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Thema

A framework for unobstructedness of Galois deformation rings

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Abstract

Let F be a number field, S a finite set of places of F and $\mathrm{Gal}_{F,S}$ the Galois group of the maximal unramified outside S extension of F. Let k be a finite field. Deformation theory of Galois representations is a technique introduced by Mazur [Maz89] in the 1980's in order to study lifts of a given residual Galois representation $\overline{\rho} : \mathrm{Gal}_{F,S} \to \mathrm{GL}_n(k)$. Mazur posed the question under which conditions the functor parametrizing the deformations of $\overline{\rho}$ to complete local Noetherian $W(k)$-algebras is unobstructed, i.e. when $H^2(\mathrm{Gal}_{F,S}, \mathrm{ad}\,\overline{\rho})$ vanishes. This unobstructedness implies the formal smoothness of the corresponding universal deformation ring. In this thesis we present a general framework to deduce unobstructedness from a list of standard assumptions (including a suitable R=T theorem). This framework is developed more generally in terms of a smooth linear algebraic group G over $W(k)$, replacing GL_n as the target of $\overline{\rho}$. We apply the framework to deduce that almost all entries in the compatible system of Galois representations associated to a Hilbert modular form admit an unobstructed deformation functor, reproving a result of Gamzon [Gam13]. We also apply this framework to a RACSDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, deducing from standard conjectures that a subset of Dirichlet density 1 of the entries of the associated $\mathcal{G}_n$-valued family of Galois representations admits unobstructed deformation functors, where $\mathcal{G}_n$ is the group scheme from Clozel, Harris and Taylor [CHT08].

Zusammenfassung

Sei F ein Zahlkörper, S eine endliche Menge von Stellen von F und $\mathrm{Gal}_{F,S}$ die Galoisgruppe der maximalen, außerhalb von S unverzweigten Erweiterung von F. Sei k ein endlicher Körper. Die Deformationstheorie von Galoisdarstellungen wurde in den 1980er Jahren von Mazur [Maz89] entwickelt um die Lifts einer gegebenen residuellen Galoisdarstellung $\overline{\rho} : \mathrm{Gal}_{F,S} \to \mathrm{GL}_n(k)$ zu untersuchen. Mazur stellte die Frage unter welchen Bedingungen der Funktor, der die Deformationen von $\overline{\rho}$ zu vollständigen Noetherschen $W(k)$-Algebren beschreibt, unobstruiert ist, d.h. wann $H^2(\mathrm{Gal}_{F,S}, \mathrm{ad}\,\overline{\rho}) = 0$ gilt. Diese Unobstruiertheit impliziert die formale Glattheit des zugehörigen universellen Deformationsringes. In der vorliegenden Arbeit wird eine Methode vorgestellt um Unobstruiertheit aus einer Liste von Standardvermutungen abzuleiten, unter anderem von einem entsprechenden R=T-Satz. Diese Methode wird allgemeiner für eine glatte algebraische Gruppe G über $W(k)$ anstelle von GL_n als Wertebereich von $\overline{\rho}$ entwickelt. Mithilfe der Methode zeigen wir, dass fast alle Einträge in dem kompatiblen System von Galoisdarstellung zu einer Hilbertschen Modulform einen unobstruierten Deformationsfunktor besitzen und erhalten damit ein Resultat von Gamzon [Gam13]. Des Weiteren wenden wir die Methode auf eine RACSDC automorphe Darstellung von $\mathrm{GL}_n(\mathbb{A}_F)$ an und erhalten, unter Ausnutzung von Standardvermutungen, dass eine Teilmenge von Dirichlet-Dichte 1 der Einträge der assoziierten $\mathcal{G}_n$-wertigen Familie von Galoisdarstellungen einen unobstruierten Deformationsfunktor besitzt, wobei $\mathcal{G}_n$ das Gruppenschema von Clozel, Harris und Taylor [CHT08] bezeichnet.

Contents

1 Introduction

Let F be a number field, S a finite set of places of F and $\mathrm{Gal}_{F,S}$ the Galois group of the maximal unramified outside S extension of F. Let k be a finite field of characteristic $\ell > 0$. The deformation theory of Galois representations is a technique introduced by Mazur in the article [Maz89] in order to study the lifts of a representation

$$\overline{\rho} : \mathrm{Gal}_{F,S} \to \mathrm{GL}_n(k).$$

More precisely, let $\mathcal{C}^{\circ}_{W(k)}$ be the category of Artinian local $W(k)$-algebras with residue field k and consider the functor

$$D(\overline{\rho}) : \mathcal{C}^{\circ}_{W(k)} \longrightarrow \underline{\mathrm{Ens}} \qquad A \longmapsto \left\{\rho : \mathrm{Gal}_{F,S} \to \mathrm{GL}_n(A) \;\middle|\; \rho \text{ reduces mod } \mathfrak{m}_A \text{ to } \overline{\rho}\right\}/\sim$$

where two lifts are equivalent if they are conjugate by an element of $\ker(\mathrm{GL}_n(A) \to \mathrm{GL}_n(k))$. Under suitable conditions, this functor is pro-representable by a complete Noetherian local $W(k)$-algebra $R(\overline{\rho})$, i.e. the conjugacy classes of lifts (which are called the deformations) to A are parametrized by morphisms $\varphi : R(\overline{\rho}) \to A$.

Assume for the moment that $F = \mathbb{Q}, n = 2, \ell > 2$ and let us fix a cuspidal eigenform $f \in \mathcal{S}_k(\Gamma_0(N))$ of some weight k and level N (which we assume to be relatively prime to ℓ and square-free). By the work of Deligne [Del71] (for $k > 2$), Eichler and Shimura [Shi71] (for $k = 2$) and Deligne and Serre [DS74] (for $k = 1$) we can attach to f an ℓ-adic representation $\rho_{f,\ell}$ of $\mathrm{Gal}_{\mathbb{Q}}$ and we will take for $\overline{\rho}$ its reduction modulo ℓ. Then $\overline{\rho}$ is absolutely irreducible and crystalline at ℓ (so, in particular, $\overline{\rho}$ is odd and $\det \circ \overline{\rho}$ equals a tensor power of the mod-ℓ cyclotomic character) for all but finitely many choices for ℓ, cf. [Rib95, GK11] and the references therein. Moreover, there exists a quotient $\widetilde{R}(\overline{\rho})$ of $R(\overline{\rho})$ which parametrizes lifts whose determinant equals the cyclotomic character, which are unramified outside N, ordinary at N and crystalline of fixed weight at ℓ.

At the heart of the celebrated proof of Wiles and Taylor-Wiles [Wil95, TW95] of Fermat's Last Theorem lies the fact that the canonical surjection

$$\widetilde{R}(\overline{\rho}) \twoheadrightarrow \mathbb{T} \tag{1.1}$$

is an isomorphism, where $\mathbb{T}$ denotes a certain localization of a Hecke algebra and parametrizes those lifts which come from modular forms. In particular, the R=T-theorem (1.1) implies the Taniyama-Shimura conjecture for semistable elliptic curves, stating that any such curve is modular, i.e. comes from a modular form. This leads to the formulation of more general modularity lifting statements, an area which is being extensively studied by contemporary number theory.

Inspired by the observation that universal deformation rings in the classical setting ($n = 2$) are often isomorphic to a power series ring over $W(k)$ in three variables, one says that $D(\overline{\rho})$ is *unobstructed* if

$$H^2(\mathrm{Gal}_{F,S}, \mathrm{ad}\,\overline{\rho}) = 0. \tag{1.2}$$

It is easily seen that this implies that $R(\overline{\rho})$ is formally smooth over $W(k)$, hence isomorphic to a power series ring over $W(k)$. We also remark that this implies a partial solution to a conjecture of Jannsen for $\rho = \rho_{f,\lambda}$ with f as above: The Frobenius eigenvalues of ρ are Weil-numbers of some fixed weight w, i.e. ρ is pure of weight w. Hence, $\mathrm{ad}\,\rho \cong \rho \otimes \check{\rho}$ is pure of weight $w - w = 0$. A conjecture of Jannsen [Jan89, Conjecture 1] (see also [Bel09, Conjecture 5.1]) predicts that $H^2(\mathrm{Gal}_{F,S}, \mathrm{ad}\,\rho)$ vanishes. This implies that $H^2(\mathrm{Gal}_{F,S}, \Lambda)$ is finite and torsion, where $\Lambda \subset \mathrm{ad}\,\rho$ denotes an integral $\mathrm{Gal}_{F,S}$-stable lattice. On the other hand, our residual H^2-vanishing (1.2) implies the vanishing of $H^2(\mathrm{Gal}_{F,S},\Lambda)$ by Nakayama's Lemma. This, in turn, implies the vanishing of $H^2(\mathrm{Gal}_{F,S},\operatorname{ad}\rho)$, as predicted by Jannsen's conjecture.

Now ℓ-adic and ℓ-modular Galois representation often come in compatible systems, i.e. as families

$$\mathcal{R} = \left(\overline{\rho}_\lambda : \mathrm{Gal}_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_{\ell(\lambda)})\right)_{\lambda\in\mathrm{Pl}_E^{\mathrm{fin}}},$$

where λ runs through the set $\mathrm{Pl}_E^{\mathrm{fin}}$ of finite places of another number field E, and the $\overline{\rho}_\lambda$ share certain properties, e.g. a common ramification set S, see Section 3.3 below for a precise definition. (Here and in the following, $\ell(\lambda)$ denotes the rational prime lying below λ.) In this setting, write $S_\ell \subset \mathrm{Pl}_F$ for the set of all places which are in S or lie above ℓ or ∞. Then we say that the deformation functor $D(\overline{\rho}_\lambda)$ of a member $\overline{\rho}_\lambda$ of $\mathcal{R}$ is unobstructed if $D(\overline{\rho}_\lambda|\,\mathrm{Gal}_{F,S_\ell})$ is unobstructed, i.e. if

$$H^2(\mathrm{Gal}_{F,S_\ell},\operatorname{ad}\overline{\rho}_\lambda) = 0.$$

The following question was then posed by Mazur in [Maz89]:

Question 1.1. When is $D(\overline{\rho}_\lambda)$ unobstructed for almost all λ. (Alternatively: When is $D(\overline{\rho}_\lambda)$ unobstructed for all λ in a subset of $\mathrm{Pl}_E^{\mathrm{fin}}$ of Dirichlet density 1.)

This question was answered affirmatively (under different technical assumptions) in the following cases:

- Mazur [Maz97a]: $\mathcal{R} = \mathcal{R}_E$, the compatible system attached to an elliptic curve E over $F = \mathbb{Q}$;
- Weston [Wes04] (see also [Yam04, Hat15]): $\mathcal{R} = \mathcal{R}_f$, the compatible system attached to a newform f of weight $k \geq 3$ over $F = \mathbb{Q}$;
- Gamzon [Gam13] (following the approach of Weston): $\mathcal{R} = \mathcal{R}_f$, the compatible system attached to a Hilbert eigenform f over a totally real field F.

Weston uses Poitou-Tate duality and results on Selmer groups to deduce the H^2-vanishing of (1.2) for almost all $\overline{\rho}_\lambda$ from the following two statements:

1) For fixed p, $H^0(\mathbb{Q}_p,\operatorname{ad}\overline{\rho}_\lambda(1))$ vanishes for almost all λ;
2) For almost all λ, $H^0(\mathbb{Q}_{\ell(\lambda)},\operatorname{ad}\overline{\rho}_\lambda(1))$ vanishes.

Statement 1) is proved by using the local Langlands correspondence if the local part π_p of the automorphic representation $\pi = \langle f\rangle$ attached to f is supercuspidal or Steinberg and by a global argument (suggested by Ribet) if π_p is special, see [Wes04, Sections 3 and 5.2]. Concerning Statement 2), Weston performs a local calculation at the level of Fontaine-Laffaille modules, see [Wes04, Sections 4].

This thesis provides a framework for proving unobstructedness and for answering Question 1.1 more generally. **Section 2** is devoted to the deformation theory for G-valued representations, i.e. for morphisms

$$\overline{\rho} : \mathrm{Gal}_F \to G(k),$$

where G is a smooth linear algebraic group over $W(k)$, generalizing the classical approach where $G = \mathrm{GL}_n$. We start by collecting several preliminary results on the occurring coefficient rings and continue in Section 2.2 with an adapted version of Kisin's framed deformation functor (or: lifting functor) and a study of its relatively representable subfunctors. We continue with deformations and deformation conditions and give a G-valued version of Schur's Lemma (Lemma 2.50) and of the presentability of multiply framed global deformations rings over local ones following Balaji [Bal12], see Corollary 2.66. Like [CDT99, Appendix A], to easily derive results on the change of the base ring for universal deformation rings, cf. e.g. Lemma 2.22.2., we also consider deformations valued in the category ${}^{\star}\mathcal{C}_{W(k)}$ of complete Noetherian local $W(k)$-algebras A where we do *not* assume $k_A = k$.

In **Section 3** we start by giving a suitable definition of unobstructedness for global deformation conditions which are composed from local ones. The reason why we have to generalize the H^2-vanishing of (1.2) is that such a vanishing is connected to the unobstructedness of the full (i.e. unconditioned) deformation ring $R(\overline{\rho})$. As our framework uses crucially an $R = T$-theorem similar to (1.1) (and as such results are currently only within reach in a *minimally ramified* situation), we can a priori not hope for unobstructedness of $R(\overline{\rho})$ but rather of a quotient

$$R(\overline{\rho}) \twoheadrightarrow R^{\chi,\mathtt{min}}(\overline{\rho}), \tag{1.3}$$

parametrizing all deformations which fulfill a local condition $\mathtt{min}$ at all places inside a fixed set of places S and whose determinant equals a fixed lift χ of the determinant of $\overline{\rho}$ (where S is usually the ramification set of the system $\mathcal{R}$ and $\mathtt{min}$ denotes usually the condition of being minimally ramified). If $\mathcal{L}^{\chi} = (L_{\nu}^{\chi})_{\nu}$ denotes the system of local conditions associated to this choice (cf. Definition 2.69), then we call $R^{\chi,\mathtt{min}}(\overline{\rho})$ *globally unobstructed* (Definition 3.7) if

- the local framed deformation rings $R_{\nu}^{\square,\chi_{\nu},\mathtt{min}}(\overline{\rho})$ are formally smooth over $W(k)$ (of predictable dimension) for all $\nu \in S$;
- the dual Selmer group vanishes:
$$H^1_{\mathcal{L}^{\chi,\vee}}(F, \mathfrak{g}^{\mathtt{der},\vee}) = 0, \tag{1.4}$$
where $\mathfrak{g}^{\mathtt{der}} = \mathrm{Lie}(G^{\mathtt{der}})$ with the adjoint representation of Gal_F via $\overline{\rho}$.

If $R^{\chi,\mathtt{min}}(\overline{\rho})$ is globally unobstructed, it follows that it is isomorphic to a power series ring over $W(k)$, see Remark 3.8.

Our main result (Theorem 3.12) is the crucial step to deduce the vanishing of the dual Selmer group in (1.4). It depends on seven standard assertions, as listed at the beginning of Section 3.1. In the situation described above[1], the main assertions to be mentioned are items 3., 4. and 7.:

3. For each place ν of F above $\ell = \ell(\lambda)$, there is a local deformation condition $\mathtt{crys}$ such that the associated framed deformation functor $D^{\square,\chi,\mathtt{crys}}(\overline{\rho}_{\nu})$ is relatively smooth over $D^{\square,\chi}(\overline{\rho}_{\nu})$ and such that the representing object is formally smooth of relative dimension
$$\dim(\mathfrak{g}^{\mathtt{der}}) + (\dim(\mathfrak{g}^{\mathtt{der}}) - \dim(\mathfrak{b}^{\mathtt{der}}))[F_{\nu} : \mathbb{Q}_{\ell}].$$
(Here, we fix a Borel subgroup $B \subset G$ and we denote by $\mathfrak{g}^{\mathtt{der}}$ (resp. $\mathfrak{b}^{\mathtt{der}}$) the Lie algebra of the derived subgroup $G^{\mathtt{der}}$ of G (resp. the Lie algebra of $B \cap G^{\mathtt{der}}$).)
4. For each place $\nu \in S$, the local deformation ring $R^{\square,\chi,\mathtt{min}}(\overline{\rho}_{\nu})$ is formally smooth of dimension $\mathfrak{g}^{\mathtt{der}}$.

[1] For simplicity, we take the condition $\mathtt{sm}$ of Section 3.1 to be the unconditioned deformation condition during this introduction.

7. The multiply framed global deformation ring $R_{S_\ell}^{\Box_{S_\ell},\chi,\mathtt{min},\mathtt{crys}}$, parametrizing deformations of $\overline{\rho}$ which
   - fulfill condition $\mathtt{min}$ at all $\nu \in S$,
   - fulfill condition $\mathtt{crys}$ at all ν above ℓ,
   - are of fixed determinant χ and
   - are unramified outside S_ℓ

   is formally smooth of relative dimension $\dim(\mathfrak{g}).\#S_\ell - \dim(\mathfrak{g}^{\mathtt{ab}})$. Here, S_ℓ denotes the set of places which are contained in S or lie above $\ell.\infty$ and $\mathfrak{g}^{\mathtt{ab}}$ denotes the Lie algebra of $G^{\mathtt{ab}}$.

Although this is not a critical assumption, let us presume for the ease of exposition during this introduction that $\overline{\rho}$ is absolutely irreducible. Under these conditions, we obtain

Theorem A (Theorem 3.12.1). *$R_{S_\ell}^{\chi,\mathtt{min}}$ is formally smooth.*

If we additionally suppose that each local framed deformation ring $R^{\Box,\chi}(\overline{\rho}_\nu)$ is formally smooth of dimension $\dim(\mathfrak{g}^{\mathtt{der}})([F_\nu : \mathbb{Q}_\ell] + 1)$ for ν above ℓ, we even get

Theorem B (Theorem 3.12.2). *$R_{S_\ell}^{\chi,\mathtt{min}}$ is formally smooth of dimension $[F : \mathbb{Q}].\dim(\mathfrak{b}^{\mathtt{der}})$.*

This can be used to deduce

Theorem C (Corollary 3.16). *Assume (in addition to the requirements of Theorem B) the following:*

- *$\ell \gg 0$, so that $\mathfrak{g} = \mathfrak{g}^{\mathtt{der}} \oplus \mathfrak{g}^{\mathtt{ab}}$;*
- *$H^0(\mathrm{Gal}_F, \mathfrak{g}^{\mathtt{der},\vee}) = 0$ (this holds automatically for $G = \mathrm{GL}_n$ and $\ell \gg 0$);*
- *For $\nu \in S$, $\dim(L_\nu) = h^0(\mathrm{Gal}_{F_\nu}, \mathfrak{g}^{\mathtt{der}})$;*

Then $D_{S_\ell}^{\chi,\mathtt{min}}(\overline{\rho})$ has vanishing dual Selmer group.

Theorems A, B and C are proved by calculations using Galois cohomology and basic facts from commutative algebra, introduced in Section 2.1.

We can now state our strategy to answer analogues of Question 1.1: We check that for a density-1 set of places λ in our system $\mathcal{R}$ the representation $\overline{\rho}_\lambda$ fulfills the requirements of Theorem C and the local conditions in the "globally unobstructed"-notion.

In practice, we will often not succeed in establishing the requirements of Theorem C for the representations $\overline{\rho}_\lambda$ themselves but only for restrictions $\overline{\rho}_\lambda|\,\mathrm{Gal}_{F^{(\lambda)}}$, where each $F^{(\lambda)}$ is a suitable finite extension of F, chosen in dependence of λ. To this end, we develop in Section 3.2 a potential version of the above. More precisely, let us consider a finite extension F' of F and a deformation condition $\mathcal{D}'$ for $\overline{\rho}|\,\mathrm{Gal}_{F'}$ (with the associated system of local conditions $\mathcal{L}'^{\chi} = (L'^{\chi}_{\nu})_\nu$). Let us assume that $\mathcal{D}'$ fulfills $\mathrm{res}^{\perp}_{\nu'}(L^{\perp}_{\nu}) \subset L'^{\perp}_{\nu'}$ for all pairs $(\nu', \nu) \in \mathrm{Pl}^{\mathrm{fin}}_{F'} \times \mathrm{Pl}^{\mathrm{fin}}_{F}$ with $\nu'|\nu$, where

$$\mathrm{res}^{\perp}_{\nu'} : H^1(F_\nu, \mathfrak{g}^{\mathtt{der},\vee}) \to H^1(F'_{\nu'}, \mathfrak{g}^{\mathtt{der},\vee})$$

is the usual restriction map. (We call such a $\mathcal{D}'$ a dual-pre-$(\chi, \mathtt{min})$-condition, cf. Definition 3.19.) Let S'_ℓ denote the places of F' above S_ℓ. We obtain

Theorem D (Lemma 3.21)**.** *Assume that the functor* $D_{S'_\ell}^{\mathcal{D}'}(\overline{\rho}|\operatorname{Gal}_{F'})$ *fulfills the conditions of Theorem C, and hence has vanishing dual Selmer group. Then also* $D_{S_\ell}^{\chi,\mathtt{min}}(\overline{\rho})$ *has vanishing dual Selmer group for* $\ell \gg 0$*.*

Section 4 is devoted to a study of several local deformation conditions for $G = \mathrm{GL}_n$. We first recall the basic notions of Fontaine-Laffaille theory as normalized in [CHT08]. The main result here is the following generalization of condition 2) of Weston:

Theorem E (Corollary 4.7)**.** *Let* K, L *be finite extensions of* $\mathbb{Q}_\ell$ *and let*

$$\rho : \operatorname{Gal}_K \to \mathrm{GL}_n(L)$$

be a crystalline representation in the Fontaine-Laffaille range. Assume that the Hodge-Tate numbers of ρ *are non-consecutive: if* τ *is an embedding* $K \hookrightarrow \overline{\mathbb{Q}}_\ell$ *and two numbers* a, b *occur in* $\mathrm{HT}_\tau(\rho)$*, then either* $a = b$ *or* $|a - b| \geq 2$*. Then*

$$H^2(K, \operatorname{ad}\overline{\rho}) = 0.$$

By this corollary, we deduce

$$R_\Lambda^\square(\overline{\rho}) \cong \Lambda[[x_1, \dots, x_m]] \tag{1.5}$$

with $m = n^2.([K : \mathbb{Q}_\ell]+1)$, if ρ meets the "no consecutive weights"-assumption of Theorem E, cf. Lemma 4.11. In Section 4.3 we compile results about the crystalline deformation condition from [CHT08, Section 2.4.1], including a smoothness property similar to (1.5) and a compatibility with the corestriction map, see Lemma 4.14 and Lemma 4.15. In Section 4.4 we study the minimally ramified deformation condition from [CHT08, Section 2.4.4]. After recalling a smoothness property similar to (1.5) (Lemma 4.23) from [CHT08] we restrict to the case of unipotent ramification, i.e. we consider a local Galois representation[2]

$$\rho : \operatorname{Gal}_K \to \mathrm{GL}_n(L)$$

where ρ is trivial on the kernel of one (hence, any) surjection $I_K \twoheadrightarrow \mathbb{Z}_\ell$. The two main results for the minimal ramification condition are Theorem 4.30, where we identify the corresponding deformation ring with a certain "fixed-type" deformation ring of Shotton [Sho15], and Corollary 4.47, where we show that under sufficient assumptions on the system $\mathcal{R}$ an arbitrary deformation is locally almost always automatically minimally ramified. The latter result can be expressed as a local equality

$$R = R^{\mathtt{min}},$$

so that the restriction to the minimal ramification in (1.3) is a posteriori waived.

In **Section 5** we apply the developed framework to Hilbert modular forms. We prove

Theorem F (Corollary 5.7)**.** *Let* F *be a totally real number field and* f *a Hilbert modular newform such that each weight is* ≥ 3 *and such that all weights have the same parity. Let* K_f *denote the coefficient field of* f *and let*

$$\mathcal{R}_f = \Big(\rho_{f,\lambda} : \operatorname{Gal}_F \to \mathrm{GL}_2(K_{f,\lambda})\Big)_{\lambda \in \mathrm{Pl}_{K_f}^{fin}}$$

be the compatible system of Galois representations attached to f *with ramification set* S*.*

[2] In the minimal case we have $\ell \neq p$, i.e. K (resp. L) is a p-adic (resp. ℓ-adic) field with $\ell \neq p$.

Let K denote the composite of all coefficient fields of Hilbert newforms of the same weight and level as f. Assume that for almost all places λ we can choose a place δ of K above λ such that

$$R^{\chi,\mathtt{crys}}_{S_\ell,\mathcal{O}_{K_\delta}}(\overline{\rho}_{f,\lambda}) \cong \mathbb{T}_\delta, \tag{1.6}$$

where

- *$R^{\chi,\mathtt{crys}}_{S_\ell,\mathcal{O}_{K_\delta}}(\overline{\rho}_{f,\lambda})$ is the universal deformation ring which parametrizes deformations of $\overline{\rho}_{f,\lambda}$ to $\mathcal{O}_{K_\delta}$ which are crystalline above $\ell = \ell(\lambda)$ and unramified outside S_ℓ;*
- *$\mathbb{T}_\delta$ denotes the $\mathcal{O}_{K_\delta}$-subalgebra of $\prod_{g\in X(f,\lambda)} \mathcal{O}_{K_\delta}$ which is generated by all $(a_\nu(g))_{g\in X(f,\lambda)}$, where ν runs through the places outside S and where $X(f,\lambda)$ denotes the set of all Hilbert newforms of the same weight and level as f.*

Then, for almost all primes λ, $R^{\chi}_{S_\ell}(\overline{\rho}_{f,\lambda})$ is globally unobstructed.

The proof of this theorem proceeds by checking the preconditions of Theorem C with `min` being the unconditioned deformation condition. The crucial $R = T$ assertion (1.6) is used to check precondition 7. of Theorem A. This assertion is widely believed to hold true and is available in the literature in several cases, cf. Remark 5.4. This gives a new proof of Theorem 1.1 of [Gam13] (assuming the $R = T$-assumption (1.6)).

In **Section 6** we apply the framework to the following situation: Let F be a CM-field and Π be a RACSDC (regular algebraic conjugate self-dual cuspidal) automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. Then there exists a number field $\mathcal{E}$ and an $\mathcal{E}$-rational strictly compatible and pure of weight $n-1$ system of semisimple ℓ-adic Galois representations attached to Π,

$$\mathcal{R}_\Pi = \Big(\rho_\lambda : \mathrm{Gal}_F \to \mathrm{GL}_n(\mathcal{E}_\lambda)\Big)_{\lambda\in\mathrm{Pl}^{\mathrm{fin}}_{\mathcal{E}}},$$

with finite ramification set $S := \{\nu \in \mathrm{Pl}_F \mid \Pi_\nu \text{ is ramified }\}$. As introduced in [CHT08], let $\mathcal{G}_n$ be the group scheme over $\mathbb{Z}$ given by

$$\big(\mathrm{GL}_n \times \mathrm{GL}_1\big) \rtimes \{1, j\}$$

where j acts as $j(g,\mu)j = (\mu\,{}^tg^{-1}, \mu)$. Let $\Lambda^1_{\mathcal{E}}$ be the set of those $\lambda \in \mathrm{Pl}^{\mathrm{fin}}_{\mathcal{E}}$ for which each $\overline{\rho}_\lambda$ (as well as any other $\overline{\rho}_{\lambda'}$ with $\ell(\lambda') = \ell(\lambda)$) is absolutely irreducible. Then each $\overline{\rho}_\lambda$ with $\lambda \in \Lambda^1_{\mathcal{E}}$ extends to a representation

$$\overline{r}_\lambda : \mathrm{Gal}_{F^+,\overline{S}_\ell} \to \mathcal{G}_n(k_\lambda),$$

where F^+ denotes the maximal totally real subfield of F and where $\overline{S}$ denotes the set of places of F^+ below S, cf. Lemma 6.21. Assume that every place of $\overline{S}$ is split in the extension $F|F^+$. Our main result is

Theorem G (Theorem 6.56)**.** *Assume the following (Assumption 6.55):*

1. ***(Irreducibility):*** *The set $\Lambda^1_{\mathcal{E}}$ of those $\lambda \in \mathrm{Pl}^{fin}_{\mathcal{E}}$ for which each $\overline{\rho}_\lambda$ (as well as any other $\overline{\rho}_{\lambda'}$ with $\ell(\lambda') = \ell(\lambda)$) is absolutely irreducible has Dirichlet density 1 in $\mathrm{Pl}_{\mathcal{E}}$;*
2. ***(Availability of a minimal R=T-theorem):*** *(Cf. Conjecture 6.37) For each $\lambda \in \Lambda^1_{\mathcal{E}}$ there exists a finite extension $\mathcal{K}_\lambda$ of $\mathcal{E}_\lambda$ and an isomorphism*

   $$R^{\mathtt{min},\mathtt{crys}}_{\mathcal{O}_{\mathcal{K}_\lambda}}(\overline{r}_\lambda) \cong {}^{\mathcal{O}_{\mathcal{K}_\lambda}}_{\underline{\sigma}}\mathbb{T}^{\mathcal{T}_\ell}_{\boldsymbol{\omega}_\lambda}(U)_{\mathfrak{n}},$$

   where

- $R^{\mathtt{min,crys}}_{\mathcal{O}_{\mathcal{K}_\lambda}}(\overline{r}_\lambda)$ *is the universal deformation ring parametrizing those deformations of* $\overline{r}_\lambda$ *to* $\mathcal{O}_{\mathcal{K}_\lambda}$*-algebras which are crystalline above* $\ell = \ell(\lambda)$*, minimally ramified above* $\overline{S}$*, unramified outside* $\overline{S}_\ell$ *and of fixed determinant;*
- ${}^{\mathcal{O}_{\mathcal{K}_\lambda}}_{\underline{\sigma}}\mathbb{T}^{\mathcal{T}_\ell}_{\boldsymbol{\omega}_\lambda}(U)_{\mathfrak{n}}$ *denotes the Hecke algebra with respect to all automorphic forms of the same level* U *and weight* $\boldsymbol{\omega}_\lambda$ *as* Π *and of "minimal type"* $\underline{\sigma}$ *(as explained in Section 6.4.1), localized at a maximal ideal* $\mathfrak{n}$.

3. ***(No consecutive weights):*** *Let* $\lambda \in \Lambda^1_{\mathcal{E}}$ *and* $\nu \in \mathrm{Pl}_F$ *with* $\ell(\lambda) = \ell(\nu)$*. Let moreover* $\tau : F_\nu \hookrightarrow \overline{\mathbb{Q}}_\ell$ *be an embedding and denote by* HT_τ *the corresponding multiset of Hodge-Tate weights of* $\rho_\lambda|\,\mathrm{Gal}_{F_\nu}$*. Then, if two numbers* a, b *occur in* HT_τ*, we must have* $a = b$ *or* $|a-b| \geq 2$*;*

4. ***(Disjoint*** q***-orbits):*** *For* $\nu \in S$*, let* (r_ν, N_ν) *be the Weil-Deligne representation associated to* Π_ν *via the local Langlands correspondence. Write*

$$r_\nu(\mathrm{Frob}_\nu) \sim \begin{pmatrix} \mathcal{H}^\nu_{l^\nu_1}(\alpha^\nu_1) & & & \\ & \mathcal{H}^\nu_{l^\nu_2}(\alpha^\nu_2) & & \\ & & \ddots & \\ & & & \mathcal{H}^\nu_{l^\nu_{k^\nu}}(\alpha^\nu_{k^\nu}) \end{pmatrix} \text{ with } \mathcal{H}^\nu_m(\alpha) = \begin{pmatrix} \alpha & & & \\ & \alpha q_\nu & & \\ & & \ddots & \\ & & & \alpha q_\nu^{m-1} \end{pmatrix}.$$

Then for all $\nu \in S$ *and for all* $0 \leq i \neq j \leq k^\nu$*, the q-orbits*

$$q_\nu^{\mathbb{Z}}\alpha_i^\nu = \{q_\nu^a.\alpha_i^\nu \mid a \in \mathbb{Z}\} \quad \text{and} \quad q_\nu^{\mathbb{Z}}\alpha_j^\nu = \{q_\nu^a.\alpha_j^\nu \mid a \in \mathbb{Z}\}$$

are disjoint.

Then, the deformation ring $R^{\mathtt{min}}(\overline{r}_\lambda)$ *that parametrizes all minimally ramified, fixed-determinant deformations of* $\overline{r}_\lambda$ *is globally unobstructed for all* λ *in a subset of* $\mathrm{Pl}^{fin}_{\mathcal{E}}$ *of Dirichlet density* 1.

The proof of Theorem G is the content of Section 6.5.2 and consists again of checking the preconditions of Theorem C, but here with `min` being the minimally ramified deformation condition. The main difficulty here is that we cannot apply Theorem C directly, but that we have to introduce for each λ a finite, solvable extension $L^{(\lambda)}$ such that we can show that $D^{\mathtt{min}}(\overline{r}_\lambda|\,\mathrm{Gal}_{L^{(\lambda)}})$ has vanishing dual Selmer group. This can be used in conjunction with the potential unobstructedness result Theorem D to deduce that the original functor has vanishing dual Selmer groups. The following issues arise in this approach:

- In order to apply Theorem D, we need that the minimally ramified deformation condition for $\overline{r}_\lambda$ is a dual-pre-condition for the the minimally ramified deformation condition for $\overline{r}_\lambda|\,\mathrm{Gal}_{L^{(\lambda)}}$. This amounts to a certain calculation involving the tangent spaces for the minimally ramified condition (similar to Lemma 4.15.2 in the crystalline case), which we do not perform in this thesis. This local calculation is circumvented by the aforementioned local $R = R^{\mathtt{min}}$ result.
- Theorem D fails at a finite set of primes of $\mathcal{E}$, depending on the extension $L^{(\lambda)}|L$. This is harmless in a static setting, but as we choose for each λ an extension $L^{(\lambda)}$, the approach could a priori break down if each λ happens to be contained in the respective failure set. This will be handled as follows: We specify a certain tower of extensions

$$F = L_0 \subset L_1 \subset L_2 \subset \ldots$$

with $[L_{i+1} : L_i] = 2$ and such that $L_i|F$ is Galois for all i. Then we show that the set

$$\Psi_i := \{\lambda \mid L^{(\lambda)} \text{ can be chosen such that } L^{(\lambda)} \subset L_i\}$$

has Dirichlet density $1 - \frac{1}{2^i}$. This allows us to use (for a given i) Theorem D in the harmless static setting and deduce the desired result by a limit process.

Let us also comment on the assumptions of Theorem G:

1. Condition (Irreducibility) becomes necessary at an early part in our arguments, as the transition $\overline{\rho}_\lambda \rightsquigarrow \overline{r}_\lambda$ is only possible for those λ for which $\overline{\rho}_\lambda$ is absolutely irreducible. This condition is conjectured to hold true in general, while a proof is only available in the literature if Π is extremely regular [BLGGT14] or if $n \leq 5$ [CG13].

2. The availability of a minimal R=T-theorem takes a similar crucial role as the assumption (1.6) in Theorem F and is necessary to check precondition 7. of Theorem A. While we treat this condition as a conjecture during this thesis, we believe that it should be possible to give a proof using standard patching techniques and [CHT08].

3. The assumption that there are no consecutive weights is used to prove that a certain set of homomorphisms between two Fontaine-Laffaille modules vanishes (Corollary 4.7), which in turn is needed to verify precondition 3. of Theorem A. As stated, the assumption presents a technically simple sufficient condition for this vanishing and it is certain that there are finer criteria. We expect that this precondition of Theorem G can be replaced by a condition on Hecke polynomials after a more careful study of the morphisms in the Fontaine-Laffaille category.

4. The assumption on the disjointness of q-orbits is needed for the local $R = R^{\mathtt{min}}$ result Corollary 4.47, and thus it is needed to apply the potential unobstructedness of Theorem D. Improvements should be possible in two directions: On one hand, the $R = R^{\mathtt{min}}$ result is believed true (almost everywhere) without this technical assumption, presuming a natural condition of genericness (cf. [All14]). On the other hand, the $R = R^{\mathtt{min}}$ result is used to circumvent the usage of the respective pre-dual property for the minimal deformation condition, similar to Lemma 4.15.2 for the crystalline deformation condition. We expect that such an analogue derives from a careful study of the minimal deformation condition, superseding the necessity of an $R = R^{\mathtt{min}}$ result altogether.

The last three items suggest promising questions for future research. It seems also promising to use the described approach to establish certain missing cases in the treatment of Gamzon [Gam13], where his assumptions on the weights and on a base-change property are not met. Another worthwhile project would be to apply the presented framework to the conjectural association of (compatible systems of) Galois representations to automorphic representations on more general groups, as predicted by Langlands functoriality and described e.g. in [BG11].

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Notation

- The category of sets is denoted by <u>Ens</u>.
- If $\mathcal{C}$ is a category, we will use "$c \in \mathcal{C}$" as a shortcut notation for "c is an object of $\mathcal{C}$", acknowledging but notationally suppressing the intricacies if $\mathcal{C}$ is not small.
- For each rational prime ℓ, we fix an isomorphism $\iota_\ell : \mathbb{C} \xrightarrow{\cong} \overline{\mathbb{Q}}_\ell$.
- For a local ring R, we denote by $\mathfrak{m}_R$ the maximal ideal and by k_R the residue field of R. If R is an integral domain, we denote by $\mathrm{Quot}(R)$ the quotient field of R.
- For a number field F, we denote by $\widetilde{F}$ the Galois closure of F over $\mathbb{Q}$.
- Let F be a number field and L a finite extension of $\mathbb{Q}_\ell$. Then we say that L is *F-big enough* if L contains the image of any field embedding $F \hookrightarrow \overline{L}$, i.e. if L contains the Galois closure of F over $\mathbb{Q}_\ell$.
- For a number field F, we denote by $\mathcal{O}_F$ the ring of integers of F and by Pl_F (resp. $\mathrm{Pl}_F^{\mathrm{fin}}$) the set of places (resp. finite places) of F. We denote by $\mathbb{A}_F$ the ring of adeles of F and by $\mathbb{A}_F^\infty$ (resp. by $\mathbb{A}_{F,\infty}$) the finite (resp. infinite) adeles. If ν is a finite place of F, the (unique) rational prime lying below ν is denoted by $\ell(\nu)$.
- For a number field F, we will write $\Omega_\infty^F \subset \mathrm{Pl}_F$ for the set of archimedean places of F. If ℓ is a rational prime, we will write $\Omega_\ell^F \subset \mathrm{Pl}_F$ for the set of all places ν fulfilling $\ell(\nu) = \ell$. (We will also use the notation Ω_∞ and Ω_ℓ if there is no risk of confusion.) If $S \subset \mathrm{Pl}_F$ is some set of places of F and ℓ a rational prime, we write S_ℓ for $S \cup \Omega_\infty \cup \Omega_\ell$.
- If k is a field, we denote by $k[\epsilon] = k[X]/X^2$ the ring of dual numbers of k (cf. [Har77, Ex. 9.13.1]).
- If G is a group (or a group scheme), we will denote by Z_G the center of G.
- When referring to the dimension of cohomology groups, we will abbreviate $h^i(\cdot,\cdot)$ for $\dim H^i(\cdot,\cdot)$.
- Let $\Gamma = \mathrm{Gal}_F$ be the absolute Galois group of a number field F, k a finite field of characteristic ℓ and M a continuous $k\Gamma$-module. We denote by M^* the Pontryagin dual of M, by $M(m)$ (for $m \in \mathbb{N}$) the twist $\overline{\epsilon}_\ell^m \otimes M$ (where $\overline{\epsilon}_\ell$ denotes the mod-ℓ cyclotomic character) and by $M^\vee = M^*(1)$ the Tate (or Cartier) dual of M.
- We will often use parentheses to simplify the notation for simultaneous statements, in particular for deformation rings (cf. the "Notational convention" at the beginning of Section 3 of [Böc07]). If necessary, we will iterate this with squared brackets. For example, the (nonsense) statement $\dim R^{(\square),[\chi]}(\overline{\rho}) = 3 + (4) - [1]$ is to be read as

$$\big(\dim R(\overline{\rho}) = 3\big) \wedge \big(\dim R^{\square}(\overline{\rho}) = 7\big) \wedge \big(\dim R^{\chi}(\overline{\rho}) = 2\big) \wedge \big(\dim R^{\square,\chi}(\overline{\rho}) = 6\big).$$

- If $A \hookrightarrow A'$ is a (previously fixed) ring extension, we will write $\iota^n_{A'|A}$ for the associated morphism

$$\mathrm{GL}_n(A) \to \mathrm{GL}_n(A').$$

- For a ring R, we denote by $\mathbb{M}_{n\times n}(R)$ the ring of $n \times n$ matrices with entries in R.

2 Liftings and Deformations

Throughout this section, let us fix a finite field k of characteristic $\ell > 0$. We will denote the ring of Witt vectors over k by $W(k)$. Let us moreover consider a profinite group Γ which fulfills the following ℓ-finiteness condition:

Assumption 2.1 (Condition Φ_ℓ from [Maz89]). For any open subgroup $H \subset \Gamma$, the maximal pro-ℓ quotient of H is topologically finitely generated.

As remarked e.g. in [Böc13a, Ex. 1.2.2], absolute Galois groups of local fields and the Galois groups of extensions $F_S|F$ (with F being a number field and F_S being the maximal extension of F unramified outside a finite set of places S) fulfill this assumption for all primes ℓ.

Let G be a smooth linear algebraic group over $W(k)$ and let

$$\overline{\rho} : \Gamma \longrightarrow G(k)$$

be a continuous group homomorphism, where $G(k)$ carries the discrete topology. We will commonly refer to $\overline{\rho}$ as a *residual representation*. The purpose of this introductory section is to describe the deformation theory of $\overline{\rho}$ to complete Noetherian local $W(k)$-algebras, building up on the expositions of Tilouine [Til96], Mauger [Mau00], Levin [Lev13], Balaji [Bal12] and Bleher and Chinburg [BC03]. Historically, deformation theory was first studied by Mazur [Maz89, Maz97b] and others in the case $G = \mathrm{GL}_n$.

We remark that the material of this section could be analogously developed for a linear algebraic group over a discrete valuation ring which is finite over $W(k)$, but we don't need this.

2.1 Coefficient rings

Let Λ be the valuation ring of a finite extension of $\mathbb{Q}_\ell$ with residue field $k = k_\Lambda$.

Definition 2.2. Denote by

- ${}^\star\mathcal{C}_\Lambda$: the category of complete Noetherian local Λ-algebras A such that $[k_A : k]$ is finite;
- ${}^\star\mathcal{C}_\Lambda^\circ$: the (fully faithful) subcategory of ${}^\star\mathcal{C}_\Lambda$ consisting of those A which are Artinian;
- $\mathcal{C}_\Lambda$: the (fully faithful) subcategory of ${}^\star\mathcal{C}_\Lambda$ consisting of those A which fulfill $k_A = k$;
- $\mathcal{C}_\Lambda^\circ$: The intersection of ${}^\star\mathcal{C}_\Lambda^\circ$ and $\mathcal{C}_\Lambda$.

As morphisms we consider local maps which induce the identity on residue fields. To be more precise (cf. footnote 3 in [Gou01, Lecture 2]), we take as objects of ${}^\star\mathcal{C}_\Lambda$ pairs (A, ι_A), where A is a complete Noetherian local Λ-algebra and where ι_A is an embedding $A/\mathfrak{m}_A \hookrightarrow \overline{k}$. Then we consider as morphisms $f : (A, \iota_A) \to (B, \iota_B)$ those local maps $f : A \to B$ whose induced map $\overline{f} : A/\mathfrak{m}_A \to B/\mathfrak{m}_B$ on residue fields makes the following diagram commute:

$$\begin{array}{ccc} A/\mathfrak{m}_A & \xrightarrow{\overline{f}} & B/\mathfrak{m}_B \\ {\scriptstyle \iota_A}\downarrow & & \downarrow{\scriptstyle \iota_B} \\ \overline{k} & \xrightarrow[\mathrm{id}]{} & \overline{k} \end{array}$$

Observe that, for $\mathcal{C}_\Lambda$, this is just the usual notion of "local maps which induce the identity on residue fields".

Remark 2.3. Our main interest is in the categories $\mathcal{C}_\Lambda$ and ${}^\star\mathcal{C}_\Lambda$. The main problem is that pullbacks do not exist in these categories, but they do exist in $\mathcal{C}_\Lambda^\circ$ and ${}^\star\mathcal{C}_\Lambda^\circ$, cf. [Gou01, Lecture 2].

Let us recall the definition of the completed tensor product (see e.g. [Maz97b, §12]): For objects A, B, C of $\mathcal{C}_\Lambda$ and maps $A \to B, A \to C$, we define

$$B\hat{\otimes}_A C := \varprojlim_{i,j} \left(B/\mathfrak{m}_B^i\right) \otimes_A \left(C/\mathfrak{m}_C^j\right) \in \mathcal{C}_\Lambda.$$

This construction realizes the pushout of B and C along A (cf. Section 0.3 of Gabriel's Exposé VII_B in [DG70]). Using the co-continuity of Hom-functors, we immediately get the following proposition on representing objects:

Proposition 2.4. *Let*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \\ C & & \end{array}$$

be a diagram in $\mathcal{C}_\Lambda$. Then $\mathrm{Hom}_{\mathcal{C}_\Lambda}(B\hat{\otimes}_A C, \_\,)$ is the pullback of the diagram of functors

$$\begin{array}{ccc} & & \mathrm{Hom}_{\mathcal{C}_\Lambda}(B, \_\,) \\ & & \downarrow \\ \mathrm{Hom}_{\mathcal{C}_\Lambda}(C, \_\,) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}_\Lambda}(A, \_\,)\ . \end{array}$$

Corollary 2.5. *Let $F_1, \ldots, F_m$ be finitely many representable functors from $\mathcal{C}_\Lambda$ to* <u>Ens</u>*. Let R_i denote the representing object of F_i. Then*

$$\prod_{i=1,\ldots,m} F_i : \mathcal{C}_\Lambda \longrightarrow \mathrm{Ens}$$

is representable by $R_1\hat{\otimes} \ldots \hat{\otimes} R_m$.

As a preparation for the next proposition, let us consider a pushout diagram in $\mathcal{C}_\Lambda$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle\pi}\downarrow & & \vdots \\ C & \underset{g}{\dashrightarrow} & P\ , \end{array}$$

such that f is surjective. This implies that g is surjective, so by taking $I := \ker(f)$ and $J := \ker(g)$ we can extend π to a map of short exact sequences (of Λ-modules):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & A & \xrightarrow{f} & B & \longrightarrow & 0 \\ & & {\scriptstyle\pi|I}\downarrow & & {\scriptstyle\pi}\downarrow & & \downarrow & & \\ 0 & \longrightarrow & J & \longrightarrow & C & \underset{g}{\longrightarrow} & P & \longrightarrow & 0 \end{array} \tag{2.1}$$

This diagram can be extended to

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & A & \longrightarrow & B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & C\hat{\otimes}_A I & \longrightarrow & C\hat{\otimes}_A A & \longrightarrow & C\hat{\otimes}_A B & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \| & & \\
0 & \longrightarrow & J & \longrightarrow & C & \longrightarrow & P & \longrightarrow & 0
\end{array}$$

Definition 2.6. For a finitely generated ideal $\mathfrak{I}$ of A we define

$$\operatorname{gen}(\mathfrak{I}) := \dim_k \mathfrak{I}/\mathfrak{m}_A\mathfrak{I}.$$

Then $\operatorname{gen}(\mathfrak{I})$ is the cardinality of a minimal set of generators for $\mathfrak{I}$.

Then we have:

Proposition 2.7. *In diagram (2.1),*

$$\operatorname{gen}(J) \leq \operatorname{gen}(I).$$

Proof. This follows from the above extended diagram, using that both the map $I \to C\hat{\otimes}_A I$ induced by base change from A to C and the surjective module homomorphism $C\hat{\otimes}_A I \twoheadrightarrow J$ send systems of generators to systems of generators. □

Recall the following elementary facts about regular systems of parameters:

Proposition 2.8 ([Ser00, Proposition 22] and the subsequent corollary)**.**

a) Let $x_1, \dots, x_l$ be l elements of the maximal ideal $\mathfrak{m}_A$ of a regular local ring A. Then the following is equivalent:

i. $x_1, \dots, x_l$ is a subset of a regular system of parameters of A;

ii. The images of $x_1, \dots, x_l$ in $\mathfrak{m}_A/\mathfrak{m}_A^2$ are linearly independent over k;

iii. The local ring $A/(x_1, \dots, x_l)$ is regular and has dimension $\dim A - l$. *(In particular, $(x_1, \dots, x_l)$ is a prime ideal.)*

b) If $\mathfrak{I}$ is an ideal of a regular local ring A, the following properties are equivalent:

i. $A/\mathfrak{I}$ is a regular local ring;

ii. $\mathfrak{I}$ is generated by a subset of a regular system of parameters of A.

Proposition 2.9. *Let $A = \Lambda[[x_1, \dots, x_a]], B = \Lambda[[y_1, \dots, y_b]]$ be objects of $\mathcal{C}_\Lambda$ and assume that there exists a surjective morphism $f : A \to B$ whose kernel we denote by I. Then* $\operatorname{gen}(I) = a - b \geq 0$.

Proof. It is clear that there cannot be a negative number of generators of I. By Proposition 2.8.b), the ideal I can be generated by a subset (of, say, cardinality r) of a regular system of parameters of A. By part a) of said proposition, the quotient A/I has dimension $\dim A - r = a + 1 - r$. We get $r = a - b$,

which is thus an upper bound on $\mathrm{gen}(I)$.
In order to derive a lower bound, consider the canonical surjection

$$\pi : A/\mathfrak{m}_A.I \twoheadrightarrow A/\mathfrak{m}_A^2.$$

The image of $I/\mathfrak{m}_A.I$ under π is $(I+\mathfrak{m}_A^2)/\mathfrak{m}_A^2 \cong I/(I\cap\mathfrak{m}_A^2)$. This implies $\mathrm{gen}(I) = \dim_k I/\mathfrak{m}_A I \geq \dim_k I/I\cap\mathfrak{m}_A^2 = r$, where the last equality is taken from the proof of [Ser00, Proposition 22]. □

Lemma 2.10. *Let $A = \Lambda[[x_1,\dots,x_a]], B = \Lambda[[y_1,\dots,y_b]]$ be objects of $\mathcal{C}_\Lambda$ and let $J \subset A$ be an ideal of the form $J = (\varphi_1,\dots,\varphi_u)$ with $\varphi_i \in A$ and $u \leq a$. Suppose moreover, that there exists a surjective morphism $f : A/J \twoheadrightarrow B$ and denote its kernel by I.*
Then $A/J \cong \Lambda[[x_1,\dots,x_{a-u}]]$ if and only if $\mathrm{gen}(I) = a-u-b$.

