in the diameter of a circle equidistant from the centre C: through X any chord PXQ is drawn, and its extremities are joined to Y; shew that the sum of the squares on the sides of the triangle PYQ is constant. [See p. 147, Ex. 24.]

PROBLEMS ON TANGENCY.

21. *To describe a circle to pass through two given points and to touch a given straight line.*



Let A and B be the given points, and CD the given st. line: it is required to describe a circle to pass through A and B and to touch CD.

Join BA, and produce it to meet CD at P.

Describe a square equal to the rect. PA, PB; II. 14.

and from PD (or PC) cut off PQ equal to a side of this square.

Through A, B and Q describe a circle. Ex. 4, p. 156.

Then since the rect. PA, PB = the sq. on PQ,

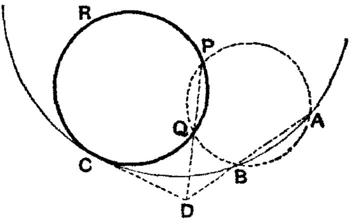
∴ the ⊙ ABQ touches CD at Q. III. 37.

Q. E. F.

NOTE. (i) Since PQ may be taken on either side of P, it is clear that there are in general two solutions of the problem.

(ii) When AB is parallel to the given line CD, the above method is not applicable. In this case a simple construction follows from III. 1, Cor. and III. 16, and it will be found that only one solution exists.

22. *To describe a circle to pass through two given points and to touch a given circle.*



Let A and B be the given points, and CRP the given circle: it is required to describe a circle to pass through A and B, and to touch the ⊙CRP.

Through A and B describe *any* circle to cut the given circle at P and Q.

Join AB, PQ, and produce them to meet at D.

From D draw DC to touch the given circle, and let C be the point of contact.

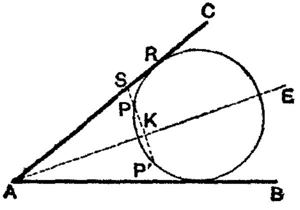
Then the circle described through A, B, C will touch the given circle.

For, from the ⊙ABQP, the rect. DA, DB = the rect. DP, DQ:
and from the ⊙PQC, the rect. DP, DQ = the sq. on DC; III. 36.
∴ the rect. DA, DB = the sq. on DC:
∴ DC touches the ⊙ABC at C. III. 37.
But DC touches the ⊙PQC at C; *Constr.*
∴ the ⊙ABC touches the given circle, and it passes through the given points A and B. Q.E.F.

NOTE. (i) Since two tangents may be drawn from D to the given circle, it follows that there will be two solutions of the problem.

(ii) The general construction fails when the straight line bisecting AB at right angles passes through the centre of the given circle: the problem then becomes symmetrical, and the solution is obvious.

23. *To describe a circle to pass through a given point and to touch two given straight lines.*



Let P be the given point, and AB, AC the given straight lines: it is required to describe a circle to pass through P and to touch AB, AC.

Now the centre of every circle which touches AB and AC must lie on the bisector of the ∠BAC. Ex. 7, p. 183.

Hence draw AE bisecting the ∠BAC.

From P draw PK perp. to AE, and produce it to P', making KP' equal to PK.

Then every circle which has its centre in AE, and passes through P, must also pass through P'. Ex. 1, p. 215.

Hence the problem is now reduced to drawing a circle through P and P' to touch *either* AC or AB. Ex. 21, p. 235.

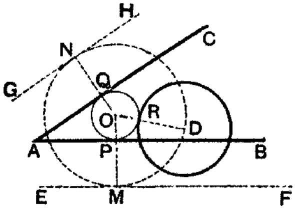
Produce P'P to meet AC at S.
Describe a square equal to the rect. SP, SP'; II. 14.
and cut off SR equal to a side of the square.
Describe a circle through the points P', P, R.
then since the rect. SP, SP' = the sq. on SR, *Constr.*
∴ the circle touches AC at R; III. 37.
and since its centre is in AE, the bisector of the ∠ BAC,
it may be shewn also to touch AB. Q. E. F.

NOTE. (i) Since SR may be taken on either side of S, it follows that there will be two solutions of the problem.

(ii) If the given straight lines are parallel, the centre lies on the parallel straight line mid-way between them, and the construction proceeds as before.

24. *To describe a circle to touch two given straight lines and a given circle.*

Let AB, AC be the two given st. lines, and D the centre of the given circle: it is required to describe a circle to touch AB, AC and the circle whose centre is D.



Draw EF, GH par^l to AB and AC respectively, on the sides remote from D, and at distances from them equal to the radius of the given circle.

Describe the ⊙ MND to touch EF and GH at M and N, and to pass through D. Ex. 23, p. 236.

Let O be the centre of this circle.

Join OM, ON, OD meeting AB, AC and the given circle at P, Q and R.

Then a circle described from centre O with radius OP will touch AB, AC and the given circle.

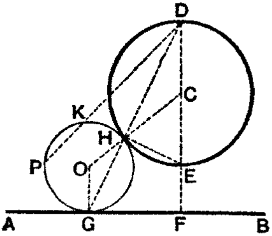
For since O is the centre of the ⊙ MND,
∴ OM = ON = OD.
But PM = QN = RD; *Constr.*
∴ OP = OQ = OR.
∴ a circle described from centre O, with radius OP, will pass through Q and R.
And since the ∠s at M and N are rt. angles, III. 18.
∴ the ∠s at P and Q are rt. angles; I. 29.
∴ the ⊙ PQR touches AB and AC.

And since R, the point in which the circles meet, is on the line of centres OD,

∴ the ⊙ PQR touches the given circle. Q. E. F.

NOTE. There will be two solutions of this problem, since two circles may be drawn to touch EF, GH and to pass through D.

25. *To describe a circle to pass through a given point and touch a given straight line and a given circle.*



Let P be the given point, AB the given st. line, and DHE the given circle, of which C is the centre: it is required to describe a circle to pass through P, and to touch AB and the ⊙ DHE.

Through C draw DCEF perp. to AB, cutting the circle at the points D and E, of which E is between C and AB.

Join DP;

and by describing a circle through F, E, and P, find a point K in DP (or DP produced) such that the rect. DE, DF = the rect. DK, DP.

Describe a circle to pass through P, K and touch AB: Ex. 21, p. 235. This circle shall also touch the given ⊙ DHE.

For let G be the point at which this circle touches AB.
Join DG, cutting the given circle DHE at H.

Join HE.

Then the ∠ DHE is a rt. angle, being in a semicircle. III. 31.
also the angle at F is a rt. angle; *Constr.*
∴ the points E, F, G, H are concyclic:
∴ the rect. DE, DF = the rect. DH, DG: III. 36.
but the rect. DE, DF = the rect. DK, DP: *Constr.*
∴ the rect. DH, DG = the rect. DK, DP:
∴ the point H is on the ⊙ PKG.

Let O be the centre of the ⊙ PHG.

Join OG, OH, CH.

Then OG and DF are par[l], since they are both perp. to AB; and DG meets them.
∴ the ∠ OGD = the ∠ GDC. I. 29.
But since OG = OH, and CD = CH,
∴ the ∠ OGH = the ∠ OHG; and the ∠ CDH = the ∠ CHD:
∴ the ∠ OHG = the ∠ CHD;
∴ OH and CH are in one st. line.
∴ the ⊙ PHG touches the given ⊙ DHE. Q. E. F.

NOTE. (i) Since two circles may be drawn to pass through P, K and to touch AB, it follows that there will be two solutions of the present problem.

(ii) Two more solutions may be obtained by joining PE, and proceeding as before.

The student should examine the nature of the contact between the circles in each case.

26. Describe a circle to pass through a given point, to touch a given straight line, and to have its centre on another given straight line.

27. Describe a circle to pass through a given point, to touch a given circle, and to have its centre on a given straight line.

28. Describe a circle to pass through two given points, and to intercept an arc of given length on a given circle.

29. Describe a circle to touch a given circle and a given straight line at a given point.

30. Describe a circle to touch two given circles and a given straight line.

V. ON MAXIMA AND MINIMA.

We gather from the Theory of Loci that the position of an angle, line or figure is capable under suitable conditions of gradual change; and it is usually found that change of *position* involves a corresponding and gradual change of *magnitude.*

Under these circumstances we may be required to note if any situations exist at which the magnitude in question, after increasing, begins to decrease; or after decreasing, to increase: in such situations the Magnitude is said to have reached a **Maximum** or a **Minimum** value; for in the former case it is greater, and in the latter case less than in adjacent situations on either side. In the geometry of the circle and straight line we only meet with such cases of continuous change as admit of *one* transition from an increasing to a decreasing state—or vice versâ—so that in all the problems with which we have to deal (where a single circle is involved) there can be only one Maximum and one Minimum—the Maximum being the greatest, and the Minimum being the least value that the variable magnitude is capable of taking.

Thus a variable geometrical magnitude reaches its maximum or minimum value at a *turning point*, towards which the magnitude may mount or descend from either side: it is natural therefore to expect a maximum or minimum value to occur when, in the course of its change, the magnitude assumes a *symmetrical* form or position; and this is usually found to be the case.

This general connection between a symmetrical form or position and a maximum or minimum value is not exact enough to constitute a *proof* in any particular problem; but by means of it a situation is suggested, which on further examination may be shewn to give the maximum or minimum value sought for.

For example, suppose it is required
to determine the greatest straight line that may be drawn perpendicular to the chord of a segment of a circle and intercepted between the chord and the arc:
we immediately anticipate that the greatest perpendicular is that which occupies a *symmetrical* position in the figure, namely the perpendicular which passes through the middle point of the chord; and on further examination this may be proved to be the case by means of I. 19, and I. 34.

Again we are able to find at what point a geometrical magnitude, varying under certain conditions, assumes its Maximum or Minimum value, if we can discover a construction for drawing the magnitude so that it may have an *assigned* value: for we may then examine between what limits the assigned value must lie in order that the construction may be possible; and the higher or lower limit will give the Maximum or Minimum sought for.

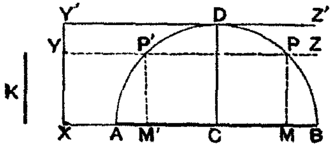
It was pointed out in the chapter on the Intersection of Loci, [see page 119] that if under certain conditions existing among the data, *two* solutions of a problem are possible, and under other conditions, *no* solution exists, there will always be some intermediate condition under which *one* and *only one* distinct solution is possible.

Under these circumstances this single or limiting solution will always be found to correspond to the maximum or minimum value of the magnitude to be constructed.

1. For example, suppose it is required
to divide a given straight line so that the rectangle contained by the two segments may be a maximum.

We may first attempt to divide the given straight line so that the rectangle contained by its segments may have a *given* area—that is, be equal to the square on a given straight line.

Let AB be the given straight line, and K the side of the given square:



it is required to divide the st. line AB at a point M, so that the rect. AM, MB may be equal to the sq. on K.

Adopting a construction suggested by II. 14,

describe a semicircle on AB; and at any point X in AB, or AB produced, draw XY perp. to AB, and equal to K.

Through Y draw YZ par[l] to AB, to meet the arc of the semicircle at P.

Then if the perp. PM is drawn to AB, it may be shewn after the manner of II. 14, or by III. 35 that

the rect. AM, MB = the sq. on PM.
= the sq. on K.

So that the rectangle AM, MB increases as K increases.

Now if K is less than the radius CD, then YZ will meet the arc of the semicircle in two points P, P'; and it follows that AB may be divided at *two* points, so that the rectangle contained by its segments may be equal to the square on K. If K increases, the st. line YZ will recede from AB, and the points of intersection P, P' will continually approach one another; until, when K is equal to the radius CD, the st. line YZ (now in the position Y'Z') will meet the arc in *two coincident points*, that is, will touch the semicircle at D; and there will be only *one* solution of the problem.

If K is greater than CD, the straight line YZ will not meet the semicircle, and the problem is impossible.

Hence the greatest length that K may have, in order that the construction may be possible, is the radius CD.

∴ the rect. AM, MB is a maximum, when it is equal to the square on CD;

that is, when PM coincides with DC, and consequently when M is the middle point of AB.

Obs. The special feature to be noticed in this problem is that the maximum is found at the transitional point between *two* solutions and *no* solution; that is, when the two solutions coincide and become identical.

The following example illustrates the same point.

2. *To find at what point in a given straight line the angle subtended by the line joining two given points, which are on the same side of the given straight line, is a maximum.*

Let CD be the given st. line, and A, B the given points on the same side of CD:

it is required to find at what point in CD the angle subtended by the st. line AB is a maximum.

First determine at what point in CD, the st. line AB subtends a *given* angle.

This is done as follows:—

On AB describe a segment of a circle containing an angle equal to the given angle. III. 33.

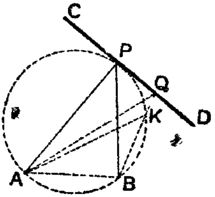
If the arc of this segment intersects CD, *two* points in CD are found at which AB subtends the given angle: but if the arc does not meet CD, *no* solution is given.

In accordance with the principles explained above, we expect that a maximum angle is determined at the limiting position, that is, when the arc touches CD; or meets it at two coincident points.

[See page 213.]

This we may prove to be the case.

Describe a circle to pass through A and B, and to touch the st. line CD. [Ex. 21, p. 235.]



Let P be the point of contact.

Then shall the ∠ APB be greater than any other angle subtended by AB at a point in CD on the same side of AB as P.

For take Q, any other point in CD, on the same side of AB as P;

and join AQ, QB.

Since Q is a point in the tangent other than the point of contact, it must be without the circle,

∴ either BQ or AQ must meet the arc of the segment APB.

Let BQ meet the arc at K: join AK.

Then the ∠ APB = the ∠ AKB, in the same segment:

but the ext. ∠ AKB is greater than the int. opp. ∠ AQB.

∴ the ∠ APB is greater than AQB.

Similarly the ∠ APB may be shewn to be greater than any other angle subtended by AB at a point in CD on the same side of AB:

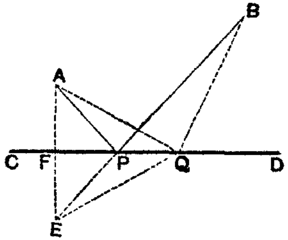
that is, the ∠ APB is the greatest of all such angles. Q. E. D.

NOTE. Two circles may be described to pass through A and B, and to touch CD, the points of contact being on opposite sides of AB:

hence two points in CD may be found such that the angle subtended by AB at each of them is greater than the angle subtended at any other point in CD *on the same side of* AB.

We add two more examples of considerable importance.

3. *In a straight line of indefinite length find a point such that the sum of its distances from two given points, on the same side of the given line, shall be a minimum.*



Let CD be the given st. line of indefinite length, and A, B the given points on the same side of CD: it is required to find a point P in CD such that the sum of AP, PB is a minimum.

Draw AF perp. to CD;
and produce AF to E, making FE equal to AF.
Join EB, cutting CD at P.
Join AP, PB.

Then of all lines drawn from A and B to a point in CD,
the sum of AP, PB shall be the least.

For, let Q be any other point in CD.
Join AQ, BQ, EQ.

Now in the $\triangle^s$ AFP, EFP,

Because { AF = EF, *Constr.*
and FP is common,
and the $\angle$ AFP = the $\angle$ EFP, being rt. angles.

$\therefore$ AP = EP. I. 4.

Similarly it may be shewn that
AQ = EQ.

Now in the $\triangle$ EQB, the two sides EQ, QB are together greater than EB;

hence, AQ, QB are together greater than EB,
that is, greater than AP, PB.

Similarly the sum of the st. lines drawn from A and B to any other point in CD may be shewn to be greater than AP, PB.

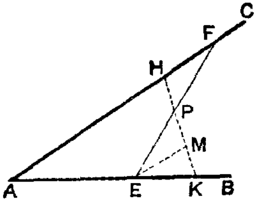
$\therefore$ the sum of AP, PB is a minimum.

Q. E. D.

NOTE. It follows from the above proof that
the $\angle$ APF = the $\angle$ EPF I. 4.
= the $\angle$ BPD. I. 15.

Thus the sum of AP, PB is a minimum, when these lines are *equally inclined* to CD.

4. *Given two intersecting straight lines* AB, AC, *and a point* P *between them; shew that of all straight lines which pass through* P *and are terminated by* AB, AC, *that which is bisected at* P *cuts off the triangle of minimum area.*



Let EF be the st. line, terminated by AB, AC, which is bisected at P:
then the △ FAE shall be of minimum area.

For let HK be any other st. line passing through P:
through E draw EM par[1] to AC.

Then in the △s HPF, MPE,

Because { the ∠ HPF = the ∠ MPE, I. 15.
and the ∠ HFP = the ∠ MEP, I. 29.
and FP = EP, *Hyp.*
∴ the △ HPF = the △ MPE. I. 26, Cor.

But the △ MPE is less than the △ KPE,
∴ the △ HPF is less than the △ KPE:
to each add the fig. AHPE;
then the △ FAE is less than the △ HAK.

Similarly it may be shewn that the △ FAE is less than any other triangle formed by drawing a st. line through P:
that is, the △ FAE is a minimum.

Examples.

1. Two sides of a triangle are given in length; how must they be placed in order that the area of the triangle may be a maximum?

2 *Of all triangles of given base and area, the isosceles is that which has the least perimeter.*

3. Given the base and vertical angle of a triangle, construct it so that its area may be a maximum.

4. Find a point in a given straight line such that the tangents drawn from it to a given circle contain the greatest angle possible.

5. A straight rod slips between two straight rulers placed at right angles to one another; in what position is the triangle intercepted between the rulers and rod a maximum?

6. Divide a given straight line into two parts, so that the sum of the squares on the segments may

(i) be equal to a given square,

(ii) may be a minimum.

7. Through a point of intersection of two circles draw a straight line terminated by the circumferences,

(i) so that it may be of given length,

(ii) so that it may be a maximum.

8. Two tangents to a circle cut one another at right angles; find the point on the intercepted arc such that the sum of the perpendiculars drawn from it to the tangents may be a minimum.

9. Straight lines are drawn from two given points to meet one another on the convex circumference of a given circle: prove that their sum is a minimum when they make equal angles with the tangent at the point of intersection.

10. Of all triangles of given vertical angle and altitude, the isosceles is that which has the least area.

11. Two straight lines CA, CB of indefinite length are drawn from the centre of a circle to meet the circumference at A and B; then of all tangents that may be drawn to the circle at points on the arc AB, that whose intercept is bisected at the point of contact cuts off the triangle of minimum area.

12. Given two intersecting tangents to a circle, draw a tangent to the convex arc so that the triangle formed by it and the given tangents may be of maximum area.

13. Of all triangles of given base and area, the isosceles is that which has the greatest vertical angle.

14. Find a point on the circumference of a circle at which the straight line joining two given points (of which both are within, or both without the circle) subtends the greatest angle.

15. A bridge consists of three arches, whose spans are 49 ft., 32 ft. and 49 ft. respectively: shew that the point on either bank of the river at which the middle arch subtends the greatest angle is 63 feet distant from the bridge.

16. From a given point P without a circle whose centre is C, draw a straight line to cut the circumference at A and B, so that the triangle ACB may be of maximum area.

17. Shew that the greatest rectangle which can be inscribed in a circle is a square.

18. A and B are two fixed points without a circle: find a point P on the circumference such that the sum of the squares on AP, PB may be a minimum. [See p. 147, Ex. 24.]

19. A segment of a circle is described on the chord AB: find a point C on its arc so that the sum of AC, BC may be a maximum.

20. *Of all triangles that can be inscribed in a circle that which has the greatest perimeter is equilateral.*

21. *Of all triangles that can be inscribed in a given circle that which has the greatest area is equilateral.*

22. *Of all triangles that can be inscribed in a given triangle that which has the least perimeter is the triangle formed by joining the feet of the perpendiculars drawn from the vertices on opposite sides.*

23. Of all rectangles of given area, the square has the least perimeter.

24. Describe the triangle of maximum area, having its angles equal to those of a given triangle, and its sides passing through three given points.

VI. HARDER MISCELLANEOUS EXAMPLES.

1. AB is a diameter of a given circle; and AC, BD, two chords on the same side of AB, intersect at E: shew that the circle which passes through D, E, C cuts the given circle orthogonally.

2. Two circles whose centres are C and D intersect at A and B, and a straight line PAQ is drawn through A and terminated by the circumferences: prove that

(i) the angle PBQ = the angle CAD

(ii) the angle BPC = the angle BQD.

3. Two chords AB, CD of a circle whose centre is O intersect at right angles at P: shew that

(i) $PA^2 + PB^2 + PC^2 + PD^2 = 4\ (\text{radius})^2$.

(ii) $AB^2 + CD^2 + 4OP^2 = 8\ (\text{radius})^2$.

4. Two parallel tangents to a circle intercept on any third tangent a portion which is so divided at its point of contact that the rectangle contained by its two parts is equal to the square on the radius.

5. Two equal circles move between two straight lines placed at right angles, so that each straight line is touched by one circle, and the two circles touch one another: find the locus of the point of contact.

6. AB is a given diameter of a circle, and CD is any parallel chord: if any point X in AB is joined to the extremities of CD, shew that

$$XC^2 + XD^2 = XA^2 + XB^2.$$

7. PQ is a fixed chord in a circle, and PX, QY any two parallel chords through P and Q: shew that XY touches a fixed concentric circle.

8. Two equal circles intersect at A and B; and from C any point on the circumference of one of them a perpendicular is drawn to AB, meeting the other circle at O and O′: shew that either O or O′ is the orthocentre of the triangle ABC. Distinguish between the two cases.

9. Three equal circles pass through the same point A, and their other points of intersection are B, C, D: shew that of the four points A, B, C, D, each is the orthocentre of the triangle formed by joining the other three.

10. From a given point without a circle draw a straight line to the concave circumference so as to be bisected by the convex circumference. When is this problem impossible?

11. Draw a straight line cutting two concentric circles so that the chord intercepted by the circumference of the greater circle may be double of the chord intercepted by the less.

12. ABC is a triangle inscribed in a circle, and A′, B′, C′ are the middle points of the arcs subtended by the sides (remote from the opposite vertices): find the relation between the angles of the two triangles ABC, A′B′C′; and prove that the pedal triangle of A′B′C′ is equiangular to the triangle ABC.

13. The opposite sides of a quadrilateral inscribed in a circle are produced to meet: shew that the bisectors of the two angles so formed are perpendicular to one another.

14. If a quadrilateral can have one circle inscribed in it, and another circumscribed about it; shew that the straight lines joining the opposite points of contact of the inscribed circle are perpendicular to one another.

15. Given the base of a triangle and the sum of the remaining sides; find the locus of the foot of the perpendicular from one extremity of the base on the bisector of the exterior vertical angle.

16. Two circles touch each other at C, and straight lines are drawn through C at right angles to one another, meeting the circles at P, P′ and Q, Q′ respectively: if the straight line which joins the centres is terminated by the circumferences at A and A′, shew that

$$P'P^2 + Q'Q^2 = A'A^2.$$

17. Two circles cut one another orthogonally at A and B; P is any point on the arc of one circle intercepted by the other, and PA, PB are produced to meet the circumference of the second circle at C and D: shew that CD is a diameter.

18. ABC is a triangle, and from any point P perpendiculars PD, PE, PF are drawn to the sides: if S_1, S_2, S_3 are the centres of the circles circumscribed about the triangles EPF, FPD, DPE, shew that the triangle $S_1S_2S_3$ is equiangular to the triangle ABC, and that the sides of the one are respectively half of the sides of the other.

19. Two tangents PA, PB are drawn from an external point P to a given circle, and C is the middle point of the chord of contact AB: if XY is any chord through P, shew that AB bisects the angle XCY.

20. Given the sum of two straight lines and the rectangle contained by them (equal to a given square): find the lines.

21. Given the sum of the squares on two straight lines and the rectangle contained by them: find the lines.

22. Given the sum of two straight lines and the sum of the squares on them: find the lines.

23. Given the difference between two straight lines, and the rectangle contained by them: find the lines.

24. Given the sum or difference of two straight lines and the difference of their squares: find the lines.

25. ABC is a triangle, and the internal and external bisectors of the angle A meet BC, and BC produced, at P and P′: if O is the middle point of PP′, shew that OA is a tangent to the circle circumscribed about the triangle ABC.

26. ABC is a triangle, and from P, any point on the circumference of the circle circumscribed about it, perpendiculars are drawn to the sides BC, CA, AB meeting the circle again in A′, B′, C′; prove that

(i) the triangle A′B′C′ is identically equal to the triangle ABC.

(ii) AA′, BB′, CC′ are parallel.

27. Two equal circles intersect at fixed points A and B, and from any point in AB a perpendicular is drawn to meet the circumferences on the same side of AB at P and Q: shew that PQ is of constant length.

28. The straight lines which join the vertices of a triangle to the centre of its circumscribed circle, are perpendicular respectively to the sides of the pedal triangle.

29. P is any point on the circumference of a circle circumscribed about a triangle ABC; and perpendiculars PD, PE are drawn from P to the sides BC, CA. Find the locus of the centre of the circle circumscribed about the triangle PDE.

30. P is any point on the circumference of a circle circumscribed about a triangle ABC: shew that the angle between SIMSON'S Line for the point P and the side BC, is equal to the angle between AP and the diameter of the circumscribed circle through A.

31. Shew that the circles circumscribed about the four triangles formed by two pairs of intersecting straight lines meet in a point.

32. Shew that the orthocentres of the four triangles formed by two pairs of intersecting straight lines are collinear.

ON THE CONSTRUCTION OF TRIANGLES.

33. Given the vertical angle, one of the sides containing it, and the length of the perpendicular from the vertex on the base: construct the triangle.

34. Given the feet of the perpendiculars drawn from the vertices on the opposite sides: construct the triangle.

35. Given the base, the altitude, and the radius of the circumscribed circle: construct the triangle.

36. Given the base, the vertical angle, and the sum of the squares on the sides containing the vertical angle: construct the triangle.

37. Given the base, the altitude and the sum of the squares on the sides containing the vertical angle: construct the triangle.

38. Given the base, the vertical angle, and the difference of the squares on the sides containing the vertical angle: construct the triangle.

39. Given the vertical angle, and the lengths of the two medians drawn from the extremities of the base: construct the triangle.

40. Given the base, the vertical angle, and the difference of the angles at the base: construct the triangle.

41. Given the base, and the position of the bisector of the vertical angle: construct the triangle.

42. Given the base, the vertical angle, and the length of the bisector of the vertical angle: construct the triangle.

43. Given the perpendicular from the vertex on the base, the bisector of the vertical angle, and the median which bisects the base: construct the triangle.

44. Given the bisector of the vertical angle, the median bisecting the base, and the difference of the angles at the base: construct the triangle.

BOOK IV.

Book IV. consists entirely of problems, dealing with various rectilineal figures in relation to the circles which pass through their angular points, or are touched by their sides.

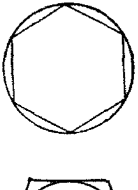
DEFINITIONS.

1. A **Polygon** is a rectilineal figure bounded by more than four sides.

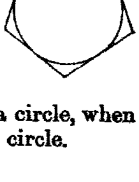
A Polygon of	*five* sides	is called a	**Pentagon,**
,,	*six* sides	,,	**Hexagon,**
,,	*seven* sides	,,	**Heptagon,**
,,	*eight* sides	,,	**Octagon,**
,,	*ten* sides	,,	**Decagon,**
,,	*twelve* sides	,,	**Dodecagon,**
,,	*fifteen* sides	,,	**Quindecagon.**

2. A Polygon is **Regular** when all its sides are equal, and all its angles are equal.

3. A rectilineal figure is said to be **inscribed** in a circle, when all its angular points are on the circumference of the circle: and a circle is said to be **circumscribed about** a rectilineal figure, when the circumference of the circle passes through all the angular points of the figure.



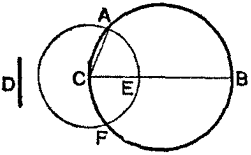
4. A rectilineal figure is said to be **circumscribed about** a circle, when each side of the figure is a tangent to the circle: and a circle is said to be **inscribed in** a rectilineal figure, when the circumference of the circle is touched by each side of the figure.



5. A straight line is said to be **placed in** a circle, when its extremities are on the circumference of the circle.

PROPOSITION 1. PROBLEM.

In a given circle to place a chord equal to a given straight line, which is not greater than the diameter of the circle.



Let ABC be the given circle, and D the given straight line not greater than the diameter of the circle:
it is required to place in the ⊙ ABC a chord equal to D.

Draw CB, a diameter of the ⊙ ABC.
Then if CB = D, the thing required is done.
But if not, CB must be greater than D. *Hyp.*
From CB cut off CE equal to D: I. 3.
and from centre C, with radius CE, describe the ⊙ AEF, cutting the given circle at A.
Join CA.
Then CA shall be the chord required.

For CA = CE, being radii of the ⊙ AEF:
and CE = D: *Constr.*
∴ CA = D.

Q. E. F.

EXERCISES.

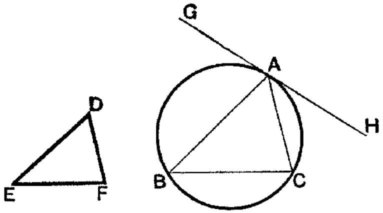
1. In a given circle place a chord of given length so as to pass through a given point (i) without, (ii) within the circle.

When is this problem impossible?

2. In a given circle place a chord of given length so that it may be parallel to a given straight line.

PROPOSITION 2. PROBLEM.

In a given circle to inscribe a triangle equiangular to a given triangle.



Let ABC be the given circle, and DEF the given triangle: it is required to inscribe in the ⊙ ABC a triangle equiangular to the △ DEF.

At any point A, on the ○ᶜᵉ of the ⊙ ABC, draw the tangent GAH. III. 17.

At A make the ∠ GAB equal to the ∠ DFE; I. 23.

and make the ∠ HAC equal to the ∠ DEF. I. 23.

Join BC.

Then ABC shall be the triangle required.

Because GH is a tangent to the ⊙ ABC, and from A its point of contact the chord AB is drawn,

∴ the ∠ GAB = the ∠ ACB in the alt. segment: III. 32.

∴ the ∠ ACB = the ∠ DFE. *Constr.*

Similarly the ∠ HAC = the ∠ ABC, in the alt. segment:

∴ the ∠ ABC = the ∠ DEF. *Constr.*

Hence the third ∠ BAC = the third ∠ EDF,

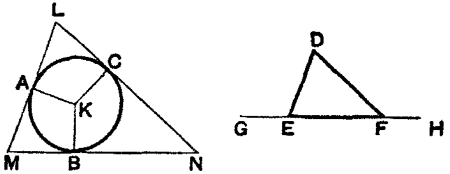
for the three angles in each triangle are together equal to two rt. angles. I. 32.

∴ the △ ABC is equiangular to the △ DEF, and it is inscribed in the ⊙ ABC.

Q. E. F.

PROPOSITION 3. PROBLEM.

About a given circle to circumscribe a triangle equiangular to a given triangle.



Let ABC be the given circle, and DEF the given △: it is required to circumscribe about the ⊙ABC a triangle equiangular to the △DEF.

Produce EF both ways to G and H.
Find K the centre of the ⊙ABC, III. 1.
and draw any radius KB.
At K make the ∠BKA equal to the ∠DEG; I. 23.
and make the ∠BKC equal to the ∠DFH.
Through A, B, C draw LM, MN, NL perp. to KA, KB, KC.
Then LMN shall be the triangle required.

Because LM, MN, NL are drawn perp. to radii at their extremities,
∴ LM, MN, NL are tangents to the circle. III. 16.

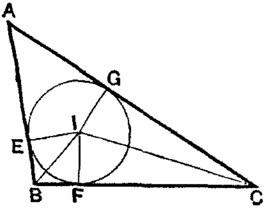
And because the four angles of the quadrilateral AKBM together = four rt. angles; I. 32. Cor.
and of these, the ∠s KAM, KBM, are rt. angles; *Constr.*
∴ the ∠s AKB, AMB, together = two rt. angles.
But the ∠s DEG, DEF together = two rt. angles; I. 13.
∴ the ∠s AKB, AMB = the ∠s DEG, DEF;
and of these, the ∠AKB = the ∠DEG; *Constr.*
∴ the ∠AMB = the ∠DEF.

Similarly it may be shewn that the ∠LNM = the ∠DFE.
∴ the third ∠MLN = the third ∠EDF. I. 32.

∴ the △LMN is equiangular to the △DEF, and it is circumscribed about the ⊙ABC. Q.E.F.

PROPOSITION 4. PROBLEM.

To inscribe a circle in a given triangle.



Let ABC be the given triangle:
it is required to inscribe a circle in the △ABC.

Bisect the ∠s ABC, ACB by the st. lines BI, CI, which intersect at I, I. 9.

From I draw IE, IF, IG perp. to AB, BC, CA. I. 12.

Then in the △s EIB, FIB,

Because { the ∠EBI = the ∠FBI; *Constr.*
and the ∠BEI = the ∠BFI, being rt. angles;
and BI is common;

∴ IE = IF. I. 26.

Similarly it may be shewn that IF = IG.

∴ IE, IF, IG are all equal.

From centre I, with radius IE, describe a circle:
this circle must pass through the points E, F, G;
and it will be inscribed in the △ABC.

For since IE, IF, IG are radii of the ⊙EFG;
and since the ∠s at E, F, G are rt. angles;
∴ the ⊙EFG is touched at these points by AB, BC, CA: III. 16.

∴ the ⊙EFG is inscribed in the △ABC.

Q.E.F.

NOTE. From page 103 it is seen that if AI be joined, then AI bisects the angle BAC.

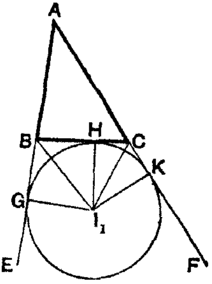
Hence it follows that the bisectors of the angles of a triangle are concurrent, the point of intersection being the centre of the inscribed circle.

The centre of the circle inscribed in a triangle is sometimes called its **in-centre**.

DEFINITION.

A circle which touches one side of a triangle and the other two sides produced is said to be an **escribed** circle of the triangle.

To draw an escribed circle of a given triangle.



Let ABC be the given triangle, of which the two sides AB, AC are produced to E and F:

it is required to describe a circle touching BC, and AB, AC produced.

Bisect the $\angle$ˢ CBE, BCF by the st. lines BI_1, CI_1, which intersect at I_1. I. 9.

From I_1 draw I_1G, I_1H, I_1K perp. to AE, BC, AF. I. 12.

Then in the $\triangle$ˢ I_1BG, I_1BH,

Because { the $\angle I_1BG$ = the $\angle I_1BH$, *Constr.*
and the $\angle I_1GB$ = the $\angle I_1HB$, being rt. angles;
also I_1B is common;

$\therefore$ $I_1G = I_1H$.

Similarly it may be shewn that $I_1H = I_1K$;

$\therefore$ I_1G, I_1H, I_1K are all equal.

From centre I_1 with radius I_1G, describe a circle:

this circle must pass through the points G, H, K:

and it will be an escribed circle of the $\triangle$ABC.

For since I_1H, I_1G, I_1K are radii of the $\odot$ HGK,

and since the angles at H, G, K are rt. angles,

$\therefore$ the $\odot$ GHK is touched at these points by BC, and by AB, AC produced:

$\therefore$ the $\odot$ GHK is an escribed circle of the $\triangle$ABC. Q.E.F.

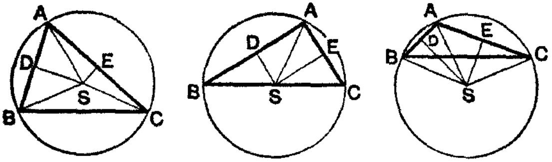
It is clear that every triangle has *three* escribed circles.

NOTE. From page 104 it is seen that if AI_1 be joined, then AI_1 bisects the angle BAC: hence it follows that

The bisectors of two exterior angles of a triangle and the bisector of the third angle are concurrent, the point of intersection being the centre of an escribed circle.

PROPOSITION 5. PROBLEM.

To circumscribe a circle about a given triangle.



Let ABC be the given triangle:
it is required to circumscribe a circle about the △ABC.

Draw DS bisecting AB at rt. angles; I. 11.
and draw ES bisecting AC at rt. angles;
then since AB, AC are neither par[l], nor in the same st. line,
∴ DS and ES must meet at some point S.

Join SA;
and if S be not in BC, join SB, SC.

Then in the △[s] ADS, BDS,

Because { AD = BD
and DS is common to both;
and the ∠ ADS = the ∠ BDS, being rt. angles;

∴ SA = SB.

Similarly it may be shewn that SC = SA.

∴ SA, SB, SC are all equal.

From centre S, with radius SA, describe a circle: this circle must pass through the points A, B, C, and is therefore circumscribed about the △ABC. Q.E.F.

It follows that

(i) when the centre of the circumscribed circle falls *within* the triangle, each of its angles must be acute, for each angle is then in a segment greater than a semicircle:

(ii) when the centre falls *on one of the sides* of the triangle, the angle opposite to this side must be a right angle, for it is the angle in a semicircle:

(iii) when the centre falls *without* the triangle, the angle opposite to the side beyond which the centre falls, must be obtuse, for it is the angle in a segment less than a semicircle.

Therefore, conversely, if the given triangle be acute-angled, the centre of the circumscribed circle falls within it: if it be a right-angled triangle, the centre falls on the hypotenuse: if it be an obtuse-angled triangle, the centre falls without the triangle.

NOTE. From page 103 it is seen that if S be joined to the middle point of BC, then the joining line is perpendicular to BC.