Proof. The "only if" direction follows from the above proposition. For the other direction, assume $\mathrm{gen}(I) = l := a-u-b$ and write $I = (\psi_1 + J,\dots,\psi_l + J)$ for suitable $\psi_i \in A$. Write

$$K := (\varphi_1,\dots,\varphi_u,\psi_1,\dots,\psi_l) \subset A.$$

It follows from the third isomorphism theorem for rings [Bou89, I.§8.9 Corollary] that

$$A/K \cong (A/J)/(K/J) \cong (A/J)/I \cong B.$$

Because $K \subset \mathfrak{m}_A$, we can apply the implication *iii.* ⇒ *i.* of Proposition 2.8.a), which tells us that there exists a regular system of of parameters of A which extends the system $\varphi_1,\dots,\varphi_u,\psi_1,\dots,\psi_l$. But then this system also extends $\varphi_1,\dots,\varphi_u$, hence (by Proposition 2.8.a), implication *i.* ⇒ *iii.*) A/J is regular of dimension $a-u+1$. Thus, we can apply Cohen's structure theorem (see [Ser00], p. 108) to finish the proof as soon as we can show that A/J is unramified, i.e. that $\ell \notin \mathfrak{m}_{A/J}^2$. But this is clear: f is a surjection onto the unramified regular ring B, so $\mathfrak{m}_{A/J} = f^{-1}(\mathfrak{m}_B)$ and $\ell \notin \mathfrak{m}_B^2$. □

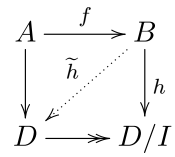
Remark 2.11. Retain the notation from Lemma 2.10. Then it follows from the above proof together with Proposition 2.9 that $\mathrm{gen}(I)$ cannot be smaller than $a-u-b$. Thus, if we want to apply the lemma in order to prove that A/J is isomorphic to a ring of power series, it suffices to show that there exists a generating set for I of cardinality not exceeding $a-u-b$. This implies that the number of variables is precisely $a-u$.

Proposition 2.12. *Let $m \in \mathbb{N}$. Then an object A of $\mathcal{C}_\Lambda$ is regular if and only if $A[[x_1,\dots,x_m]]$ is regular.*

Proof. It is clearly sufficient to consider the case $m=1$. The "only if" part is [Mat80, Proposition 24D]. For the other direction, assume that $A[[x]]$ is regular. It is clear that x is not contained in $\mathfrak{m}_{A[[x]]}^2 = (\mathfrak{m}_A, x)^2$, so implication *ii.* ⇒ *iii.* of Proposition 2.8.a) yields regularity of $A[[x]]/(x) \cong A$. □

Recall the following definition from [Gro64, §19] (see also [Ser06, Appendix C]):

Definition 2.13. A morphism $f : A \to B$ of rings is called *formally smooth* if the following lifting property is fulfilled for any commutative A-algebra D and any nilpotent ideal $I \subset D$: Any A-algebra morphism $h : B \to D/I$ factors through the projection $D \twoheadrightarrow D/I$. Written as a diagram: For any h there exists an $\widetilde{h}$ such that



commutes.

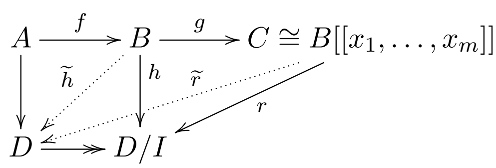
One of the reasons why we are interested in this notion is the following result:

Proposition 2.14. *Let $f : A \to B$ be a morphism in $\mathcal{C}_\Lambda$. Then f is formally smooth if and only if B is isomorphic to a formal power series ring over A.*

Proof. This is the equivalence $(i) \Leftrightarrow (ii)$ of [Ser06, Proposition C.6]. □

Lemma 2.15. *Consider morphisms $f : A \to B$ and $g : B \to C$ in $\mathcal{C}_\Lambda$. If g and $g \circ f$ are formally smooth, then f is formally smooth.*

Proof. Using the formal smoothness of g and Proposition 2.14, we can consider the following diagram:



Here, we start with a morphism $h : B \to D/I$ and want to show the existence of a suitable $\widetilde{h}$. For this, take r as the unique map satisfying $r \circ g = h$ and $r(x_i) = 0$ for all $i \in \{1, \ldots, m\}$. Using the assumption that $g \circ f$ is formally smooth, we see that r lifts to a map $\widetilde{r}$. But then $\widetilde{h} := \widetilde{r} \circ g$ yields the desired lift of h. □

We next prove a rather general lemma: Consider a ring of the form

$$R = \Lambda[[x_1, \ldots, x_a]]/(f_1, \ldots, f_b) \tag{2.2}$$

where this is a minimal presentation, i.e. $a \in \mathbb{N}_0$ is minimal among all possibilities to write R as a quotient of a power series ring over Λ and $b = \mathrm{gen}(f_1, \ldots, f_b)$. Also consider

$$R' := \Delta \otimes_\Lambda R = \Delta[[x_1, \ldots, x_a]]/(f_1, \ldots, f_b) \tag{2.3}$$

for some $\Delta \in {}^\star\mathcal{C}_\Lambda$ such that the structure morphism $\Lambda \to \Delta$ is flat.

Remark 2.16. We will mainly be interested in the case where Δ is a discrete valuation ring extending Λ, where we suppose $[\mathrm{Quot}(\Delta) : \mathrm{Quot}(\Lambda)] < \infty$. In this case, the flatness condition is fulfilled.

Lemma 2.17. *The presentation in (2.3) is minimal. In particular, R is formally smooth over Λ of dimension a if and only if R' is formally smooth over Δ of dimension a.*

Proof. Minimality of (2.2) amounts to the inclusion $I := (f_1, \ldots, f_b) \subset (\mathfrak{m}^2, \ell)$, where $\mathfrak{m}$ denotes the maximal ideal of $\Lambda[[x_1, \ldots, x_a]]$ and $b = \dim I/\mathfrak{m}.I$. By the flatness of Δ, we can compare the exact sequences

$$0 \longrightarrow I \longrightarrow \Lambda[[x_1, \ldots, x_a]] \longrightarrow R \longrightarrow 0$$

and

$$0 \longrightarrow \Delta \otimes_\Lambda I \longrightarrow \Delta[[x_1, \ldots, x_a]] \longrightarrow R' \longrightarrow 0.$$

It remains to show that the latter gives a minimal presentation. We easily see that $\Delta \otimes_\Lambda I \subset (\mathfrak{m}'^2, \ell)$, where $\mathfrak{m}'$ is the maximal ideal of $\Delta[[x_1, \ldots, x_a]]$: Exactness of

$$0 \to I \to (\mathfrak{m}^2, \ell)$$

implies exactness of

$$0 \to \Delta \otimes_\Lambda I \to \Delta \otimes_\Lambda (\mathfrak{m}^2, \ell) = (\mathfrak{m}'^2, \ell).$$

It remains to check that b equals $b' := \dim_{\Delta/\mathfrak{m}'} \Delta \otimes_\Lambda I/\mathfrak{m}'.\Delta \otimes_\Lambda I$. But this follows directly from the isomorphism

$$\Delta \otimes_\Lambda I/\mathfrak{m}'.\Delta \otimes_\Lambda I \cong I/\mathfrak{m}.I \otimes_{\Lambda/\mathfrak{m}} \Delta/\mathfrak{m}'$$

and the fact that $\Lambda/\mathfrak{m} \to \Delta/\mathfrak{m}'$ is a monomorphism of fields:

$$b = \dim_{\Lambda/\mathfrak{m}} I/\mathfrak{m}.I = \dim_{\Delta/\mathfrak{m}'} \Delta \otimes_\Lambda I/\mathfrak{m}'.\Delta \otimes_\Lambda I = b'.$$ □

We conclude this section with two general lemmas which will be useful for comparing two deformation rings:

Lemma 2.18. *Let $R, R' \in \mathcal{C}_\Lambda$ and let*

$$\varphi : R \twoheadrightarrow R'$$

be a surjective morphism. Assume moreover that R is formally smooth over Λ of relative dimension d. Then φ is an isomorphism if $\dim(R') = d + 1$.

Proof. Assume that φ is not injective, i.e. $\ker \varphi \neq 0$. Then it follows that $\dim(R/\ker \varphi) < \dim R$. The claim now follows from the additivity of the dimension. □

Lemma 2.19. *Let $R \in \mathcal{C}_\Lambda$ such that*

$$\Lambda[[x_1, \ldots, x_m]] \cong R \,\hat{\otimes}_\Lambda\, \Lambda[[x]]$$

for some $m \in \mathbb{N}$. Then

$$R \cong \Lambda[[x_1, \ldots, x_{m-1}]].$$

Proof. Let ϖ be a uniformizing element of Λ. Clearly, the indeterminant

$$x \in (R/\varpi.R)[[x]] \cong k[[x_1, \ldots, x_m]]$$

is contained in a regular system of parameters, so

$$R/\varpi.R \cong k[[x_1, \ldots, x_{m-1}]]. \tag{2.4}$$

Now consider the diagram

$$\begin{array}{ccc} \Lambda & \longrightarrow & \Lambda[[x_1, \ldots, x_{m-1}]] \\ \downarrow & \overset{\widetilde{h}}{\swarrow} & \downarrow h \\ R & \underset{g}{\twoheadrightarrow} & R/\varpi.R \end{array}$$

where h and g are the projection maps modulo ϖ. As $\Lambda[[x_1, \ldots, x_{m-1}]]$ is formally smooth over Λ, there exists a dotted map $\widetilde{h}$. Because of the isomorphism (2.4), R modulo the maximal ideal of $\Lambda[[x_1, \ldots, x_{m-1}]]$ is k and hence, by Nakayama's Lemma, the map $\widetilde{h}$ is surjective.

Now we see that $\widetilde{h}$ must be an isomorphism: Assume, this is not the case. Then $\dim R < m$, which is in conflict with the isomorphism $\Lambda[[x_1, \ldots, x_m]] \cong R \hat{\otimes}_\Lambda \Lambda[[x]] \cong R[[x]]$. □

2.2 Liftings of G-valued representations

For an object A of ${}^\star\mathcal{C}_\Lambda$ with residue field k_A we consider the following maps induced by reduction modulo the maximal ideal and by the structure map $\Lambda \to A$, respectively:

$$\mathrm{mod}_{\mathfrak{m}_A} : G(A) \longrightarrow G(A/\mathfrak{m}_A) = G(k_A), \qquad \iota_{k \subset k_A} : G(k) \longrightarrow G(k_A). \tag{2.5}$$

Definition 2.20. Let

$$\overline{\rho} : \Gamma \longrightarrow G(k)$$

be a residual representation and A be an object of ${}^\star\mathcal{C}_\Lambda$. Then a *lifting* of $\overline{\rho}$ to A is a continuous group homomorphism

$$\rho : \Gamma \longrightarrow G(A)$$

which fulfills

$$\mathrm{mod}_{\mathfrak{m}_A} \circ \rho = \iota_{k \subset k_A} \circ \overline{\rho}.$$

Definition 2.21 (Lifting functor)**.** Retaining the notation from the above definition, let

$${}^\star D^\square_\Lambda(\overline{\rho}) : {}^\star\mathcal{C}_\Lambda \longrightarrow \underline{\mathrm{Ens}}$$

be the functor which assigns to an object A of ${}^\star\mathcal{C}_\Lambda$ the set of all liftings of $\overline{\rho}$ to A. The restriction of ${}^\star D^\square_\Lambda(\overline{\rho})$ to $\mathcal{C}_\Lambda$ is denoted by $D^\square_\Lambda(\overline{\rho})$.

Theorem 2.22. *1. Both ${}^\star D^\square_\Lambda(\overline{\rho})$ and $D^\square_\Lambda(\overline{\rho})$ are representable by the same object $R^\square_\Lambda(\overline{\rho})$ which lies in $\mathcal{C}_\Lambda$.*

2. Let Λ' be the ring of integers of a finite extension of $\mathrm{Quot}(\Lambda)$ *with residue field $k' := k_{\Lambda'}$ and abbreviate $\overline{\rho}'$ for $\iota_{k \subset k'} \circ \overline{\rho}$. Then*

$$R^\square_{\Lambda'}(\overline{\rho}') \cong \Lambda' \otimes_\Lambda R^\square_\Lambda(\overline{\rho}).$$

We will call $R^\square_\Lambda(\overline{\rho})$ the *universal lifting ring* (or universal framed deformation ring, cf. Proposition 2.61 below) of $\overline{\rho}$. The afforded morphism

$$\rho^\square : \Gamma \to G(R^\square_\Lambda(\overline{\rho}))$$

is called the *universal lifting* of $\overline{\rho}$.

Proof. Representability of $D^{\square}_{\Lambda}(\overline{\rho})$ is the content of Theorem 1.2.2 of Balaji's thesis [Bal12], where a representing object is explicitly constructed. Moreover, it is easily seen that the proof is applicable to the functor ${}^{\star}D^{\square}_{\Lambda}(\overline{\rho})$ without any changes (so in particular with the same constructed representing object). An alternative proof using an embedding of G into GL_N is given in [Lev13, Proposition 7.2.1]. Also, the second claim can be deduced by comparing the construction of the representing object of $D^{\square}_{\Lambda}(\overline{\rho})$ and the representing object of $D^{\square}_{\Lambda'}(\overline{\rho}')$ in Balaji's proof: Using his notation, $R^{\square}_{\Lambda}(\overline{\rho})$ is constructed as the I-adic completion of a quotient R_2 of a power series ring over Λ (for an explicitly described ideal $I \subset R_2$). Similarly, $R^{\square}_{\Lambda'}(\overline{\rho}')$ can be constructed as the I'-adic completion of a quotient R'_2 of a power series ring over Λ'. The rings R_2 and R'_2 depend only on Γ and G (resp. on Γ and the extension of scalars of G to Λ'), which immediately implies

$$R'_2 \cong \Lambda' \otimes_{\Lambda} R_2.$$

The ideal I (resp. I') is defined using the residual representation, and it follows directly from the identity $\overline{\rho}' = \iota_{k'|k} \circ \overline{\rho}$ that $I' \cong \Lambda' \otimes_{\Lambda} I$, from which we conclude the claim. □

Remark 2.23. It is easy to check that the Noetherian objects of the completion $\widehat{{}^{(\star)}\mathcal{C}^{\circ}_{\Lambda}}$ lie in ${}^{(\star)}\mathcal{C}_{\Lambda}$ (i.e. that any object R of ${}^{(\star)}\mathcal{C}_{\Lambda}$ fulfills

$$R \cong \varprojlim_{i} R/\mathfrak{m}_R^i$$

with $R/\mathfrak{m}_R^i \in {}^{(\star)}\mathcal{C}^{\circ}_{\Lambda}$), see [Gou01, Problem 2.3]. Moreover, the functors ${}^{(\star)}D^{\square}_{\Lambda}(\overline{\rho})$ are continuous in the following sense: For any object A of ${}^{(\star)}\mathcal{C}_{\Lambda}$ with maximal ideal $\mathfrak{m}_A$ we have

$${}^{(\star)}D^{\square}_{\Lambda}(\overline{\rho})(A) \;\cong\; \varprojlim_{i} {}^{(\star)}D^{\square}_{\Lambda}(\overline{\rho})(A/\mathfrak{m}_A^i).$$

Thus, ${}^{(\star)}D^{\square}_{\Lambda}(\overline{\rho})$ is already determined by its restriction ${}^{(\star)}D^{\circ,\square}_{\Lambda}(\overline{\rho})$ to ${}^{(\star)}\mathcal{C}^{\circ}_{\Lambda}$. The first part of Theorem 2.22 may therefore be rephrased as follows: The functors ${}^{\star}D^{\circ,\square}_{\Lambda}(\overline{\rho})$ and $D^{\circ,\square}_{\Lambda}(\overline{\rho})$ are both pro-representable by the same object $R^{\square}_{\Lambda}(\overline{\rho})$.

Remark 2.24. The things said so far imply two *extension principles*:

1. A pro-representable functor on ${}^{(\star)}\mathcal{C}^{\circ}_{\Lambda}$ can be extended to a continuous functor on ${}^{(\star)}\mathcal{C}_{\Lambda}$ in a unique way. This extension is representable by the same object.

2. A pro-representable functor $F = \mathrm{Hom}_{\mathcal{C}^{\circ}_{\Lambda}}(R, \_)$ on $\mathcal{C}^{\circ}_{\Lambda}$ can be extended to a pro-representable functor ${}^{\star}F = \mathrm{Hom}_{{}^{\star}\mathcal{C}^{\circ}_{\Lambda}}(R, \_)$ on ${}^{\star}\mathcal{C}^{\circ}_{\Lambda}$. Moreover, this extension is unique up to natural isomorphism: Assume ${}^{\star}F' = \mathrm{Hom}_{{}^{\star}\mathcal{C}^{\circ}_{\Lambda}}(R', \_)$ is another pro-representable extension of F, then the unicity of the pro-representing object implies a unique isomorphism $R \cong R'$. Using the first part of this remark, this can equivalently be stated as follows: A representable functor on $\mathcal{C}_{\Lambda}$ can be extended to a representable functor on ${}^{\star}\mathcal{C}_{\Lambda}$ in a way which is unique up to natural isomorphism.

We will later generalize the extension principles of Remark 2.24.2, cf. Observation 2.34.

Relatively representable subfunctors and lifting conditions For this paragraph, let $\mathcal{C}$ be either ${}^{\star}\mathcal{C}^{\circ}_{\Lambda}$ or $\mathcal{C}^{\circ}_{\Lambda}$ and consider a functor

$$F : \mathcal{C} \longrightarrow \underline{\mathrm{Ens}},$$

which fulfills

$$\#F(k') = 1 \tag{2.6}$$

for any finite field k' in $\mathcal{C}$.

Theorem 2.25 (Grothendieck's criterion [Gro95a])**.** *The functor F is pro-representable if and only if the following conditions are met:*

1. *Mayer-Vietoris property: F respects fiber products, i.e. for any two morphisms $f : A \to E, g : B \to E$ in $\mathcal{C}$, the canonical map*

$$h^F_{f,g} : F(A \times_E B) \longrightarrow F(A) \times_{F(E)} F(B)$$

   is an isomorphism of sets;

2. *Finitude of tangent spaces: For any finite field k' which is contained in $\mathcal{C}$, the set $F(k'[\epsilon])$ is finite.*

Definition 2.26 ([Maz97b, §19])**.** A subfunctor H of F is called *relatively representable*, if

1. $H(k) = F(k)$;
2. For any two morphisms $f : A \to E, g : B \to E$ in $\mathcal{C}$, the following is a pullback diagram in Ens:

$$\begin{array}{ccc} H(A \times_E B) & \xrightarrow{h^H_{f,g}} & H(A) \times_{H(E)} H(B) \\ \big\downarrow & & \big\downarrow \\ F(A \times_E B) & \xrightarrow[h^F_{f,g}]{} & F(A) \times_{F(E)} F(B) \end{array} \tag{2.7}$$

Let $f : A \to B, g : C \to B$ be maps of sets. Then the pullback in Ens is explicitly given by

$$A \times_{f,B,g} C = \{(a, c) \in A \times C \mid f(a) = g(c)\}.$$

Similarly, if $F, G, H : \mathcal{C} \to \underline{\text{Ens}}$ are functors together with natural transformations

$$\tau : F \to G, \psi : H \to G,$$

we can characterize (or *define* – as done e.g. in [Maz97b] – if we don't want to refer to the general limit construction in functor categories) the pullback functor

$$F \times_{\tau,G,\psi} H : \mathcal{C} \to \underline{\text{Ens}}$$

by sending $C \in \mathcal{C}$ to the set $F(C) \times_{\tau_C,G(C),\psi_C} H(C)$.

Lemma 2.27. *Consider a diagram of functors and natural transformations*

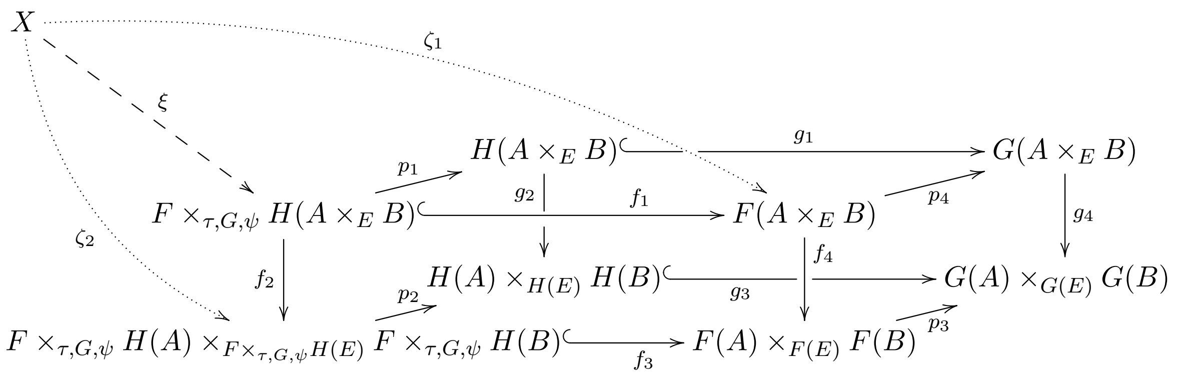
$$\begin{array}{ccc} & & F \\ & & \big\downarrow \tau \\ H & \xrightarrow[\psi]{} & G \end{array}$$

and assume that ψ is injective (i.e. all components ψ_A are monic), so that we can view H as a subfunctor of G. Assume that H is relatively representable in G via ψ. Then the induced natural transformation to the first factor,

$$\pi_F : F \times_{\tau,G,\psi} H \to F,$$

allows us to view $F \times_{\tau,G,\psi} H$ as a subfunctor of F and, as such, $F \times_{\tau,G,\psi} H$ is relatively representable in F.

Proof. Consider the commutative diagram



where $X \in \mathcal{C}$ is arbitrary and will be used as a test object. In order to prove the claim, we have to show that the square in the foreground $(f_1 - f_2 - f_3 - f_4)$ is a pullback diagram, given that the square in the background $(g_1 - g_2 - g_3 - g_4)$ is one. So for any two maps ζ_1, ζ_2 fulfilling $f_4 \circ \zeta_1 = f_3 \circ \zeta_2$, we have to show that there exists a unique ξ fulfilling

$$\zeta_1 = f_1 \circ \xi \text{ and } \zeta_2 = f_2 \circ \xi. \tag{2.8}$$

By assumption on the relative representability of H in G, we know that there exists a unique map $\eta : X \to H(A \times_E B)$, such that $g_1 \circ \eta = p_4 \circ \zeta_1$ and $g_2 \circ \eta = p_2 \circ \zeta_2$. Thus we can define

$$\xi : X \longrightarrow F \times_{\tau,G,\psi} H(A \times_E B) \cong F(A \times_E B) \times_{g_1,G(A \times_E B),p_4} H(A \times_E B)$$

by sending x to $\big(\zeta_1(x), \eta(x)\big)$. Checking the requirements (2.8) is obvious for ζ_1, and for ζ_2 we can use the following observation: An element of

$$F \times_{\tau,G,\psi} H(A) \times_{F \times_{\tau,G,\psi} H(E)} F \times_{\tau,G,\psi} H(B)$$

is uniquely determined by its image under p_2 and f_3. Thus, in order to show $\zeta_2 = f_2 \circ \xi$, it is sufficient to show

$$p_2 \circ \zeta_2 = p_2 \circ f_2 \circ \xi \text{ and } f_3 \circ \zeta_2 = f_3 \circ f_2 \circ \xi.$$

The first identity follows from the definition of ξ and commutativity of the square p_1-g_2-f_2-p_2, and the second identity follows from $f_4 \circ \zeta_1 = f_3 \circ \zeta_2$, $\zeta_1 = f_1 \circ \xi$ and commutativity of the foreground square $f_1 - f_2 - f_3 - f_4$.

It remains to check uniqueness of ξ: Let ξ' be another map fulfilling the requirements (2.8). We use the observation that an element of $F \times_{\tau,G,\psi} H(A \times_E B)$ is uniquely determined by its image under p_1 and f_1. Hence, it suffices to show

$$f_1 \circ \xi = f_1 \circ \xi' \text{ and } p_1 \circ \xi = p_1 \circ \xi'.$$

The first identity is fulfilled by the requirements (2.8) made on ξ and ξ', and the second identity follows from commutativity of the square p_1-g_2-f_2-p_2 together with the assumption that the background square $g_1 - g_2 - g_3 - g_4$ is a pullback diagram: This implies that the map η as above is unique with respect to the requirements $g_1 \circ \eta = p_4 \circ \zeta_1$ and $g_2 \circ \eta = p_2 \circ \zeta_2$. By commutativity of the above diagram, $p_1 \circ \xi$ and $p_1 \circ \xi'$ fulfill these requirements and we get

$$p_1 \circ \xi = \eta = p_1 \circ \xi'$$

and the lemma follows. □

Lemma 2.28. *Let H be a relatively representable subfunctor of F and assume that F is pro-representable by a suitable object $R \in \mathcal{C}_\Lambda$. Then there exists an ideal $I \subset R$ such that H is pro-representable by R/I.*

Proof. We first check that H fulfills the Grothendieck criterion, provided that F does. Finitude of tangent spaces is obvious, and the Mayer-Vietoris property can be read off from diagram (2.7), using that $h^F_{f,g}$ is an isomorphism and using the formal property $Y \times_Y Z \cong Z$ of the fiber product. That the pro-representing object of H is a quotient of the pro-representing object of F follows from [Maz97b, §19, Lemma]. (We remark that Lemma 2.28 is a standard fact, often proved via Schlessinger's criterion instead of Grothendieck's criterion, see [Gou01, Problem 3.5] or, for more details, [Har07, Proposition 1.3].) □

Remark 2.29. Although we will not make extensive use of this in the sequel, let us remark that the following strengthening of Lemma 2.28 holds: If F is pro-representable, then a subfunctor H of F is pro-representable if and only if it is relatively representable: One implication was proved in Lemma 2.28, so assume that H is pro-representable. This implies that both F and H fulfill the Mayer-Vietoris property (part 1. of Theorem 2.25). Using again the formal property $Y \times_Y Z \cong Z$, it is clear that the diagram (2.7) is a pullback diagram. That condition 1. of Definition 2.26 is fulfilled follows from assumption (2.6) together with the pro-representability of H. Thus, H is relatively representable. (We also remark that this strengthening appeared in [Gro95b] as Proposition 3.7, albeit for contravariant functors.)

We make this explicit for the choice $\mathcal{C} = {}^\star\mathcal{C}^\circ_\Lambda$ and $F = {}^\star D^{\circ,\square}_\Lambda(\overline{\rho})$:

Definition 2.30. A *lifting condition* is a family ${}^\star\mathcal{D} = (S(A))_{A \in {}^\star\mathcal{C}^\circ_\Lambda}$, where each $S(A)$ is a set of A-valued liftings of $\overline{\rho}$ such that

1. $\overline{\rho} \in S(k)$;
2. Let $f : A \to A'$ be a morphism in ${}^\star\mathcal{C}^\circ_\Lambda$ and $\rho \in S(A)$. Then $\rho' := G(f) \circ \rho$ is in $S(A')$;
3. Let $f_1 : A_1 \to A, f_2 : A_2 \to A$ be morphisms in ${}^\star\mathcal{C}^\circ_\Lambda$ and let ρ_3 be a lifting of $\overline{\rho}$ to $A_3 := A_1 \times_A A_2$. For $i \in \{1, 2\}$ denote by $\pi_i : A_3 \to A_i$ the canonical projection and by ρ_i the lifting $G(\pi_i) \circ \rho_3$ of $\overline{\rho}$ to A_i. Then, $\rho_3 \in S(A_3)$ if and only if $\rho_1 \in S(A_1)$ and $\rho_2 \in S(A_2)$.

By condition 2. of this definition, the assignment $A \mapsto S(A)$ defines a subfunctor ${}^\star D^{\circ,\square,{}^\star\mathcal{D}}_\Lambda(\overline{\rho})$ of ${}^\star D^{\circ,\square}_\Lambda(\overline{\rho})$.

Proposition 2.31. ${}^\star D^{\circ,\square,{}^\star\mathcal{D}}_\Lambda(\overline{\rho})$ *is a relatively representable subfunctor of* ${}^\star D^{\circ,\square}_\Lambda(\overline{\rho})$.

Proof. Observe that condition 3. of Definition 2.30 is just the Mayer-Vietoris property spelled out. As condition 1. of Definition 2.30 is the same as condition 1. of Definition 2.26, we can use Remark 2.29 (or check condition 2. of Definition 2.26 by hand) to verify the claim. $\square$

There exists a converse to this proposition:

Proposition 2.32. *Let H be a relatively representable subfunctor of ${}^{\star}D^{\circ,\square}_{\Lambda}(\overline{\rho})$. Then $\big(H(A)\big)_{A\in {}^{\star}\mathcal{C}^{\circ}_{\Lambda}}$ is a lifting condition.*

Proof. Let $I \subset R^{\square}_{\Lambda}(\overline{\rho})$ be the ideal corresponding to H via Lemma 2.28, so that we have

$$H(A) = \Big\{\varphi \circ \rho^{\square} \,\Big|\, \varphi \in \mathrm{Hom}_{\Lambda}\big(R^{\square}_{\Lambda}(\overline{\rho}), A\big), \varphi(I) = 0\Big\} \text{ for } A \in {}^{\star}\mathcal{C}^{\circ}_{\Lambda}. \tag{2.9}$$

It is clear that $\overline{\rho} \in H(k)$. For condition 2. of Definition 2.30, let $\rho \in H(A)$ for some $A \in {}^{\star}\mathcal{C}^{\circ}_{\Lambda}$. Then $\rho = \varphi \circ \rho^{\square}$ for a suitable φ as in (2.9). If $f : A \to A'$ is a morphism in ${}^{\star}\mathcal{C}^{\circ}_{\Lambda'}$, we have to check $f \circ \rho \in H(A')$. But this is obvious from the characterization (2.9).
Recall the notation from condition 3. of 2.30. That $\rho_3 \in H(A_3)$ implies $\rho_1 \in H(A_1), \rho_2 \in H(A_2)$ follows from the same argument we used for condition 2. For the reverse implication, assume that $\rho_1 \in H(A_1)$ and $\rho_2 \in H(A_2)$ and let φ_1, φ_2 be the respective maps as in (2.9). Let $A_4 = A_1 \times A_2$ and $\rho_4 = \rho_1 \times \rho_2$ and observe that A_3 embeds into A_4. We see that $\rho_4 : \Gamma \to G(A_4)$ factors as

$$\rho_4 = (\varphi_1 \circ \rho^{\square}) \times (\varphi_2 \circ \rho^{\square}) = \varphi_4 \circ \rho^{\square}$$

for $\varphi_4 = (\varphi_1, \varphi_2)$ which fulfills $\varphi_4(I) = 0$ (but observe that φ_4 is *not* a lifting of $\overline{\rho}$, since A_4 is not local). On the other hand, we have a commutative diagram

$$\begin{array}{ccccc} & & & & A_3 \\ & & & \nearrow^{\varphi_3} & \big\downarrow \\ \Gamma & \xrightarrow{\rho^{\square}} & R^{\square}_{\Lambda}(\overline{\rho}) & & \\ & & & \searrow_{\varphi_4} & \big\downarrow \\ & & & & A_4 \end{array}$$

for a suitable map φ_3. It follows that $\varphi_3(I) = 0$ and hence $\rho_3 \in H(A_3)$. $\square$

This justifies that we will not distinguish between the terms “relatively representable subfunctor of ${}^{\star}D^{\circ,\square}_{\Lambda}(\overline{\rho})$” and “lifting condition”.

Proposition 2.33. *Let ${}^{\star}\mathcal{D} = (S(A))_{A\in {}^{\star}\mathcal{C}^{\circ}_{\Lambda}}$ and ${}^{\star}\mathcal{D}' = (S'(A))_{A\in {}^{\star}\mathcal{C}^{\circ}_{\Lambda}}$ be two deformation conditions. Then the assignment*

$${}^{\star}\mathcal{C}^{\circ}_{\Lambda} \to \underline{\mathrm{Ens}} \qquad A \mapsto S(A) \cap S'(A)$$

defines a lifting condition denoted ${}^{\star}\mathcal{D} \wedge {}^{\star}\mathcal{D}'$ or ${}^{\star}\mathcal{D}, {}^{\star}\mathcal{D}'$.

Proof. Conditions 1.-3. of Definition 2.30 are easily checked for ${}^{\star}\mathcal{D} \wedge {}^{\star}\mathcal{D}'$. Alternatively, we can observe that the pullback of the diagram

$$\begin{array}{ccc} & & {}^{\star}D^{\circ,\square,{}^{\star}\mathcal{D}}_{\Lambda}(\overline{\rho}) \\ & & \big\downarrow \\ {}^{\star}D^{\circ,\square,{}^{\star}\mathcal{D}'}_{\Lambda}(\overline{\rho}) & \longrightarrow & {}^{\star}D^{\circ,\square}_{\Lambda}(\overline{\rho}) \end{array}$$

corresponds precisely to the condition ${}^\star\mathcal{D} \wedge {}^\star\mathcal{D}'$, so the claim follows from Lemma 2.27. □

We come now to our second *extension principle*:

Observation 2.34. Let $\mathcal{D}$ be a family $(S(A))_{A\in\mathcal{C}^\circ_\Lambda}$ fulfilling the conditions of Definition 2.30, then the afforded subfunctor $D^{\circ,\square,\mathcal{D}}_\Lambda(\overline{\rho})$ of $D^{\circ,\square}_\Lambda(\overline{\rho})$ is relatively representable, hence gives rise to a quotient $R^{\square,\mathcal{D}}_\Lambda(\overline{\rho})$ of $R^{\square}_\Lambda(\overline{\rho})$ as pro-representing object. Let us assume that $R^{\square,\mathcal{D}}_\Lambda(\overline{\rho})$ is reduced[3]. Then we can extend the family $\mathcal{D}$ to a lifting condition ${}^\star\mathcal{D}$ (in the sense of Definition 2.30) by setting

$$S(A) := \Big\{\varphi\circ\pi_{\mathcal{D}}\circ\rho^\square \,\Big|\, \varphi\in\mathrm{Hom}_\Lambda\big(R^{\square,\mathcal{D}}_\Lambda(\overline{\rho}),A\big)\Big\},$$

where $\pi_{\mathcal{D}}: R^{\square}_\Lambda(\overline{\rho})\to R^{\square,\mathcal{D}}_\Lambda(\overline{\rho})$ is the canonical quotient map and $A\in{}^\star\mathcal{C}^\circ_\Lambda$. Moreover, by the unicity of the pro-representing object, this is the unique lifting condition which extends $\mathcal{D}$. This sets up a bijection between lifting conditions (denoted ${}^\star\mathcal{D}$) and $\mathcal{C}^\circ_\Lambda$-truncated lifting conditions (denoted $\mathcal{D}$) and justifies the omittance of the star in the notation of lifting conditions from now on.

Let us fix some consequences:

Corollary 2.35. *Let $\mathcal{D}=(S(A))_{A\in\mathcal{C}^\circ_\Lambda}$ be a lifting condition. Then:*

1. *There is an ideal $I_{\mathcal{D}}\subset R^{\square}_\Lambda(\overline{\rho})$ such that both $D^{\circ,\square,\mathcal{D}}_\Lambda(\overline{\rho})$ and ${}^\star D^{\circ,\square,\mathcal{D}}_\Lambda(\overline{\rho})$ are pro-representable by $R^{\square,\mathcal{D}}_\Lambda(\overline{\rho})=R^{\square}_\Lambda(\overline{\rho})/I_{\mathcal{D}}$;*
2. *$D^{\circ,\square,\mathcal{D}}_\Lambda(\overline{\rho})$ and ${}^\star D^{\circ,\square,\mathcal{D}}_\Lambda(\overline{\rho})$ extend to continuous subfunctors $D^{\square,\mathcal{D}}_\Lambda(\overline{\rho})$ of $D^{\square}_\Lambda(\overline{\rho})$ and ${}^\star D^{\square,\mathcal{D}}_\Lambda(\overline{\rho})$ of ${}^\star D^{\square}_\Lambda(\overline{\rho})$. Both $D^{\square,\mathcal{D}}_\Lambda(\overline{\rho})$ and ${}^\star D^{\square,\mathcal{D}}_\Lambda(\overline{\rho})$ are representable by $R^{\square}_\Lambda(\overline{\rho})/I_{\mathcal{D}}$;*
3. *Let Λ' be an object of ${}^\star\mathcal{C}_\Lambda$ with residue field $k':=k_{\Lambda'}$, and observe that, via the structure map $\Lambda\to\Lambda'$, we can understand ${}^{(\star)}\mathcal{C}^{[\circ]}_{\Lambda'}$ as a subcategory of ${}^{(\star)}\mathcal{C}^{[\circ]}_{\Lambda}$. Abbreviate $\overline{\rho}'$ for $\iota_{k\subset k'}\circ\overline{\rho}$ and $\mathcal{D}'$ for the truncation of $\mathcal{D}$ to those $S(A)$ for which A is in ${}^\star\mathcal{C}^\circ_{\Lambda'}$. Then, both $D^{\square,\mathcal{D}'}_{\Lambda'}(\overline{\rho}')$ and ${}^\star D^{\square,\mathcal{D}'}_{\Lambda'}(\overline{\rho}')$ are representable by*
$$R^{\square,\mathcal{D}'}_{\Lambda'}(\overline{\rho}')\cong\Lambda'\otimes_\Lambda R^{\square,\mathcal{D}}_\Lambda(\overline{\rho}).$$

(We remark that a special case of part 3. can also be found in [BLGGT14, Lemma 1.2.1]. We also remark at this point that an unframed version of these assertions hold true, i.e. anticipating the language of Section 2.3 below we have
$$R^{\mathcal{D}'}_{\Lambda'}(\overline{\rho}')\cong\Lambda'\otimes_\Lambda R^{\mathcal{D}}_\Lambda(\overline{\rho}).$$
if $\mathcal{D}$ is a deformation condition and if $R^{\mathcal{D}}_\Lambda(\overline{\rho})$ is representable.)

Proof. Only the last part requires an explanation: Let $I_{\mathcal{D}}$ denote the ideal associated to $\mathcal{D}$. Then $S(A)$ consists precisely of those $f\in\mathrm{Hom}_\Lambda(R^{\square}_\Lambda(\overline{\rho}),A)$ which vanish on $I_{\mathcal{D}}$. If A is an object of ${}^\star\mathcal{C}^\circ_{\Lambda'}$, $S(A)$ is in canonical bijection with the set of those $f\in\mathrm{Hom}_{\Lambda'}(\Lambda'\otimes_\Lambda R^{\square}_\Lambda(\overline{\rho}),A)$ which vanish on $\Lambda'\otimes_\Lambda I_{\mathcal{D}}$. It follows that $D^{\square,\mathcal{D}'}_{\Lambda'}(\overline{\rho}')$ and ${}^\star D^{\square,\mathcal{D}'}_{\Lambda'}(\overline{\rho}')$ are representable by

$$\big(\Lambda'\otimes_\Lambda R^{\square}_\Lambda(\overline{\rho})\big)/\big(\Lambda'\otimes_\Lambda I_{\mathcal{D}}\big)=\Lambda'\otimes_\Lambda\big(R^{\square}_\Lambda(\overline{\rho})/I_{\mathcal{D}}\big)=\Lambda'\otimes_\Lambda R^{\square,\mathcal{D}}_\Lambda(\overline{\rho}).$$

(Note that this also gives an alternative proof for the second part of Theorem 2.22.) □

In the remainder of this section, we introduce several examples for lifting conditions.

[3]The reducedness is a sufficient condition for the extension ${}^\star\mathcal{D}$ to define a deformation condition, cf. the corrections to [CHT08] in [BLGHT11, Lemma 3.2].

Liftings with a constraint on the kernel Let us fix another profinite group Δ together with an inclusion $\iota : \Delta \hookrightarrow \Gamma$. Consider the family $\mathcal{D}^{\ker \supset \Delta} = (S(A))_{A \in \mathcal{C}_\Lambda^\circ}$ with

$$S(A) = \left\{ \rho \in D_\Lambda^{\circ,\square}(\overline{\rho})(A) \;\middle|\; \rho|\Delta = \mathbf{1}_A \right\},$$

where $\mathbf{1}_A : \Delta \to G(A)$ denotes the trivial map (sending everything to the neutral element of $G(A)$) and where we abbreviate $\rho|\Delta$ for $\rho \circ \iota$.

Proposition 2.36. *Assume that $\overline{\rho}|\Delta$ is trivial, so that $\overline{\rho} \in S(k)$. Then $\mathcal{D}^{\ker \supset \Delta}$ defines a lifting condition.*

Proof. We easily observe that $D_\Lambda^{\circ,\square,\mathcal{D}^{\ker \supset \Delta}}(\overline{\rho})$ is the pullback of the diagram

$$\begin{array}{ccc} & & D_\Lambda^{\circ,\square}(\overline{\rho}) \\ & & \big\downarrow{\scriptstyle \iota^*} \\ T & \xrightarrow[\varphi]{} & D_\Lambda^{\circ,\square}(\overline{\rho}|\Delta) \end{array},$$

where $T : \mathcal{C}_\Lambda^\circ \to \underline{\mathrm{Ens}}$ is the functor sending any coefficient ring A to the one point set $\{*\}$ and where φ_A sends $\{*\}$ to the set $\{\mathrm{triv}_A\}$ containing only the trivial lift

$$\mathrm{triv}_A : \Delta \to G(A) \qquad \delta \mapsto 1_{G(A)}$$

of $\overline{\rho}|\Delta$. Thus, the claim follows once again from Lemma 2.27. $\square$

We will be mainly interested in the case where Γ is the absolute Galois group of a local field and where Δ is the inertia subgroup. We will then denote the afforded subfunctor by $D_\Lambda^{\circ,\square,\mathtt{nr}}(\overline{\rho})$ and refer to the parametrized liftings as *unramified liftings.*

Liftings of fixed factorization type In order to give another example of a deformation condition, let G' be a smooth linear algebraic group over $W(k)$ together with a morphism $d : G \to G'$ of algebraic groups. Let $\chi : \Gamma \to G'(\Lambda)$ be a fixed representation such that the following diagram commutes:

$$\begin{array}{ccccc} & & G'(\Lambda) & & \\ & \overset{\chi}{\nearrow} & & \overset{G'(\mathrm{mod}_{\mathfrak{m}_\Lambda})}{\searrow} & \\ \Gamma & & & & G'(k) \\ & \underset{\overline{\rho}}{\searrow} & & \underset{d_k}{\nearrow} & \\ & & G(k) & & \end{array}$$

Consider the family $\mathcal{D}^{d=\chi} = (S(A))_{A \in \mathcal{C}_\Lambda^\circ}$ with

$$S(A) = \{ \rho \in D_\Lambda^{\circ,\square}(\overline{\rho})(A) \mid d_A \circ \rho = \iota_A \circ \chi \},$$

where $\iota_A : G'(\Lambda) \to G'(A)$ is the homomorphism induced by the canonical structure morphism $\tilde{\iota}_A : \Lambda \to A$.

Proposition 2.37. *$\mathcal{D}^{d=\chi}$ is a lifting condition.*

Proof. Let us first treat the special case $G' = G$, $d = \mathrm{id}$ (and, hence, $\overline{\rho} = \overline{\chi}$): As $S(A) = \{\iota_A \circ \chi\}$, both conditions of Definition 2.26 are trivially fulfilled. Thus, $D_\Lambda^{\circ,\square,\mathcal{D}^{d=\chi}}(\overline{\chi})$ is a relatively representable subfunctor of $D_\Lambda^{\circ,\square}(\overline{\chi})$ and $\mathcal{D}^{d=\chi}$ is a lifting condition. The general case now follows immediately from Lemma 2.27, as $D_\Lambda^{\circ,\square,\mathcal{D}^{d=\chi}}(\overline{\rho})$ is the pullback of the diagram

$$\begin{array}{ccc} & & D_\Lambda^{\circ,\square}(\overline{\rho}) \\ & & \downarrow \\ D_\Lambda^{\circ,\square,\mathcal{D}^{d=\chi}}(\overline{\chi}) & \hookrightarrow & D_\Lambda^{\circ,\square}(\overline{\chi}) \, , \end{array}$$

where the horizontal map is the canonical inclusion and the (A-component of the) vertical map sends a lift ρ of $\overline{\rho}$ to the lift $d_A \circ \rho$ of $\overline{\chi}$. □

We will be mainly interested in the case where $G' = G^{\mathsf{ab}}$ and $d : G \to G^{\mathsf{ab}}$ is the projection modulo the derived subgroup G^{der}, where we abbreviate $D_\Lambda^{(\circ),\square,\chi}(\overline{\rho})$ for the subfunctor of $D_\Lambda^{(\circ),\square}(\overline{\rho})$ afforded by $\mathcal{D}^{d=\chi}$. We call $D_\Lambda^{(\circ),\square,\chi}(\overline{\rho})$ the *universal fixed determinant lifting ring.*

Ramakrishna lifting functor We will continue with a categorical description of certain lifting conditions in the case $G = \mathrm{GL}_n$, first considered in [Ram93]: Let $\underline{\mathrm{Rep}}_\Lambda^\circ(\Gamma)$ be the category of finite length Λ-modules together with a continuous action of Γ.

Definition 2.38. A full subcategory $\mathcal{R}$ of $\underline{\mathrm{Rep}}_\Lambda^\circ(\Gamma)$ which is stable under taking subobjects, quotients and finite direct sums is called a *Ramakrishna subcategory.*

The choice of a Ramakrishna subcategory $\mathcal{R}$ gives rise to a functor

$${}^\star D_\Lambda^{\circ,\square,\mathcal{R}}(\overline{\rho}) : \ {}^\star\mathcal{C}_\Lambda^\circ \longrightarrow \underline{\mathrm{Ens}}$$

characterized by

$${}^\star D_\Lambda^{\circ,\square,\mathcal{R}}(\overline{\rho})(A) = \{\rho \in {}^\star D_\Lambda^{\circ,\square}(\overline{\rho})(A) \mid \rho \in \mathcal{R}\},$$

where ρ is considered as an object of $\underline{\mathrm{Rep}}_\Lambda^\circ(\Gamma)$ via the structure morphism $\Lambda \to A$.

Proposition 2.39. *Suppose that $\overline{\rho}$ is in $\mathcal{R}$ (and that $G = \mathrm{GL}_n$). Then $\left({}^\star D_\Lambda^{\circ,\square,\mathcal{R}}(\overline{\rho})(A)\right)_{A \in {}^\star\mathcal{C}_\Lambda^\circ}$ is a lifting condition in the sense of Definition 2.30.*

Proof. Let $f : A \to A'$ be a morphism in ${}^\star\mathcal{C}_\Lambda^\circ$ and consider a diagram

$$\begin{array}{ccc} \Gamma & \xrightarrow{\ \rho\ } & \mathrm{GL}_n(A) \\ & \searrow^{\rho'} & \downarrow{\scriptstyle \mathrm{GL}_n(f)} \\ & & \mathrm{GL}_n(A') \, , \end{array}$$

where we suppose that $M = A^n$ (considered as a Λ-module via ρ) is in $\mathcal{R}$. In order to verify part 2. of Definition 2.30, we have to check that $M' := M \otimes_A A'$ is in $\mathcal{R}$. In order to do this, we may assume that

A' is of the form $A[X_1, \dots, X_m]/I$ for some $m \in \mathbb{N}$ and for a suitable ideal $I \subset A[X_1, \dots, X_m]$ which contains $(X_1, \dots, X_m)^t$ for some $t \in \mathbb{N}$. The passage

$$M \rightsquigarrow M'' := M \otimes_A A[X_1, \dots, X_m]/(X_1, \dots, X_m)^t$$

replaces M by a finite direct sum of copies of M, which is in $\mathcal{R}$. The passage

$$M'' \rightsquigarrow M' = M'' \otimes_{A[X_1,\dots,X_m]/(X_1,\dots,X_m)^t} A[X_1, \dots, X_m]/I$$

replaces M'' by a quotient of M'', which is in $\mathcal{R}$.

Let $f_1 : A_1 \to A, f_2 : A_2 \to A$ be a morphisms in ${}^\star\mathcal{C}^\circ_\Lambda$ and let ρ be a lifting of $\overline{\rho}_3$ to $A_3 := A_1 \times_A A_2$. Recall the notation ρ_1, ρ_2, ρ_3 from part 3. of Definition 2.30 and assume that ρ_3 is in $\mathcal{R}$. If M_i is the underlying module of ρ_i, it is clear that $M_1 = M_3 \otimes_{A_3} A_1$ and $M_2 = M_3 \otimes_{A_3} A_2$ are quotients of M_3, thus it follows that M_1 and M_2 are in $\mathcal{R}$. For the opposite direction, assume that M_1, M_2 are in $\mathcal{R}$. Then M_3 can be realized as a submodule of $M_1 \times M_2$, hence the claim follows.

(This proof is based on unpublished lecture notes of M. Harris [Har07].) □

Global conditions composed by local conditions For this final paragraph, fix profinite groups Δ_σ together with inclusions

$$\left(\iota_\sigma : \Delta_\sigma \hookrightarrow \Gamma\right)_{\sigma \in \Sigma},$$

for a finite index set Σ. (The example we have in mind is where Γ is the absolute Galois group of a global field F and the Δ_σ are decomposition groups at places of F.) We continue to denote by $\overline{\rho}$ a fixed G-valued residual representation of Γ, so we get natural transformations

$$f_\sigma : D^{\circ,\square,(\chi)}_\Lambda(\overline{\rho}) \to D^{\circ,\square,(\chi_\sigma)}_\Lambda(\overline{\rho}_\sigma),$$

characterized by sending a lift ρ of $\overline{\rho}$ to the lift $\rho_\sigma := \rho \circ \iota_\sigma$ of $\overline{\rho}_\sigma := \overline{\rho} \circ \iota_\sigma$. (Here, χ_σ denotes the lift $\chi \circ \iota_\sigma$ of the determinant of $\overline{\rho}_\sigma$.)

In addition to the given data, let us fix for each $\sigma \in \Sigma$ a lifting condition $D_\sigma = (S_\sigma(A))_{A \in \mathcal{C}^{(\circ)}_\Lambda}$ for $\overline{\rho}_\sigma$. We define a family $\mathcal{D} = (S(A))_{A \in \mathcal{C}^{(\circ)}_\Lambda}$ by

$$S(A) = \left\{\rho \in D^{\circ,\square,(\chi)}_\Lambda(\overline{\rho})(A) \;\middle|\; \rho_\sigma \in S_\sigma(A) \text{ for all } \sigma \in \Sigma\right\}.$$

Lemma 2.40. *$\mathcal{D}$ is a lifting condition for $\overline{\rho}$.*

Proof. We argue as in the proof of Proposition 2.37: First observe that the f_σ glue to a natural transformation

$$f : D^{\circ,\square,(\chi)}_\Lambda(\overline{\rho}) \to \prod_{\sigma \in \Sigma} D^{\circ,\square,(\chi_\sigma)}_\Lambda(\overline{\rho}_\sigma). \tag{2.10}$$

If we denote by g_σ the inclusion transformation $D^{\circ,\square,(\chi_\sigma),D_\sigma}_\Lambda \hookrightarrow D^{\circ,\square,(\chi_\sigma)}_\Lambda$, we get another natural transformation

$$g = \prod_{\sigma \in \Sigma} g_\sigma : \prod_{\sigma \in \Sigma} D^{\circ,\square,(\chi_\sigma),D_\sigma}_\Lambda \hookrightarrow \prod_{\sigma \in \Sigma} D^{\circ,\square,(\chi_\sigma)}_\Lambda(\overline{\rho}_\sigma).$$

It is easy to check that this inclusion is relatively representable, using our assumption that each inclusion g_σ is relatively representable (use Corollary 2.5 together with Remark 2.29). Thus we get a diagram of natural transformations

$$\begin{array}{ccc} & & D_\Lambda^{\circ,\square,(\chi)}(\overline{\rho}) \\ & & \downarrow f \\ \prod_{\sigma\in\Sigma} D_\Lambda^{\circ,\square,(\chi_\sigma),D_\sigma} & \overset{}{\underset{g}{\hookrightarrow}} & \prod_{\sigma\in\Sigma} D_\Lambda^{\circ,\square,(\chi_\sigma)}(\overline{\rho}_\sigma) \end{array}$$

and the subfunctor $D_\Lambda^{\circ,\square,(\chi),\mathcal{D}}(\overline{\rho})$ of $D_\Lambda^{\circ,\square,(\chi)}(\overline{\rho})$ corresponding to $\mathcal{D}$ is its pullback. Thus, the claim follows immediately from Lemma 2.27 and Proposition 2.32. □

(We remark that there is a similar statement in [Böc99, Lemma 2.3].)

2.3 Deformations and deformation conditions

For $A \in \mathcal{C}_\Lambda$, recall the reduction map $\mathrm{mod}_{\mathfrak{m}_A} : G(A) \to G(k)$ from (2.5) and consider the following subsets of $G(A)$:

$$\hat{G}(A) = \ker(\mathrm{mod}_{\mathfrak{m}_A}) \qquad \text{and} \qquad Z_{A,\overline{\rho}} = \{g \in G(A) \mid \mathrm{mod}_{\mathfrak{m}_A}(g).\overline{\rho}.\,\mathrm{mod}_{\mathfrak{m}_A}(g)^{-1} = \overline{\rho}\}.$$

Definition 2.41. Two liftings $\rho_1, \rho_2 \in D_\Lambda^{\square,}(\overline{\rho})(A)$ are called *equivalent* (in symbols: $\rho_1 \sim \rho_2$) if they are conjugate by an element of $Z_{A,\overline{\rho}}$. They are called *strictly equivalent* (in symbols: $\rho_1 \overset{\mathrm{st}}{\sim} \rho_2$) if they are conjugate by an element of $\hat{G}(A)$.

We will usually impose the following two conditions:

(Centr)$_k$: $Z_{G,k}$ contains the centralizer $C_G(\overline{\rho}(\Gamma))$ as schemes over k;
(SmCtr) : Z_G is formally smooth over Λ.

(Here, $Z_{G,k}$ denotes the base change of Z_G to k.)

Proposition 2.42. *If* ***(Centr)***$_k$ *and* ***(SmCtr)*** *are fulfilled, then ρ_1 and ρ_2 are equivalent if and only if they are strictly equivalent.*

Proof. See [Til96, Section 3.2]. □

Definition 2.43 (Deformation functor)**.** Let

$$D_\Lambda(\overline{\rho}) : \mathcal{C}_\Lambda \longrightarrow \underline{\mathrm{Ens}}$$

denote the functor which assigns to A the set of strict equivalence classes of lifts of $\overline{\rho}$ to A:

$$D_\Lambda(\overline{\rho})(A) = D_\Lambda^{\square}(\overline{\rho})(A)/\hat{G}(A).$$

For ease of notation, we will refer to elements of $D_\Lambda(\overline{\rho})(A)$ by representatives, i.e. we will write ρ instead of $[\rho]_{\overset{\mathrm{st}}{\sim}}$.

Remark 2.44. It is possible to develop a theory of deformations of $\overline{\rho}$ to rings in ${}^{\star}\mathcal{C}_\Lambda$ entirely analogous to Section 2.2. However, we do not need this and refer the reader to Appendix A of [CDT99] where this is carried out in detail. Another reference for (partial) results in that direction is [Maz97b, §12].

Let us denote by $\mathfrak{g} = \hat{G}(k[\epsilon])$ (resp. by $\mathfrak{z}$) the Lie algebra of the special fiber of G (resp. of the center Z_G of G). With reference to our fixed residual representation $\overline{\rho}$, we will regard $\mathfrak{g}$ as a Γ-module via the adjoint representation, i.e.

$$\gamma.X := \mathrm{Ad}(\overline{\rho}(\gamma)).X \qquad (\gamma \in \Gamma, X \in \mathfrak{g}).$$

This operation restricts to the subalgebra $\mathfrak{g}^{\mathtt{der}}$. In the case $G = \mathrm{GL}_n$, we will also use the more familiar notation $\mathrm{ad}\,\overline{\rho}$ (resp. $\mathrm{ad}\,\overline{\rho}^0$) instead of $\mathfrak{g}$ (resp. $\mathfrak{g}^{\mathtt{der}}$).

Now, consider the condition

(Centr) : $\quad H^0(\Gamma, \mathfrak{g}) = \mathfrak{z}.$

Remark 2.45.

1. It is shown in [Mau00, Lemma 2.4(i)] that **(Centr)** is equivalent to the following condition: For any $A \in \mathcal{C}_\Lambda$ and any deformation ρ of $\overline{\rho}$ to A, we have an inclusion

$$\big(Z_{A,\rho} \cap \hat{G}(A)\big) \subset Z_G(A),$$

where

$$Z_{A,\rho} = \{g \in G(A) \mid g.\rho.g^{-1} = \rho\}.$$

2. Assume **(SmCtr)** and assume that we have an equality

$$Z_G^\circ = C_G(\overline{\rho}(\Gamma))^\circ \text{ (as varieties).}$$

Then an equality

$$Z_G^\circ = C_G(\overline{\rho}(\Gamma))^\circ \text{ (as group schemes)} \tag{2.11}$$

follows if we suppose that the closed subgroup $\overline{\rho}(\Gamma) \subset G$ is *separable* in G (in the sense of [BMR05, Definition 3.27]).

3. Assume **(SmCtr)**. Then the equality (2.11) is equivalent to **(Centr)**, cf. [Til96], Comment 2 following Theorem 3.3.

Definition 2.46 ([BMRT10])**.** Fix a maximal torus $T \subset G$ and a Borel subgroup $B \subset G$ containing T. Let $\Psi = \Psi(G,T)$ denote the set of roots of G with respect to T, let $\Sigma = \Sigma(G,T)$ denote the set of simple roots of Ψ defined by B and and let $\Psi^+ = \Psi(B,T)$ denote the set of positive roots of G. If $\beta \in \Psi^+$, we write

$$\beta = \sum_{\alpha \in \Sigma} c_{\alpha,\beta}\alpha$$

for suitable $c_{\alpha,\beta} \in \mathbb{N}_0$. A prime ℓ is then called *good* for G if it does not divide any non-zero $c_{\alpha,\beta}$. A prime ℓ is called *very good* for G if ℓ is good for G and if ℓ does not divide $n+1$ for any simple component of G of type A_n.

Lemma 2.47 ([BMRT10, Theorem 1.2 and Corollary 2.13])**.**

1. *If $\ell = \mathrm{char}\, k$ is very good for G, then any closed subgroup $H \subset G$ is separable in G;*

2. *If there exists an embedding $G \hookrightarrow \mathrm{GL}(V)$ such that $(\mathrm{GL}(V), G)$ is a reductive pair, then any closed subgroup $H \subset G$ is separable in G.*

Moreover, we have the following result:

Theorem 2.48 ([Til96, Theorem 3.3])**.** *Assume that **(SmCtr)** and **(Centr)** are fulfilled. Then $D_\Lambda(\overline{\rho})$ is representable by some ring $R_\Lambda(\overline{\rho})$ in $\mathcal{C}_\Lambda$.*

We call $R_\Lambda(\overline{\rho})$ the *universal deformation ring* of $\overline{\rho}$ and the afforded deformation $\rho : \Gamma \to G(R_\Lambda(\overline{\rho}))$ the corresponding *universal deformation.*

Observe that in the case $G = \mathrm{GL}_n$, **(Centr)** corresponds to the usual centralizer condition

$$\mathrm{Hom}_{k[\Gamma]}(\overline{\rho}, \overline{\rho}) = k.$$

In practice, this condition is often deduced from absolute irreducibility of $\overline{\rho}$ by Schur's Lemma. In order to generalize this implication for more general groups, we first make a definition following [Ser98]:

Definition 2.49 (Irreducibility)**.** We say that $\overline{\rho}$ is *absolutely irreducible* if there does not exist a proper parabolic subgroup $P \subsetneq G$ over $\overline{k}$ such that $\overline{\rho}(\Gamma) \subset P$.

Lemma 2.50 (Schur's Lemma)**.** *Assume **(SmCtr)** and assume that ℓ is very good for G or that there exists an embedding $G \hookrightarrow \mathrm{GL}(V)$ such that $(\mathrm{GL}(V), G)$ is a reductive pair. Then **(Centr)** is fulfilled if $\overline{\rho}$ is absolutely irreducible.*

Proof. As the image $\overline{\rho}(\Gamma)$ is finite, it follows from [BMR05, Proposition 2.13] that $Z_G^\circ = C_G(\overline{\rho}(\Gamma))^\circ$ (as varieties) if $\overline{\rho}$ is absolutely irreducible. The claim now follows from Remark 2.45 and Lemma 2.47. □

In the sequel, we will say that $\overline{\rho}$ is *Schur* if the conditions of Lemma 2.50 are fulfilled. (Observe that Clozel, Harris and Taylor give a different definition of "Schur" in [CHT08, Definition 2.1.6].)

Tangent spaces

Definition 2.51. The *tangent spaces* of $D_\Lambda^\square(\overline{\rho})$ and $D_\Lambda(\overline{\rho})$ are the finite-dimensional k-vector spaces[4]

$$t_{D_\Lambda^\square(\overline{\rho})} = D_\Lambda^\square(\overline{\rho})(k[\epsilon]) \qquad \text{and} \qquad t_{D_\Lambda(\overline{\rho})} = D_\Lambda(\overline{\rho})(k[\epsilon]).$$

Proposition 2.52. 1. *There are canonical isomorphisms*

$$t_{D_\Lambda^\square(\overline{\rho})} \cong Z^1(\Gamma, \mathfrak{g}) \qquad \textit{and} \qquad t_{D_\Lambda(\overline{\rho})} \cong H^1(\Gamma, \mathfrak{g});$$

2. *The natural transformation*

$$\eta : D_\Lambda^\square(\overline{\rho}) \to D_\Lambda(\overline{\rho})$$

defined by $\eta_A(\rho) = [\rho] \in D_\Lambda(\overline{\rho})(A)$ for $A \in \mathcal{C}_\Lambda$ and $\rho \in D_\Lambda^\square(\overline{\rho})(A)$ is formally smooth[5]*.*

[4] For an explanation how $D_\Lambda^\square(\overline{\rho})(k[\epsilon])$ and $D_\Lambda(\overline{\rho})(k[\epsilon])$ are regarded as k-vector spaces, see e.g. [Gou01, Lecture 2].

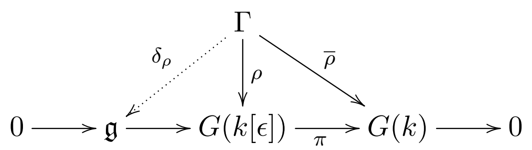
[5] For the definition of the term "formally smooth" in this context see [Böc13a, Definition 1.4.5].

3. *If* **(Centr)** *and* **(SmCtr)** *hold, then there is an isomorphism*

$$R^{\square}_{\Lambda}(\overline{\rho}) \cong R_{\Lambda}(\overline{\rho})[[X_1, \ldots, X_m]]$$

with $m = \dim \mathfrak{g} - h^0(\Gamma, \mathfrak{g}) = \dim \mathfrak{g} - \dim \mathfrak{z}$.

Proof. For the first part, consider the diagram



If $\rho \in D^{\square}_{\Lambda}(\overline{\rho})(k[\epsilon])$, we set

$$\delta_{\rho} : \Gamma \to \mathfrak{g} = \ker(\pi), \qquad \gamma \mapsto \rho(\gamma)\overline{\rho}(\gamma)^{-1}.$$

We easily check that this defines a 1-cocycle:

$$\delta_{\rho}(\gamma_1\gamma_2) = \rho(\gamma_1\gamma_2)\overline{\rho}(\gamma_1\gamma_2)^{-1} = \rho(\gamma_1)\overline{\rho}(\gamma_1)^{-1}\overline{\rho}(\gamma_1)\rho(\gamma_2)\overline{\rho}(\gamma_2)^{-1}\overline{\rho}(\gamma_1)^{-1} = \delta_{\rho}(\gamma_1)\,\delta_{\rho}(\gamma_2)^{\gamma_1}.$$

On the other hand, let $\delta \in Z^1(\Gamma, \mathfrak{g})$. We define a map

$$\rho_{\delta} : \Gamma \to G(k[\epsilon]), \qquad \gamma \mapsto \delta(\gamma)\overline{\rho}(\gamma),$$

and we easily check that this defines a representation (which obviously lifts $\overline{\rho}$):

$$\rho_{\delta}(\gamma_1\gamma_2) = \delta(\gamma_1)\delta(\gamma_2)^{\gamma_1}\overline{\rho}(\gamma_1\gamma_2) = \delta(\gamma_1)\overline{\rho}(\gamma_1)\delta(\gamma_2)\overline{\rho}(\gamma_1)^{-1}\overline{\rho}(\gamma_1)\overline{\rho}(\gamma_2) = \rho_{\delta}(\gamma_1)\rho_{\delta}(\gamma_2)$$

Therefore, the assignments $\rho \mapsto \delta_{\rho}$ and $\delta \mapsto \rho_{\delta}$ provide the desired identification between $Z^1(\Gamma, \mathfrak{g})$ and $D^{\square}_{\overline{\rho}}(k[\epsilon])$.

For the claim in the unframed situation (which is also treated in Chapter 3 of [Til96]), we easily check that conjugating a lift ρ with elements of $\hat{G}(k[\epsilon])$ amounts to multiplying δ_{ρ} with coboundaries. (Further references are: [Mau00, Theorem 2.6] and [Böc07, Theorem 2.2 (c)].)

Formal smoothness is proved in [Lev13, Proposition 7.2.5] (using smoothness of G) for the corresponding natural transformation between deformation groupoids[6]

$$\tilde{\eta} : \mathcal{D}^{\square}(\overline{\rho}) \to \mathcal{D}(\overline{\rho}).$$

It remains to check that for a surjection $A \to A'$ in $\mathcal{C}_{\Lambda}$, the map

$$\mathcal{D}^{\square}(\overline{\rho})(A') \times_{\mathcal{D}(\overline{\rho})(A')} \mathcal{D}(\overline{\rho})(A) \longrightarrow D^{\square}(\overline{\rho})(A') \times_{D(\overline{\rho})(A')} D(\overline{\rho})(A)$$

is surjective, which is straightforward.

In [Lev13] it is also explained that the fiber $\mathcal{F} := \tilde{\eta}^{-1}_{k[\epsilon]}(\overline{\rho})$ of $\tilde{\eta}_{k[\epsilon]}$ is a principal homogeneous space for $\hat{G}(k[\epsilon]) = \mathfrak{g}$. If $|\cdot|$ denotes the canonical map from $\mathcal{D}^{\square}(\overline{\rho})$ to $D^{\square}(\overline{\rho})$, it is clear that the kernel K in

$$0 \to K \to \mathcal{F} \to |\mathcal{F}| \to 0$$

can be identified with $\{g \in \hat{G}(k[\epsilon]) \mid g\overline{\rho}g^{-1} = \overline{\rho}\} \cong H^0(\Gamma, \mathfrak{g})$. Therefore, the fiber $\eta^{-1}_{k[\epsilon]}(\overline{\rho})$ is a principal homogeneous space for $\mathfrak{g}/H^0(\Gamma, \mathfrak{g})$ and the claims follow. (This can also be found in the proof of [Bal12, Proposition 4.1.5].) □

[6] Cf. [Böc13a, Section 1.6].

From now on, let us assume conditions **(Centr)** and **(SmCtr)**.

Theorem 2.53. *There exists a presentation*

$$0 \to J \to \Lambda[[X_1, \dots, X_h]] \to R_\Lambda(\overline{\rho}) \to 0$$

with $h = h^1(\Gamma, \mathfrak{g})$ and where the number of generators of J is bounded from above by $h^2(\Gamma, \mathfrak{g})$.

Proof. See [Böc07, Theorem 2.2 (d)]. □

Deformation conditions

Definition 2.54. Let $\mathcal{D} = (S(A))_{A \in \mathcal{C}^\circ_\Lambda}$ be a lifting condition in the sense of Definition 2.30 (and Observation 2.34). Then $\mathcal{D}$ is called a *deformation condition* if the following additional condition is fulfilled:

4. Let $\rho \in S(A)$ and $g \in \hat{G}(A)$ for some $A \in \mathcal{C}^\circ_\Lambda$. Then $g\rho g^{-1} \in S(A)$.

A deformation condition $\mathcal{D}$ defines a subfunctor $D^{\mathcal{D}}_\Lambda(\overline{\rho})$ of $D_\Lambda(\overline{\rho})$ which is relatively representable:

Lemma 2.55. *$D^{\mathcal{D}}_\Lambda(\overline{\rho})$ is representable by a quotient $R^{\mathcal{D}}_\Lambda(\overline{\rho})$ of $R_\Lambda(\overline{\rho})$.*

Proof. Using condition 4., this can be deduced as in the framed case (Proposition 2.31). □

It is straightforward to check that the conditions from the end of Section 2.2,

- Liftings with a constraint on the kernel,
- Liftings of fixed factorization type,
- Ramakrishna liftings,
- Global conditions composed by local conditions,

fulfill the additional property 4. Moreover, it is clear that an assertion analogous to Proposition 2.33 holds, i.e. that if $\mathcal{D}$ and $\mathcal{D}'$ are deformation conditions, then so is $\mathcal{D} \wedge \mathcal{D}'$. We introduce another condition:

Example 2.56 (Deformations unramified outside Σ)**.** Let Γ be the absolute Galois group of a global field F and let Σ be a finite subset of Pl_F such that $\overline{\rho}$ is unramified outside Σ. We denote by $\mathcal{D}^{\Sigma-\mathrm{nr}}$ the condition on deformations of being unramified outside Σ, i.e. parametrizing those deformations ρ of $\overline{\rho}$ for which ρ_ν is unramified if $\nu \notin \Sigma$. It is easily checked by hand that this defines a deformation condition, but this can also be achieved by the following characterization: A lift ρ of $\overline{\rho}$ which is unramified outside Σ can be regarded as a (unconditioned) lift of $\overline{\rho}|\,\mathrm{Gal}(F_\Sigma|F)$ and vice versa, where F_Σ denotes the maximal extension of F unramified outside Σ. In this way, we get a natural isomorphism of functors

$$D^{\mathcal{D}^{\Sigma-\mathrm{nr}}}_\Lambda(\overline{\rho}) \cong D_\Lambda(\overline{\rho}|\,\mathrm{Gal}(F_\Sigma|F)).$$

In the sequel, we will not distinguish between these two and refer to them as $D_{\Lambda,\Sigma}(\overline{\rho})$.

Let $\mathfrak{g}^{\mathrm{der}}$ denote the Lie algebra of G^{der}, $\mathfrak{g}^{\mathrm{ab}}$ the Lie algebra of G^{ab} and let $H^1(\Gamma, \mathfrak{g}^{\mathrm{der}})'$ denote the image of the map

$$H^1(\Gamma, \mathfrak{g}^{\mathrm{der}}) \to H^1(\Gamma, \mathfrak{g}).$$

(We remark that for $\ell \gg 0$, we have $H^1(\Gamma, \mathfrak{g}^{\mathrm{der}})' = H^1(\Gamma, \mathfrak{g}^{\mathrm{der}})$.)
Then we have the following variation of Proposition 2.52.1 for the fixed determinant condition:

Proposition 2.57. *There are canonical isomorphisms of k-vector spaces*

$$t_{D_\Lambda^{\square,\chi}(\overline{\rho})} \cong Z^1(\Gamma, \mathfrak{g}^{\mathrm{der}}) \qquad \textit{and} \qquad t_{D_\Lambda^{\chi}(\overline{\rho})} \cong H^1(\Gamma, \mathfrak{g}^{\mathrm{der}})'.$$

Proof. The first part is shown as in the proof of Proposition 2.52, except that we have to check that $\rho_\delta = \delta\overline{\rho}$ is of type $\mathcal{D}^{d=\chi}$ precisely if the values of δ lie in the subspace $\mathfrak{g}^{\mathrm{der}} \subset \mathfrak{g}$. Assume first that δ fulfills this condition, then it follows from the fact that $\delta(\gamma) \in G^{\mathrm{der}}(k[\epsilon])$ (for any $\gamma \in \Gamma$) that

$$d_{k[\epsilon]} \circ (\delta\overline{\rho})(\gamma) = d_{k[\epsilon]} \circ \overline{\rho}(\gamma) = \chi_{k[\epsilon]},$$

where $d : G \to G^{\mathrm{ab}}$ is the projection map modulo G^{der} and where $\chi_{k[\epsilon]}$ is the concatenation of χ with the canonical map $\Lambda^\times \to k[\epsilon]^\times$. On the other hand, suppose that ρ is of type $\mathcal{D}^{d=\chi}$, then the corresponding $\delta = \delta_\rho$ is given by $\gamma \mapsto \rho(\gamma)\overline{\rho}(\gamma)^{-1}$. Thus, we have that $d_{k[\epsilon]} \circ \delta = 1$, i.e. that δ has values in $\mathfrak{g}^{\mathrm{der}}$.
The second claim is proved in [Til96, Proposition 3.2]. We also refer to [Bal12], where this isomorphism is explicitly stated (following the proof of Proposition 4.2.4). □

Proposition 2.58. *There is an exact sequence*

$$0 \longrightarrow \mathfrak{g}/\mathfrak{g}^\Gamma \longrightarrow t_{D_\Lambda^{\square,(\chi)}(\overline{\rho})} \longrightarrow t_{D_\Lambda^{(\chi)}(\overline{\rho})} \longrightarrow 0.$$

Proof. The fibers of $t_{D_\Lambda^{\square,(\chi)}(\overline{\rho})} \longrightarrow t_{D_\Lambda^{(\chi)}(\overline{\rho})}$ are isomorphic to $\hat{G}(k[\epsilon])/\hat{G}(k[\epsilon])^\Gamma$, so the claim follows by applying the exponential map. (See also [Bal12, proof of Proposition 4.1.5], and the remarks following Proposition 4.2.4 in the fixed determinant case.) □

We remark that for $\ell \gg 0$, we have a decomposition $\mathfrak{g} = \mathfrak{g}^{\mathrm{der}} \oplus \mathfrak{g}^{\mathrm{ab}}$. Thus $\mathfrak{g}^\Gamma = (\mathfrak{g}^{\mathrm{der}})^\Gamma \oplus \mathfrak{g}^{\mathrm{ab}}$ and we get the following alternative version:

$$0 \longrightarrow \mathfrak{g}^{\mathrm{der}}/(\mathfrak{g}^{\mathrm{der}})^\Gamma \longrightarrow t_{D_\Lambda^{\square,(\chi)}(\overline{\rho})} \longrightarrow t_{D_\Lambda^{(\chi)}(\overline{\rho})} \longrightarrow 0. \tag{2.12}$$

2.4 Multiply framed deformations

Let F be a number field and fix for each place ν of F an embedding $\iota_\nu : \mathrm{Gal}(F_\nu) \hookrightarrow \mathrm{Gal}(F)$. For this section, we take $\Gamma = \mathrm{Gal}(F)$ or $\Gamma = \mathrm{Gal}(F_S)$, the absolute Galois group of the maximal extension of F which is unramified outside a finite set S of places of F. With respect to a residual representation

$$\overline{\rho} : \Gamma \to G(k)$$

fulfilling **(Centr)** and **(SmCtr)** we consider the *local representations* $\overline{\rho}_\nu := \overline{\rho} \circ \iota_\nu$. Likewise, we will denote the deformation and lifting functors with respect to $\overline{\rho}$ by $D_\star^\star(\overline{\rho})$ if $\Gamma = \mathrm{Gal}(F)$, by $D_{S,\star}^\star(\overline{\rho})$ if $\Gamma = \mathrm{Gal}(F_S)$, and with respect to $\overline{\rho}_\nu$ by $D_\star^\star(\overline{\rho}_\nu)$.

Definition 2.59. For a finite set Σ of places of F define the functor $D^{\square_\Sigma}_{S,\Lambda}(\overline{\rho})$ by the assignment

$$A \mapsto \left\{(\rho, (\rho_\nu, \beta_\nu)_{\nu\in\Sigma}) \;\middle|\; \substack{\rho\in D^{\square}_{S,\Lambda}(\overline{\rho})(A),\ \rho_\nu\in D^{\square}_{\Lambda}(\overline{\rho}_\nu)(A),\ \beta_\nu\in\hat{G}(A) \\ \text{s.t. } \rho|\operatorname{Gal}(F_\nu)=\beta_\nu\rho_\nu\beta_\nu^{-1}}\right\}_{/\sim},$$

where $(\rho, (\rho_\nu, \beta_\nu)_{\nu\in\Sigma})$ and $(\rho', (\rho'_\nu, \beta'_\nu)_{\nu\in\Sigma})$ are equivalent if $\rho_\nu = \rho'_\nu$ for all ν and if there is a $\gamma \in \hat{G}(A)$ such that $\rho' = \gamma\rho\gamma^{-1}$ and $\beta'_\nu = \gamma^{-1}\beta_\nu$ for all ν.

Remark 2.60. Note that specifying the ρ_ν is not strictly necessary, as they can be obtained from ρ and β_ν.

We will consider three types of subfunctors of $D^{\square_\Sigma}_{S,\Lambda}(\overline{\rho})$, all of which are familiar from the previous subsections (and which define deformation conditions, cf. the remark preceding Example 2.56):

- **Fixed determinant liftings**: Fix a lift $\chi : \Gamma \to G^{\mathsf{ab}}(\Lambda)$ of the determinant map

  $$\overline{\chi} : \Gamma \xrightarrow{\overline{\rho}} G(k) \longrightarrow G^{\mathsf{ab}}(k),$$

  then we define the subfunctor $D^{\square_\Sigma}_{S,\Lambda}(\overline{\rho})$ via

  $$D^{\square_\Sigma,\chi}_{S,\Lambda}(\overline{\rho})(A) = \left\{[\rho, (\rho_\nu, \beta_\nu)_{\nu\in\Sigma}] \in D^{\square_\Sigma}_{S,\Lambda}(\overline{\rho}) \;\middle|\; \rho \in D^{\square,\chi}_{S,\Lambda}(\overline{\rho})(A)\right\}.$$

- **Liftings constraint by local conditions**: Fix a family $\mathcal{D} = (\mathcal{D}_\nu)_{\nu\in\Sigma}$ of local deformation conditions for $\overline{\rho}_\nu$. Define the subfunctor $D^{\square_\Sigma,\mathcal{D}}_{S,\Lambda}(\overline{\rho})$ of $D^{\square_\Sigma}_{S,\Lambda}(\overline{\rho})$ by

  $$D^{\square_\Sigma,\mathcal{D}}_{S,\Lambda}(\overline{\rho})(A) = \left\{[\rho, (\rho_\nu, \beta_\nu)_{\nu\in\Sigma}] \in D^{\square_\Sigma}_{S,\Lambda}(\overline{\rho}) \;\middle|\; \rho_\nu \in D^{\square,\mathcal{D}_\nu}_{\Lambda}(\overline{\rho}_\nu)(A) \text{ for all } \nu \in \Sigma\right\}.$$

- **A combination of the two**: Let $\chi, \mathcal{D}$ be as above, then we define the subfunctor $D^{\square_\Sigma,\chi,\mathcal{D}}_{S,\Lambda}(\overline{\rho})$ of $D^{\square_\Sigma}_{S,\Lambda}(\overline{\rho})$ by

  $$D^{\square_\Sigma,\chi,\mathcal{D}}_{S,\Lambda}(\overline{\rho})(A) = \left\{[\rho, (\rho_\nu, \beta_\nu)_{\nu\in\Sigma}] \in D^{\square_\Sigma}_{S,\Lambda}(\overline{\rho}) \;\middle|\; \rho \in D^{\square_\Sigma,\mathcal{D}}_{S,\Lambda}(\overline{\rho})(A) \cap D^{\square_\Sigma,\chi}_{S,\Lambda}(\overline{\rho})(A)\right\}.$$

  Note that we have an equality

  $$D^{\square_\Sigma,\chi,\mathcal{D}}_{S,\Lambda}(\overline{\rho}) = D^{\square_\Sigma,\chi,\mathcal{D}^\chi}_{S,\Lambda}(\overline{\rho}),$$

  where $\mathcal{D}^\chi = (\mathcal{D}^\chi_\nu)_{\nu\in\Sigma}$ is the family where each $\mathcal{D}^\chi_\nu$ parametrizes those lifts of $\overline{\rho}_\nu$ which are of type $\mathcal{D}_\nu$ *and* of determinant χ_ν.

Proposition 2.61. *1. $D^{\square_\Sigma,(\chi),\mathcal{D}}_{S,\Lambda}(\overline{\rho})$ is representable;*

2. If $\#\Sigma = 1$, then the functors $D^{\square_\Sigma,(\chi),\mathcal{D}}_{S,\Lambda}(\overline{\rho})$ and $D^{\square,(\chi),\mathcal{D}}_{S,\Lambda}(\overline{\rho})$ are naturally isomorphic;

3. If $\Sigma \neq \emptyset$, then $D^{\square_\Sigma,(\chi),\mathcal{D}}_{S,\Lambda}(\overline{\rho})$ is formally smooth over $D^{\square,(\chi),\mathcal{D}}_{S,\Lambda}(\overline{\rho})$.

We denote the afforded deformation ring as $R^{\square_\Sigma,(\chi),\mathcal{D}}_{S,\Lambda}(\overline{\rho})$ and the universal deformation by $\rho^{\square_\Sigma,(\chi),\mathcal{D}}_{S,\Lambda}$. If $\mathcal{D}$ is the unconditioned deformation condition, we abbreviate this as $R^{\square_\Sigma,(\chi)}_{S,\Lambda}(\overline{\rho})$ and $\rho^{\square_\Sigma,(\chi)}_{S,\Lambda}$.

Proof. For part 1, let us first assume that $\Sigma = \emptyset$. Then $D^{\square_\Sigma,(\chi),\mathcal{D}}_{S,\Lambda}(\overline{\rho}) = D^{(\chi),\mathcal{D}}_{S,\Lambda}(\overline{\rho})$, and it follows from Proposition 2.40 and the material below Remark 2.60 that $\mathcal{D}$ (resp. $\chi \wedge \mathcal{D}$) defines a global deformation condition (i.e. a deformation condition in the sense of Definition 2.54 for the global residual representation $\overline{\rho}$). Thus the claim follows from Lemma 2.55.

For the remaining claims, fix a place $\nu_0 \in \Sigma$ and consider the natural transformation

$$\eta : D^{\square_\Sigma,(\chi),\mathcal{D}}_{S,\Lambda}(\overline{\rho}) \longrightarrow D^{\square,(\chi),\mathcal{D}}_{S,\Lambda}(\overline{\rho}) \times \prod_{\nu\in\Sigma,\nu\neq\nu_0} \hat{G}\ ,$$

where the components η_A are given by sending $(\rho,(\rho_\nu,\beta_\nu)_{\nu\in\Sigma})_{/\sim}$ to $(\beta_{\nu_0}\rho\beta_{\nu_0}^{-1},(\beta_{\nu_0}^{-1}\beta_\nu)_{\nu\in\Sigma,\nu\neq\nu_0})$. Remark that the target of η is representable (by Proposition 2.31 and Corollary 2.5). We readily check that η_A is a bijection, with inverse given by sending $(\rho,(\beta_\nu)_{\nu\in\Sigma,\nu\neq\nu_0})$ to the equivalence class of $(\rho,(\beta_\nu^{-1}\rho\beta_\nu|\operatorname{Gal}_{F_\nu},\beta_\nu)_{\nu\in\Sigma})$ with $\beta_{\nu_0} := 1$. So η provides a natural isomorphism and the claims follow. □

We remark that condition 1. of the proposition is not true if $\Sigma = \emptyset$ and $\overline{\rho}$ is not Schur.

Proposition 2.62. *Assume $\Sigma \neq \emptyset$. Then*

$$R^{\square}_{S,\Lambda}(\overline{\rho}) \cong R_{S,\Lambda}(\overline{\rho})[[x_1,\ldots,x_u]] \text{ and } R^{\square_\Sigma}_{S,\Lambda}(\overline{\rho}) \cong R^{\square}_{S,\Lambda}(\overline{\rho})[[x_1,\ldots,x_t]]$$

and

$$R^{\square,(\chi),\mathcal{D}}_{S,\Lambda}(\overline{\rho}) \cong R^{(\chi),\mathcal{D}}_{S,\Lambda}(\overline{\rho})[[x_1,\ldots,x_u]] \text{ and } R^{\square_\Sigma,(\chi),\mathcal{D}}_{S,\Lambda}(\overline{\rho}) = R^{\square,(\chi),\mathcal{D}}_{S,\Lambda}(\overline{\rho})[[x_1,\ldots,x_t]]$$

with

- $t = \dim(\mathfrak{g}).(\#\Sigma - 1)$,
- $u = \dim(\mathfrak{g}) - \dim(\mathfrak{z}) = \dim(\mathfrak{g}^{\mathtt{der}})$.

(Note that the first set of formulae follows from the second one if we leave χ out and take for $\mathcal{D}$ the unconditioned deformation condition.)

(We remark that the two isomorphisms on the right hold even if the (unframed) deformation functors are not representable. We will not use this in the sequel, however.)

Proof. The first (upper left) isomorphism is Proposition 2.52.3. Its proof can be easily generalized to a non-trivial deformation condition $\mathcal{D}$: Recall that deformation conditions are in correspondence with (certain) ideals of $R_{S,\Lambda}(\overline{\rho})$ and denote by $I^{(\chi)}$ the ideal corresponding to $\mathcal{D},(\chi)$. Then

$$R^{(\chi),\mathcal{D}}_{S,\Lambda}(\overline{\rho}) \cong R_{S,\Lambda}(\overline{\rho})/I^{(\chi)}$$

and

$$\begin{aligned} R^{\square,(\chi),\mathcal{D}}_{S,\Lambda}(\overline{\rho}) &\cong R^{\square}_{S,\Lambda}(\overline{\rho})/R^{\square}_{S,\Lambda}(\overline{\rho}).I^{(\chi)} = R_{S,\Lambda}(\overline{\rho})[[x_1,\ldots,x_u]]/I^{(\chi)}[[x_1,\ldots,x_u]] \\ &\cong \big(R_{S,\Lambda}(\overline{\rho})/I^{(\chi)}\big)[[x_1,\ldots,x_u]] \cong R^{(\chi),\mathcal{D}}_{S,\Lambda}(\overline{\rho})[[x_1,\ldots,x_u]]. \end{aligned}$$

This gives the lower left isomorphism. Now we can finish the proof if we can provide an isomorphism

$$R^{\square_\Sigma,(\chi),\mathcal{D}}_{S,\Lambda}(\overline{\rho}) \cong R^{\square,(\chi),\mathcal{D}}_{S,\Lambda}(\overline{\rho})[[x_1,\ldots,x_t]].$$

This can be done in an entirely analogous way, citing from Proposition 2.61 that the natural transformation

$$D^{\square_\Sigma}_{S,\Lambda}(\overline{\rho}) \to D^{\square}_{S,\Lambda}(\overline{\rho})$$

is formally smooth and observing that consequently the fibers of $D^{\square_\Sigma}_{S,\Lambda}(\overline{\rho})(k[\epsilon]) \to D^{\square}_{S,\Lambda}(\overline{\rho})(k[\epsilon])$ are $\prod_{i=1}^{\#\Sigma-1} \mathfrak{g}$ -torsors (cf. also [KW09, Proposition 4.1], where this is carried out for $G = \mathrm{GL}_2$). □

Presentations over local deformation rings For this section, we will suppose

Assumption 2.63.

$$H^0(\mathrm{Gal}_{F,S}, \mathfrak{g}^{(\mathtt{der}),\vee}) = 0.$$

Definition 2.64. Let $\Sigma \subset S$ be finite sets of places of F containing $\Omega_\infty \sqcup \Omega_\ell$, then we define

$$R^{\mathtt{loc}_\Sigma,(\chi)}_{\Lambda}(\overline{\rho}) = \widehat{\bigotimes}_{\nu\in\Sigma} R^{\square,(\chi_\nu)}_{\Lambda}(\overline{\rho}_\nu).$$

There is an obvious map $R^{\mathtt{loc}_\Sigma,(\chi)}_{S,\Lambda}(\overline{\rho}) \to R^{\square,(\chi)}_{\Lambda}(\overline{\rho})$ induced from f in (2.10) and we have

Theorem 2.65.

$$R^{\square,(\chi)}_{S,\Lambda}(\overline{\rho}) \cong R^{\mathtt{loc}_\Sigma,(\chi)}_{\Lambda}(\overline{\rho})[[x_1,\dots,x_a]]_{/(f_1,\dots,f_{a+b})}$$

for a suitable $a \in \mathbb{N}_0$ and

$$b = \begin{cases} (\#\Sigma-1).\dim \mathfrak{g}^{\mathtt{der}} & \textit{(determinant fixed);} \\ (\#\Sigma-1).\dim \mathfrak{g} & \textit{(determinant not fixed).} \end{cases}$$

(The set S does not show up in the definition of the object on the right side of the isomorphism. However, we remark that this does *not* imply that $R^{\square,(\chi)}_{S,\Lambda}(\overline{\rho}) \cong R^{\square,(\chi)}_{S',\Lambda}(\overline{\rho})$ for $S \neq S'$, as the number a of variables and the elements f_i can differ in either case.)

Proof. For the fixed determinant, this is (a special case of) [Bal12, Proposition 4.2.5]. The case where the determinant is not fixed can be proved analogously. □

Corollary 2.66. *Assume that the unframed deformation functor $D^{\chi}_{\Lambda}(\overline{\rho})$ is representable. Then*

$$R^{\square_\Sigma,(\chi)}_{S,\Lambda}(\overline{\rho}) \cong R^{\mathtt{loc}_\Sigma,(\chi)}_{\Lambda}(\overline{\rho})[[x_1,\dots,x_{a+b}]]_{/(f_1,\dots,f_a)}$$

for a suitable $a \in \mathbb{N}_0$ and

$$b = \begin{cases} (\#\Sigma-1).\dim \mathfrak{g}^{\mathtt{ab}} & \textit{(determinant fixed);} \\ 0 & \textit{(determinant not fixed).} \end{cases}$$

Proof. By Proposition 2.62, we have

$$R^{\square_\Sigma,(\chi)}_{S,\Lambda}(\overline{r}) \cong R^{\square,(\chi)}_{S,\Lambda}(\overline{\rho})[[x_1,\dots,x_c]] \tag{2.13}$$

with $c = (\#\Sigma-1).\dim \mathfrak{g}$. The claim follows immediately from the identity $\dim \mathfrak{g} = \dim \mathfrak{g}^{\mathtt{der}} + \dim \mathfrak{g}^{\mathtt{ab}}$. □

Now, for each $\nu \in \Sigma$ fix a local deformation condition $\mathcal{D}_\nu$. This gives rise to a global condition $\mathcal{D}$ in the sense that a global lift ρ of $\overline{\rho}$ is of type $\mathcal{D}$ if and only if each local component ρ_ν is of type $\mathcal{D}_\nu$ for $\nu \in \Sigma$. This gives rise to a global functor $D^{\square_\Sigma,(\chi),\mathcal{D}}_{S,\Lambda}(\overline{\rho})$ with representing object $R^{\square_\Sigma,(\chi),\mathcal{D}}_{S,\Lambda}(\overline{\rho})$ and we define

$$R^{\mathtt{loc}_\Sigma,(\chi),\mathcal{D}}_{\Lambda}(\overline{\rho}) = \widehat{\bigotimes}_{\nu\in\Sigma} R^{\square,(\chi_\nu),\mathcal{D}_\nu}_{\Lambda}(\overline{\rho}_\nu).$$

Corollary 2.67.

$$R^{\square_\Sigma,(\chi),\mathcal{D}}_{S,\Lambda}(\overline{\rho}) \cong R^{\mathtt{loc}_\Sigma,(\chi),\mathcal{D}}_{\Lambda}(\overline{\rho}) \otimes_{R^{\mathtt{loc}_\Sigma,(\chi)}_{\Lambda}(\overline{\rho})} R^{\square_\Sigma,(\chi)}_{S,\Lambda}(\overline{\rho}).$$

Proof. We only give a proof for the fixed determinant case (the other case being analogous): Write $D^{\mathtt{loc}_\Sigma,\chi,(\mathcal{D})}_{\Lambda}(\overline{\rho})$ for the functor

$$\prod_{\nu\in\Sigma} D^{\square,\chi_\nu,(\mathcal{D}_\nu)}_{\Lambda}(\overline{\rho}_\nu)$$

with representing object $R^{\mathtt{loc}_\Sigma,\chi,(\mathcal{D})}_{\Lambda}(\overline{\rho})$. The claim then follows from Proposition 2.4 as

$$\begin{array}{ccc} D^{\square_\Sigma,\chi,\mathcal{D}}_{S,\Lambda}(\overline{\rho}) & \dashrightarrow & D^{\square_\Sigma,\chi}_{S,\Lambda}(\overline{\rho}) \\ \vdots & & \downarrow \\ D^{\mathtt{loc}_\Sigma,\chi,\mathcal{D}}_{\Lambda}(\overline{\rho}) & \longrightarrow & D^{\mathtt{loc}_\Sigma,\chi}_{\Lambda}(\overline{\rho}) \end{array}$$

is a pull-back diagram of functors. □

Thus by tensoring the claim of Corollary 2.66 with $R^{\mathtt{loc}_\Sigma,\chi,\mathcal{D}}_{\Lambda}(\overline{\rho})$ we get

Corollary 2.68.

$$R^{\square_\Sigma,(\chi),\mathcal{D}}_{S,\Lambda}(\overline{\rho}) \cong R^{\mathtt{loc}_\Sigma,(\chi),\mathcal{D}}_{\Lambda}(\overline{\rho})[[x_1,\dots,x_{a+b}]]_{/(f_1,\dots,f_a)}$$

for a suitable $a \in \mathbb{N}_0$ and

$$b = \begin{cases} (\#\Sigma - 1).\dim \mathfrak{g}^{\mathtt{ab}} & \textit{(determinant fixed);} \\ 0 & \textit{(determinant not fixed).} \end{cases}$$

We will conclude this subsection with another characterization of a composed global deformation condition:

Definition 2.69 (System of local conditions)**.** Let $\overline{\rho}, \chi$ be as above, then a system of local conditions $\mathcal{L}^{(\chi)} = (L^{(\chi)}_\nu)_{\nu\in\mathrm{Pl}_F}$ consists of a choice of subspaces $L_\nu \subset H^1(F_\nu, \mathfrak{g})$ (resp. $L^\chi_\nu \subset H^1(F_\nu, \mathfrak{g}^{\mathtt{der}})'$) such that

$$L_\nu = H^1(\mathrm{Gal}(F_\nu)/I_{F_\nu}, \mathfrak{g}) \qquad (\text{resp. } L^\chi_\nu = H^1(\mathrm{Gal}(F_\nu)/I_{F_\nu}, \mathfrak{g}^{\mathtt{der}})'\ ) \tag{2.14}$$

holds for almost all ν.

Now let T be a finite set of finite places of F containing the ramification set of $\overline{\rho}$. Let $\mathcal{D}^{(\chi)}$ be a global deformation condition composed of local conditions $\mathcal{D}^{(\chi)}_\nu$ for $\nu \in T$. Then, for each ν, we have an inclusion of tangent spaces

$$t_{D_\nu^{\mathcal{D}_\nu^{(\chi)}}(\overline{\rho}_\nu)} \hookrightarrow t_{D_\nu^{(\chi)_\nu}(\overline{\rho}_\nu)} \cong \begin{cases} H^1(F_\nu, \mathfrak{g}) & \text{determinant not fixed;} \\ H^1(F_\nu, \mathfrak{g}^{\mathtt{der}})' & \text{determinant fixed.} \end{cases}$$

Thus we can define a system $\mathcal{L}^{(\chi)}$ attached to $\mathcal{D}$ and T by decreeing $L_\nu^{(\chi)} = t_{D_\nu^{\mathcal{D}_\nu^{(\chi)}}(\overline{\rho}_\nu)}$ for $\nu \in T$ and as in (2.14) for $\nu \notin T$. This gives rise to a map from the set of composed global deformation conditions to the set of systems of local conditions.

2.5 Liftings at infinity

Proposition 2.70. *Consider a representation*

$$\overline{\rho} : \mathbb{Z}/2\mathbb{Z} = \{1, c\} \to G(\mathbb{F})$$

and assume that $\ell = \mathrm{char}(\mathbb{F}) \neq 2$. *Then*

$$R_\Lambda^\square(\overline{\rho}) \cong \Lambda[[x_1, \dots, x_m]] \text{ with } m = \dim(\mathfrak{g}^{c=-1}).$$

If χ *denotes a lift of the determinant, then the same result holds for* $R_\Lambda^{\square,\chi}(\overline{\rho})$ *after replacing* $\mathfrak{g}$ *by* $\mathfrak{g}^{\mathtt{der}}$.

Proof. Let $k \in \mathbb{N}$. We use the general formula $H^2(\mathbb{Z}/k\mathbb{Z}, M) \cong M^{\mathbb{Z}/k\mathbb{Z}}/\operatorname{im}(\varphi)$ for a $\mathbb{Z}/k\mathbb{Z}$-module M and with

$$\varphi : M \to M, \qquad m \mapsto \sum_{i=0}^{k-1} i.m.$$

Now, if $x \in \mathfrak{g}^{\{1,c\}}$, we see that $(c+1)(\frac{1}{2}x) = x \in \operatorname{im}(c+1)$, hence $H^2(\{1,c\}, \mathfrak{g}) = 0$ and the lifting ring is unobstructed. To get the number of variables we have to evaluate

$$Z^1(\{1,c\}, \mathfrak{g}) = \{f : \{1,c\} \to \mathfrak{g} \mid f(xy) = f(x) + {}^x f(y)\}.$$

Looking at $x = y = c$, we see that f is uniquely determined by the vector $v = f(c)$. Looking at $x = 1, y = c$, we see that $f(1) = v + {}^c v = 0$, i.e. that $v \in \mathfrak{g}^{c=-1}$. On the other hand, any such v defines an $f \in Z^1$ via $1 \mapsto 0, c \mapsto v$.
The modifications of this argument for the fixed-determinant case are straight-forward. □

2.6 A criterion for vanishing of cohomology groups

Recall that if Γ is the absolute Galois group of a non-archimedean local field, then by Tate local duality [NSW08, Theorem 7.2.6] we have

$$H^2(\Gamma, \mathfrak{g})^* \cong H^0(\Gamma, \mathfrak{g}^\vee) = (\mathfrak{g}^\vee)^\Gamma \tag{2.15}$$

where $*$ denotes the Pontryagin dual and $\vee$ denotes the Tate dual. We now give a simple (and presumably well-known) criterion to determine if $H^2(\Gamma, \mathfrak{g}^{(\mathtt{der})})$ vanishes in the case $G = \mathrm{GL}_n$.

Lemma 2.71 (Local case)**.** *Let* Γ *be the absolute Galois group of a non-archimedean local field,* k *be a finite field of characteristic* ℓ *and*

$$\overline{\rho} : \Gamma \to \mathrm{GL}_n(k)$$

a representation. Then

1. $\mathrm{Hom}_\Gamma(\overline{\rho}, \overline{\rho}(1))$ *vanishes if and only if* $H^2(\Gamma, \mathrm{ad}\,\overline{\rho})$ *vanishes;*
2. *Assume that* $\ell \nmid n$. *Then, if* $\mathrm{Hom}_\Gamma(\overline{\rho}, \overline{\rho}(1))$ *vanishes, also* $H^2(\Gamma, \mathrm{ad}\,\overline{\rho}^0)$ *vanishes.*

Proof. By (2.15), we need to show that $H^0(\Gamma, (\mathrm{ad}\,\overline{\rho}^{(0)})^\vee)$ vanishes. As explained[7] in [Böc07, Example

[7] Remark that there is a mistake in [Böc07]: In Example 4.1 is should say $\mathrm{ad}_{\overline{\rho}}^{(0)} \cong (\mathrm{ad}_{\overline{\rho}}^{(0)})^*$ instead of $\mathrm{ad}_{\overline{\rho}}^{(0)} \cong (\mathrm{ad}_{\overline{\rho}}^{(0)})^\vee$.

4.1], the trace pairing allows us to identify $(\operatorname{ad}\overline{\rho}^{(0)})^\vee$ and $(\operatorname{ad}\overline{\rho}^{(0)})(1)$, where we have to assume $\ell \not| n$ for the $\operatorname{ad}\overline{\rho}^0$-case so that we have $\operatorname{ad}\overline{\rho} \cong \operatorname{ad}\overline{\rho}^0 \oplus k$. We thus see that

$$H^0(\Gamma, (\operatorname{ad}\overline{\rho}^0)(1)) \subset H^0(\Gamma, (\operatorname{ad}\overline{\rho})(1)) \cong \operatorname{Hom}_\Gamma(\overline{\rho}, \overline{\rho}(1))$$

and the claim follows. □

In the global case, there is no duality theorem and we record the following:

Lemma 2.72 (Global case)**.** *Let $\Gamma = \operatorname{Gal}_{F,S}$ for a number field F and a finite set S of places of F. Let $k, \overline{\rho}$ be as in Lemma 2.71 above.*

1. $\operatorname{Hom}_\Gamma(\overline{\rho}, \overline{\rho}(1)) \cong H^0(\Gamma, (\operatorname{ad}\overline{\rho})^\vee)$;
2. *Assume that $\ell \not| n$. Then $H^0(\Gamma, (\operatorname{ad}\overline{\rho}^0)^\vee)$ is a direct summand of* $\operatorname{Hom}_\Gamma(\overline{\rho}, \overline{\rho}(1))$.