Hence the perpendiculars drawn to the sides of a triangle from their middle points are concurrent, the point of intersection being the centre of the circle circumscribed about the triangle.

The centre of the circle circumscribed about a triangle is sometimes called its **circum-centre.**

EXERCISES.

ON THE INSCRIBED, CIRCUMSCRIBED, AND ESCRIBED CIRCLES OF A TRIANGLE.

1. An equilateral triangle is inscribed in a circle, and tangents are drawn at its vertices, prove that

   (i) the resulting figure is an equilateral triangle:

   (ii) its area is four times that of the given triangle.

2. Describe a circle to touch two parallel straight lines and a third straight line which meets them. Shew that two such circles can be drawn, and that they are equal.

3. *Triangles which have equal bases and equal vertical angles have equal circumscribed circles.*

4. I *is the centre of the circle inscribed in the triangle* ABC, *and* I_1 *is the centre of the circle which touches* BC *and* AB, AC *produced: shew that* A, I, I_1 *are collinear.*

5. *If the inscribed and circumscribed circles of a triangle are concentric, shew that the triangle is equilateral; and that the diameter of the circumscribed circle is double that of the inscribed circle.*

6. ABC is a triangle; and I, S are the centres of the inscribed and circumscribed circles; if A, I, S are collinear, shew that AB = AC.

7. The sum of the diameters of the inscribed and circumscribed circles of a right-angled triangle is equal to the sum of the sides containing the right angle.

8. If the circle inscribed in a triangle ABC touches the sides at D, E, F, shew that the triangle DEF is acute-angled; and express its angles in terms of the angles at A, B, C.

9. If I is the centre of the circle inscribed in the triangle ABC, and I_1 the centre of the escribed circle which touches BC; shew that I, B, I_1, C are concyclic.

10. In any triangle the difference of two sides is equal to the difference of the segments into which the third side is divided at the point of contact of the inscribed circle.

11. In the triangle ABC the bisector of the angle BAC meets the base at D, and from I the centre of the inscribed circle a perpendicular IE is drawn to BC: shew that the angle BID is equal to the angle CIE.

12. In the triangle ABC, I and S are the centres of the inscribed and circumscribed circles: shew that IS subtends at A an angle equal to half the difference of the angles at the base of the triangle.

13. In a triangle ABC, I and S are the centres of the inscribed and circumscribed circles, and AD is drawn perpendicular to BC: shew that AI is the bisector of the angle DAS.

14. Shew that the area of a triangle is equal to the rectangle contained by its semi-perimeter and the radius of the inscribed circle.

15. The diagonals of a quadrilateral ABCD intersect at O: shew that the centres of the circles circumscribed about the four triangles AOB, BOC, COD, DOA are at the angular points of a parallelogram.

16. In any triangle ABC, if I is the centre of the inscribed circle, and if AI is produced to meet the circumscribed circle at O; shew that O is the centre of the circle circumscribed about the triangle BIC.

17. Given the base, altitude, and the radius of the circumscribed circle; construct the triangle.

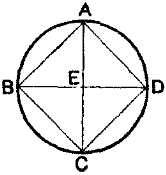
18. Describe a circle to intercept equal chords of given length on three given straight lines.

19. In an equilateral triangle the radii of the circumscribed and escribed circles are respectively double and treble of the radius of the inscribed circle.

20. Three circles whose centres are A, B, C touch one another externally two by two at D, E, F: shew that the inscribed circle of the triangle ABC is the circumscribed circle of the triangle DEF.

PROPOSITION 6. PROBLEM.

To inscribe a square in a given circle.



Let ABCD be the given circle:
it is required to inscribe a square in the ⊙ABCD.

Find E the centre of the circle: III. 1.
and draw two diameters AC, BD perp. to one another. I. 11.
Join AB, BC, CD, DA.

Then the fig. ABCD shall be the square required.

For in the △s BEA, DEA,

Because { BE = DE,
and EA is common;
and the ∠ BEA = the ∠ DEA, being rt. angles;

∴ BA = DA. I. 4.

Similarly it may be shewn that CD = DA, and that BC = CD.
∴ the fig. ABCD is equilateral.

And since BD is a diameter of the ⊙ABCD,
∴ BAD is a semicircle;
∴ the ∠ BAD is a rt. angle. III. 31.

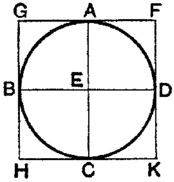
Similarly the other angles of the fig. ABCD are rt. angles.
∴ the fig. ABCD is a square,
and it is inscribed in the given circle.

Q. E. F.

[For Exercises see page 263.]

Proposition 7. Problem.

To circumscribe a square about a given circle.



Let ABCD be the given circle:
it is required to circumscribe a square about it.

Find E the centre of the ⊙ ABCD: III. 1.
and draw two diameters AC, BD perp. to one another. I. 11.
Through A, B, C, D draw FG, GH, HK, KF perp. to EA, EB, EC, ED.

Then the fig. GK shall be the square required.

Because FG, GH, HK, KF are drawn perp. to radii at their extremities,

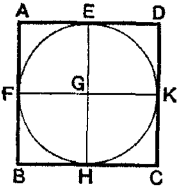
∴ FG, GH, HK, KF are tangents to the circle. III. 16.

And because the ∠s AEB, EBG are both rt. angles, *Constr.*
∴ GH is par[l] to AC. I. 28.
Similarly FK is par[l] to AC:
and in like manner GF, BD, HK are par[l].
Hence the figs. GK, GC, AK, GD, BK, GE are par[ms].
∴ GF and HK each = BD;
also GH and FK each = AC:
but AC = BD;
∴ GF, FK, KH, HG are all equal:
that is, the fig. GK is equilateral.
And since the fig. GE is a par[m],
∴ the ∠ BGA = the ∠ BEA; I. 34.
but the ∠ BEA is a rt. angle; *Constr.*
∴ the ∠ at G is a rt. angle.
Similarly the ∠s at F, K, H are rt. angles.

∴ the fig. GK is a square, and it has been circumscribed about the ⊙ ABCD. Q.E.F.

PROPOSITION 8. PROBLEM.

To inscribe a circle in a given square.



Let ABCD be the given square:
it is required to inscribe a circle in the sq. ABCD.

Bisect the sides AB, AD at F and E. I. 10.
Through E draw EH par[l] to AB or DC: I. 31.
and through F draw FK par[l] to AD or BC, meeting EH at G.

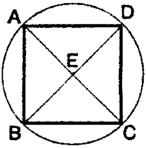
Now AB = AD, being the sides of a square;
and their halves are equal; *Constr.*
∴ AF = AE. *Ax.* 7.
But the fig. AG is a par[m]; *Constr.*
∴ AF = GE, and AE = GF;
∴ GE = GF.
Similarly it may be shewn that GE = GK, and GK = GH:
∴ GF, GE, GK, GH are all equal.
From centre G, with radius GE, describe a circle;
this circle must pass through the points F, E, K, H:
and it will be touched by BA, AD, DC, CB; III. 16.
for GF, GE, GK, GH are radii;
and the angles at F, E, K, H are rt. angles. I. 29.
Hence the ⊙ FEKH is inscribed in the sq. ABCD.

Q. E. F.

[For Exercises see p. 263.]

PROPOSITION 9. PROBLEM.

To circumscribe a circle about a given square.



Let ABCD be the given square:
it is required to circumscribe a circle about the sq. ABCD.

Join AC, BD, intersecting at E.

Then in the $\triangle^s$ BAC, DAC,

Because { BA = DA, — I. *Def.* 28.
and AC is common;
and BC = DC; — I. *Def.* 28.

∴ the ∠ BAC = the ∠ DAC: — I. 8.

that is, the diagonal AC bisects the ∠ BAD.

Similarly the remaining angles of the square are bisected by the diagonals AC or BD.

Hence each of the ∠ˢ EAD, EDA is half a rt. angle;

∴ the ∠ EAD = the ∠ EDA:

∴ EA = ED. — I. 6.

Similarly it may be shewn that ED = EC, and EC = EB.

∴ EA, EB, EC, ED are all equal.

From centre E, with radius EA, describe a circle: this circle must pass through the points A, B, C, D, and is therefore circumscribed about the sq. ABCD. Q.E.F.

DEFINITION. A rectilineal figure about which a circle may be described is said to be **Cyclic.**

EXERCISES ON PROPOSITIONS 6—9.

1. *If a circle can be inscribed in a quadrilateral, shew that the sum of one pair of opposite sides is equal to the sum of the other pair.*

2. *If the sum of one pair of opposite sides of a quadrilateral is equal to the sum of the other pair, shew that a circle may be inscribed in the figure.*

[Bisect two adjacent angles of the figure, and so describe a circle to touch three of its sides. Then prove indirectly by means of the last exercise that this circle must also touch the fourth side.]

3. *Prove that a rhombus and a square are the only parallelograms in which a circle can be inscribed.*

4. *All cyclic parallelograms are rectangular.*

5. *The greatest rectangle which can be inscribed in a given circle is a square.*

6. Circumscribe a rhombus about a given circle.

7. All squares circumscribed about a given circle are equal.

8. The area of a square circumscribed about a circle is double of the area of the inscribed square.

9. ABCD is a square inscribed in a circle, and P is any point on the arc AD: shew that the side AD subtends at P an angle three times as great as that subtended at P by any one of the other sides.

10. Inscribe a square in a given square ABCD so that one of its angular points should be at a given point X in AB.

11. In a given square inscribe the square of minimum area.

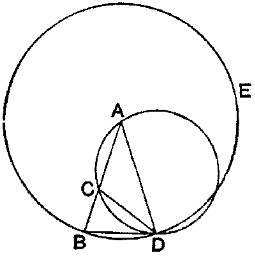
12. Describe (i) a circle, (ii) a square about a given rectangle.

13. Inscribe (i) a circle, (ii) a square in a given quadrant.

14. ABCD is a square inscribed in a circle, and P is any point on the circumference; shew that the sum of the squares on PA, PB, PC, PD is double the square on the diameter. [See Ex. 24, p. 147.]

Proposition 10. Problem.

To describe an isosceles triangle having each of the angles at the base double of the third angle.



Take any straight line AB.

Divide AB at C, so that the rect. BA, BC = the sq. on AC. II. 11.

From centre A, with radius AB, describe the ⊙ BDE;
and in it place the chord BD equal to AC. IV. 1.

Join DA.

Then ABD shall be the triangle required.

Join CD;
and about the △ ACD circumscribe a circle. IV. 5.

Then the rect. BA, BC = the sq. on AC *Constr.*
= the sq. on BD. *Constr.*

Hence BD is a tangent to the ⊙ ACD: III. 37.
and from the point of contact D a chord DC is drawn;
∴ the ∠ BDC = the ∠ CAD in the alt. segment. III. 32.

To each of these equals add the ∠ CDA:
then the whole ∠ BDA = the sum of the ∠s CAD, CDA.

But the ext. ∠ BCD = the sum of the ∠s CAD, CDA; I. 32.
∴ the ∠ BCD = the ∠ BDA.

And since AB = AD, being radii of the ⊙ BDE,
∴ the ∠ DBA = the ∠ BDA: I. 5.
∴ the ∠ DBC = the ∠ DCB;

$\therefore$ DC = DB ; I. 6.
that is, DC = CA : *Constr.*
$\therefore$ the $\angle$ CAD = the $\angle$ CDA ; I. 5.
$\therefore$ the sum of the $\angle$s CAD, CDA = twice the angle at A.
But the $\angle$ ADB = the sum of the $\angle$s CAD, CDA ; *Proved.*
$\therefore$ each of the $\angle$s ABD, ADB = twice the angle at A.

Q. E. F.

EXERCISES ON PROPOSITION 10.

1. In an isosceles triangle in which each of the angles at the base is double of the vertical angle, shew that the vertical angle is one-fifth of two right angles.

2. *Divide a right angle into five equal parts.*

3. Describe an isosceles triangle whose vertical angle shall be three times either angle at the base. Point out a triangle of this kind in the figure of Proposition 10.

4. *In the figure of Proposition* 10, *if the two circles intersect at* F, *shew that* BD = DF.

5. *In the figure of Proposition* 10, *shew that the circle* ACD *is equal to the circle circumscribed about the triangle* ABD.

6. In the figure of Proposition 10, if the two circles intersect at F, shew that

(i) BD, DF are sides of a regular decagon inscribed in the circle EBD.

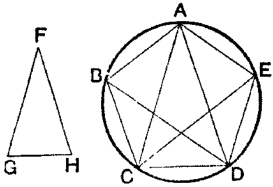
(ii) AC, CD, DF are sides of a regular pentagon inscribed in the circle ACD.

7. In the figure of Proposition 10, shew that the centre of the circle circumscribed about the triangle DBC is the middle point of the arc CD.

8. In the figure of Proposition 10, if I is the centre of the circle inscribed in the triangle ABD, and I′, S′ the centres of the inscribed and circumscribed circles of the triangle DBC, shew that S′I = S′I′.

PROPOSITION 11. PROBLEM.

To inscribe a regular pentagon in a given circle.



Let ABC be a given circle:
it is required to inscribe a regular pentagon in the ⊙ABC.

Describe an isosceles △FGH, having each of the angles at G and H double of the angle at F. IV. 10.

In the ⊙ABC inscribe the △ACD equiangular to the △FGH, IV. 2.
so that each of the ∠s ACD, ADC is double of the ∠CAD.

Bisect the ∠s ACD, ADC by CE and DB, which meet the ○ce at E and B. I. 9.

Join AB, BC, AE, ED.

Then ABCDE shall be the required regular pentagon.

Because each of the ∠s ACD, ADC = twice the ∠CAD;
and because the ∠s ACD, ADC are bisected by CE, DB,
∴ the five ∠s ADB, BDC, CAD, DCE, ECA are all equal.
∴ the five arcs AB, BC, CD, DE, EA are all equal. III. 26.
∴ the five chords AB, BC, CD, DE, EA are all equal. III. 29.
∴ the pentagon ABCDE is equilateral.

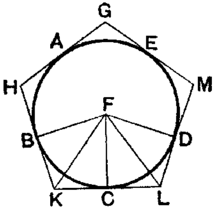
Again the arc AB = the arc DE; *Proved.*
to each of these equals add the arc BCD;
∴ the whole arc ABCD = the whole arc BCDE:
hence the angles at the ○ce which stand upon these equal arcs are equal; III. 27.
that is, the ∠AED = the ∠BAE.

In like manner the remaining angles of the pentagon may be shewn to be equal;
∴ the pentagon is equiangular.

Hence the pentagon, being both equilateral and equiangular, is regular; and it is inscribed in the ⊙ABC. Q. E. F.

PROPOSITION 12. PROBLEM.

To circumscribe a regular pentagon about a given circle.



Let ABCD be the given circle:
it is required to circumscribe a regular pentagon about it.

Inscribe a regular pentagon in the ⊙ ABCD, IV. 11.
and let A, B, C, D, E be its angular points.

At the points A, B, C, D, E draw GH, HK, KL, LM, MG, tangents to the circle. III. 17.

Then shall GHKLM be the required regular pentagon.

Find F the centre of the ⊙ ABCD; III. 1.
and join FB, FK, FC, FL, FD.

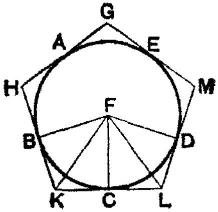
Then in the two △s BFK, CFK,

Because { BF = CF, being radii of the circle,
and FK is common:
and KB = KC, being tangents to the circle from the same point K. III. 17. *Cor.*

∴ the ∠ BFK = the ∠ CFK, I. 8.
also the ∠ BKF = the ∠ CKF. I. 8. *Cor.*
Hence the ∠ BFC = twice the ∠ CFK,
and the ∠ BKC = twice the ∠ CKF.

Similarly it may be shewn
that the ∠ CFD = twice the ∠ CFL,
and that the ∠ CLD = twice the ∠ CLF.

But since the arc BC = the arc CD, IV. 11.
∴ the ∠ BFC = the ∠ CFD; III. 27.
and the halves of these angles are equal,
that is, the ∠ CFK = the ∠ CFL.



Then in the $\triangle^s$ CFK, CFL,

Because { the $\angle$ CFK = the $\angle$ CFL, *Proved.*
and the $\angle$ FCK = the $\angle$ FCL, being rt. angles, III. 18.
and FC is common ;

$\therefore$ CK = CL, I. 26.

and the $\angle$ FKC = the $\angle$ FLC.

Hence KL is double of KC; similarly HK is double of KB.

And since KC = KB, III. 17. *Cor.*

$\therefore$ KL = HK.

In the same way it may be shewn that every two consecutive sides are equal;

$\therefore$ the pentagon GHKLM is equilateral.

Again, it has been proved that the $\angle$ FKC = the $\angle$ FLC, and that the $\angle^s$ HKL, KLM are respectively double of these angles:

$\therefore$ the $\angle$ HKL = the $\angle$ KLM.

In the same way it may be shewn that every two consecutive angles of the figure are equal;

$\therefore$ the pentagon GHKLM is equiangular.

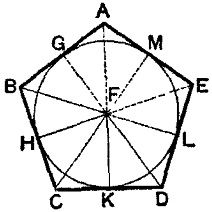
$\therefore$ the pentagon is regular, and it is circumscribed about the $\odot$ABCD. Q.E.F.

COROLLARY. *Similarly it may be proved that if tangents are drawn at the vertices of any regular polygon inscribed in a circle, they will form another regular polygon of the same species circumscribed about the circle.*

[For Exercises see p. 276.]

PROPOSITION 13. PROBLEM.

To inscribe a circle in a given regular pentagon.



Let ABCDE be the given regular pentagon:
it is required to inscribe a circle within it.

Bisect two consecutive $\angle^s$ BCD, CDE by CF and DF which intersect at F. I. 9.

Join FB;
and draw FH, FK perp. to BC, CD. I. 12.

Then in the $\triangle^s$ BCF, DCF,

Because { BC = DC, *Hyp.*
and CF is common to both;
and the $\angle$ BCF = the $\angle$ DCF; *Constr.*

∴ the $\angle$ CBF = the $\angle$ CDF. I. 4.

But the $\angle$ CDF is half an angle of the regular pentagon:
∴ also the $\angle$ CBF is half an angle of the regular pentagon:
that is, FB bisects the $\angle$ ABC.

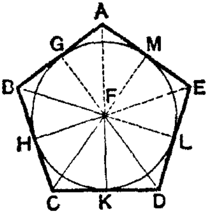
So it may be shewn that if FA, FE were joined, these lines would bisect the $\angle^s$ at A and E.

Again, in the $\triangle^s$ FCH, FCK,

Because { the $\angle$ FCH = the $\angle$ FCK, *Constr.*
and the $\angle$ FHC = the $\angle$ FKC being rt. angles;
also FC is common;

∴ FH = FK. I. 26.

Similarly if FG, FM, FL be drawn perp. to BA, AE, ED, it may be shewn that the five perpendiculars drawn from F to the sides of the pentagon are all equal.



From centre F, with radius FH, describe a circle; this circle must pass through the points H, K, L, M, G; and it will be touched at these points by the sides of the pentagon, for the ∠s at H, K, L, M, G are rt. ∠s. *Constr.*

∴ the ⊙HKLMG is inscribed in the given pentagon. Q.E.F.

COROLLARY. *The bisectors of the angles of a regular pentagon meet at a point.*

In the same way it may be shewn that the bisectors of the angles of any regular polygon meet at a point. [See Ex. 1, p. 274.]

[For Exercises on Regular Polygons see p. 276.]

MISCELLANEOUS EXERCISES.

1. Two tangents AB, AC are drawn from an external point A to a given circle: describe a circle to touch AB, AC and the convex arc intercepted by them on the given circle.

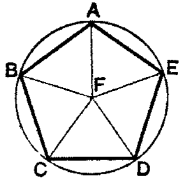
2. ABC is an isosceles triangle, and from the vertex A a straight line is drawn to meet the base at D and the circumference of the circumscribed circle at E: shew that AB is a tangent to the circle circumscribed about the triangle BDE.

3. An equilateral triangle is inscribed in a given circle: shew that twice the square on one of its sides is equal to three times the area of the square inscribed in the same circle.

4. ABC is an isosceles triangle in which each of the angles at B and C is double of the angle at A: shew that the square on AB is equal to the rectangle AB, BC with the square on BC.

PROPOSITION 14. PROBLEM.

To circumscribe a circle about a given regular pentagon.



Let ABCDE be the given regular pentagon :
it is required to circumscribe a circle about it.

Bisect the $\angle^s$ BCD, CDE by CF, DF intersecting at F. I. 9.
Join FB, FA, FE.

Then in the $\triangle^s$ BCF, DCF,

Because { BC = DC, *Hyp.*
and CF is common to both ;
and the $\angle$ BCF = the $\angle$ DCF ; *Constr.*

$\therefore$ the $\angle$ CBF = the $\angle$ CDF. I. 4.

But the $\angle$ CDF is half an angle of the regular pentagon :
$\therefore$ also the $\angle$ CBF is half an angle of the regular pentagon :
that is, FB bisects the $\angle$ ABC.

So it may be shewn that FA, FE bisect the $\angle^s$ at A and E.

Now the $\angle^s$ FCD, FDC are each half an angle of the given regular pentagon ;

$\therefore$ the $\angle$ FCD = the $\angle$ FDC, IV. *Def.*

$\therefore$ FC = FD. I. 6.

Similarly it may be shewn that FA, FB, FC, FD, FE are all equal.

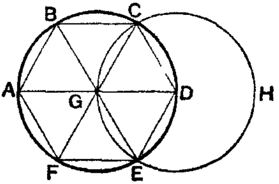
From centre F, with radius FA describe a circle :
this circle must pass through the points A, B, C, D, E,
and therefore is circumscribed about the given pentagon.

Q. E. F.

In the same way a circle may be circumscribed about any regular polygon.

Proposition 15. Problem.

To inscribe a regular hexagon in a given circle.



Let ABDF be the given circle:
it is required to inscribe a regular hexagon in it.

Find G the centre of the ⊙ ABDF; III. 1.
and draw a diameter AGD.

From centre D, with radius DG, describe the ⊙ EGCH.

Join CG, EG, and produce them to cut the $\bigcirc^{ce}$ of the given circle at F and B.

Join AB, BC, CD, DE, EF, FA.

Then ABCDEF shall be the required regular hexagon.

Now GE = GD, being radii of the ⊙ ACE;
and DG = DE, being radii of the ⊙ EHC:
∴ GE, ED, DG are all equal, and the △ EGD is equilateral.
Hence the ∠ EGD = one-third of two rt. angles. I. 32.
Similarly the ∠ DGC = one-third of two rt. angles.
But the ∠s EGD, DGC, CGB together = two rt. angles; I. 13.
∴ the remaining ∠ CGB = one-third of two rt. angles.
∴ the three ∠s EGD, DGC, CGB are equal to one another.
And to these angles the vert. opp. ∠s BGA, AGF, FGE are respectively equal:
∴ the ∠s EGD, DGC, CGB, BGA, AGF, FGE are all equal;
∴ the arcs ED, DC, CB, BA, AF, FE are all equal; III. 26.
∴ the chords ED, DC, CB, BA, AF, FE are all equal: III. 29.
∴ the hexagon is equilateral.

Again the arc FA = the arc DE: *Proved.*
to each of these equals add the arc ABCD;
then the whole arc FABCD = the whole arc ABCDE:
hence the angles at the $\bigcirc^{ce}$ which stand on these equal arcs are equal,

that is, the ∠ FED = the ∠ AFE. III. 27.

In like manner the remaining angles of the hexagon may be shewn to be equal.

∴ the hexagon is equiangular:

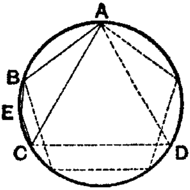
∴ the hexagon is regular, and it is inscribed in the ⊙ ABDF.

Q. E. F.

COROLLARY. *The side of a regular hexagon inscribed in a circle is equal to the radius of the circle.*

PROPOSITION 16. PROBLEM.

To inscribe a regular quindecagon in a given circle.



Let ABCD be the given circle:
it is required to inscribe a regular quindecagon in it.

In the ⊙ ABCD inscribe an equilateral triangle, IV. 2.
and let AC be one of its sides.

In the same circle inscribe a regular pentagon, IV. 11.
and let AB be one of its sides.

Then of such equal parts as the whole ○ᶜᵉ contains fifteen,
the arc AC, which is one-third of the ○ᶜᵉ, contains five;
and the arc AB, which is one-fifth of the ○ᶜᵉ, contains three;
∴ their difference, the arc BC, contains two.

Bisect the arc BC at E: III. 30.
then each of the arcs BE, EC is one-fifteenth of the ○ᶜᵉ.

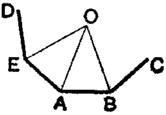
∴ if BE, EC be joined, and st. lines equal to them be placed successively round the circle, a regular quindecagon will be inscribed in it.

Q. E. F.

NOTE ON REGULAR POLYGONS.

The following propositions, proved by Euclid for a regular pentagon, hold good for all regular polygons.

1. *The bisectors of the angles of any regular polygon are concurrent.*



Let D, E, A, B, C be consecutive angular points of a regular polygon of any number of sides.

Bisect the ∠ˢ EAB, ABC by AO, BO, which intersect at O.

Join EO.

It is required to prove that EO bisects the ∠ DEA.

For in the △ˢ EAO, BAO,

Because { EA = BA, being sides of a regular polygon; and AO is common; and the ∠ EAO = the ∠ BAO; *Constr.*

∴ the ∠ OEA = the ∠ OBA. I. 4.

But the ∠ OBA is half the ∠ ABC; *Constr.*

also the ∠ ABC = the ∠ DEA, since the polygon is regular;

∴ the ∠ OEA is half the ∠ DEA:

that is, EO bisects the ∠ DEA.

Similarly if O be joined to the remaining angular points of the polygon, it may be proved that each joining line bisects the angle to whose vertex it is drawn.

That is to say, the bisectors of the angles of the polygon meet at the point O. Q. E. D.

COROLLARIES. Since the ∠ EAB = the ∠ ABC; *Hyp.*
and since the ∠ˢ OAB, OBA are respectively half of the ∠ˢ EAB, ABC;

∴ the ∠ OAB = the ∠ OBA.

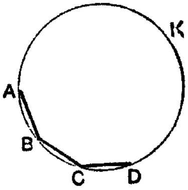
∴ OA = OB. I. 6.

Similarly OE = OA.

Hence *The bisectors of the angles of a regular polygon are all equal:* and a circle described from the centre O, with radius OA, will be circumscribed about the polygon.

Also it may be shewn, as in Proposition 13, that perpendiculars drawn from O to the sides of the polygon are all equal; therefore a circle described from centre O with any one of these perpendiculars as radius will be inscribed in the polygon.

2. *If a polygon inscribed in a circle is equilateral, it is also equiangular.*



Let AB, BC, CD be consecutive sides of an equilateral polygon inscribed in the $\odot$ ADK;
then shall this polygon be equiangular.

Because the chord AB = the chord DC, *Hyp.*
∴ the minor arc AB = the minor arc DC. III. 28.
To each of these equals add the arc AKD:
then the arc BAKD = the arc AKDC;
∴ the angles at the $\bigcirc^{ce}$, which stand on these equal arcs, are equal;
that is, the ∠ BCD = the ∠ ABC. III. 27.

Similarly the remaining angles of the polygon may be shewn to be equal:

∴ the polygon is equiangular. Q.E.D.

3. *If a polygon inscribed in a circle is equiangular, it is also equilateral, provided that the number of its sides is odd.*

[Observe that Theorems 2 and 3 are only true of polygons *inscribed in a circle.*

The accompanying figures are sufficient to shew that otherwise a polygon may be equilateral without being equiangular, Fig. 1; or equiangular without being equilateral, Fig. 2.]

Fig 1

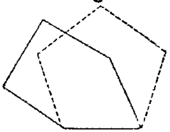
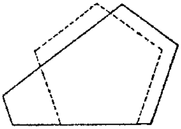


Fig. 2



NOTE. The following extensions of Euclid's constructions for Regular Polygons should be noticed.

By continual bisection of arcs, we are enabled to divide the circumference of a circle,

by means of Proposition 6, into 4, 8, 16,..., $2 \cdot 2^n$,... equal parts;
by means of Proposition 15, into 3, 6, 12, .., $3 \cdot 2^n$,.. equal parts;
by means of Proposition 11, into 5, 10, 20,.. , $5 \cdot 2^n$,... equal parts;
by means of Proposition 16, into 15, 30, 60,..., $15 \cdot 2^n$, . equal parts.

Hence we can inscribe in a circle a regular polygon the number of whose sides is included in any one of the formulæ $2 \cdot 2^n$, $3 \cdot 2^n$, $5 \cdot 2^n$, $15 \cdot 2^n$, n being any positive integer. In addition to these, it has been shewn that a regular polygon of $2^n + 1$ sides, provided $2^n + 1$ is a prime number, may be inscribed in a circle.

EXERCISES ON PROPOSITIONS 11—16.

1. Express in terms of a right angle the magnitude of an angle of the following *regular* polygons:

(i) a pentagon, (ii) a hexagon, (iii) an octagon,
(iv) a decagon, (v) a quindecagon.

2. The angle of a regular pentagon is trisected by the straight lines which join it to the opposite vertices.

3. In a polygon of n sides the straight lines which join any angular point to the vertices not adjacent to it, divide the angle into $n-2$ equal parts.

4. Shew how to construct on a given straight line
(i) a regular pentagon, (ii) a regular hexagon, (iii) a regular octagon.

5. An equilateral triangle and a regular hexagon are inscribed in a given circle; shew that

(i) the area of the triangle is half that of the hexagon;

(ii) the square on the side of the triangle is three times the square on the side of the hexagon.

6. ABCDE is a regular pentagon, and AC, BE intersect at H: shew that

(i) AB = CH = EH.

(ii) AB is a tangent to the circle circumscribed about the triangle BHC.

(iii) AC and BE cut one another in medial section.

7. The straight lines which join alternate vertices of a regular pentagon intersect so as to form another regular pentagon.

8. The straight lines which join alternate vertices of a regular polygon of n sides, intersect so as to form another regular polygon of n sides.

If $n=6$, shew that the area of the resulting hexagon is one-third of the given hexagon.

9. By means of IV. 16, inscribe in a circle a triangle whose angles are as the numbers 2, 5, 8.

10. Shew that the area of a regular hexagon inscribed in a circle is three-fourths of that of the corresponding circumscribed hexagon.

THEOREMS AND EXAMPLES ON BOOK IV.

I. ON THE TRIANGLE AND ITS CIRCLES.

1. D, E, F *are the points of contact of the inscribed circle of the triangle* ABC, *and* D_1, E_1, F_1 *the points of contact of the escribed circle, which touches* BC *and the other sides produced:* a, b, c *denote the lengths of the sides* BC, CA, AB; s *the semi-perimeter of the triangle, and* r, r_1 *the radii of the inscribed and escribed circles.*

Prove the following equalities:—

(i) $AE = AF = s - a$,
$BD = BF = s - b$,
$CD = CE = s - c$.

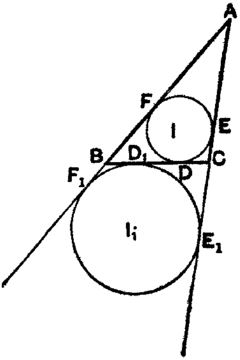
(ii) $AE_1 = AF_1 = s$.

(iii) $CD_1 = CE_1 = s - b$,
$BD_1 = BF_1 = s - c$.

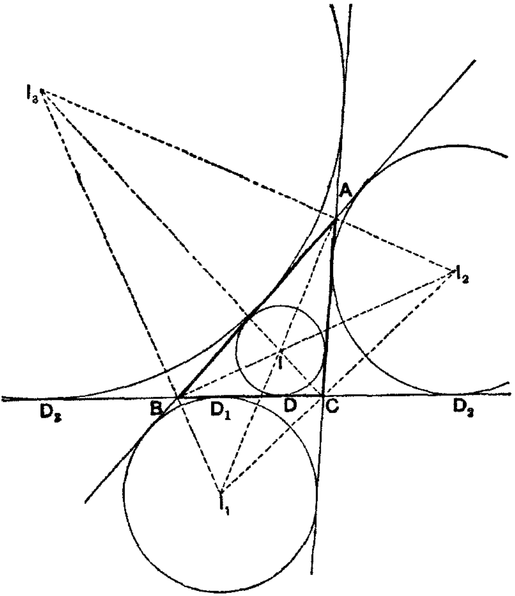
(iv) $CD = BD_1$ and $BD = CD_1$.

(v) $EE_1 = FF_1 = a$.

(vi) The area of the $\triangle ABC$
$= rs = r_1 (s - a)$.



2. *In the triangle* ABC, I *is the centre of the inscribed circle, and* I_1, I_2, I_3 *the centres of the escribed circles touching respectively the sides* BC, CA, AB *and the other sides produced.*



Prove the following properties :—

(i) *The points* A, I, I_1 *are collinear; so are* B, I, I_2; *and* C, I, I_3.

(ii) *The points* I_2, A, I_3 *are collinear; so are* I_3, B, I_1; *and* I_1, C, I_2.

(iii) *The triangles* BI_1C, CI_2A, AI_3B *are equiangular to one another.*

(iv) *The triangle* $I_1I_2I_3$ *is equiangular to the triangle formed by joining the points of contact of the inscribed circle.*

(v) *Of the four points* I, I_1, I_2, I_3 *each is the orthocentre of the triangle whose vertices are the other three.*

(vi) *The four circles, each of which passes through three of the points* I, I_1, I_2, I_3, *are all equal.*

3. With the notation of page 277, shew that in a triangle ABC, if the angle at C is a right angle,

$$r = s - c; \quad r_1 = s - b; \quad r_2 = s - a; \quad r_3 = s.$$

4. With the figure given on page 278, shew that if the circles whose centres are I, I_1, I_2, I_3 touch BC at D, D_1, D_2, D_3, then

(i) $DD_2 = D_1D_3 = b$. (ii) $DD_3 = D_1D_2 = c$.

(iii) $D_2D_3 = b + c$. (iv) $DD_1 = b \sim c$.

5. *Shew that the orthocentre and vertices of a triangle are the centres of the inscribed and escribed circles of the pedal triangle.* [See Ex. 20, p. 225.]

6. *Given the base and vertical angle of a triangle, find the locus of the centre of the inscribed circle.* [See Ex. 36, p. 228.]

7. *Given the base and vertical angle of a triangle, find the locus of the centre of the escribed circle which touches the base.*

8. *Given the base and vertical angle of a triangle, shew that the centre of the circumscribed circle is fixed.*

9. Given the base BC, and the vertical angle A of a triangle, find the locus of the centre of the escribed circle which touches AC.

10. Given the base, the vertical angle, and the radius of the inscribed circle; construct the triangle.

11. Given the base, the vertical angle, and the radius of the escribed circle, (i) which touches the base, (ii) which touches one of the sides containing the given angle; construct the triangle.

12. Given the base, the vertical angle, and the point of contact with the base of the inscribed circle; construct the triangle.

13. Given the base, the vertical angle, and the point of contact with the base, or base produced, of an escribed circle; construct the triangle.

14. From an external point A two tangents AB, AC are drawn to a given circle; and the angle BAC is bisected by a straight line which meets the circumference in I and I_1: shew that I is the centre of the circle inscribed in the triangle ABC, and I_1 the centre of one of the escribed circles.

15. *I is the centre of the circle inscribed in a triangle, and I_1, I_2, I_3 the centres of the escribed circles; shew that II_1, II_2, II_3 are bisected by the circumference of the circumscribed circle.*

16. ABC is a triangle, and I_2, I_3 the centres of the escribed circles which touch AC, and AB respectively: shew that the points B, C, I_2, I_3 lie upon a circle whose centre is on the circumference of the circle circumscribed about ABC.

17. With three given points as centres describe three circles touching one another two by two. How many solutions will there be?

18. Two tangents AB, AC are drawn to a given circle from an external point A; and in AB, AC two points D and E are taken so that DE is equal to the sum of DB and EC: shew that DE touches the circle.

19. Given the perimeter of a triangle, and one angle in magnitude and position: shew that the opposite side always touches a fixed circle.

20. Given the centres of the three escribed circles; construct the triangle.

21. Given the centre of the inscribed circle, and the centres of two escribed circles; construct the triangle.

22. Given the vertical angle, perimeter, and the length of the bisector of the vertical angle; construct the triangle.

23. Given the vertical angle, perimeter, and altitude; construct the triangle.

24. Given the vertical angle, perimeter, and radius of the inscribed circle; construct the triangle.

25. Given the vertical angle, the radius of the inscribed circle, and the length of the perpendicular from the vertex to the base; construct the triangle.

26. Given the base, the difference of the sides containing the vertical angle, and the radius of the inscribed circle; construct the triangle. [See Ex. 10, p. 258.]

27. Given a vertex, the centre of the circumscribed circle, and the centre of the inscribed circle, construct the triangle.

28. In a triangle ABC, I is the centre of the inscribed circle; shew that the centres of the circles circumscribed about the triangles BIC, CIA, AIB lie on the circumference of the circle circumscribed about the given triangle.

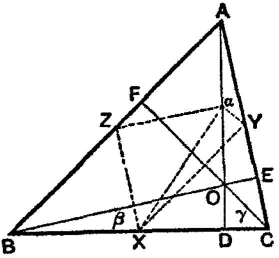
29. In a triangle ABC, the inscribed circle touches the base BC at D; and r, r_1 are the radii of the inscribed circle and of the escribed circle which touches BC: shew that $r \, . \, r_1 =$ BD . DC.