Proof. The proof is identical to the proof of Lemma 2.71. □

We deduce the following result, which also implies the vanishing of the error term δ in [Böc13a, Remark 5.2.3.(d)] for large ℓ:

Corollary 2.73. *There exists a constant C, depending only on n and F, such that Assumption 2.63 holds if* $\operatorname{char}(k) > C$, $G = \operatorname{GL}_n$ *and $\overline{\rho}$ is irreducible.*

In preparation for a proof, let us first consider the following:

Lemma 2.74. *Let F be a number field and denote by $\zeta \in \overline{\mathbb{Q}}$ a primitive ℓ-th root of unity. Assume that ℓ does not ramify in F. Then $[F(\zeta) : F] = \ell - 1$.*

Proof. Write $F(\zeta)$ for the composite field of F and $\mathbb{Q}(\zeta)$. If we can show that F and $\mathbb{Q}(\zeta)$ are linearly disjoint, then

$$[F(\zeta) : F] = [\mathbb{Q}(\zeta) : \mathbb{Q}] = \ell - 1$$

by [Bou89, A.V.14, §2, Prop. 5.a]. For this, by [Coh91, Proposition 5.4 (on p. 188)], it suffices to check that $F \cap \mathbb{Q}(\zeta) = \mathbb{Q}$. Assume that this does not hold. Then ℓ ramifies in $F \cap \mathbb{Q}(\zeta)$, hence in F. This was excluded in the claim of the lemma. □

Corollary 2.75. *Let F be a number field and consider the mod-ℓ cyclotomic character $\overline{\epsilon}_\ell : \operatorname{Gal}_{F,S} \to \mathbb{F}_\ell^\times$. Then, for $\ell \gg 0$, $\overline{\epsilon}_\ell$ is surjective.*

Proof. By definition, $\operatorname{im}(\overline{\epsilon}_\ell) \cong \operatorname{Gal}(F(\zeta)|F)$. Thus, if we take ℓ large enough (so that ℓ does not ramify in F), the result follows from Lemma 2.74. □

Proof of Corollary 2.73. Let us first exclude all ℓ which divide n. Because $\overline{\rho}$ is irreducible, by Lemma 2.72 we have to check in what situations we can have an isomorphism $\overline{\rho} \cong \overline{\rho}(1)$. Assume that ℓ is big enough such that the cyclotomic character $\overline{\epsilon}_\ell$ is surjective, according to Corollary 2.75. Let $\alpha \in \overline{\mathbb{F}}$ be a non-zero eigenvalue of $\overline{\rho}(x)$, where $x \in \Gamma$ is some element which maps to a generator β of $\mathbb{F}_\ell^\times$ under $\overline{\epsilon}_\ell$. Then, $\overline{\rho} \cong \overline{\rho}(1) \cong \overline{\rho}(2) \cong \ldots$ implies that there are $\ell - 1$ distinct eigenvalues $\alpha, \beta\alpha, \beta^2\alpha, \ldots$ of $\overline{\rho}(x)$. This can only happen if $n \geq \ell - 1$. Thus, C can be taken to be the maximum of $n + 2$ and the bound from Corollary 2.75. □

Remark 2.76. We expect Corollary 2.73 to hold more generally for any linear group scheme G over $\mathbb{Z}$ and any absolutely irreducible representation $\overline{\rho}$ (in the sense of Definition 2.49) by embedding $G \hookrightarrow \mathrm{GL}_N$ over $\mathbb{Z}[\frac{1}{N}]$ for a suitable $N \in \mathbb{N}$, but we did not check the details.

3 Unobstructedness of universal deformation rings

As in Section 2.1, let Λ be the valuation ring of a finite extension of $\mathbb{Q}_\ell$ with residue field k. We consider a residual representation $\overline{\rho} : \Gamma \to G(k)$ together with a fixed lift χ of the determinant.

Definition 3.1. The functors $D_\Lambda^{\square,(\chi)}(\overline{\rho})$ and $D_\Lambda^{(\chi)}(\overline{\rho})$ are called *unobstructed* if $h^2(\Gamma, \mathfrak{g}^{(\mathtt{der})}) = 0$.

We will apply this mainly for $\Gamma = \mathrm{Gal}_{F,S}$ and $\Gamma = \mathrm{Gal}_{F_\nu}$, where F is a number field and $S, \{\nu\}$ are sets of places of F.

Proposition 3.2. *Assume that $D_\Lambda^{\square,(\chi)}(\overline{\rho})$ is unobstructed and, in the fixed-determinant case, assume that $\ell \gg 0$. Let $b = h^1(\Gamma, \mathfrak{g})$ (resp. $h^1(\Gamma, \mathfrak{g}^{\mathrm{der}})'$ if the determinant is fixed) and $a = b + \dim(\mathfrak{g}^{(\mathtt{der})}) - h^0(\Gamma, \mathfrak{g}^{(\mathtt{der})})$.*

Then

$$R_\Lambda^{\square,(\chi)}(\overline{\rho}) \cong \Lambda[[x_1, \ldots, x_a]].$$

If in addition the conditions **(SmCtr)** *and* **(Centr)** *are fulfilled, then*

$$R_\Lambda^{(\chi)}(\overline{\rho}) \cong \Lambda[[x_1, \ldots, x_b]].$$

Proof. Assume first the conditions **(SmCtr)** and **(Centr)** (so that $D_\Lambda^{(\chi)}(\overline{\rho})$ is representable), then the isomorphism $R_\Lambda^{(\chi)}(\overline{\rho}) \cong \Lambda[[x_1, \ldots, x_b]]$ follows from Theorem 2.2 (resp. Theorem 2.4 in the fixed-determinant case) of [Böc07]. (Observe that we already cited Theorem 2.2 of ibid. as Theorem 2.53). Using the decomposition following Proposition 2.58, we see that

$$a - b = \begin{cases} \dim(\mathfrak{g}) - \dim(\mathfrak{z}) = \dim(\mathfrak{g}^{\mathrm{der}}) & \text{determinant not fixed,} \\ \dim(\mathfrak{g}^{\mathrm{der}}) + \dim(\mathfrak{g}^{\mathrm{ab}}) - h^0(\Gamma, \mathfrak{g}^{\mathrm{der}}) - \dim(\mathfrak{g}^{\mathrm{ab}}) & \\ \qquad = \dim(\mathfrak{g}) - \dim(\mathfrak{z}) = \dim(\mathfrak{g}^{\mathrm{der}}) & \text{determinant fixed.} \end{cases}$$

Thus, the isomorphism $R_\Lambda^{\square,(\chi)}(\overline{\rho}) \cong \Lambda[[x_1, \ldots, x_a]]$ follows from Proposition 2.62. This isomorphism can be proved without representability by an analogous argument as in [Böc07]: We get a presentation

$$f : \Lambda[[x_1, \ldots, x_c]] \to R_\Lambda^{\square,(\chi)}(\overline{\rho})$$

with $c = \dim D_\Lambda^{\square,(\chi)}(\overline{\rho})(k[\epsilon])$ and such that $\mathrm{Hom}(\ker(f)/\mathfrak{m}_{\Lambda[[x_1,\ldots,x_c]]} \ker(f), k) \hookrightarrow H^2(\Gamma, \mathfrak{g}^{(\mathtt{der})})$. Thus, the claim follows from Proposition 2.57 and Proposition 2.58. $\square$

Remark 3.3. If, for example, $R_\Lambda(\overline{\rho})/(\ell)$ is known to have Krull dimension $h^1(\Gamma, \mathfrak{g}) - h^2(\Gamma, \mathfrak{g})$, then it follows that $R_\Lambda(\overline{\rho})$ is of relative dimension $h^1(\Gamma, \mathfrak{g}) - h^2(\Gamma, \mathfrak{g})$ over Λ. Thus, in this situation, a converse to the above proposition holds: An isomorphism

$$R_\Lambda(\overline{\rho}) \cong \Lambda[[x_1, \ldots, x_{h^1(\Gamma,\mathfrak{g})}]]$$

implies the vanishing of $h^2(\Gamma, \mathfrak{g})$. However, the assumption on $R_\Lambda(\overline{\rho})/(\ell)$ holds (conjecturally) only in certain circumstances, cf. [Böc07, Remark 2.3]. Positive results exist in the local Galois case for $G = \mathrm{GL}_n$ [Sho15, Theorem 2.4] (cf. also [Böc07, Remark 6.2] for $n = 2$). (An analogous remark holds for $R_\Lambda^\chi(\overline{\rho})$.)

Let us resume the assumptions and notations from the beginning of Section 2.4, i.e. that $\Gamma = \mathrm{Gal}_{F,S}$ and that we have fixed embeddings $\iota_\nu : \mathrm{Gal}_{F_\nu} \hookrightarrow \mathrm{Gal}_{F,S}$ (now, for all places $\nu \in \mathrm{Pl}_F$). If $\overline{\rho}$ denotes our fixed global residual representation, we denote the afforded restriction to Gal_{F_ν} by $\overline{\rho}_\nu$.

Definition 3.4. A relatively representable subfunctor of $D_\Lambda^{\square,(\chi)}(\overline{\rho}_\nu)$ or $D_\Lambda^{(\chi)}(\overline{\rho}_\nu)$ is called *smooth* (of dimension m) if its representing object is isomorphic to $\Lambda[[x_1, \dots, x_m]]$.

Now, let $\mathcal{L}^{(\chi)} = (L_\nu^{(\chi)})_{\nu \in \mathrm{Pl}_F}$ be a system of local conditions in the sense of Definition 2.69 and denote the corresponding global deformation condition by $\mathcal{D}^{(\chi)} = (\mathcal{D}_\nu^{(\chi)})_{\nu \in \mathrm{Pl}_F}$.

Definition 3.5 (Dual Selmer group)**.** Denote by $\mathfrak{g}^{(\mathtt{der}),\vee}$ the Tate dual of $\mathfrak{g}^{(\mathtt{der})}$ and by $L_\nu^{(\chi),\perp}$ the annihilator of $L_\nu^{(\chi)}$ under the Tate pairing

$$H^i(F_\nu, \mathfrak{g}^{(\mathtt{der}),\vee}) \times H^{2-i}(F_\nu, \mathfrak{g}^{(\mathtt{der})}) \longrightarrow H^2(F_\nu, k(1)) \cong \mathbb{Q}/\mathbb{Z}$$

for $i = 1$, cf. [NSW08, (7.2.6) Theorem]. Then we denote by $H^1_{\mathcal{L}^{(\chi)},\perp}(F, \mathfrak{g}^{(\mathtt{der}),\vee})$ the kernel of the map

$$\bigoplus_{\nu \in \mathrm{Pl}_F} \mathrm{res}_\nu : H^1(F, \mathfrak{g}^{(\mathtt{der}),\vee}) \longrightarrow \bigoplus_{\nu \in \mathrm{Pl}_F} H^1(F_\nu, \mathfrak{g}^{(\mathtt{der}),\vee})/L_\nu^{(\chi),\perp}.$$

For the next definitions we assume that S contains all places at which $\overline{\rho}$ ramifies and that $\mathcal{D}_\nu^{(\chi)}$ parametrizes unramified deformations for $\nu \notin S$.

Definition 3.6. We say that $D_{S,\Lambda}^{\mathcal{D}^{(\chi)}}(\overline{\rho})$ (or $D_{S,\Lambda}^{\square,\mathcal{D}^{(\chi)}}(\overline{\rho})$, or $D_{S,\Lambda}^{\square_S,\mathcal{D}^{(\chi)}}(\overline{\rho})$) *has vanishing dual Selmer group* if $H^1_{\mathcal{L}^{(\chi)},\perp}(F, \mathfrak{g}^{(\mathtt{der}),\vee}) = 0$.

Definition 3.7. Let $\mathbf{m} = (m_\nu)_{\nu \in S} \in \mathbb{N}_0^S$. We say that $D_{S,\Lambda}^{\mathcal{D}^{(\chi)}}(\overline{\rho})$ (resp. $D_{S,\Lambda}^{\square,\mathcal{D}^{(\chi)}}(\overline{\rho})$, resp. $D_{S,\Lambda}^{\square_S,\mathcal{D}^{(\chi)}}(\overline{\rho})$) is *globally unobstructed* (of local dimension $\mathbf{m}$) if its dual Selmer group vanishes and if each $D_\Lambda^{\square,\mathcal{D}_\nu^{(\chi)}}(\overline{\rho}_\nu)$, for $\nu \in S$, is smooth (of dimension m_ν).

To simplify the exposition, the following remark is stated in the unframed setting. Analogous statements hold in the framed setting as well.

Remark 3.8. If $\widetilde{D}_\Lambda^{(\psi)}(\overline{\rho}) := D_{S,\Lambda}^{\mathcal{D}^{(\chi)}}(\overline{\rho})$ is globally unobstructed, it follows that the ideals $\widetilde{J}_\nu^{(\eta)}$ from [Böc07], equation (6), vanish. Hence [Böc07, Theorem 5.2] implies that $R_{S,\Lambda}^{\mathcal{D}^{(\chi)}}(\overline{\rho})$ is isomorphic to a power series ring in $h^1_{\mathcal{L}}(F, \mathfrak{g}^{(\mathtt{der})})^{(\,\prime\,)}$ variables. For general profinite groups Γ, the converse direction is known not to hold, i.e. the formal smoothness of $\widetilde{R}^{(\psi)}(\overline{\rho})$ does not imply that the deformation problem is globally unobstructed, see [Spr13].

For the next proposition, we consider the system of local conditions $\mathcal{L}^{(\chi)}$ parametrizing all deformations which are unramified outside S with corresponding deformation condition $(\mathcal{D}^{(\chi)} \wedge) \mathcal{D}^{S-\mathrm{nr}}$.

Proposition 3.9. *Assume that*

1. $D_\Lambda^{(\chi)}(\overline{\rho}_\nu)$ *is unobstructed (in the sense of Definition 3.1) for all* $\nu \in S$*;*
2. $D_{S,\Lambda}^{(\chi)}(\overline{\rho})$ *is globally unobstructed (and we don't make an assumption on the dimension).*

Then $D_{S,\Lambda}^{(\chi)}(\overline{\rho})$ is unobstructed (in the sense of Definition 3.1).

Proof. As $Ш_S^2(\mathfrak{g}^{(\mathtt{der})}) := H^1_{\mathcal{L}^\perp}(F, \mathfrak{g}^{(\mathtt{der}),\vee})^*$ vanishes[8] by Assumption 2., we can deduce this directly from the following exact sequence (see p. 7 of [Böc07]):

$$0 \to Ш_S^2(\mathfrak{g}^{(\mathtt{der})}) \to H^2(F, \mathfrak{g}^{(\mathtt{der})}) \to \bigoplus_{\nu \in S} H^2(F_\nu, \mathfrak{g}^{(\mathtt{der})}) \to H^0(F, \mathfrak{g}^{(\mathtt{der}),\vee})^* \to 0. \qquad \square \quad (3.1)$$

If we are in a situation where Remark 3.3 applies locally (so e.g. when $G = \mathrm{GL}_n$), we can replace Assumptions 1. and 2. by

1'. $D_{S,\Lambda}^{(\chi)}(\overline{\rho})$ is globally unobstructed of local dimension $\mathbf{m}_0 = (h^1(F_\nu, \mathfrak{g}^{(\mathtt{der})})^{(\,\prime\,)})_{\nu \in S}$.

On the other hand, assume that $D_{\Lambda,S}^{(\chi)}(\overline{\rho})$ is unobstructed. Then the sequence (3.1) implies that $Ш_S^2(\mathfrak{g}^{(\mathtt{der})})$ vanishes. We see that the vanishing of the local dimensions $h^2(F_\nu, \mathfrak{g}^{(\mathtt{der})})$ is equivalent to the vanishing of $H^0(F, \mathfrak{g}^{(\mathtt{der}),\vee})^*$, or alternatively, to the vanishing of $H^0(F, \mathfrak{g}^{(\mathtt{der})}(1))$. If we suppose this vanishing (which was proved for almost all ℓ given that $G = \mathrm{GL}_n$, see Section 2.6), the condition that $D_{\Lambda,S}^{(\chi)}(\overline{\rho})$ is unobstructed therefore implies that $D_{\Lambda,S}^{(\chi)}(\overline{\rho})$ is globally unobstructed of dimension $\mathbf{m}_0$ and that each $D_\Lambda^{(\chi)_\nu}(\overline{\rho}_\nu)$ (for $\nu \in S$) is unobstructed (and, hence, locally smooth of dimension $h^1(F_\nu, \mathfrak{g}^{(\mathtt{der})})^{(\,\prime\,)}$).

3.1 A general framework for unobstructedness

We will retain the notations and conventions from the previous sections. In particular, we fix a representation

$$\overline{\rho} : \mathrm{Gal}_{F,S} \to G(k),$$

together with a lift $\chi : \mathrm{Gal}_{F,S} \to G^{\mathtt{ab}}(\Lambda)$ of the determinant, where F is a totally real number field, $S \subset \mathrm{Pl}_F^\infty$ is a finite set of finite places and k is a finite field of characteristic $\ell := \mathrm{char}(k)$. We suppose that $\ell \notin S \cup \{2\}$. At this point, we will consider only liftings and deformations with values in $\mathcal{C}_\Lambda$ for $\Lambda = W := W(k)$, so we will suppress the specification of Λ in the index of the deformation functors and rings. As we will not vary the residual representation $\overline{\rho}$, we will also suppress "$(\overline{\rho})$" in the notion of the deformation rings.

Let us fix a Borel subgroup $B \subset G$ and denote by $\mathfrak{g}^{\mathtt{der}}$ (resp. $\mathfrak{b}^{\mathtt{der}}$) the Lie algebra of the derived subgroup $G^{\mathtt{der}}$ of G (resp. the Lie algebra of $B \cap G^{\mathtt{der}}$). Consider the following assumptions:

1. **(Representability)**: The S_ℓ-framed deformation functor

$$D_{S_\ell}^{\square_{S_\ell},\chi}(\overline{\rho})$$

is representable (by an object $R_{S_\ell}^{\square_{S_\ell},\chi}$).

[8] We remark that the vanishing of the "Tate-Shafarevich group" $Ш_S^2(\mathfrak{g}^{(\mathtt{der})})$ implies that all obstructions for $D_\Lambda^{(\chi)}(\overline{\rho}_\nu)$ come from local obstructions, see [Böc07, Theorem 3.1].

2. **(sm/k)**: For each $\nu \in \Omega_\ell$, there exists a relatively representable subfunctor

$$D^{\square,\chi,\mathtt{sm}}(\overline{\rho}_\nu) \hookrightarrow D^{\square,\chi}(\overline{\rho}_\nu)$$

   such that the representing object $R_\nu^{\square,\chi,\mathtt{sm}}$ is formally smooth (and we denote the relative dimension by $d_\nu^{\square,\mathtt{sm}}$).

3. **(crys)**: For each $\nu \in \Omega_\ell$, there exists a subfunctor

$$D^{\square,\chi,\mathtt{crys}}(\overline{\rho}_\nu) \hookrightarrow D^{\square,\chi,\mathtt{sm}}(\overline{\rho}_\nu)$$

   which is relatively representable over $D^{\square,\chi}(\overline{\rho}_\nu)$ and such that the representing object $R_\nu^{\square,\chi,\mathtt{crys}}$ is formally smooth over W of relative dimension

$$d_\nu^{\square,\mathtt{crys}} = \dim(\mathfrak{g}^{\mathtt{der}}) + (\dim(\mathfrak{g}^{\mathtt{der}}) - \dim(\mathfrak{b}^{\mathtt{der}}))[F_\nu : \mathbb{Q}_\ell].$$

   In other words,

$$R_\nu^{\square,\chi,\mathtt{crys}} \cong W[[x_1, \dots, x_{d_\nu^{\square,\mathtt{crys}}}]].$$

4. **(min)**: For each $\nu \in S$, there exists a relatively representable subfunctor

$$D^{\square,\chi,\mathtt{min}}(\overline{\rho}_\nu) \hookrightarrow D^{\square,\chi}(\overline{\rho}_\nu)$$

   such that the representing ring $R_\nu^{\square,\chi,\mathtt{min}}$ is formally smooth over W of relative dimension

$$d_\nu^{\square,\mathtt{min}} = \dim(\mathfrak{g}^{\mathtt{der}}).$$

5. **(∞)**: For each $\nu \in \Omega_\infty$, the local deformation ring $R_\nu^{\square,\chi}$ is formally smooth of relative dimension $d_\nu^{\square} = \dim(\mathfrak{b}^{\mathtt{der}})$. (As $\ell > 2 \geq \#\operatorname{Gal}_{F_\nu}$, the strict ℓ-cohomological dimension $\operatorname{scd}_\ell(\operatorname{Gal}_{F_\nu})$ is zero, i.e. $R_\nu^{\square,\chi}$ is automatically unobstructed.)

6. **(Presentability)**: Consider the ring

$$R^{\mathtt{loc,sm}} := \widehat{\bigotimes_{\nu \in S_\ell}} \widetilde{R}_\nu \qquad \text{with } \widetilde{R}_\nu = \begin{cases} R_\nu^{\square,\chi,\mathtt{min}} & \text{if } \nu \in S; \\ R_\nu^{\square,\chi,\mathtt{sm}} & \text{if } \nu \in \Omega_\ell; \\ R_\nu^{\square,\chi} & \text{if } \nu \in \Omega_\infty. \end{cases} \tag{3.2}$$

   Then there exists a presentation

$$R_{S_\ell}^{\square_{S_\ell},\chi,\mathtt{min,sm}} \cong R^{\mathtt{loc}}[[x_1, \dots, x_a]]_{/(f_1,\dots,f_b)}$$

   for suitable $a, b \in \mathbb{N}$ with $a - b = (\#S_\ell - 1).\dim(\mathfrak{g}^{\mathtt{ab}})$.

7. **(R=T)**: The ring $R_{S_\ell}^{\square_{S_\ell},\chi,\mathtt{min,crys}}$ is formally smooth of relative dimension

$$r_0 := \dim(\mathfrak{g}).\#S_\ell - \dim(\mathfrak{g}^{\mathtt{ab}}).$$

Remark 3.10. Assume $G = \mathrm{GL}_n$. For `crys` we will mainly consider the crystalline deformation condition which will be introduced in Section 4.3 and for `sm` the unconditioned deformation condition. However, the presented general framework is also applicable to `sm` = `ord`, with `ord` parametrizing ordinary deformations (cf. [Hid89a, Hid89b, CM14, Ger10a, Til96]). For $n = 2$, this corresponds to twisted Hida families and there exists a criterion for smoothness of `ord`, cf. [Sno11]. (Remark that what we call ordinary is called nearly ordinary by Tilouine [Til96].) For $n > 2$ there exists no such criterion[9]. It seems worthwhile to investigate whether a sufficient condition for the smoothness of `ord` can be derived from properties of the mod-ℓ reduction of the Hecke polynomial in the fixed weight case.

Remark 3.11 (Taylor-Wiles condition). From condition **(∞)**, $\mathrm{scd}_\ell(\mathrm{Gal}_{F_\nu}) = 0$ for $\ell > 2$ and Proposition 3.2 it follows that, for $\nu \in \Omega_\infty$, we have

$$\dim(\mathfrak{b}^{\mathtt{der}}) = \dim_W(R_\nu^\square) = h^1(\mathrm{Gal}_{F_\nu}, \mathfrak{g}^{\mathtt{der}})' + \dim(\mathfrak{g}^{\mathtt{der}}) - h^0(\mathrm{Gal}_{F_\nu}, \mathfrak{g}^{\mathtt{der}})$$

$$= \dim(\mathfrak{g}^{\mathtt{der}}) - h^0(\mathrm{Gal}_{F_\nu}, \mathfrak{g}^{\mathtt{der}}).$$

This implies

$$\sum_{\nu \in \Omega_\infty} h^0(\mathrm{Gal}_{F_\nu}, \mathfrak{g}^{\mathtt{der}}) = [F : \mathbb{Q}].\big(\dim(\mathfrak{g}^{\mathtt{der}}) - \dim(\mathfrak{b}^{\mathtt{der}})\big). \tag{3.3}$$

Our main result is now as follows:

Theorem 3.12. *1. If assumptions 1-7 are met, then $R_{S_\ell}^{\square_{S_\ell},\chi,\mathtt{min},\mathtt{sm}}$ is formally smooth (i.e. isomorphic to a ring of power series over W). If the unframed deformation functor $D_{S_\ell}^{\chi,\mathtt{min},\mathtt{sm}}$ is representable, then the representing object $R_{S_\ell}^{\chi,\mathtt{min},\mathtt{sm}}$ is also formally smooth.*

2. For $\nu \in \Omega_\ell$, write $d_\nu^{\square,\mathtt{sm}} = \dim(\mathfrak{g}^{\mathtt{der}})([F_\nu : \mathbb{Q}_\ell] + 1) - \delta_\nu$ for suitable numbers $\delta_\nu \in \mathbb{N}_0$. Then $R_{S_\ell}^{\square_{S_\ell},\chi,\mathtt{min},\mathtt{sm}}$ is formally smooth of dimension

$$\#S_\ell.\dim(\mathfrak{g}) - \dim(\mathfrak{g}^{\mathtt{ab}}) + [F : \mathbb{Q}].\dim(\mathfrak{b}^{\mathtt{der}}) - \sum_{\nu \in \Omega_\ell} \delta_\nu.$$

If the unframed deformation functor $D_{S_\ell}^{\chi,\mathtt{min},\mathtt{sm}}$ is representable, then $R_{S_\ell}^{\chi,\mathtt{min},\mathtt{sm}}$ is formally smooth of dimension $[F : \mathbb{Q}].\dim(\mathfrak{b}^{\mathtt{der}}) - \sum_{\nu \in \Omega_\ell} \delta_\nu$.

Remark 3.13. As the deformation conditions in **(min)** and **(sm/k)** were chosen as relatively representable, $D_{S_\ell}^{\chi,\mathtt{min},\mathtt{sm}}$ is representable if $D_{S_\ell}^{\chi}$ is representable. For example, this is the case if $\overline{\rho}$ is Schur (i.e. fulfills conditions **(SmCtr)** and **(Centr)** of Section 2.3).

Remark 3.14. If $D_\nu^{\square,\chi}$ is unobstructed for $\nu \in \Omega_\ell$, the condition in part 2. (with $\delta_\nu = 0$) amounts to `sm` being the unrestricted deformation condition.

Proof of Theorem 3.12. First remark that the second claim of part 1. follows directly from Lemma 2.15, as we know that $R_{S_\ell}^{\square_{S_\ell},\chi,\mathtt{min},\mathtt{sm}}$ is a power series ring over $R_{S_\ell}^{\chi,\mathtt{min},\mathtt{sm}}$ by Proposition 2.62. The same reasoning (together with the formula $\dim \mathfrak{g} = \dim \mathfrak{g}^{\mathtt{der}} + \dim \mathfrak{g}^{\mathtt{ab}}$) also shows the second claim of part 2.

[9] But cf. [Ger10a].

For the first sentences (of 1. and 2.), we use the shorthand notation $d_T^\star = \sum_{\nu \in T} d_\nu^\star$ if T denotes a subset of Pl_F. Moreover we write $d_\infty^\square$ for $d_{\Omega_\infty}^\square$ and $d_\ell^\star$ for $d_{\Omega_\ell}^\star$. Consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & R^{\mathtt{loc,sm}} & \xrightarrow{f} & R^{\mathtt{loc,crys}} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} & & \\
0 & \longrightarrow & J & \longrightarrow & R_{S_\ell}^{\square_{S_\ell},\chi,\mathtt{min,sm}} & \xrightarrow[g]{} & R_{S_\ell}^{\square_{S_\ell},\chi,\mathtt{min,crys}} & \longrightarrow & 0\,,
\end{array}$$

where

- the right square is a pushout diagram;
- $R^{\mathtt{loc,crys}}$ is defined as in (3.2) with $\mathtt{crys}$ in place of $\mathtt{sm}$;
- f and g are the canonical projections;
- $\pi = \otimes_{\nu \in S_\ell} \pi_\nu$ is induced by the natural transformations

$$D_{S_\ell}^{\square_{S_\ell},\chi,\mathtt{min,sm}} \to \widetilde{D}_\nu,$$

  where $\widetilde{D}_\nu$ is the deformation functor corresponding to (i.e. being represented by) the ring $\widetilde{R}_\nu$ in (3.2);
- Analogously, $\pi' = \otimes_{\nu \in S_\ell} \pi'_\nu$ is induced by the natural transformations

$$D_{S_\ell}^{\square_{S_\ell},\chi,\mathtt{min,crys}} \to \widetilde{D}'_\nu,$$

  where $\widetilde{D}'_\nu = \widetilde{D}_\nu$ for ν coprime to ℓ and $\widetilde{D}'_\nu$ being the crystalline deformation functor for $\nu|\ell$.

Using the assumptions, we can rewrite this as

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & W[[x_1,\dots,x_{d_\ell^{\square,\mathtt{sm}}+d_\infty^\square+d_S^{\square,\mathtt{min}}}]] & \xrightarrow{f} & W[[x_1,\dots,x_{d_\ell^{\square,\mathtt{crys}}+d_\infty^\square+d_S^{\square,\mathtt{min}}}]] & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow & & \\
0 & \longrightarrow & J & \longrightarrow & W[[x_1,\dots,x_m]]/(f_1,\dots,f_{m-\gamma}) & \xrightarrow[g]{} & W[[x_1,\dots,x_{r_0}]] & \longrightarrow & 0
\end{array}$$

with $\gamma = (\#S_\ell - 1).\dim(\mathfrak{g}^{\mathtt{ab}}) + d_\ell^{\square,\mathtt{sm}} + d_\infty^\square + d_S^{\square,\mathtt{min}}$. Using Lemma 2.10, we are good as soon as we can show that $\mathrm{gen}(J) \le m - (m-\gamma) - r_0 = \gamma - r_0$. By Proposition 2.7 and Proposition 2.9, we can replace this inequality by

$$d_\ell^{\square,\mathtt{sm}} - d_\ell^{\square,\mathtt{crys}} \le \gamma - r_0 = (\#S_\ell - 1).\dim(\mathfrak{g}^{\mathtt{ab}}) + d_\ell^{\square,\mathtt{sm}} + d_\infty^\square + d_S^{\square,\mathtt{min}} - \dim(\mathfrak{g}).\#S_\ell + \dim(\mathfrak{g}^{\mathtt{ab}})$$

$$= \#S_\ell.(\dim(\mathfrak{g}^{\mathtt{ab}}) - \dim(\mathfrak{g})) + d_\ell^{\square,\mathtt{sm}} + d_\infty^\square + d_S^{\square,\mathtt{min}}.$$

Using assumptions **(min)**, **(∞)** and the identity $\dim(\mathfrak{g}^{\mathtt{der}}) + \dim(\mathfrak{g}^{\mathtt{ab}}) = \dim(\mathfrak{g})$, this amounts to the inequality

$$d_\ell^{\square,\mathtt{crys}} \ge \dim(\mathfrak{g}^{\mathtt{der}}).(\#\Omega_\ell + [F:\mathbb{Q}]) - \dim(\mathfrak{b}^{\mathtt{der}})[F:\mathbb{Q}].$$

Now assumption **(crys)** amounts precisely to the fact that this inequality is fulfilled (with equality), which completes part 1.

Concerning (the remaining first sentence of) part 2., we use Remark 2.11 which tells us that the relative dimension of $R_{S_\ell}^{\square_{S_\ell},\chi,\mathtt{min},\mathtt{sm}}$ is given by

$$\gamma = (\#S_\ell - 1).\dim(\mathfrak{g}^{\mathtt{ab}}) + d_\ell^{\square,\mathtt{sm}} + d_\infty^{\square} + d_S^{\square,\mathtt{min}}$$

$$= \#S_\ell.\dim(\mathfrak{g}^{\mathtt{ab}}) - \dim(\mathfrak{g}^{\mathtt{ab}}) + \dim(\mathfrak{g}^{\mathtt{der}})([F:\mathbb{Q}] + \#\Omega_\ell) - \sum_{\nu\in\Omega_\ell}\delta_\nu + [F:\mathbb{Q}].\dim(\mathfrak{b}^{\mathtt{der}}) + \#S.\dim(\mathfrak{g}^{\mathtt{der}})$$

$$= \#S_\ell.\dim(\mathfrak{g}) + [F:\mathbb{Q}].\dim(\mathfrak{b}^{\mathtt{der}}) - \dim(\mathfrak{g}^{\mathtt{ab}}) - \sum_{\nu\in\Omega_\ell}\delta_\nu.$$

□

Corollary 3.15. *Assume that $\ell \gg 0$, so that $\mathfrak{g} = \mathfrak{g}^{\mathtt{der}} \oplus \mathfrak{g}^{\mathtt{ab}}$. Assume that the requirements of Theorem 3.12.2 are met (with $\delta_\nu = 0$) for the trivial choice in **(min)** and **(sm/k)**, i.e. $D^{\square,\chi,\mathtt{sm}}(\overline{\rho}_\nu) = D^{\square,\chi}(\overline{\rho}_\nu)$ (for $\nu\in\Omega_\ell$) and $D^{\square,\chi,\mathtt{min}}(\overline{\rho}_\nu) = D^{\square,\chi}(\overline{\rho}_\nu)$ (for $\nu \in S$). Then the deformation functor $D_{S_\ell}^{(\square_{S_\ell}),\chi,\mathtt{min},\mathtt{sm}} = D_{S_\ell}^{(\square_{S_\ell}),\chi}$ is unobstructed.*

Proof. Recall (e.g. from [NSW08, (8.7.4)]) the global Euler-Poincaré formula

$$\chi(\mathrm{Gal}_{F,S},\mathfrak{g}^{\mathtt{der}}) := h^1(\mathrm{Gal}_{F,S},\mathfrak{g}^{\mathtt{der}}) - h^0(\mathrm{Gal}_{F,S},\mathfrak{g}^{\mathtt{der}}) - h^2(\mathrm{Gal}_{F,S},\mathfrak{g}^{\mathtt{der}})$$

$$= [F:\mathbb{Q}].\dim(\mathfrak{g}^{\mathtt{der}}) - \sum_{\nu\in\Omega_\infty} h^0(\mathrm{Gal}_{F_\nu},\mathfrak{g}^{\mathtt{der}}).$$

Using the Taylor-Wiles condition (3.3), this implies

$$h^1(\mathrm{Gal}_{F,S},\mathfrak{g}^{\mathtt{der}}) - h^0(\mathrm{Gal}_{F,S},\mathfrak{g}^{\mathtt{der}}) - h^2(\mathrm{Gal}_{F,S},\mathfrak{g}^{\mathtt{der}}) = [F:\mathbb{Q}].\dim(\mathfrak{b}^{\mathtt{der}}).$$

By Theorem 3.12.2 we know that $D_{S_\ell}^{\chi} \cong W[[x_1,\ldots,x_r]]$ with $r := [F:\mathbb{Q}].\dim(\mathfrak{b}^{\mathtt{der}})$. But this implies $h^1(G_{F,S},\mathfrak{g}^{\mathtt{der}}) = r$. As both $h^0(G_{F,S},\mathfrak{g}^{\mathtt{der}})$ and $h^2(G_{F,S},\mathfrak{g}^{\mathtt{der}})$ are non-negative, they must vanish and the claim follows. (This argument is easily seen to be adaptable to the framed situation, so the case where $D_{S_\ell}^{\chi}$ is not representable is handled in the same way.) □

Corollary 3.16. *Let $\mathcal{L} := \mathcal{L}^{\chi} = (L_\nu^{\chi})_\nu$ be the system of local conditions corresponding to the deformation functor $D^{\chi,\mathtt{min},\mathtt{sm}}(\overline{\rho})$ (see Definition 2.69). Assume (in addition to the requirements of Theorem 3.12.2 with $\delta_\nu = 0$) the following:*

- *$\ell \gg 0$, so that $\mathfrak{g} = \mathfrak{g}^{\mathtt{der}} \oplus \mathfrak{g}^{\mathtt{ab}}$;*
- *$H^0(\mathrm{Gal}_F,\mathfrak{g}^{\mathtt{der},\vee}) = 0$ (this holds automatically for $G = \mathrm{GL}_n$ and $\ell \gg 0$, see Lemma 2.72);*
- *For $\nu\in S$, $\dim(L_\nu) = h^0(\mathrm{Gal}_{F_\nu},\mathfrak{g}^{\mathtt{der}})$.*

Then $D_{S_\ell}^{(\square_{S_\ell}),\chi,\mathtt{min},\mathtt{sm}}(\overline{\rho})$ has vanishing dual Selmer group (i.e. $H^1_{\mathcal{L}^\perp}(\mathrm{Gal}_{F,S},\mathfrak{g}^{\mathtt{der},\vee}) = 0$, cf. Definition 3.6). Moreover,

$$h^0(\mathrm{Gal}_{F,S},\mathfrak{g}^{\mathtt{der}}) = 0.$$

Let us first mention that the last condition ($\dim(L_\nu) = h^0(\mathrm{Gal}_{F_\nu}, \mathfrak{g}^{\mathtt{der}})$) holds automatically for $\nu \notin S_\ell$ if $\ell \gg 0$ (so that $\mathfrak{g} = \mathfrak{g}^{\mathtt{der}} \oplus \mathfrak{g}^{\mathtt{ab}}$). We also remind the reader that the deformation condition $\mathtt{sm}$ is assumed to fulfill $d_\nu^{\square,\mathtt{sm}} = \dim(\mathfrak{g}^{\mathtt{der}})([F_\nu : \mathbb{Q}_\ell] + 1)$ for $\nu \in \Omega_\ell$, as demanded by Theorem 3.12.2 with $\delta_\nu = 0$.

Proof. Using (a $\mathtt{sm}$-conditioned version of) the exact sequence (2.12), we see for $\nu \in \Omega_\ell$:

$$\dim(L_\nu) = \dim t_{D^{\chi,\mathtt{sm}}(\overline{\rho}_\nu)} \overset{3.12.2}{=} h^0(\mathrm{Gal}_{F_\nu}, \mathfrak{g}^{\mathtt{der}}) + [F_\nu : \mathbb{Q}_\ell].\dim(\mathfrak{g}^{\mathtt{der}}).$$

Recall the Wiles-Formula (e.g. from [NSW08, Theorem 8.7.9]):

$$\begin{aligned}&\dim H^1_{\mathcal{L}}(\mathrm{Gal}_{F,S}, \mathfrak{g}^{\mathtt{der}}) - \dim H^1_{\mathcal{L}^\perp}(\mathrm{Gal}_{F,S}, \mathfrak{g}^{\mathtt{der},\vee}) \\ &= h^0(\mathrm{Gal}_{F,S}, \mathfrak{g}^{\mathtt{der}}) - h^0(\mathrm{Gal}_{F,S}, \mathfrak{g}^{\mathtt{der},\vee}) + \sum_{\nu \in S_\ell} \big(\dim(L_\nu) - h^0(\mathrm{Gal}_{F_\nu}, \mathfrak{g}^{\mathtt{der}})\big)\end{aligned}$$

By [Böc07, Section 5], we know that $H^1_{\mathcal{L}}(\mathrm{Gal}_{F,S}, \mathfrak{g}^{\mathtt{der}})$ can be identified with the tangent space of $D_{S_\ell}^{\chi,\mathtt{min},\mathtt{sm}}$. Thus, it follows from Theorem 3.12.2 that $\dim H^1_{\mathcal{L}}(\mathrm{Gal}_{F,S}, \mathfrak{g}^{\mathtt{der}}) = [F : \mathbb{Q}].\dim(\mathfrak{b}^{\mathtt{der}})$. On the other hand, $h^0(\mathrm{Gal}_{F,S}, \mathfrak{g}^{\mathtt{der},\vee})$ was assumed to vanish. Concerning the places in Ω_∞, we know that $L_\nu \subset H^1(\mathrm{Gal}_{F,S}, \mathfrak{g}^{\mathtt{der}}) = 0$. Thus, using the Taylor-Wiles formula (3.3), the sum on the right evaluates to

$$\sum_{\nu \in S_\ell} \big(\dim(L_\nu) - h^0(\mathrm{Gal}_{F_\nu}, \mathfrak{g}^{\mathtt{der}})\big) = [F : \mathbb{Q}].\dim(\mathfrak{g}^{\mathtt{der}}) - [F : \mathbb{Q}].\big(\dim(\mathfrak{g}^{\mathtt{der}}) - \dim(\mathfrak{b}^{\mathtt{der}})\big).$$

Therefore we get

$$-\dim H^1_{\mathcal{L}^\perp}(\mathrm{Gal}_{F,S}, \mathfrak{g}^{\mathtt{der},\vee}) = h^0(\mathrm{Gal}_{F,S}, \mathfrak{g}^{\mathtt{der}}).$$

As neither quantity can be negative, they must both vanish and the result follows. □

Corollary 3.17. *Retain the assumptions of Corollary 3.16. Then* $Ш^2_{S_\ell}(\mathfrak{g}^{\mathtt{der}}) = 0$. *In particular, the unrestricted deformation functor* $D_{S_\ell}^{(\square_{S_\ell}),\chi}(\overline{\rho})$ *is globally unobstructed precisely if the local deformation functors* $D^{(\square),\chi}(\overline{\rho}_\nu)$ *(for* $\nu \in S_\ell$*) are relatively smooth. (This is automatic for* Ω_∞*, so it has only to be checked for* $S \sqcup \Omega_\ell$*.)*

Proof. This follows from the exact sequence

$$\dim H^1_{\mathcal{L}^\perp}(\mathrm{Gal}_{F,S}, \mathfrak{g}^{\mathtt{der},\vee})^* \to Ш^2_{S_\ell}(\mathfrak{g}^{\mathtt{der}}) \to 0$$

(see e.g. equation (9) on p. 10 of [Böc07]). □

3.2 Potential unobstructedness

In this short subsection, we will investigate how unobstructedness of a deformation functor for $\overline{\rho}$ can be deduced from unobstructedness of the restricted functor to the base change of $\overline{\rho}$ to a finite extension F' of F. Let us first start with the easy case of unconditioned deformations:

Lemma 3.18. *Let*

$$\overline{\rho} : \mathrm{Gal}_{F,S} \to G(k)$$

be a global Galois representation as considered previously and let F' *be a finite extension of* F *such that* $[F' : F]$ *is coprime to* ℓ.

1. *Assume that* $D_{S'_\ell}^{(\square_{S'_\ell}),\chi}(\overline{\rho}|\operatorname{Gal}_{F',S'})$ *is unobstructed, where* S' *denotes the set of places of* F' *lying over* S*. Then* $D_{S_\ell}^{(\square_{S_\ell}),\chi}(\overline{\rho})$ *is unobstructed.*
2. *Let* ν *be a place of* F *and assume that there exists a place* ν' *of* F' *which lies above* ν *such that the local deformation functor* $D^{(\square),\chi_\nu}(\overline{\rho}_\nu|\operatorname{Gal}_{F'_{\nu'}})$ *is unobstructed. Then also* $D^{(\square),\chi_\nu}(\overline{\rho}_\nu)$ *is unobstructed.*

Proof. For the first part, we have to show that the vanishing of $H^2(\operatorname{Gal}_{F',S'_\ell},\mathfrak{g}^{\mathtt{der}})$ implies the vanishing of $H^2(\operatorname{Gal}_{F,S_\ell},\mathfrak{g}^{\mathtt{der}})$. This is seen using [NSW08, Corollary (1.5.7)] and our assumption that

$$(\operatorname{Gal}_{F',S'_\ell}:\operatorname{Gal}_{F,S_\ell})=[F':F]$$

is invertible in k, therefore the restriction map

$$H^2(\operatorname{Gal}_{F,S_\ell},\mathfrak{g}^{\mathtt{der}})\to H^2(\operatorname{Gal}_{F',S'_\ell},\mathfrak{g}^{\mathtt{der}})$$

is injective and the claim follows.

For the second part, we can argue analogously by considering the local restriction map

$$H^2(F_\nu,\mathfrak{g}^{\mathtt{der}})\to H^2(F'_{\nu'},\mathfrak{g}^{\mathtt{der}})$$

and using that $[F'_{\nu'}:F_\nu]$ is a divisor of $[F':F]$, hence is also invertible in k. □

We continue to denote by F' a finite extension of F. For any G_F-module M and any pair of primes ν,ν' with $\nu\in\operatorname{Pl}_F,\nu'\in\operatorname{Pl}_{F'}$ such that ν' divides ν, the diagram

$$\begin{array}{ccc} H^1(F,M) & \longrightarrow & H^1(F_\nu,M) \\ \downarrow & & \downarrow \\ H^1(F',M) & \longrightarrow & H^1(F'_{\nu'},M) \end{array}$$

is commutative, where all maps are the respective restriction maps. If now S is some finite set of primes of F, the diagram

$$\begin{array}{ccc} H^1(F,M) & \longrightarrow & \bigoplus_{\nu\in S} H^1(F_\nu,M) \\ \downarrow & & \downarrow \\ H^1(F',M) & \longrightarrow & \bigoplus_{\nu'\in S'} H^1(F'_{\nu'},M) \end{array}$$

is commutative as well.

Definition 3.19 (Dual-pre system)**.** Let $\mathcal{L}'=(L'_{\nu'})_{\{\nu'\in\operatorname{Pl}_{F'}\}}$ be a system of local conditions for F' (i.e. $L_{\nu'}$ is a subgroup of $H^1(F'_{\nu'},\mathfrak{g}^{\mathtt{der}})$, cf. Definition 2.69). We say that a system $\mathcal{L}=(L_\nu)_{\{\nu\in\operatorname{Pl}_F\}}$ of conditions for F is *dual-pre-*$\mathcal{L}'$ if $\operatorname{res}^\perp_{\nu'}(L^\perp_\nu)\subset L'^\perp_{\nu'}$ for all pairs ν,ν' with $\nu\in\operatorname{Pl}_F,\nu'\in\operatorname{Pl}_{F'}$, such that ν' divides ν, where

$$\operatorname{res}^\perp_{\nu'}:H^1(F_\nu,\mathfrak{g}^{\mathtt{der},\vee})\to H^1(F'_{\nu'},\mathfrak{g}^{\mathtt{der},\vee})$$

is the usual restriction map.

Remark 3.20*.* We now give two criteria for Definition 3.19:

1. Let
$$\mathrm{res}_{\nu'} : H^1(F_\nu, \mathfrak{g}^{\mathtt{der}}) \to H^1(F'_{\nu'}, \mathfrak{g}^{\mathtt{der}})$$
denote the restriction map and let $\mathcal{L}$ be a system of conditions of F. If $\mathrm{res}_{\nu'}(L_\nu)$ contains $L'_{\nu'}$ for all pairs ν, ν' with $\nu \in \mathrm{Pl}_F, \nu' \in \mathrm{Pl}_{F'}$, such that ν' divides ν, then $\mathcal{L}$ is dual-pre-$\mathcal{L}'$. This can be seen by using the fact that Tate duality is given by the cup product: We have to check that $\mathrm{res}^{\perp}_{\nu'}(l) \cup l' = 0$ for any $l \in L^{\perp}_{\nu}$ and any $l' \in L'_{\nu'}$. By our assumption, we can write $l' = \mathrm{res}_{\nu'}(\tilde{l})$ for some $\tilde{l} \in L_\nu$. But then the claim follows from the formula
$$(\mathrm{res}\, x) \cup (\mathrm{res}\, y) = \mathrm{res}(x \cup y).$$

2. Let
$$\mathrm{cor}_{\nu'} : H^1(F'_{\nu'}, \mathfrak{g}^{\mathtt{der}}) \to H^1(F_\nu, \mathfrak{g}^{\mathtt{der}})$$
denote the corestriction map and let $\mathcal{L}$ be a system of conditions of F. If L_ν contains $\mathrm{cor}_{\nu'}(L'_{\nu'})$ for all pairs ν, ν' with $\nu \in \mathrm{Pl}_F, \nu' \in \mathrm{Pl}_{F'}$, such that ν' divides ν, then $\mathcal{L}$ is dual-pre-$\mathcal{L}'$. As above, we argue with the cup product and check the equivalent condition $\mathrm{res}^{\perp}_{\nu'}(l) \cup l' = 0$ for any $l \in L^{\perp}_{\nu}$ and any $l' \in L'_{\nu'}$. As the corestriction map on the H^2-level
$$\mathrm{cor}_{\nu'} : H^2(F'_{\nu'}, k(1)) \to H^2(F_\nu, k(1))$$
is an isomorphism, this is equivalent to
$$\mathrm{cor}_{\nu'}(\mathrm{res}^{\perp}_{\nu'}(l) \cup l') = l \cup \mathrm{cor}_{\nu'}(l') = 0$$
for any $l \in L^{\perp}_{\nu}$ and any $l' \in L'_{\nu'}$. The claim follows.

For the next theorem, we again assume $\ell \gg 0$.

Lemma 3.21. *Let*
$$\overline{\rho} : \mathrm{Gal}_{F,S} \to G(k)$$
be a global residual representation together with a finite extension F' of F of degree coprime to ℓ. Furthermore, let $\mathtt{min}, \mathtt{crys}, \mathtt{sm}$ *be suitable deformation conditions for the functor* $D^{(\square_{S'_\ell}),\chi}_{S'_\ell}(\overline{\rho}|\,\mathrm{Gal}_{F',S'})$ *as demanded by the framework of Theorem 3.12 and Corollary 3.16, such that* $D^{(\square_{S'_\ell}),\chi,\mathtt{min},\mathtt{sm}}_{S'_\ell}(\overline{\rho}|\,\mathrm{Gal}_{F',S'})$ *has vanishing dual Selmer group. Let $\mathcal{L}$ be a dual-pre-($\chi, \mathtt{min}, \mathtt{sm}$) system for F (with corresponding deformation condition $\mathcal{D}_{\mathcal{L}}$), then $D^{(\square_{S_\ell}),\mathcal{D}_{\mathcal{L}}}_{S_\ell}(\overline{\rho})$ has vanishing dual Selmer group.*

Proof. Analogous to the proof of Lemma 3.18, because $[F' : F]$ is invertible in k, the map
$$H^1(\mathrm{Gal}_{F,S_\ell}, \mathfrak{g}^{\mathtt{der},(\vee)}) \to H^1(\mathrm{Gal}_{F',S'_\ell}, \mathfrak{g}^{\mathtt{der},(\vee)})$$
is injective. We consider the following diagram
$$\begin{array}{ccccc}
H^1_{\mathcal{L}^{\perp}}(\mathrm{Gal}_{F,S_\ell}, \mathfrak{g}^{\mathtt{der},\vee}) & \hookrightarrow & H^1(\mathrm{Gal}_{F,S_\ell}, \mathfrak{g}^{\mathtt{der},\vee}) & \longrightarrow & \bigoplus_{\nu \in S_\ell} H^1(F_\nu, \mathfrak{g}^{\mathtt{der},\vee})/L^{\perp}_{\nu} \\
\downarrow{\scriptstyle\varphi} & & \downarrow & & \downarrow \\
H^1_{\mathcal{L}(\mathtt{min})^{\perp}}(\mathrm{Gal}_{F',S'_\ell}, \mathfrak{g}^{\mathtt{der},\vee}) & \hookrightarrow & H^1(\mathrm{Gal}_{F',S'_\ell}, \mathfrak{g}^{\mathtt{der},\vee}) & \longrightarrow & \bigoplus_{\nu' \in S'_\ell} H^1(F'_{\nu'}, \mathfrak{g}^{\mathtt{der},\vee})/L(\mathtt{min})^{\perp}_{\nu'} ,
\end{array}$$
where $\mathcal{L}(\mathtt{min}) = (L(\mathtt{min})_{\nu'})_{\nu'}$ is the local system of conditions associated to the deformation condition $\chi, \mathtt{min}, \mathtt{sm}$. The vertical map on the right is defined because $\mathcal{L}$ is dual-pre-$\mathcal{L}(\mathtt{min})$, and this implies the well-definedness of φ. A simple diagram chase implies injectivity of φ, from which the claim follows. □

3.3 Compatible systems of Galois representations

In the sequel, we will apply the framework of Theorem 3.12 not for a fixed residue field k, but we will rather consider systems of Galois representations valued in various residue fields of characteristic running through all rational primes ℓ. For this, let us first recall the relevant notions from p-adic Hodge theory, where we follow [Gue11, Section 0.3] and [Böc13b, Section 5.2]:

Definition 3.22. Let L, K be finite extensions of $\mathbb{Q}_p$ (with maximal unramified subfield L_0 of L and where K is L-big enough) and let

$$\rho : \mathrm{Gal}_L \to \mathrm{GL}_n(K)$$

be a continuous representation. Recall the p-adic period rings $B_{\mathtt{HT}}, B_{\mathtt{dR}}$ and $B_{\mathtt{crys}}$ of Fontaine [Fon94]. We say that ρ is

- *Hodge-Tate*, if $(\rho \otimes_{\mathbb{Q}_p} B_{\mathtt{HT}})^{\mathrm{Gal}_L}$ is free over $L \otimes_{\mathbb{Q}_p} K$ of rank $\dim \rho$;
- *de Rham*, if $(\rho \otimes_{\mathbb{Q}_p} B_{\mathtt{dR}})^{\mathrm{Gal}_L}$ is free over $L \otimes_{\mathbb{Q}_p} K$ of rank $\dim \rho$;
- *crystalline*, if $(\rho \otimes_{\mathbb{Q}_p} B_{\mathtt{crys}})^{\mathrm{Gal}_L}$ is free over $L_0 \otimes_{\mathbb{Q}_p} K$ of rank $\dim \rho$.

We remark that there is a chain of implications

$$\text{crystalline} \;\Rightarrow\; \text{de Rham} \;\Rightarrow\; \text{Hodge-Tate}.$$

If ρ is Hodge-Tate, it follows that also $(\rho \otimes_{L,\tau} B_{\mathtt{HT}})^{\mathrm{Gal}_L}$ is free over $L \otimes_{\mathbb{Q}_p} K$ of rank $\dim \rho$ for any embedding $\tau : L \hookrightarrow K$. This space inherits a grading from $B_{\mathtt{HT}}$, and we define the *Hodge-Tate weights* of ρ as the multiset $\mathrm{HT}_\tau(\rho)$ consisting of those $m \in \mathbb{Z}$ for which

$$\mathrm{gr}^{-m}(\rho \otimes_{L,\tau} B_{\mathtt{HT}})^{\mathrm{Gal}_L} \neq 0.$$

The multiplicity of such an m is then taken as $\dim \mathrm{gr}^{-m}(\rho \otimes_{L,\tau} B_{\mathtt{HT}})^{\mathrm{Gal}_L}$.

With this convention, the p-adic cyclotomic character $\overline{\epsilon}_p$ has Hodge-Tate weight -1.

Recall from [Tat79] the notions of the Weil group W_F of F, of Weil-Deligne representations and their link to Galois representations.

Definition 3.23 (Compatible system of Galois representations for $G = \mathrm{GL}_n$, [BLGGT14], [Böc13b])**.** A *weakly (E-rational) compatible system (with ramification set S and defect set T)* is a tuple

$$\mathcal{R} = \big(F, E, S, T, (\rho_\lambda)_{\lambda \in \Lambda}, (Q_\nu)_{\nu \in \mathrm{Pl}_F - S}, (\mathrm{HT}_\tau)_{\tau \in I}\big)$$

with $\Lambda \coloneqq \mathrm{Pl}_E - T$, where

- F, E are number fields;
- S is a finite subset of Pl_F and T is a subset of Pl_E;
- each

$$\rho_\lambda : \mathrm{Gal}_F \to \mathrm{GL}_n(E_\lambda)$$

is a continuous, semisimple representation;

- each $Q_\lambda \in E[X]$ is a monic polynomial of degree n;
- I denotes the set of embeddings $F \hookrightarrow E$ and each HT_τ is a multiset of n integers.

We impose the following list of compatibilities:

- For $\lambda \in \Lambda$ and $\nu \in \mathrm{Pl}_F - (S \cup \Omega_{\ell(\lambda)})$, the representation ρ_λ is unramified at ν. Moreover, the characteristic polynomial of $\rho_\lambda(\mathrm{Frob}_\nu)$ equals Q_ν for all $\nu \notin S \cup \Omega_{\ell(\lambda)}$;
- For $\lambda \in \mathrm{Pl}_E$ and $\nu \in \Omega_{\ell(\lambda)}$, ρ_λ is de Rham; if and $\nu \notin S$, then ρ_λ is even crystalline;
- The set of Hodge-Tate weights $\mathrm{HT}_{\iota_\lambda \circ \tau}(\rho_\lambda)$ coincides with HT_τ for any place $\lambda \in \Lambda$ and any embedding $\tau \in I$.

Additionally, we say that $\mathcal{R}$ is

- *regular*, if every element of H_τ (for any τ) has multiplicity one;
- *strict*, if, for any $\nu \in \mathrm{Pl}_F$, there exists a Weil-Deligne representation WD_ν of the Weil group W_{F_ν} of F_ν over $\overline{E}$ such that the following holds: For any choice of places $\nu \in \mathrm{Pl}_F, \lambda \in \mathrm{Pl}_E$ with $\ell(\nu) \neq \ell(\lambda)$, the Frobenius-semi-simplification of the Weil-Deligne representation attached to $\rho_\lambda | \mathrm{Gal}_{F_\nu}$ is isomorphic to WD_ν;
- *pure* of weight $w \in \mathbb{R}$: We define pureness only for strictly compatible systems, and here we additionally suppose
  - for any $\nu \notin S$, any root α of the characteristic polynomial of $\rho_\lambda | \mathrm{Gal}_{F_\nu}$ (which is independent of the place λ coprime to ν) and any embedding $\iota : \overline{E} \hookrightarrow \mathbb{C}$, we have
  $$|\iota(\alpha)|^2 = q_\nu^w; \tag{3.4}$$
  - for any $\tau : F \hookrightarrow \overline{E}$ and any complex conjugation $c \in \mathrm{Gal}_\mathbb{Q}$, we have
  $$\mathrm{HT}_{\tau^c} = \{w - h | h \in \mathrm{HT}_\tau\}.$$

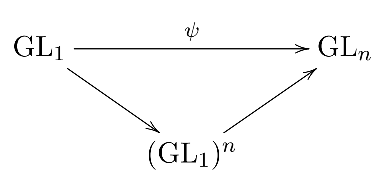
Definition 3.24. Let $\psi : \mathrm{GL}_1 \to (\mathrm{GL}_1)^n$ be a cocharacter and let $(e_1, \dots, e_n)$ be the standard basis of $\overline{E}^n$. Then we define the multiset weights(ψ) consisting of all integers j which fulfill

$$\psi(x) e_i = x^j e_i \tag{3.5}$$

for all $x \in \overline{E}$ and for a suitable $i \in \{1, \dots, n\}$. The multiplicity of j is

$$\#\{i' \in \{1, \dots, n\} \big| (3.5) \text{ holds for } i = i'\}.$$

We will also use the notation weights(ψ) for a GL_n-valued cocharacter. Any such cocharacter factorizes as in the following diagram where one chooses a maximal split torus that contains the image of ψ:



Now we want to stretch the notion of a compatible system to cover families of Galois representations with values in a more general (connected and reductive) group G:

Definition 3.25 (Compatible system of Galois representations, cf. [BG11, Pat14]). A *weakly (E-rational) compatible system (with ramification set S and defect set T)* is a tuple

$$\mathcal{R} = \big(F, E, S, T, (\rho_\lambda)_{\lambda\in\Lambda}, ([\varphi_\nu])_{\nu\in \mathrm{Pl}_F - S}, (\mu_\tau)_{\tau\in I}\big) \tag{3.6}$$

with $\Lambda \coloneqq \mathrm{Pl}_E - T$ and $I \coloneqq \{\tau \in \mathrm{Hom}_{\mathrm{fields}}(F, \overline{E}) \mid \tau \text{ injective } \}$, where

- F, E are number fields;
- S is a finite subset of Pl_F and T is a subset of Pl_E;
- each
$$\rho_\lambda : \mathrm{Gal}_F \to G(E_\lambda)$$
is a continuous, semisimple representation;
- each $[\varphi_\nu]$ is a semisimple G-conjugacy class in $G(E)$;
- each $\mu_\tau : \mathrm{GL}_1 /\overline{E} \to G/\overline{E}$ is a Hodge-Tate cocharacter.

We impose the following list of compatibilities:

- For $\lambda \in \Lambda$ and $\nu \in \mathrm{Pl}_F - (S \cup \Omega_{\ell(\lambda)})$, the representation ρ_λ is unramified at ν. Moreover, the semi-simplification $\rho_\lambda(\mathrm{Frob}_\nu)^{\mathsf{ss}}$ is contained in $[\varphi_\nu]$ for all $\nu \notin S \cup \Omega_{\ell(\lambda)}$;
- For $\lambda \in \Lambda$,$\nu \in \Omega_{\ell(\lambda)}$ and any faithful representation $\eta : G \to \mathrm{GL}_n$ of algebraic groups, $\eta_{E_\lambda} \circ \rho_\lambda$ is de Rham; if $\nu \notin S$, then $\eta_{E_\lambda} \circ \rho_\lambda$ is even crystalline;
- For any choice of
  - a place $\lambda \in \Lambda$,
  - a place $\nu \in \Omega_{\ell(\lambda)}$,
  - a faithful representation $\eta : G \to \mathrm{GL}_n$ of algebraic groups,
  - an embedding $\tau : F \hookrightarrow \overline{E}$

  we have
$$\mathrm{HT}_\tau(\eta_{E_\lambda} \circ \rho_\lambda) = \mathrm{weights}(\eta \circ \mu_\tau).$$

In the sequel, we will often use the abbreviatory notation

$$\mathcal{R} = (\rho_\lambda)_{\lambda\in\Lambda}$$

and suppress the remaining data of the compatible system if there is no risk of confusion.

4 Local deformation conditions

In the course of this section, we will consider certain deformation conditions in the local case for $G = \mathrm{GL}_n$. To this end, let us denote by

$$\overline{\rho} : \mathrm{Gal}_K \to \mathrm{GL}_n(k)$$

a residual representation, where K is a finite extension of $\mathbb{Q}_p$ (for some prime number p) and k is a finite field of characteristic ℓ. In our presentation of the various deformation conditions we will distinguish between the cases $\ell = p$ and $\ell \neq p$. Before we start, we will need to generalize a key computation of Weston [Wes04, Proposition 4.4] using Fontaine-Laffaille theory.

4.1 Fontaine-Laffaille theory

In this subsection, we will recall the main results of Fontaine-Laffaille theory [FL82] as normalized in [CHT08, Section 2.4.1] and draw conclusions about the vanishing of a certain H^2-group. Our main reference for this material is [BLGGT14, Section 1.4].

Let K, k, ℓ, p be as before and assume $\ell = p$. Let moreover L be a finite extension of $\mathbb{Q}_\ell$ with ring of integers $\mathcal{O}_L$, such that the residue field of L is isomorphic to k. We assume furthermore that the extension $K|\mathbb{Q}_\ell$ is unramified. The ring of integers of K is denoted by $\mathcal{O}_K$.

As in [CHT08, Section 2.4.1], we also make the following bigness assumption (which will be revoked later on):

Assumption 4.1. L contains the images of all embeddings $K \hookrightarrow \overline{\mathbb{Q}}_\ell$, i.e. $L \supset K$ since K is unramified over $\mathbb{Q}_\ell$.

We denote by $\sigma : \mathcal{O}_K \to \mathcal{O}_K$ the arithmetic Frobenius morphism.

Definition 4.2. We define the category $\underline{\mathrm{MF}}_{\mathcal{O}_K,\mathcal{O}_L}$ as follows: An object $\mathbf{M} = (M, (\mathrm{Fil}^i M)_{i\in\mathbb{Z}}, (\varphi_{M,i})_{i\in\mathbb{Z}})$ of $\underline{\mathrm{MF}}_{\mathcal{O}_K,\mathcal{O}_L}$ consists of

- an $\mathcal{O}_K \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L$-module M of finite type;
- A decreasing filtration $(\mathrm{Fil}^i M)_{i\in\mathbb{Z}}$ of M by $\mathcal{O}_K \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L$-submodules which are $\mathcal{O}_K$-direct summands and fulfill $\mathrm{Fil}^0 M = M$ and $\mathrm{Fil}^{\ell-1} M = 0$;
- A family of $\sigma \otimes 1$-linear maps $\varphi_{M,i} : \mathrm{Fil}^i M \to M$, such that

$$\begin{array}{ccc} \mathrm{Fil}^{i+1} M & \hookrightarrow & \mathrm{Fil}^i M \\ \downarrow{\scriptstyle \varphi_{M,i+1}} & & \downarrow{\scriptstyle \varphi_{M,i}} \\ M & \xrightarrow[\text{multiplication with } \ell]{} & M \end{array}$$

  commutes for all $i \in \mathbb{Z}$ and such that

$$\sum_{i\in\mathbb{Z}} \varphi_{M,i}(\mathrm{Fil}^i M) = M.$$

A morphism
$$f : \mathbf{M} \longrightarrow \mathbf{N} = (N, (\operatorname{Fil}^i N)_{i\in\mathbb{Z}}, (\varphi_{N,i})_{i\in\mathbb{Z}})$$
is an $\mathcal{O}_K \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L$-linear map $M \to N$ satisfying $f(\operatorname{Fil}^i M) \subset \operatorname{Fil}^i N$ and making

$$\begin{array}{ccc} \operatorname{Fil}^i M & \xrightarrow{f|\operatorname{Fil}^i M} & \operatorname{Fil}^i N \\ {\scriptstyle \varphi_{M,i}}\downarrow & & \downarrow{\scriptstyle \varphi_{N,i}} \\ M & \xrightarrow[f]{} & N \end{array} \tag{4.1}$$

commutative.

We remark that M can be understood as a strongly admissible lattice in the filtered φ-module $M \otimes_{\mathcal{O}_L} L$, cf. [Böc13a, Section 4.6.3].

We also consider the following categories:

- $\underline{\mathrm{MF}}^{\mathtt{proj}}_{\mathcal{O}_K,\mathcal{O}_L}$: The full subcategory of $\underline{\mathrm{MF}}_{\mathcal{O}_K,\mathcal{O}_L}$ consisting of projective objects;
- $\underline{\mathrm{MF}}^{\circ}_{\mathcal{O}_K,\mathcal{O}_L}$: The full subcategory of $\underline{\mathrm{MF}}_{\mathcal{O}_K,\mathcal{O}_L}$ consisting of objects of finite length;
- $\underline{\mathrm{MF}}_{\mathcal{O}_K,k}$: The full subcategory of $\underline{\mathrm{MF}}^{\circ}_{\mathcal{O}_K,\mathcal{O}_L}$ consisting of objects annihilated by the maximal ideal $\varpi_L.\mathcal{O}_L$ of $\mathcal{O}_L$;
- $\underline{\mathrm{Rep}}_{\mathcal{O}_L}(\mathrm{Gal}_K)$: The category of $\mathcal{O}_L$-modules of finite type together with a continuous Gal_K-action;
- $\underline{\mathrm{Rep}}^{\circ}_{\mathcal{O}_L}(\mathrm{Gal}_K)$: The full subcategory of $\mathrm{Rep}_{\mathcal{O}_L}(\mathrm{Gal}_K)$ consisting of objects of finite length;
- $\underline{\mathrm{Rep}}^{\circ}_{k}(\mathrm{Gal}_K)$: The full subcategory of $\mathrm{Rep}^{\circ}_{\mathcal{O}_L}(\mathrm{Gal}_K)$ consisting of finite k-modules together with a continuous Gal_K-action;

Let $\mathbb{E}_K$ denote the set of all embeddings $K \hookrightarrow \overline{\mathbb{Q}}_\ell$. For $\tau \in \mathbb{E}_K$ and $V \in \underline{\mathrm{Rep}}_{\mathcal{O}_L}(\mathrm{Gal}_K)$ being Hodge-Tate we will denote by $\mathrm{HT}_\tau(V)$ the multiset of Hodge-Tate numbers with respect to τ, counted with multiplicity. For $\mathbf{M} \in \underline{\mathrm{MF}}_{\mathcal{O}_K,k}$, we denote by $\mathrm{FL}_\tau(\mathbf{M})$ the multiset of integers i such that

$$\mathrm{gr}^i(\mathbf{M}^\tau) := \operatorname{Fil}^i M \otimes_{\mathcal{O}_K\otimes_{\mathbb{Z}_\ell}\mathcal{O}_L,\tau\otimes 1} \mathcal{O}_L / \operatorname{Fil}^{i+1} M \otimes_{\mathcal{O}_K\otimes_{\mathbb{Z}_\ell}\mathcal{O}_L,\tau\otimes 1} \mathcal{O}_L$$

does not vanish, where i is counted with multiplicity $\dim_k \mathrm{gr}^i(\mathbf{M}^\tau)$.

Theorem 4.3 (Fontaine-Laffaille)**.**

1. *There is an exact, fully faithful, covariant and $\mathcal{O}_L$-linear functor*
$$\mathtt{G}_K : \underline{\mathrm{MF}}_{\mathcal{O}_K,\mathcal{O}_L} \longrightarrow \underline{\mathrm{Rep}}_{\mathcal{O}_L}(\mathrm{Gal}_K).$$
The essential image of $\mathtt{G}_K$ is closed under taking subobjects and quotients. Moreover, the functor $\mathtt{G}_K$ maps $\underline{\mathrm{MF}}^{\circ}_{\mathcal{O}_K,\mathcal{O}_L}$ to $\underline{\mathrm{Rep}}^{\circ}_{\mathcal{O}_L}(\mathrm{Gal}_K)$.

2. *Let*
$$\rho : \mathrm{Gal}_K \longrightarrow \mathrm{GL}_L(V)$$
be a crystalline representation with Hodge-Tate weights in the range $[0, \ell-2]$. Then any Gal_K-stable $\mathcal{O}_L$-lattice $\Lambda \subset V$ is in the image of $\mathtt{G}_K$, and so is its reduction $\Lambda/\varpi_F.\Lambda$.

3. *Let* $\mathbf{M} \in \underline{\mathrm{MF}}^{\circ}_{\mathcal{O}_K,\mathcal{O}_L}$, *then*

$$\mathrm{length}_{\mathcal{O}_L-\underline{\mathrm{Mod}}}(\mathbf{M}) = [K:\mathbb{Q}_\ell].\,\mathrm{length}_{\mathcal{O}_L-\underline{\mathrm{Mod}}}(\mathtt{G}_K(\mathbf{M})).$$

4. $\mathtt{G}_K$ *restricts to a functor*

$$\underline{\mathrm{MF}}_{\mathcal{O}_K,k} \longrightarrow \underline{\mathrm{Rep}}^{\circ}_k(\mathrm{Gal}_K).$$

For $\mathbf{M} \in \underline{\mathrm{MF}}^{\mathtt{proj}}_{\mathcal{O}_K,\mathcal{O}_L}$ *we have*

$$\mathrm{HT}_\tau(\mathtt{G}_K(\mathbf{M}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell) = \mathrm{FL}_\tau(\mathbf{M} \otimes_{\mathcal{O}_L} k)$$

for all $\tau \in \mathbb{E}_K$.

A representation ρ as in part 2. will be called *crystalline in the Fontaine-Laffaille range* or *FL-crystalline.*

Proof. See [FL82], [CHT08, Section 2.4.1] or [BLGGT14, Section 1.4]. □

Remark 4.4. The functor $\mathtt{G}_K$ is compatible with the tensor product in the following sense [DFG04, p. 670]: If $\mathbf{M}, \mathbf{N}$ are as in Definition 4.2, we can define a filtered module $\mathbf{M} \otimes_{\mathcal{O}_L} \mathbf{N}$ by taking the m-th filtration step as

$$\mathrm{Fil}^m(M \otimes_{\mathcal{O}_L} N) := \mathrm{im}\Big(\bigoplus_{i+j=m} \mathrm{Fil}^i M \otimes_{\mathcal{O}_L} \mathrm{Fil}^j N \to M \otimes_{\mathcal{O}_L} N\Big).$$

Then, if $\mathbf{M} \otimes_{\mathcal{O}_L} \mathbf{N} \in \underline{\mathrm{MF}}_{\mathcal{O}_K,\mathcal{O}_L}$, we have

$$\mathtt{G}_K(\mathbf{M} \otimes_{\mathcal{O}_L} \mathbf{N}) \cong \mathtt{G}_K(\mathbf{M}) \otimes_{\mathcal{O}_L} \mathtt{G}_K(\mathbf{N}).$$

Remark that the requirement $\mathbf{M} \otimes_{\mathcal{O}_L} \mathbf{N} \in \underline{\mathrm{MF}}_{\mathcal{O}_K,\mathcal{O}_L}$ boils down to the following condition on the vanishing of the filtration steps: $\mathrm{Fil}^m(\mathbf{M} \otimes_{\mathcal{O}_L} \mathbf{N}) = 0$ holds for all $m \geq \ell - 1$. Thus, by parts 2. and 4. of Theorem 4.3, we can state this compatibility in the following, equivalent form: Assume that $V, W \in \underline{\mathrm{Rep}}^{\circ}_{\mathcal{O}_L}(\mathrm{Gal}_K)$ are FL-crystalline and assume that $V \otimes_{\mathcal{O}_L} W$ has Hodge-Tate weights in the Fontaine-Laffaille range $[0, \ell-2]$. Then $V \otimes_{\mathcal{O}_L} W$ is FL-crystalline. In other words, the property of being FL-crystalline is stable with respect to taking tensor products, as long as the Hodge-Tate weights stay in the Fontaine-Laffaille range.

Proposition 4.5. *Morphisms in* $\underline{\mathrm{MF}}_{\mathcal{O}_K,k}$ *are strict with filtrations: Let* $\mathbf{M}, \mathbf{N} \in \underline{\mathrm{MF}}_{\mathcal{O}_K,k}$ *and let* $f \in \mathrm{Hom}_{\underline{\mathrm{MF}}_{\mathcal{O}_K,k}}(\mathbf{M}, \mathbf{N})$, *then*

$$f(\mathrm{Fil}^i M) = f(M) \cap \mathrm{Fil}^i N$$

for all $i \in \mathbb{Z}$.

Proof. In general, a morphism f in an additive category with filtered objects is strict if and only if the canonical morphism

$$\mathrm{coim}(f) \to \mathrm{im}(f)$$

is an isomorphism [CZGT14, Section 3.2.1.3]. Therefore the claim follows from the abelianness of $\underline{\mathrm{MF}}_{\mathcal{O}_K,k}$, cf. e.g. [Alu09, Proof of Theorem 1.12 on p. 573]. (Abelianness follows from [FL82, 1.10]. Alternatively we can use the embedding

$$\underline{\mathrm{MF}}_{\mathcal{O}_K,k} \overset{\text{full}}{\subset} \underline{\mathrm{MF}}^{\circ}_{\mathcal{O}_K,\mathcal{O}_L},$$

and refer to [GL14, Section 2.2] for the abelianness of $\underline{\mathrm{MF}}^{\circ}_{\mathcal{O}_K,\mathcal{O}_L}$ and conclude that also $\underline{\mathrm{MF}}_{\mathcal{O}_K,k}$ must be abelian using [Rot79, Proposition 5.92].) □

For $\mathbf{M} \in \underline{\mathrm{MF}}_{\mathcal{O}_K,k}$ and $\tau \in \mathbb{E}_K$, we get a decreasing filtration

$$\ldots \supset M_i^\tau \supset M_{i+1}^\tau \supset \ldots \qquad (i \in \mathbb{Z}) \tag{4.2}$$

of M^τ, where

$$M_i^\tau := \mathrm{Fil}^i M \otimes_{\mathcal{O}_K \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L, \tau\otimes 1} \mathcal{O}_L \text{ and } M^\tau := M \otimes_{\mathcal{O}_K \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L, \tau\otimes 1} \mathcal{O}_L$$

arise from the homomorphism $\tau \otimes 1 : \mathcal{O}_K \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L \to \mathcal{O}_L$ (which uses $K \subset L$). As explained before, the jumps in this filtration correspond to the entries of the multiset $\mathrm{FL}_\tau(\mathbf{M})$. Moreover (cf. [CHT08, proof of Corollary 2.4.3]),

$$M = \bigoplus_{\tau\in\mathbb{E}_K} M^\tau \text{ and } M_i = \bigoplus_{\tau\in\mathbb{E}_K} M_i^\tau$$

and morphisms respect this decomposition: Let $f : \mathbf{M} \to \mathbf{N}$, then

$$f = \bigoplus_{\tau\in\mathbb{E}_K} f^\tau \quad \text{ with } f^\tau := f|M^\tau : M^\tau \to N^\tau.$$

Then each f^τ respects the filtration (4.2) and it follows from Proposition 4.5 that it does so strictly:

$$f^\tau(M_i^\tau) = f^\tau(M^\tau) \cap N_i^\tau. \tag{4.3}$$

Thus we get:

Proposition 4.6. *Let $\mathbf{M}, \mathbf{N} \in \underline{\mathrm{MF}}_{\mathcal{O}_K,k}$ such that for all $\tau \in \mathbb{E}_K$ we have*

$$\mathrm{FL}_\tau(\mathbf{M}) \cap \mathrm{FL}_\tau(\mathbf{N}) = \emptyset. \tag{4.4}$$

Then $\mathrm{Hom}_{\underline{\mathrm{MF}}_{\mathcal{O}_K,k}}(\mathbf{M}, \mathbf{N}) = 0$.

Proof. Let $f \in \mathrm{Hom}_{\underline{\mathrm{MF}}_{\mathcal{O}_K,k}}(\mathbf{M}, \mathbf{N})$. By (4.3), we are clearly done if we can show

$$f^\tau(M_i^\tau) = f^\tau(M_{i+1}^\tau) \tag{4.5}$$

for all $i \in \mathbb{Z}, \tau \in \mathbb{E}_K$: If this is the case, then

$$f^\tau(M^\tau) = f^\tau(M_0^\tau) = f^\tau(M_{\ell-1}^\tau) = f^\tau(0) = 0,$$

hence $f = \oplus_{\tau\in\mathbb{E}_K} 0 = 0$.

For i, τ with $M_i^\tau = M_{i+1}^\tau$, equation (4.5) holds trivially. For i, τ with $M_i^\tau \supsetneq M_{i+1}^\tau$ our assumption (4.4) on the filtration jumps implies $N_i^\tau = N_{i+1}^\tau$. Thus we can use (4.3) to conclude

$$f^\tau(M_i^\tau) = f^\tau(M^\tau) \cap N_i^\tau = f^\tau(M^\tau) \cap N_{i+1}^\tau = f^\tau(M_{i+1}^\tau). \qquad \square$$

Corollary 4.7. *Let K and L be finite field extensions of $\mathbb{Q}_\ell$ and assume that K is unramified (but we do not impose Assumption 4.1). Let*

$$\rho : \mathrm{Gal}_K \longrightarrow \mathrm{GL}_n(L)$$

be a crystalline representation and assume that

1. *There exists an $\alpha \in \mathbb{Z}$ such that all Hodge-Tate weights of ρ lie in the range $[\alpha, \alpha + \ell - 3]$;*

2. The Hodge-Tate weights of ρ are non-consecutive: if $\tau \in \mathbb{E}_K$ and two numbers a, b occur in $\mathrm{HT}_\tau(\rho)$, then either $a = b$ or $|a-b| \geq 2$.

Then

$$H^2(K, \mathrm{ad}\,\overline{\rho}) = 0.$$

Proof. Because the assertion is invariant under any change of the finite coefficient field $k \rightsquigarrow k'$, we may apply coefficient change to L (by replacing ρ by $\rho \otimes_L L'$) and then assume that L satisfies Assumption 4.1 and the "sufficiently ramified"-hypothesis of Lemma 4.10 below.

Making use of Lemma 2.71, we are good if we can show that

$$\mathrm{Hom}_{\mathrm{Gal}_K}(\overline{\rho}, \overline{\rho}(1))$$

vanishes. Because

$$\mathrm{Hom}_{\mathrm{Gal}_K}(\overline{\rho}, \overline{\rho}(1)) = \mathrm{Hom}_{\mathrm{Gal}_K}(\overline{\rho}(1-\alpha), \overline{\rho}(2-\alpha))$$

we can assume without loss of generality that $\alpha = 1$.

Let Λ be a Gal_K-stable $\mathcal{O}_L$-lattice in ρ and recall that $\overline{\rho}$ is defined as the semi-simplification of the reduction $\Lambda/\varpi_L.\Lambda$ of Λ. By Lemma 4.10 (postponed to the end of this section), we can choose Λ in a way such that $\Lambda/\varpi_L.\Lambda$ is already semisimple. By our first assumption that all weights of ρ lie in the range $[1, \ell-2]$ it thus follows from Theorem 4.3, parts 1. and 2., that $\overline{\rho}$ is of the form $\mathtt{G}_K(\mathbf{M})$ for a suitable $\mathbf{M} \in \underline{\mathrm{MF}}_{\mathcal{O}_K,k}$. By the same argument, $\overline{\rho}(1) = \mathtt{G}_K(\mathbf{N})$ for a suitable $\mathbf{N} \in \underline{\mathrm{MF}}_{\mathcal{O}_K,k}$.

Using Theorem 4.3, part 4, and the fact that twisting by the cyclotomic character shifts the Hodge-Tate numbers by -1, we see that our second condition on the weights of ρ translates precisely to condition (4.4) of Proposition 4.6. Thus, using fully faithfulness of $\mathtt{G}_K$, we get

$$0 = \mathrm{Hom}_{\underline{\mathrm{MF}}_{\mathcal{O}_K,k}}(\mathbf{M}, \mathbf{N}) \cong \mathrm{Hom}_{\mathrm{Gal}_K}(\overline{\rho}, \overline{\rho}(1)). \qquad \square$$

Example 4.8. Let $f = \sum_i a_i q^i$ be a newform of some weight $k \geq 2$ and level N as considered in [Wes04, Example 4.3]. Let $E|\mathbb{Q}$ be a finite extension which contains all Hecke eigenvalues of f and fix a place λ of E. Assume moreover that $\ell(\lambda)$ does not divide N and $\ell(\lambda) > k+1$. Then the associated representation

$$\rho_{f,\lambda} : \mathrm{Gal}_{\mathbb{Q}_\ell} \longrightarrow \mathrm{GL}_2(E_\lambda)$$

is crystalline with Hodge-Tate weights $0, k-1$ and Corollary 4.7 yields an alternative proof of [Wes04, Proposition 4.4]:

$$H^2(\mathbb{Q}_\ell, \mathrm{ad}\,\overline{\rho}_{f,\lambda}) = 0.$$

Remark 4.9. Let $A \in \mathcal{C}_{\mathcal{O}_L}$ and $M \in \underline{\mathrm{Rep}}^\circ_A(\mathrm{Gal}_K)$. Via the canonical map $\mathcal{O}_L \to A$ we can understand M as an $\mathcal{O}_L$-module. In this way, we can talk about M being "FL-crystalline" or "in the image of the functor $\mathtt{G}_K$" even if A is not the ring of integers of a finite extension of $\mathbb{Q}_\ell$. On the other hand, we can consider a subcategory $\underline{\mathrm{MF}}_{\mathcal{O}_K,A} \subset \underline{\mathrm{MF}}_{\mathcal{O}_K,\mathcal{O}_L}$ consisting of $\mathcal{O}_K \otimes_{\mathbb{Z}_\ell} A$-modules of finite type together with additional data analogous to Definition 4.2. The embedding of categories is again via the canonical map $\mathcal{O}_L \to A$. The essential image of the restriction of $\mathtt{G}_K$ to $\underline{\mathrm{MF}}_{\mathcal{O}_K,A}$ consists then precisely of the FL-crystalline objects of $\underline{\mathrm{Rep}}^\circ_A(\mathrm{Gal}_K)$. An analogue of Theorem 4.3 holds for this restricted functor.

On the existence of a suitable lattice Let $(L, \mathcal{O}, k)$ denote an ℓ-modular system, i.e. $\mathcal{O}$ is a discrete valuation ring with uniformizing element ϖ, maximal ideal $\mathfrak{m} = (\varpi)$, field of fractions L and residue field $k = \mathcal{O}/\mathfrak{m}$ of characteristic ℓ. Let

$$\rho : \Gamma \to \mathrm{GL}_n(L)$$

be a representation of a compact group Γ. It is well-known (and we already used this several times) that there exists a Γ-invariant $\mathcal{O}$-lattice $\Lambda \subset L^n$ and that the semi-simplification of the reduction

$$\overline{\rho}_\Lambda : \Gamma \curvearrowright \Lambda/\varpi.\Lambda \cong k^n$$

does not depend on the choice of Λ (as a corollary of the Brauer-Nesbitt Theorem). In our notation, $\overline{\rho} = \overline{\rho}_\Lambda^{\mathsf{ss}}$. If $m_{\bar{\rho}}$ denotes the length of the Γ-module $\overline{\rho}$, we say that L is *sufficiently ramified* for ρ if there exists a subfield $L^\star \subset L$, such that

La.1) there exists a representation
$$\rho^\star : \Gamma \to \mathrm{GL}_n(L^\star),$$
such that $\iota^n_{L^\star|L} \circ \rho^\star = \rho$;

La.2) the extension $L|L^\star$ is totally ramified of degree $m_{\bar{\rho}}$ (so we can fix a uniformizer $\varpi^\star$ of $\mathcal{O}_{L^\star}$ for which we can assume $\varpi^\star = \varpi^{m_{\bar{\rho}}}$);

La.3) $m_{\bar{\rho}} = m_{\overline{\rho^\star}}$.

Lemma 4.10. *Assume that L is sufficiently ramified for ρ, then we can chose a lattice Λ such that $\overline{\rho}_\Lambda$ is semisimple.*

This fact is essentially well known (cf. the closely related result [Fei82, Lemma 18.2], and the usage at the end of Section 2.6 in [Böc13a]). However, in lack of a citeable reference, we include a proof which is based on [Dat05, proof of Lemma 6.11]. We also remark that condition La.3 was added to technically simplify the proof and can certainly be weakened (but this would offer no additional benefits for our purposes).

Proof. Let us first write
$$\overline{\rho} = \oplus_{i=1}^{m_{\bar{\rho}}} \overline{\sigma}_i,$$
where each $\overline{\sigma}_i$ is a d_i-dimensional irreducible representation of Γ. Let $\Lambda^\star \subset (K^\star)^n$ be a Γ-stable $\mathcal{O}_{L^\star}$-lattice. By the Brauer-Nesbitt theorem (and La.3), we can assume (up to rearranging the components $\overline{\sigma}_i$) that there is a basis $\mathbf{B}^\star = (e_1^\star, \dots, e_n^\star)$ of $\Lambda^\star$, such that $\rho^\star$ factors through the standard parahoric subgroup associated to the partition $(d_1, \dots, d_{m_{\bar{\rho}}})$ of n (see e.g. [Gui13, Section 2.2]). We make this explicit: For $0 \le j \le m_{\bar{\rho}}$ denote $\langle j \rangle = \sum_{i=1}^{j} d_j$ (with $\langle 0 \rangle := 0$). Then our choice of $\mathbf{B}^\star$ is such that for i with $\langle r-1 \rangle \le i < \langle r \rangle$ we have

$$\rho(\gamma)(e_i^\star) \in \bigoplus_{j=1}^{\langle r \rangle} \mathcal{O}_{L^\star}.e_j^\star \oplus \bigoplus_{j=\langle r \rangle+1}^{n} \mathcal{O}_{L^\star}.\varpi^\star.e_j^\star \qquad \text{for all } \gamma \in \Gamma.$$

Now consider the $\mathcal{O}_L$-lattice $\Lambda \subset L^n$ spanned by

$$\mathbf{B} = (e_1, \dots, e_n) := \left(\frac{1}{\varpi^{m_{\bar{\rho}}-1}}.e_1^\star, \dots, \frac{1}{\varpi^{m_{\bar{\rho}}-1}}.e_{\langle 1 \rangle}^\star, \frac{1}{\varpi^{m_{\bar{\rho}}-2}}.e_{\langle 1 \rangle+1}^\star, \dots, \frac{1}{\varpi^{m_{\bar{\rho}}-2}}.e_{\langle 2 \rangle}^\star, \dots, e_n^\star\right).$$

It is a straight-forward computation (using crucially property La.2) to check the following: For i with $\langle r-1\rangle \leq i < \langle r\rangle$ we have

$$\rho(\gamma)(e_i) \in \bigoplus_{j=\langle r-1\rangle+1}^{\langle r\rangle} \mathcal{O}_L.e_j \oplus \bigoplus_{j\notin\{\langle r-1\rangle+1,\dots,\langle r\rangle\}} \mathcal{O}_L.\varpi.e_j \qquad \text{for all } \gamma\in\Gamma.$$

This implies the claim. □

Observe that for given ρ and L, the preconditions of this lemma can always be achieved after a totally ramified, finite coefficient base change, i.e. after (if necessary, repeatedly until La.3 is fulfilled) adjoining a suitable root of a uniformizer to L.

4.2 $\ell = p$: Unconditioned deformations

Let Λ be the ring of integers of a finite extension of $\mathrm{Quot}(W(k))$ such that $k_\Lambda = k$.

Lemma 4.11. *Assume that $\rho : \mathrm{Gal}_K \to \mathrm{GL}_n(L)$ fulfills the conditions of Corollary 4.7. Then*

$$R^\square_\Lambda(\overline{\rho}) \cong \Lambda[[x_1,\dots,x_m]]$$

with $m = n^2.([K:\mathbb{Q}_\ell]+1)$.

Proof. As $H^2(K,\mathrm{ad}\,\overline{\rho})$ vanishes (Corollary 4.7), this follows from Proposition 2.52 (part 2) and Theorem 2.53 with

$$m = h^1(K,\mathrm{ad}\,\overline{\rho}) + \dim\mathfrak{g} - \dim\mathfrak{z} = h^1(K,\mathrm{ad}\,\overline{\rho}) + (n^2-1).$$

After replacing $h^1(K,\mathrm{ad}\,\overline{\rho})$ by $h^0(K,\mathrm{ad}\,\overline{\rho}) + h^2(K,\mathrm{ad}\,\overline{\rho}) - \chi(K,\mathrm{ad}\,\overline{\rho})$, the claim becomes a simple consequence of the local Euler-Poincaré formula [Böc13a, Chapter 5.3] and condition **(Centr)**:

$$m = h^0(K,\mathrm{ad}\,\overline{\rho})+h^2(K,\mathrm{ad}\,\overline{\rho})-\chi(K,\mathrm{ad}\,\overline{\rho})+(n^2-1) = 1+0+n^2.[K:\mathbb{Q}_\ell]+(n^2-1) = n^2.([K:\mathbb{Q}_\ell]+1).$$

□

There is a variation for the fixed determinant deformation ring, which we will only formulate in the case $\Lambda = W := W(k)$:

Corollary 4.12. *Retain all notation from above and fix a lift χ of the determinant. Then*

$$R^{\square,\chi}(\overline{\rho}) \cong W[[x_1,\dots,x_{m'}]]$$

with $m' = n^2.[K:\mathbb{Q}_\ell]$.

Proof. By [Böc98, Proposition 2.1], we have an isomorphism

$$R^\square(\overline{\rho}) \cong R^{\square,\chi}(\overline{\rho})\hat{\otimes}_W W[[x]].$$

Thus, by Lemma 4.11, we have

$$R^{\square,\chi}(\overline{\rho})[[x]] = R^{\square,\chi}(\overline{\rho})\hat{\otimes}_W W[[x]] \cong W[[x_1,\dots,x_m]]. \tag{4.6}$$

But this implies $R^{\square,\chi}(\overline{\rho}) \cong W[[x_1,\dots,x_{m-1}]]$ by Lemma 2.19. □

4.3 $\ell = p$: Crystalline deformations

Consider again a representation $\rho : \mathrm{Gal}_K \to \mathrm{GL}_n(L)$ which fulfills the conditions of Corollary 4.7. We will also make the additional regularity assumption that all occurring Hodge-Tate weights of $\overline{\rho}$ have multiplicity one. We will consider the deformation problem $\mathtt{crys}$ of $\overline{\rho}$ consisting of those lifts $\widetilde{\rho} : \mathrm{Gal}_K \to \mathrm{GL}_n(A)$ of $\overline{\rho}$ for which $\widetilde{\rho} \otimes_A A'$ lies in the essential image of $\mathtt{G}_K$ for all Artinian quotients A' of A (cf. [CHT08], Section 2.4.1). We refer to those lifts as *(FL)-crystalline lifts* of $\overline{\rho}$.

That $\mathtt{crys}$ defines a lifting condition in the sense of Definition 2.30 follows from the Ramakrishna framework[10]: We already remarked that the essential image of $\mathtt{G}_K$ is closed under subobjects and quotients (Theorem 4.3). That the essential image is closed under direct sums follows immediately from the exactness of $\mathtt{G}_K$, since then $\mathtt{G}_K$ preserves direct sums (see [Fre64, Theorem 3.12$^{(*)}$]). Thus Proposition 2.39 and Corollary 2.35 yield the following lemma:

Lemma 4.13. *Let Λ be the ring of integers of a finite extension E of $\mathrm{Quot}(W(k))$, such that $k_\Lambda = k$. Let Λ' be the ring of integers of a finite extension of E. Denote the residue field of Λ' by k' and set $\overline{\rho}' = \iota_{k'|k} \circ \overline{\rho}$. Then:*

1. *The functor $D_\Lambda^{\square,\mathtt{crys}}(\overline{\rho})$ is representable by a quotient $R_\Lambda^{\square,\mathtt{crys}}(\overline{\rho})$ of $R_\Lambda^{\square}(\overline{\rho})$.*

2. *The functor $D_{\Lambda'}^{\square,\mathtt{crys}}(\overline{\rho}')$ is representable by*

$$R_{\Lambda'}^{\square,\mathtt{crys}}(\overline{\rho}') \cong \Lambda' \otimes_\Lambda R_\Lambda^{\square,\mathtt{crys}}(\overline{\rho}).$$

Lemma 4.14. *Under the above hypotheses,*

$$R_\Lambda^{\square,\mathtt{crys}}(\overline{\rho}) \cong \Lambda[[x_1, \ldots, x_m]]$$

with $m = n^2 + [K : \mathbb{Q}_\ell]\frac{n.(n-1)}{2}$.

Proof. This is a part of the statement of [CHT08, Corollary 2.4.3]. □

Let us also note the following useful compatibility with base change:

Lemma 4.15. *Let K' be a finite unramified extension of K with associated inclusion map $\iota_{K'|K} : \mathrm{Gal}_{K'} \to \mathrm{Gal}_K$. Set $\overline{\rho}' = \overline{\rho} \circ \iota_{K'|K}$. Let $\widetilde{\rho}$ be a crystalline lift of $\overline{\rho}$. Then the following holds:*

1. *$\widetilde{\rho}' = \widetilde{\rho} \circ \iota_{K'|K}$ is a crystalline lift of $\overline{\rho}'$.*

   In particular, the restriction map $\mathrm{res} : H^1(K, \mathrm{ad}\,\overline{\rho}) \to H^1(K', \mathrm{ad}\,\overline{\rho}')$ maps the tangent subspace associated to the crystalline deformation condition for $\overline{\rho}$ into the tangent subspace associated to the crystalline deformation condition for $\overline{\rho}'$.

2. *The corestriction map $\mathrm{cor} : H^1(K', \mathrm{ad}\,\overline{\rho}') \to H^1(K, \mathrm{ad}\,\overline{\rho})$ maps the tangent subspace associated to the crystalline deformation condition for $\overline{\rho}'$ into the tangent subspace associated to the crystalline deformation condition for $\overline{\rho}$.*

[10] This was already noticed in [CHT08] (see the remark preceding Lemma 2.4.1), albeit without explanation.

Proof. The first part is a direct consequence of the following compatibility of the Fontaine-Laffaille functor with base change: Let $\mathbf{M} \in \underline{\mathrm{MF}}_{\mathcal{O}_K,\mathcal{O}_L}$, then $\mathcal{O}_{K'} \otimes_{\mathcal{O}_K} \mathbf{M}$ defines an object of $\underline{\mathrm{MF}}_{\mathcal{O}_{K'},\mathcal{O}_L}$. It follows from the definition of the functors $\mathtt{G}_K, \mathtt{G}_{K'}$ and a calculation analogous to the one in Section 3.11 of [FL82] that $\mathtt{G}_K(\mathbf{M})$ and $\mathtt{G}_{K'}(\mathcal{O}_{K'} \otimes_{\mathcal{O}_K} \mathbf{M})$ are isomorphic as $\mathcal{O}_L$-modules and that this isomorphism commutes with the action of $\mathrm{Gal}_{K'}$. In other words,

$$r^K_{K'}\big(\mathtt{G}_K(\mathbf{M})\big) \cong \mathtt{G}_{K'}(\mathcal{O}_{K'} \otimes_{\mathcal{O}_K} \mathbf{M}),$$

where $r^K_{K'}$ denotes the restriction to $\mathrm{Gal}_{K'}$.

For the second part, we need the following assertions:

- (The induction functor respects the property of being FL-crystalline.) Let $\rho_0 : \mathrm{Gal}_{K'} \to \mathrm{GL}_n(L)$ be an FL-crystalline representation. Then $\bar{\rho}_0$ is in the essential image of $\mathtt{G}_{K'}$. Then we claim that

  $$\mathrm{ind}^K_{K'} \bar{\rho}_0 : \mathrm{Gal}_K \to \mathrm{GL}_{n.[K':K]}(k)$$

  is in the essential image of $\mathtt{G}_K$. There are several ways to see this. By definition, $(\rho_0 \otimes_{\mathbb{Q}_\ell} B_{\mathtt{crys}})^{\mathrm{Gal}_{K'}}$ is free over $K_0' \otimes_{\mathbb{Q}_\ell} L$ of rank n and has Hodge-Tate weights in the Fontaine-Laffaille range $[0, \ell-2]$. We see that

  $$(\mathrm{ind}^K_{K'} \rho_0 \otimes_{\mathbb{Q}_\ell} B_{\mathtt{crys}})^{\mathrm{Gal}_K} \cong (\rho_0 \otimes_{\mathbb{Q}_\ell} B_{\mathtt{crys}})^{\mathrm{Gal}_{K'}}$$

  as free $K_0' \otimes_{\mathbb{Q}_\ell} L$-modules. As $[K' : K] = [K_0' : K_0]$, it follows that $\mathrm{ind}^K_{K'} \rho_0$ is crystalline. A similar observation for the B_{HT}-filtration shows that the Hodge-Tate weights of $\mathrm{ind}^K_{K'} \rho_0$ are again in the Fontaine-Laffaille range, thus the claim follows from Theorem 4.3 and the fact that the reduction functor and the induction functor commute with each other. Alternatively we can explicitly describe the Fontaine-Laffaille module $\mathbf{M}_K = \mathtt{G}_K^{-1}(\mathrm{ind}^K_{K'} \rho_0)$ in terms of $\mathbf{M} = (M, (\mathrm{Fil}^i M)_{i\in\mathbb{Z}}, (\varphi_{M,i})_{i\in\mathbb{Z}}) = \mathtt{G}_{K'}^{-1}(\rho_0)$: We take $\mathbf{M}_K = (M', (\mathrm{Fil}^i M')_{i\in\mathbb{Z}}, (\varphi_{M',i})_{i\in\mathbb{Z}})$ with

  - $M' := M$ (understood as an $\mathcal{O}_K \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L$-module);
  - $\mathrm{Fil}^i M' := \mathrm{Fil}^i M$ (understood as $\mathcal{O}_K \otimes_{\mathbb{Z}_\ell} \mathcal{O}_L$-submodules of M');
  - $\varphi_{M',i} = \varphi_{M,i}$.

  For $\mathbf{N} \in \underline{\mathrm{MF}}_{\mathcal{O}_K,\mathcal{O}_L}$, we can check the Frobenius-like reciprocity

  $$\mathrm{Hom}_{\underline{\mathrm{MF}}_{\mathcal{O}_K,\mathcal{O}_L}}(\mathbf{M}_K, \mathbf{N}) \cong \mathrm{Hom}_{\underline{\mathrm{MF}}_{\mathcal{O}_{K'},\mathcal{O}_L}}(\mathbf{M}, \mathcal{O}_{K'} \otimes_{\mathcal{O}_K} \mathbf{N}),$$

  showing that the functor $\mathbf{M} \rightsquigarrow \mathbf{M}_K$ is left-adjoint to $\mathbf{N} \rightsquigarrow \mathcal{O}_{K'} \otimes_{\mathcal{O}_K} \mathbf{N}$. Using that the functor $\mathtt{G}_K$ (resp. $\mathtt{G}_{K'}$) establishes an equivalence between $\underline{\mathrm{MF}}_{\mathcal{O}_K,\mathcal{O}_L}$ and a full subcategory of $\underline{\mathrm{Rep}}_{\mathcal{O}_L}(\mathrm{Gal}_K)$ (resp. between $\underline{\mathrm{MF}}_{\mathcal{O}_{K'},\mathcal{O}_L}$ and a full subcategory of $\underline{\mathrm{Rep}}_{\mathcal{O}_L}(\mathrm{Gal}_{K'})$) and the adjointness relation between the induction- and restriction-functors on representations, the claim follows from the first part of this lemma.