30. ABC is a triangle, D, E, F the points of contact of its inscribed circle; and D′E′F′ is the pedal triangle of the triangle DEF: shew that the sides of the triangle D′E′F′ are parallel to those of ABC.

31. In a triangle ABC the inscribed circle touches BC at D. Shew that the circles inscribed in the triangles ABD, ACD touch one another.

On the Nine-Points Circle.

32. *In any triangle the middle points of the sides, the feet of the perpendiculars drawn from the vertices to the opposite sides, and the middle points of the lines joining the orthocentre to the vertices are concyclic.*



In the $\triangle$ ABC, let X, Y, Z be the middle points of the sides BC, CA, AB; let D, E, F be the feet of the perp^s drawn to these sides from A, B, C; let O be the orthocentre, and α, β, γ the middle points of OA, OB, OC:
then shall the nine points X, Y, Z, D, E, F, α, β, γ be concyclic.

Join XY, XZ, Xα, Yα, Zα.
Now from the $\triangle$ ABO, since AZ = ZB, and Aα = αO, *Hyp.*
$\therefore$ Zα is par^l to BO. Ex. 2, p. 96.
And from the $\triangle$ ABC, since BZ = ZA, and BX = XC, *Hyp.*
$\therefore$ ZX is par^l to AC.
But BO makes a rt. angle with AC; *Hyp.*
$\therefore$ the $\angle$ XZα is a rt. angle.
Similarly, the $\angle$ XYα is a rt. angle. I. 29.
$\therefore$ the points X, Z, α, Y are concyclic:
that is, α lies on the O^ce of the circle, which passes through X, Y, Z; and Xα is a diameter of this circle.

Similarly it may be shewn that β and γ lie on the O^ce of the circle which passes through X, Y, Z.

Again, since αDX is a rt. angle, *Hyp.*
$\therefore$ the circle on Xα as diameter passes through D.

Similarly it may be shewn that E and F lie on the circumference of the same circle.

$\therefore$ the points X, Y, Z, D, E, F, α, β, γ are concyclic. Q.E.D.

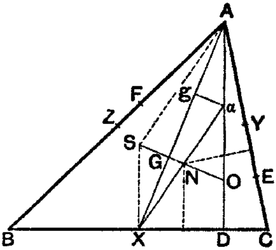
From this property the circle which passes through the middle points of the sides of a triangle is called the **Nine-Points Circle**; many of its properties may be derived from the fact of its being the circle circumscribed about the pedal triangle.

33. *To prove that*

(i) *the centre of the nine-points circle is the middle point of the straight line which joins the orthocentre to the circumscribed centre:*

(ii) *the radius of the nine-points circle is half the radius of the circumscribed circle:*

(iii) *the centroid is collinear with the circumscribed centre, the nine-points centre, and the orthocentre.*



In the △ABC, let X, Y, Z be the middle points of the sides; D, E, F the feet of the perps; O the orthocentre; S and N the centres of the circumscribed and nine-points circles respectively.

(i) To prove that N is the middle point of SO.

It may be shewn that the perp. to XD from its middle point bisects SO; Ex. 14, p. 98.

Similarly the perp. to EY at its middle point bisects SO:

that is, these perps intersect at the middle point of SO:

And since XD and EY are chords of the nine-points circle,

∴ the intersection of the lines which bisect XD and EY at rt. angles is its centre: III. 1.

∴ the centre N is the middle point of SO.

(ii) To prove that the radius of the nine-points circle is half the radius of the circumscribed circle.

By the last Proposition, Xα is a diameter of the nine-points circle.

∴ the middle point of Xα is its centre:

but the middle point of SO is also the centre of the nine-points circle. (*Proved.*)

Hence Xα and SO bisect one another at N.

Then from the △s SNX, ONα

Because { SN = ON,
and NX = Nα,
and the ∠ SNX = the ∠ ONα; I. 15.

∴ SX = Oα I. 4.

= Aα.

And SX is also parl to Aα,

∴ SA = Xα. I. 33.

But SA is a *radius* of the circumscribed circle;

and Xα is a *diameter* of the nine-points circle;

∴ the radius of the nine-points circle is half the radius of the circumscribed circle.

(iii) To prove that the centroid is collinear with points S, N, O.

Join AX and draw ag par[l] to SO.

Let AX meet SO at G.

Then from the $\triangle$ AGO, since $Aa = aO$ and ag is par[l] to OG,

$\therefore Ag = gG.$ Ex. 13, p. 98.

And from the $\triangle Xag$, since $aN = NX$, and NG is par[l] to ag,

$\therefore gG = GX.$ Ex. 13, p. 98.

$\therefore AG = \frac{2}{3}$ of AX;

$\therefore$ G is the centroid of the triangle ABC.

That is, the centroid is collinear with the points S, N, O. Q.E.D.

34. *Given the base and vertical angle of a triangle, find the locus of the centre of the nine-points circle.*

35. The nine-points circle of any triangle ABC, whose orthocentre is O, is also the nine-points circle of each of the triangles AOB, BOC, COA.

36. If I, I_1, I_2, I_3 are the centres of the inscribed and escribed circles of a triangle ABC, then the circle circumscribed about ABC is the nine-points circle of each of the four triangles formed by joining three of the points I, I_1, I_2, I_3.

37. All triangles which have the same orthocentre and the same circumscribed circle, have also the same nine points circle.

38. Given the base and vertical angle of a triangle, shew that one angle and one side of the pedal triangle are constant.

39. Given the base and vertical angle of a triangle, find the locus of the centre of the circle which passes through the three escribed centres.

NOTE. For another important property of the Nine-points Circle see Ex. 60, p. 382.

II. MISCELLANEOUS EXAMPLES.

1. If four circles are described to touch every three sides of a quadrilateral, shew that their centres are concyclic.

2. If the straight lines which bisect the angles of a rectilineal figure are concurrent, a circle may be inscribed in the figure.

3. Within a given circle describe three equal circles touching one another and the given circle.

4. The perpendiculars drawn from the centres of the three escribed circles of a triangle to the sides which they touch, are concurrent.

5. Given an angle and the radii of the inscribed and circumscribed circles; construct the triangle.

6. Given the base, an angle at the base, and the distance between the centre of the inscribed circle and the centre of the escribed circle which touches the base; construct the triangle.

7. In a given circle inscribe a triangle such that two of its sides may pass through two given points, and the third side be of given length.

8. In any triangle ABC, I, I_1, I_2, I_3 are the centres of the inscribed and escribed circles, and S_1, S_2, S_3 are the centres of the circles circumscribed about the triangles BIC, CIA, AIB: shew that the triangle $S_1S_2S_3$ has its sides parallel to those of the triangle $I_1I_2I_3$, and is one-fourth of it in area: also that the triangles ABC and $S_1S_2S_3$ have the same circumscribed circle.

9. O is the orthocentre of a triangle ABC: shew that

$$AO^2 + BC^2 = BO^2 + CA^2 = CO^2 + AB^2 = d^2,$$

where d is the diameter of the circumscribed circle.

10. If from any point within a regular polygon of n sides perpendiculars are drawn to the sides the sum of the perpendiculars is equal to n times the radius of the inscribed circle.

11. The sum of the perpendiculars drawn from the vertices of a regular polygon of n sides on any straight line is equal to n times the perpendicular drawn from the centre of the inscribed circle.

12. The area of a cyclic quadrilateral is independent of the order in which the sides are placed in the circle.

13. Given the orthocentre, the centre of the nine-points circle, and the middle point of the base; construct the triangle.

14. Of all polygons of a given number of sides, which may be inscribed in a given circle, that which is regular has the maximum area and the maximum perimeter.

15. Of all polygons of a given number of sides circumscribed about a given circle, that which is regular has the minimum area and the minimum perimeter.

16. Given the vertical angle of a triangle in position and magnitude, and the sum of the sides containing it: find the locus of the centre of the circumscribed circle.

17. P is any point on the circumference of a circle circumscribed about an equilateral triangle ABC: shew that $PA^2 + PB^2 + PC^2$ is constant.

BOOK V.

Book V. treats of Ratio and Proportion.

INTRODUCTORY.

The first four books of Euclid deal with the absolute equality or inequality of Geometrical magnitudes. In the Fifth Book magnitudes are compared by considering their *ratio*, or relative greatness.

The meaning of the words *ratio* and *proportion* in their simplest arithmetical sense, as contained in the following definitions, is probably familiar to the student:

The ratio of one number to another is the multiple, part, or parts that the first number is of the second; and it may therefore be measured by the fraction of which the first number is the numerator and the second the denominator.

Four numbers are in proportion when the ratio of the first to the second is equal to that of the third to the fourth.

But it will be seen that these definitions are inapplicable to Geometrical magnitudes for the following reasons:

(1) Pure Geometry deals only with *concrete magnitudes*, represented by diagrams, but not referred to any common unit in terms of which they are measured: in other words, it makes no use of *number* for the purpose of comparison between different magnitudes.

(2) It commonly happens that Geometrical magnitudes of the same kind are *incommensurable*, that is, they are such that it is impossible to express them *exactly* in terms of some common unit.

For example, we can make comparison between the side and diagonal of a square, and we may form an idea of their relative greatness, but it can be shewn that it is impossible to divide either of them into equal parts *of which the other contains an exact number*. And as the magnitudes we meet with in Geometry are more often incommensurable than not, it is clear that it would not always be possible to exactly represent such magnitudes by *numbers*, even if reference to a common unit were not foreign to the principles of Euclid.

It is therefore necessary to establish the Geometrical Theory of Proportion on a basis quite independent of Arithmetical principles. This is the aim of Euclid's Fifth Book.

We shall employ the following notation.

Capital letters, A, B, C,... will be used to denote the magnitudes themselves, *not any numerical or algebraical measures of them*, and small letters, m, n, p,... will be used to denote whole numbers. Also it will be assumed that multiplication, in the sense of repeated addition, can be applied to any magnitude, so that $m.A$ or mA will denote the magnitude A taken m times.

The symbol $>$ will be used for the words *greater than*, and $<$ for *less than*.

DEFINITIONS.

1. A greater magnitude is said to be a **multiple** of a less, when the greater contains the less an *exact* number of times.

2. A less magnitude is said to be a **submultiple** of a greater, when the less is contained an *exact* number of times in the greater.

The following properties of multiples will be assumed as self-evident.

(1) $mA > = \text{ or } < mB$ according as $A > = \text{ or } < B$; and conversely.

(2) $mA + mB + ... = m(A + B + ...)$.

(3) If $A > B$, then $mA - mB = m(A - B)$.

(4) $mA + nA + ... = (m + n + ...)A$.

(5) If $m > n$, then $mA - nA = (m - n)A$.

(6) $m.nA = mn.A = nm.A = n.mA$.

3. The **Ratio** of one magnitude to another of the same kind is the relation which the first bears to the second in respect of *quantuplicity*.

The ratio of A to B is denoted thus, $A : B$; and A is called the **antecedent**, B the **consequent** of the ratio.

The term *quantuplicity* denotes the capacity of the first magnitude to contain the second with or without remainder. If the magnitudes are commensurable, their quantuplicity may be expressed *numerically* by observing what multiples of the two magnitudes are equal to one another.

Thus if $A = ma$, and $B = na$, it follows that $nA = mB$. In this case $A = \frac{m}{n}B$, and the quantuplicity of A with respect to B is the arithmetical fraction $\frac{m}{n}$.

But if the magnitudes are incommensurable, no multiple of the first can be equal to any multiple of the second, and therefore the quantuplicity of one with respect to the other cannot exactly be expressed numerically: in this case it is determined by examining how the multiples of one magnitude are distributed among the multiples of the other.

Thus, let all the multiples of A be formed, the scale extending *ad infinitum;* also let all the multiples of B be formed and placed in their proper order of magnitude among the multiples of A. This forms the relative scale of the two magnitudes, and the quantuplicity of A with respect to B is estimated by examining how the multiples of A are distributed among those of B in their relative scale.

In other words, the ratio of A to B is known, if for all integral values of m we know the multiples nB and $(n+1)$ B between which mA lies.

In the case of two given magnitudes A and B, the relative scale of multiples is definite, and is different from that of A to C, if C differs from B by any magnitude however small.

For let D be the difference between B and C; then however small D may be, it will be possible to find a number m such that $m\mathrm{D} > \mathrm{A}$. In this case, mB and mC would differ by a magnitude greater than A, and therefore could not lie between the same two multiples of A; so that after a certain point the relative scale of A and B would differ from that of A and C.

[It is worthy of notice that we can always estimate the arithmetical ratio of two incommensurable magnitudes *within any required degree of accuracy.*

For suppose that A and B are incommensurable; divide B into m equal parts each equal to β, so that $\mathrm{B} = m\beta$, where m is an integer. Also suppose β is contained in A more than n times and less than $(n+1)$ times; then

$$\frac{\mathrm{A}}{\mathrm{B}} > \frac{n\beta}{m\beta} \text{ and } < \frac{(n+1)\beta}{m\beta},$$

$$\text{that is, } \frac{\mathrm{A}}{\mathrm{B}} \text{ lies between } \frac{n}{m} \text{ and } \frac{n+1}{m};$$

so that $\frac{\mathrm{A}}{\mathrm{B}}$ differs from $\frac{n}{m}$ by a quantity less than $\frac{1}{m}$. And since we can choose β (our unit of measurement) as small as we please, m can be made as great as we please. Hence $\frac{1}{m}$ can be made as small as we please, and two integers n and m can be found whose ratio will express that of a and b to any required degree of accuracy.]

4. The ratio of one magnitude to another is equal to that of a third magnitude to a fourth, when if any equimultiples whatever of the antecedents of the ratios are taken, and also any equimultiples whatever of the consequents, the multiple of one antecedent is greater than, equal to, or less than that of its consequent, according as the multiple of the other antecedent is greater than, equal to, or less than that of its consequent.

Thus the ratio A to B is equal to that of C to D when $m\text{C} > = \text{or} < n\text{D}$ according as $m\text{A} > = \text{or} < n\text{B}$, whatever whole numbers m and n may be.

Again, let m be any whole number whatever, and n another whole number determined in such a way that either $m\text{A}$ is equal to $n\text{B}$, or $m\text{A}$ lies between $n\text{B}$ and $(n+1)\,\text{B}$; then the definition asserts that the ratio of A to B is equal to that of C to D if $m\text{C} = n\text{D}$ when $m\text{A} = n\text{B}$; or if $m\text{C}$ lies between $n\text{D}$ and $(n+1)\,\text{D}$ when $m\text{A}$ lies between $n\text{B}$ and $(n+1)\,\text{B}$.

In other words, the ratio of A to B is equal to that of C to D when the multiples of A are distributed among those of B in the same manner as the multiples of C are distributed among those of D.

5. When the ratio of A to B is equal to that of C to D the four magnitudes are called **proportionals.** This is expressed by saying "A *is to* B *as* C *is to* D", and the **proportion** is written

$$\text{A} : \text{B} :: \text{C} : \text{D},$$

or

$$\text{A} : \text{B} = \text{C} : \text{D}.$$

A and D are called the **extremes**, B and C the **means**; also D is said to be a **fourth proportional** to A, B, and C.

Two terms in a proportion are said to be **homologous** when they are both antecedents, or both consequents of the ratios.

[It will be useful here to compare the algebraical and geometrical definitions of proportion, and to shew that each may be deduced from the other.

According to the geometrical definition A, B, C, D are in proportion, when $m\text{C} > = < n\text{D}$ according as $m\text{A} > = < n\text{B}$, m and n being *any positive integers whatever.*

According to the algebraical definition A, B, C, D are in proportion when $\frac{\text{A}}{\text{B}} = \frac{\text{C}}{\text{D}}$.

(i) To deduce the geometrical definition of proportion from the algebraical definition.

Since $\frac{A}{B} = \frac{C}{D}$, by multiplying both sides by $\frac{m}{n}$, we obtain

$$\frac{mA}{nB} = \frac{mC}{nD};$$

hence from the nature of fractions,

$$mC >=< nD \text{ according as } mA >=< nB,$$

which is the geometrical test of proportion.

(ii) To deduce the algebraical definition of proportion from the geometrical definition.

Given that $mC >=< nD$ according as $mA >=< nB$, to prove

$$\frac{A}{B} = \frac{C}{D}.$$

If $\frac{A}{B}$ is not equal to $\frac{C}{D}$, one of them must be the greater.

Suppose $\frac{A}{B} > \frac{C}{D}$; then it will be possible to find some fraction $\frac{n}{m}$ which lies between them, n and m being positive integers.

Hence $$\frac{A}{B} > \frac{n}{m} \quad \ldots \quad (1);$$

and $$\frac{C}{D} < \frac{n}{m} \quad \ldots \quad (2).$$

From (1), $mA > nB$;
from (2), $mC < nD$;
and these contradict the hypothesis.

Therefore $\frac{A}{B}$ and $\frac{C}{D}$ are not unequal; that is, $\frac{A}{B} = \frac{C}{D}$; which proves the proposition.]

6. The ratio of one magnitude to another is greater than that of a third magnitude to a fourth, when it is possible to find equimultiples of the antecedents and equimultiples of the consequents such that while the multiple of the antecedent of the first ratio is greater than, or equal to, that of its consequent, the multiple of the antecedent of the second is not greater, or is less, than that of its consequent.

This definition asserts that if whole numbers m and n can be found such that while mA is greater than nB, mC is not greater than nD, or while $m\text{A}=n\text{B}$, mC is less than nD, then the ratio of A to B is greater than that of C to D.

7. If A is equal to B, the ratio of A to B is called a **ratio of equality.**

If A is greater than B, the ratio of A to B is called a **ratio of greater inequality.**

If A is less than B, the ratio of A to B is called a **ratio of less inequality.**

8. Two ratios are said to be **reciprocal** when the antecedent and consequent of one are the consequent and antecedent of the other respectively; thus B : A is the reciprocal of A : B.

9. Three magnitudes of the same kind are said to be proportionals, when the ratio of the first to the second is equal to that of the second to the third.

Thus A, B, C are proportionals if

A : B :: B : C.

B is called a **mean proportional** to A and C, and C is called a **third proportional** to A and B.

10. Three or more magnitudes are said to be **in continued proportion** when the ratio of the first to the second is equal to that of the second to the third, and the ratio of the second to the third is equal to that of the third to the fourth, and so on.

11. When there are any number of magnitudes of the same kind, the first is said to have to the last the **ratio compounded** of the ratios of the first to the second, of the second to the third, and so on up to the ratio of the last but one to the last magnitude.

For example, if A, B, C, D, E be magnitudes of the same kind, A : E is the ratio compounded of the ratios A : B, B : C, C : D, and D : E.

This is sometimes expressed by the following notation:

$$A : E = \begin{cases} A : B \\ B : C \\ C : D \\ D : E. \end{cases}$$

12. If there are any number of ratios, and a set of magnitudes is taken such that the ratio of the first to the second is equal to the first ratio, and the ratio of the second to the third is equal to the second ratio, and so on, then the first of the set of magnitudes is said to have to the last the **ratio compounded** of the given ratios.

Thus, if A : B, C : D, E : F be given ratios, and if P, Q, R, S be magnitudes taken so that

$$P : Q :: A : B,$$
$$Q : R :: C : D,$$
$$R : S :: E : F;$$

then

$$P : S = \begin{cases} A : B \\ C : D \\ E : F. \end{cases}$$

13. When three magnitudes are proportionals, the first is said to have to the third the **duplicate ratio** of that which it has to the second.

Thus if $A : B :: B : C,$

then A is said to have to C the duplicate ratio of that which it has to B.

Since $A : C = \begin{cases} A : B \\ B : C. \end{cases}$

it is clear that the ratio compounded of two equal ratios is the duplicate ratio of either of them.

14. When four magnitudes are in *continued proportion*, the first is said to have to the fourth the **triplicate ratio** of that which it has to the second.

It may be shewn as above that the ratio compounded of three equal ratios is the triplicate ratio of any one of them.

Although an algebraical treatment of ratio and proportion when applied to geometrical magnitudes cannot be considered exact, it will perhaps be useful here to summarise in algebraical form the principal theorems of proportion contained in Book V. The student will then perceive that its leading propositions do not introduce new ideas, but merely supply rigorous proofs, based on the geometrical definition of proportion, of results already familiar in the study of Algebra.

We shall only here give those propositions which are afterwards referred to in Book VI. It will be seen that in their algebraical form many of them are so simple that they hardly require proof.

Summary of Principal Theorems of Book V.

Proposition 1.

Ratios which are equal to the same ratio are equal to one another.

That is, if $A : B = X : Y$ and $C : D = X : Y$;

then $A : B = C : D$.

Proposition 3.

If four magnitudes are proportionals, they are also proportionals when taken inversely.

That is, if $A : B = C : D$,

then $B : A = D : C$.

This inference is referred to as **invertendo** or **inversely**.

Proposition 4.

(i) *Equal magnitudes have the same ratio to the same magnitude.*

For if $A = B$,

then $A : C = B : C$.

(ii) *The same magnitude has the same ratio to equal magnitudes.*

For if $A = B$,

then $C : A = C : B$.

Proposition 6.

(i) *Magnitudes which have the same ratio to the same magnitude are equal to one another.*

That is, if $A : C = B : C$,

then $A = B$.

(ii) *Those magnitudes to which the same magnitude has the same ratio are equal to one another.*

That is, if $C : A = C : B$,

then $A = B$.

Proposition 8.

Magnitudes have the same ratio to one another which their equimultiples have.

That is, $A : B = mA : mB$,

where m is any whole number.

Proposition 11.

If four magnitudes of the same kind are proportionals, they are also proportionals when taken alternately.

If $A : B = C : D$,

then shall $A : C = B : D$.

For since $\frac{A}{B} = \frac{C}{D}$;

$\therefore$ multiplying by $\frac{B}{C}$, we have $\frac{A}{B} \cdot \frac{B}{C} = \frac{C}{D} \cdot \frac{B}{C}$;

that is, $\frac{A}{C} = \frac{B}{D}$,

or $A : C = B : D$.

This inference is referred to as **alternando** or **alternately.**

Proposition 12.

If any number of magnitudes of the same kind are proportionals, then as one of the antecedents is to its consequent, so is the sum of the antecedents to the sum of the consequents.

Let $\qquad A : B = C : D = E : F = \ldots;$

then shall $\qquad A : B = A + C + E + \ldots : B + D + F + \ldots.$

For put each of the equal ratios $\frac{A}{B}, \frac{C}{D}, \frac{E}{F}, \ldots$ equal to k;

then $\qquad A = Bk,\ C = Dk,\ E = Fk, \ldots$

$$\therefore \frac{A + C + E + \ldots}{B + D + F + \ldots} = \frac{Bk + Dk + Fk + \ldots}{B + D + F + \ldots} = k = \frac{A}{B} = \frac{C}{D} = \frac{E}{F} = \ldots;$$

$$\therefore A : B = A + C + E + \ldots : B + D + F + \ldots.$$

This inference is sometimes referred to as **addendo.**

Proposition 13.

(i) *If four magnitudes are proportionals, the sum of the first and second is to the second as the sum of the third and fourth is to the fourth.*

Let $\qquad A : B = C : D,$

then shall $\qquad A + B : B = C + D : D.$

For since $\qquad \frac{A}{B} = \frac{C}{D};$

$$\therefore \frac{A}{B} + 1 = \frac{C}{D} + 1;$$

that is, $\qquad \frac{A + B}{B} = \frac{C + D}{D},$

or $\qquad A + B : B = C + D : D.$

This inference is referred to as **componendo.**

(ii) *If four magnitudes are proportionals, the difference of the first and second is to the second as the difference of the third and fourth is to the fourth.*

That is, if $\qquad A : B = C : D,$

then $\qquad A \sim B : B = C \sim D : D.$

The proof is similar to that of the former case.

This inference is referred to as **dividendo.**

PROPOSITION 14.

If there are two sets of magnitudes, such that the first is to the second of the first set as the first to the second of the other set, and the second to the third of the first set as the second to the third of the other, and so on to the last magnitude: then the first is to the last of the first set as the first to the last of the other.

First let there be three magnitudes, A, B, C, of one set, and three, P, Q, R, of another set,

and let $$A : B = P : Q,$$

and $$B : C = Q : R;$$

then shall $$A : C = P : R.$$

For since $$\frac{A}{B} = \frac{P}{Q}, \text{ and } \frac{B}{C} = \frac{Q}{R};$$

$$\therefore \frac{A}{B} \cdot \frac{B}{C} = \frac{P}{Q} \cdot \frac{Q}{R};$$

that is, $$\frac{A}{C} = \frac{P}{R};$$

or $$A : C = P : R.$$

Similarly if
$$A : B = P : Q,$$
$$B : C = Q : R,$$
$$\ldots\ldots = \ldots\ldots,$$
$$L : M = Y : Z;$$

it can be proved that $$A : M = P : Z.$$

This inference is referred to as **ex æquali.**

COROLLARY. If $$A : B = P : Q,$$

and $$B : C = R : P;$$

then shall $$A : C = R : Q.$$

For since $$\frac{A}{B} = \frac{P}{Q}, \text{ and } \frac{B}{C} = \frac{R}{P};$$

$$\therefore \frac{A}{B} \cdot \frac{B}{C} = \frac{P}{Q} \cdot \frac{R}{P};$$

$$\therefore \frac{A}{C} = \frac{R}{Q};$$

or $$A : C = R : Q.$$

PROPOSITION 15.

If $A : B = C : D$,

and $E : B = F : D$;

then shall $A + E . B = C + F : D$.

For since $\frac{A}{B} = \frac{C}{D}$, and $\frac{E}{B} = \frac{F}{D}$;

$$\therefore \frac{A+E}{B} = \frac{C+F}{D};$$

that is, $A + E : B = C + F : D$.

PROPOSITION 16.

If two ratios are equal, their duplicate ratios are equal; and conversely.

Let $A : B = C : D$;

then shall the duplicate ratio of $A : B$ be equal to the duplicate ratio of $C : D$.

Let X be a third proportional to A, B;

so that $A : B = B : X$;

$$\therefore \frac{B}{X} = \frac{A}{B};$$

$$\therefore \frac{B}{X} \cdot \frac{A}{B} = \frac{A}{B} \cdot \frac{A}{B};$$

that is, $\frac{A}{X} = \frac{A^2}{B^2}$.

But $A : X$ is the duplicate ratio of $A : B$;

$\therefore$ the duplicate ratio of $A : B = A^2 : B^2$.

But since $A : B = C : D$;

$$\therefore \frac{A}{B} = \frac{C}{D},$$

$$\therefore \frac{A^2}{B^2} = \frac{C^2}{D^2},$$

or $A^2 : B^2 = C^2 : D^2$;

that is, the duplicate ratio of $A : B$ = the duplicate ratio of $C : D$.

Conversely, let the duplicate ratio of $A : B$ be equal to the duplicate ratio of $C : D$;

then shall $A : B = C : D$,

for since $A^2 : B^2 = C^2 : D^2$,

$$\therefore A : B = C : D.$$

PROOFS OF THE PROPOSITIONS OF BOOK V. DERIVED FROM THE GEOMETRICAL DEFINITION OF PROPORTION.

Obs. The Propositions of Book V. are all theorems.

PROPOSITION 1.

Ratios which are equal to the same ratio are equal to one another.

Let A : B :: P : Q, and also C : D :: P : Q; then shall A : B :: C : D.

For it is evident that two scales or arrangements of multiples which agree in every respect with a third scale, will agree with one another.

PROPOSITION 2.

If two ratios are equal, the antecedent of the second is greater than, equal to, or less than its consequent according as the antecedent of the first is greater than, equal to, or less than its consequent.

Let A : B :: C : D,

then $C >= \text{ or } < D$,

according as $A >= \text{ or } < B$.

This follows at once from Def. 4, by taking m and n each equal to unity.

Proposition 3.

If two ratios are equal, their reciprocal ratios are equal.

Let A : B :: C : D,

then shall B : A :: D : C.

For, by hypothesis, the multiples of A are distributed among those of B in the same manner as the multiples of C are among those of D;

therefore also, the multiples of B are distributed among those of A in the same manner as the multiples of D are among those of C.

That is, B : A :: D : C.

Note. This proposition is sometimes enunciated thus

If four magnitudes are proportionals, they are also proportionals when taken inversely,

and the inference is referred to as **invertendo** or **inversely**.

Proposition 4.

Equal magnitudes have the same ratio to the same magnitude; and the same magnitude has the same ratio to equal magnitudes.

Let A, B, C be three magnitudes of the same kind, and let A be equal to B;

then shall A : C :: B : C

and C : A :: C : B.

Since A = B, their multiples are identical and therefore are distributed in the same way among the multiples of C.

∴ A : C :: B : C, *Def.* 4.

∴ also, *invertendo*, C : A :: C : B. v. 3.

PROPOSITION 5.

Of two unequal magnitudes, the greater has a greater ratio to a third magnitude than the less has; and the same magnitude has a greater ratio to the less of two magnitudes than it has to the greater.

First, let A be $>$ B;

then shall A : C be $>$ B : C.

Since A $>$ B, it will be possible to find m such that mA exceeds mB by a magnitude greater than C;

hence if mA lies between nC and $(n+1)$C, mB $<$ nC:

and if mA $=$ nC, then mB $<$ nC;

$\therefore$ A : C $>$ B : C. *Def.* 6.

Secondly, let B be $<$ A;

then shall C : B be $>$ C : A.

For taking m and n as before,

nC $>$ mB, while nC is not $>$ mA;

$\therefore$ C : B $>$ C : A. *Def.* 6.

PROPOSITION 6.

Magnitudes which have the same ratio to the same magnitude are equal to one another; and those to which the same magnitude has the same ratio are equal to one another.

First, let A : C :: B : C;

then shall A $=$ B.

For if A $>$ B, then A : C $>$ B : C,

and if B $>$ A, then B : C $>$ A : C, v. 5.

which contradict the hypothesis;

$\therefore$ A $=$ B.

Secondly, let C : A :: C : B;

then shall A = B.

Because C : A :: C : B,

∴, *invertendo,* A : C :: B : C, v. 3.

A = B,

by the first part of the proof.

Proposition 7.

That magnitude which has a greater ratio than another has to the same magnitude is the greater of the two; and that magnitude to which the same has a greater ratio than it has to another magnitude is the less of the two.

First, let A : C be > B : C;

then shall A be > B.

For if A = B, then A : C :: B : C, v. 4.

which is contrary to the hypothesis.

And if A < B, then A : C < B : C; v. 5.

which is contrary to the hypothesis;

∴ A > B.

Secondly, let C : A be > C : B;

then shall A be < B.

For if A = B, then C : A :: C : B, v. 4.

which is contrary to the hypothesis.

And if A > B, then C : A < C : B; v. 5.

which is contrary to the hypothesis;

∴ A < B.

PROPOSITION 8.

Magnitudes have the same ratio to one another which their equimultiples have.

Let A, B be two magnitudes;

then shall $A : B :: mA : mB$.

If p, q be any two whole numbers,

then $m \cdot pA >= \text{or} < m \cdot qB$

according as $pA >= \text{or} < qB$.

But $m \cdot pA = p \cdot mA$, and $m \cdot qB = q \cdot mB$;

$\therefore p \cdot mA >= \text{or} < q \cdot mB$

according as $pA >= \text{or} < qB$;

$\therefore A : B :: mA : mB$. *Def.* 4.

COR. Let $A : B :: C : D$.

Then since $A : B :: mA : mB$,

and $C : D :: nC : nD$;

$\therefore mA : mB :: nC : nD$. V. 1.

PROPOSITION 9.

If two ratios are equal, and any equimultiples of the antecedents and also of the consequents are taken, the multiple of the first antecedent has to that of its consequent the same ratio as the multiple of the other antecedent has to that of its consequent.

Let $A : B :: C : D$;

then shall $mA : nB :: mC : nD$.

Let p, q be any two whole numbers,

then because $A : B :: C : D$,

$pm \cdot C >= \text{or} < qn \cdot D$

according as $pm \cdot A >= \text{or} < qn \cdot B$, *Def.* 4.

that is, $p \cdot mC >= \text{or} < q \cdot nD$,

according as $p \cdot mA >= \text{or} < q \cdot nB$;

$\therefore mA : nB :: mC : nD$. *Def.* 4.

PROPOSITION 10.

If four magnitudes of the same kind are proportionals, the first is greater than, equal to, or less than the third, according as the second is greater than, equal to, or less than the fourth.

Let A, B, C, D be four magnitudes of the same kind such that

$$A : B :: C : D;$$

then $A >=$ or $< C$

according as $B >=$ or $< D$.

If $B > D$, then $A : B < A : D$; v. 5.

but $A : B :: C : D$;

$\therefore C : D < A : D$;

$\therefore A : D > C : D$;

$\therefore A > C$. v. 7.

Similarly it may be shewn that

if $B < D$, then $A < C$,

and if $B = D$, then $A = C$.

PROPOSITION 11.

If four magnitudes of the same kind are proportionals, they are also proportionals when taken alternately.

Let A, B, C, D be four magnitudes of the same kind such that

$$A : B :: C : D;$$

then shall $A : C :: B : D$.

Because $A : B :: mA : mB$, v. 8.

and $C : D :: mC : nD$;

$\therefore mA : mB :: nC : nD$. v. 1.

$\therefore mA >=$ or $< nC$

according as $mB >=$ or $< nD$: v. 10.

and m and n are any whole numbers;

$\therefore A : C :: B : D$. *Def.* 4.

NOTE. This inference is usually referred to as **alternando** or **alternately**.

PROPOSITION 12.

If any number of magnitudes of the same kind are proportionals, as one of the antecedents is to its consequent, so is the sum of the antecedents to the sum of the consequents.

Let A, B, C, D, E, F,... be magnitudes of the same kind such that

$$A : B :: C : D :: E : F :: \ldots\ldots;$$

then shall $A : B :: A + C + E + \ldots : B + D + F + \ldots.$

Because $A : B :: C : D :: E : F :: \ldots,$

$\therefore$ according as $mA >= \text{or} < nB,$

so is $mC >= \text{or} < nD,$

and $mE >= \text{or} < nF,$

....

$\therefore$ so is $mA + mC + mE + \ldots >= \text{or} < nB + nD + nF + \ldots$

or $m(A + C + E + \ldots) >= \text{or} < n(B + D + F + \ldots);$

and m and n are any whole numbers;

$\therefore A : B :: A + C + E + \ldots : B + D + F + \ldots.$ *Def.* 4.

NOTE. This inference is usually referred to as **addendo.**

PROPOSITION 13.

If four magnitudes are proportionals, the sum or difference of the first and second is to the second as the sum or difference of the third and fourth is to the fourth.

Let $A : B :: C : D;$

then shall $A + B : B :: C + D : D,$

and $A \sim B : B :: C \sim D : D.$

If m be any whole number, it is possible to find another number n such that $mA = nB$, or lies between nB and $(n+1)B$,

$\therefore mA + mB = mB + nB$, or lies between $mB + nB$ and $mB + (n+1)B.$

But $mA + mB = m(A + B)$, and $mB + nB = (m + n)B$;
$\therefore m(A + B) = (m + n)B$, or lies between $(m + n)B$ and $(m + n + 1)B$.

Also because $A : B :: C : D$,
$\therefore mC = nD$, or lies between nD and $(n + 1)D$; *Def.* 4.
$\therefore m(C + D) = (m + n)D$ or lies between $(m + n)D$ and $(m + n + 1)D$;

that is, the multiples of $C + D$ are distributed among those of D in the same way as the multiples of $A + B$ among those of B;

$$\therefore A + B : B :: C + D : D.$$

In the same way it may be proved that

$$A - B : B :: C - D : D,$$

or $B - A : B :: D - C : D$,

according as A is $>$ or $<$ B.

NOTE. These inferences are referred to as **componendo** and **dividendo** respectively.

PROPOSITION 14.

If there are two sets of magnitudes, such that the first is to the second of the first set as the first to the second of the other set, and the second to the third of the first set as the second to the third of the other, and so on to the last magnitude: then the first is to the last of the first set as the first to the last of the other.

First, let there be three magnitudes A, B, C, of one set and three, P, Q, R, of another set,

and let $A : B :: P : Q$,
and $B : C :: Q : R$;
then shall $A : C :: P : R$.

Because $A : B :: P : Q$,
$\therefore mA : mB :: mP : mQ$; v. 8, *Cor.*
and because $B : C :: Q : R$,
$\therefore mB : nC :: mQ : nR$, v. 9.
$\therefore$, *invertendo*, $nC : mB :: nR : mQ$. v. 3.

Now, if $mA > nC$,

then $mA : mB > nC : mB$; v. 5.

$\therefore mP : mQ > nR : mQ$,

and $\therefore mP > nR$. v. 7.

Similarly it may be shewn that $mP =$ or $< nR$,

according as $mA =$ or $< nC$,

$\therefore A : C :: P : R$. *Def.* 4.

Secondly, let there be any number of magnitudes, A, B, C, ... L, M, of one set, and the same number P, Q, R, ... Y, Z, of another set, such that

$A : B :: P : Q$,

$B : C :: Q : R$,

...... :: ......

$L : M :: Y : Z$;

then shall $A : M :: P : Z$.

For $A : C :: P : R$, *Proved.*

and $C : D :: R : S$; *Hyp.*

$\therefore$ by the first case $A : D :: P : S$,

and so on, until finally

$A : M = P : Z$.

NOTE. This inference is referred to as **ex æquali.**

COROLLARY. If $A : B :: P : Q$,

and $B : C :: R : P$;

then $A : C :: R : Q$.

PROPOSITION 15.

If $A : B :: C : D$,

and $E : B :: F : D$;

then shall $A + E : B :: C + F : D$.

For since $E : B :: F : D$, *Hyp.*

$\therefore$, *invertendo*, $B : E :: D : F$. v. 3.

Also $A : B :: C : D$,

$\therefore$, *ex æquali*, $A : E :: C : F$, v. 14.

∴, *componendo*, A + E : E :: C + F : F. v. 13.
Again, E : B :: F : D, *Hyp.*
∴, *ex æquali*, A + E : B :: C + F : D. v. 14.

Proposition 16.

If two ratios are equal, their duplicate ratios are equal; and conversely, if the duplicate ratios of two ratios are equal, the ratios themselves are equal.

Let A : B :: C : D;
then shall the duplicate ratio of A to B be equal to that of C to D.

Let X be a third proportional to A and B, and Y a third proportional to C and D,

so that A : B :: B : X, and C : D :: D : Y;
then because A : B :: C : D,
∴ B : X :: D : Y;
∴, *ex æquali*, A : X :: C : Y.

But A : X and C : Y are respectively the duplicate ratios of
A : B and C : D, *Def.* 13.
∴ the duplicate ratio of A : B = that of C : D.

Conversely, let the duplicate ratio of A : B = that of C : D;
then shall A : B :: C : D.

Let P be such that A : B :: C : P,
∴, *invertendo*, B : A :: P : C.
Also, by hypothesis, A : X :: C : Y,
∴, *ex æquali*, B : X :: P : Y;
but A : B :: B : X,
∴ A : B :: P : Y; v. 1.
∴ C : P :: P : Y; v. 1.

that is, P is the mean proportional between C and Y.
∴ P = D,
∴ A : B :: C : D.

BOOK VI.

Definitions.

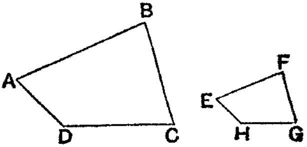
1. Two rectilineal figures are said to be **equiangular** when the angles of the first, taken in order, are equal respectively to those of the second, taken in order. Each angle of the first figure is said to **correspond** to the angle to which it is equal in the second figure, and sides **opposite** to corresponding angles are called **corresponding sides.**

2. Rectilineal figures are said to be **similar** when they are equiangular and have the sides about the equal angles proportionals, the corresponding sides being **homologous.**

[See Def. 5, page 288.]

Thus the two quadrilaterals ABCD, EFGH are similar if the angles at A, B, C, D are respectively equal to those at E, F, G, H, and if the following proportions hold

AB : BC :: EF : FG,
BC : CD :: FG : GH,
CD : DA :: GH : HE,
DA : AB :: HE : EF.



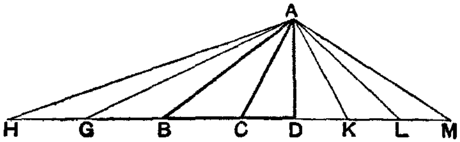
3. Two figures are said to have their sides about two of their angles **reciprocally proportional** when a side of the first is to a side of the second as the remaining side of the second is to the remaining side of the first.

4. A straight line is said to be divided **in extreme and mean ratio** when the whole is to the greater segment as the greater segment is to the less.

5. Two similar rectilineal figures are said to be **similarly situated** with respect to two of their sides when these sides are homologous.

PROPOSITION 1. THEOREM.

The areas of triangles of the same altitude are to one another as their bases.



Let ABC, ACD be two triangles of the same altitude, namely the perpendicular from A to BD:

then shall the △ ABC : the △ ACD :: BC : CD.

Produce BD both ways,

and from CB produced cut off any number of parts BG, GH, each equal to BC;

and from CD produced cut off any number of parts DK, KL, LM each equal to CD.

Join AH, AG, AK, AL, AM.

Then the △ˢ ABC, ABG, AGH are equal in area, for they are of the same altitude and stand on the equal bases CB, BG, GH, I. 38.

∴ the △ AHC is the same multiple of the △ ABC that HC is of BC;

Similarly the △ ACM is the same multiple of ACD that CM is of CD.

And if HC = CM,

the △ AHC = the △ ACM; I. 38.

and if HC is greater than CM,

the △ AHC is greater than the △ ACM; I. 38, *Cor.*

and if HC is less than CM,

the △ AHC is less than the △ ACM. I. 38, *Cor.*

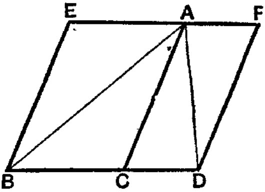
Now since there are four magnitudes, namely, the △ˢ ABC, ACD, and the bases BC, CD; and of the antecedents, any equimultiples have been taken, namely, the △ AHC and the base HC; and of the consequents, any equimultiples have been taken, namely the △ ACM and the base CM; and since it has been shewn that the △ AHC is greater than, equal to, or less than the △ ACM, according as HC is greater than, equal to, or less than CM;

∴ the four original magnitudes are proportionals, v. *Def.* 4.

that is,

the △ ABC : the △ ACD :: the base BC : the base CD. Q.E.D.

COROLLARY. *The areas of parallelograms of the same altitude are to one another as their bases.*

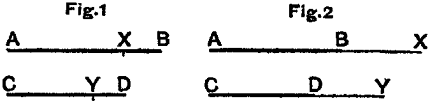


Let EC, CF be par^ms of the same altitude;
then shall the par^m EC : the par^m CF :: BC : CD.

Join BA, AD.

Then the △ ABC : the △ ACD :: BC : CD; *Proved.*
but the par^m EC is double of the △ ABC,
and the par^m CF is double of the △ ACD;
∴ the par^m EC : the par^m CF :: BC : CD. v. 8.

NOTE. Two straight lines are **cut proportionally** when the segments of one line are in the same ratio as the corresponding segments of the other. [See definition, page 131.]



Thus AB and CD are cut proportionally at X and Y, if

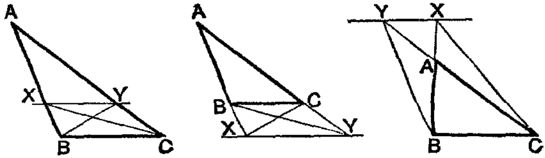
AX : XB :: CY : YD.

And the same definition applies equally whether X and Y divide AB, CD internally as in Fig. 1 or externally as in Fig. 2.

Proposition 2. Theorem.

If a straight line be drawn parallel to one side of a triangle, it shall cut the other sides, or those sides produced, proportionally:

Conversely, if the sides or the sides produced be cut proportionally, the straight line which joins the points of section, shall be parallel to the remaining side of the triangle.



Let XY be drawn par[1] to BC, one of the sides of the △ ABC:

then shall BX : XA :: CY : YA.

Join BY, CX.

Then the △ BXY = the △ CXY, being on the same base XY and between the same parallels XY, BC; I. 37.

and AXY is another triangle;

∴ the △ BXY : the △ AXY :: the △ CXY : the △ AXY. V. 4.

But the △ BXY : the △ AXY :: BX : XA, VI. 1.

and the △ CXY : the △ AXY :: CY : YA,

∴ BX : XA :: CY : YA. V. 1.

Conversely, let BX : XA :: CY : YA, and let XY be joined: then shall XY be par[1] to BC.

As before, join BY, CX.

By hypothesis BX : XA :: CY : YA;

but BX : XA :: the △ BXY : the △ AXY, VI. 1.

and CY : YA :: the △ CXY : the △ AXY;

∴ the △ BXY : the △ AXY :: the △ CXY : the △ AXY. V. 1.

∴ the △ BXY = the △ CXY; V. 6.

and they are triangles on the same base and on the same side of it.

∴ XY is par[1] to BC. I. 39.

Q.E.D.

EXERCISES.

1. Shew that every quadrilateral is divided by its diagonals into four triangles proportional to each other.

2. *If any two straight lines are cut by three parallel straight lines, they are cut proportionally.*

3. From a point E in the common base of two triangles ACB, ADB, straight lines are drawn parallel to AC, AD, meeting BC, BD at F, G: shew that FG is parallel to CD.

4. In a triangle ABC the straight line DEF meets the sides BC, CA, AB at the points D, E, F respectively, and it makes equal angles with AB and AC: prove that

BD : CD :: BF : CE.

5. If the bisector of the angle B of a triangle ABC meets AD at right angles, shew that a line through D parallel to BC will bisect AC.

6. From B and C, the extremities of the base of a triangle ABC, lines BE, CF are drawn to the opposite sides so as to intersect on the median from A: shew that EF is parallel to BC.

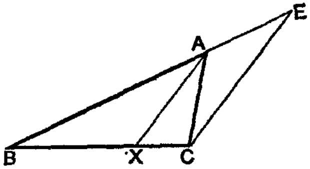
7. From P, a given point in the side AB of a triangle ABC, draw a straight line to AC produced, so that it will be bisected by BC.

8. Find a point within a triangle such that, if straight lines be drawn from it to the three angular points, the triangle will be divided into three equal triangles.

Proposition 3. Theorem.

If the vertical angle of a triangle be bisected by a straight line which cuts the base, the segments of the base shall have to one another the same ratio as the remaining sides of the triangle:

Conversely, if the base be divided so that its segments have to one another the same ratio as the remaining sides of the triangle have, the straight line drawn from the vertex to the point of section shall bisect the vertical angle.



In the △ ABC let the ∠ BAC be bisected by AX, which meets the base at X;

then shall BX : XC :: BA : AC.

Through C draw CE par[l] to XA, to meet BA produced at E. I. 31.

Then because XA and CE are par[l],

∴ the ∠ BAX = the int. opp. ∠ AEC, I. 29.

and the ∠ XAC = the alt. ∠ ACE. I. 29.

But the ∠ BAX = the ∠ XAC; *Hyp.*

∴ the ∠ AEC = the ∠ ACE;

∴ AC = AE. I. 6.

Again, because XA is par[l] to CE, a side of the △ BCE,

∴ BX : XC :: BA : AE; VI. 2.

that is, BX : XC :: BA : AC.

Conversely, let BX : XC :: BA : AC; and let AX be joined:
then shall the ∠ BAX = ∠ XAC.

For, with the same construction as before,
because XA is par[l] to CE, a side of the △ BCE,
∴ BX : XC :: BA : AE. VI. 2.
But by hypothesis BX : XC :: BA : AC;
∴ BA : AE :: BA : AC; V. 1.
∴ AE = AC;
∴ the ∠ ACE = the ∠ AEC. I. 5.
But because XA is par[l] to CE,
∴ the ∠ XAC = the alt. ∠ ACE. I. 29.
and the ext. ∠ BAX = the int. opp. ∠ AEC; I. 29.
∴ the ∠ BAX = the ∠ XAC.

Q.E.D.

EXERCISES.

1. The side BC of a triangle ABC is bisected at D, and the angles ADB, ADC are bisected by the straight lines DE, DF, meeting AB, AC at E, F respectively: shew that EF is parallel to BC.

2. Apply Proposition 3 to trisect a given finite straight line.

3. If the line bisecting the vertical angle of a triangle be divided into parts which are to one another as the base to the sum of the sides, the point of division is the centre of the inscribed circle.

4. ABCD is a quadrilateral: shew that if the bisectors of the angles A and C meet in the diagonal BD, the bisectors of the angles B and D will meet on AC.

5. *Construct a triangle having given the base, the vertical angle, and the ratio of the remaining sides.*

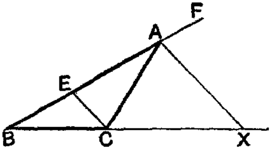
6. Employ this proposition to shew that the bisectors of the angles of a triangle are concurrent.

7. AB is a diameter of a circle, CD is a chord at right angles to it, and E any point in CD: AE and BE are drawn and produced to cut the circle in F and G: shew that the quadrilateral CFDG has any two of its adjacent sides in the same ratio as the remaining two.

Proposition A. Theorem.

If one side of a triangle be produced, and the exterior angle so formed be bisected by a straight line which cuts the base produced, the segments between the bisector and the extremities of the base shall have to one another the same ratio as the remaining sides of the triangle have:

Conversely, if the segments of the base produced have to one another the same ratio as the remaining sides of the triangle have, the straight line drawn from the vertex to the point of section shall bisect the exterior vertical angle.



In the △ ABC let BA be produced to F, and let the exterior ∠ CAF be bisected by AX which meets the base produced at X:

then shall BX : XC :: BA : AC.

Through C draw CE par^l to XA, I. 31.
and let CE meet BA at E.

Then because AX and CE are par^l,
∴ the ext. ∠ FAX = the int. opp. ∠ AEC,
and the ∠ XAC = the alt. ∠ ACE. I. 29.
But the ∠ FAX = the ∠ XAC; *Hyp.*
∴ the ∠ AEC = the ∠ ACE;
∴ AC = AE. I. 6.

Again, because XA is par^l to CE, a side of the △ BCE, *Constr.*

∴ BX : XC :: BA : AE; VI. 2.
that is, BX : XC :: BA : AC.

Conversely, let BX : XC :: BA : AC, and let AX be joined:
then shall the ∠ FAX = the ∠ XAC.

For, with the same construction as before,
because AX is par[l] to CE, a side of the △ BCE,
∴ BX : XC :: BA : AE. VI. 2.
But by hypothesis BX : XC :: BA : AC;
∴ BA : AE :: BA : AC; V. 1.
∴ AE = AC,
∴ the ∠ ACE = the ∠ AEC. I. 5.
But because AX is par[l] to CE,
∴ the ∠ XAC = the alt. ∠ ACE,
and the ext. ∠ FAX = the int. opp. ∠ AEC; I. 29.
∴ the ∠ FAX = the ∠ XAC. Q.E.D.

Propositions 3 and A may be both included in one enunciation as follows:

If the interior or exterior vertical angle of a triangle be bisected by a straight line which also cuts the base, the base shall be divided internally or externally into segments which have the same ratio as the sides of the triangle:

Conversely, if the base be divided internally or externally into segments which have the same ratio as the sides of the triangle, the straight line drawn from the point of division to the vertex will bisect the interior or exterior vertical angle.

EXERCISES.

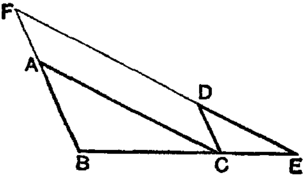
1. In the circumference of a circle of which AB is a diameter, a point P is taken; straight lines PC, PD are drawn equally inclined to AP and on opposite sides of it, meeting AB in C and D;
shew that AC : CB :: AD : DB.

2. From a point A straight lines are drawn making the angles BAC, CAD, DAE, each equal to half a right angle, and they are cut by a straight line BCDE, which makes BAE an isosceles triangle: shew that BC or DE is a mean proportional between BE and CD.

3. By means of Propositions 3 and A, prove that the straight lines bisecting one angle of a triangle internally, and the other two externally, are concurrent.

PROPOSITION 4. THEOREM.

If two triangles be equiangular to one another, the sides about the equal angles shall be proportionals, those sides which are opposite to equal angles being homologous.



Let the △ ABC be equiangular to the △ DCE, having the ∠ ABC equal to the ∠ DCE, the ∠ BCA equal to the ∠ CED, and consequently the ∠ CAB equal to the ∠ EDC: I. 32.
then shall the sides about these equal angles be proportionals, namely

AB : BC :: DC : CE,
BC : CA :: CE : ED,
and AB : AC :: DC : DE.

Let the △ DCE be placed so that its side CE may be contiguous to BC, and in the same straight line with it.

Then because the ∠ s ABC, ACB are together less than two rt. angles, I. 17.
and the ∠ ACB = the ∠ DEC; *Hyp.*
∴ the ∠ s ABC, DEC are together less than two rt. angles;
∴ BA and ED will meet if produced. *Ax.* 12.
Let them be produced and meet at F.

Then because the ∠ ABC = the ∠ DCE, *Hyp.*
∴ BF is parl to CD; I. 28.
and because the ∠ ACB = the ∠ DEC, *Hyp.*
∴ AC is parl to FE, I. 28.
∴ FACD is a parm;
∴ AF = CD, and AC = FD. I. 34.

Again, because CD is par[l] to BF, a side of the $\triangle$ EBF,

$\therefore$ BC : CE :: FD : DE; VI. 2.

but FD = AC;

$\therefore$ BC : CE :: AC : DE;

and, *alternately*, BC : CA :: CE : ED. V. 11.

Again, because AC is par[l] to FE, a side of the $\triangle$ FBE,

$\therefore$ BA : AF :: BC : CE; VI. 2.

but AF = CD;

$\therefore$ BA : CD :: BC : CE;

and, *alternately*, AB : BC :: DC : CE. V. 11.

Also BC : CA :: CE : ED; *Proved.*

$\therefore$, *ex æquali*, AB : AC :: DC : DE. V. 14.

Q. E. D.

[For Alternative Proof see Page 320.]

EXERCISES.

1. If one of the parallel sides of a trapezium is double the other, shew that the diagonals intersect one another at a point of trisection.

2. In the side AC of a triangle ABC any point D is taken: shew that if AD, DC, AB, BC are bisected in E, F, G, H respectively, then EG is equal to HF.

3. AB and CD are two parallel straight lines; E is the middle point of CD; AC and BE meet at F, and AE and BD meet at G: shew that FG is parallel to AB.

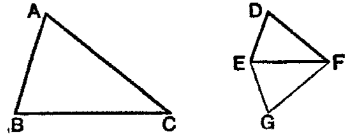
4. ABCDE is a regular pentagon, and AD and BE intersect in F: shew that AF : AE :: AE : AD.

5. In the figure of I. 43 shew that EH and GF are parallel, and that FH and GE will meet on CA produced.

6. Chords AB and CD of a circle are produced towards B and D respectively to meet in the point E, and through E, the line EF is drawn parallel to AD to meet CB produced in F. Prove that EF is a mean proportional between FB and FC.

PROPOSITION 5. THEOREM.

If the sides of two triangles, taken in order about each of their angles, be proportionals, the triangles shall be equiangular to one another, having those angles equal which are opposite to the homologous sides.



Let the $\triangle^s$ ABC, DEF have their sides proportionals, so that

AB : BC :: DE : EF,
BC : CA :: EF : FD,

and consequently, *ex æquali*,

AB : CA :: DE : FD.

Then shall the triangles be equiangular.

At E in FE make the ∠ FEG equal to the ∠ ABC;
and at F in EF make the ∠ EFG equal to the ∠ BCA; I. 23.
then the remaining ∠ EGF = the remaining ∠ BAC. I. 32.
∴ the $\triangle$ GEF is equiangular to the $\triangle$ ABC;
∴ GE : EF :: AB : BC. VI. 4.
But AB : BC :: DE : EF; *Hyp.*
∴ GE : EF :: DE : EF; V. 1.
∴ GE = DE.
Similarly GF = DF.

Then in the triangles GEF, DEF

Because { GE = DE,
GF = DF,
and EF is common; }

∴ the ∠ GEF = the ∠ DEF, I. 8.
and the ∠ GFE = the ∠ DFE,
and the ∠ EGF = the ∠ EDF.

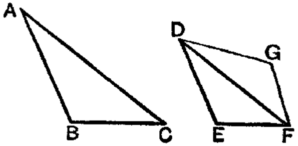
But the ∠ GEF = the ∠ ABC; *Constr.*
∴ the ∠ DEF = the ∠ ABC.
Similarly, the ∠ EFD = the ∠ BCA,

∴ the remaining ∠ FDE = the remaining ∠ CAB; I. 32.
that is, the △ DEF is equiangular to the △ ABC.

Q. E. D.

PROPOSITION 6. THEOREM.

If two triangles have one angle of the one equal to one angle of the other, and the sides about the equal angles proportionals, the triangles shall be similar.



In the △s ABC, DEF let the ∠ BAC = the ∠ EDF,
and let BA : AC :: ED : DF.

Then shall the △s ABC, DEF be similar.

At D in FD make the ∠ FDG equal to one of the ∠s EDF, BAC:
at F in DF make the ∠ DFG equal to the ∠ ACB; I. 23.
∴ the remaining ∠ FGD = the remaining ∠ ABC. I. 32.

Then the △ ABC is equiangular to the △ DGF;
∴ BA : AC :: GD : DF. VI. 4.
But BA : AC :: ED : DF; *Hyp.*
∴ GD : DF :: ED : DF,
∴ GD = ED.

Then in the △s GDF, EDF,

Because { GD = ED,
and DF is common;
and the ∠ GDF = the ∠ EDF; *Constr.*

∴ the △s GDF, EDF are equal in all respects, I. 4.
so that the △ EDF is equiangular to the △ GDF;
but the △ GDF is equiangular to the △ BAC; *Constr.*
∴ the △ EDF is equiangular to the △ BAC;
∴ their sides about the equal angles are proportionals, VI. 4.
that is, the △s ABC, DEF are similar.

Q. E. D.

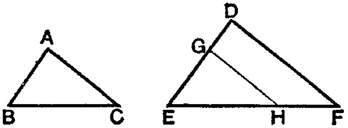
NOTE 1. From Definition 2 it is seen that *two* conditions are necessary for similarity of rectilineal figures, namely (1) the figures must be equiangular, and (2) the sides about the equal angles must be proportionals. In the case of *triangles* we learn from Props. 4 and 5 that each of these conditions follows from the other: this however is not necessarily the case with rectilineal figures of more than three sides.

NOTE 2. We have given Euclid's demonstrations of Propositions 4, 5, 6; but these propositions also admit of easy proof by the method of superposition.

As an illustration, we will apply this method to Proposition 4.

PROPOSITION 4. [ALTERNATIVE PROOF.]

If two triangles be equiangular to one another, the sides about the equal angles shall be proportionals, those sides which are opposite to equal angles being homologous.



Let the △ ABC be equiangular to the △ DEF, having the ∠ ABC equal to the ∠ DEF, the ∠ BCA equal to the ∠ EFD, and consequently the ∠ CAB equal to the ∠ FDE: I. 32.

then shall the sides about these equal angles be proportionals.

Apply the △ ABC to the △ DEF, so that B falls on E and BA along ED:

then BC will fall along EF, since the ∠ ABC = the ∠ DEF. *Hyp.*

Let G and H be the points in ED and EF, on which A and C fall.

Join GH.

Then because the ∠ EGH = the ∠ EDF, *Hyp.*

∴ GH is par[l] to DF:

∴ DG : GE :: FH : HE;

∴, *componendo*, DE : GE :: FE : HE, V. 18.

∴, *alternately*, DE : FE :: GE : HE, V. 11.

that is, DE : EF :: AB : BC.

Similarly by applying the △ ABC to the △ DEF, so that the point C may fall on F, it may be proved that

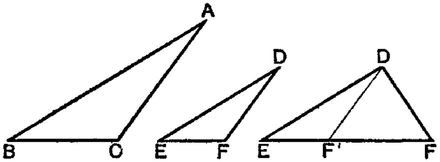
EF : FD :: BC : CA.

∴, *ex æquali*, DE : DF :: AB : AC.

Q. E. D.

Proposition 7. Theorem.

If two triangles have one angle of the one equal to one angle of the other and the sides about one other angle in each proportional, so that the sides opposite to the equal angles are homologous, then the third angles are either equal or supplementary; and in the former case the triangles are similar.



Let ABC, DEF be two triangles having the ∠ ABC equal to the ∠ DEF, and the sides about the angles at A and D proportional, so that

BA : AC :: ED : DF;

then shall the ∠s ACB, DFE be either equal or supplementary, and in the former case the triangles shall be similar.

If the ∠ BAC = the ∠ EDF,
then the ∠ BCA = the ∠ EFD; I. 32.
and the △s are equiangular and therefore similar. VI. 4.

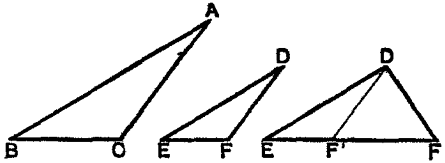
But if the ∠ BAC is not equal to the ∠ EDF, one of them must be the greater.

Let the ∠ EDF be greater than the ∠ BAC.

At D in ED make the ∠ EDF′ equal to the ∠ BAC. I. 23.
Then the △s BAC, EDF′ are equiangular, *Constr.*
∴ BA : AC :: ED : DF′; VI. 4.
but BA : AC :: ED : DF; *Hyp.*
∴ ED : DF :: ED : DF′, V. 1.
∴ DF = DF′,
∴ the ∠ DFF′ = the ∠ DF′F. I. 5.
But the ∠s DF′F, DF′E are supplementary, I. 13.
∴ the ∠s DFF′, DF′E are supplementary:
that is, the ∠s DFE, ACB are supplementary.

Q. E. D.

Corollaries to Proposition 7.



Three cases of this theorem deserve special attention.

It has been proved that if the angles ACB, DFE are not *supplementary*, they are *equal:*
and we know that of angles which are supplementary and unequal, one must be acute and the other obtuse.

Hence, in addition to the hypothesis of this theorem,

(i) If the angles ACB, DFE, opposite to the two homologous sides AB, DE are both acute, both obtuse, or if one of them is a right angle,
it follows that these angles are equal;
and therefore the triangles are similar.

(ii) If the two given angles are right angles or obtuse angles, it follows that the angles ACB, DFE must be both acute, and therefore equal, by (i):
so that the triangles are similar.

(iii) If in each triangle the side opposite the given angle is not less than the other given side; that is, if AC and DF are not less than AB and DE respectively, then
the angles ACB, DFE cannot be greater than the angles ABC, DEF, respectively;
therefore the angles ACB, DFE, are both acute;
hence, as above, they are equal;
and the triangles ABC, DEF similar.

EXERCISES.

ON PROPOSITIONS 1 TO 7.

1. Shew that the diagonals of a trapezium cut one another in the same ratio.

2. If three straight lines drawn from a point cut two parallel straight lines in A, B, C and P, Q, R respectively, prove that

AB : BC :: PQ : QR.

3. From a point O, a tangent OP is drawn to a given circle, and OQR is drawn cutting it in Q and R; shew that

OQ : OP :: OP : OR.

4. *If two triangles are on equal bases and between the same parallels, any straight line parallel to their bases will cut off equal areas from the two triangles.*

5. *If two straight lines* PQ, XY *intersect in a point* O, *so that* PO : OX :: YO : OQ, *prove that* P, X, Q, Y *are concyclic.*

6. On the same base and on the same side of it two equal triangles ACB, ADB are described; AC and BD intersect in O, and through O lines parallel to DA and CB are drawn meeting the base in E and F. Shew that AE = BF.

7. BD, CD are perpendicular to the sides AB, AC of a triangle ABC, and CE is drawn perpendicular to AD, meeting AB in E: shew that the triangles ABC, ACE are similar.

8. AC and BD are drawn perpendicular to a given straight line CD from two given points A and B; AD and BC intersect in E, and EF is perpendicular to CD: shew that AF and BF make equal angles with CD.

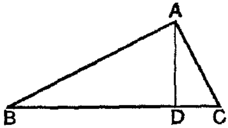
9. ABCD is a parallelogram; P and Q are points in a straight line parallel to AB; PA and QB meet at R, and PD and QC meet at S: shew that RS is parallel to AD.

10. In the sides AB, AC of a triangle ABC two points D, E are taken such that BD is equal to CE; if DE, BC produced meet at F, shew that AB : AC :: EF : DF.

11. Find a point the perpendiculars from which on the sides of a given triangle shall be in a given ratio.

Proposition 8. Theorem.

In a right-angled triangle if a perpendicular be drawn from the right angle to the hypotenuse, the triangles on each side of it are similar to the whole triangle and to one another.



Let ABC be a triangle right-angled at A, and let AD be perp. to BC:
then shall the △s DBA, DAC be similar to the △ABC and to one another.

In the △s DBA, ABC,
the ∠BDA = the ∠BAC, being rt. angles,
and the ∠ABC is common to both;
∴ the remaining ∠BAD = the remaining ∠BCA, I. 32.
that is, the △s DBA, ABC are equiangular;
∴ they are similar. VI. 4.

In the same way it may be proved that the △s DAC, ABC are similar.

Hence the △s DBA, DAC, being equiangular to the same △ABC, are equiangular to one another;
∴ they are similar. VI. 4.

Q.E.D.

Corollary. Because the △s BDA, ADC are similar,
∴ BD : DA :: DA : DC;
and because the △s CBA, ABD are similar,
∴ CB : BA :: BA : BD;
and because the △s BCA, ACD are similar,
∴ BC : CA :: CA : CD.

EXERCISES.

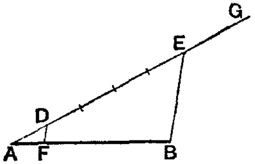
1. Prove that the hypotenuse is to one side as the second side is to the perpendicular.

2. *Shew that the radius of a circle is a mean proportional between the segments of any tangent between its point of contact and a pair of parallel tangents.*

DEFINITION. A less magnitude is said to be a **submultiple** of a greater, when the less is contained an *exact* number of times in the greater. [Book v. Def. 2.]

PROPOSITION 9. PROBLEM.

From a given straight line to cut off any required submultiple.



Let AB be the given straight line.

It is required to cut off a certain submultiple from AB.

From A draw a straight line AG of indefinite length making any angle with AB.

In AG take any point D; and, by cutting off successive parts each equal to AD, make AE to contain AD as many times as AB contains the required submultiple.

Join EB.

Through D draw DF par[l] to EB, meeting AB in F.

Then shall AF be the required submultiple.

Because DF is par[l] to EB, a side of the $\triangle$ AEB,

∴ BF : FA :: ED : DA; vi. 2.

∴, *componendo*, BA : AF :: EA : AD. v. 13.

But AE contains AD the required number of times; *Constr.*

∴ AB contains AF the required number of times;

that is, AF is the required submultiple. Q.E.F.

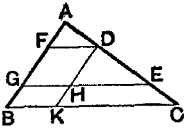
EXERCISES.

1. Divide a straight line into five equal parts.

2. Give a geometrical construction for cutting off two-sevenths of a given straight line.

PROPOSITION 10. PROBLEM.

To divide a straight line similarly to a given divided straight line.



Let AB be the given straight line to be divided, and AC the given straight line divided at the points D and E.

It is required to divide AB similarly to AC.

Let AB, AC be placed so as to form any angle.

Join CB.

Through D draw DF par[l] to CB, I. 31.

and through E draw EG par[l] to CB,

and through D draw DHK par[l] to AB.

Then AB shall be divided at F and G similarly to AC.

For by construction each of the figs. FH, HB is a par[m];

∴ DH = FG, and HK = GB. I. 34.

Now since HE is par[l] to KC, a side of the △DKC,

∴ KH : HD :: CE : ED. VI. 2.

But KH = BG, and HD = GF;

∴ BG : GF :: CE : ED. V. 1.

Again, because FD is par[l] to GE, a side of the △AGE,

∴ GF : FA :: ED : DA, VI. 2.

and it has been shewn that

BG : GF :: CE : ED,

∴, *ex æquali*, BG : FA :: CE : DA : V. 14.

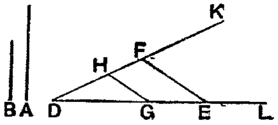
∴ AB is divided similarly to AC. Q. E. F.

EXERCISE.

Divide a straight line internally and externally in a given ratio. Is this always possible?

PROPOSITION 11. PROBLEM.

To find a third proportional to two given straight lines.



Let A, B be two given straight lines.
It is required to find a third proportional to A and B.

Take two st. lines DL, DK of indefinite length, containing any angle:
from DL cut off DG equal to A, and GE equal to B;
and from DK cut off DH equal to B. I. 3.
Join GH.
Through E draw EF par[1] to GH, meeting DK in F. I. 31.
Then shall HF be a third proportional to A and B.

Because GH is par[1] to EF, a side of the △ DEF;
∴ DG : GE :: DH : HF. VI. 2.
But DG = A; and GE, DH each = B; *Constr.*
∴ A : B :: B : HF;
that is, HF is a third proportional to A and B.

Q. E. F.

EXERCISES.

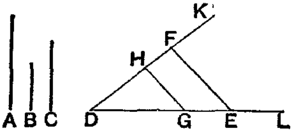
1. AB is a diameter of a circle, and through A any straight line is drawn to cut the circumference in C and the tangent at B in D: shew that AC is a third proportional to AD and AB.

2. ABC is an isosceles triangle having each of the angles at the base double of the vertical angle BAC; the bisector of the angle BCA meets AB at D. Shew that AB, BC, BD are three proportionals.

3. Two circles intersect at A and B; and at A tangents are drawn, one to each circle, to meet the circumferences at C and D: shew that if CB, BD are joined, BD is a third proportional to CB, BA.

PROPOSITION 12. PROBLEM.

To find a fourth proportional to three given straight lines.



Let A, B, C be the three given straight lines.
It is required to find a fourth proportional to A, B, C.

Take two straight lines DL, DK containing any angle:
from DL cut off DG equal to A, GE equal to B;
and from DK cut off DH equal to C. I. 3.
Join GH.
Through E draw EF par[l] to GH. I. 31.
Then shall HF be a fourth proportional to A, B, C.

Because GH is par[l] to EF, a side of the $\triangle$DEF;
$\therefore$ DG : GE :: DH : HF. VI. 2.
But DG = A, GE = B, and DH = C; *Constr.*
$\therefore$ A : B :: C : HF;
that is, HF is a fourth proportional to A, B, C.

Q.E.F.

EXERCISES.

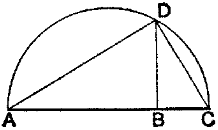
1. If from D, one of the angular points of a parallelogram ABCD, a straight line is drawn meeting AB at E and CB at F; shew that CF is a fourth proportional to EA, AD, and AB.

2. In a triangle ABC the bisector of the vertical angle BAC meets the base at D and the circumference of the circumscribed circle at E: shew that BA, AD, EA, AC are four proportionals.

3. From a point P tangents PQ, PR are drawn to a circle whose centre is C, and QT is drawn perpendicular to RC produced: shew that QT is a fourth proportional to PR, RC, and RT.

PROPOSITION 13. PROBLEM.

To find a mean proportional between two given straight lines.



Let AB, BC be the two given straight lines.
It is required to find a mean proportional between them.

Place AB, BC in a straight line, and on AC describe the semicircle ADC.

From B draw BD at rt. angles to AC. I. 11.

Then shall BD be a mean proportional between AB and BC.

Join AD, DC.

Now the ∠ ADC being in a semicircle is a rt. angle; III. 31. and because in the right-angled △ ADC, DB is drawn from the rt. angle perp. to the hypotenuse,

∴ the △s ABD, DBC are similar; VI. 8.

∴ AB : BD :: BD : BC;

that is, BD is a mean proportional between AB and BC.

Q. E. F.

EXERCISES.

1. If from one angle A of a parallelogram a straight line be drawn cutting the diagonal in E and the sides in P, Q, shew that AE is a mean proportional between PE and EQ.

2. A, B, C are three points in order in a straight line: find a point P in the straight line so that PB may be a mean proportional between PA and PC.

3. The diameter AB of a semicircle is divided at any point C, and CD is drawn at right angles to AB meeting the circumference in D; DO is drawn to the centre, and CE is perpendicular to OD: shew that DE is a third proportional to AO and DC.

4. AC is the diameter of a semicircle on which a point B is taken so that BC is equal to the radius: shew that AB is a mean proportional between BC and the sum of BC, CA.

5. A is any point in a semicircle on BC as diameter; from D any point in BC a perpendicular is drawn meeting AB, AC, and the circumference in E, G, F respectively; shew that DG is a third proportional to DE and DF.

6. Two circles touch externally, and a common tangent touches them at A and B: prove that AB is a mean proportional between the diameters of the circles. [See Ex. 21, p. 219.]

7. If a straight line be divided in two given points, determine a third point such that its distances from the extremities may be proportional to its distances from the given points.

8. AB is a straight line divided at C and D so that AB, AC, AD are in continued proportion; from A a line AE is drawn in any direction and equal to AC; shew that BC and CD subtend equal angles at E.

9. In a given triangle draw a straight line parallel to one of the sides, so that it may be a mean proportional between the segments of the base.

10. On the radius OA of a quadrant OAB, a semicircle ODA is described, and at A a tangent AE is drawn; from O any line ODFE is drawn meeting the circumferences in D and F and the tangent in E: if DG is drawn perpendicular to OA, shew that OE, OF, OD, and OG are in continued proportion.

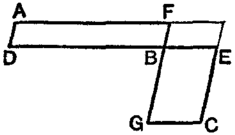
11. From any point A, in the circumference of the circle ABE, as centre, and with any radius, a circle BDC is described cutting the former circle in B and C; from A any line AFE is drawn meeting the chord BC in F, and the circumferences BDC, ABE in D, E respectively: shew that AD is a mean proportional between AF and AE.

Definition. Two figures are said to have their sides about two of their angles **reciprocally proportional**, when a side of the first is to a side of the second as the remaining side of the second is to the remaining side of the first.
[Book VI. Def. 3.]

Proposition 14. Theorem.

Parallelograms which are equal in area, and which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional:

Conversely, parallelograms which have one angle of the one equal to one angle of the other, and the sides about these angles reciprocally proportional, are equal in area.



Let the par^{ms} AB, BC be of equal area, and have the ∠ DBF equal to the ∠ GBE:
then shall the sides about these equal angles be reciprocally proportional,

that is, DB : BE :: GB : BF.

Place the par^{ms} so that DB, BE may be in the same straight line;

∴ FB, BG are also in one straight line. I. 14.

Complete the par^{m} FE.