- (Explicit characterization of the corestriction map.) We use the identifications

  $$H^1(K', \mathrm{ad}\,\overline{\rho}') \cong \mathrm{Ext}^1_{k[\mathrm{Gal}_{K'}]}(\overline{\rho}', \overline{\rho}') \text{ and } H^1(K, \mathrm{ad}\,\overline{\rho}) \cong \mathrm{Ext}^1_{k[\mathrm{Gal}_K]}(\overline{\rho}, \overline{\rho})$$

  and the corresponding characterization of the corestriction map as the concatenation

  $$\mathrm{Ext}^1_{k[\mathrm{Gal}_{K'}]}(\overline{\rho}', \overline{\rho}') \xrightarrow{\cong} \mathrm{Ext}^1_{k[\mathrm{Gal}_K]}(\overline{\rho}, \mathrm{ind}^K_{K'} \overline{\rho}') \longrightarrow \mathrm{Ext}^1_{k[\mathrm{Gal}_K]}(\overline{\rho}, \overline{\rho}), \tag{4.7}$$

where the first map is the isomorphism from the Eckman-Shapiro Lemma [Ben98, Cor. 2.8.4] and the second map is induced from

$$\mathrm{can} : \mathrm{ind}_{K'}^{K} \overline{\rho}' = k[\mathrm{Gal}_K] \otimes_{k[\mathrm{Gal}_{K'}]} \overline{\rho}' \longrightarrow \overline{\rho}, \qquad \sigma \otimes v \longmapsto \sigma v.$$

Now, as explained following the proof of Corollary 2.8.4 in [Ben98] (in the dual situation), the Eckman-Shapiro isomorphism can be explicitly characterized by sending an extension

$$0 \longrightarrow \overline{\rho}' \longrightarrow M \longrightarrow \overline{\rho}' \longrightarrow 0$$

to the extension

$$0 \longrightarrow \mathrm{ind}_{K'}^{K} \overline{\rho}' \longrightarrow X \longrightarrow \overline{\rho} \longrightarrow 0,$$

where X is the pullback as in the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{ind}_{K'}^{K} \overline{\rho}' & \longrightarrow & \mathrm{ind}_{K'}^{K} M & \longrightarrow & \mathrm{ind}_{K'}^{K} \overline{\rho}' & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow \scriptstyle{\mathrm{can}'} & & \\ 0 & \longrightarrow & \mathrm{ind}_{K'}^{K} \overline{\rho}' & \longrightarrow & X & \longrightarrow & \overline{\rho} & \longrightarrow & 0 \end{array} \tag{4.8}$$

Here, the vertical map on the right is defined as

$$\mathrm{can}' : \overline{\rho} \longrightarrow \mathrm{ind}_{K'}^{K} \overline{\rho}' = k[\mathrm{Gal}_K] \otimes_{k[\mathrm{Gal}_{K'}]} \overline{\rho}', \qquad v \longmapsto 1 \otimes v.$$

The map on the right hand of (4.7) now maps the extension X to the the pushout Y in the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{ind}_{K'}^{K} \overline{\rho}' & \longrightarrow & X & \longrightarrow & \overline{\rho} & \longrightarrow & 0 \\ & & \scriptstyle{\mathrm{can}} \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \overline{\rho} & \longrightarrow & Y & \longrightarrow & \overline{\rho} & \longrightarrow & 0 \end{array}$$

We can now complete the proof: Start with an $M \in \mathrm{Ext}^1_{k[\mathrm{Gal}_K]}(\overline{\rho}, \overline{\rho})$ which is FL-crystalline. By the first bullet point applied to $\rho_0 = \rho$, it follows that $\mathrm{ind}_{K'}^{K} \overline{\rho}'$ is FL-crystalline. Moreover, we know that the universal lifting ring of M is formally smooth over Λ (Lemma 4.14), so there exists a lift

$$\widetilde{M} : \mathrm{Gal}_{K'} \longrightarrow \mathrm{GL}_{2.\dim \overline{\rho}}(\Lambda)$$

of M. Applying the first bullet point to $\rho_0 = \widetilde{M} \otimes_{\mathcal{O}_L} L$, we see that $\mathrm{ind}_{K'}^{K} M$ is FL-crystalline. Thus, all objects in (4.8) (except for possibly X) are FL-crystalline. But the category $\underline{\mathrm{MF}}_{\mathcal{O}_K, \mathcal{O}_L}$ is abelian, hence it is closed under taking finite limits and colimits. It follows that also X must be FL-crystalline. The same argument applied to (4.8) shows that $Y = \mathrm{cor}(X)$ is FL-crystalline. □

4.4 $\ell \neq p$: Minimally ramified deformations

We continue to denote by Λ be the ring of integers of a finite extension of $\mathrm{Quot}(W(k))$ such that $k_\Lambda = k$, but this time we consider an (absolutely irreducible) residual representation

$$\overline{\rho} : \mathrm{Gal}_K \to \mathrm{GL}_n(k),$$

where K is a finite extension of $\mathbb{Q}_p$ with $p \neq \ell = \operatorname{char} k$. Denote by

$$\rho : \mathrm{Gal}_K \to \mathrm{GL}_n(A) \qquad (\text{ with } A \in \mathcal{C}_\Lambda\,)$$

a lift of $\overline{\rho}$. Let us shortly recall from [CHT08, Section 2.4.4], what it means for ρ to be minimally ramified: Let P_K denote the kernel of one (hence, any) surjection $I_K \twoheadrightarrow \mathbb{Z}_\ell$ and set $T_K = \mathrm{Gal}_K / P_K$. For an integer q coprime to ℓ, define the group

$$T_q := \mathbb{Z}_\ell \rtimes \hat{\mathbb{Z}},$$

where we denote by σ_q a generator of the factor $\mathbb{Z}_\ell$, by φ_q a generator of the factor $\hat{\mathbb{Z}}$, and where the semi-direct product is defined by $\varphi_q \sigma_q \varphi_q^{-1} = \sigma_q^q$. Then the sequence

$$0 \longrightarrow P_K \longrightarrow \mathrm{Gal}_K \longrightarrow T_K \longrightarrow 0 \tag{4.9}$$

splits (so that $\mathrm{Gal}_K \cong P_K \rtimes T_K$) and $T_K \cong T_{\#k}$.

Now let τ be an irreducible P_K-representation over k and set $d := \dim \tau$. Set

$$G_\tau = \{\sigma \in \mathrm{Gal}_K \,|\tau^\sigma \sim \tau\}, \quad T_\tau = G_\tau / P_K.$$

We have an isomorphism $\xi_\tau : T_\tau \cong T_{q(\tau)}$ (with $q(\tau) = (\#k)^{[\mathrm{Gal}_K : G_\tau . I_K]}$) and the splitting from (4.9) restricts to a splitting $T_\tau \hookrightarrow G_\tau$. It is shown in [CHT08, Lemma 2.4.11] that τ admits a unique lift $\widetilde{\tau} : P_K \to \mathrm{GL}_n(\Lambda)$. For M a finite Λ-module with a continuous action of Gal_K, we set

$$M_\tau = \mathrm{Hom}_{P_K}(\widetilde{\tau}, M)$$

and regard M_τ as a (continuous) T_τ-module. Finally, let $\Psi_{K,k}$ (or Ψ_K, if k is understood) denote the set of equivalence classes of irreducible P_K-representations over k.

Proposition 4.16. *For $A \in \mathcal{C}_\Lambda^\circ$, the association*

$$\rho \mapsto (\rho_\tau)_{[\tau]\in\Psi_K}$$

provides a bijection between the deformations of $\overline{\rho}$ (as a Gal_K-representation) to A and the tuples of deformations of $\overline{\rho}_\tau$ (as T_τ-representations) to A.

Proof. For a proof, we refer to [CHT08, Corollary 2.4.13]. We remark however that the representation ρ can be can be reconstructed from the tuple $(\rho_\tau)_{[\tau]\in\Psi_K}$ as

$$\rho = \bigoplus_{[\tau]\in\Psi_K} \mathrm{ind}_{G_\tau}^{\mathrm{Gal}_K}(\widetilde{\tau} \otimes \rho_\tau), \tag{4.10}$$

cf. [CHT08, Lemma 2.4.12]. □

Definition 4.17. 1. Let τ be in Ψ_K and let π be a lift of τ to $A \in \mathcal{C}_\Lambda$. Then π is *minimally ramified* if the natural map

$$\ker(\pi(\zeta_\tau) - \mathbf{1})^i \otimes_A k \longrightarrow \ker(\tau(\zeta_\tau) - \mathbf{1})^i$$

(with $\zeta_\tau = \xi_\tau^{-1}(\sigma_{q(\tau)})$) is an isomorphism for all $i \in \mathbb{N}$.

2. Let ρ be a lift of $\overline{\rho}$ to $A \in \mathcal{C}_\Lambda$. Then ρ is *minimally ramified* if ρ_τ is a minimally ramified lift of $\overline{\rho}_\tau$ for each $\tau \in \Psi_K$.

Remark 4.18. If $\overline{\rho}$ is unramified, then a lift ρ is minimally ramified if and only if it is unramified (see the remark in [CHT08] after Definition 2.4.14).

Remark 4.19. Let $A \in \mathcal{C}_\Lambda^\circ$, then a lift π of τ to A is minimally ramified in the sense of Definition 4.17.1 if and only if each $\ker(\pi(\zeta_\tau) - \mathbf{1})^i$ is free over A and

$$\mathrm{rk}_A \operatorname{im}(\pi(\zeta_\tau) - \mathbf{1})^i = \mathrm{length}(A).\, \mathrm{rk}_k \operatorname{im}(\tau(\zeta_\tau) - \mathbf{1})^i. \tag{4.11}$$

Moreover, we always have an inequality in the direction $\leq$, i.e. (4.11) can only fail via a *loss* of rank upon reduction (in the non-minimally ramified case). (This all follows from [CHT08, Lemma 2.4.15]).

For convenience, we will denote the set of all $\tau \in \Psi_K$ with $\overline{\rho}_\tau \neq 0$ by $\Delta_{\overline{\rho}}$. Consider the following assumption:

Assumption 4.20. Any $\tau \in \Delta_{\overline{\rho}}$ is absolutely irreducible.

(In Lemma 4.24 we will give a simple criterion from which we can deduce Assumption 4.20 in many situations.)

Now, let Λ' be the ring of integers of a finite extension of $\mathrm{Quot}(\Lambda)$ with residue field k' and denote the induced embedding by $\iota^n_{k'|k} : \mathrm{GL}_n(k) \hookrightarrow \mathrm{GL}_n(k')$. This gives rise to a residual representation

$$\overline{\rho}' = \iota^n_{k'|k} \circ \overline{\rho} : \mathrm{Gal}_K \to \mathrm{GL}_n(k').$$

Similarly, if $A \in \mathcal{C}_\Lambda, A' \in \mathcal{C}_{\Lambda'}$ and $A \hookrightarrow A'$ is an inclusion which induces the inclusion of k into k' as above, we get an embedding $\iota^n_{A'|A} : \mathrm{GL}_n(A) \hookrightarrow \mathrm{GL}_n(A')$. Thus, if ρ is an A-valued lift of $\overline{\rho}$ as above, we can define

$$\rho' = \iota^n_{A'|A} \circ \rho : \mathrm{Gal}_K \to \mathrm{GL}_n(A')$$

which then is a lift of $\overline{\rho}'$. Thanks to Assumption 4.20, the sets $\Delta_{\overline{\rho}}$ and $\Delta_{\overline{\rho}'}$ are in correspondence via $[\tau] \leftrightarrow [\tau']$ with $\tau' = \iota^{\dim \tau}_{k'|k} \circ \tau$. Moreover, as is easily extractable from Section 2.4.4 of [CHT08], the tuples $(\rho_\tau)_{[\tau] \in \Delta_{\overline{\rho}}}$ and $(\rho'_{\tau'})_{[\tau'] \in \Delta_{\overline{\rho}'}}$ correspond to each other via $\rho'_{\tau'} = \iota^{\dim(\rho_\tau)}_{A'|A} \circ \rho_\tau$.

We have

Proposition 4.21. *Under Assumption 4.20, ρ is a minimally ramified lift of $\overline{\rho}$ if and only if ρ' is a minimally ramified lift of $\overline{\rho}'$.*

Proof. By Assumption 4.20, we have $\Psi_{K,k} = \Psi_{K,k'}$, i.e. no difficulties arise in terms of irreducible τ becoming reducible as we go from k to k'. We write this bijection as $\tau \leftrightarrow \tau'$, i.e. we write τ' instead of τ when we consider $\tau \in \Psi_{K,k}$ as an element of $\Psi_{K,k'}$. Moreover, for dimension reasons (cf. (4.10)), we then also have an equality $\Delta_{\overline{\rho}} = \Delta_{\overline{\rho}'}$.

We will check the claim by checking the first part of Definition 4.17 for any $\tau \in \Delta_{\overline{\rho}}$. For this, write $X_\tau = (\rho_\tau(\zeta_\tau) - \mathbf{1})$ which we consider as an element of $\mathbb{M}_{\dim \rho_\tau, \dim \rho_\tau}(A)$ and $X_{\tau'} = (\rho'_{\tau'}(\zeta_{\tau'}) - \mathbf{1})$ for X_τ if considered as an element of $\mathbb{M}_{\dim \rho_\tau, \dim \rho_\tau}(A')$. Write $\overline{X}_\tau$ (resp. $\overline{X}_{\tau'}$) for the reduction of X_τ (resp. of $X_{\tau'}$), which we consider as an element of $\mathbb{M}_{\dim \rho_\tau, \dim \rho_\tau}(k)$ (resp. $\mathbb{M}_{\dim \rho_\tau, \dim \rho_\tau}(k')$). The claim becomes obvious when we write down the commuting diagram of k'-vector spaces

$$\begin{array}{ccc} \ker(X^i_{\tau'}) \otimes_{A'} k' & \longrightarrow & \ker(\overline{X}^i_{\tau'}) \\ \cong \Big| & & \Big| \cong \\ (\ker(X^i_\tau) \otimes_A k) \otimes_k k' & \longrightarrow & \ker(\overline{X}^i_\tau) \otimes_k k'. \end{array}$$

The vertical arrows are isomorphisms by construction of ρ from ρ'. Now we use the following simple fact: Let V, W be k-vector spaces together with a k-linear map $f : V \to W$. Then f is an isomorphism if and only if $f \otimes_k k' : V \otimes_k k' \to W \otimes_k k'$ is an isomorphism. $\square$

We have the following consequence of Proposition 4.21:

Lemma 4.22. *Let $\overline{\rho}$ fulfill Assumption 4.20. Then the condition of being minimally ramified defines a lifting condition, denoted* `min`*. Moreover, we have*

$$R^{\square,\mathtt{min}}_{\Lambda'}(\overline{\rho}') \cong \Lambda' \otimes_\Lambda R^{\square,\mathtt{min}}_{\Lambda}(\overline{\rho}). \tag{4.12}$$

Proof. It suffices to prove that `min` defines a lifting condition in the sense of Definition 2.30, as the isomorphism (4.12) is then a direct consequence of Corollary 2.35. That `min` defines a lifting condition on $\mathcal{C}_\Lambda$ follows from [CHT08, Corollary 2.4.18], and by Observation 2.34 this extends to a lifting condition, say ${}^\star\mathcal{D}$, on ${}^\star\mathcal{C}_\Lambda$. It remains to show that ${}^\star\mathcal{D}$ indeed parametrized minimally ramified lifts of $\overline{\rho}$ to ${}^\star\mathcal{C}_\Lambda$. For this, let $A' \in {}^\star\mathcal{C}_\Lambda$ with residue field k' and consider a lift ρ' of $\overline{\rho}$ to A' which is in the image of ${}^\star\mathcal{D}$, i.e. such that the corresponding map

$$\varphi_{\rho'} : R^{\square}_{\Lambda}(\overline{\rho}) \to A'$$

factors through $R^{\square,\mathtt{min}}_{\Lambda}(\overline{\rho})$. We have to show that ρ' is a minimally ramified lift of $\overline{\rho}$ (i.e. of $\overline{\rho}' = \iota^n_{k'|k} \circ \overline{\rho}$). If there exists an $A \in \mathcal{C}_\Lambda$, an embedding $\psi : A \hookrightarrow A'$ which is a morphism of ${}^\star\mathcal{C}_\Lambda$ and a lift ρ of $\overline{\rho}'$ to A such that $\rho' = \iota^n_{A'|A} \circ \rho$, then this is clear: The map corresponding to ρ,

$$\varphi_\rho : R^{\square}_{\Lambda}(\overline{\rho}) \to A,$$

fits in the equation $\psi \circ \varphi_\rho = \varphi_{\rho'}$, so φ_ρ also factors through $R^{\square,\mathtt{min}}_{\Lambda}(\overline{\rho})$. Hence, by definition, ρ is a minimally ramified lift of $\overline{\rho}$ and the claim follows from Proposition 4.21. We now claim that there always exist such an A and a lift ρ: First, it is sufficient to consider the case where A is Artinian, cf. Remark 2.23. Now define $A := p^{-1}(k)$, where $p : A' \to k'$ is the projection map. As A is the pullback of the diagram

$$\begin{array}{ccc} & & A' \\ & & \big\downarrow{\scriptstyle p} \\ k & \hookrightarrow & k' \end{array}$$

and as ${}^\star\mathcal{C}^\circ_\Lambda$ is closed with respect to pullbacks (Remark 2.3), we see that A is an object of $\mathcal{C}_\Lambda$. Moreover $\rho := \rho'$ clearly has values in A and fulfills $\rho' = \iota^n_{A'|A} \circ \rho$. The claim follows. $\square$

Lemma 4.23.

$$R^{\square,\mathtt{min}}_{\Lambda}(\overline{\rho}) \cong \Lambda[[X_1, \ldots, X_{n^2}]].$$

Proof. This is part of the statement of [CHT08, Corollary 2.4.21]. $\square$

We will now give a criterion for Assumption 4.20 to hold. For this, as the image of $\overline{\rho}$ is finite, we can understand $\mathrm{res}^{\mathrm{Gal}_K}_{P_K}(\overline{\rho})$ as a representation of a finite quotient $\mathfrak{G}$ of P_K. Let us write the exponent of $\mathfrak{G}$ in the form $\exp(\mathfrak{G}) = \ell^a.m$ with $(\ell, m) = 1$.

Lemma 4.24. *Assume that k contains all m-th roots of unity. Then Assumption 4.20 is fulfilled.*

Proof. Under the above assumptions, a theorem of Brauer (see [DH92], Corollary (5.21) and the preceding remarks) guarantees that k is a splitting field for $\mathfrak{G}$, i.e. that a k-valued representation of $\mathfrak{G}$ is irreducible if and only if it is absolutely irreducible. $\square$

4.4.1 Unipotent ramification and fixed-type lifting rings

During this paragraph, we will study the case where $\overline{\rho}$ fulfills the following condition:

Definition 4.25. We say that $\overline{\rho}$ has *unipotent ramification* if $\overline{\rho}(P_K)$ is trivial.

Remark 4.26. This notion is explained by the following observation: $\overline{\rho}$ has unipotent ramification if and only if $\overline{\rho}|I_K$ has values in a conjugate of the standard unipotent subgroup

$$U_n(k) = \begin{pmatrix} 1 & * & * & * & \cdots \\ & 1 & * & * & \cdots \\ & & \ddots & & \\ & & & 1 & * \\ & & & & 1 \end{pmatrix} \subset \mathrm{GL}_n(k).$$

Clearly, if $\overline{\rho}$ is unipotently ramified we have $\Delta_{\overline{\rho}} = \{\mathrm{triv}\}$ and Assumption 4.20 is automatically fulfilled. Moreover, in the unipotent case we have a strong connection between minimally ramified liftings and liftings of prescribed type as considered in [Sho15]. In order to make this precise, let E denote the quotient field of Λ and $\overline{E}$ its algebraic closure (considered with the ℓ-adic topology).

Definition 4.27 (Def. 2.10 of [Sho15])**.** Let $\tau : I_K \to \mathrm{GL}_n(\overline{E})$ be a representation which extends to a continuous representation of the Weil group W_K of K. Then the isomorphism class of τ is called an *inertial type*. (*Warning*: 1. This differs from the usual definition of an inertial type as e.g. in [GK14]. 2. There is no connection with the elements of Ψ_K, but the usage of the letter τ seems to be so common in both cases that we are reluctant to use a differing notation.)

Let ρ be a lift of $\overline{\rho}$ which has values in $\overline{E}$, then we say that ρ "is of type τ" if $\rho|I_K$ is isomorphic to τ.

For the following we consider a τ which is defined over E. Then we say that a morphism

$$x : \mathrm{Spec}\,\overline{E} \to \mathrm{Spec}\,R^{\square}_{\Lambda}(\overline{\rho})$$

is of type τ if the associated $\overline{E}$-valued representation ρ_x is of type τ. This notion depends only on the image of x (because τ is defined over E).

Definition 4.28 (Fixed type deformation ring, [Sho15, Def. 2.14])**.** Let $R^{\square,\tau}_{\Lambda}(\overline{\rho})$ be the reduced quotient of $R^{\square}_{\Lambda}(\overline{\rho})$ which is characterized by the requirement that $\mathrm{Spec}\,R^{\square,\tau}_{\Lambda}(\overline{\rho})$ is the Zariski closure of the $\overline{E}$-points of type τ in $\mathrm{Spec}\,R^{\square}_{\Lambda}(\overline{\rho})$.

A general classification of inertial types is given in Section 2.2.1 of [Sho15]. Under the unipotent ramification assumption, this becomes particularly simple: The set $\mathcal{I}^{\mathtt{uni}}$ of those inertial types is in bijection with the set $\mathcal{Y}_n$ of Young diagrams of size n. The partition $(l_1, \ldots, l_k)$ corresponds (using the notation of [Sho15]) to the type given by the restriction of the Weil-Deligne representation

$$\bigoplus_{i=1}^{k} \mathrm{Sp}(\mathbf{1}, l_i)$$

to I_K. We can express this differently: Each member of $\mathcal{I}^{\texttt{uni}}$ is uniquely characterized by (the conjugacy class of) its value on the generator $\zeta := \zeta_{\mathrm{triv}}$, and a bijection ∇ with $\mathcal{Y}_n$ is given by

$$(l_1, \dots, l_k) \overset{\nabla}{\longleftrightarrow} \left(1 + \begin{pmatrix} \mathcal{B}_{l_1} & & & \\ & \mathcal{B}_{l_2} & & \\ & & \ddots & \\ & & & \mathcal{B}_{l_k} \end{pmatrix}\right) \text{ with } \mathcal{B}_m = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix} \in \mathbb{M}_{m\times m}(E). \quad (4.13)$$

On the other hand, we can associate to a $\tau \in \mathcal{I}^{\texttt{uni}}$ a partition of n by considering the kernel sequence, i.e. we have a map

$$\Theta : \mathcal{I}^{\texttt{uni}} \to \mathcal{Y}_n \qquad \tau \mapsto (s_1, \dots, s_r)$$

with

$$s_i := \dim\ker(\tau(\zeta) - \mathbf{1})^i - \dim\ker(\tau(\zeta) - \mathbf{1})^{i-1}$$

and

$$r := \min\big\{i \big| \dim\ker(\tau(\zeta) - \mathbf{1})^i = \dim\ker(\tau(\zeta) - \mathbf{1})^{i+1}\big\} = \min\big\{i \big| \ker(\tau(\zeta) - \mathbf{1})^i = V\big\}.$$

(We use the convention that f^0 is the identity map for any f.) It follows easily from the characterization of $\mathcal{I}^{\texttt{uni}}$ in (4.13) that $s_i \geq s_{i+1}$, i.e. that Θ has values in $\mathcal{Y}_n$.

It is an easy combinatorial calculation to check that τ is uniquely characterized by its value under Θ and that each Young diagram occurs as a kernel sequence (i.e. that Θ is a bijection). More precisely, we have

Lemma 4.29. *The map $\Theta \circ \nabla^{-1} : \mathcal{Y}_n \to \mathcal{Y}_n$ is given by the conjugation operation on Young diagrams (cf. [FH91, §4.1] or [HHM08, Section 2.8]). In particular, for a given $\tau \in \mathcal{I}^{\texttt{uni}}$, the block matrix structure of $\tau(\zeta)$ (up to reordering blocks) as in (4.13) determines its kernel sequence and vice versa.*

Proof. Retaining the notation used in (4.13), we first remark that for $i \in \mathbb{N}_0$ we have

$$\dim\ker\mathcal{B}_m^i = \min(i, m).$$

Thus, setting $\mathcal{B} = \mathrm{diag}(\mathcal{B}_{l_1}, \dots, \mathcal{B}_{l_k})$, we get

$$\dim\ker\mathcal{B}^i = \sum_{j=1}^{k} \min(i, l_j).$$

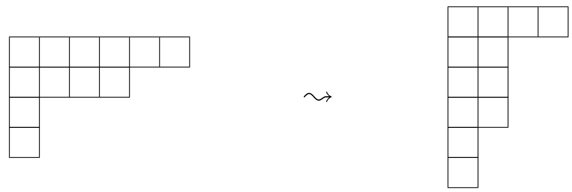
Consequently, the kernel sequence $(s_1, \dots, s_r)$ associated to $(l_1, \dots, l_k)$ is given by

$$s_i = \sum_{j=1}^{k} \min(i, l_j) - \min(i-1, l_j) = \#\{j | l_j \geq i\} = \max\{j | l_j \geq i\}$$

and

$$r = \max\{l_j | j = 1, \dots, k\} = l_1.$$

Hence, the transition $(l_1, \dots, l_k) \rightsquigarrow (s_1, \dots, s_r)$ is precisely the conjugation operation of reflecting a Young diagram at the main diagonal (cf. [HHM08, Section 2.8]), e.g.



$\square$

In order to state the desired comparison result, let us recap that we consider a residual representation $\overline{\rho} : \mathrm{Gal}_K \to \mathrm{GL}_n(k)$ with unipotent ramification. Let $\underline{\lambda} = (l_1, \dots, l_k) \in \mathcal{Y}_n$ such that

$$\overline{\rho}(\zeta) \sim \mathbf{1} + \mathrm{diag}(\mathcal{B}_{l_1}, \dots, \mathcal{B}_{l_k}).$$

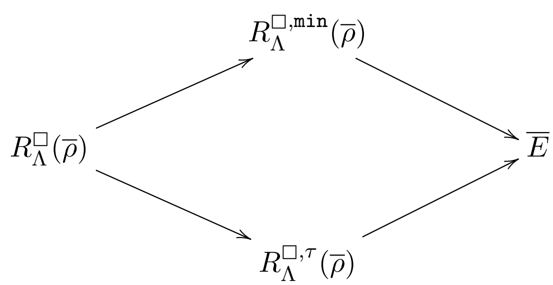
Let $\tau = \nabla(\underline{\lambda}) \in \mathcal{I}^{\mathtt{uni}}$.

Theorem 4.30. *Assume $\overline{\rho}$ is unipotently ramified and let τ be as above. Then there is an isomorphism of the quotients*

$$R_\Lambda^{\square,\mathtt{min}}(\overline{\rho}) \cong R_\Lambda^{\square,\tau}(\overline{\rho})$$

of $R_\Lambda^\square(\overline{\rho})$, i.e. a lifting of $\overline{\rho}$ is minimally ramified if and only if it is of type τ.

Proof. The diagram



allows us to consider the $\overline{E}$-points of $\mathrm{Spec}\, R_\Lambda^{\square,\mathtt{min}}(\overline{\rho})$ and $\mathrm{Spec}\, R_\Lambda^{\square,\tau}(\overline{\rho})$ as subsets of the $\overline{E}$-points of $\mathrm{Spec}\, R_\Lambda^\square(\overline{\rho})$. We claim that they are equal: Translated into terms of $\overline{E}$-valued representations, we have to compare the sets

$$\Xi^{\mathtt{min}} = \left\{ \rho : \mathrm{Gal}_K \to \mathrm{GL}_n(\overline{E}) \;\middle|\; \substack{\rho \text{ lifts } \overline{\rho} \text{ and has values in } \mathcal{O}_{\overline{E}}, \\ \dim \ker(\rho(\zeta) - \mathbf{1})^{i-1} - \dim \ker(\rho(\zeta) - \mathbf{1})^i = l_i \ \forall i} \right\}$$

and

$$\Xi^{\tau} = \left\{ \rho : \mathrm{Gal}_K \to \mathrm{GL}_n(\overline{E}) \;\middle|\; \substack{\rho \text{ lifts } \overline{\rho} \text{ and has values in } \mathcal{O}_{\overline{E}}, \\ \rho|I_K \cong \tau} \right\}.$$

Lemma 4.29 implies that $\Xi^{\mathtt{min}} = \Xi^\tau$.

Now by definition of the ring $R_\Lambda^{\square,\tau}(\overline{\rho})$ (as the schematic closure of the points in Ξ^τ) we have

$$\ker\bigl(R_\Lambda^\square(\overline{\rho}) \to R_\Lambda^{\square,\tau}(\overline{\rho})\bigr) = \bigcap_{\rho \in \Xi^\tau} \ker(\varphi_\rho),$$

where

$$\varphi_\rho : R_\Lambda^\square(\overline{\rho}) \to \overline{E}$$

is the map corresponding to the lift ρ. Moreover, we clearly have

$$\ker\big(R_\Lambda^\square(\overline{\rho}) \to R_\Lambda^{\square,\mathtt{min}}(\overline{\rho})\big) \subseteq \bigcap_{\rho\in\Xi^{\mathtt{min}}} \ker(\varphi_\rho).$$

Hence, by $\Xi^\tau = \Xi^{\mathtt{min}}$, we get a factorization

$$R_\Lambda^\square(\overline{\rho}) \twoheadrightarrow R_\Lambda^{\square,\mathtt{min}}(\overline{\rho}) \overset{\varphi}{\twoheadrightarrow} R_\Lambda^{\square,\tau}(\overline{\rho}),$$

where the middle and the right ring have the same spectrum as topological spaces. Now we know by Lemma 4.23 that $R_\Lambda^{\square,\mathtt{min}}(\overline{\rho})$ is formally smooth over Λ of relative dimension n^2 and that $\dim R_\Lambda^{\square,\tau}(\overline{\rho})$ equals n^2+1 (combine Theorem 2.4 with Proposition 2.15 of [Sho15]). Thus, φ is an isomorphism by Lemma 2.18 and the claim follows. □

Minimal ramification and base change For this paragraph, consider two finite extensions K', K of $\mathbb{Q}_p$ with $K \subset K'$. Moreover let (r, N) be a Weil-Deligne representation of W_K. Let $\ell \neq p$ be a rational prime and recall that we fixed an isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_\ell$, so let us denote by

$$\rho : \mathrm{Gal}_K \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$$

the ℓ-adic Galois representation associated to (r, N) and by V the underlying vector space. After base change to K' we get a Weil-Deligne representation (r', N) with $r' := r|W_{K'}$ and associated Galois representation $\rho' := \rho|\,\mathrm{Gal}_{K'}$. Let us denote the corresponding mod-ℓ reductions by $\overline{\rho}, \overline{\rho}'$ and make the following assumption (which is independent of our choice of ℓ for $\ell \gg 0$):

Assumption 4.31. $\overline{\rho}'$ is unipotently ramified.

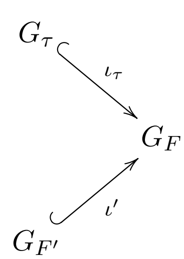
Under this assumption, $(r', N) \cong \bigoplus_{i=1}^k \mathrm{Sp}(\chi_i, l_i)$ for a suitable partition $(l_1, \dots, l_k)$ of n and suitable unramified characters χ_i. Thus, we can choose a basis $B = (b_1, \dots, b_n)$ of V such that N has the block matrix form

$$N = \begin{pmatrix} \mathcal{B}_{l_1} & & & \\ & \mathcal{B}_{l_2} & & \\ & & \ddots & \\ & & & \mathcal{B}_{l_k} \end{pmatrix}$$

with $\mathcal{B}_m$ as in (4.13). Assumption 4.31 implies $\Delta_{\overline{\rho}'} = \{\mathrm{triv}\}$. As before, let us write $\zeta_\tau = \xi_\tau^{-1}(\sigma_{q(\tau)})$ for $\tau \in \Delta_{\overline{\rho}}$.

Proposition 4.32. *Assume $\ell \gg 0$, then the generators $\sigma_{q(\tau)}$ (for $\tau \in \Delta_{\overline{\rho}}$) can be chosen such that $\zeta' := \zeta_\tau$ is contained in $\mathrm{Gal}_{F'}$, does not depend on τ and generates the first factor in $\mathrm{Gal}_{F'}/P_{F'} \cong \mathbb{Z}_\ell \rtimes \hat{\mathbb{Z}}$.*

Proof. Consider the inclusions



and let F_τ denote the fixed field of G_τ. As $\ell \gg 0$, we can assume that $\ell \nmid [K' : K]$. Assume moreover that the extensions $F_\tau|F$ are unramified, then the canonical maps

$$I_{K'}/P_{K'} \to I_K/P_K \qquad \text{and} \qquad I_{K_\tau}/P_{K_\tau} \to I_K/P_K$$

are isomorphisms (for all τ). It follows that the generators ζ', ζ_τ can be chosen such that $\iota'(\zeta') = \iota_\tau(\zeta_\tau)$ and the claim follows.

Thus, we are left to show that $F_\tau|F$ is unramified. For this, let us take $s = \#\rho(I_K)$ and observe that $s = \overline{\rho}(I_K)$ as long as $\ell > s$. For such an ℓ, we also see that the operation of $\mathbb{Z}_\ell$ and of P_K commute on any deformation of $\overline{\rho}$. Hence, $\overline{\rho}|I_K$ factorizes through $\Gamma_\ell \times \Gamma_s$, where Γ_ℓ is a suitable ℓ-group and Γ_s is a suitable group of order s. But this implies that $F_\tau|F$ is unramified. □

For $\tau \in \Delta_{\overline{\rho}}$, denote by $V_\tau \subset V$ the underlying vector space of $\overline{\rho}_\tau$. Then we can decompose V into isotypic components,

$$V = \bigoplus_{\tau \in \Delta_{\overline{\rho}}} V_\tau = \bigoplus_{i=1}^{s} V_{\tau_i},$$

where (for ease of notation) we choose a numbering $\Delta_{\overline{\rho}} = \{\tau_1, \ldots, \tau_s\}$ with $s = \#\Delta_{\overline{\rho}}$. Moreover, possibly after re-arranging the blocks $\mathcal{B}_m$, we can assume that there exists a disjoint partition

$$(1, \ldots, k) = (a_1 = 1, \ldots, e_1) \sqcup (a_2 = e_1 + 1, \ldots, e_2) \sqcup \cdots \sqcup (a_s = e_{s-1} + 1, \ldots, e_s = k)$$

for suitable $a_i \leq e_i \in \mathbb{N}$ such that ζ_{τ_i} acts on V_{τ_i} as $\mathbf{1}_{\dim V_{\tau_i}} + \operatorname{diag}(\mathcal{B}_{l_{a_i}}, \ldots, \mathcal{B}_{l_{b_i}})$. (This all follows from Proposition 4.32, the shape of N and the fact that $\mathbf{1}_n + N$ respects the decomposition of $\bar{\rho}$ into P_K-isotypic components, see [CHT08, Lemma 2.4.12].)

Now, let $\widetilde{\rho}$ be a $k[\epsilon]$-valued lift of $\bar{\rho}$ and assume that $\widetilde{\rho}' := \widetilde{\rho}|\operatorname{Gal}_{F'}$ is minimally ramified (as a lift of $\bar{\rho}'$).

Lemma 4.33. *Let $\ell \gg 0$ and presume Assumption 4.31. Then $\widetilde{\rho}$ is a minimally ramified lift of $\bar{\rho}$.*

Proof. We first remark that the minimal-ramified assumption for $\widetilde{\rho}'$ can be expressed by the identity

$$\operatorname{rk}_k \operatorname{im}(\widetilde{\rho}'(\zeta') - \mathbf{1}_n)^m = 2.\operatorname{rk}_k \operatorname{im} N^m \tag{4.14}$$

for all $m \in \mathbb{N}$, where we always have an inequality in the direction $\leq$, cf. Remark 4.19. (The factor on the right side comes from $\dim_k k[\epsilon] = 2$ and is necessary because we take rk_k instead of $\operatorname{rk}_{k[\epsilon]}$ on the left side.)

We want to show that $\widetilde{\rho}$ is minimally ramified. Therefore, we have to show that for each i we have

$$\operatorname{rk}_k \operatorname{im}(\widetilde{\rho}_{\tau_i}(\zeta_{\tau_i}) - \mathbf{1}_{\dim V_{\tau_i}})^m = 2.\operatorname{rk}_k \operatorname{im} \operatorname{diag}(\mathcal{B}_{l_{a_i}}, \ldots, \mathcal{B}_{l_{b_i}})^m, \tag{4.15}$$

where we again always have an inequality $\leq$. As

$$\operatorname{rk}_k \operatorname{im} N^m = \sum_{i=1}^{s} \operatorname{rk}_k \operatorname{im} \operatorname{diag}(\mathcal{B}_{l_{a_i}}, \ldots, \mathcal{B}_{l_{b_i}})^m,$$

we see that the equality (4.14) can only be fulfilled if (4.15) is fulfilled for all i. The claim follows. □

Remark 4.34. We expect that Lemma 4.33 holds without presuming Assumption 4.31

We immediately get the following corollary:

Corollary 4.35. *Let $\ell \gg 0$ and presume Assumption 4.31. Denote by $L \subset H^1(G_L, \mathrm{ad}(\overline{\rho}))$ and $L' \subset H^1(G_{L'}, \mathrm{ad}(\overline{\rho}))$ the subspaces parametrizing minimally ramified deformations. Then $\mathrm{res}^{-1}(L') \subset L$, where*

$$\mathrm{res} : H^1(K, \mathrm{ad}(\overline{\rho})) \longrightarrow H^1(K', \mathrm{ad}(\overline{\rho}|G_{K'})).$$

denotes the restriction map.

4.4.2 An $R = R^{\mathtt{min}}$-theorem

For this short section, let F, E be number fields and consider a strictly compatible system of irreducible E-rational Galois representations

$$\mathcal{R} = \big(\rho_\lambda : \mathrm{Gal}_F \to \mathrm{GL}_n(E_\lambda)\big)_{\lambda \in \mathrm{Pl}_E}.$$

Let (r, N) be the associated Weil-Deligne representation at a fixed place ν of F. We suppose the following assumption:

Assumption 4.36. (r, N) is Frobenius-semisimple and $r(I_{F_\nu}) = 1$, i.e. $\mathcal{R}$ is unipotently ramified at ν.

Thus, we can write

$$(r, N) \cong \bigoplus_{i=1}^{k} \mathrm{Sp}(\widetilde{r}_i, l_i)$$

for a suitable partition $(l_1, \ldots, l_k)$ of n and 1-dimensional W_{F_ν}-representations $\widetilde{r}_i$. Therefore, we can assume that the generator ζ of T_{F_ν} acts on the underlying vector space V of r as

$$1 + N = \left(1 + \begin{pmatrix} \mathcal{B}_{l_1} & & & \\ & \mathcal{B}_{l_2} & & \\ & & \ddots & \\ & & & \mathcal{B}_{l_k} \end{pmatrix}\right) \text{ with } \mathcal{B}_m = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix} \in \mathbb{M}_{m \times m}(\mathbb{C})$$

and Frob_ν acts on V as

$$\mathcal{H} = \begin{pmatrix} \mathcal{H}_{l_1}(\alpha_1) & & & \\ & \mathcal{H}_{l_2}(\alpha_2) & & \\ & & \ddots & \\ & & & \mathcal{H}_{l_k}(\alpha_k) \end{pmatrix} \text{ with } \mathcal{H}_m(\alpha) = \begin{pmatrix} \alpha & & & \\ & \alpha q & & \\ & & \ddots & \\ & & & \alpha q^{m-1} \end{pmatrix} \in \mathbb{M}_{m \times m}(\mathbb{C})$$

for suitable numbers $\alpha_i \in \mathbb{C}$ and where q denotes the cardinality of the residue field of F_ν. Let us additionally assume

Assumption 4.37. For $i \neq j$, the q-orbits

$$q^{\mathbb{Z}}\alpha_i = \{q^a.\alpha_i \mid a \in \mathbb{Z}\} \quad \text{and} \quad q^{\mathbb{Z}}\alpha_j = \{q^a.\alpha_j \mid a \in \mathbb{Z}\}$$

are disjoint.

Remark that Assumption 4.37 implies the first part of Assumption 4.36, i.e. that (r, N) is Frobenius-semisimple.

Definition 4.38 ([All14, Definition 1.1.2])**.** The Weil-Deligne representation (r, N) is called *generic* if

$$\mathrm{Hom}_{\text{WD-Rep}}\big((r,N),(r(1),N)\big) = 0.$$

Lemma 4.39. *Presume Assumptions 4.36 and 4.37, then (r, N) is generic.*

Proof. Let $f \in \mathrm{Hom}_{\text{WD-Rep}}\big((r,N),(r(1),N)\big)$ and let A_f be the corresponding $n \times n$ matrix, then we have

$$qA_f.\mathcal{H} = \mathcal{H}.A_f, \tag{4.16}$$

and

$$A_f.N = N.A_f. \tag{4.17}$$

Let us write

$$A_f = (A^{(i,j)})_{1\le i,j\le k} \text{ with } A^{(i,j)} \in \mathbb{M}_{l_i\times l_j}(\mathbb{C})$$

and

$$\mathcal{H}.A_f.\mathcal{H}^{-1} = (B^{(i,j)})_{1\le i,j\le k} \text{ with } B^{(i,j)} \in \mathbb{M}_{l_i\times l_j}(\mathbb{C}).$$

We claim that $A_{i,j} = 0$ if $i \neq j$: By the shape of $\mathcal{H}$ we first see that

$$B^{(i,j)}_{u,v} = \frac{\alpha_i}{\alpha_j} q^{v-u} A^{(i,j)}_{u,v} \qquad (0 \le u \le l_i, 0 \le v \le l_j)$$

and by (4.16) it follows that

$$qA^{(i,j)}_{u,v} = \frac{\alpha_i}{\alpha_j} q^{v-u} A^{(i,j)}_{u,v}. \tag{4.18}$$

By Assumption 4.37 this implies $A^{(i,j)}_{u,v} = 0$.

Thus, we can assume w.l.o.g. that $k = 1$. Comparing again the coefficients in (4.18) then yields $A^{(1,1)}_{u,v} = 0$ whenever $u \neq v + 1$, in other words

$$A_f = \begin{pmatrix} 0 & & & & \\ \beta_1 & 0 & & & \\ 0 & \beta_2 & 0 & & \\ \vdots & \ddots & & \ddots & \\ 0 & \cdots & 0 & \beta_{n-1} & 0 \end{pmatrix} \qquad \text{for suitable } \beta_i \in \mathbb{C}.$$

By (4.17), we get

$$A_f.N = \mathrm{diag}(0, \beta_1, \ldots, \beta_{n-1}) = \mathrm{diag}(\beta_1, \ldots, \beta_{n-1}, 0) = N.A_f.$$

Hence, all β_i vanish and the claim follows. □

We also have:

Lemma 4.40 ([All14, Lemma 1.1.3])**.** *Let π be an admissible complex representation of $\mathrm{GL}_n(F)$ such that π and (r, N) correspond to each other via the local Langlands correspondence. Then π is generic (in the sense of [GK75]) if and only if (r, N) is generic.*

Consider the following assumption, which is met for example if we know that the α_i are Weil numbers, which follows if we impose that $\mathcal{R}$ is pure:

Assumption 4.41. All the occurring numbers α_i are algebraic integers.

We will restrict our exposition to those $\lambda \in \mathrm{Pl}_E$ for which $\ell(\lambda)$ is large enough such that the following analogue of Assumption 4.37 holds:

Assumption 4.42. For $i \neq j$, the q-orbits

$$q^{\{1,\ldots,n\}}\alpha_i = \{q^a.\bar{\alpha}_i \mid a \in \{1,\ldots,n\}\} \subset \overline{\mathbb{F}}_{\ell(\lambda)} \quad \text{and} \quad q^{\{1,\ldots,n\}}\alpha_j = \{q^a.\bar{\alpha}_j \mid a \in \{1,\ldots,n\}\} \subset \overline{\mathbb{F}}_{\ell(\lambda)}$$

are disjoint.

Lemma 4.43. *Let λ be such that $\ell(\lambda) > q$ and presume Assumptions 4.36, 4.41 and 4.42. Assume moreover that $\rho_{\lambda,\nu}$ is a minimal lift of its reduction $\overline{\rho}_{\lambda,\nu}$ in the sense of Definition 4.17. Then*

$$\mathrm{Hom}(\overline{\rho}_{\lambda,\nu}, \overline{\rho}_{\lambda,\nu}(1)) = 0.$$

Proof. The argument used in the proof of Lemma 4.39 carries over: Let $f \in \mathrm{Hom}\big(\overline{\rho}_{\lambda,\nu}, \overline{\rho}_{\lambda,\nu}(1)\big)$ and let A_f be the corresponding $n \times n$ matrix, then A_f must again fulfill (4.16) and (4.17) with the reductions $\overline{\mathcal{H}}$ and $\overline{N}$ instead of $\mathcal{H}$ and N. As $\rho_{\lambda,\nu}$ is a minimal lift of $\overline{\rho}_{\lambda,\nu}$, $\overline{N}$ and N have the same shape, so thanks to Assumption 4.42 we can compare the coefficients and conclude the claim as in the proof of Lemma 4.39. □

At this point we remark that we conjecture the vanishing of $\mathrm{Hom}\big(\overline{\rho}_{\lambda,\nu}, \overline{\rho}_{\lambda,\nu}(1)\big)$ if the Weil-Deligne representation (r, N) at ν is generic. In other words (and using Lemma 4.40) we expect the following to hold:

Conjecture 4.44. Let π be a generic admissible representation of $\mathrm{GL}_n(F)$ with associated Weil-Deligne representation (r, N) and Galois representation $\rho : \mathrm{Gal}_F \to \mathrm{GL}_n(\mathbb{C})$. Then the reduction $\overline{\rho}_\ell$ of $\mathrm{GL}_n(\iota_\ell) \circ \rho$ fulfills

$$\mathrm{Hom}\big(\overline{\rho}_\ell, \overline{\rho}_\ell(1)\big) = 0$$

for all $\ell \gg 0$.

We expect this to be provable by methods of this thesis under certain standard hypotheses (Assumption 6.6) on the reductions of compatible systems of Galois representations.

We make another assumption on our compatible system $\mathcal{R}$ which will be verified later if $\mathcal{R} = \mathcal{R}_\Pi$ for a RACSDC automorphic representation Π of a general linear group over a totally real field, see Section 6.4.4.

Assumption 4.45. There exists a finite failure set $X \subset \mathrm{Pl}_E^{\mathrm{fin}}$ such that for all $\lambda \in \mathrm{Pl}_E^{\mathrm{fin}} - X$, $\rho_{\lambda,\nu}$ is a minimally ramified lift of $\overline{\rho}_{\lambda,\nu}$ (for all $\nu \in \mathrm{Pl}_F^{\mathrm{fin}}$).

Corollary 4.46. *Presume Assumptions 4.36, 4.37, 4.41 and 4.45, then for almost all λ we have*

$$\mathrm{Hom}\big(\overline{\rho}_{\lambda,\nu}, \overline{\rho}_{\lambda,\nu}(1)\big) = 0.$$

Corollary 4.47. *Presume Assumptions 4.36, 4.37 4.41 and 4.45, then for almost all λ the canonical surjection*

$$R_\Lambda^{\square,(\chi_\nu)}(\overline{\rho}_{\lambda,\nu}) \twoheadrightarrow R_\Lambda^{\square,(\chi_\nu),\mathtt{min}}(\overline{\rho}_{\lambda,\nu}) \tag{4.19}$$

is an isomorphism.

Proof. Let us first treat the case where the determinant is not fixed: By Lemma 4.23, the right hand side of (4.19) is isomorphic to $\Lambda[[x_1, \ldots, x_{n^2}]]$. By Corollary 4.46 and Lemma 2.71, for almost all λ we have

$$H^2(F_\nu, \mathrm{ad}(\overline{\rho}_\lambda)) = 0.$$

It follows (cf. [Kis09, Maz97b] and our Proposition 2.58) that $R^{\square}_{\Lambda}(\overline{\rho}_{\lambda,\nu})$ is formally smooth and hence isomorphic to a power series ring over Λ in

$$h^1(F_\nu, \mathrm{ad}(\overline{\rho}_\lambda)) + n^2 - h^0(F_\nu, \mathrm{ad}(\overline{\rho}_\lambda)) = n^2$$

variables, where the vanishing of $h^1(F_\nu, \mathrm{ad}(\overline{\rho}_\lambda)) - h^0(F_\nu, \mathrm{ad}(\overline{\rho}_\lambda))$ follows from the local Euler-Poincaré formula. Therefore, Lemma 2.18 implies the claim.

As we have shown that any lifting of $\overline{\rho}_\lambda$ is minimally ramified (subject to $\ell(\lambda) \gg 0$), the claim in the fixed determinant case is tautologically true. $\square$

5 Unobstructedness for Hilbert modular forms

Let F be a totally real number field and let I denote the set of embeddings $F \hookrightarrow \mathbb{R}$. Let $f \in S_k(\mathfrak{n})$ be a Hilbert modular newform of weight $\boldsymbol{\omega} = (\omega_\tau)_{\tau \in I} \in \mathbb{Z}^I$ and of level $\mathfrak{n} \subset \mathcal{O}_F$. We include that f is normalized (in the sense of the definition on p. 7 of [SW93]) in the definition of a newform. We demand that $\omega_\tau \geq 2$ and $\omega_\tau \equiv \omega_{\tau'} \bmod 2$ for all τ, τ'. Denote by $K_f = \mathbb{Q}(\{a_\nu(f) \,|\, \nu \nmid \mathfrak{n}\})$ the number field generated by the eigenvalues $a_\nu(f)$ of f under the Hecke operators T_ν, for all $\nu \nmid \mathfrak{n}$.

For each $\lambda \in \mathrm{Pl}^{\mathrm{fin}}_{K_f}$, we denote by $K_{f,\lambda}$ the completion of K_f at λ (with ring of integers $\mathcal{O}_{f,\lambda}$) and by $k_{f,\lambda}$ the residue field of $K_{f,\lambda}$. According to [Car86] (a more explicit reference is [Böc13b, Theorem 4.12]), we can associate to f a strictly compatible system

$$\Big(\rho_{f,\lambda} : \mathrm{Gal}_F \to \mathrm{GL}_2(K_{f,\lambda})\Big)_{\lambda \in \mathrm{Pl}^{\mathrm{fin}}_{K_f}}$$

of Galois representations with ramification set $S_0 = \{\lambda | \lambda \text{ divides } \mathfrak{n}\}$. When we have specified a prime λ and a finite set of places S of F containing S_0, we will denote by

$$\overline{\rho}_{f,\lambda} : \mathrm{Gal}_{F,S} \to \mathrm{GL}_2(k_{f,\lambda})$$

the (semi-simplification of the) reduction modulo λ. It is known, that there exists a cofinite subset $Q^{(\mathtt{irr})} \subset \mathrm{Pl}^{\mathrm{fin}}_{K_f}$ such that $\overline{\rho}_{f,\lambda}$ is absolutely irreducible for $\lambda \in Q^{(\mathtt{irr})}$ (see item 1. below; this will later also be deductible from Remark 6.8).

Now, fix such a finite set of non-archimedean places S which contains S_0. We want to describe a set $Q^1 \subset \mathrm{Pl}^{\mathrm{fin}}_{K_f} - S$ of places λ where the framework of Theorem 3.12 applies to $G = \mathrm{GL}_2, \overline{\rho} = \overline{\rho}_{f,\lambda}$ and for the following choices: `min` and `sm` are both the condition parametrizing arbitrary (fixed-determinant) lifts and `crys` parametrizes (fixed-determinant) lifts which are FL-crystalline in the sense of Section 4.3.

1. By [Dim05, Proposition 3.1], there exists a cofinite subset $Q^{(\mathtt{irr})} \subset \mathrm{Pl}^{\mathrm{fin}}_{K_f}$ such that condition **(Representability)** of Section 3.1 is fulfilled. (Cf. also [Tay95, Prop. 1.2].)