Then because the par^{m} AB = the par^{m} BC, *Hyp.*
and FE is another par^{m},
∴ the par^{m} AB : the par^{m} FE :: the par^{m} BC : the par^{m} FE;
but the par^{m} AB : the par^{m} FE :: DB : BE, VI. 1.
and the par^{m} BC : the par^{m} FE :: GB : BF,
∴ DB : BE :: GB : BF. V. 1.

Conversely, let the ∠ DBF be equal to the ∠ GBE,
and let DB : BE :: GB : BF.
Then shall the par^{m} AB be equal in area to the par^{m} BC.

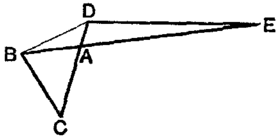
For, with the same construction as before,
by hypothesis DB : BE :: GB : BF;
but DB : BE :: the par^{m} AB : the par^{m} FE, VI. 1.
and GB : BF :: the par^{m} BC : the par^{m} FE,
∴ the par^{m} AB : the par^{m} FE :: the par^{m} BC : the par^{m} FE; V. 1.
∴ the par^{m} AB = the par^{m} BC.

Q. E. D.

PROPOSITION 15. THEOREM.

Triangles which are equal in area, and which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional:

Conversely, triangles which have one angle of the one equal to one angle of the other, and the sides about these angles reciprocally proportional, are equal in area.



Let the $\triangle^s$ ABC, ADE be of equal area, and have the $\angle$ CAB equal to the $\angle$ EAD:
then shall the sides of the triangles about these angles be reciprocally proportional,

that is, CA : AD :: EA : AB.

Place the $\triangle^s$ so that CA and AD may be in the same st. line;
∴ BA, AE are also in one st. line. I. 14.

Join BD.

Then because the $\triangle$ CAB = the $\triangle$ EAD, *Hyp.*
and ABD is another triangle;
∴ the $\triangle$ CAB : the $\triangle$ ABD :: the $\triangle$ EAD : the $\triangle$ ABD;
but the $\triangle$ CAB : the $\triangle$ ABD :: CA : AD, VI. 1.
and the $\triangle$ EAD : the $\triangle$ ABD :: EA : AB,
∴ CA : AD :: EA : AB. V. 1.

Conversely, let the $\angle$ CAB be equal to the $\angle$ EAD,
and let CA : AD :: EA : AB.
Then shall the $\triangle$ CAB = $\triangle$ EAD.

For, with the same construction as before,
by hypothesis CA : AD :: EA : AB;
but CA : AD :: the $\triangle$ CAB : the $\triangle$ ABD, VI. 1.
and EA : AB :: the $\triangle$ EAD : the $\triangle$ ABD,
∴ the $\triangle$ CAB : the $\triangle$ ABD :: the $\triangle$ EAD : the $\triangle$ ABD; V. 1.
∴ the $\triangle$ CAB = the $\triangle$ EAD. Q. E. D.

EXERCISES.

ON PROPOSITIONS 14 AND 15.

1. *Parallelograms which are equal in area and which have their sides reciprocally proportional, have their angles respectively equal.*

2. *Triangles which are equal in area, and which have the sides about a pair of angles reciprocally proportional, have those angles equal or supplementary.*

3. AC, BD are the diagonals of a trapezium which intersect in O; if the side AB is parallel to CD, use Prop. 15 to prove that the triangle AOD is equal to the triangle BOC.

4. From the extremities A, B of the hypotenuse of a right-angled triangle ABC lines AE, BD are drawn perpendicular to AB, and meeting BC and AC produced in E and D respectively: employ Prop. 15 to shew that the triangles ABC, ECD are equal in area.

5. On AB, AC, two sides of any triangle, squares are described externally to the triangle. If the squares are ABDE, ACFG, shew that the triangles DAG, FAE are equal in area.

6. ABCD is a parallelogram; from A and C any two parallel straight lines are drawn meeting DC and AB in E and F respectively; EG, which is parallel to the diagonal AC, meets AD in G: shew that the triangles DAF, GAB are equal in area.

7. Describe an isosceles triangle equal in area to a given triangle and having its vertical angle equal to one of the angles of the given triangle.

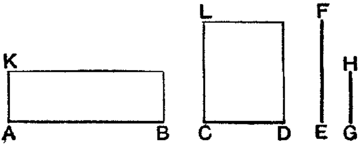
8. Prove that the equilateral triangle described on the hypotenuse of a right-angled triangle is equal to the sum of the equilateral triangles described on the sides containing the right angle.

[Let ABC be the triangle right-angled at C; and let BXC, CYA, AZB be the equilateral triangles. Draw CD perpendicular to AB; and join DZ. Then shew by Prop. 15 that the $\triangle$AYC = the $\triangle$DAZ; and similarly that the $\triangle$BXC = the $\triangle$BDZ.]

Proposition 16. Theorem.

If four straight lines are proportional, the rectangle contained by the extremes is equal to the rectangle contained by the means:

Conversely, if the rectangle contained by the extremes is equal to the rectangle contained by the means, the four straight lines are proportional.



Let the st. lines AB, CD, EF, GH be proportional, so that
AB : CD :: EF : GH.
Then shall the rect. AB, GH = the rect. CD, EF.

From A draw AK perp. to AB, and equal to GH. I. 11, 3.
From C draw CL perp. to CD, and equal to EF.
Complete the par^ms KB, LD.

Then because AB : CD :: EF : GH; *Hyp.*
and EF = CL, and GH = AK; *Constr.*
∴ AB : CD :: CL : AK;
that is, the sides about equal angles of par^ms KB, LD are reciprocally proportional;
∴ KB = LD. VI. 14.
But KB is the rect. AB, GH, for AK = GH, *Constr.*
and LD is the rect. CD, EF, for CL = EF;
∴ the rect. AB, GH = the rect. CD, EF.

Conversely, let the rect. AB, GH = the rect. CD, EF:
then shall AB : CD :: EF : GH.
For, with the same construction as before,
because the rect. AB, GH = the rect. CD, EF; *Hyp.*
and the rect. AB, GH = KB, for GH = AK, *Constr.*
and the rect. CD, EF = LD, for EF = CL;
∴ KB = LD;

that is, the par^{ms} KB, LD, which have the angle at A equal to the angle at C, are equal in area;

∴ the sides about the equal angles are reciprocally proportional:

that is, AB : CD :: CL : AK;

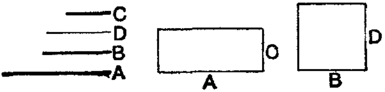
∴ AB : CD :: EF : GH.

Q. E. D.

PROPOSITION 17. THEOREM.

If three straight lines are proportional the rectangle contained by the extremes is equal to the square on the mean:

Conversely, if the rectangle contained by the extremes is equal to the square on the mean, the three straight lines are proportional.



Let the three st. lines A, B, C be proportional, so that

A : B :: B : C.

Then shall the rect. A, C be equal to the sq. on B.

Take D equal to B.

Then because A : B :: B : C, and D = B;

∴ A : B :: D : C;

∴ the rect. A, C = the rect. B, D; VI. 16.

but the rect. B, D = the sq. on B, for D = B;

∴ the rect. A, C = the sq. on B.

Conversely, let the rect. A, C = the sq. on B:

then shall A : B :: B : C.

For, with the same construction as before,

because the rect. A, C = the sq. on B, *Hyp.*

and the sq. on B = the rect. B, D, for D = B;

∴ the rect. A, C = the rect. B, D,

∴ A : B :: D : C, VI. 16.

that is, A : B :: B : C.

Q. E. D.

EXERCISES.

ON PROPOSITIONS 16 AND 17.

1. Apply Proposition 16 to prove that if two chords of a circle intersect, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.

2. Prove that the rectangle contained by the sides of a right-angled triangle is equal to the rectangle contained by the hypotenuse and the perpendicular on it from the right angle.

3. On a given straight line construct a rectangle equal to a given rectangle.

4. ABCD is a parallelogram; from B any straight line is drawn cutting the diagonal AC at F, the side DC at G, and the side AD produced at E: shew that the rectangle EF, FG is equal to the square on BF.

5. On a given straight line as base describe an isosceles triangle equal to a given triangle.

6. AB is a diameter of a circle, and any line ACD cuts the circle in C and the tangent at B in D; shew by Prop. 17 that the rectangle AC, AD is constant.

7. The exterior angle A of a triangle ABC is bisected by a straight line which meets the base in D and the circumscribed circle in E: shew that the rectangle BA, AC is equal to the rectangle EA, AD.

8. If two chords AB, AC drawn from any point A in the circumference of the circle ABC be produced to meet the tangent at the other extremity of the diameter through A in D and E, shew that the triangle AED is similar to the triangle ABC.

9. At the extremities of a diameter of a circle tangents are drawn; these meet the tangent at a point P in Q and R: shew that the rectangle QP, PR is constant for all positions of P.

10 A is the vertex of an isosceles triangle ABC inscribed in a circle, and ADE is a straight line which cuts the base in D and the circle in E; shew that the rectangle EA, AD is equal to the square on AB.

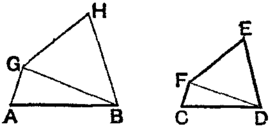
11. Two circles touch one another externally in A; a straight line touches the circles at B and C, and is produced to meet the straight line joining their centres at S: shew that the rectangle SB, SC is equal to the square on SA.

12. Divide a triangle into two equal parts by a straight line at right angles to one of the sides.

DEFINITION. Two similar rectilineal figures are said to be **similarly situated** with respect to two of their sides when these sides are homologous. [Book VI. Def. 5.]

PROPOSITION 18. PROBLEM.

On a given straight line to describe a rectilineal figure similar and similarly situated to a given rectilineal figure.



Let AB be the given st. line, and CDEF the given rectil. figure: first suppose CDEF to be a quadrilateral.

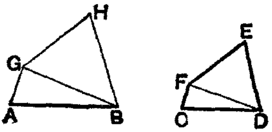
It is required to describe on the st. line AB, a rectil. figure similar and similarly situated to CDEF.

Join DF.

At A in BA make the ∠ BAG equal to the ∠ DCF, I. 23.
and at B in AB make the ∠ ABG equal to the ∠ CDF;
∴ the remaining ∠ AGB = the remaining ∠ CFD; I. 32.
and the △AGB is equiangular to the △CFD.

Again at B in GB make the ∠ GBH equal to the ∠ FDE,
and at G in BG make the ∠ BGH equal to the ∠ DFE; I. 23.
∴ the remaining ∠ BHG = the remaining ∠ DEF; I. 32.
and the △BHG is equiangular to the △DEF.
Then shall ABHG be the required figure.

(i) To prove that the quadrilaterals are equiangular.
Because the ∠ AGB = the ∠ CFD,
and the ∠ BGH = the ∠ DFE; *Constr.*
∴ the whole ∠ AGH = the whole ∠ CFE. *Ax.* 2.
Similarly the ∠ ABH = the ∠ CDE;
and the angles at A and H are respectively equal to the angles at C and E; *Constr.*
∴ the fig. ABHG is equiangular to the fig. CDEF.



(ii) To prove that the quadrilaterals have the sides about their equal angles proportional.

Because the $\triangle^s$ BAG, DCF are equiangular;

$\therefore$ AG : GB :: CF : FD. VI. 4.

And because the $\triangle^s$ BGH, DFE are equiangular;

$\therefore$ BG : GH :: DF : FE,

$\therefore$, *ex æquali*, AG : GH :: CF : FE. V. 14.

Similarly it may be shewn that

AB : BH :: CD : DE.

Also BA : AG :: DC : CF, VI. 4.

and GH : HB :: FE : ED;

$\therefore$ the figs. ABHG, CDEF have their sides about the equal angles proportional;

$\therefore$ ABHG is similar to CDEF. *Def.* 2.

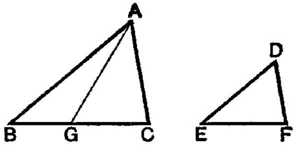
In like manner the process of construction may be extended to a figure of five or more sides.

Q. E. F.

Definition. When three magnitudes are proportionals the first is said to have to the third the **duplicate ratio** of that which it has to the second. [Book V. Def. 13.]

PROPOSITION 19. THEOREM.

Similar triangles are to one another in the duplicate ratio of their homologous sides.



Let ABC, DEF be similar triangles, having the ∠ ABC equal to the ∠ DEF, and let BC and EF be homologous sides: then shall the △ABC be to the △DEF in the duplicate ratio of BC to EF.

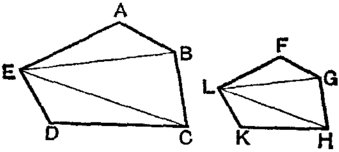
To BC and EF take a third proportional BG,
so that BC : EF :: EF : BG. VI. 11.
Join AG.

Then because the △s ABC, DEF are similar, *Hyp.*
∴ AB : BC :: DE : EF;
∴, *alternately*, AB : DE :: BC : EF; V. 11.
but BC : EF :: EF : BG; *Constr.*
∴ AB : DE :: EF : BG; V. 1.
that is, the sides of the △s ABG, DEF about the equal angles at B and E are reciprocally proportional;
∴ the △ABG = the △DEF. VI. 15.

Again, because BC : EF :: EF : BG, *Constr.*
∴ BC : BG in the duplicate ratio of BC to EF. *Def.*
But the △ABC : the △ABG :: BC : BG, VI. 1.
∴ the △ABC : the △ABG in the duplicate ratio of BC to EF: V. 1.
and the △ABG = the △DEF; *Proved.*
∴ the △ABC : the △DEF in the duplicate ratio of BC : EF.
Q.E.D.

PROPOSITION 20. THEOREM.

Similar polygons may be divided into the same number of similar triangles, having the same ratio each to each that the polygons have; and the polygons are to one another in the duplicate ratio of their homologous sides.



Let ABCDE, FGHKL be similar polygons, and let AB be the side homologous to FG;

then (i) the polygons may be divided into the same number of similar triangles;

(ii) these triangles shall have each to each the same ratio that the polygons have;

(iii) the polygon ABCDE shall be to the polygon FGHKL in the duplicate ratio of AB to FG.

Join EB, EC, LG, LH.

(i) Then because the polygon ABCDE is similar to the polygon FGHKL, *Hyp.*

∴ the ∠ EAB = the ∠ LFG,

and EA : AB :: LF : FG; VI. *Def.* 2.

∴ the △ EAB is similar to the △ LFG; VI. 6.

∴ the ∠ ABE = the ∠ FGL.

But, because the polygons are similar, *Hyp.*

∴ the ∠ ABC = the ∠ FGH, VI. *Def.* 2.

∴ the remaining ∠ EBC = the remaining ∠ LGH.

And because the △s ABE, FGL are similar, *Proved.*

∴ EB : BA :: LG : GF;

and because the polygons are similar, *Hyp.*

∴ AB : BC = FG : GH; VI. *Def.* 2.

∴, *ex æquali*, EB : BC :: LG : GH, V. 14.

that is, the sides about the equal ∠s EBC, LGH are proportionals;

∴ the △ EBC is similar to the △ LGH. VI. 6.

In the same way it may be proved that the △ECD is similar to the △LHK.

∴ the polygons have been divided into the same number of similar triangles.

(ii) Again, because the △ABE is similar to the △FGL,

∴ the △ABE is to the △FGL in the duplicate ratio of EB : LG; VI. 19.

and, in like manner,

the △EBC is to the △LGH in the duplicate ratio of EB to LG;

∴ the △ABE : the △FGL :: the △EBC : the △LGH. V. 1.

In like manner it can be shewn that

the △EBC : the △LGH :: the △EDC : the △LKH.

∴ the △ABE : the △FGL :: the △EBC : the △LGH

:: the △EDC : the △LKH.

But when any number of ratios are equal, as each antecedent is to its consequent so is the sum of all the antecedents to the sum of all the consequents; V. 12.

∴ the △ABE : the △LFG :: the fig. ABCDE : the fig. FGHKL.

(iii) Now the △EAB : the △LFG in the duplicate ratio of AB : FG,

and the △EAB : the △LFG :: the fig. ABCDE : the fig. FGHKL;

∴ the fig. ABCDE : the fig. FGHKL in the duplicate ratio of AB : FG. Q.E.D.

COROLLARY 1. Let a third proportional X be taken to AB and FG,

then AB is to X in the duplicate ratio of AB : FG;

but the fig. ABCDE : the fig. FGHKL in the duplicate ratio of AB : FG.

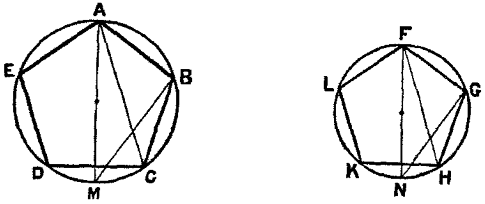
Hence, *if three straight lines are proportionals, as the first is to the third, so is any rectilineal figure described on the first to a similar and similarly described rectilineal figure on the second.*

COROLLARY 2. It follows that *similar rectilineal figures are to one another as the squares on their homologous sides.* For squares are similar figures and therefore are to one another in the duplicate ratio of their sides.

NOTE. The following theorem, taken from Euclid's Twelfth Book, is an important application of the preceding proposition.

BOOK XII. PROPOSITION 1.

The areas of similar polygons inscribed in circles are to one another as the squares on their diameters.



Let ABCDE and FGHKL be two similar polygons, inscribed in the circles ACE, FHL, of which AM, FN are diameters: then shall

the fig. ABCDE : the fig. FGHKL :: the sq. on AM : the sq. on FN.

Join BM, AC and GN, FH.

Then since the polygon ABCDE is similar to the polygon FGHKL,

∴ the ∠ABC = the ∠FGH,

and AB : BC :: FG : GH;

∴ the △ ABC is similar to the △ FGH; vi. 6.

∴ the ∠ACB = the ∠FHG.

But the ∠ACB = the ∠AMB; iii. 21.

and the ∠FHG = the ∠FNG;

∴ the ∠AMB = the ∠FNG.

Also in the △s ABM, FGN, the ∠s ABM, FGN are equal, being rt. angles; iii. 31.

hence the remaining ∠s BAM, GFN are equal; i. 32.

and the △s ABM, FGN are similar: vi. 4.

∴ AB : FG :: AM : FN.

But the fig. ABCDE : the fig. FGHKL in the duplicate ratio of AB : FG, vi. 20.

that is, in the duplicate ratio of AM : FN. v. 16.

Hence

the fig. ABCDE : the fig. FGHKL :: the sq. on AM : the sq. on FN. vi. 20, *Cor.* 2.

The following theorem, which forms the 3rd proposition of Euclid's Twelfth Book, may be derived as a corollary from the preceding proof.

COROLLARY. *The areas of circles are to one another as the squares on their diameters.*

It has been shewn that

the fig. ABCDE : the fig. FGHKL :: the sq. on AM : the sq. on FN : and this is true however many sides the two polygons may have.

Suppose the polygons are *regular*; then by sufficiently increasing the number of their sides, we may make their areas differ from the areas of their circumscribed circles by quantities smaller than any that can be named; hence ultimately the area of the ⊙ ACE : the area of the ⊙ FHL :: the sq. on AM : the sq. on FN.

EXERCISES ON PROPOSITIONS 19, 20.

1. If ABC is a triangle right-angled at A, and AD is drawn perpendicular to BC, shew that

   (i) CB : BD in the duplicate ratio of CB to BA;

   (ii) The square on CB : the square on BA :: CB : BD;

   (iii) The △ABD : the △CAD in the duplicate ratio of BA to AC.

2. In any triangle ABC, the sides AB, AC are cut by a line XY drawn parallel to BC. If AX is one-third of AB, what part is the triangle AXY of the triangle ABC?

3. A trapezium ABCD has its sides AB, CD parallel, and its diagonals intersect at O. If AB is double of CD, find the ratio of the triangle AOB to the triangle COD.

4. ABC and XYZ are two similar triangles whose areas are respectively 245 and 5 square inches. If AB is 21 inches in length, find XY.

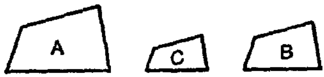
5. Shew how to draw a straight line XY parallel to the base BC of a triangle ABC, so that the area of the triangle AXY may be four-ninths of the triangle ABC.

6. Two circles intersect at A and B, and at A tangents are drawn, one to each circle, meeting the circumferences at C and D. If AB, CB and BD are joined, shew that

   the △CBA : the △ABD :: CB : BD.

PROPOSITION 21. THEOREM.

Rectilineal figures which are similar to the same rectilineal figure, are also similar to each other.



Let each of the rectilineal figures A and B be similar to C: then shall A be similar to B.

For because A is similar to C, *Hyp.*
∴ A is equiangular to C, and the sides about their equal angles are proportionals. VI. *Def.* 2.

Again, because B is similar to C, *Hyp.*
∴ B is equiangular to C, and the sides about their equal angles are proportionals. VI. *Def.* 2.

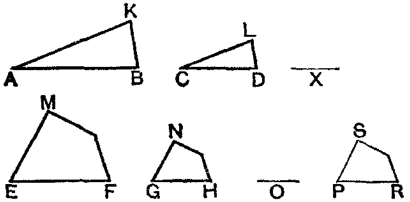
∴ A and B are each of them equiangular to C, and have the sides about the equal angles proportional to the corresponding sides of C;
∴ A is equiangular to B, and the sides about their equal angles are proportionals; V. 1.
∴ A is similar to B.

Q. E. D.

PROPOSITION 22. THEOREM.

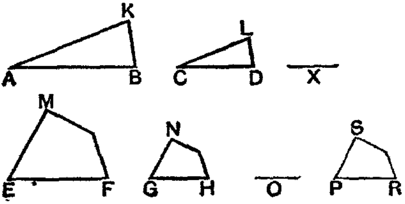
If four straight lines be proportional and a pair of similar rectilineal figures be similarly described on the first and second, and also a pair on the third and fourth, these figures shall be proportional:

Conversely, if a rectilineal figure on the first of four straight lines be to the similar and similarly described figure on the second as a rectilineal figure on the third is to the similar and similarly described figure on the fourth, the four straight lines shall be proportional.



Let AB, CD, EF, GH be proportionals,
so that AB : CD :: EF : GH;
and let similar figures KAB, LCD be similarly described on AB, CD, and also let similar figs. MF, NH be similarly described on EF, GH:
then shall
the fig. KAB : the fig. LCD :: the fig. MF : the fig. NH.

To AB and CD take a third proportional X, VI. 11.
and to EF and GH take a third proportional O;
then AB : CD :: CD : X, *Constr.*
and EF : GH :: GH : O.
But AB : CD :: EF : GH; *Hyp.*
∴ CD : X :: GH : O, V. 1.
∴, *ex æquali*, AB : X :: EF : O. V. 14.
But AB : X :: the fig. KAB : the fig. LCD, VI. 20, *Cor.*
and EF : O :: the fig. MF : the fig. NH;
∴ the fig. KAB : the fig. LCD :: the fig. MF : the fig. NH.
V. 1.



Conversely,

let the fig. KAB : the fig. LCD :: the fig. MF : the fig. NH;
then shall AB : CD :: EF : GH.

To AB, CD, and EF take a fourth proportional PR: vi. 12.
and on PR describe the fig. SR similar and similarly situated to either of the figs. MF, NH. vi. 18.

Then because AB : CD :: EF : PR, *Constr.*

∴, by the former part of the proposition,

the fig. KAB : the fig. LCD :: the fig. MF : the fig. SR.

But

the fig. KAB : the fig. LCD :: the fig. MF : the fig. NH. *Hyp.*

∴ the fig. MF : the fig. SR :: the fig. MF : the fig. NH, v. 1.

∴ the fig. SR = the fig. NH.

And since the figs. SR and NH are similar and similarly situated,

∴ PR = GH*.

Now AB : CD :: EF : PR; *Constr.*

∴ AB : CD :: EF : GH.

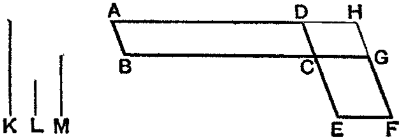
Q. E. D.

* Euclid here assumes that *if two similar and similarly situated figures are equal, their homologous sides are equal.* The proof is easy and may be left as an exercise for the student.

DEFINITION. When there are any number of magnitudes of the same kind, the first is said to have to the last the **ratio compounded** of the ratios of the first to the second, of the second to the third, and so on up to the ratio of the last but one to the last magnitude. [Book v. Def. 12.]

PROPOSITION 23. THEOREM.

Parallelograms which are equiangular to one another have to one another the ratio which is compounded of the ratios of their sides.



Let the par^m AC be equiangular to the par^m CF, having the $\angle$ BCD equal to the $\angle$ ECG:
then shall the par^m AC have to the par^m CF the ratio compounded of the ratios BC : CG and DC : CE.

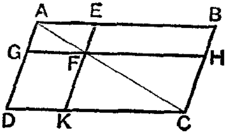
Let the par^{ms} be placed so that BC and CG are in a st. line;
then DC and CE are also in a st. line. I. 14.
Complete the par^m DG.
Take any st. line K,
and to BC, CG, and K find a fourth proportional L; VI. 12.
and to DC, CE, and L take a fourth proportional M;
then BC : CG :: K : L,
and DC : CE :: L : M.
But K : M is the ratio compounded of the ratios
K : L and L : M, V. *Def.* 12.
that is, K : M is the ratio compounded of the ratios
BC : CG and DC : CE.
Now the par^m AC : the par^m CH :: BC : CG VI. 1.
:: K : L, *Constr.*
and the par^m CH : the par^m CF :: DC : CE VI. 1.
:: L : M, *Constr.*
$\therefore$, *ex æquali*, the par^m AC : the par^m CF :: K : M. V. 14.
But K : M is the ratio compounded of the ratios of the sides;
$\therefore$ the par^m AC has to the par^m CF the ratio compounded of the ratios of the sides. Q.E.D.

EXERCISE.

The areas of two triangles or parallelograms are to one another in the ratio compounded of the ratios of their bases and of their altitudes.

PROPOSITION 24. THEOREM.

Parallelograms about a diagonal of any parallelogram are similar to the whole parallelogram and to one another.



Let ABCD be a par^m of which AC is a diagonal;
and let EG, HK be par^{ms} about AC:
then shall the par^{ms} EG, HK be similar to the par^m ABCD, and to one another.

For, because DC is par^l to GF,
∴ the ∠ ADC = the ∠ AGF; I. 29.
and because BC is par^l to EF,
∴ the ∠ ABC = the ∠ AEF; I. 29.
and each of the ∠^{s} BCD, EFG is equal to the opp. ∠ BAD,
∴ the ∠ BCD = the ∠ EFG; [I. 34.
∴ the par^m ABCD is equiangular to the par^m AEFG.

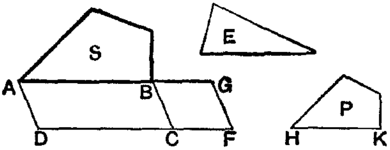
Again in the △^{s} BAC, EAF,
because the ∠ ABC = the ∠ AEF, I. 29.
and the ∠ BAC is common;
∴ △^{s} BAC, EAF are equiangular to one another; I. 32.
∴ AB : BC :: AE : EF. VI. 4.
But BC = AD, and EF = AG; I. 34.
∴ AB : AD :: AE : AG;
and DC : CB :: GF : FE,
and CD : DA :: FG : GA,
∴ the sides of the par^{ms} ABCD, AEFG about their equal angles are proportional;
∴ the par^m ABCD is similar to the par^m AEFG. VI. *Def.* 2.

In the same way it may be proved that the par^m ABCD is similar to the par^m FHCK,
∴ each of the par^{ms} EG, HK is similar to the whole par^m;
∴ the par^m EG is similar to the par^m HK. VI. 21.

Q. E. D.

Proposition 25. Problem.

To describe a rectilineal figure which shall be equal to one and similar to another rectilineal figure.



Let E and S be two rectilineal figures:
it is required to describe a figure equal to the fig. E and similar to the fig. S.

On AB a side of the fig. S describe a par^m ABCD equal to S, and on BC describe a par^m CBGF equal to the fig. E, and having the ∠ CBG equal to the ∠ DAB: I. 45.

then AB and BG are in one st. line, and also DC and CF in one st. line.

Between AB and BG find a mean proportional HK; VI. 13. and on HK describe the fig. P, similar and similarly situated to the fig. S: VI. 18.

then P shall be the figure required.

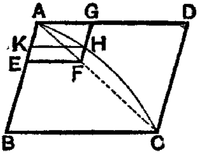
Because AB : HK :: HK : BG, *Constr.*
∴ AB : BG :: the fig. S : the fig. P. VI. 20, *Cor.*
But AB : BG :: the par^m AC : the par^m BF;
∴ the fig. S : the fig. P :: the par^m AC : the par^m BF; V. I.
and the fig. S = the par^m AC; *Constr.*
∴ the fig. P = the par^m BF
= the fig. E. *Constr.*

And since, by construction, the fig. P is similar to the fig. S,
∴ P is the rectil. figure required.

Q. E. F.

PROPOSITION 26. THEOREM.

If two similar parallelograms have a common angle, and be similarly situated, they are about the same diagonal.



Let the par^ms ABCD, AEFG be similar and similarly situated, and have the common angle BAD:

then shall these par^ms be about the same diagonal.

Join AC.

Then if AC does not pass through F, let it cut FG, or FG produced, at H.

Through H draw HK par^l to AD or BC. I. 31.

Then the par^ms BD and KG are similar, since they are about the same diagonal AHC; VI. 24.

∴ DA : AB :: GA : AK.

But because the par^ms BD and EG are similar; *Hyp.*

∴ DA : AB :: GA : AE; VI. *Def.* 2.

∴ GA : AK :: GA : AE;

∴ AK = AE, which is impossible;

∴ AC must pass through F;

that is, the par^ms BD, EG are about the same diagonal.

Q.E.D.

Obs. Propositions 27, 28, 29 being cumbrous in form and of little value as geometrical results are now very generally omitted.

DEFINITION. A straight line is said to be divided **in extreme and mean ratio,** when the whole is to the greater segment as the greater segment is to the less.

[Book VI. Def. 4.]

PROPOSITION 30. PROBLEM.

To divide a given straight line in extreme and mean ratio.

Let AB be the given st. line:
it is required to divide it in extreme and mean ratio.

Divide AB in C so that the rect. AB, BC may be equal to the sq. on AC. II. 11.

Then because the rect. AB, BC = the sq. on AC,

∴ AB : AC :: AC : BC. VI. 17.

Q. E. F.

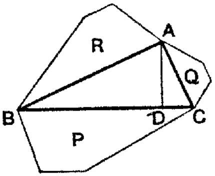
EXERCISES.

1. ABCDE is a regular pentagon; if the lines BE and AD intersect in O, shew that each of them is divided in extreme and mean ratio.

2. If the radius of a circle is cut in extreme and mean ratio, the greater segment is equal to the side of a regular decagon inscribed in the circle.

Proposition 31. Theorem.

In a right-angled triangle, any rectilineal figure described on the hypotenuse is equal to the sum of the two similar and similarly described figures on the sides containing the right angle.



Let ABC be a right-angled triangle of which BC is the hypotenuse; and let P, Q, R be similar and similarly described figures on BC, CA, AB respectively:
then shall the fig. P be equal to the sum of the figs. Q and R.

Draw AD perp. to BC.

Then the $\triangle^s$ CBA, ABD are similar; VI. 8.
∴ CB : BA :: BA : BD;
∴ CB : BD :: the fig. P : the fig. R, VI. 20, *Cor.*
∴, *inversely*, BD : BC :: the fig. R : the fig. P. V. 2.
In like manner DC : BC :: the fig. Q : the fig. P;
∴ the sum of BD, DC : BC :: the sum of figs. R, Q : fig. P; V. 15.
but BC = the sum of BD, DC;
∴ the fig. P = the sum of the figs. R and Q.

Q.E.D.

NOTE. This proposition is a generalization of the 47th Prop. of Book I. It will be a useful exercise for the student to deduce the general theorem from the particular case with the aid of Prop. 20, Cor. 2.

EXERCISES.

1. In a right-angled triangle if a perpendicular be drawn from the right angle to the opposite side, the segments of the hypotenuse are in the duplicate ratio of the sides containing the right angle.

2. If, in Proposition 31, the figure on the hypotenuse is equal to the given triangle, the figures on the other two sides are each equal to one of the parts into which the triangle is divided by the perpendicular from the right angle to the hypotenuse.

3. AX and BY are medians of the triangle ABC which meet in G: if XY be joined, compare the areas of the triangles AGB, XGY.

4. *Shew that similar triangles are to one another in the duplicate ratio of* (i) *corresponding medians,* (ii) *the radii of their inscribed circles,* (iii) *the radii of their circumscribed circles.*

5. DEF is the pedal triangle of the triangle ABC; prove that the triangle ABC is to the triangle DBF in the duplicate ratio of AB to BD. Hence shew that

the fig. AFDC : the △ BFD :: $AD^2 : BD^2$.

6. The base BC of a triangle ABC is produced to a point D such that BD : DC in the duplicate ratio of BA : AC. Shew that AD is a mean proportional between BD and DC.

7. Bisect a triangle by a line drawn parallel to one of its sides.

8. Shew how to draw a line parallel to the base of a triangle so as to form with the other two sides produced a triangle double of the given triangle.

9. If through any point within a triangle lines be drawn from the angles to cut the opposite sides, the segments of any one side will have to each other the ratio compounded of the ratios of the segments of the other sides.

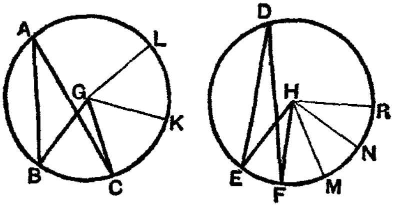
10. Draw a straight line parallel to the base of an isosceles triangle so as to cut off a triangle which has to the whole triangle the ratio of the base to a side.

11. Through a given point, between two straight lines containing a given angle, draw a line which shall cut off a triangle equal to a given rectilineal figure.

Obs. The 32nd Proposition as given by Euclid is defective, and as it is never applied, we have omitted it.

PROPOSITION 33. THEOREM.

In equal circles, angles, whether at the centres or the circumferences, have the same ratio as the arcs on which they stand: so also have the sectors.



Let ABC and DEF be equal circles, and let BGC, EHF be angles at the centres, and BAC and EDF angles at the ○^{ces}; then shall

(i) the ∠ BGC : the ∠ EHF :: the arc BC : the arc EF,
(ii) the ∠ BAC : the ∠ EDF :: the arc BC : the arc EF,
(iii) the sector BGC : the sector EHF :: the arc BC : the arc EF.

Along the ○^{ce} of the ⊙ABC take any number of arcs CK, KL each equal to BC; and along the ○^{ce} of the ⊙DEF take any number of arcs FM, MN, NR each equal to EF.

Join GK, GL, HM, HN, HR.

(i) Then the ∠^{s} BGC, CGK, KGL are all equal,
for they stand on the equal arcs BC, CK, KL: III. 27.
∴ the ∠ BGL is the same multiple of the ∠ BGC that the arc BL is of the arc BC.

Similarly the ∠ EHR is the same multiple of the ∠ EHF that the arc ER is of the arc EF.

And if the arc BL = the arc ER,
the ∠ BGL = the ∠ EHR; III. 27.
and if the arc BL is greater than the arc ER,
the ∠ BGL is greater than the ∠ EHR;
and if the arc BL is less than the arc ER,
the ∠ BGL is less than the ∠ EHR.

Now since there are four magnitudes, namely the ∠^{s} BGC, EHF and the arcs BC, EF; and of the antecedents any equimultiples have been taken, namely the ∠ BGL and the arc BL; and of the consequents any equimultiples have been taken, namely the ∠ EHR and the arc ER: and it has been proved that the ∠ BGL is greater than, equal to, or less than the ∠ EHR according as BL is greater than, equal to, or less than ER;

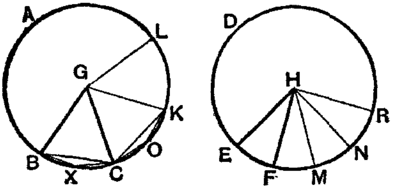
∴ the four magnitudes are proportionals; v. *Def.* 4.

that is, the ∠ BGC : the ∠ EHF :: the arc BC : the arc EF.

(ii) And since the ∠ BGC = twice the ∠ BAC, III. 20.

and the ∠ EHF = twice the ∠ EDF;

∴ the ∠ BAC : the ∠ EDF :: the arc BC : the arc EF. v. 8.



(iii) Join BC, CK; and in the arcs BC, CK take any points X, O.

Join BX, XC, CO, OK.

Then in the △^{s} BGC, CGK,

Because { BG = CG, GC = GK, and the ∠ BGC = the ∠ CGK; III. 27.

∴ BC = CK; I. 4.

and the △ BGC = the △ CGK.

And because the arc BC = the arc CK, *Constr.*

∴ the remaining arc BAC = the remaining arc CAK:

∴ the ∠ BXC = the ∠ COK; III. 27.

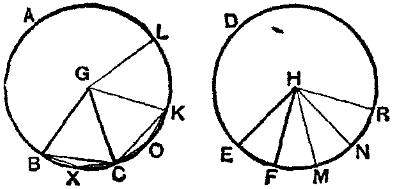
∴ the segment BXC is similar to the segment COK; III. *Def.*

and they stand on equal chords BC, CK;

∴ the segment BXC = the segment COK. III. 24.

And the △ BGC = the △ CGK;

∴ the sector BGC = the sector CGK.