2. It is a key computation of Gamzon [Gam13, Proposition 4.4] that there exists a cofinite subset $Q^{(\mathtt{sm})} \subset \mathrm{Pl}^{\mathrm{fin}}_{K_f}$ such that $H^0(F_\nu, \mathrm{ad}^0\,\overline{\rho}(1))$ vanishes for all $\nu \in \Omega^F_{\ell(\lambda)}$ if we suppose[11] that $\omega_\tau > 2$ for all τ. By local Tate duality, this implies that $D_\nu^{\square,\chi,\mathtt{sm}} = D_\nu^{\square,\chi}$ is formally smooth, hence condition **(sm/k)** is fulfilled. Using the local Euler characteristic formula, we get

$$h^1(F_\nu, \mathrm{ad}^0\,\overline{\rho}) = h^0(F_\nu, \mathrm{ad}^0\,\overline{\rho}) + \dim(\mathrm{ad}^0\,\overline{\rho})[F_\nu : \mathbb{Q}_\ell].$$

   We can assume w.l.o.g. that the exact sequence (2.12) is available for all $\lambda \in Q^{(\mathtt{sm})}$ (if this is not the case, we exclude the finitely many places where this fails from the set $Q^{(\mathtt{sm})}$). Thus we get

$$d_\nu^{\square,\mathtt{sm}} = \dim t_{D_\nu^{\square,\chi}} = \dim t_{D_\nu^\chi} + \dim \mathrm{ad}^0\,\overline{\rho} - h^0(F_\nu, \mathrm{ad}^0\,\overline{\rho})$$

$$= h^0(F_\nu, \mathrm{ad}^0\,\overline{\rho}) + \dim(\mathrm{ad}^0\,\overline{\rho})[F_\nu : \mathbb{Q}_\ell] + \dim(\mathrm{ad}^0\,\overline{\rho}) - h^0(F_\nu, \mathrm{ad}^0\,\overline{\rho}) = 3.([F_\nu : \mathbb{Q}_\ell] + 1).$$

   Hence the additional condition in Theorem 3.12.2 is fulfilled with $\delta_\nu = 0$.

[11] In the recent version of [Gam13] (as accessed on 29 November 2015 via `http://www.mtholyoke.edu/~agamzon/homepage.html`), Proposition 4.4 is formulated under the weaker condition that $\omega_\tau > 2$ for at least one τ. However, we cannot follow his argument and believe that this is a mistake.

3. By [Tay95, Theorem 1.4] (see also [Bre99, Dim09]), there is a cofinite subset $Q^{(\mathtt{crys})} \subset Q^{(\mathtt{irr})}$ such that the restriction $\rho_{f,\lambda,\nu}$ of $\rho_{f,\lambda}$ to a decomposition group at ν is crystalline for all $\nu \in \Omega_\ell$ (with $\ell = \ell(\lambda)$). By Lemma 4.14 it follows that

$$R^{\square,\mathtt{crys}}(\overline{\rho}_\nu) \cong W[[x_1, \ldots, x_m]]$$

with $m = 4 + [K_\nu : \mathbb{Q}_\ell]$. We are, however, interested in a fixed-determinant version:

Theorem 5.1. *Fix a lift χ of the determinant. Then, for all $\lambda \in Q^{(\mathtt{crys})}$ with $\ell(\lambda) > 2$,*

$$R^{\square,\chi,\mathtt{crys}}(\overline{\rho}_\nu) \cong W[[x_1, \ldots, x_k]]$$

with $k = 3 + [K_\nu : \mathbb{Q}_\ell]$.

Before we can prove this theorem, we need a preparatory lemma (where we again take $\ell = \ell(\lambda)$):

Lemma 5.2. *Recall that we abbreviate "FL-crystalline" for "crystalline with Hodge-Tate weights in the Fontaine-Laffaille range $[0, \ell - 2]$". We have:*

1. *Let $\overline{\rho} : \mathrm{Gal}_{F_\nu} \to \mathrm{GL}_2(k)$ be an FL-crystalline representation (where k is a finite field of characteristic $\ell = \ell(\lambda) = \ell(\nu)$) and let $\rho \in D^{(\square),\mathtt{crys}}(\overline{\rho})(A)$ be an FL-crystalline lift to some coefficient ring $A \in \mathcal{C}_{W(k)}$. Assume moreover that all Hodge-Tate weights of $\overline{\rho}$ lie in $[0, \lfloor \frac{\ell-1}{2} \rfloor]$. Then $\det(\overline{\rho})$ is FL-crystalline and $\det(\rho) \in D^{(\square),\mathtt{crys}}(\det(\overline{\rho}))(A)$.*
2. *Let $\overline{\rho} : \mathrm{Gal}_{F_\nu} \to \mathrm{GL}_2(k)$ and $\overline{\psi} : \mathrm{Gal}_{F_\nu} \to k^\times$ be FL-crystalline and assume that $\kappa + \kappa' \in [0, \ell - 2]$ for any Hodge-Tate weight κ of $\overline{\rho}$ and any Hodge-Tate weight κ' of $\overline{\psi}$. Let ρ, ψ be FL-crystalline lifts to some coefficient ring A. Then $\psi \otimes \rho$ is FL-crystalline.*
3. *A lift of the trivial character $\mathbf{1} : \mathrm{Gal}_{F_\nu} \to k^\times$ is FL-crystalline if and only if it is unramified. In other words,*
$$R^{(\square),\mathtt{crys}}(\mathbf{1}) = R^{(\square),\mathtt{nr}}(\mathbf{1}),$$
where the object on the right denotes the universal deformation ring parametrizing unramified deformations (resp. liftings) of $\mathbf{1}$.
4. *Assume $\ell > 2$. Let A be a coefficient ring as above and consider two FL-crystalline characters $\chi, \psi : \mathrm{Gal}_{F_\nu} \to A^\times$ such that $\overline{\chi} = \overline{\psi}$. Then $(\chi.\psi^{-1})^{1/2}$ is an FL-crystalline lift of $\mathbf{1}$.*

(We believe the bound on the Hodge-Tate weights in part 1. to be unnecessary, but it enables us to give a very simple proof. We remind the reader that we are ultimately interested in $\ell \gg 0$. We also emphasize that with our conventions, ϵ_ℓ^0 is FL-crystalline, while $\epsilon_\ell^{\ell-1}$ is not.)

Proof. Part 2. is a direct application of Remark 4.4.

For part 1., we apply the same argument (i.e. Remark 4.4) to the choice $\rho_1 = \rho_2 = \overline{\rho}$. Then the bound on the Hodge-Tate weights implies that $\overline{\rho} \otimes \overline{\rho}$ is crystalline. We already saw that the condition of being FL-crystalline passes over to quotients (see Theorem 4.3). Thus $\det(\overline{\rho})$ is FL-crystalline, as it can be realized as the one-dimensional quotient $\Lambda^2(\overline{\rho})$ of $\overline{\rho} \otimes \overline{\rho}$. By the same argument, $\det(\rho)$ is seen to be an FL-crystalline lift of $\det(\overline{\rho})$.

For part 3., remark that any unramified lift is automatically FL-crystalline (see e.g. [Wes04, Example 4.2], where the pre-image of an unramified character under the Fontaine-Laffaille functor is described). Thus we get a canonical surjection

$$\varphi : R^{(\square),\mathtt{crys}}(\mathbf{1}) \longrightarrow R^{(\square),\mathtt{nr}}(\mathbf{1}) \tag{5.1}$$

of W-algebras. By our Lemma 4.14 and [BLGG11, Proof of Lemma 3.4.2], both objects in (5.1) are isomorphic to $W[[x]]$. It follows by Lemma 2.18 that φ is an isomorphism.

Concerning part 4., we first note that ψ^{-1} is crystalline (albeit with Hodge-Tate weights in $[-(\ell-2), 0]$, i.e. generally not in the essential image of the Fontaine-Laffaille functor), cf. Section 2 of [Niz93]. Analogously to the proof of part 2., we conclude that $\chi.\psi^{-1} = \chi \otimes \psi^{-1}$ is an FL-crystalline, hence (by part 3.) unramified lift of $\mathbf{1}$. But then, the square root of $\chi.\psi^{-1}$ is also unramified, hence FL-crystalline. (Observe that we had to assume $\ell > 2$ in order to be able to take the square root character.) □

Proof of Theorem 5.1. We use an argument analogous to [Böc98, Proposition 2.1] to get an isomorphism

$$D^{\square,\mathtt{crys}}(\overline{\rho}_\nu) \cong D^{\square,\chi,\mathtt{crys}}(\overline{\rho}_\nu) \times D^{\square,\mathtt{crys}}(\mathbf{1})$$

by identifying a crystalline lift ρ of $\overline{\rho}_\nu$ with the pair

$$\big((\chi.\det(\rho)^{-1})^{1/2} \otimes \rho, (\chi.\det(\rho)^{-1})^{1/2}\big).$$

(We use Lemma 5.2 here to assure that both entries of this pair are *crystalline* lifts of $\overline{\rho}_\nu$ and $\mathbf{1}$, respectively.) Hence, by Proposition 2.5,

$$R^{\square,\mathtt{crys}}(\overline{\rho}_\nu) \cong R^{\square,\chi,\mathtt{crys}}(\overline{\rho}_\nu) \hat{\otimes}_W R^{\square,\mathtt{crys}}(\mathbf{1}).$$

Thus, after another application of Lemma 4.14 we have

$$W[[x_1,\dots,x_m]] \cong R^{\square,\chi,\mathtt{crys}}(\overline{\rho}_\nu) \hat{\otimes}_W W[[x]] = R^{\square,\chi,\mathtt{crys}}(\overline{\rho}_\nu)[[x]]. \tag{5.2}$$

But this implies $R^{\square,\chi,\mathtt{crys}}(\overline{\rho}_\nu) \cong W[[x_1,\dots,x_{m-1}]]$ by Lemma 2.19. □

4. Condition **(min)** is fulfilled for a cofinite subset $Q^{(\mathtt{min})} \subset \mathrm{Pl}^{\mathrm{fin}}_{K_f}$. This follows from [Gam13, Corollary 9], local Tate duality and the finiteness of S. We have to see that $d_\nu^{\square,\mathtt{min}} = \dim(\mathfrak{sl}_2) = 3$ for $\nu \in \Omega_\ell$. Using the formal smoothness, this follows from the fact that the Krull dimension of $R^{\square,\chi}(\overline{\rho}_\nu)$ is 4, see [Böc13a, Theorem 3.3.1(h)].

5. Condition **(∞)** demands that $R^{\square,\chi}(\overline{\rho}_\nu) = R^{\square}(\overline{\rho}_\nu)$ is formally smooth with $u_\nu = 2$ for $\nu|\infty$, which is verified (independently of the choice of λ) in [Böc13a], Section 5.5 (see also our Proposition 2.70).

6. Condition **(Presentability)** is a direct consequence of our Corollary 2.68. However, we can alternatively deduce it from the more classical (i.e. GL_n-bound) literature: The condition follows from Key Lemma 5.2.2 and Lemma 5.3.1 of [Böc13a], as soon as we can show that the error term δ from loc. cit. vanishes. In both cases, we have to check that Assumption 2.63 holds for a cofinite subset $Q^\delta \subset Q^{(\mathtt{crys})}$, which is the case thanks to Corollary 2.73.

7. Considering condition **(R=T)**, we note that using Proposition 2.62 it is sufficient to show that $R^{\chi,\mathtt{crys}}_{S_\ell} \cong W$. The main ingredient for this is the following (conjectural) $R = T$-theorem:

   - Let $X(f, \lambda)$ denote the set of all Hilbert modular newforms g of weight $\boldsymbol{\omega}$ such that $\rho_{g,\lambda}$ (understood with values in $\mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$) is a lift of $\overline{\rho}_{f,\lambda}$ (understood with values in $\mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$) which fulfills $\mathtt{crys}$, has determinant χ and is unramified outside S_ℓ.

- Let K denote the composite of all the fields of definition K_g, where g runs through $X(f,\lambda)$. Denote by $\mathcal{O}$ the ring of integers of K.
- Let us chose a place δ of K above λ. Then denote by $\mathbb{T}_\delta$ the $\mathcal{O}_{K_\delta}$-subalgebra of $\prod_{g\in X(f,\lambda)} \mathcal{O}_{K_\delta}$ which is generated by all $(a_\nu(g))_{g\in X(f,\lambda)}$, where ν runs through the places of F outside S.

Conjecture 5.3. Suppose that the places λ, δ are chosen such that $\ell = \ell(\lambda) = \ell(\delta)$ is odd, coprime to S and such that $F_\nu|\mathbb{Q}_\ell$ is unramified for all ν above ℓ. Suppose moreover that the image of $\overline{\rho}_{f,\lambda}$ is adequate in the sense of Section 6.4.1 and that $\overline{\rho}_{f,\lambda}$ is totally odd: $\det\overline{\rho}_{f,\lambda}(c) = -1$ for all complex conjugations c. Then

$$R^{\chi,\mathtt{crys}}_{S_\ell,\mathcal{O}_{K_\delta}}(\overline{\rho}_{f,\lambda}) \cong \mathbb{T}_\delta.$$

Remark 5.4. 1) It follows from Theorem 6.6 (with $\Sigma = S$) and Remark 1.2 of [Dim09] that Conjecture 5.3 holds under the conditions that F is Galois over $\mathbb{Q}$ and that f is not a theta series nor a twist of a base change of a newform on some $E \subsetneq F$.

2) If we let $\mathtt{crys}$ be the deformation condition parametrizing nearly ordinary or flat deformations, a suitable $R = T$-theorem is due to Fujiwara [Fuj06, Theorem 11.1].

3) If we let $\mathtt{crys}$ be the deformation condition parametrizing potentially Barsotti-Tate deformations (and we correspondingly restrict the previous exposition to forms of parallel weight 2), there exists a suitable $R = T$-theorem (see [Che13, Theorem 4.1]) under the following additional assumption:

$$\forall\, \nu \in \Omega_\ell : \mathrm{End}_{\overline{\mathbb{F}}_\ell[\mathrm{Gal}_{F_\nu}]}(\overline{\rho}_{f,\lambda,\nu} \otimes \overline{\mathbb{F}}_\ell) = \overline{\mathbb{F}}_\ell.$$

Corollary 5.5. *Assume Conjecture 5.3. Then there exists a cofinite subset* $Q^{(\mathtt{R=T})} \subset Q^{(\mathtt{irr})}$ *such that, for all* $\lambda \in Q^{(\mathtt{R=T})}$*, we have*

$$R^{\chi,\mathtt{crys}}_{S_\ell}(\overline{\rho}_{f,\lambda}) \cong W(k_{f,\lambda}).$$

Observe that Proposition 2.62 then implies that $R^{\square_{S_\ell},\chi,\mathtt{crys}}_{S_\ell}(\overline{\rho}_{f,\lambda})$ is formally smooth of dimension $4.\#S_\ell - 1 = \dim(\mathfrak{gl}_2).\#S_\ell - \dim(\mathfrak{gl}_2^{\mathrm{ab}})$.

Proof. First, observe that the oddness requirement of Conjecture 5.3 is automatically fulfilled (cf. [Dim05, Section 0.1] and the references therein). After excluding finitely many places λ, also the adequateness requirement is automatically fulfilled, cf. [GHTT12].

Let us first show that for almost all λ we can chose a place δ of the field K above λ such that

$$R^{\chi,\mathtt{crys}}_{S_\ell,\mathcal{O}_{K_\delta}}(\overline{\rho}_{f,\lambda}) \cong \mathcal{O}_{K_\delta}. \tag{5.3}$$

We are clearly done if we can show $X(f,\lambda) = \{f\}$ for almost all λ.

Now if $g_1, g_2 \in X(f,\lambda)$, we see that

$$a_\nu(g_1) \equiv a_\nu(g_2) \mod \lambda \qquad \forall \nu \notin S_\ell,$$

by the construction of $\rho_{g_i,\lambda}$ from g_i (see equation (1) in [Dim09]). Therefore, the assumption $g_1, g_2 \in X(f,\lambda)$ for infinitely many primes λ implies that $a_\nu(g_1)$ equals $a_\nu(g_2)$ for all $\nu \notin S_\ell$. But this implies $g_1 = g_2$ by a suitable multiplicity one theorem, see e.g. [SW93, Theorem 3.5].

Therefore, for a given newform $g \neq f$ there exist only finitely many λ such that $g \in X(f, \lambda)$. The claim now follows as there are only finitely many newforms of a given level and weight (see [BJ79, Section 4.3]).

The claim now follows from (5.3) by (the unframed version of) Corollary 2.35.3 and Lemma 2.17. □

Observe that the first two conditions of Corollary 3.16 are automatically fulfilled for a cofinite subset $Q^{(\mathtt{sp})} \subset \mathrm{Pl}_{K_f}^{\mathrm{fin}}$ and that the third condition follows from [CHT08, Corollary 2.4.21]. Thus, considering the intersection $Q^1 := Q^{(\mathtt{sp})} \cap Q^{(\mathtt{R=T})} \cap Q^{\delta} \cap Q^{(\mathtt{min})} \cap Q^{\mathtt{sm}} \cap (\mathrm{Pl}_{K_f}^{\mathrm{fin}} - S)$ and applying Corollary 3.16, we get

Theorem 5.6. *Assume Conjecture 5.3 and $\omega_\tau > 2$ for all τ. Then, for almost all primes λ, $D_{S_\ell}^{\chi}(\overline{\rho}_{f,\lambda})$ has vanishing dual Selmer group.*

Corollary 5.7. *Assume Conjecture 5.3 and $\omega_\tau > 2$ for all τ. Then, for almost all primes λ, $R_{S_\ell}^{\chi}(\overline{\rho}_{f,\lambda})$ is globally unobstructed.*

Proof. The local parts of the "globally unobstructed" notion (cf. Definition 3.7), i.e. the relative smoothness of the local deformation rings $R^{\square,\chi}(\overline{\rho}_\nu)$ for $\nu \in S_\ell$, follow from Proposition 2.70, Corollary 4.12 and item 4. of this section.

□

6 Unobstructedness for RACSDC automorphic representations

Let F be a CM field with maximal real subfield F^+ and recall from [CHT08] the following definition:

Definition 6.1 (RACSDC automorphic representation)**.** An automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$ is called RACSDC (regular, algebraic, conjugate self dual, cuspidal) if it is cuspidal and fulfills

- $\Pi^\vee \cong \Pi^c$, where c denotes the non-trivial element of $\mathrm{Gal}(F|F^+)$;
- Π_∞ has the same infinitesimal character as some irreducible algebraic representation of the restriction of scalars from F to $\mathbb{Q}$ of GL_n.

Remark 6.2. As we follow largely the article [CHT08], it would be natural to suppose additionally that there is a prime ν_0 of F^+ which splits as $w_0 w_0^c$ in F such that at least one of $\Pi_{w_0}, \Pi_{w_0^c}$ is square integrable (cf. condition 5. of [CHT08, Theorem 4.4.2]). By the recent developments ([BLGGT14, Theorem 2.1.1], but see also [Shi11, CHLN11, Gue11]), this restriction is not necessary.

Remark 6.3. We remark that (by the same references as mentioned in Remark 6.2) it is possible to treat the material of this section for RAECSDC automorphic representations, i.e. for such Π as above where the "conjugate self-dual"-condition is weakened to the following "essentially conjugate self-dual"-condition:

- $\Pi^\vee \cong \Pi^c \otimes (\chi \circ N_{F|F^+} \circ \det)$, where $\chi : \mathbb{A}_{F^+}^\times/(F^+)^\times \to \mathbb{C}^\times$ is a continuous character such that χ_ν is independent of $\nu|\infty$.

For the remainder, let us fix such a RACSDC automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$. Then we can associate to Π a compatible system of ℓ-adic Galois representations:

Theorem 6.4. *There exists a number field $\mathcal{E}_\Pi$ and an $\mathcal{E}_\Pi$-rational strictly compatible and pure of weight $n-1$ system of semisimple ℓ-adic Galois representations attached to Π*

$$\mathcal{R}_\Pi = \Big(\rho_\lambda : \mathrm{Gal}_F \to \mathrm{GL}_n(\mathcal{E}_{\Pi,\lambda})\Big)_{\lambda \in \mathrm{Pl}^{\mathrm{fin}}_{\mathcal{E}_\Pi}}$$

with finite ramification set $S_\Pi := \{\nu \in \mathrm{Pl}_F \mid \Pi_\nu$ is ramified $\}$ such that (in addition to the compatibility requirements of Definition 3.23) the following holds:

- *The L-functions match: For all λ, $L(s, \Pi) = L(s, \rho_\lambda)$;*
- *The ρ_λ are polarized, i.e. $\rho_\lambda^\vee \otimes \epsilon^{1-n} \cong \rho_\lambda^c$, where ϵ is the λ-adic cyclotomic character.*

Proof. See [HT01, TY07] and Section 5 of [BLGGT14]. Purity is proved in [Clo13]. □

Remark 6.5. Before we continue, let us add a remark on the role of the field of rationality: We will in the sequel frequently consider finite extensions $\mathcal{E}$ of $\mathcal{E}_\Pi$. We can understand $\mathcal{R}_\Pi$ as an $\mathcal{E}$-rational family by the following convention: If λ is a place of $\mathcal{E}$, then write λ' for the place of $\mathcal{E}_\Pi$ below λ. Then we get from $\mathcal{R}_\Pi$ a family

$$\mathcal{R}'_\Pi = \Big(\rho'_\lambda : \mathrm{Gal}_F \to \mathrm{GL}_n(\mathcal{E}_\lambda)\Big)_{\lambda \in \mathrm{Pl}^{\mathrm{fin}}_{\mathcal{E}}}$$

with

$$\rho'_\lambda : \mathrm{Gal}_F \xrightarrow{\rho_{\lambda'}} \mathrm{GL}_n(\mathcal{E}_{\Pi,\lambda'}) \hookrightarrow \mathrm{GL}_n(\mathcal{E}_\lambda).$$

We will not distinguish in our notation between $\mathcal{R}_\Pi$ and $\mathcal{R}'_\Pi$ or between ρ_λ and ρ'_λ if their target is clear from the context, i.e. we will suppress the prime symbol $'$.

Now, let $\mathcal{E}$ be a finite extension of $\mathcal{E}_\Pi$ as in Remark 6.5. Replacing the entries of $\mathcal{R}_\Pi$ by their mod-λ reductions, we get a family of modular Galois representations

$$\Big(\overline{\rho}_\lambda : \mathrm{Gal}_F \to \mathrm{GL}_n(k_\lambda)\Big)_{\lambda\in \mathrm{Pl}_{\mathcal{E}}^{\mathrm{fin}}}, \tag{6.1}$$

where k_λ denotes the residue field of $\mathcal{E}_\lambda$, which is a finite extension of $\mathbb{F}_{\ell(\lambda)}$. We assume the following:

Assumption 6.6. The set

$$\Lambda^1 := \{\lambda \in \mathrm{Pl}_{\mathcal{E}}^{\mathrm{fin}} \mid \overline{\rho}_\lambda \text{ is absolutely irreducible } \}$$

has Dirichlet density 1 in $\mathrm{Pl}_{\mathcal{E}}^{\mathrm{fin}}$. Consequently, the set

$$\Lambda^1_{\mathbb{Q}} := \{\ell \in \mathrm{Pl}_{\mathbb{Q}}^{\mathrm{fin}} \mid \lambda \in \Lambda^1 \text{ for all } \lambda \in \mathrm{Pl}_{\mathcal{E}} \text{ above } \ell \ \}$$

has Dirichlet density 1 in the set of rational primes. Finally, we take $\Lambda^1_{\mathcal{E}}$ to be the set of primes of E above $\Lambda^1_{\mathbb{Q}}$.

We remark that the validity of Assumption 6.6 does not depend on the choice of $\mathcal{E}$. Observe moreover that we can have $\Lambda^1_{\mathcal{E}} \subsetneq \Lambda^1$, but still the following holds:

Lemma 6.7. *The set $\Lambda^1_{\mathcal{E}}$ has Dirichlet density 1 in $\mathrm{Pl}_{\mathcal{E}}^{fin}$.*

Proof. Recall that a subset $X \subset \mathrm{Pl}_E^{\mathrm{fin}}$ of a number field E has Dirichlet density $\delta(X) \in [0,1]$ if

$$\sum_{\nu\in X} \frac{1}{\mathcal{N}(\nu)^s} \sim \delta(X).\log\frac{1}{s-1} \text{ as } s \searrow 0. \tag{6.2}$$

Recall moreover [Mil13, Proposition 4.4 (d) and (e)]:

- Let $X_1, X_2 \subset \mathrm{Pl}_E^{\mathrm{fin}}$ be disjoint, then if any two of $\delta(X_1), \delta(X_2), \delta(X_1 \sqcup X_2)$ are defined, so is the third and we have
$$\delta(X_1 \sqcup X_2) = \delta(X_1) + \delta(X_2).$$
- Let $X_1 \subset X_2 \subset \mathrm{Pl}_E^{\mathrm{fin}}$, then $\delta(X_1) \leq \delta(X_2)$.

Write $\Lambda^1 = \mathrm{Pl}_{\mathcal{E}}^{\mathrm{fin}} - \mathscr{D}$ for a suitable defect set $\mathscr{D}$ of Dirichlet density 0. We first check the claim under the assumption that $\mathcal{E}|\mathbb{Q}$ is Galois: As $\mathrm{Gal}(\mathcal{E}|\mathbb{Q})$ permutes the places above each rational prime, we have

$$\Lambda^1_{\mathcal{E}} = \mathrm{Pl}_{\mathcal{E}}^{\mathrm{fin}} - \Big(\bigcup_{\sigma\in\mathrm{Gal}(\mathcal{E}|\mathbb{Q})} \mathscr{D}^\sigma\Big).$$

But $\mathrm{Gal}(\mathcal{E}|\mathbb{Q})$ is finite and by the characterization in (6.2) we have $\delta(\mathscr{D}^\sigma) = \delta(\mathscr{D}) = 0$ for each $\sigma \in \mathrm{Gal}(\mathcal{E}|\mathbb{Q})$, so $\delta(\Lambda^1_{\mathcal{E}}) = 1$.

For the general case, we argue as follows: Let $\widetilde{\Lambda^1}$ and $\widetilde{\mathscr{D}}$ be the sets of places of the Galois closure $\widetilde{\mathcal{E}}$ of $\mathcal{E}$ above Λ^1 and $\mathscr{D}$, so that we have $\mathrm{Pl}_{\widetilde{\mathcal{E}}}^{\mathrm{fin}} = \widetilde{\Lambda^1} \sqcup \widetilde{\mathscr{D}}$. By assumption, $\delta(\Lambda^1) = 1$, but comparing the left hand side of (6.2) for $E = \mathcal{E}, X = \Lambda^1$ and for $E = \widetilde{\mathcal{E}}, X = \widetilde{\Lambda^1}$, we see that $\delta(\widetilde{\Lambda^1}) = 1$ and, hence, $\delta(\widetilde{\mathscr{D}})$ must vanish. By applying the above argument to $\widetilde{\mathcal{E}}$ instead of $\mathcal{E}$ we end up with a decomposition $\mathrm{Pl}_{\widetilde{\mathcal{E}}}^{\mathrm{fin}} = \widetilde{\Lambda^1_{\widetilde{\mathcal{E}}}} \sqcup \widetilde{\mathscr{D}}_{\widetilde{\mathcal{E}}}$ such that $\delta(\widetilde{\Lambda^1_{\widetilde{\mathcal{E}}}}) = 1$, $\widetilde{\mathscr{D}}_{\widetilde{\mathcal{E}}} = 0$ and such that $\widetilde{\Lambda^1_{\widetilde{\mathcal{E}}}}$ contains precisely the places above $\Lambda^1_{\mathcal{E}}$. Comparing the left hand side of (6.2) for $E = \mathcal{E}, X = \mathrm{Pl}_{\mathcal{E}}^{\mathrm{fin}} - \Lambda^1_{\mathcal{E}}$ and for $E = \widetilde{\mathcal{E}}, X = \widetilde{\mathscr{D}}_{\widetilde{\mathcal{E}}}$, we see that $\delta(\mathrm{Pl}_{\mathcal{E}}^{\mathrm{fin}} - \Lambda^1_{\mathcal{E}}) = 0$, i.e. that $\delta(\Lambda^1_{\mathcal{E}}) = 1$. □

Remark 6.8. Assumption 6.6 is known e.g. if Π is extremely regular [BLGGT14] or if $n \leq 5$ [CG13]. Results in this direction are also contained in [PT15], but they are not directly applicable to our situation.

This implies that we can extend $\overline{\rho}$ to the group $\mathcal{G}_n$ (see Section 6.1.4 below, in particular Lemma 6.21) at all $\lambda \in \Lambda^1_{\mathcal{E}}$, i.e. there is a family of Galois representations

$$\Big(\overline{r}_\lambda : \mathrm{Gal}_{F^+} \to \mathcal{G}_n(k_\lambda)\Big)_{\lambda \in \Lambda^1_{\mathcal{E}}} \tag{6.3}$$

which fulfill the Schur-conditions **(SmCtr)** and **(Centr)** from Section 2.3. The purpose of this section is to prove (in Theorem 6.56 below) that certain deformation rings associated to these representations $\overline{r}_\lambda$ are unobstructed for a large class of places λ (and for the choice $\mathcal{E} = \mathcal{E}_\Pi$).

6.1 Preparations

6.1.1 Algebraic representations of GL_n and U_n

Recall the following characterization of representations of the unitary and general linear groups in terms of their highest weight character:

Theorem 6.9. *1. The isomorphism classes of complex, irreducible, continuous representations of $U_n(\mathbb{R})$ can be parametrized by the set*

$$\mathbb{Z}^{n,+} = \big\{(\omega_1, \dots, \omega_n) \in \mathbb{Z}^n \;\big|\; \omega_1 \geq \dots \geq \omega_n \big\}.$$

For such a tuple $\underline{\omega} \in \mathbb{Z}^{n,+}$ denote the corresponding representation by

$$\xi^u_{\underline{\omega}} : U_n(\mathbb{R}) \longrightarrow \mathrm{GL}(W^u_{\underline{\omega}}),$$

where $W^u_{\underline{\omega}}$ denotes a suitable $\mathbb{C}$-vector space.

2. Let K be a finite extension of $\mathbb{Q}_\ell$ with ring of integers $\mathcal{O}_K$. Then the isomorphism classes of irreducible, algebraic $\mathrm{GL}_n(K)$-representations over K can be parametrized by $\mathbb{Z}^{n,+}$. For such a tuple $\underline{\omega} \in \mathbb{Z}^{n,+}$ denote the corresponding representation by

$$\xi^K_{\underline{\omega}} : \mathrm{GL}_n(K) \longrightarrow \mathrm{GL}(W^K_{\underline{\omega}}),$$

where $W^K_{\underline{\omega}}$ denotes a suitable K-vector space.

3. The representations $\xi^K_{\underline{\omega}}$ from part 2. admit integral models: For each $\underline{\omega}$ there exists a finite free $\mathcal{O}_K$-module $M^{\mathcal{O}_K}_{\underline{\omega}}$ and a representation

$$\xi^{\mathcal{O}_K}_{\underline{\omega}} : \mathrm{GL}_n(\mathcal{O}_K) \longrightarrow \mathrm{GL}(M^{\mathcal{O}_K}_{\underline{\omega}})$$

such that

$$\xi^K_{\underline{\omega}} | \, \mathrm{GL}_n(\mathcal{O}_K) \;=\; \xi^{\mathcal{O}_K}_{\underline{\omega}} \otimes_{\mathcal{O}_K} K.$$

Proof. Parts 1. and 2. are taken almost verbatim from [BC09] and the last part can be found e.g. in [Gue11] or [Ger10a]. □

6.1.2 A lemma on prime densities in non-Galois extensions

For this paragraph, consider the following setup:

- As before, F denotes a CM field with totally real subfield F^+;
- $L^+ = F^+(\sqrt{d_1}, \dots, \sqrt{d_k})$ denotes a totally real extension of F^+ of degree 2^k, obtained by adjoining the square roots of k elements $d_1, \dots, d_k \in \mathbb{N}$.

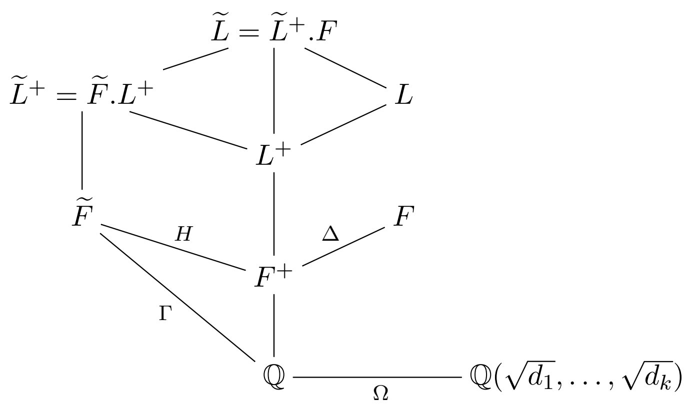
Let us also assume that each d_i is a non-square in the Galois closure $\widetilde{F}^+$ of F^+. Then we have

Lemma 6.10. *Let $\Xi_{\mathbb{Q}}$ be the set of all those rational primes ℓ with the following property: For any place $\wp$ of L^+,*

$$\left[\wp|\ell\right] \implies \left[\wp \text{ splits in } L|L^+\right].$$

Then the density $\delta(\Xi_{\mathbb{Q}})$ of $\Xi_{\mathbb{Q}}$ in the set of all rational primes is at least $1 - \frac{1}{2^k}$.

Proof. Consider the following diagram of fields



with corresponding Galois groups

- $\Delta = \mathbb{Z}/2\mathbb{Z}$;
- $\Omega = \left(\mathbb{Z}/2\mathbb{Z}\right)^k$;
- Γ and H, for which we don't make an assumption.

By the assumption that the d_i are not squares we have

$$\mathrm{Gal}(\widetilde{L}^+|\mathbb{Q}) \cong \Gamma \times \Omega,$$

and hence

$$\mathrm{Gal}(\widetilde{L}|\mathbb{Q}) \cong \Gamma \times \Omega \times \Delta.$$

Let $\mathfrak{P}$ be a place of $\widetilde{L}$ with corresponding Frobenius element $(\gamma, \omega, \delta) \in \mathrm{Gal}(\widetilde{L}|\mathbb{Q})$. As Ω and Δ are abelian, the conjugacy class of $\mathfrak{P}$ can be written as

$$\{(u\gamma u^{-1}, \omega, \delta) \mid u \in \Gamma\}$$

and consists precisely of the Frobenii of the places of L lying over the same rational prime p as $\mathfrak{P}$.

Let $\wp$ be the place of L below $\mathfrak{P}$. Its Frobenius element is given by

$$(\gamma, \omega, \delta)^{e_{\gamma,\omega,\delta}} \in H \times \{1\} \times \Delta = \mathrm{Gal}(\widetilde{L}|L^+)$$

for $e_{\gamma,\omega,\delta} \in \mathbb{N}$ minimal such that $(\gamma, \omega, \delta)^{e_{\gamma,\omega,\delta}} \in H \times \{1\} \times \Delta$.

The condition that $\wp$ splits in $L|L^+$ then amounts precisely to

$$(\gamma, \omega, \delta)^{e_{\gamma,\omega,\delta}} \in H \times \{1\} \times \{1\},$$

or, written in a more sophisticated way, that $q((\gamma, \omega, \delta)^{e_{\gamma,\omega,\delta}}) = 1$, where

$$q : \mathrm{Gal}(\widetilde{L}|L^+) \to \mathrm{Gal}(\widetilde{L}|L^+)/\,\mathrm{Gal}(\widetilde{L}|\widetilde{L}^+)$$

is the quotient map.

If $\omega \neq 1$, we clearly must have $2|e_{\gamma,\omega,\delta}$, which implies that $\wp$ splits in $L|L^+$. It is also important to note that the condition $\omega \neq 1$ is not destroyed by conjugation inside $\mathrm{Gal}(\widetilde{L}|\mathbb{Q})$.

Now, set

$$\Xi^* = \{(\gamma, \omega, \delta) \in \mathrm{Gal}(\widetilde{L}|\mathbb{Q}) \mid q((\gamma, \omega, \delta)^{e_{\gamma,\omega,\delta}}) = 1\}$$

and consider the subset $\Xi \subset \Xi^*$ which consists of those $g \in \Xi^*$ for which the complete conjugacy class is contained in Ξ^*, i.e.

$$\Xi = \{g \in \Xi^* \mid \langle g \rangle \subset \Xi^*\}.$$

We can give another characterization of this set: Ξ is the union of all conjugacy classes $\langle g \rangle \subset \mathrm{Gal}(\widetilde{L}|\mathbb{Q})$ with the following property: If $\mathbf{P}_g$ denotes the set of all places $\mathfrak{P}$ of $\widetilde{L}$ such that $\mathrm{Frob}_{\mathfrak{P}} \in \langle g \rangle$, then for any place $\wp$ of L^+ the following holds:

$$\Big[\exists \mathfrak{P} \in \mathbf{P}_g \text{ such that } \mathfrak{P} \text{ divides } \wp\Big] \implies \Big[\wp \text{ splits in } L|L^+\Big]$$

Then we have

$$\#\Xi \geq \#\{(\gamma, \omega, \delta) \in \mathrm{Gal}(\widetilde{L}|\mathbb{Q}) \mid \omega \neq 1\} = (2^k - 1).2.\#\Gamma.$$

As

$$\Xi_{\mathbb{Q}} = \{\ell \in \mathrm{Pl}_{\mathbb{Q}} \mid \exists g \in \Xi \text{ such that } \mathfrak{P}|\ell \text{ for all } \mathfrak{P} \in \mathbf{P}_g\},$$

it follows from Chebotarev's density theorem that

$$\delta(\Xi_{\mathbb{Q}}) \geq \frac{(2^k - 1).2.\#\Gamma}{\mathrm{Gal}(\widetilde{L}|\mathbb{Q})} = 1 - \frac{1}{2^k}. \qquad \square$$

6.1.3 Cuspidality and base change

We start with a general lemma:

Lemma 6.11. *Let π_ν be an irreducible unramified representation of $\mathrm{GL}_n(F_\nu)$ for a local non-archimedean field F_ν of characteristic 0. Let χ_ν be a smooth character of $F_\nu^\times$ such that $\pi_\nu \cong \chi_\nu \otimes \pi_\nu$.*

Then χ_ν is unramified.

Proof. As π_ν is unramified, it follows from a result of Satake [Rog97, Thm. 4 on p. 337] that there exist unramified characters

$$\psi_i : F_\nu^\times \to \mathbb{C}^\times \qquad (i \in \{1, \dots, n\})$$

such that π_ν is a subquotient of

$$\mathrm{ind}_{\mathrm{Borel}}^{\mathrm{GL}_n(F_\nu)}(\psi_1 \otimes \ldots \otimes \psi_n).$$

But then (by [Vig96, part d) of I.5.2]), $\chi_\nu \otimes \pi_\nu$ is a subquotient of

$$\chi_\nu \otimes \mathrm{ind}_{\mathrm{Borel}}^{\mathrm{GL}_n(F_\nu)}(\psi_1 \otimes \ldots \otimes \psi_n) \cong \mathrm{ind}_{\mathrm{Borel}}^{\mathrm{GL}_n(F_\nu)}\big((\chi_\nu \otimes \psi_1) \otimes \ldots \otimes (\chi_\nu \otimes \psi_n)\big).$$

Assume that χ_ν is ramified, then each $\chi_\nu \otimes \psi_i$ is ramified. Then an isomorphism $\pi_\nu \cong \chi_\nu \otimes \pi_\nu$ would be in conflict with unicity of supercuspidal support (see [GH11, Corollary 14.5.6] for $F_\nu = \mathbb{Q}_p$ and [Vig98, Chapter V.4] for the general case). □

Now, let π be an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. For a prime p in

$$\Delta_F \coloneqq \{p \in \mathrm{Pl}_{\mathbb{Q}}^{\mathrm{fin}} \mid \sqrt{p} \notin F\},$$

let us denote by Σ_p the set of Hecke characters

$$\chi : I_F / F^\times \to \mathbb{C}^\times$$

which fulfill

$$N_{F(\sqrt{p})|F}(I_{F(\sqrt{p})}) = \ker(\chi).$$

Our aim here is to prove:

Lemma 6.12. *For almost all $p \in \Delta_F$, the set*

$$\Theta_p \coloneqq \Sigma_p \cap \{\chi \mid \pi \cong \chi \otimes \pi\}$$

is empty.

We are interested in this lemma because $\Theta_p = \emptyset$ implies that the base change $\mathrm{BC}_{F(\sqrt{p})|F}(\pi)$ of π to $F(\sqrt{p})$ is cuspidal (provided that π is cuspidal), cf. [AC89, Thm. 4.2].

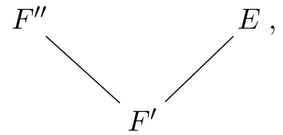
Proof. Denote by $\mathrm{Pl}_{\mathbb{Q}}^{F-\mathtt{ram}}$ the set of rational primes which ramify in the extension $F|\mathbb{Q}$. Then for an odd p in $\Delta_F - \mathrm{Pl}_{\mathbb{Q}}^{F-\mathtt{ram}}$ and a place ν of F above p, the extension $F_\nu(\sqrt{p})|F_\nu$ is ramified (as can easily be deduced from the multiplicativity of the ramification index).
Hence, the image $N_{F_\nu(\sqrt{p})|F_\nu}(F_\nu(\sqrt{p})^\times)$ equals $\varpi_{F_\nu}.N_{F_\nu(\sqrt{p})|F_\nu}(\mathcal{O}_{F_\nu(\sqrt{p})}^\times)$ and we have

$$\left[\mathcal{O}_{F_\nu}^\times : N_{F_\nu(\sqrt{p})|F_\nu}(\mathcal{O}_{F_\nu(\sqrt{p})}^\times)\right] = 2.$$

In particular, any $\chi \in \Theta_p$ is ramified at ν.

By the Lemma 6.11, this is only possible if π ramifies at ν. The claim follows. □

The remainder of this section is aimed at showing that the initial base change in our main argument later (Theorem 6.56) can be chosen such that cuspidality of the automorphic representation will not get destroyed. We start again with a general lemma. For this, consider a number field F' and two finite extensions



where we assume that $E|F'$ is Galois with Galois group

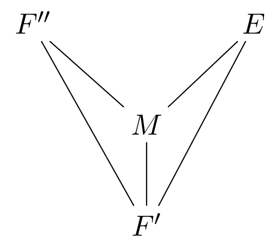
$$\mathrm{Gal}(E|F') = \langle \mathrm{Frob}_{\nu_1}, \dots, \mathrm{Frob}_{\nu_k} \rangle$$

for suitable places $\nu_1, \dots, \nu_k$ of E which are unramified in $E|F'$. Let us denote by $\Psi_{F'}(\nu_1, \dots, \nu_k)$ the set of places of F' below $\{\nu_1, \dots, \nu_k\}$.

Lemma 6.13. *Assume that $F''|F'$ splits completely at each place $w \in \Psi_{F'}$. Then*

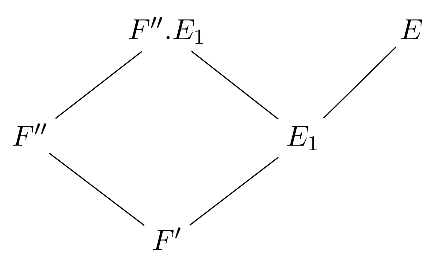
$$F'' \cap E = F'.$$

Proof. The case $E = F'$ is trivial, so we can assume that $k \geq 1$. Let us first start with the case $k = 1$: Then, clearly, $w \in \Psi_{F'}(\nu_1) = \{w\}$ is totally inert in the extension $E|F'$. Let us consider the intermediate extension



with $M \coloneqq F'' \cap E$. By the multiplicativity of the inertial degree it follows that w is totally inert in $M|F'$, but on the other hand w must also split completely in $M|F'$ (as, otherwise, it could not split completely in $F''|F'$). It follows that $M = F'$.

This result can easily be extended to arbitrary k, and we only carry out the step $k = 1 \rightsquigarrow k = 2$. For this, consider the intermediate field $E_1 = E^{\mathrm{Frob}_{\nu_2}}$ and the diagram



Let $\tilde{\nu}_i$ be the place of E_1 below ν_i, for $i \in \{1, 2\}$. Then the Galois group of the extension $E_1|F'$ is generated by $\mathrm{Frob}_{\tilde{\nu}_1}$, so it follows by the previous argument that $F'' \cap E_1 = F'$. Now, Frob_{ν_2} acts trivially on E_1, so the place w_2 of F' below ν_2 does not split in $E_1|F'$. Moreover, w_2 is not ramified as it is not ramified in $E|F'$. If follows that w_2 is totally inert in $E|F'$ and splits completely in $F''|F'$. By multiplicativity of the inertial degree, it follows that $\tilde{\nu}_2$ is totally split in the extension $F''.E_1|E_1$. It follows from the previous argument that $F''.E_1 \cap E = E_1$. We can now conclude

$$F'' \cap E = F'' \cap F'.E_1 \cap E = F'' \cap E_1 = F'. \qquad \square$$

Let us recall the following well-known[12] weak version of the Grunwald-Wang theorem:

Theorem 6.14. *Let $\Sigma = \{w_1, \dots, w_m\}$ be a finite set of places of F'. For each $i \in \{1, \dots, n\}$ choose a finite separable extension C_i of F'_{w_i}. Then there exists a finite separable extension F'' of F' and places ν_i of F'' above w_i such that F''_{ν_i} is isomorphic to C_i for every i. Moreover:*

- *Set $d_i = [C'_i : F'_{w_i}]$, then F'' can be chosen such that $[F'' : F'] = \max\{d_1, \dots, d_m\}$;*
- *If S is a finite set of non-archimedean places of F' with $S \cap \Sigma = \emptyset$, then F'' can be chosen to be unramified at all places in S;*
- *If all extensions $C_i|F'_{w_i}$ are Galois, then F'' can be chosen such that $F''|F'$ is Galois and solvable;*
- *If all extensions $C_i|F'_{w_i}$ are abelian (resp. cyclic), then F'' can be chosen such that $F''|F'$ is abelian (resp. cyclic).*

Proof. This is taken almost verbatim from [Con05, Theorem 3.1]. There, the first bullet point is only stated for the case that all d_i are equal, but our formulation is easily deductible from the proof in [Con05, Theorem 3.1]. $\square$

Recall that we have chosen a CM-field F together with an automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$ in the beginning and that we denote by $(\overline{\rho}_\lambda)_{\lambda \in \mathrm{Pl}_\mathcal{E}}$ the associated compatible family of residual GL_n-valued Galois representations. Let $S \subset \mathrm{Pl}_F$ denote the ramification set of Π and assume that any place in S is unramified in the extension $F|\mathbb{Q}$. Then there exists a finite solvable extension K of F which is a CM-field and such that the restriction of $\overline{\rho}_\lambda$ to Gal_K has unipotent ramification (in the sense of Definition 4.25) at each $\nu \in \mathrm{Pl}_K$ which lies above S if $\ell(\lambda) \neq \ell(\nu)$. Write K^+ for the maximal totally real subfield of K, then we have $K = K^+.F$.

Lemma 6.15. *There exists a finite solvable Galois extension K' of F which is a CM-field and such that*

- *The base change of Π to K' remains cuspidal;*
- *The restriction of $\overline{\rho}_\lambda$ to $\mathrm{Gal}_{K'}$ has unipotent ramification at each place $\nu \in \mathrm{Pl}^{fin}_{K'}$ above S in the sense of Definition 4.25 if $\ell(\lambda) \neq \ell(\nu)$.*

Proof. First, recall from [AC89] that there exists a finite extension E of F such that for any extension K' of F we have the following implication: If $E \cap K' = F$, then the base change of Π to K' remains cuspidal. This implication remains true after replacing E by its Galois closure, so we can assume that $E|F$ is Galois. By Chebotarev's density theorem, we can assume that $\mathrm{Gal}(E|F) = \langle \mathrm{Frob}_{\nu_1}, \dots, \mathrm{Frob}_{\nu_k} \rangle$ with $\Psi_F(\nu_1, \dots, \nu_k) \cap S = \emptyset$. Using Lemma 6.13, we have also the following implication: If for each

[12] Cf. the usage in [SW01, Section 2]: "Here and throughout the paper we use the well-known fact that one can always find a totally real cyclic extension of F with prescribed splitting and ramification at any given finite set of primes of F."

pair of places $(\nu, w) \in \mathrm{Pl}_{K'}^{\mathrm{fin}} \times \mathrm{Pl}_F^{\mathrm{fin}}$ with $\nu | w, w \in \Psi_F(\nu_1, \dots, \nu_k)$ we have $[K'_\nu : F_w] = 1$, then the base change of Π to K' remains cuspidal.

Now apply Theorem 6.14 to

- $F' = F^+$;
- $\Sigma = \{w_1, \dots, w_m\}$ is the set of places of F^+ which divide ∞ or lie below $S \sqcup \Psi_F(\nu_1, \dots, \nu_k)$,
- $C_i = \begin{cases} F^+_{w_i} & \text{if } w_i \text{ divides } \infty \text{ or lies below } \Psi_F(\nu_1, \dots, \nu_k); \\ \widetilde{K}_{\tilde{\nu}_i} & \text{if } w_i \text{ lies below } S. \end{cases}$

(Here, $\tilde{\nu}_i$ denotes an arbitrary choice of a place of the Galois closure $\widetilde{K}$ of K which lies above w_i.) This yields a finite solvable totally real Galois extension F'' of $F' = F^+$, and the extension $K' := F''.F$ fulfills the conditions of the lemma. □

We now give a slight generalization of this result. First, consider the following general lemma:

Lemma 6.16. *Consider a field K together with a fixed algebraic closure $\overline{K}$. Let E, F be finite extensions of K, both contained in $\overline{K}$, such that $F|K$ is separable and such that E contains the Galois closure $\widetilde{F}$ of F. Then*

$$F \otimes_K E = E^{[F:K]}.$$

Proof. Write $F = K[x]/(f(x))$ with $f \in K[x]$ irreducible. Over E, f decomposes completely in linear factors, i.e.

$$f = \prod_{i=1}^{[F:K]} (x - \alpha_i).$$

Using the Chinese remainder theorem, we get

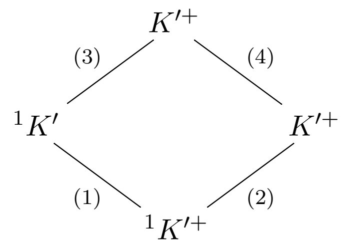
$$F \otimes_K E = E[x]/(f(x)) = \prod_{i=1}^{[F:K]} E[x]/(x - \alpha_i) = E^{[F:K]}.$$ □

Corollary 6.17. *The extension K' in Lemma 6.15 can be chosen such that any place $\nu \in \mathrm{Pl}_{K'+}$ above S is split in the extension $K'|K'^+$. Here, $K'^+ = F''$ (in the notation of the proof of Lemma 6.15) denotes the maximal totally real subfield of K'.*

Proof. Let us denote the extension yielded by Lemma 6.15 temporarily by ${}^1K'$ and the maximal totally real subfield by ${}^1K'^+$. We apply Theorem 6.14 once again with

- $F' =^1 K'^+$;
- $\Sigma = \{w_1, \dots, w_m\}$ is the set of places of ${}^1K'^+$ which lie above the subset of Pl_{F^+} which was denoted by Σ in the proof of Lemma 6.15, i.e. the set of places above the set $\overline{S} \subset \mathrm{Pl}_{F^+}$ of places below S, above ∞ or above the set $\overline{\Psi} \subset \mathrm{Pl}_{F^+}$ of places below $\Psi_F(\nu_1, \dots, \nu_k)$.
- $C_i = \begin{cases} {}^1K'^+_{w_i} & \text{if } w_i \text{ divides } \infty \text{ or lies above } \overline{\Psi}, \\ {}^1K'_{\tilde{w}_i} & \text{if } w_i \text{ lies above } \overline{S}, \end{cases}$
  where $\tilde{w}_i$ denotes an arbitrary place of ${}^1K'$ above w_i.