Similarly it may be shewn that the sectors BGC, CGK, KGL are all equal;
and likewise the sectors EHF, FHM, MHN, NHR are all equal.
∴ the sector BGL is the same multiple of the sector BGC that the arc BL is of the arc BC;
and the sector EHR is the same multiple of the sector EHF that the arc ER is of the arc EF:

And if the arc BL = the arc ER,
the sector BGL = the sector EHR: *Proved.*
and if the arc BL is greater than the arc ER,
the sector BGL is greater than the sector EHR:
and if the arc BL is less than the arc ER,
the sector BGL is less than the sector EHR.

Now since there are four magnitudes, namely, the sectors BGC, EHF and the arcs BC, EF; and of the antecedents any equimultiples have been taken, namely the sector BGL and the arc BL; and of the consequents any equimultiples have been taken, namely the sector EHR and the arc ER: and it has been shewn that the sector BGL is greater than, equal to, or less than the sector EHR according as the arc BL is greater than, equal to, or less than the arc ER;

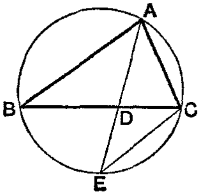
∴ the four magnitudes are proportionals; v. *Def.* 4.

that is,
the sector BGC : the sector EHF :: the arc BC : the arc EF.

Q. E. D.

Proposition B. Theorem.

If the vertical angle of a triangle be bisected by a straight line which cuts the base, the rectangle contained by the sides of the triangle shall be equal to the rectangle contained by the segments of the base, together with the square on the straight line which bisects the angle.



Let ABC be a triangle having the ∠ BAC bisected by AD, then shall

the rect. BA, AC = the rect. BD, DC, with the sq. on AD.

Describe a circle about the △ ABC, IV. 5.
and produce AD to meet the ○ce in E.
Join EC.

Then in the △s BAD, EAC,
because the ∠ BAD = the ∠ EAC, *Hyp.*
and the ∠ ABD = the ∠ AEC in the same segment; III. 21.
∴ the △ BAD is equiangular to the △ EAC. I. 32.
∴ BA : AD :: EA : AC; VI. 4.
∴ the rect. BA, AC = the rect. EA, AD, VI. 16.
= the rect. ED, DA, with the sq. on AD. II. 3.
But the rect. ED, DA = the rect. BD, DC; III. 35.
∴ the rect. BA, AC = the rect. BD, DC, with the sq. on AD.

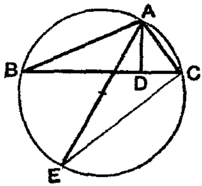
Q. E. D

EXERCISE.

If the vertical angle BAC be externally bisected by a straight line which meets the base in D, shew that the rectangle contained by BA, AC together with the square on AD is equal to the rectangle contained by the segments of the base.

PROPOSITION C. THEOREM.

If from the vertical angle of a triangle a straight line be drawn perpendicular to the base, the rectangle contained by the sides of the triangle shall be equal to the rectangle contained by the perpendicular and the diameter of the circle described about the triangle.



Let ABC be a triangle, and let AD be the perp. from A to BC:
then the rect. BA, AC shall be equal to the rect. contained by AD and the diameter of the circle circumscribed about the △ABC.

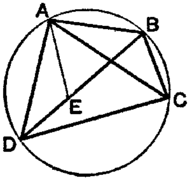
Describe a circle about the △ABC; IV. 5.
draw the diameter AE, and join EC.

Then in the △s BAD, EAC,
the rt. angle BDA = the rt. angle ACE, in the semicircle ACE,
and the ∠ ABD = the ∠ AEC, in the same segment; III. 21.
∴ the △BAD is equiangular to the △EAC; I. 32.
∴ BA : AD :: EA : AC; VI. 4.
∴ the rect. BA, AC = the rect. EA, AD. VI. 16.

Q.E.D.

PROPOSITION D. THEOREM.

The rectangle contained by the diagonals of a quadrilateral inscribed in a circle is equal to the sum of the two rectangles contained by its opposite sides.



Let ABCD be a quadrilateral inscribed in a circle, and let AC, BD be its diagonals:
then the rect. AC, BD shall be equal to the sum of the rectangles AB, CD and BC, AD.

Make the ∠ DAE equal to the ∠ BAC; I. 23.
to each add the ∠ EAC,
then the ∠ DAC = the ∠ BAE.

Then in the △s EAB, DAC,
the ∠ EAB = the ∠ DAC,
and the ∠ ABE = the ∠ ACD in the same segment; III. 21.
∴ the triangles are equiangular to one another; I. 32.
∴ AB : BE :: AC : CD; VI. 4.
∴ the rect. AB, CD = the rect. AC, EB. VI 16.

Again in the △s DAE, CAB,
the ∠ DAE = the ∠ CAB, *Constr.*
and the ∠ ADE = the ∠ ACB, in the same segment, III. 21.
∴ the triangles are equiangular to one another; I. 32.
∴ AD : DE :: AC : CB; VI. 4.
∴ the rect. BC, AD = the rect. AC, DE. VI. 16.
But the rect. AB, CD = the rect. AC, EB. *Proved.*
∴ the sum of the rects. BC, AD and AB, CD = the sum of the rects. AC, DE and AC, EB;
that is, the sum of the rects. BC, AD and AB, CD
= the rect. AC, BD. II. 1.

Q. E. D.

NOTE. Propositions B, C, and D do not occur in Euclid, but were added by Robert Simson.

Prop. D is usually known as Ptolemy's theorem, and it is the particular case of the following more general theorem:

The rectangle contained by the diagonals of a quadrilateral is less than the sum of the rectangles contained by its opposite sides, unless a circle can be circumscribed about the quadrilateral, in which case it is equal to that sum.

EXERCISES.

1. ABC is an isosceles triangle, and on the base, or base produced, any point X is taken: shew that the circumscribed circles of the triangles ABX, ACX are equal.

2. From the extremities B, C of the base of an isosceles triangle ABC, straight lines are drawn perpendicular to AB, AC respectively, and intersecting at D: shew that the rectangle BC, AD is double of the rectangle AB, DB.

3. If the diagonals of a quadrilateral inscribed in a circle are at right angles, the sum of the rectangles of the opposite sides is double the area of the figure.

4. ABCD is a quadrilateral inscribed in a circle, and the diagonal BD bisects AC: shew that the rectangle AD, AB is equal to the rectangle DC, CB.

5. If the vertex A of a triangle ABC be joined to any point in the base, it will divide the triangle into two triangles such that their circumscribed circles have radii in the ratio of AB to AC.

6. Construct a triangle, having given the base, the vertical angle, and the rectangle contained by the sides.

7. Two triangles of equal area are inscribed in the same circle: shew that the rectangle contained by any two sides of the one is to the rectangle contained by any two sides of the other as the base of the second is to the base of the first.

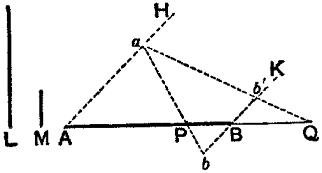
8. A circle is described round an equilateral triangle, and from any point in the circumference straight lines are drawn to the angular points of the triangle: shew that one of these straight lines is equal to the sum of the other two.

9. ABCD is a quadrilateral inscribed in a circle, and BD bisects the angle ABC: if the points A and C are fixed on the circumference of the circle and B is variable in position, shew that the sum of AB and BC has a constant ratio to BD.

THEOREMS AND EXAMPLES ON BOOK VI.

I. ON HARMONIC SECTION.

1. *To divide a given straight line internally and externally so that its segments may be in a given ratio.*



Let AB be the given st. line, and L, M two other st. lines which determine the given ratio: it is required to divide AB internally and externally in the ratio L : M.

Through A and B draw any two par[1] st. lines AH, BK.

From AH cut off Aa equal to L,

and from BK cut off Bb and Bb' each equal to M, Bb' being taken in the *same* direction as Aa, and Bb in the opposite direction.

Join ab, cutting AB in P;

join ab', and produce it to cut AB externally at Q.

Then AB is divided internally at P and externally at Q,

so that $$AP : PB = L : M,$$

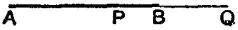
and $$AQ : QB = L : M.$$

The proof follows at once from Euclid VI. 4.

Obs. The solution is *singular;* that is, only *one* internal and *one* external point can be found that will divide the given straight line into segments which have the given ratio.

DEFINITION.

A finite straight line is said to be **cut harmonically** when it is divided internally and externally into segments which have the same ratio.



Thus AB is divided harmonically at P and Q, if

$$AP : PB = AQ : QB.$$

P and Q are said to be **harmonic conjugates** of A and B.

If P and Q divide AB internally and externally in the same ratio, it is easy to shew that A and B divide PQ internally and externally in the same ratio: hence A and B are harmonic conjugates of P and Q.

Example. The base of a triangle is divided harmonically by the internal and external bisectors of the vertical angle: for in each case the segments of the base are in the ratio of the other sides of the triangle. [Euclid VI. 3 and A.]

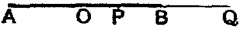
Obs. We shall use the terms *Arithmetic*, *Geometric*, and *Harmonic Means* in their ordinary Algebraical sense.

1. *If* AB *is divided internally at* P *and externally at* Q *in the same ratio, then* AB *is the harmonic mean between* AQ and AP.

For by hypothesis $AQ : QB = AP : PB$;
$\therefore$, *alternately*, $AQ : AP = QB : PB$,
that is, $AQ : AP = AQ - AB : AB - AP$,
which proves the proposition.

2. *If* AB *is divided harmonically at* P *and* Q, *and* O *is the middle point of* AB;

then shall $OP \cdot OQ = OA^2$.



For since AB is divided harmonically at P and Q,

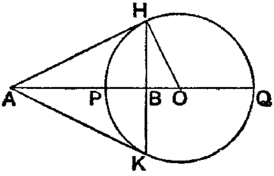
$$\therefore AP : PB = AQ : QB;$$
$$\therefore AP - PB : AP + PB = AQ - QB : AQ + QB,$$
or, $$2OP : 2OA = 2OA : 2OQ;$$
$$\therefore OP \cdot OQ = OA^2.$$

Conversely, if $OP \cdot OQ = OA^2$,
it may be shewn that

$$AP : PB = AQ : QB;$$

that is, that AB is divided harmonically at P and Q.

3. *The Arithmetic, Geometric and Harmonic means of two straight lines may be thus represented graphically.*



In the adjoining figure, two tangents AH, AK are drawn from any external point A to the circle PHQK; HK is the chord of contact, and the st. line joining A to the centre O cuts the $\bigcirc^{ce}$ at P and Q.

Then (i) AO is the Arithmetic mean between AP and AQ: for clearly

$$AO = \tfrac{1}{2}(AP + AQ).$$

(ii) AH is the Geometric mean between AP and AQ:

for $AH^2 = AP \cdot AQ$. III. 36.

(iii) AB is the Harmonic mean between AP and AQ:

for $OA \cdot OB = OP^2$. Ex. 1, p. 238.

$\therefore$ AB is cut harmonically at P and Q. Ex. 1, p. 360.

That is, AB is the Harmonic mean between AP and AQ.

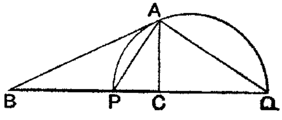
And from the similar triangles OAH, HAB,

$$OA : AH = AH : AB,$$

$$\therefore AO \cdot AB = AH^2;$$ VI. 17.

$\therefore$ *the Geometric mean between two straight lines is the mean proportional between their Arithmetic and Harmonic means.*

4. *Given the base of a triangle and the ratio of the other sides, to find the locus of the vertex.*



Let BC be the given base, and let BAC be any triangle standing upon it, such that BA : AC = the given ratio:

it is required to find the locus of A.

Bisect the $\angle$ BAC internally and externally by AP, AQ.

Then BC is divided internally at P, and externally at Q,

so that BP : PC = BQ : QC = the given ratio;

$\therefore$ P and Q are fixed points.

And since AP, AQ are the internal and external bisectors of the $\angle$ BAC,

$\therefore$ the $\angle$ PAQ is a rt. angle;

$\therefore$ the locus of A is a circle described on PQ as diameter.

EXERCISE. *Given three points* B, P, C *in a straight line: find the locus of points at which* BP *and* PC *subtend equal angles.*

DEFINITIONS.

1. A series of points in a straight line is called a **range.** If the range consists of four points, of which one pair are harmonic conjugates with respect to the other pair, it is said to be a **harmonic range.**

2. A series of straight lines drawn through a point is called a **pencil.**

The point of concurrence is called the **vertex** of the pencil, and each of the straight lines is called a **ray**.

A pencil of four rays drawn from any point to a harmonic range is said to be a **harmonic pencil.**

3. A straight line drawn to cut a system of lines is called a **transversal.**

4. A system of four straight lines, no three of which are concurrent, is called a **complete quadrilateral.**

These straight lines will intersect two and two in six points, called the **vertices** of the quadrilateral; the three straight lines which join opposite vertices are **diagonals.**

THEOREMS ON HARMONIC SECTION.

1. *If a transversal is drawn parallel to one ray of a harmonic pencil, the other three rays intercept equal parts upon it: and conversely.*

2. *Any transversal is cut harmonically by the rays of a harmonic pencil.*

3. *In a harmonic pencil, if one ray bisect the angle between the other pair of rays, it is perpendicular to its conjugate ray. Conversely if one pair of rays form a right angle, then they bisect internally and externally the angle between the other pair.*

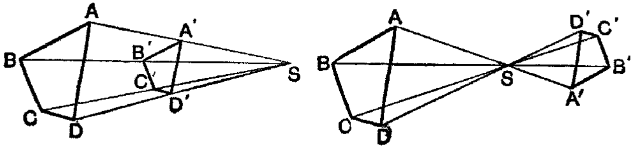
4. *If* A, P, B, Q *and* a, p, b, q *are harmonic ranges, one on each of two given straight lines, and if* Aa, Pp, Bb, *the straight lines which join three pairs of corresponding points, meet at* S ; *then will* Qq *also pass through* S.

5. *If two straight lines intersect at* A, *and if* A, P, B, Q *and* A, p, b, q *are two harmonic ranges one on each straight line (the points corresponding as indicated by the letters), then* Pp, Bb, Qq *will be concurrent: also* Pq, Bb, Qp *will be concurrent.*

6. *Use Theorem 5 to prove that in a complete quadrilateral in which the three diagonals are drawn, the straight line joining any pair of opposite vertices is cut harmonically by the other two diagonals.*

II. On centres of similarity and similitude.

1. *If any two unequal similar figures are placed so that their homologous sides are parallel, the lines joining corresponding points in the two figures meet in a point, whose distances from any two corresponding points are in the ratio of any pair of homologous sides.*



Let ABCD, A′B′C′D′ be two similar figures, and let them be placed so that their homologous sides are parallel; namely, AB, BC, CD, DA parallel to A′B′, B′C′, C′D′, D′A′ respectively:
then shall AA′, BB′, CC′, DD′ meet in a point, whose distances from any two corresponding points shall be in the ratio of any pair of homologous sides.

Let AA′ meet BB′, produced if necessary, in S.

Then because AB is par^{l} to A′B′; *Hyp.*

∴ the Δ^s SAB, SA′B′ are equiangular;

∴ SA : SA′ = AB : A′B′; VI. 4.

∴ AA′ divides BB′, externally or internally, in the ratio of AB to A′B′.

Similarly it may be shewn that CC′ divides BB′ in the ratio of BC to B′C′.

But since the figures are similar,

BC : B′C′ = AB : A′B′;

∴ AA′ and CC′ divide BB′ in the same ratio;

that is, AA′, BB′, CC′ meet in the same point S.

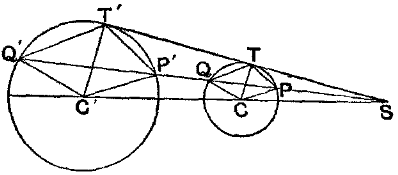
In like manner it may be proved that DD′ meets CC′ in the point S.

∴ AA′, BB′, CC′, DD′ are concurrent, and each of these lines is divided at S in the ratio of a pair of homologous sides of the two figures. Q. E. D.

Cor. *If any line is drawn through* S *meeting any pair of homologous sides in* K *and* K′, *the ratio* SK : SK′ *is constant, and equal to the ratio of any pair of homologous sides.*

Note. It will be seen that the lines joining corresponding points are divided externally or internally at S according as the corresponding sides are drawn in the same or in opposite directions. In either case the point of concurrence S is called **a centre of similarity** of the two figures.

2. *A common tangent* STT′ *to two circles whose centres are* C, C′, *meets the line of centres in* S. *If through* S *any straight line is drawn meeting these two circles in* P, Q, *and* P′, Q′, *respectively, then the radii* CP, CQ *shall be respectively parallel to* C′P′, C′Q′. *Also the rectangles* SQ . SP′, SP . SQ′ *shall each be equal to the rectangle* ST . ST′.



Join CT, CP, CQ and C′T′, C′P′, C′Q′.

Then since each of the ∠s CTS, C′T′S is a right angle, III. 18.

∴ CT is par[l] to C′T′;

∴ the △s SCT, SC′T′ are equiangular;

∴ SC : SC′ = CT : C′T′

= CP : C′P′;

∴ the △s SCP, SC′P′ are similar; VI. 7.

∴ the ∠ SCP = the ∠ SC′P′;

∴ CP is par[l] to C′P′.

Similarly CQ is par[l] to C′Q′.

Again, it easily follows that TP, TQ are par[l] to T′P′, T′Q′ respectively;

∴ the △s STP, ST′P′ are similar.

Now the rect. SP . SQ = the sq. on ST; III. 36.

∴ SP : ST = ST : SQ, VI. 16.

and SP : ST = SP′ : ST′;

∴ ST : SQ = SP′ : ST′;

∴ the rect. ST . ST′ = SQ . SP′.

In the same way it may be proved that

the rect. SP . SQ′ = the rect. ST . ST′.

Q. E. D.

Cor. 1. It has been proved that

SC : SC′ = CP : C′P′;

thus the external common tangents to the two circles meet at a point S which divides the line of centres externally in the ratio of the radii.

Similarly it may be shewn that the transverse common tangents meet at a point S′ which divides the line of centres internally in the ratio of the radii.

Cor. 2. CC′ is divided harmonically at S and S′.

Definition. The points S and S′ which divide externally and internally the line of centres of two circles in the ratio of their radii are called the **external and internal centres of similitude** respectively.

EXAMPLES.

1. Inscribe a square in a given triangle.

2. In a given triangle inscribe a triangle similar and similarly situated to a given triangle.

3. Inscribe a square in a given sector of circle, so that two angular points shall be on the arc of the sector and the other two on the bounding radii.

4. *In the figure on page* 278, *if* DI *meets the inscribed circle in* X, *shew that* A, X, D_1 *are collinear. Also if* AI_1 meets the base in Y *shew that* II_1 *is divided harmonically at* Y *and* A.

5. *With the notation on page* 282 *shew that* O *and* G *are respectively the external and internal centres of similitude of the circumscribed and nine-points circle.*

6. *If a variable circle touches two fixed circles, the line joining their points of contact passes through a centre of similitude. Distinguish between the different cases.*

7. *Describe a circle which shall touch two given circles and pass through a given point.*

8. *Describe a circle which shall touch three given circles.*

9. C_1, C_2, C_3 *are the centres of three given circles;* S'_1, S_1, *are the internal and external centres of similitude of the pair of circles whose centres are* C_2, C_3, *and* S'_2, S_2, S'_3, S_3, *have similar meanings with regard to the other two pairs of circles: shew that*

(i) S'_1C_1, S'_2C_2, S'_3C_3 *are concurrent;*

(ii) *the six points* S_1, S_2, S_3, S'_1, S'_2, S'_3, *lie three and three on four straight lines.* [See Ex. 1 and 2, pp. 375, 376.]

III. ON POLE AND POLAR.

DEFINITIONS.

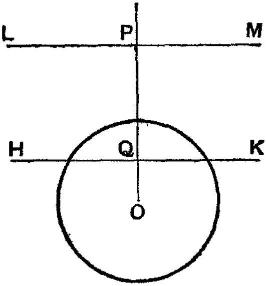
(i) If in any straight line drawn from the centre of a circle two points are taken such that the rectangle contained by their distances from the centre is equal to the square on the radius, each point is said to be the **inverse** of the other:

Thus in the figure given below, if O is the centre of the circle, and if OP . OQ = (radius)^2, then each of the points P and Q is the inverse of the other.

It is clear that if one of these points is within the circle the other must be without it.

(ii) The **polar** of a given point with respect to a given circle is the straight line drawn through the inverse of the given point at right angles to the line which joins the given point to the centre: and with reference to the polar the given point is called the **pole**.

Thus in the adjoining figure, if OP . OQ = (radius)^2, and if through



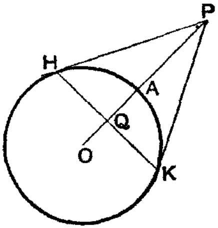
P and Q, LM and HK are drawn perp. to OP; then HK is the polar of the point P, and P is the pole of the st. line HK: also LM is the polar of the point Q, and Q the pole of LM.

It is clear that the polar of an *external* point must intersect the circle, and that the polar of an *internal* point must fall without it: also that the polar of a point *on the circumference* is the tangent at that point.

1. Now it has been proved [see Ex. 1, page 233] that if from an external point P two tangents PH, PK are drawn to a circle, of which O is the centre, then OP cuts the chord of contact HK at right angles at Q, so that

OP . OQ = (radius)^2,

∴ HK is the polar of P with respect to the circle. *Def.* 2.

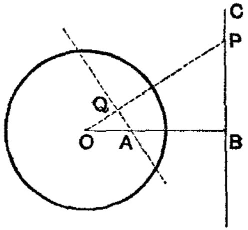


Hence we conclude that

The Polar of an external point with reference to a circle is the chord of contact of tangents drawn from the given point to the circle.

The following Theorem is known as the **Reciprocal Property of Pole and Polar.**

2. *If* A *and* P *are any two points, and if the polar of* A *with respect to any circle passes through* P, *then the polar of* P *must pass through* A.



Let BC be the polar of the point A with respect to a circle whose centre is O, and let BC pass through P:
then shall the polar of P pass through A.

Join OP; and from A draw AQ perp. to OP. We shall shew that AQ is the polar of P.

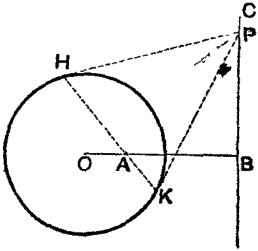
Now since BC is the polar of A,
∴ the ∠ ABP is a rt. angle; *Def.* 2, page 360.
and the ∠ AQP is a rt. angle: *Constr.*
∴ the four points A, B, P, Q are concyclic;
∴ OQ . OP = OA . OB III. 36.
= (radius)^2, for CB is the polar of A:
∴ P and Q are inverse points with respect to the given circle.

And since AQ is perp. to OP,
∴ AQ is the polar of P.
That is, the polar of P passes through A.

Q. E. D.

A similar proof applies to the case when the given point A is without the circle, and the polar BC cuts it.

3. *To prove that the locus of the intersection of tangents drawn to a circle at the extremities of all chords which pass through a given point within the circle is the polar of that point.*



Let A be the given point within the circle. Let HK be *any* chord passing through A; and let the tangents at H and K intersect at P:
it is required to prove that the locus of P is the polar of the point A.

I. To shew that P lies on the polar of A.

Since HK is the chord of contact of tangents drawn from P,
∴ HK is the polar of P. Ex. 1, p. 366.

But HK, the polar of P, passes through A;

∴ the polar of A passes through P: Ex. 2, p. 367.

that is, the point P lies on the polar of A.

II. To shew that any point on the polar of A satisfies the given conditions.

Let BC be the polar of A, and let P be any point on it.

Draw tangents PH, PK, and let HK be the chord of contact.

Now from Ex. 1, p. 366, we know that the chord of contact HK is the polar of P,

and we also know that the polar of P must pass through A; for P is on BC, the polar of A: Ex. 2, p. 367.

that is, HK passes through A.

∴ P is the point of intersection of tangents drawn at the extremities of a chord passing through A.

From I. and II. we conclude that the required locus is the polar of A.

Note. If A is *without* the circle, the theorem demonstrated in Part I. of the above proof still holds good; but the converse theorem in Part II. is not true for *all* points in BC. For if A is without the circle, the polar BC will intersect it; and no point on that part of the polar which is within the circle can be the point of intersection of tangents.

We now see that

(i) *The Polar of an* external *point with respect to a circle is the chord of contact of tangents drawn from it.*

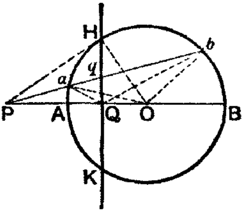
(ii) *The Polar of an* internal *point is the locus of the intersections of tangents drawn at the extremities of all chords which pass through it.*

(iii) *The Polar of a* point on the circumference *is the tangent at that point.*

The following theorem is known as the Harmonic Property of Pole and Polar.

4. *Any straight line drawn through a point is cut harmonically by the point, its polar, and the circumference of the circle.*

Let AHB be a circle, P the given point and HK its polar; let Paqb be any straight line drawn through P meeting the polar at q and the O^{ce} of the circle at a and b:
then shall P, a, q, b be a harmonic range.



In the case here considered, P is an external point.

Join P to the centre O, and let PO cut the O^{ce} at A and B: let the polar of P cut the O^{ce} at H and K, and PO at Q.

Then PH is a tangent to the ⊙ AHB. Ex. 1, p. 366.

From the similar triangles OPH, HPQ,

$$OP : PH = PH : PQ,$$
$$\therefore PQ . PO = PH^2$$
$$= Pa . Pb.$$

∵ the points O, Q, a, b are concyclic:

∴ the ∠ aQA = the ∠ abO
= the ∠ Oab I. 5.
= the ∠ OQb, in the same segment.

And since QH is perp. to AB,
∴ the ∠ aQH = the ∠ bQH.

∴ Qq and QP are the internal and external bisectors of the ∠ aQb:
∴ P, a, q, b is a harmonic range. Ex. 1, p. 360.

The student should investigate for himself the case when P is an internal point.

Conversely, it may be shewn that if through a fixed point P *any secant is drawn cutting the circumference of a given circle at* a *and* b, *and if* q *is the harmonic conjugate at* P *with respect to* a, b; *then the locus of* q *is the polar of* P *with respect to the given circle.*

[For Examples on Pole and Polar, see p. 370.]

DEFINITION.

A triangle so related to a circle that each side is the polar of the opposite vertex is said to be **self-conjugate with respect** to the circle.

EXAMPLES ON POLE AND POLAR.

1. *The straight line which joins any two points is the polar with respect to a given circle of the point of intersection of their polars.*

2. *The point of intersection of any two straight lines is the pole of the straight line which joins their poles.*

3. *Find the locus of the poles of all straight lines which pass through a given point.*

4. *Find the locus of the poles, with respect to a given circle, of tangents drawn to a concentric circle.*

5. *If two circles cut one another orthogonally and* PQ *be any diameter of one of them; shew that the polar of* P *with regard to the other circle passes through* Q.

6. *If two circles cut one another orthogonally, the centre of each circle is the pole of their common chord with respect to the other circle.*

7. *Any two points subtend at the centre of a circle an angle equal to one of the angles formed by the polars of the given points.*

8. O *is the centre of a given circle, and* AB *a fixed straight line.* P *is* any *point in* AB; *find the locus of the point inverse to* P *with respect to the circle.*

9. *Given a circle, and a fixed point* O *on its circumference:* P *is* any *point on the circle: find the locus of the point inverse to* P *with respect to any circle whose centre is* O.

10. *Given two points* A *and* B, *and a circle whose centre is* O; *shew that the rectangle contained by* OA *and the perpendicular from* B *on the polar of* A *is equal to the rectangle contained by* OB *and the perpendicular from* A *on the polar of* B.

11. *Four points* A, B, C, D *are taken in order on the circumference of a circle;* DA, CB *intersect at* P, AC, BD *at* Q *and* BA, CD *in* R; *shew that the triangle* PQR *is self-conjugate with respect to the circle.*

12. *Give a* linear *construction for finding the polar of a given point with respect to a given circle. Hence find a linear construction for drawing a tangent to a circle from an external point.*

13. *If a triangle is self-conjugate with respect to a circle, the centre of the circle is at the orthocentre of the triangle.*

14. *The polars, with respect to a given circle, of the four points of a harmonic range form a harmonic pencil: and conversely.*

IV. ON THE RADICAL AXIS.

1. *To find the locus of points from which the tangents drawn to two given circles are equal.*

Fig. 1.

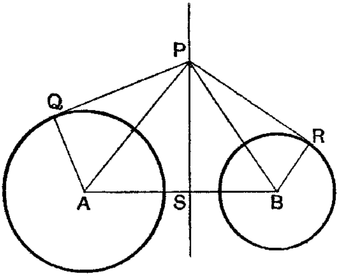
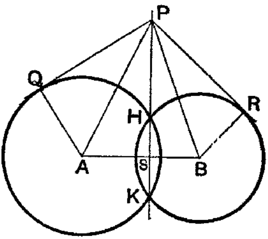


Fig. 2.



Let A and B be the centres of the given circles, whose radii are a and b; and let P be any point such that the tangent PQ drawn to the circle (A) is equal to the tangent PR drawn to the circle (B):

it is required to find the locus of P.

Join PA, PB, AQ, BR, AB; and from P draw PS perp. to AB.

Then because $PQ = PR$, $\therefore PQ^2 = PR^2$.

But $PQ^2 = PA^2 - AQ^2$; and $PR^2 = PB^2 - BR^2$: I. 47.

$\therefore PA^2 - AQ^2 = PB^2 - BR^2$;

that is, $PS^2 + AS^2 - a^2 = PS^2 + SB^2 - b^2$; I. 47.

or, $AS^2 - a^2 = SB^2 - b^2$.

Hence AB is divided at S, so that $AS^2 - SB^2 = a^2 - b^2$:

$\therefore$ S is a *fixed point.*

Hence all points from which equal tangents can be drawn to the two circles lie on the straight line which cuts AB at rt. angles, so that the difference of the squares on the segments of AB is equal to the difference of the squares on the radii.

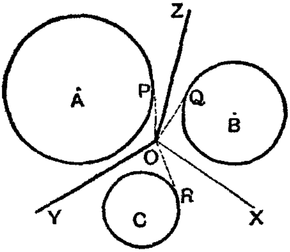
Again, by simply retracing these steps, it may be shewn that in Fig. 1 every point in SP, and in Fig. 2 every point in SP exterior to the circles, is such that tangents drawn from it to the two circles are equal.

Hence we conclude that in Fig. 1 the whole line SP is the required locus, and in Fig. 2 that part of SP which is without the circles.

In either case SP is said to be the **Radical Axis** of the two circles.

COROLLARY. *If the circles cut one another as in* Fig. 2, *it is clear that the Radical Axis is identical with the straight line which passes through the points of intersection of the circles;* for it follows readily from III. 36 that tangents drawn to two intersecting circles from any point in the common chord produced are equal.

2. *The Radical Axes of three circles taken in pairs are concurrent.*



Let there be three circles whose centres are A, B,

Let OZ be the radical axis of the ⊙ˢ (A) and (B);
and OY the Radical Axis of the ⊙ˢ (A) and (C), O being the point of their intersection:
then shall the radical axis of the ⊙ˢ (B) and (C) pass through O.

It will be found that the point O is either *without* or *within all the circles.*

I. When O is without the circles.

From O draw OP, OQ, OR tangents to the ⊙ˢ (A), (B), (C).

Then because O is a point on the radical axis of (A) and (B); *Hyp.*

∴ OP = OQ.

And because O is a point on the radical axis of (A) and (C), *Hyp.*

∴ OP = OR,

∴ OQ = OR;

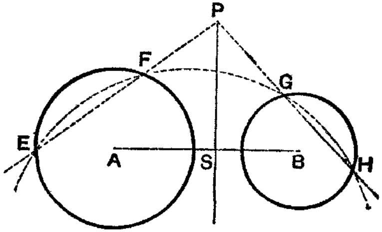
∴ O is a point on the radical axis of (B) and (C),

i.e. the radical axis of (B) and (C) passes through O.

II. If the circles intersect in such a way that O is within them all;
the radical axes are then the common chords of the three circles taken two and two; and it is required to prove that these common chords are concurrent. This may be shewn indirectly by III. 35.

DEFINITION. The point of intersection of the radical axes of three circles taken in pairs is called the radical centre.

8. *To draw the radical axis of two given circles.*



Let A and B be the centres of the given circles: it is required to draw their radical axis.

If the given circles intersect, then the st. line drawn through their points of intersection will be the radical axis. [Ex. 1, Cor. p. 372.]

But if the given circles do not intersect,

describe any circle so as to cut them in E, F and G, H:

Join EF and HG, and produce them to meet in P.

Join AB; and from P draw PS perp. to AB.

Then PS shall be the radical axis of the ⊙s (A), (B).

DEFINITION. If each pair of circles in a given system have the same radical axis, the circles are said to be **co-axal.**

EXAMPLES.

1. *Shew that the radical axis of two circles bisects any one of their common tangents.*

2. *If tangents are drawn to two circles from any point on their radical axis; shew that a circle described with this point as centre and any one of the tangents as radius, cuts both the given circles orthogonally.*

3. O *is the radical centre of three circles, and from* O *a tangent* OT *is drawn to any one of them: shew that a circle whose centre is* O *and radius* OT *cuts all the given circles orthogonally.*

4. *If three circles touch one another, taken two and two, shew that their common tangents at the points of contact are concurrent.*

5. *If circles are described on the three sides of a triangle as diameter, their radical centre is the orthocentre of the triangle.*

6. *All circles which pass through a fixed point and cut a given circle orthogonally, pass through a second fixed point.*

7. *Find the locus of the centres of all circles which pass through a given point and cut a given circle orthogonally.*

8. *Describe a circle to pass through two given points and cut a given circle orthogonally.*

9. *Find the locus of the centres of all circles which cut two given circles orthogonally.*

10. *Describe a circle to pass through a given point and cut two given circles orthogonally.*

11. *The difference of the squares on the tangents drawn from any point to two circles is equal to twice the rectangle contained by the straight line joining their centres and the perpendicular from the given point on their radical axis.*

12. *In a system of co-axal circles which do not intersect, any point is taken on the radical axis; shew that a circle described from this point as centre with radius equal to the tangent drawn from it to any one of the circles, will meet the line of centres in two* fixed *points.*

[*These fixed points are called the* **Limiting Points** *of the system.*]

13. *In a system of co-axal circles the two limiting points and the points in which any one circle of the system cuts the line of centres form a harmonic range.*

14. *In a system of co-axal circles a limiting point has the same polar with regard to all the circles of the system.*

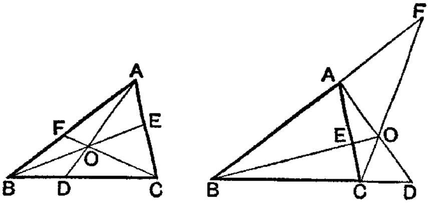
15. *If two circles are orthogonal any diameter of one is cut harmonically by the other.*

Obs. In the two following theorems we are to suppose that the segments of straight lines are expressed numerically in terms of some common unit; and the ratio of one such segment to another will be denoted by the fraction of which the first is the numerator and the second the denominator.

V. ON TRANSVERSALS.

DEFINITION. A straight line drawn to cut a given system of lines is called a **transversal.**

1. *If three concurrent straight lines are drawn from the angular points of a triangle to meet the opposite sides, then the product of three alternate segments taken in order is equal to the product of the other three segments.*



Let AD, BE, CF be drawn from the vertices of the $\triangle$ ABC to intersect at O, and cut the opposite sides at D, E, F:
then shall BD . CE . AF = DC . EA . FB.

By similar triangles it may be shewn that

BD : DC = the alt. of $\triangle$AOB : the alt. of $\triangle$AOC;

$$\therefore \frac{BD}{DC} = \frac{\triangle AOB}{\triangle AOC};$$

similarly,
$$\frac{CE}{EA} = \frac{\triangle BOC}{\triangle BOA};$$

and
$$\frac{AF}{FB} = \frac{\triangle COA}{\triangle COB}.$$

Multiplying these ratios, we have

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1;$$

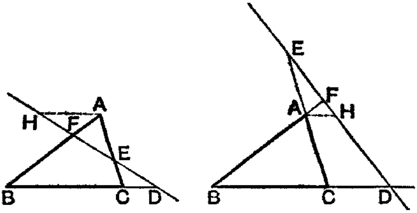
or, BD . CE . AF = DC . EA . FB. Q. E. D.

The converse of this theorem, which may be proved indirectly, is very important: it may be enunciated thus:

If three straight lines drawn from the vertices of a triangle cut the opposite sides so that the product of three alternate segments taken in order is equal to the product of the other three, then the three straight lines are concurrent.

That is, if BD . CE . AF = DC . EA . FB,
then AD, BE, CF *are concurrent.*

2. *If a transversal is drawn to cut the sides, or the sides produced, of a triangle, the product of three alternate segments taken in order is equal to the product of the other three segments.*



Let ABC be a triangle, and let a transversal meet the sides BC, CA, AB, or these sides produced, at D, E, F:

then shall BD . CE . AF = DC . EA . FB.

Draw AH par[1] to BC, meeting the transversal at H.

Then from the similar △[s] DBF, HAF,

$$\frac{BD}{FB} = \frac{HA}{AF};$$

and from the similar △[s] DCE, HAE,

$$\frac{CE}{DC} = \frac{EA}{HA};$$

∴, by multiplication, $$\frac{BD}{FB} \cdot \frac{CE}{DC} = \frac{EA}{AF};$$

that is, $$\frac{BD \,.\, CE \,.\, AF}{DC \,.\, EA \,.\, FB} = 1,$$

or, $$BD \,.\, CE \,.\, AF = DC \,.\, EA \,.\, FB.$$

Q. E. D.

NOTE. In this theorem the transversal must either meet two sides and the third side produced, as in Fig. 1; or all three sides produced, as in Fig. 2.

The converse of this Theorem may be proved indirectly:

If three points are taken in two sides of a triangle and the third side produced, or in all three sides produced, so that the product of three alternate segments taken in order is equal to the product of the other three segments, the three points are collinear.

The propositions given on pages 103—106 relating to the concurrence of straight lines in a triangle, may be proved by the method of transversals, and in addition to these the following important theorems may be established.

DEFINITIONS.

(i) If two triangles are such that three straight lines joining corresponding vertices are concurrent, they are said to be **co-polar.**

(ii) If two triangles are such that the points of intersection of corresponding sides are collinear, they are said to be **co-axial.**

THEOREMS TO BE PROVED BY TRANSVERSALS.

1. *The straight lines which join the vertices of a triangle to the points of contact of the inscribed circle (or any of the three escribed circles) are concurrent.*

2. *The middle points of the diagonals of a complete quadrilateral are collinear.*

3. *Co-polar triangles are also co-axial; and conversely co-axial triangles are also co-polar.*

4. *The six centres of similitude of three circles lie three by three on four straight lines.*

MISCELLANEOUS EXAMPLES ON BOOK VI.

1. Through D, any point in the base of a triangle ABC, straight lines DE, DF are drawn parallel to the sides AB, AC, and meeting the sides at E, F: shew that the triangle AEF is a mean proportional between the triangles FBD, EDC.

2. If two triangles have one angle of the one equal to one angle of the other, and a second angle of the one supplementary to a second angle of the other, then the sides about the third angles are proportional.

3. AE bisects the vertical angle of the triangle ABC and meets the base in E; shew that if circles are described about the triangles ABE, ACE, the diameters of these circles are to each other in the same ratio as the segments of the base.

4. Through a fixed point O draw a straight line so that the parts intercepted between O and the perpendiculars drawn to the straight line from two other fixed points may have a given ratio.

5. The angle A of a triangle ABC is bisected by AD meeting BC in D, and AX is the median bisecting BC: shew that XD has the same ratio to XB as the difference of the sides has to their sum.

6. AD and AE bisect the vertical angle of a triangle internally and externally, meeting the base in D and E; shew that if O is the middle point of BC, then OB is a mean proportional between OD and OE.

7. P and Q are fixed points; AB and CD are fixed parallel straight lines; any straight line is drawn from P to meet AB at M, and a straight line is drawn from Q parallel to PM meeting CD at N: shew that the ratio of PM to QN is constant, and thence shew that the straight line through M and N passes through a fixed point.

8. C is the middle point of an arc of a circle whose chord is AB; D is any point in the conjugate arc: shew that

$$AD + DB : DC :: AB : AC.$$

9. In the triangle ABC the side AC is double of BC. If CD, CE bisect the angle ACB internally and externally meeting AB in D and E, shew that the areas of the triangles CBD, ACD, ABC, CDE are as 1, 2, 3, 4.

10. AB, AC are two chords of a circle; a line parallel to the tangent at A cuts AB, AC in D and E respectively: shew that the rectangle AB, AD is equal to the rectangle AC, AE.

11. If from any point on the hypotenuse of a right-angled triangle perpendiculars are drawn to the two sides, the rectangle contained by the segments of the hypotenuse will be equal to the sum of the rectangles contained by the segments of the sides.

12. D is a point in the side AC of the triangle ABC, and E is a point in AB. If BD, CE divide each other into parts in the ratio 4 : 1, then D, E divide CA, BA in the ratio 3 : 1.

13. If the perpendiculars from two fixed points on a straight line passing between them be in a given ratio, the straight line must pass through a third fixed point.

14. PA, PB are two tangents to a circle; PCD any chord through P: shew that the rectangle contained by one pair of opposite sides of the quadrilateral ACBD is equal to the rectangle contained by the other pair.

15. A, B, C are any three points on a circle, and the tangent at A meets BC produced in D: shew that the diameters of the circles circumscribed about ABD, ACD are as AD to CD.

16. AB, CD are two diameters of the circle ADBC at right angles to each other, and EF is any chord; CE, CF are drawn meeting AB produced in G and H: prove that the rect. CE, HG = the rect. EF, CH.

17. From the vertex A of any triangle ABC draw a line meeting BC produced in D so that AD may be a mean proportional between the segments of the base.

18. Two circles touch internally at O; AB a chord of the larger circle touches the smaller in C which is cut by the lines OA, OB in the points P, Q: shew that OP : OQ :: AC : CB.

19. AB is any chord of a circle; AC, BC are drawn to any point C in the circumference and meet the diameter perpendicular to AB at D, E: if O be the centre, shew that the rect. OD, OE is equal to the square on the radius.

20. YD is a tangent to a circle drawn from a point Y in the diameter AB produced; from D a perpendicular DX is drawn to the diameter: shew that the points X, Y divide AB internally and externally in the same ratio.

21. Determine a point in the circumference of a circle, from which lines drawn to two other given points shall have a given ratio.

22. O is the centre and OA a radius of a given circle, and V is a fixed point in OA; P and Q are two points on the circumference on opposite sides of A and equidistant from it; QV is produced to meet the circle in L: shew that, whatever be the length of the arc PQ, the chord LP will always meet OA produced in a fixed point.

23. EA, EA′ are diameters of two circles touching each other externally at E; a chord AB of the former circle, when produced, touches the latter at C′, while a chord A′B′ of the latter touches the former at C: prove that the rectangle, contained by AB and A′B′, is four times as great as that contained by BC′ and B′C.

24. If a circle be described touching externally two given circles, the straight line passing through the points of contact will intersect the line of centres of the given circles at a fixed point.

25. Two circles touch externally in C; if any point D be taken without them so that the radii AC, BC subtend equal angles at D, and DE, DF be tangents to the circles, shew that DC is a mean proportional between DE and DF.

26. If through the middle point of the base of a triangle any line be drawn intersecting one side of the triangle, the other produced, and the line drawn parallel to the base from the vertex, it will be divided harmonically.

27. If from either base angle of a triangle a line be drawn intersecting the median from the vertex, the opposite side, and the line drawn parallel to the base from the vertex, it will be divided harmonically.

28. Any straight line drawn to cut the arms of an angle and its internal and external bisectors is cut harmonically.

29. P, Q are harmonic conjugates of A and B, and C is an external point: if the angle PCQ is a right angle, shew that CP, CQ are the internal and external bisectors of the angle ACB.

30. From C, one of the base angles of a triangle, draw a straight line meeting AB in G, and a straight line through A parallel to the base in E, so that CE may be to EG in a given ratio.

31. P is a given point outside the angle formed by two given lines AB, AC: shew how to draw a straight line from P such that the parts of it intercepted between P and the lines AB, AC may have a given ratio.

32. Through a given point within a given circle, draw a straight line such that the parts of it intercepted between that point and the circumference may have a given ratio. How many solutions does the problem admit of?

33. If a common tangent be drawn to any number of circles which touch each other internally, and from any point of this tangent as a centre a circle be described, cutting the other circles; and if from this centre lines be drawn through the intersections of the circles, the segments of the lines within each circle shall be equal.

34. APB is a quadrant of a circle, SPT a line touching it at P; C is the centre, and PM is perpendicular to CA: prove that

the △ SCT : the △ ACB :: the △ ACB : the △ CMP.

35. ABC is a triangle inscribed in a circle, AD, AE are lines drawn to the base BC parallel to the tangents at B, C respectively; shew that AD = AE, and BD : CE :: $AB^2 : AC^2$.

36. AB is the diameter of a circle, E the middle point of the radius OB; on AE, EB as diameters circles are described; PQL is a common tangent touching the circles at P and Q, and AB produced at L: shew that BL is equal to the radius of the smaller circle.

37. The vertical angle C of a triangle is bisected by a straight line which meets the base at D, and is produced to a point E, such that the rectangle contained by CD and CE is equal to the rectangle contained by AC and CB: shew that if the base and vertical angle be given, the position of E is invariable.

38. ABC is an isosceles triangle having the base angles at B and C each double of the vertical angle: if BE and CD bisect the base angles and meet the opposite sides in E and D, shew that DE divides the triangle into figures whose ratio is equal to that of AB to BC.

39. If AB, the diameter of a semicircle, be bisected in C and on AC and CB circles be described, and in the space between the three circumferences a circle be inscribed, shew that its diameter will be to that of the equal circles in the ratio of two to three.

40. O is the centre of a circle inscribed in a quadrilateral ABCD; a line EOF is drawn and making equal angles with AD and BC, and meeting them in E and F respectively: shew that the triangles AEO, BOF are similar, and that

$$AE : ED = CF : FB.$$

41. From the last exercise deduce the following: The inscribed circle of a triangle ABC touches AB in F; XOY is drawn through the centre making equal angles with AB and AC, and meeting them in X and Y respectively: shew that BX : XF = AY : YC.

42. Inscribe a square in a given semicircle.

43. Inscribe a square in a given segment of a circle.

44. Describe an equilateral triangle equal to a given isosceles triangle.

45. Describe a square having given the difference between a diagonal and a side.

46. Given the vertical angle, the ratio of the sides containing it, and the diameter of the circumscribing circle, construct the triangle.

47. Given the vertical angle, the line bisecting the base, and the angle the bisector makes with the base, construct the triangle.

48. In a given circle inscribe a triangle so that two sides may pass through two given points and the third side be parallel to a given straight line.

49. In a given circle inscribe a triangle so that the sides may pass through three given points.

50. A, B, X, Y are four points in a straight line, and O is such a point in it that the rectangle OA, OY is equal to the rectangle OB, OX: if a circle be described with centre O and radius equal to a mean proportional between OA and OY, shew that at every point on this circle AB and XY will subtend equal angles.

51. O is a fixed point, and OP is any line drawn to meet a fixed straight line in P; if on OP a point Q is taken so that OQ to OP is a constant ratio, find the locus of Q.

52. O is a fixed point, and OP is any line drawn to meet the circumference of a fixed circle in P; if on OP a point Q is taken so that OQ to OP is a constant ratio, find the locus of Q.

53. If from a given point two straight lines are drawn including a given angle, and having a fixed ratio, find the locus of the extremity of one of them when the extremity of the other lies on a fixed straight line.

54. On a straight line PAB, two points A and B are marked and the line PAB is made to revolve round the fixed extremity P. C is a fixed point in the plane in which PAB revolves; prove that if CA and CB be joined and the parallelogram CADB be completed, the locus of D will be a circle.

55. Find the locus of a point whose distances from two fixed points are in a given ratio.

56. Find the locus of a point from which two given circles subtend the same angle.

57. Find the locus of a point such that its distances from two intersecting straight lines are in a given ratio.

58. In the figure on page 364, shew that QT, P′T′ meet on the radical axis of the two circles.

59. ABC is any triangle, and on its sides equilateral triangles are described externally: if X, Y, Z are the centres of their inscribed circles, shew that the triangle XYZ is equilateral.

60. If S, I are the centres, and R, r the radii of the circumscribed and inscribed circles of a triangle, and if N is the centre of its nine-points circle,

prove that (i) $SI^2 = R^2 - 2Rr$,

(ii) $NI = \frac{1}{2}R - r$.

Establish corresponding properties for the escribed circles, and hence prove that the nine-points circle touches the inscribed and escribed circles of a triangle.

SOLID GEOMETRY.

EUCLID. BOOK XI.

DEFINITIONS.

FROM the Definitions of Book I. it will be remembered that

(i) A **line** is that which has *length*, without breadth or thickness.

(ii) A **surface** is that which has *length* and *breadth*, without thickness.

To these definitions we have now to add:

(iii) **Space** is that which has *length*, *breadth*, and *thickness*.

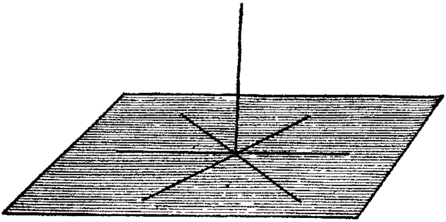
Thus a line is said to be of **one dimension**;
a surface is said to be of **two dimensions**;
and space is said to be of **three dimensions**.

The Propositions of Euclid's Eleventh Book here given establish the first principles of the *geometry of space*, or *solid geometry*. They deal with the properties of straight lines which are not all in the same plane, the relations which straight lines bear to planes which do not contain those lines, and the relations which two or more planes bear to one another. Unless the contrary is stated the straight lines are supposed to be of indefinite length, and the planes of infinite extent.

Solid geometry then proceeds to discuss the properties of solid figures, of surfaces which are not planes, and of lines which can not be drawn on a plane surface.

LINES AND PLANES.

1. A straight line is **perpendicular to a plane** when it is perpendicular to *every* straight line which meets it in that plane.

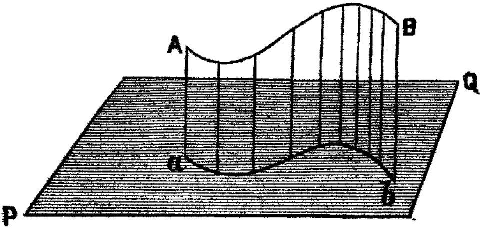


NOTE. It will be proved in Proposition 4 that if a straight line is perpendicular to *two* straight lines which meet it in a plane, it is also perpendicular to *every* straight line which meets it in that plane.

A straight line drawn perpendicular to a plane is said to be a **normal** to that plane.

2. The foot of the perpendicular let fall from a given point on a plane is called the **projection of that point** on the plane.

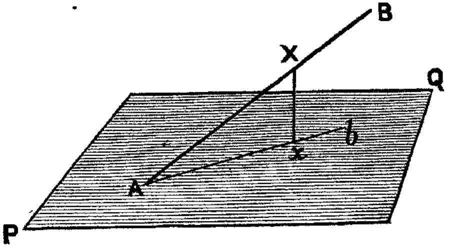
3. The **projection of a line** on a plane is the locus of the feet of perpendiculars drawn from all points in the given line to the plane.



Thus in the above figure the line *ab* is the projection of the line AB on the plane PQ.

It will be proved hereafter (see page 420) that the projection of a straight line on a plane is also a straight line.

4. **The inclination of a straight line to a plane** is the acute angle contained by that line and another drawn from the point at which the first line meets the plane to the point at which a perpendicular to the plane let fall from any point of the first line meets the plane.

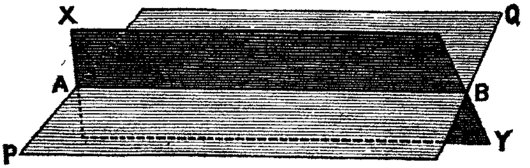


Thus in the above figure, if from any point X in the given straight line AB, which intersects the plane PQ at A, a perpendicular Xx is let fall on the plane, and the straight line Axb is drawn from A through x, then the inclination of the straight line AB to the plane PQ is measured by the acute angle BAb. In other words :—

The inclination of a straight line to a plane is the acute angle contained by the given straight line and its *projection on the plane.*

Axiom. If two surfaces intersect one another, they meet in a *line or lines.*

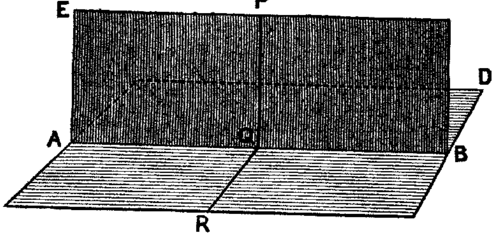
5. The **common section** of two intersecting surfaces is the line (or lines) in which they meet.



Note. It is proved in Proposition 3 that the common section of two planes is a straight line.

Thus AB, the common section of the two planes PQ, XY is proved to be a straight line.

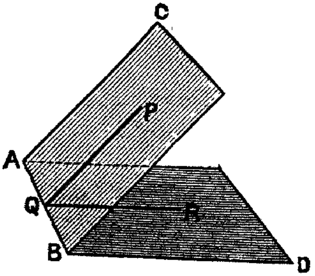
6. One plane is perpendicular to another plane when *any* straight line drawn in one of the planes perpendicular to the common section is also perpendicular to the other plane.



Thus in the adjoining figure, the plane EB is perpendicular to the plane CD, if *any* straight line PQ, drawn in the plane EB at right angles to the common section AB, is also at right angles to the plane CD.

7. **The inclination of a plane to a plane** is the acute angle contained by two straight lines drawn from any point in the common section at right angles to it, one in one plane and one in the other.

Thus in the adjoining figure, the straight line AB is the common section of the two intersecting planes BC, AD; and from Q, *any point* in AB, two straight lines QP, QR are drawn perpendicular to AB, one in each plane: then the inclination of the two planes is measured by the acute angle PQR.



NOTE. This definition assumes that the angle PQR is of constant magnitude whatever point Q is taken in AB: the truth of which assumption is proved in Proposition 10.

The angle formed by the intersection of two planes is called a **dihedral angle.**

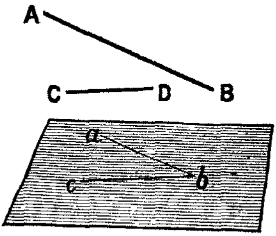
It may be proved that two planes are perpendicular to one another when the dihedral angle formed by them is a right angle.

8. **Parallel planes** are such as do not meet when produced.

9. A straight line is **parallel to a plane** if it does not meet the plane when produced.

10. The angle between two straight lines which do not meet is the angle contained by two *intersecting* straight lines respectively parallel to the two non-intersecting lines.

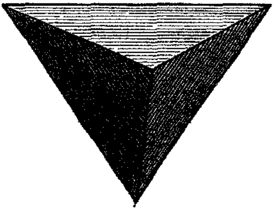
Thus if AB and CD are two straight lines which do not meet, and *ab*, *bc* are two intersecting lines parallel respectively to AB and CD; then the angle between AB and CD is measured by the angle *abc*.



11. A **solid angle** is that which is made by three or more plane angles which have a common vertex, but are not in the same plane.

A solid angle made by *three* plane angles is said to be **trihedral**; if made by more than three, it is said to be **polyhedral**.

A solid angle is sometimes called **a corner**.



12. A solid figure is any portion of space bounded by one or more surfaces, plane or curved.

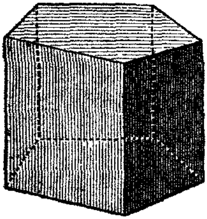
These surfaces are called the **faces** of the solid, and the intersections of adjacent faces are called **edges**.

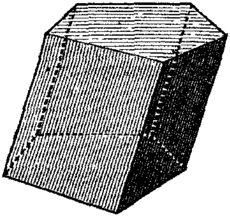
POLYHEDRA.

13. A **polyhedron** is a solid figure bounded by plane faces.

Obs. A plane rectilineal figure must at least have *three* sides; or *four*, if two of the sides are parallel. A polyhedron must at least have *four* faces; or, if two faces are parallel, it must at least have *five* faces.

14. A **prism** is a solid figure bounded by plane faces, of which two that are opposite are similar and equal polygons in parallel planes, and the other faces are parallelograms.





The polygons are called the **ends** of the prism. A prism is said to be **right** if the edges formed by each pair of adjacent parallelograms are perpendicular to the two ends; if otherwise the prism is **oblique.**

15. A **parallelepiped** is a solid figure bounded by three pairs of parallel plane faces.

Fig. 1.

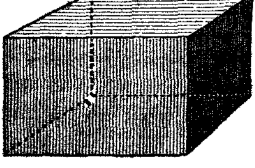
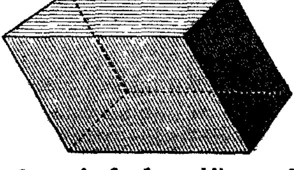
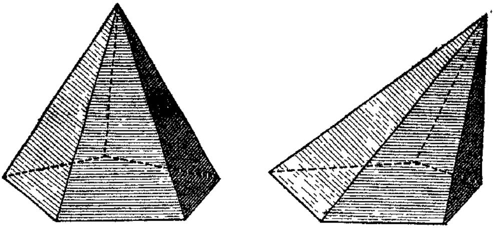


Fig. 2.



A parallelepiped may be rectangular as in fig. 1, or oblique as in fig. 2.

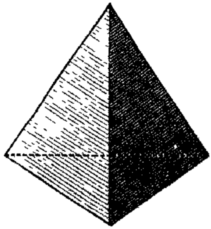
16. A **pyramid** is a solid figure bounded by plane faces, of which one is a polygon, and the rest are triangles having as bases the sides of the polygon, and as a common vertex some point not in the plane of the polygon.



The polygon is called the **base** of the pyramid.

A pyramid having for its base a *regular* polygon is said to be **right** when the vertex lies in the straight line drawn perpendicular to the base from its central point (the centre of its inscribed or circumscribed circle).

17. A **tetrahedron** is a pyramid on a triangular base: it is thus contained by *four* triangular faces.



18. Polyhedra are classified according to the number of their *faces*:

thus a **hexahedron** has *six* faces;

an **octahedron** has *eight* faces;

a **dodecahedron** has *twelve* faces.

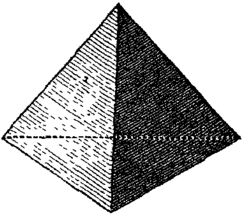
19. **Similar polyhedra** are such as have all their solid angles equal, each to each, and are bounded by the same number of similar faces.

20. A Polyhedron is **regular** when its faces are similar and equal regular polygons.

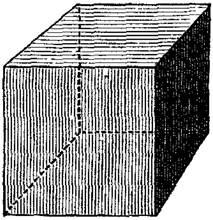
21. It will be proved (see page 425) that there can only be *five* regular polyhedra.

They are defined as follows.

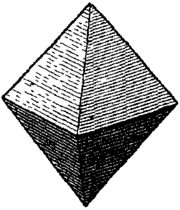
(i) **A regular tetrahedron** is a solid figure bounded by *four* plane faces, which are equal and equilateral *triangles.*



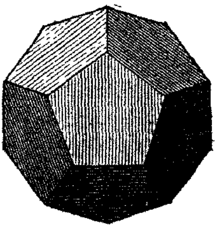
(ii) A **cube** is a solid figure bounded by *six* plane faces, which are equal *squares.*



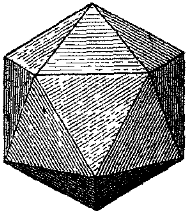
(iii) **A regular octahedron** is a solid figure bounded by *eight* plane faces, which are equal and equilateral *triangles.*



(iv) **A regular dodecahedron** is a solid figure bounded by *twelve* plane faces, which are equal and regular *pentagons.*



(v) A **regular icosahedron** is a solid figure bounded by *twenty* plane faces, which are equal and equilateral *triangles*.



SOLIDS OF REVOLUTION.

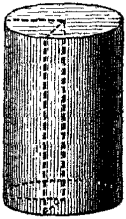
22. A **sphere** is a solid figure described by the revolution of a semicircle about its diameter, which remains fixed.

The **axis** of the sphere is the fixed straight line about which the semicircle revolves.

The **centre** of the sphere is the same as the centre of the semicircle.

A diameter of a sphere is any straight line which passes through the centre, and is terminated both ways by the surface of the sphere.

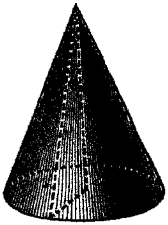
23. A **right cylinder** is a solid figure described by the revolution of a rectangle about one of its sides which remains fixed.



The **axis** of the cylinder is the fixed straight line about which the rectangle revolves.

The **bases, or ends** of the cylinder are the circular faces described by the two revolving opposite sides of the rectangle.

24. **A right cone** is a solid figure described by the revolution of a right-angled triangle about one of the sides containing the right angle which remains fixed.



The **axis** of the cone is the fixed straight line about which the triangle revolves.

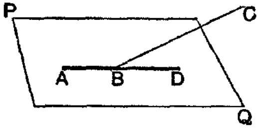
The **base** of the cone is the circular face described by that side which revolves.

The hypotenuse of the right-angled triangle in any one of its positions is called a **generating line** of the cone.

25. **Similar cones and cylinders** are those which have their axes and the diameters of their bases proportionals.

PROPOSITION 1. THEOREM.

One part of a straight line cannot be in a plane and another part outside it.



If possible, let AB, part of the st. line ABC, be in the plane PQ, and the part BC without it.

Then since the st. line AB is in the plane PQ,
∴ it can be produced in that plane. I. *Post.* 2.

Produce AB to D;
and let any other plane which passes through AD be turned about AD until it passes also through C.

Then because the points B and C are in this plane,
∴ the st. line BC is in it: I. *Def.* 5.

∴ ABC and ABD are in the same plane and are both st. lines; which is impossible. I. *Def.* 3.

∴ the st. line ABC has not one part AB in the plane PQ, and another part BC outside it. Q. E. D.

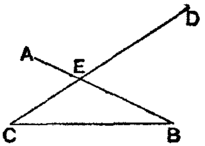
NOTE. This proposition scarcely needs proof, for the truth of it follows almost immediately from the definitions of a straight line and a plane.

It should be observed that the method of proof used in this and the next proposition rests upon the following axiom.

If a plane of unlimited extent turns about a fixed straight line as an axis, it can be made to pass through any point in space.

Proposition 2. Theorem.

Any two straight lines which cut one another are in one plane: and any three straight lines, of which each pair intersect one another, are in one plane.



Let the two st. lines AB and CD intersect at E;
and let the st. line BC be drawn cutting AB and CD at B and C:

then (i) AB and CD shall lie in one plane.
(ii) AB, BC, CD shall lie in one plane.

(i) Let any plane pass through AB;
and let this plane be turned about AB until it passes through C.

Then, since C and E are points in this plane,
∴ the whole st. line CED is in it. I. *Def.* 5 and XI. 1.
That is, AB and CD lie in one plane.

(ii) And since B and C are points in the plane which contains AB and CD,
∴ also the st. line BC lies in this plane. Q. E. D.

Corollary. *One, and only one, plane can be made to pass through two given intersecting straight lines.*

Hence the position of a plane is fixed,

(i) if it passes through a given straight line and a given point outside it; *Ax.* p. 393.

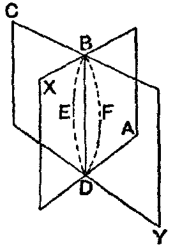
(ii) if it passes through two intersecting straight lines; XI. 2.

(iii) if it passes through three points not collinear; XI. 2.

(iv) if it passes through two parallel straight lines. I. *Def.* 25.

Proposition 3. Theorem.

If two planes cut one another their common section is a straight line.



Let the two planes XA, CY cut one another, and let BD be their common section:

then shall BD be a st. line.

For if not, from B to D in the plane XA draw the st. line BED;

and in the plane CY draw the st. line BFD.

Then the st. lines BED, BFD have the same extremities;

∴ they include a space;

but this is impossible.

∴ the common section BD cannot be otherwise than a st. line. Q. E. D.

Or, more briefly thus—

Let the planes XA, CY cut one another, and let B and D be two points in their common section.

Then because B and D are two points in the plane XA,

∴ the st. line joining B, D lies in that plane. I. *Def.* 5.

And because B and D are two points in the plane CY,

∴ the st. line joining B, D lies in that plane.

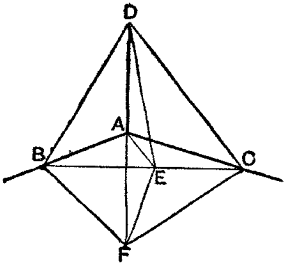
Hence the st. line BD lies in *both planes*,

and is therefore their common section.

That is, the common section of the two planes is a *straight* line. Q. E. D.

PROPOSITION 4. THEOREM. [Alternative Proof.]

If a straight line is perpendicular to each of two straight lines at their point of intersection, it shall also be perpendicular to the plane in which they lie.



Let the straight line AD be perp. to each of the st. lines AB, AC at A their point of intersection:
then shall AD be perp. to the plane in which AB and AC lie.

Produce DA to F, making AF equal to DA.

Draw any st. line BC in the plane of AB, AC, to cut AB, AC at B and C;
and in the same plane draw through A any st. line AE to cut BC at E.

It is required to prove that AD is perp. to AE.

Join DB, DE, DC; and FB, FE, FC.

Then in the $\triangle^s$ BAD, BAF,
because DA = FA, *Constr.*
and the common side AB is perp. to DA, FA:
∴ BD = BF. I. 4.
Similarly CD = CF.

Now if the $\triangle$BFC be turned about its base BC until the vertex F comes into the plane of the $\triangle$BDC,
then F will coincide with D,
since the conterminous sides of the triangles are equal. I. 7.

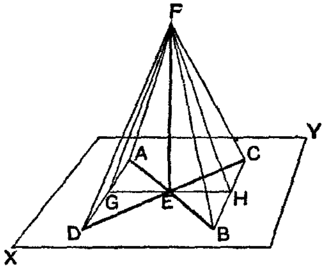
∴ EF will coincide with ED,
that is, EF = ED.

Hence in the △s DAE, FAE,
since DA, AE, ED = FA, AE, EF respectively,
∴ the ∠ DAE = the ∠ FAE. I. 8.
That is, DA is perp. to AE.

Similarly it may be shewn that DA is perp. to every st. line which meets it in the plane of AB, AC;
∴ DA is perp. to this plane. Q.E.D.

PROPOSITION 4. THEOREM. [Euclid's Proof.]

If a straight line is perpendicular to each of two straight lines at their point of intersection, it shall also be perpendicular to the plane in which they lie.



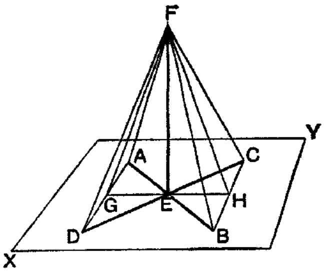
Let the st. line EF be perp. to each of the st. lines AB, DC at E their point of intersection:
then shall EF be also perp. to the plane XY, in which AB and DC lie.

Make EA, EC, EB, ED all equal, and join AD, BC.

Through E in the plane XY draw any st. line cutting AD and BC in G and H.

Take any pt. F in EF, and join FA, FG, FD, FB, FH, FC.

Then in the △s AED, BEC,
because AE, ED = BE, EC respectively, *Constr.*
and the ∠ AED = the ∠ BEC; I. 15.
∴ AD = BC, and the ∠ DAE = the ∠ CBE. I. 4.



In the $\triangle^s$ AEG, BEH,

because the $\angle$ GAE = the $\angle$ HBE, *Proved.*

and the $\angle$ AEG = the $\angle$ BEH, I. 15.

and EA = EB; *Constr.*

$\therefore$ EG = EH, and AG = BH. I. 26.

Again in the $\triangle^s$ FEA, FEB,

because EA = EB,

and the common side FE is perp. to EA, EB; *Hyp.*

$\therefore$ FA = FB. I. 4.

Similarly FC = FD.

Again in the $\triangle^s$ DAF, CBF,

because DA, AF, FD = CB, BF, FC, respectively,

$\therefore$ the $\angle$ DAF = the $\angle$ CBF. I. 8.

And in the $\triangle^s$ FAG, FBH,

because FA, AG, = FB, BH, respectively,

and the $\angle$ FAG = the $\angle$ FBH, *Proved.*

$\therefore$ FG = FH. I. 4.

Lastly in the $\triangle^s$ FEG, FEH,

because FE, EG, GF = FE, EH, HF, respectively,

$\therefore$ the $\angle$ FEG = the $\angle$ FEH; I. 8.

that is, FE is perp. to GH.

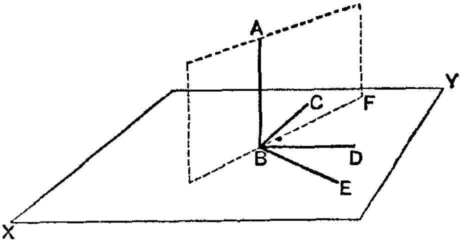
Similarly it may be shewn that FE is perp. to *every* st. line which meets it in the plane XY,

$\therefore$ FE is perp. to this plane. XI. *Def.*

Q.E.D.

PROPOSITION 5. THEOREM.

If a straight line is perpendicular to each of three concurrent straight lines at their point of intersection, these three straight lines shall be in one plane.



Let the straight line AB be perpendicular to each of the straight lines BC, BD, BE, at B their point of intersection:

then shall BC, BD, BE be in one plane.

Let XY be the plane which passes through BE, BD; XI. 2.
and, if possible, suppose that BC is not in this plane.

Let AF be the plane which passes through AB, BC; and let the common section of the two planes XY, AF be the st. line BF. XI. 3.

Then since AB is perp. to BE and BD,
and since BF is in the same plane as BE, BD,
∴ AB is also perp. to BF. XI. 4.

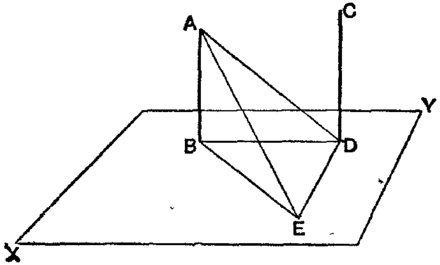
But AB is perp. to BC; *Hyp.*
∴ the ∠s ABF, ABC, which are in the same plane, are both rt. angles; which is impossible.

∴ BC is not outside the plane of BD, BE:
that is, BC, BD, BE are in one plane.

Q.E.D.

PROPOSITION 6. THEOREM.

If two straight lines are perpendicular to the same plane, they shall be parallel to one another.



Let the st. lines AB, CD be perp. to the plane XY:
then shall AB and CD be par[1].*

Let AB and CD meet the plane XY at B and D.
Join BD;
and in the plane XY draw DE perp. to BD, making DE equal to AB.
Join BE, AE, AD.

Then since AB is perp. to the plane XY, *Hyp.*
∴ AB is also perp. to BD and BE, which meet it in that plane; XI. *Def.* 1.
that is, the ∠s ABD, ABE are rt. angles.
Similarly the ∠s CDB, CDE are rt. angles.

Now in the △s ABD, EDB,
because AB, BD = ED, DB, respectively, *Constr.*
and the ∠ ABD = the ∠ EDB, being rt. angles;
∴ AD = EB. I. 4.

Again in the △s ABE, EDA,
because AB, BE = ED, DA, respectively,
and AE is common;
∴ the ∠ ABE = the ∠ EDA. I. 8.

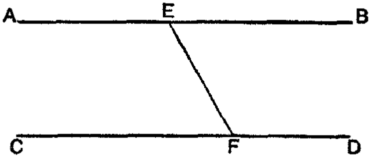
* NOTE. In order to shew that AB and CD are parallel, it is necessary to prove that (i) they are *in the same plane*, (ii) the angles ABD, CDB, are supplementary.

But the ∠ ABE is a rt. angle; *Proved.*
∴ the ∠ EDA is a rt. angle.
But the ∠ EDB is a rt. angle by construction,
and the ∠ EDC is a rt. angle, since CD is perp. to the plane XY. *Hyp.*
Hence ED is perp. to the three lines DA, DB, and DC;
∴ DA, DB, DC are in one plane. XI. 5.
But AB is in the plane which contains DA, DB; XI. 2.
∴ AB, BD, DC are in one plane.
And each of the ∠s ABD, CDB is a rt. angle; *Hyp.*
∴ AB and CD are par[l]. I. 28.

Q.E.D.

PROPOSITION 7. THEOREM.

If two straight lines are parallel, the straight line which joins any point in one to any point in the other is in the same plane as the parallels.



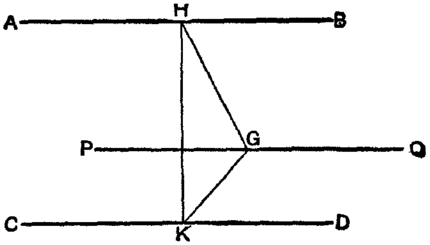
Let AB and CD be two par[l] st. lines,
and let E, F be any two points, one in each st. line:
then shall the st. line which joins E, F be in the same plane as AB, CD.

For since AB and CD are par[l],
∴ they are in one plane. I. *Def.* 25.
And since the points E and F are in this plane,
∴ the st. line which joins them lies wholly in this plane. I. *Def.* 5.
That is, EF is in the plane of the par[ls] AB, CD.

Q.E.D.

PROPOSITION 9. THEOREM.

Two straight lines which are parallel to a third straight line are parallel to one another.



Let the st. lines AB, CD be each par[l] to the st. line PQ: then shall AB be par[l] to CD.

I. If AB, CD and PQ are *in one plane*, the proposition has already been proved. I. 30.

II. But if AB, CD and PQ are not in one plane,
in PQ take any point G;
and from G, in the plane of the par[ls] AB, PQ, draw GH perp. to PQ; I. 11.
also from G, in the plane of the par[ls] CD, PQ, draw GK perp. to PQ. I. 11.

Then because PQ is perp. to GH and GK, *Constr.*
∴ PQ is perp. to the plane HGK, which contains them. XI. 4.

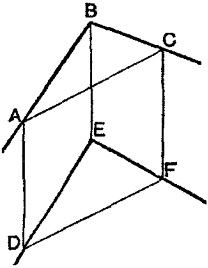
But AB is par[l] to PQ; *Hyp.*
∴ AB is also perp. to the plane HGK. XI. 8.
Similarly, CD is perp. to the plane HGK.

Hence AB and CD, being perp. to the same plane, are par[l] to one another, XI. 6.

Q.E.D.

PROPOSITION 10. THEOREM.

If two intersecting straight lines are respectively parallel to two other intersecting straight lines not in the same plane with them, then the first pair and the second pair shall contain equal angles.



Let the st. lines AB, BC be respectively par[l] to the st. lines DE, EF, which are not in the same plane with them: then shall the ∠ ABC = the ∠ DEF.

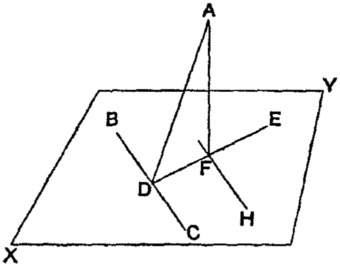
In BA and ED, make BA equal to ED;
and in BC and EF, make BC equal to EF.
Join AD, BE, CF, AC, DF.

Then because BA is equal and par[l] to ED,
Hyp. and Constr.
∴ AD is equal and par[l] to BE. I. 33.
And because BC is equal and par[l] to EF,
∴ CF is equal and par[l] to BE. I. 33.
∴ AD is equal and par[l] to CF; XI. 9.
hence it follows that AC is equal and par[l] to DF. I. 33.
Then in the △[s] ABC, DEF,
because AB, BC, AC = DE, EF, DF, respectively,
∴ the ∠ ABC = the ∠ DEF. I. 8.

Q.E.D.

PROPOSITION 11. PROBLEM.

To draw a straight line perpendicular to a given plane from a given point outside it.



Let A be the given point outside the plane XY.

It is required to draw from A a st. line perp. to the plane XY.

Draw any st. line BC in the plane XY;

and from A draw AD perp. to BC. I. 12.

Then if AD is also perp. to the plane XY, what was required is done.

But if not, from D draw DE in the plane XY perp. to BC; I. 11.

and from A draw AF perp. to DE. I. 12.

Then AF shall be perp. to the plane XY.

Through F draw FH par[l] to BC. I. 31.

Now because CD is perp. to DA and DE, *Constr.*

∴ CD is perp. to the plane containing DA, DE. XI. 4.

And HF is par[l] to CD;

∴ HF is also perp. to the plane containing DA, DE. XI. 8.

And since FA meets HF in this plane,

∴ the ∠ HFA is a rt. angle; XI. *Def.* 1.

that is, AF is perp. to FH.

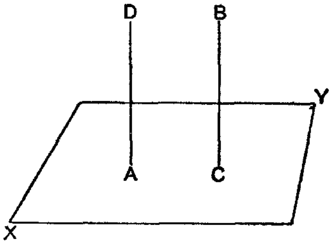
And AF is also perp. to DE; *Constr.*

∴ AF is perp. to the plane containing FH, DE;

that is, AF is perp. to the plane XY. Q.E.F.

PROPOSITION 12. PROBLEM.

To draw a straight line perpendicular to a given plane from a given point in the plane.



Let A be the given point in the plane XY.

It is required to draw from A a st. line perp. to the plane XY.

From any point B outside the plane XY draw BC perp. to the plane. XI. 11.

Then if BC passes through A, what was required is done.

But if not, from A draw AD par[l] to BC. I. 31.

Then AD shall be the perpendicular required.

For since BC is perp. to the plane XY, *Constr.*
and since AD is par[l] to BC, *Constr.*
∴ AD is also perp. to the plane XY. XI. 8.

Q.E.F.

EXERCISES.

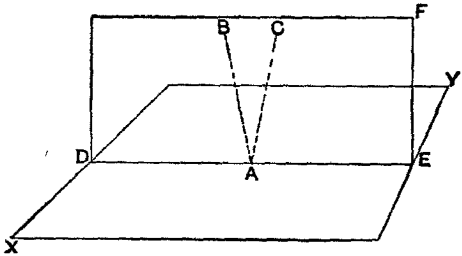
1. Equal straight lines drawn to meet a plane from a point without it are equally inclined to the plane.

2. Find the locus of the foot of the perpendicular drawn from a given point upon any plane which passes through a given straight line.

3. From a given point A a perpendicular AF is drawn to a plane XY; and from F, FD is drawn perpendicular to BC, any line in that plane: shew that AD is also perpendicular to BC.

PROPOSITION 13. THEOREM.

Only one perpendicular can be drawn to a given plane from a given point either in the plane or outside it.



CASE I. Let the given point A be in the given plane XY; and, if possible, let two perps. AB, AC be drawn from A to the plane XY.

Let DF be the plane which contains AB and AC; and let the st. line DE be the common section of the planes DF and XY. XI. 3.

Then the st. lines AB, AC, AE are in one plane.

And because BA is perp. to the plane XY, *Hyp.*

∴ BA is also perp. to AE, which meets it in this plane; XI. *Def.* 1.

that is, the ∠ BAE is a rt. angle.

Similarly, the ∠ CAE is a rt. angle.

∴ the ∠s BAE, CAE, which are in the same plane, are equal to one another.

Which is impossible.

∴ two perpendiculars cannot be drawn to the plane XY from the point A in that plane.

CASE II. Let the given point A be outside the plane XY.

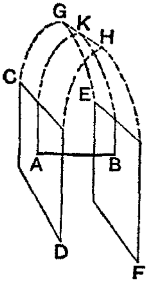
Then two perps cannot be drawn from A to the plane;

for if there could be two, they would be par^l, XI. 6.

which is absurd. Q.E.D.

PROPOSITION 14. THEOREM.

Planes to which the same straight line is perpendicular are parallel to one another.



Let the st. line AB be perp. to each of the planes CD, EF: then shall these planes be par[l].

For if not, they will meet when produced.

If possible, let the two planes meet, and let the st. line GH be their common section. XI. 3.

In GH take any point K;
and join AK, BK.

Then because AB is perp. to the plane EF,
∴ AB is also perp. to BK, which meets it in this plane;
XI. *Def.* 1.

that is, the ∠ ABK is a rt. angle.
Similarly, the ∠ BAK is a rt. angle.

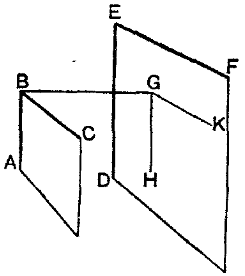
∴ in the △ KAB, the two ∠[s] ABK, BAK are together equal to two rt. angles;

which is impossible. I. 17.

∴ the planes CD, EF, though produced, do not meet:
that is, they are par[l]. Q.E.D.

PROPOSITION 15. THEOREM.

If two intersecting straight lines are parallel respectively to two other intersecting straight lines which are not in the same plane with them, then the plane containing the first pair shall be parallel to the plane containing the second pair.



Let the st. lines AB, BC be respectively par^l to the st. lines DE, EF, which are not in the same plane as AB, BC:

then shall the plane containing AB, BC be par^l to the plane containing DE, EF.

From B draw BG perp. to the plane of DE, EF; XI. 11.
and let it meet that plane at G.

Through G draw GH, GK par^l respectively to DE, EF. I. 31.

Then because BG is perp. to the plane of DE, EF,
∴ BG is also perp. to GH and GK, which meet it in that plane: XI. *Def.* 1.
that is, each of the ∠^{s} BGH, BGK is a rt. angle.

Now because BA is par^l to ED, *Hyp.*
and because GH is also par^l to ED, *Constr.*
∴ BA is par^l to GH. XI. 9.

And since the ∠ BGH is a rt. angle; *Proved.*
∴ the ∠ ABG is a rt. angle. I. 29.

Similarly the ∠ CBG is a rt. angle.

Then since BG is perp. to each of the st. lines BA, BC,

∴ BG is perp. to the plane containing them. XI. 4.

But BG is also perp. to the plane of ED, EF : *Constr.*

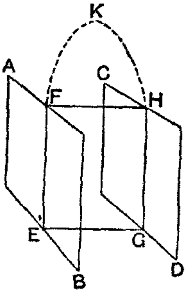
that is, BG is perp. to the two planes AC, DF :

∴ these planes are par[l]. XI. 14.

Q.E.D.

PROPOSITION 16. THEOREM.

If two parallel planes are cut by a third plane their common sections with it shall be parallel.



Let the par[l] planes AB, CD be cut by the plane EFHG, and let the st. lines EF, GH be their common sections with it:

then shall EF, GH be par[l].

For if not, EF and GH will meet if produced.

If possible, let them meet at K.

Then since the whole st. line EFK is in the plane AB, XI. 1.

and K is a point in that line,

∴ the point K is in the plane AB.

Similarly the point K is in the plane CD.

Hence the planes AB, CD when produced meet at K;

which is impossible, since they are par[l]. *Hyp.*

∴ the st. lines EF and GH do not meet;

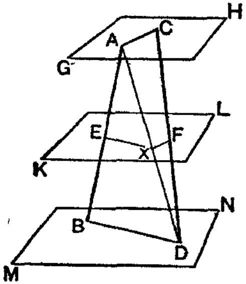
and they are in the same plane EFHG;

∴ they are par[l]. I. *Def.* 25.

Q.E.D.

Proposition 17. Theorem.

Straight lines which are cut by parallel planes are cut proportionally.



Let the st. lines AB, CD be cut by the three par[1] planes GH, KL, MN at the points A, E, B, and C, F, D:
then shall AE : EB :: CF : FD.

Join AC, BD, AD;
and let AD meet the plane KL at the point X:
join EX, XF.

Then because the two par[1] planes KL, MN are cut by the plane ABD,
∴ the common sections EX, BD are par[1]. xi. 16.

and because the two par[1] planes GH, KL are cut by the plane DAC,
∴ the common sections XF, AC are par[1]. xi. 16.

And because EX is par[1] to BD, a side of the △ABD,
∴ AE : EB :: AX : XD. vi. 2.

Again because XF is par[1] to AC, a side of the △DAC,
∴ AX : XD :: CF : FD. vi. 2.

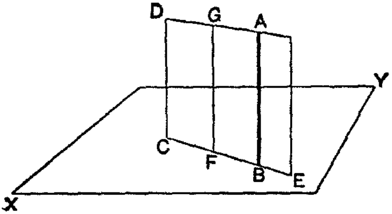
Hence AE : EB :: CF : FD. v. 1.

Q.E.D.

Definition. One plane is perpendicular to another plane, when any straight line drawn in one of the planes perpendicular to their common section is also perpendicular to the other plane. [Book xi. *Def.* 6.]

PROPOSITION 18. THEOREM.

If a straight line is perpendicular to a plane, then every plane which passes through the straight line is also perpendicular to the given plane.



Let the st. line AB be perp. to the plane XY;
and let DE be any plane passing through AB:
then shall the plane DE be perp. to the plane XY.

Let the st. line CE be the common section of the planes XY, DE. XI. 3.

From F, *any* point in CE, draw FG in the plane DE perp. to CE. I. 11.

Then because AB is perp. to the plane XY, *Hyp.*
∴ AB is also perp. to CE, which meets it in that plane, XI. *Def.* 1.
that is, the ∠ ABF is a rt. angle.
But the ∠ GFB is also a rt. angle; *Constr.*
∴ GF is par[l] to AB. I. 28.
And AB is perp. to the plane XY, *Hyp.*
∴ GF is also perp. to the plane XY. XI. 8.

Hence it has been shewn that any st. line GF drawn in the plane DE perp. to the common section CE is also perp. to the plane XY.

∴ the plane DE is perp. to the plane XY. XI. *Def.* 6.

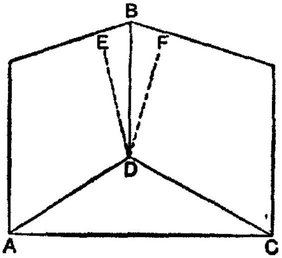
Q.E.D.

EXERCISE.

Shew that two planes are perpendicular to one another when the dihedral angle formed by them is a right angle.

PROPOSITION 19. THEOREM.

If two intersecting planes are each perpendicular to a third plane, their common section shall also be perpendicular to that plane.



Let each of the planes AB, BC be perp. to the plane ADC, and let BD be their common section:

then shall BD be perp. to the plane ADC.

For if not, from D draw in the plane AB the st. line DE perp. to AD, the common section of the planes ADB, ADC: I. 11.

and from D draw in the plane BC the st. line DF perp. to DC, the common section of the planes BDC, ADC.

Then because the plane BA is perp. to the plane ADC, *Hyp.*

and DE is drawn in the plane BA perp. to AD the common section of these planes, *Constr.*

∴ DE is perp. to the plane ADC. XI. *Def.* 6.

Similarly DF is perp. to the plane ADC.

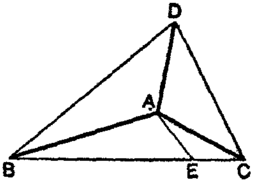
∴ from the point D two st. lines are drawn perp. to the plane ADC; which is impossible. XI. 13.

Hence DB cannot be otherwise than perp. to the plane ADC.

Q.E.D.

PROPOSITION 20. THEOREM.

Of the three plane angles which form a trihedral angle, any two are together greater than the third.



Let the trihedral angle at A be formed by the three plane ∠ˢ BAD, DAC, BAC:
then shall any two of them, such as the ∠ˢ BAD, DAC, be together greater than the third, the ∠ BAC.

CASE I. If the ∠ BAC is less than, or equal to, either of the ∠ˢ BAD, DAC;
it is evident that the ∠ˢ BAD, DAC are together greater than the ∠ BAC.

CASE II. But if the ∠ BAC is greater than either of the ∠ˢ BAD, DAC;
then at the point A in the plane BAC make the ∠ BAE equal to the ∠ BAD;

and cut off AE equal to AD.

Through E, and in the plane BAC, draw the st. line BEC cutting AB, AC at B and C:

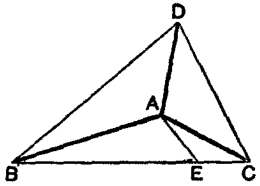
join DB, DC.

Then in the △ˢ BAD, BAE,

since BA, AD = BA, AE, respectively, *Constr.*
and the ∠ BAD = the ∠ BAE; *Constr.*
∴ BD = BE. I. 4.

Again in the △ BDC, since BD, DC are together greater than BC, I. 20.

and BD = BE, *Proved.*
∴ DC is greater than EC.



And in the △ˢ DAC, EAC,

because DA, AC = EA, AC respectively, *Constr.*

but DC is greater than EC; *Proved.*

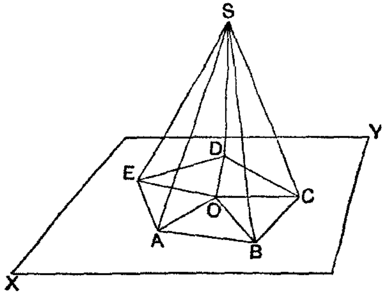
∴ the ∠ DAC is greater than the ∠ EAC. I. 25.

But the ∠ BAD = the ∠ BAE; *Constr.*

∴ the two ∠ˢ BAD, DAC are together greater than the ∠ BAC. Q.E.D.

Proposition 21. Theorem.

Every (convex) solid angle is formed by plane angles which are together less than four right angles.



Let the solid angle at S be formed by the plane ∠ˢ ASB, BSC, CSD, DSE, ESA:

then shall the sum of these plane angles be less than four rt. angles.

For let a plane XY intersect all the arms of the plane angles on the same side of the vertex at the points A, B, C, D, E: and let AB, BC, CD, DE, EA be the common sections of the plane XY with the planes of the several angles.

Within the polygon ABCDE take any point O;
and join O to each of the vertices of the polygon.

Then since the ∠s SAE, SAB, EAB form the trihedral angle A,

∴ the ∠s SAE, SAB are together greater than the ∠ EAB; XI. 20.

that is,
the ∠s SAE, SAB are together greater than the ∠s OAE, OAB.

Similarly,
the ∠s SBA, SBC are together greater than the ∠s OBA, OBC:
and so on, for each of the angular points of the polygon.

Thus by addition,
the sum of the base angles of the triangles whose vertices are at S, is greater than the sum of the base angles of the triangles whose vertices are at O.

But these two systems of triangles are equal in number;
∴ the sum of all the angles of the one system is equal to the sum of all the angles of the other.

It follows that the sum of the vertical angles at S is less than the sum of the vertical angles at O.

But the sum of the angles at O is four rt. angles;
∴ the sum of the angles at S is less than four rt. angles.

Q.E.D.

Note. This proposition was not given in this form by Euclid, who established its truth only in the case of *trihedral* angles. The above demonstration, however, applies to all cases in which the polygon ABCDE is *convex*, but it must be observed that without this condition the proposition is not necessarily true.

A solid angle is **convex** when it lies entirely on one side of each of the infinite planes which pass through its plane angles. If this is the case, the polygon ABCDE will have no *re-entrant* angle. And it is clear that it would not be possible to apply XI. 20 to a vertex at which a re-entrant angle existed.

EXERCISES ON BOOK XI.

1. Equal straight lines drawn to a plane from a point without it have equal projections on that plane.

2. If S is the centre of the circle circumscribed about the triangle ABC, and if SP is drawn perpendicular to the plane of the triangle, shew that any point in SP is equidistant from the vertices of the triangle.

3. Find the locus of points in space equidistant from three given points.

4. From Example 2 deduce a practical method of drawing a perpendicular from a given point to a plane, having given ruler, compasses, and a straight rod longer than the required perpendicular.

5. Give a geometrical construction for drawing a straight line equally inclined to three straight lines which meet in a point, but are not in the same plane.

6. In a *gauche* quadrilateral (that is, a quadrilateral whose sides are not in the same plane) if the middle points of adjacent sides are joined, the figure thus formed is a parallelogram.

7. AB and AC are two straight lines intersecting at right angles, and from B a perpendicular BD is drawn to the plane in which they are: shew that AD is perpendicular to AC.

8. If two intersecting planes are cut by two parallel planes, the lines of section of the first pair with each of the second pair contain equal angles.

9. If a straight line is parallel to a plane, shew that any plane passing through the given straight line will intersect the given plane in a line of section which is parallel to the given line.

10. Two intersecting planes pass one through each of two parallel straight lines; shew that the common section of the planes is parallel to the given lines.

11. If a straight line is parallel to each of two intersecting planes, it is also parallel to the common section of the planes.

12. Through a given point in space draw a straight line to intersect each of two given straight lines which are not in the same plane.

13. If AB, BC, CD are straight lines not all in one plane, shew that a plane which passes through the middle point of each one of them is parallel both to AC and BD.

14. From a given point A a perpendicular AB is drawn to a plane XY; and a second perpendicular AE is drawn to a straight line CD in the plane XY: shew that EB is perpendicular to CD.

15. From a point A two perpendiculars AP, AQ are drawn one to each of two intersecting planes: shew that the common section of these planes is perpendicular to the plane of AP, AQ.

16. From A, a point in one of two given intersecting planes, AP is drawn perpendicular to the first plane, and AQ perpendicular to the second: if these perpendiculars meet the second plane at P and Q, shew that PQ is perpendicular to the common section of the two planes.

17. A, B, C, D are four points not in one plane, shew that the four angles of the gauche quadrilateral ABCD are together less than four right angles.

18. OA, OB, OC are three straight lines drawn from a given point O not in the same plane, and OX is another straight line within the solid angle formed by OA, OB, OC: shew that

(i) the sum of the angles AOX, BOX, COX is greater than half the sum of the angles AOB, BOC, COA.

(ii) the sum of the angles AOX, COX is less than the sum of the angles AOB, COB.

(iii) the sum of the angles AOX, BOX, COX is less than the sum of the angles AOB, BOC, COA.

19. OA, OB, OC are three straight lines forming a solid angle at O, and OX bisects the plane angle AOB; shew that the angle XOC is less than half the sum of the angles AOC, BOC.

20. If a point be equidistant from the angles of a right-angled triangle and not in the plane of the triangle, the line joining it with the middle point of the hypotenuse is perpendicular to the plane of the triangle.

21. The angle which a straight line makes with its projection on a plane is less than that which it makes with any other straight line which meets it in that plane.

22. Find a point in a given plane such that the sum of its distances from two given points (not in the plane but on the same side of it) may be a minimum.

23. If two straight lines in one plane are equally inclined to another plane, they will be equally inclined to the common section of these planes.

24. PA, PB, PC are three concurrent straight lines each of which is at right angles to the other two: PX, PY, PZ are perpendiculars drawn from P to BC, CA, AB respectively. Shew that XYZ is the pedal triangle of the triangle ABC.

25. PA, PB, PC are three concurrent straight lines each of which is at right angles to the other two, and from P a perpendicular PO is drawn to the plane of ABC: shew that O is the orthocentre of the triangle ABC.

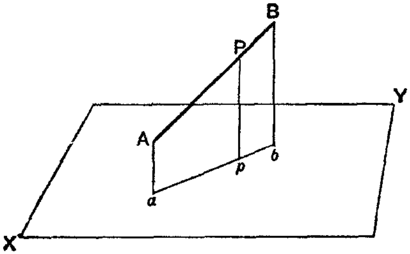
THEOREMS AND EXAMPLES ON BOOK XI.

DEFINITIONS.

(i) Lines which are drawn on a plane, or through which a plane may be made to pass, are said to be **co-planar.**

(ii) The **projection of a line** on a plane is the locus of the feet of perpendiculars drawn from all points in the given line to the plane.

THEOREM 1. *The projection of a straight line on a plane is itself a straight line.*



Let AB be the given st. line, and XY the given plane.

From P, *any* point in AB, draw P*p* perp. to the plane XY: it is required to shew that the locus of *p* is a st. line.

From A and B draw A*a*, B*b* perp. to the plane XY.

Now since A*a*, P*p*, B*b* are all perp. to the plane XY,

∴ they are par[l]. XI. 6.

And since these par[ls] all intersect AB,

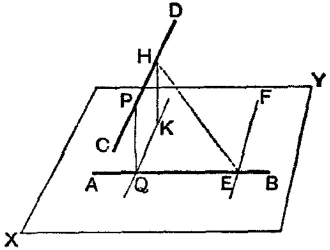
∴ they are co-planar. XI. 7.

∴ the point *p* is in the common section of the planes A*b*, XY; that is, *p* is in the st. line *ab*.

But *p* is any point in the projection of AB,

∴ the projection of AB is the st. line *ab*. Q.E.D.

THEOREM 2. *Draw a perpendicular to each of two straight lines which are not in the same plane. Prove that this perpendicular is the shortest distance between the two lines.*



Let AB and CD be the two straight lines, not in the same plane.

(i) It is required to draw a st. line perp. to each of them.

Through E, any point in AB, draw EF par[l] to CD.
Let XY be the plane which passes through AB, EF.

From H, any point in CD, draw HK perp. to the plane XY. XI. 11.
And through K, draw KQ par[l] to EF, cutting AB at Q.
Then KQ is also par[l] to CD; XI. 9.
and CD, HK, KQ are in one plane. XI. 7.

From Q, draw QP par[l] to HK to meet CD at P.
Then shall PQ be perp. to both AB and CD.

For, since HK is perp. to the plane XY, and PQ is par[l] to HK, *Constr.*
∴ PQ is perp. to the plane XY; XI. 8.
∴ PQ is perp. to AB, which meets it in that plane. *Def.* 1.
For a similar reason PQ is perp. to QK,
∴ PQ is also perp. to CD, which is par[l] to QK.

(ii) It is required to shew that PQ is the least of all st. lines drawn from AB to CD.

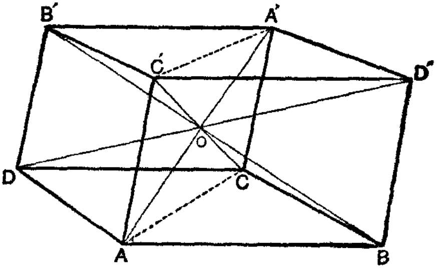
Take HE, any other st. line drawn from AB to CD.

Then HE, being oblique to the plane XY is greater than the perp. HK. p. 403, Ex. 1.
∴ HE is also greater than PQ. Q.E.D.

DEFINITION. A parallelepiped is a solid figure bounded by three pairs of parallel faces.

THEOREM 3. (i) *The faces of a parallelepiped are parallelograms, of which those which are opposite are identically equal.*

(ii) *The four diagonals of a parallelepiped are concurrent and bisect one another.*



Let ABA′B′ be a par^{ped}, of which ABCD, C′D′A′B′ are opposite faces.

(i) Then all the faces shall be par^{ms}, and the opposite faces shall be identically equal.

For since the planes DA′, AD′ are par^{l}, *Def.*
and the plane DB meets them,
∴ the common sections AB and DC are par^{l}. XI. 16.
Similarly AD and BC are par^{l}.
∴ the fig. ABCD is a par^{m},
and AB = DC; also AD = BC. I. 34.

Similarly each of the faces of the par^{ped} is a par^{m};
so that the edges AB, C′D′, B′A′, DC are equal and par^{l}:
so also are the edges AD, C′B′, D′A′, BC; and likewise AC′, BD′, CA′, DB′.

Then in the opp. faces ABCD, C′D′A′B′,
we have AB = C′D′ and BC = D′A′; *Proved.*
and since AB, BC are respectively par^{l} to C′D′, D′A′,
∴ the ∠ ABC = the ∠ C′D′A′; XI. 10.
∴ the par^{m} ABCD = the par^{m} C′D′A′B′ identically. P. 64, Ex. 11.

(ii) The diagonals AA′, BB′, CC′, DD′ shall be concurrent and bisect one another.

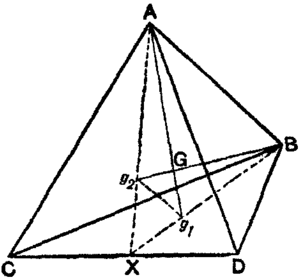
Join AC and A′C′.

Then since AC′ is equal and par[l] to A′C,
∴ the fig. ACA′C′ is a par[m];
∴ its diagonals AA′, CC′ bisect one another.
That is, AA′ passes through O, the middle point of CC′.

Similarly if BC′ and B′C were joined, the fig. BCB′C′ would be a par[m];
∴ the diagonals BB′, CC′ bisect one another.
That is, BB′ also passes through O the middle point of CC′.

Similarly it may be shewn that DD′ passes through, and is bisected at, O. Q.E.D.

THEOREM 4. *The straight lines which join the vertices of a tetrahedron to the centroids of the opposite faces are concurrent.*



Let ABCD be a tetrahedron, and let g_1, g_2, g_3, g_4 be the centroids of the faces opposite respectively to A, B, C, D.

Then shall Ag_1, Bg_2, Cg_3, Dg_4 be concurrent.

Take X the middle point of the edge CD;
then g_1 and g_2 must lie respectively in BX and AX,
so that $BX = 3 . Xg_1$, P. 105, Ex. 4.
and $AX = 3 . Xg_2$;
∴ g_1g_2 is par[l] to AB.

And Ag_1, Bg_2 must intersect one another, since they are both in the plane of the △AXB:
let them intersect at the point G.

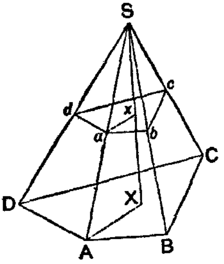
Then by similar △[s], $AG : Gg_1 = AB : g_1g_2$
$= AX : Xg_2$
$= 3 : 1.$

∴ Bg_2 cuts Ag_1 at a point G whose distance from $g_1 = \frac{1}{4} . Ag_1$.

Similarly it may be shewn that Cg_3 and Dg_4 cut Ag_1 at the same point;
∴ these lines are concurrent. Q.E.D.

THEOREM 5. (i) *If a pyramid is cut by planes drawn parallel to its base, the sections are similar to the base.*

(ii) *The areas of such sections are in the duplicate ratio of their perpendicular distances from the vertex.*



Let SABCD be a pyramid, and *abcd* the section formed by a plane drawn par[l] to the base ABCD.

(i) Then the figs. ABCD, *abcd* shall be similar.

Because the planes *abcd*, ABCD are par[l],
and the plane AB*ba* meets them,
∴ the common sections *ab*, AB are par[l].

Similarly *bc* is par[l] to BC; *cd* to CD; and *da* to DA.

And since *ab*, *bc* are respectively par[l] to AB, BC,
∴ the ∠ *abc* = the ∠ ABC. xi. 10.

Similarly the remaining angles of the fig. *abcd* are equal to the corresponding angles of the fig. ABCD.

And since the △[s] S*ab*, SAB are similar,
∴ *ab* : AB = S*b* : SB
= *bc* : BC, for the △[s] S*bc*, BC*c* are similar.

Or, *ab* : *bc* = AB : BC.

In like manner, *bc* : *cd* = BC : CD.

And so on.

∴ the figs. *abcd*, ABCD are equiangular, and have their sides about the equal angles proportional.

∴ they are similar.

(ii) From S draw SxX perp. to the planes *abcd*, ABCD and meeting them at x and X.

Then shall fig. *abcd* : fig. ABCD = Sx^2 : SX^2.

Join ax, AX.

Then it is clear that the △[s] Sax, SAX are similar.

And the fig. *abcd* : fig. ABCD = ab^2 : AB^2 vi. 20.
= aS^2 : AS^2,
= Sx^2 : SX^2. Q.E.D.

DEFINITION.

A polyhedron is *regular* when its faces are similar and equal regular polygons.

THEOREM 6. *There cannot be more than five regular polyhedra.*

This is proved by examining the number of ways in which it is possible to form a solid angle out of the plane angles of various regular polygons; bearing in mind that *three* plane angles at least are required to form a solid angle, and the sum the plane angles forming a solid angle *is less than four right angles.* XI. 21.

Suppose the faces of the regular polyhedron to be *equilateral triangles.*

Then since each angle of an equilateral triangle is $\frac{2}{3}$ of a right angle, it follows that a solid angle may be formed (i) by *three*, (ii) by *four*, or (iii) by *five* such faces; for the sums of the plane angles would be respectively (i) two right angles, (ii) $\frac{8}{3}$ of a right angle, (iii) $\frac{10}{3}$ of a right angle;

that is, in all three cases the sum of the plane angles would be less than four right angles.

But it is impossible to form a solid angle of *six* or more equilateral triangles, for then the sum of the plane angles would be equal to, or greater than four right angles.

Again, suppose that the faces of the polyhedron are *squares.*

(iv) Then it is clear that a solid angle could be formed of *three*, but not more than three, of such faces.

Lastly, suppose the faces are *regular pentagons.*

(v) Then, since each angle of a regular pentagon is $\frac{6}{5}$ of a right angle, it follows that a solid angle may be formed of *three* such faces; but the sum of more than three angles of a regular pentagon is greater than four right angles.

Further, since each angle of a *regular hexagon* is equal to $\frac{4}{3}$ of a right angle, it follows that no solid angle could be formed of such faces; for the sum of three angles of a hexagon is equal to four right angles.

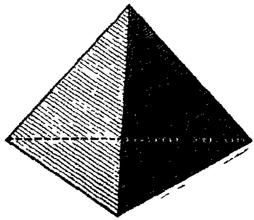
Similarly, no solid angle can be formed of the angles of a polygon of more sides than six.

Thus there can be no more than *five* regular polyhedra.

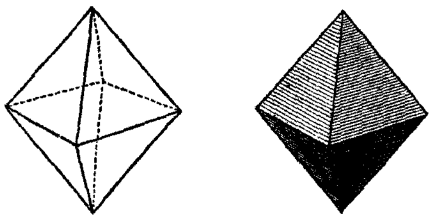
NOTE ON THE REGULAR POLYHEDRA.

(i) The polyhedron of which each solid angle is formed by *three equilateral triangles* is called a regular **tetrahedron.**

It has *four* faces,
four vertices,
six edges.

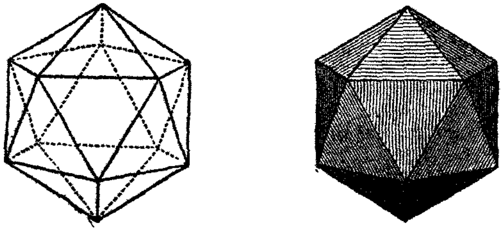


(ii) The polyhedron of which each solid angle is formed by *four equilateral triangles* is called a regular **octahedron.**



It has *eight* faces, *six* vertices, *twelve* edges.

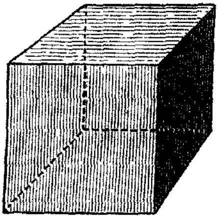
(iii) The polyhedron of which each solid angle is formed by *five equilateral triangles* is called a regular **icosahedron.**



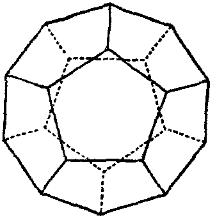
It has *twenty* faces, *twelve* vertices, *thirty* edges.

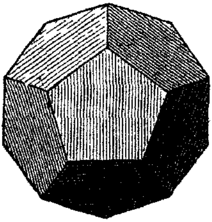
(iv) The regular polyhedron of which each solid angle is formed by *three squares* is called a **cube**.

It has *six* faces,
eight vertices,
twelve edges.



(v) The polyhedron of which each solid angle is formed by *three regular pentagons* is called a regular **dodecahedron**.





It has *twelve* faces, *twenty* vertices, *thirty* edges.

THEOREM 7. *If* F *denote the number of faces,* E *of edges, and* V *of vertices in any polyhedron, then will*

$$E+2=F+V.$$

Suppose the polyhedron to be formed by fitting together the faces in succession: suppose also that E_r denotes the number of edges, and V_r of vertices, when r faces have been placed in position, and that the polyhedron has n faces when complete.

Now when *one* face is taken there are as many vertices as edges, that is $$E_1=V_1.$$

The *second* face on being adjusted has *two* vertices and *one* edge in common with the first; therefore by adding the second face we increase the number of edges by one more than the number of vertices; $$\therefore\ E_2-V_2=1.$$

Again, the *third* face on adjustment has *three* vertices and *two* edges in common with the former two faces; therefore on adding the third face we once more increase the number of edges by one more than the number of vertices;

$$\therefore\ E_3-V_3=2.$$

Similarly, when all the faces but one have been placed in position,

$$E_{n-1}-V_{n-1}=n-2.$$

But in fitting on the last face we add no new edges nor vertices;

$$\therefore\ E=E_{n-1},\qquad V=V_{n-1},\qquad \text{and } F=n.$$

$$\text{So that } E-V=F-2,$$
$$\text{or, } E+2=F+V.$$

This is known as *Euler's Theorem.*

MISCELLANEOUS EXAMPLES ON SOLID GEOMETRY.

1. The projections of parallel straight lines on any plane are parallel.

2. If ab and cd are the projections of two parallel straight lines AB, CD on any plane, shew that AB : CD $=ab$: cd.

3. Draw two parallel planes one through each of two straight lines which do not intersect and are not parallel.

4. If two straight lines do not intersect and are not parallel, on what planes will their projections be parallel?

5. Find the locus of the middle point of a straight line of constant length whose extremities lie one on each of two non-intersecting straight lines, having directions at right angles to one another.

6. Three points A, B, C are taken one on each of the conterminous edges of a cube: prove that the angles of the triangle ABC are all acute.

7. If a parallelepiped is cut by a plane which intersects two pairs of opposite faces, the common sections form a parallelogram.

8. The square on the diagonal of a rectangular parallelepiped is equal to the sum of the squares on the three edges conterminous with the diagonal.

9. The square on the diagonal of a cube is three times the square on one of its edges.

10. The sum of the squares on the four diagonals of a parallelepiped is equal to the sum of the squares on the twelve edges.

11. If a perpendicular is drawn from a vertex of a regular tetrahedron on its triangular base, shew that the foot of the perpendicular will divide each median of the base in the ratio 2 : 1.

12. Prove that the perpendicular from the vertex of a regular tetrahedron upon the opposite face is three times that dropped from its foot upon any of the other faces.

13. If AP is the perpendicular drawn from the vertex of a regular tetrahedron upon the opposite face, shew that

$$3AP^2 = 2a^2,$$

where a is the length of an edge of the tetrahedron.

14. The straight lines which join the middle points of opposite edges of a tetrahedron are concurrent.

15. If a tetrahedron is cut by any plane parallel to two opposite edges, the section will be a parallelogram.

16. Prove that the shortest distance between two opposite edges of a regular tetrahedron is one half of the diagonal of the square on an edge.

17. In a tetrahedron if two pairs of opposite edges are at right angles, then the third pair will also be at right angles.

18. In a tetrahedron whose opposite edges are at right angles in pairs, the four perpendiculars drawn from the vertices to the opposite faces and the three shortest distances between opposite edges are concurrent.

19. In a tetrahedron whose opposite edges are at right angles, the sum of the squares on each pair of opposite edges is the same.

20. The sum of the squares on the edges of any tetrahedron is four times the sum of the squares on the straight lines which join the middle points of opposite edges.

21. In any tetrahedron the plane which bisects a dihedral angle divides the opposite edge into segments which are proportional to the areas of the faces meeting at that edge.

22. If the angles at one vertex of a tetrahedron are all right angles, and the opposite face is equilateral, shew that the sum of the perpendiculars dropped from any point in this face upon the other three faces is constant.

23. Shew that the polygons formed by cutting a prism by parallel planes are equal.

24. Three straight lines in space OA, OB, OC, are mutually at right angles, and their lengths are a, b, c: express the area of the triangle ABC in its simplest form.

25. Find the diagonal of a regular octahedron in terms of one of its edges.

26. Shew how to cut a cube by a plane so that the lines of section may form a regular hexagon.

27. Shew that every section of sphere by a plane is a circle.

28. Find in terms of the length of an edge the radius of a sphere inscribed in a regular tetrahedron.

29. Find the locus of points in a given plane at which a straight line of fixed length and position subtends a right angle.

30. A fixed point O is joined to any point P in a given plane which does not contain O; on OP a point Q is taken such that the rectangle OP, OQ is constant: shew that Q lies on a fixed sphere.