This yields a quadratic, totally real extension K'^+ of ${}^1K'^+$ and we claim that $K' = K'^+.F$ fulfills the condition of the corollary. It is clear that the two bullet points of Theorem 6.15 carry over. It remains to check that any place $\nu \in \mathrm{Pl}_{K'^+}$ above $\overline{S}$ is split in the extension $K'^+|K'^+$. Denote by $\overline{\nu}$ the place of ${}^1K'^+$ below ν and consider the following diagram



All the extensions (1),(2),(3) and (4) are quadratic.

Case 1: ($\overline{\nu}$ is split in (1).) As $\overline{\nu}$ is split in (1), it follows that ${}^1K'_{\tilde{\overline{\nu}}} = {}^1K'^+_{\overline{\nu}}$, hence that $\overline{\nu}$ is split in (2). It follows by [Neu99, Exercise 3 on p. 52] that $\overline{\nu}$ splits completely in $K'^+|{}^1K'^+$. But this implies that $\overline{\nu}$ splits in (4) by the multiplicativity of the inertial degree.

Case 2: ($\overline{\nu}$ is inert in (1).) This implies that ${}^1K'_{\tilde{\overline{\nu}}} \supsetneq {}^1K'^+_{\overline{\nu}}$ and hence that $\overline{\nu}$ is inert in (2). Moreover, we see that the conditions of Lemma 6.16 are fulfilled for the choices

- $K = {}^1K'^+_{\overline{\nu}}$,
- $E = K'^+_{\nu}$,
- $F = {}^1K'_{\tilde{\overline{\nu}}}$.

It follows that

$$ {}^1K'_{\tilde{\overline{\nu}}} \otimes_{{}^1K'^+_{\overline{\nu}}} K'^+_{\nu} = K'^+_{\nu} \prod K'^+_{\nu}, $$

i.e. that ν splits in (4). □

6.1.4 The group $\mathcal{G}_n$ from Clozel-Harris-Taylor

Let $n \in \mathbb{N}$ and recall from [CHT08] the following definition:

Definition 6.18. By $\mathcal{G}_n$ we denote the group scheme over $\mathbb{Z}$ given by

$$ \big(\mathrm{GL}_n \times \mathrm{GL}_1\big) \rtimes \{1, j\}\ , $$

where j acts as $j(g,\mu)j = (\mu\, {}^tg^{-1}, \mu)$. $\mathcal{G}_n$ defines a linear algebraic group, which can either be deduced from the definition or from the embedding $\mathcal{G}_n \hookrightarrow \mathrm{GSp}_{2n}$ given in [BLGGT14, Section 1.1]. We denote by $\mathcal{G}_n^0$ the connected component of $\mathcal{G}_n$ and by $\mathit{m}\colon \mathcal{G}_n \to \mathrm{GL}_1$ the multiplier character given by

$$ \mathit{m}\big((g,\mu) \rtimes x\big) = (-1)^{1+\mathrm{ord}(x)}.\mu. $$

We write $\mathfrak{g}_n$ for the Lie algebra of $\mathcal{G}_n$. Observe that we differ here from [CHT08], where $\mathfrak{g}_n$ is used for the Lie algebra of GL_n (to which we refer by $\mathfrak{gl}_n$ instead).

Proposition 6.19. $\mathcal{G}_n^{\mathsf{der}} = (\mathrm{GL}_n \times 1) \rtimes 1 \cong \mathrm{GL}_n$ *and* $\mathcal{G}_n^{\mathsf{ab}} \cong \mathrm{GL}_1 \times \{1, j\}$.

Proof. It is clear that $\mathcal{G}_n^{\mathtt{der}} \subset (\mathrm{GL}_n \times 1) \rtimes 1$, as any commutator in $\mathcal{G}_n$ is contained in this subgroup. For the other inclusion, by taking commutators of the form $[x, y]$ with $x, y \in (\mathrm{GL}_n \times 1) \rtimes 1$, we see that $\mathcal{G}_n^{\mathtt{der}} \supset (\mathrm{SL}_n \times 1) \rtimes 1$. So we are good if we can show that an arbitrary scalar matrix

$$z \in (\mathbb{G}_m \times 1) \rtimes 1 \subset (\mathrm{GL}_n \times 1) \rtimes 1$$

is contained in $\mathcal{G}_n^{\mathtt{der}}$. For this, consider the element $\tilde{z} \in (1 \times \mathrm{GL}_1) \rtimes 1$ corresponding to z via $\mathbb{G}_m \cong \mathrm{GL}_1$ and observe that $z = [j, \tilde{z}]$. □

Proposition 6.20. *Let P be the image of the map $\mathrm{GL}_1 \to \mathrm{GL_n} \times \mathrm{GL}_1$, sending λ to $\mathrm{diag}(\lambda, \dots, \lambda) \times \lambda^2$. Then the center $Z_{\mathcal{G}_n}$ of $\mathcal{G}_n$ fulfills*

$$Z_{\mathcal{G}_n} = P \rtimes 1 \cong \mathrm{GL}_1 .$$

Proof. First, consider a $y = (g, \mu) \rtimes j \in \mathcal{G}_n$ and compare y with ryr^{-1}, where $r = (\mathrm{diag}(\mu, \dots, \mu), 1) \rtimes 1$ is an element of $\mathcal{G}_n$ and for some μ fulfilling $\mu^2 \neq 1$. This implies that y cannot be in the center, i.e. that $Z_{\mathcal{G}_n} \subset \mathcal{G}_n^0$. So let $x = (g, \mu) \rtimes 1 \in Z_{\mathcal{G}_n}$, then we see that $g \in Z_{\mathrm{GL}_n}$ must be a diagonal matrix. Comparing x and jxj, we see that (g, μ) must be contained in P. It is easy to check that any element in $P \rtimes 1$ is central. □

Recall that we consider a CM field F with maximal real subfield denoted by F^+.

Lemma 6.21. *Let $c \in \mathrm{Gal}_{F^+}$ be a complex conjugation and fix a topological field $\mathbb{K}$ together with a continuous character $\chi : \mathrm{Gal}_{F^+} \to \mathbb{K}^\times$. Let*

$$\rho : \mathrm{Gal}_F \longrightarrow \mathrm{GL}_n(\mathbb{K})$$

be continuous and absolutely irreducible and assume $\chi\rho^\vee \cong \rho^c$. (The latter condition means that ρ is a conjugate self-dual representation.)

Then there exists a continuous representation

$$r : \mathrm{Gal}_{F^+} \longrightarrow \mathcal{G}_n(\mathbb{K}),$$

such that

- $r|\,\mathrm{Gal}_F = \rho$;
- $(\mathit{m} \circ r)|\,\mathrm{Gal}_F = \chi|\,\mathrm{Gal}_F$;
- $r(c) \in \mathcal{G}_n(\mathbb{K}) - \mathcal{G}_n^0(\mathbb{K})$.

There is a bijection between the $\mathrm{GL}_n(\mathbb{K})$-conjugacy classes of such r and $\mathbb{K}/\mathbb{K}^2$, so in particular r is uniquely determined (up to conjugacy) if $\mathbb{K}$ is algebraically closed. Moreover, if ρ is Schur, then so is r.

Proof. This is [CHT08, Lemma 1.1.4] in the formulation of [Gee11, Lemma 5.1.1]. The last part about Schurness of r follows easily from [CHT08, Lemma 2.1.3]. □

Deformations We will be interested in deformations of $\mathcal{G}_n$-valued residual representations. In the local split case, this becomes particularly simple: Let $\mathbb{K} = k$ be a finite field of positive characteristic and let $\overline{\rho}, \overline{\chi}, \overline{r}$ be as in Lemma 6.21 (but we put a bar over it to indicate that we consider them as residual objects). Let Λ be the ring of integers of a finite extension of $W(k)$.

Proposition 6.22. *Let ν be a place of F^+ which splits as $\tilde{\nu}\tilde{\nu}^c$ in F (in particular, we fix a place $\tilde{\nu}$ above ν). We denote by $\overline{r}_\nu$ the restriction of $\overline{r}$ to the decomposition group at ν and by $\overline{\rho}_{\tilde{\nu}}$ the restriction of $\overline{\rho}$ to the decomposition group at $\tilde{\nu}$. Fix a lift $\chi_\nu : \mathrm{Gal}_{F_\nu^+} \to \Lambda^\times$ of $\mathfrak{m} \circ \overline{r}_\nu$. Then*

$$R_\Lambda^{\chi_\nu,(\square)}(\overline{r}_\nu) \cong R_\Lambda^{(\square)}(\overline{\rho}_{\tilde{\nu}}) \tag{6.4}$$

and

$$H^i(F_\nu^+, \mathfrak{g}_n^{\mathrm{der}}) \cong H^i(F_{\tilde{\nu}}, \mathfrak{gl}_n),\ Z^1(F_\nu^+, \mathfrak{g}_n^{\mathrm{der}}) \cong Z^1(F_{\tilde{\nu}}, \mathfrak{gl}_n)$$

for $i \in \mathbb{N}_0$.

(As usual, in the unframed situation our claim in (6.4) implicitly assumes that $\overline{r}_\nu$ is Schur.)

Proof. As $\mathrm{Gal}_{F_\nu^+} = \mathrm{Gal}_{F_{\tilde{\nu}}}$ is contained in Gal_F, the image of $\overline{r}_\nu$ (and all of its lifts) must be contained in $\mathcal{G}_n^0$: The diagram

$$\begin{array}{ccc}
\mathrm{Gal}_{F_\nu^+} & \xrightarrow{(\overline{r}_\nu, \overline{\chi}_\nu)} & \mathrm{GL}_n(k) \times \mathrm{GL}_1(k) \\
 & & \downarrow \\
\downarrow \overline{r}_\nu & & \mathcal{G}_n(k) \\
 & & \downarrow \\
\mathcal{G}_n(k) & \longrightarrow & \mathbb{Z}/2\mathbb{Z}
\end{array}$$

commutes and the resulting map $\mathrm{Gal}_{F_\nu^+} \to \mathbb{Z}/2\mathbb{Z}$ is trivial.

As we fixed the multiplier character for the left hand side, it is therefore clear that there is a natural isomorphism of the functors

$$D_\Lambda^{\chi_\nu,(\square)}(\overline{r}_\nu) \cong D_\Lambda^{(\square)}(\overline{\rho}_{\tilde{\nu}}).$$

The last part is clear as we have an isomorphism of the Lie algebras $\mathfrak{gl}_n \cong \mathfrak{g}_n^{\mathrm{der}}$, compatible with the action of $\mathrm{Gal}_{F_\nu^+} = \mathrm{Gal}_{F_{\tilde{\nu}}}$. ∎

Let S be a finite set of finite places of F^+ which split in F. For each $\nu \in S$ fix a place $\tilde{\nu}$ of F above ν and set $\widetilde{S} = \{\tilde{\nu} | \nu \in S\}$. For a global residual representation

$$\overline{r} : \mathrm{Gal}_{F^+,S} \longrightarrow \mathcal{G}_n(k)$$

we have defined what we mean by a lifting/deformation problem in Sections 2.2 and 2.3. Proposition 6.22 justifies the following alternative characterization:

Definition 6.23 (Deformation problem, following [CHT08])**.** Fix a character $\chi : \mathrm{Gal}_{F^+,S} \to \Lambda^\times$ and set $\chi_\nu = \chi|\,\mathrm{Gal}_{F_\nu^+}$ for $\nu \in S$. Moreover, for each $\nu \in S$ fix a deformation condition $\mathcal{D}_\nu$ for the functor $D_\Lambda^{\chi_\nu,(\square)}(\overline{r}_\nu) \cong D_\Lambda^{(\square)}(\overline{\rho}_{\tilde{\nu}})$. Then the collection

$$\mathscr{S} = (F|F^+, S, \widetilde{S}, \Lambda, \overline{r}, \chi, \{\mathcal{D}_\nu\}_{\nu \in S})$$

defines a deformation problem for the functor $D_\Lambda^{(\square)}(\bar{r})$ as follows: A (framed) deformation r of $\bar{r}$ (to some $A \in \mathcal{C}_\Lambda$) is of type $\mathscr{S}$ if and only if

- $\mathpzc{m} \circ r = \chi$;
- for each $\nu \in S$, $r|\operatorname{Gal}_{F_\nu^+}$ is a deformation of $\bar{r}_\nu$ of type $\mathcal{D}_\nu$.

In this way, we get a relatively representable subfunctor $D_\Lambda^{\mathscr{S},(\square)}(\bar{r}) \subset D_\Lambda^{(\square)}(\bar{r})$ which fits in the framework of Section 2.3. For the framed functors, however, there is a slight discrepancy between our conventions and the conventions in [CHT08] which we will explain now.

Framing conventions Retain all notation from above. Let $\mathscr{S}$ be a global deformation problem in the sense of Definition 6.23. Assume that $T \subset S$ is not empty and recall our Definition 2.59 for the multiply framed deformation functor $D_\Lambda^{\square_T,\mathscr{S}}(\bar{r})$ and its representing object $R_\Lambda^{\square_T,\mathscr{S}}(\bar{r})$.

Definition 6.24. A T-framed lifting (in the sense of [CHT08] and with respect to χ) of $\bar{r}$ to $A \in \mathcal{C}_\Lambda$ is a tuple $(r, \alpha_\nu)_{\nu \in T}$, where r is an $\mathcal{G}_n(A)$-valued lift of $\bar{r}$ and $\alpha_\nu \in 1 + \mathbb{M}_{n\times n}(\mathfrak{m}_A)$ and where we demand $\mathpzc{m} \circ r = \chi$. Two framed liftings $(r, \alpha_\nu)_{\nu\in T}$ and $(r', \alpha'_\nu)_{\nu \in T}$ are equivalent if there is a $\beta \in 1 + \mathbb{M}_{n\times n}(\mathfrak{m}_A)$ such that $r' = \beta r \beta^{-1}$ and $\alpha'_\nu = \beta\alpha_\nu$ for all $\nu \in T$. An equivalence class is called a T-framed deformation and the corresponding functor is denoted by $D_\Lambda^{\boxdot_T}(\bar{r})$. If $\mathscr{S}$ is as above, this gives rise to a conditioned deformation functor $D_\Lambda^{\boxdot_T,\mathscr{S}}(\bar{r})$.

Proposition 6.25. *$D_\Lambda^{\boxdot_T,\mathscr{S}}(\bar{r})$ is representable by an object $R_\Lambda^{\boxdot_T,\mathscr{S}}(\bar{r})$ which fulfills*

$$R_\Lambda^{\square_T,\mathscr{S}}(\bar{r}) \cong R_\Lambda^{\boxdot_T,\mathscr{S}}(\bar{r})[[X_1, \dots, X_t]]$$

with $t = \#T$.

Proof. The statement about representability is contained in [CHT08, Proposition 2.2.9]. For the second claim, consider for $A \in \mathcal{C}_\Lambda$ the assignment

$$D_\Lambda^{\square_T,\mathscr{S}}(\bar{r})(A) \longrightarrow D_\Lambda^{\boxdot_T,\mathscr{S}}(\bar{r})(A) \times D_\Lambda^{\square_T,\chi}(\mathpzc{m}\circ\bar{r})(A)$$

given by

$$\big(r, (r_\nu, \beta_\nu)_T\big) \longmapsto \big(r, (r_\nu, \beta_\nu^{(1)})_T\big) \times \big(\chi, (\chi_\nu, \beta_\nu^{(2)})_T\big),$$

where we split up $\beta \in \mathcal{G}_n^0(A)$ as $\beta = (\beta^{(1)}, \beta^{(2)})$ with $\beta^{(1)} \in \mathrm{GL}_n(A)$ and $\beta^{(2)} \in \mathrm{GL}_1(A)$. It is easily checked that this provides a natural isomorphism of the functors $R_\Lambda^{\square_T,\mathscr{S}}(\bar{r})$ and $R_\Lambda^{\boxdot_T,\mathscr{S}}(\bar{r}) \times R_\Lambda^{\square_T,\chi}(\mathpzc{m}\circ\bar{r})$. We conclude from Proposition 2.4 that

$$R_\Lambda^{\square_T,\mathscr{S}}(\bar{r}) \cong R_\Lambda^{\boxdot_T,\mathscr{S}}(\bar{r}) \mathbin{\hat{\otimes}_\Lambda} R_\Lambda^{\square_T,\chi}(\mathpzc{m}\circ\bar{r}) = R_\Lambda^{\boxdot_T,\mathscr{S}}(\bar{r}) \mathbin{\hat{\otimes}_\Lambda} \Lambda[[X_1, \dots, X_t]].$$ □

6.2 Automorphic forms and Hecke algebras

Recall from the beginning of this section that we are working with a CM-field F with totally real subfield F^+ and with an automorphic representation Π of the group $\mathrm{GL}_n(\mathbb{A}_F)$. Let us impose

Assumption 6.26. 1. $F|F^+$ is unramified at all finite places;

2. If n is even, then $\frac{n}{2}.[F^+ : \mathbb{Q}]$ is even.

This allows us to fix a definite unitary group H over $\mathcal{O}_{F_+}$ as considered in [Gue11, Section 2.1] or [Ger10a, Section 1.1], whose key properties we recall here:

- The extension of scalars of H to F^+ is an outer form of GL_n / F^+ which becomes isomorphic to GL_n / F after extending scalars to F;
- H is quasi-split at every finite place of F^+;
- H is totally definite, i.e. $H(F^+_\infty)$ is compact and

$$H(F^+_\nu) \cong U_n(\mathbb{R})$$

  for all infinite places ν of F^+;
- For any finite place ν of F^+ which splits as $\tilde{\nu}\tilde{\nu}^c$ in F, we can choose an isomorphism

$$\iota_{\tilde{\nu}} : H(F^+_\nu) \xrightarrow{\cong} \mathrm{GL}_n(F_{\tilde{\nu}})$$

  whose restriction to $H(\mathcal{O}_{F^+_\nu})$ provides an isomorphism $H(\mathcal{O}_{F^+_\nu}) \cong \mathrm{GL}_n(\mathcal{O}_{F_{\tilde{\nu}}})$;

Level subgroup Let us fix two disjoint finite sets $\Sigma_{\mathtt{ram}}, \Sigma_{\mathtt{aux}}$ of finite primes of F^+ subject to the following conditions:

- each $\nu \in \Sigma_{\mathtt{ram}} \sqcup \Sigma_{\mathtt{aux}}$ is split in $F|F^+$;
- each $\nu \in \Sigma_{\mathtt{aux}}$ is unramified over $\ell(\nu)$ in $F^+|\mathbb{Q}$;
- $[F(\zeta_{\ell(\nu)}) : F] > n$ for all $\nu \in \Sigma_{\mathtt{aux}}$;.

We write $\mathcal{T} = \Sigma_{\mathtt{ram}} \sqcup \Sigma_{\mathtt{aux}}$ and fix for each $\nu \in \mathcal{T}$ a place $\tilde{\nu}$ of F above ν.

For the remainder of this section, the letter U will denote an open compact subgroup U of $H(\mathbb{A}^\infty_{F^+})$. For later applications, we will be interested in particular in the choice

$$U_{(\Sigma_{\mathtt{ram}},\Sigma_{\mathtt{aux}})} := \prod_{\nu \in \mathrm{Pl}^{fin}_{F^+}} U_\nu$$

with:

- If ν is not split in $F|F^+$, then U_ν is a hyperspecial maximal compact subgroup of $H(F^+_\nu)$;
- If $\nu \notin \mathcal{T}$ splits, then $U_\nu = H(\mathcal{O}_{F^+_\nu})$;
- If $\nu \in \Sigma_{\mathtt{aux}}$, then $U_\nu = \iota_{\tilde{\nu}}^{-1} \ker\big(\mathrm{GL}_n(\mathcal{O}_{F_{\tilde{\nu}}}) \to \mathrm{GL}_n(k_{F_{\tilde{\nu}}})\big)$;
- If $\nu \in \Sigma_{\mathtt{ram}}$, then $U_\nu = \iota_{\tilde{\nu}}^{-1}(\mathrm{Iw})$, where $\mathrm{Iw} \subset \mathrm{GL}_n(\mathcal{O}_{F_{\tilde{\nu}}})$ denotes the Iwahori subgroup.

Weight In order to characterize the weight of our automorphic forms, let us first consider the following parametrization based on Theorem 6.9 (cf. also [Gue11]):

1. Let $\boldsymbol{\omega} = (\underline{\omega}_\tau) \in (\mathbb{Z}^{n,+})^{\mathrm{Hom}(F^+,\mathbb{R})}$, then we denote by

$$\xi^u_{\boldsymbol{\omega}} : H(F^+_\infty) = \prod_{\tau\in\mathrm{Hom}(F^+,\mathbb{R})} H(F^+_\tau) \cong \prod_{\tau\in\mathrm{Hom}(F^+,\mathbb{R})} \mathrm{U}_n(\mathbb{R}) \overset{\phi}{\longrightarrow} \prod_{\tau\in\mathrm{Hom}(F^+,\mathbb{R})} \mathrm{GL}(W^u_{\underline{\omega}_\tau}) \subset \mathrm{GL}(W^u_{\boldsymbol{\omega}})$$

the (complex) representation which is given by

- $W^u_{\boldsymbol{\omega}} \coloneqq \bigotimes_\tau W^u_{\underline{\omega}_\tau}$;
- $\phi \coloneqq \prod_\tau \xi^u_{\underline{\omega}_\tau}$.

2. Let ℓ be a rational prime such that every place ν of F^+ above ℓ splits in $F|F^+$ and fix for each such ν a place $\tilde{\nu}$ of F above ν. Let $\mathcal{K}$ be a finite extension of $\mathbb{Q}_\ell$ which is F-big enough (i.e. contains the image of every embedding $F \hookrightarrow \overline{\mathcal{K}}$) and let $\boldsymbol{\omega} = (\underline{\omega}_\tau) \in (\mathbb{Z}^{n,+})^{\mathrm{Hom}(F,\mathcal{K})}$. To each $\tau \in \mathrm{Hom}(F,\mathcal{K})$ we can associate a place ν of F^+ above ℓ for which we have just fixed a place $\tilde{\nu}$. Denote this assignment $\mathrm{Hom}(F,\mathcal{K}) \to \Omega^F_\ell$ by $\tau \mapsto w_\tau$. Then denote by

$$\xi^{\mathcal{K}}_{\boldsymbol{\omega}} : H(F^+_\ell) = \prod_{\nu\in\Omega^{F^+}_\ell} H(F^+_\nu) \cong \prod_{\nu\in\Omega^{F^+}_\ell} \mathrm{GL}_n(F_{\tilde{\nu}}) \overset{\prod d_\nu}{\longrightarrow} \prod_{\nu\in\Omega^{F^+}_\ell} \prod_{\substack{\tau\in\mathrm{Hom}(F,\mathcal{K})\\ \text{s.t. } w_\tau=\tilde{\nu}}} \mathrm{GL}_n(F_{\tilde{\nu}}) = \prod_{\tau\in\mathrm{Hom}(F,\mathcal{K})} \mathrm{GL}_n(F_{\tilde{\nu}})$$

$$\overset{\psi}{\longrightarrow} \prod_{\tau\in\mathrm{Hom}(F,\mathcal{K})} \mathrm{GL}(W^{\mathcal{K}}_{\underline{\omega}_\tau}) \subset \mathrm{GL}(W^{\mathcal{K}}_{\boldsymbol{\omega}})$$

the representation which is given by

- each d_ν is the diagonal embedding;
- $W^{\mathcal{K}}_{\boldsymbol{\omega}} \coloneqq \bigotimes_\tau W^{\mathcal{K}}_{\underline{\omega}_\tau}$;
- $\psi \coloneqq \prod_\tau \xi^{\mathcal{K}}_{\underline{\omega}_\tau}$.

3. The representation $\xi^{\mathcal{K}}_{\boldsymbol{\omega}}$ from above admits an integral model: There exists a finite free $\mathcal{O}_{\mathcal{K}}$-module $M^{\mathcal{O}_{\mathcal{K}}}_{\boldsymbol{\omega}}$ and a representation

$$\xi^{\mathcal{O}_{\mathcal{K}}}_{\boldsymbol{\omega}} : H(\mathcal{O}_{F^+_\ell}) \longrightarrow \mathrm{GL}(M^{\mathcal{O}_{\mathcal{K}}}_{\boldsymbol{\omega}})$$

such that

$$\xi^{\mathcal{K}}_{\boldsymbol{\omega}}|H(\mathcal{O}_{F^+_\ell}) \;=\; \xi^{\mathcal{O}_{\mathcal{K}}}_{\boldsymbol{\omega}} \otimes_{\mathcal{O}_{\mathcal{K}}} \mathcal{K}.$$

Automorphic forms Denote by $\mathcal{A}(H)$ the the space of (complex) automorphic forms on H, such that we have a decomposition

$$\mathcal{A}(H) = \bigoplus_\pi \pi^{m(\pi)}$$

into isomorphism classes of irreducible representations of $H(\mathbb{A}_{F^+})$, each occurring with finite multiplicity $m(\pi)$ (see e.g. [Gue11]).

Definition 6.27 (Vector-valued automorphic form)**.** Let $\boldsymbol{\omega} \in (\mathbb{Z}^{n,+})^{\mathrm{Hom}(F^+,\mathbb{R})}$ be a weight, then we denote by $\mathcal{S}_{\boldsymbol{\omega}}$ the space of locally constant functions

$$f : H(\mathbb{A}_{F^+}^\infty) \longrightarrow W_{\boldsymbol{\omega}}^{u,\vee}$$

which fulfill

$$f(\gamma.h) = \gamma_\infty f(h) \qquad \forall\, h \in H(\mathbb{A}_{F^+}^\infty), \gamma \in H(F^+).$$

(We denote by γ_∞ the image of γ under the canonical embedding $H(F^+) \to H(F_\infty^+)$.) $H(\mathbb{A}_{F^+}^\infty)$ acts on $\mathcal{S}_{\boldsymbol{\omega}}$ via right translation, and for a level subgroup U we denote by $\mathcal{S}_{\boldsymbol{\omega}}(U)$ the space of U-fixed vectors.

This allows us to give an $H(\mathbb{A}_{F^+}) = H(\mathbb{A}_{F^+,\infty}) \times H(\mathbb{A}_{F^+}^\infty)$-equivariant decomposition

$$\mathcal{A}(H) = \bigoplus_{\boldsymbol{\omega} \in (\mathbb{Z}^{n,+})^{\mathrm{Hom}(F^+,\mathbb{R})}} W_{\boldsymbol{\omega}}^u \otimes \mathcal{S}_{\boldsymbol{\omega}}.$$

This, in turn, allows us to associate to an $f \in \mathcal{S}_{\boldsymbol{\omega}}$ the (irreducible) automorphic representation $\langle f \rangle$ which is uniquely characterized by the condition that it contains all vectors of $W_{\boldsymbol{\omega}}^u \otimes f$.

The main feature of H is the existence of an *avatar* (using the language of M. Harris):

Theorem 6.28. *Let Π be RACSDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ of weight $\boldsymbol{\omega} \in (\mathbb{Z}^{n,+})^{\mathrm{Hom}(F,\mathbb{C})}$ in the sense of [CHT08, Section 4]. Then there exists an automorphic representation π_0 of $H(\mathbb{A}_{F^+})$ such that Π is a base change of π_0, i.e.*

- *For each archimedean place ν of F^+ and each place $\tilde{\nu}$ of F lying above ν, we have $\pi_{0,\nu} \cong \xi^u_{\underline{\omega}_{\tilde{\nu}}}$;*
- *For each finite place ν of F^+ which splits as $\tilde{\nu}\tilde{\nu}^c$ in F, $\Pi_{\tilde{\nu}}$ is the local base change of $\pi_{0,\nu}$;*
- *If ν is a finite place of F^+ which stays inert in F and for which Π_ν is unramified, then π_ν has a fixed vector for a maximal hyperspecial compact subgroups of $H(F_\nu^+)$.*

Proof. See [Gue11, Theorem 2.2] and [Ger10b, Lemma 2.2.7]. □

Hecke algebras Fix a sets of places $\mathcal{T} = \Sigma_{\mathtt{ram}} \sqcup \Sigma_{\mathtt{aux}}$ (with corresponding level subgroup $U = U_{(\Sigma_{\mathtt{ram}},\Sigma_{\mathtt{aux}})}$) and a weight vector $\boldsymbol{\omega} \in (\mathbb{Z}^{n,+})^{\mathrm{Hom}(F^+,\mathbb{R})}$ as above. For $j \in \{1, \ldots, n\}$ and w a place of F which is split over F^+ and does not divide an element of $\mathcal{T}$, we consider the Hecke operator

$$T_{F_w}^{(j)} = \left[U.\iota_w^{-1} \begin{pmatrix} \varpi_{F_w} \mathbf{1}_j & 0 \\ 0 & \mathbf{1}_{1-j} \end{pmatrix} .U \right]$$

acting on $\mathcal{S}_{\boldsymbol{\omega}}(U)$.

Let $\mathcal{T}'$ be a finite set of places of F^+ containing $\mathcal{T}$ and let $\mathscr{R}$ be a subring of $\mathbb{C}$, then define the Hecke algebra

$${}^{\mathscr{R}}\mathbf{T}_{\boldsymbol{\omega}}^{\mathcal{T}'}(U) \coloneqq \mathrm{im}\Big(\mathscr{R}[T_{F_w}^{(j)} \,|\, j \in \{1, \ldots, n\}, w \in \mathrm{Pl}_F^{\mathtt{split},\mathcal{T}'}] \longrightarrow \mathrm{End}_{\mathbb{C}}(\mathcal{S}_{\boldsymbol{\omega}}(U)) \Big)$$

where $\mathrm{Pl}_F^{\mathtt{split},\mathcal{T}'}$ denotes the set of places of F which are split over F^+ and which do not divide an element of $\mathcal{T}'$. Besides $\mathscr{R} = \mathbb{Z}$ we will mainly be interested in the case $\mathscr{R} = \mathcal{O}_{\mathcal{E}(U)}$, where $\mathcal{O}_{\mathcal{E}(U)}$ is the ring of integers of the following field:

Definition 6.29. For $f \in \mathcal{S}_{\boldsymbol{\omega}}(U)$ an eigenform (with respect to ${}^{\mathbb{Z}}\mathbf{T}^{\mathcal{T}}_{\boldsymbol{\omega}}(U)$) denote by

$$\mathcal{E}_f := \mathbb{Q}(a_f(T^{(j)}_{F_w}) \,|\, j \in \{1, \ldots, n\}, w \in \mathrm{Pl}_F^{\mathtt{split},\mathcal{T}})$$

the number field generated by the eigenvalues $a_f(T)$ of $T \in {}^{\mathbb{Z}}\mathbf{T}^{\mathcal{T}}_{\boldsymbol{\omega}}(U)$ acting on $\mathbb{C}.f$. We denote by $\mathcal{E}(U)$ the composite of the fields $\mathcal{E}_f$, where f runs through all eigenforms of $\mathcal{S}_{\boldsymbol{\omega}}(U)$.

Note that the definition of $\mathcal{E}(U)$ depends also on the chosen weight, but we suppress this from the notation. Let us also list two well-known facts:

- There are only finitely many one-dimensional eigenspaces $\mathbb{C}.f_1, \ldots, \mathbb{C}.f_r$ contained in $\mathcal{S}_{\boldsymbol{\omega}}(U)$. In particular, $\mathcal{E}(U)$ is a number field. Moreover, $\mathcal{S}_{\boldsymbol{\omega}}(U)$ admits a basis of eigenforms, i.e. we can choose the $f_1, \ldots, f_r$ such that

$$\mathcal{S}_{\boldsymbol{\omega}}(U) \cong \mathbb{C}.f_1 \oplus \ldots \oplus \mathbb{C}.f_r. \tag{6.5}$$

  (This follows from the decomposition (3.1.1) of [Gue11] together with Proposition 6.31 below.)

- Any eigenform $f \in \mathcal{S}_{\boldsymbol{\omega}}(U)$ gives rise to a $\mathbb{Z}$-algebra homomorphism

$$\varphi_f : {}^{\mathbb{Z}}\mathbf{T}^{\mathcal{T}}_{\boldsymbol{\omega}}(U) \longrightarrow \mathcal{E}(U) \quad T^{(j)}_{F_w} \longmapsto a_f(T^{(j)}_{F_w})$$

  and it can be shown that $\mathrm{im}(\varphi_f) \subset \mathcal{O}_{\mathcal{E}(U)}$. Moreover, f is uniquely characterized by φ_f (up to $\mathbb{C}$-multiples).

ℓ-adic models of automorphic forms During the course of this paragraph (which is based strongly on Section 2.3 of [Gue11]) we will use the following static[13] setup:

- ℓ denotes a rational prime (fixed throughout this paragraph) which does not lie below $\mathcal{T}$ and such that all places of $\Omega_\ell^{F^+}$ are split in $F|F^+$;
- we fix a finite extension $\mathcal{K}$ of $\mathbb{Q}_\ell$ which is F-big enough together with an isomorphism $\iota : \overline{\mathcal{K}} \cong \mathbb{C}$;
- we fix an ℓ-adic weight $\boldsymbol{\omega}$, i.e. an element of

$$(\mathbb{Z}^{n,+})^{\mathrm{Hom}(F,\mathcal{K})}_c = \left\{\boldsymbol{\omega} \in (\mathbb{Z}^{n,+})^{\mathrm{Hom}(F,\mathcal{K})} \,\middle|\, \underline{\omega}_{\tau^c,i} = -\underline{\omega}_{\tau,n-i+1} \;\forall \tau \in \mathrm{Hom}(F,\mathcal{K}), i \in \{1,\ldots,n\}\right\}.$$

Definition 6.30. For U a compact subgroup of $H(\mathbb{A}^\infty_{F^+})$ and A an $\mathcal{O}_\mathcal{K}$-algebra, suppose that either the projection of U to $H(F_\ell^+)$ is contained in $H(\mathcal{O}_{F_\ell^+})$ or that A is a $\mathcal{K}$-algebra. Then we define $S_{\boldsymbol{\omega}}(U, A)$ to be the space of functions

$$f : H(F^+)\backslash H(\mathbb{A}^\infty_{F^+}) \longrightarrow A \otimes_{\mathcal{O}_\mathcal{K}} M_{\boldsymbol{\omega}}^{\mathcal{O}_\mathcal{K}}$$

which fulfill

$$u_\ell.f(hu) = f(h) \qquad \forall u \in U, h \in H(\mathbb{A}^\infty_{F^+}),$$

where u_ℓ denotes the image of u under the projection map $H(\mathbb{A}^\infty_{F^+}) \to H(F_\ell^+)$.

[13] With "static" we mean that we don't vary the prime ℓ, in contrast to the bigger part of this section.

As we are primarily interested in the choice $U = U_{(\Sigma_{\mathtt{ram}},\Sigma_{\mathtt{aux}})}$ with $\Sigma_{\mathtt{aux}} \neq \emptyset$, our level will be "sufficiently small" in the sense of [CHT08], i.e. there exists a place ν of F^+ such that the projection of U to $H(F^+_\nu)$ contains no element of finite order except the identity. Thus we have

$$S_{\boldsymbol{\omega}}(U, A) \cong A \otimes_{\mathcal{O}_\mathcal{K}} S_{\boldsymbol{\omega}}(U, \mathcal{O}_\mathcal{K}).$$

(This is also true without a condition on U if we suppose that A is flat as an $\mathcal{O}_\mathcal{K}$-module, cf. [Ger10a].)

The main connection with complex automorphic forms is given by the following proposition:

Proposition 6.31. *1. The isomorphism ι gives rise to a bijection*

$$\iota_*^+ : (\mathbb{Z}^{n,+})_c^{\mathrm{Hom}(F,\mathcal{K})} \xrightarrow{\cong} (\mathbb{Z}^{n,+})^{\mathrm{Hom}(F^+,\mathbb{R})};$$

2. For $\boldsymbol{\omega} \in (\mathbb{Z}^{n,+})_c^{\mathrm{Hom}(F,\mathcal{K})}$ there is an isomorphism of $\mathbb{C}$-vector spaces

$$\theta_{\boldsymbol{\omega}} : \mathbb{C} \otimes_{\mathcal{K},\iota} W_{\boldsymbol{\omega}}^{\mathcal{K}} \xrightarrow{\cong} W_{\boldsymbol{\omega}}^u;$$

3. The assignment $f \mapsto (h \mapsto \theta_{\boldsymbol{\omega}}(h_\ell.f(h)))$ provides an isomorphism of $\mathbb{C}H(\mathbb{A}_{F^+}^\infty)$-modules

$$S_{\boldsymbol{\omega}}(\{1\}, \mathbb{C}) := \bigcup_U S_{\boldsymbol{\omega}}(U, \mathbb{C}) \cong \mathcal{S}_{\iota_*^+(\boldsymbol{\omega})^\vee} \tag{6.6}$$

which restricts to an isomorphism $S_{\boldsymbol{\omega}}(U, \mathbb{C}) \cong \mathcal{S}_{\iota_*^+(\boldsymbol{\omega})^\vee}(U)$ for a level subgroup U. (In these isomorphisms $\mathbb{C}$ is understood as a $\mathcal{O}_\mathcal{K}$-algebra via ι and $\iota_*^+(\boldsymbol{\omega})^\vee$ is defined by $\iota_*^+(\boldsymbol{\omega})_{\tau,i}^\vee = -\iota_*^+(\boldsymbol{\omega})_{\tau,n+1-i}^\vee$.)

Proof. See [Gue11, Section 2.3]. □

For $w \nmid \ell$, the Hecke operators $T_{F_w}^{(j)}$ from above also act on $S_{\boldsymbol{\omega}}(U, \mathcal{O}_\mathcal{K}) \subset S_{\boldsymbol{\omega}}(U, \mathbb{C})$ and this action commutes with the isomorphism (6.6). This motivates the following definition: Let $\mathcal{T}'$ be a finite set of places of F^+ containing $\mathcal{T}^\ell := \mathcal{T} \cup \Omega_\ell^{F^+}$ and $\mathscr{R}$ a subring of $\overline{\mathcal{K}}$, then define the Hecke algebra

$${}^{\mathscr{R}}\mathbb{T}_{\boldsymbol{\omega}}^{\mathcal{T}'}(U) := \mathrm{im}\Big(\, q : \mathscr{R}[T_{F_w}^{(j)} \,|\, j \in \{1, \ldots, n\}, w \in \mathrm{Pl}_F^{\mathtt{split},\mathcal{T}'}] \longrightarrow \mathrm{End}_{\mathcal{O}_\mathcal{K}}(S_{\boldsymbol{\omega}}(U, \mathcal{O}_\mathcal{K}))\Big).$$

Let $f \in S_{\boldsymbol{\omega}}(U, \mathcal{O}_\mathcal{K})$ be an eigenform for this algebra, then we see, using the compatibility with the isomorphism (6.6), that the eigenvalue for a Hecke operator T is given by $\iota^{-1}(a_{\tilde{f}})$, where $\tilde{f} \in \mathcal{S}_{\iota_*^+(\boldsymbol{\omega})^\vee}(U)$ is the corresponding complex automorphic form. In other words, we can interpret the map $\varphi_{\tilde{f}}$ from above as

$$\varphi_f^\ell : {}^{\mathbb{Z}}\mathbb{T}_{\boldsymbol{\omega}}^{\mathcal{T}_\ell}(U) \longrightarrow \iota(\mathcal{E}(U)) \cong \mathcal{E}(U).$$

Note that we use the bold symbol $\mathbf{T}$ for complex Hecke algebras and the blackboard bold symbol $\mathbb{T}$ for ℓ-adic Hecke algebras.

Fixed type Hecke algebra We will finish this subsection by defining a slight variation of the above Hecke algebra. For this, fix a finite set $\widetilde{\Sigma} \subset (\mathcal{T}' - \Omega_\ell^F)$ of places of F together with a tuple

$$\underline{\sigma} = (\sigma_\nu)_{\nu \in \widetilde{\Sigma}}, \tag{6.7}$$

where each σ_ν is a complex representation of $\mathrm{GL}_n(\mathcal{O}_{F_\nu})$. Let

$$ {}_{\underline{\sigma}}S_{\boldsymbol{\omega}}(U,\mathcal{O}_{\mathcal{K}}) \subset S_{\boldsymbol{\omega}}(U,\mathcal{O}_{\mathcal{K}}) $$

be the subspace of those $f \in S_{\boldsymbol{\omega}}(U,\mathcal{O}_{\mathcal{K}})$ whose complex correspondents $\tilde{f}$ (via (6.6)) fulfill the following condition for all places $\nu \in \widetilde{\Sigma}$: If π_ν denotes the local component of the automorphic representation $\pi = \langle \tilde{f} \rangle$ at ν, then $\pi_\nu | \mathrm{GL}_n(\mathcal{O}_{F_\nu})$ contains σ_ν as a subrepresentation. Note that the $T^{(j)}_{F_w}$ (for w in $\mathrm{Pl}_F^{\mathtt{split},\mathcal{T}'}$) stabilize the subspace ${}_{\underline{\sigma}}S_{\boldsymbol{\omega}}(U,\mathcal{O}_{\mathcal{K}})$, so we can define

$$ {}^{\mathscr{R}}_{\underline{\sigma}}\mathbb{T}^{\mathcal{T}'}_{\boldsymbol{\omega}}(U) := \mathrm{im}\Big({}_{\underline{\sigma}}q : \mathscr{R}[T^{(j)}_{F_w} \,|\, j \in \{1,\dots,n\}, w \in \mathrm{Pl}_F^{\mathtt{split},\mathcal{T}'}] \longrightarrow \mathrm{End}_{\mathcal{O}_{\mathcal{K}}}({}_{\underline{\sigma}}S_{\boldsymbol{\omega}}(U,\mathcal{O}_{\mathcal{K}})) \Big). $$

We easily see that the assignment $q(T^{(j)}_{F_w}) \mapsto {}_{\underline{\sigma}}q(T^{(j)}_{F_w})$ defines an $\mathscr{R}$-algebra surjection

$$ {}_{\underline{\sigma}}\theta : {}^{\mathscr{R}}\mathbb{T}^{\mathcal{T}'}_{\boldsymbol{\omega}}(U) \longrightarrow {}^{\mathscr{R}}_{\underline{\sigma}}\mathbb{T}^{\mathcal{T}'}_{\boldsymbol{\omega}}(U). \tag{6.8} $$

Thus we can note the following (for $\mathscr{R} = \mathcal{O}_{\mathcal{K}}$):

Observation 6.32. 1. Assume that ${}^{\mathcal{O}_{\mathcal{K}}}\mathbb{T}^{\mathcal{T}'}_{\boldsymbol{\omega}}(U)_{\mathfrak{m}} \cong \mathcal{O}_{\mathcal{K}}$ holds for any maximal ideal $\mathfrak{m} \subset {}^{\mathcal{O}_{\mathcal{K}}}\mathbb{T}^{\mathcal{T}'}_{\boldsymbol{\omega}}(U)$. Then ${}^{\mathcal{O}_{\mathcal{K}}}_{\underline{\sigma}}\mathbb{T}^{\mathcal{T}'}_{\boldsymbol{\omega}}(U)_{\mathfrak{n}}$ is a quotient of $\mathcal{O}_{\mathcal{K}}$ for any maximal ideal $\mathfrak{n} \subset {}^{\mathcal{O}_{\mathcal{K}}}_{\underline{\sigma}}\mathbb{T}^{\mathcal{T}'}_{\boldsymbol{\omega}}(U)$. (This follows from the fact that the completion process sends surjections to surjections.)

2. In the same way as for ${}^{\mathcal{O}_{\mathcal{K}}}\mathbf{T}^{\mathcal{T}'}_{\boldsymbol{\omega}}(U)$ (see Corollary 6.41 below) we can check that ${}^{\mathcal{O}_{\mathcal{K}}}_{\underline{\sigma}}\mathbb{T}^{\mathcal{T}'}_{\boldsymbol{\omega}}(U)$ is torsion-free and finitely generated. As $\mathcal{O}_{\mathcal{K}}$ is a discrete valuation ring, it follows that ${}^{\mathcal{O}_{\mathcal{K}}}_{\underline{\sigma}}\mathbb{T}^{\mathcal{T}'}_{\boldsymbol{\omega}}(U)$ is free and finitely generated. Hence, the following strengthening of part 1. holds: Assume that ${}^{\mathcal{O}_{\mathcal{K}}}\mathbb{T}^{\mathcal{T}'}_{\boldsymbol{\omega}}(U)_{\mathfrak{m}} \cong \mathcal{O}_{\mathcal{K}}$ holds for any maximal ideal $\mathfrak{m} \subset {}^{\mathcal{O}_{\mathcal{K}}}\mathbb{T}^{\mathcal{T}'}_{\boldsymbol{\omega}}(U)$. Then ${}^{\mathcal{O}_{\mathcal{K}}}_{\underline{\sigma}}\mathbb{T}^{\mathcal{T}'}_{\boldsymbol{\omega}}(U)_{\mathfrak{n}} \cong \mathcal{O}_{\mathcal{K}}$ holds for any maximal ideal $\mathfrak{n} \subset {}^{\mathcal{O}_{\mathcal{K}}}_{\underline{\sigma}}\mathbb{T}^{\mathcal{T}'}_{\boldsymbol{\omega}}(U)$.

6.3 From automorphic forms to Galois representations

Proposition 6.33 ([CHT08, Proposition 3.4.2 and 3.4.4]). *Let $\mathfrak{m} \subset {}^{\mathcal{O}_{\mathcal{K}}}\mathbb{T}^{\mathcal{T}_\ell}_{\boldsymbol{\omega}}(U)$ be a maximal ideal. Then there exists a representation*

$$ \rho_{\mathfrak{m}} : \mathrm{Gal}_F \to \mathrm{GL}_n\Big({}^{\mathcal{O}_{\mathcal{K}}}\mathbb{T}^{\mathcal{T}_\ell}_{\boldsymbol{\omega}}(U)_{\mathfrak{m}} \Big) $$

(where the subscript-$\mathfrak{m}$ denotes the completion, so that the coefficient ring of the general linear group is an object of $\mathcal{C}_{\mathcal{O}_{\mathcal{K}}}$) with the following properties (the first two already characterize $\rho_{\mathfrak{m}}$ uniquely):

1. *$\rho_{\mathfrak{m}}$ is unramified at all but finitely many places; If a place ν of F^+ is inert and unramified in F and if U_ν is a hyperspecial maximal compact subgroup of $H(F^+_\nu)$, then $\rho_{\mathfrak{m}}$ is unramified above ν;*

2. *If a place $\nu \notin \mathcal{T}_\ell$ splits as $\tilde{\nu}\tilde{\nu}^c$ in F, then $\rho_{\mathfrak{m}}$ is unramified at $\tilde{\nu}$ and $\rho_{\mathfrak{m}}(\mathrm{Frob}_{\tilde{\nu}})$ has characteristic polynomial*

$$ X^n - T^{(1)}_{\tilde{\nu}} X^{n-1} + \ldots + (-1)^j (\mathbf{N}\tilde{\nu})^{j(j-1)/2} T^{(j)}_{\tilde{\nu}} X^{n-j} + \ldots + (-1)^n (\mathbf{N}\tilde{\nu})^{n(n-1)/2} T^{(n)}_{\tilde{\nu}}. $$

3. *$\rho^c_{\mathfrak{m}} \cong \rho_{\mathfrak{m}} \otimes \epsilon_\ell^{1-n}$;*

4. *Fix a set of primes* $\widetilde{\Omega}_\ell^{F^+}$ *of* F *such that* $\widetilde{\Omega}_\ell^{F^+} \sqcup \widetilde{\Omega}_\ell^{F^+,c} = \Omega_\ell^F$ *and denote by* $\tilde{I}_\ell$ *the set of embeddings* $F \hookrightarrow \mathcal{K}$ *which give rise to an element of* $\widetilde{\Omega}_\ell^{F^+}$. *Suppose that* $w \in \widetilde{\Omega}_\ell^{F^+}$ *is unramified over* ℓ, *that* $U_{\overline{w}} = H(\mathcal{O}_{F^+,\overline{w}})$ *(where* $\overline{w}$ *denotes the place of* F^+ *below* w*) and that for each* $\tau \in \tilde{I}_\ell$ *above* w *we have*

$$\ell - 1 - n \geq \underline{\omega}_{\tau,1} \geq \ldots \geq \underline{\omega}_{\tau,n} \geq 0.$$

Then, for each open ideal $I \subset {}^{\mathcal{O}_\mathcal{K}}\mathbb{T}_{\boldsymbol{\omega}}^{\mathcal{T}_\ell}(U)$

$$\left(\rho_\mathfrak{m} \otimes_{\mathcal{O}_\mathcal{K}\mathbb{T}_{\boldsymbol{\omega}}^{\mathcal{T}_\ell}(U)} {}^{\mathcal{O}_\mathcal{K}}\mathbb{T}_{\boldsymbol{\omega}}^{\mathcal{T}_\ell}(U)/I\right)\Big|\,\mathrm{Gal}(F_w) \cong \mathcal{G}_{F_w}(M_{\mathfrak{m},I,w})$$

for some object $M_{\mathfrak{m},I,w}$ *of* $\underline{\mathrm{MF}}_{\mathcal{O}_{F_w},\mathcal{O}_\mathcal{K}}$.

If $\mathfrak{m}$ *is non-Eisenstein in the sense of [CHT08, Definition 3.4.3], then both* $\rho_\mathfrak{m}$ *and its reduction extend to*

$$r_\mathfrak{m} : \mathrm{Gal}_{F^+} \to \mathcal{G}_n\left({}^{\mathcal{O}_\mathcal{K}}\mathbb{T}_{\boldsymbol{\omega}}^{\mathcal{T}_\ell}(U)_\mathfrak{m}\right)$$

and

$$\overline{r}_\mathfrak{m} : \mathrm{Gal}_{F^+} \to \mathcal{G}_n\left({}^{\mathcal{O}_\mathcal{K}}\mathbb{T}_{\boldsymbol{\omega}}^{\mathcal{T}_\ell}(U)/\mathfrak{m}\right),$$

where the coefficient ring of the group in the last case is a finite extension of $k_{\mathcal{O}_\mathcal{K}}$, *hence of* $\mathbb{F}_\ell$.

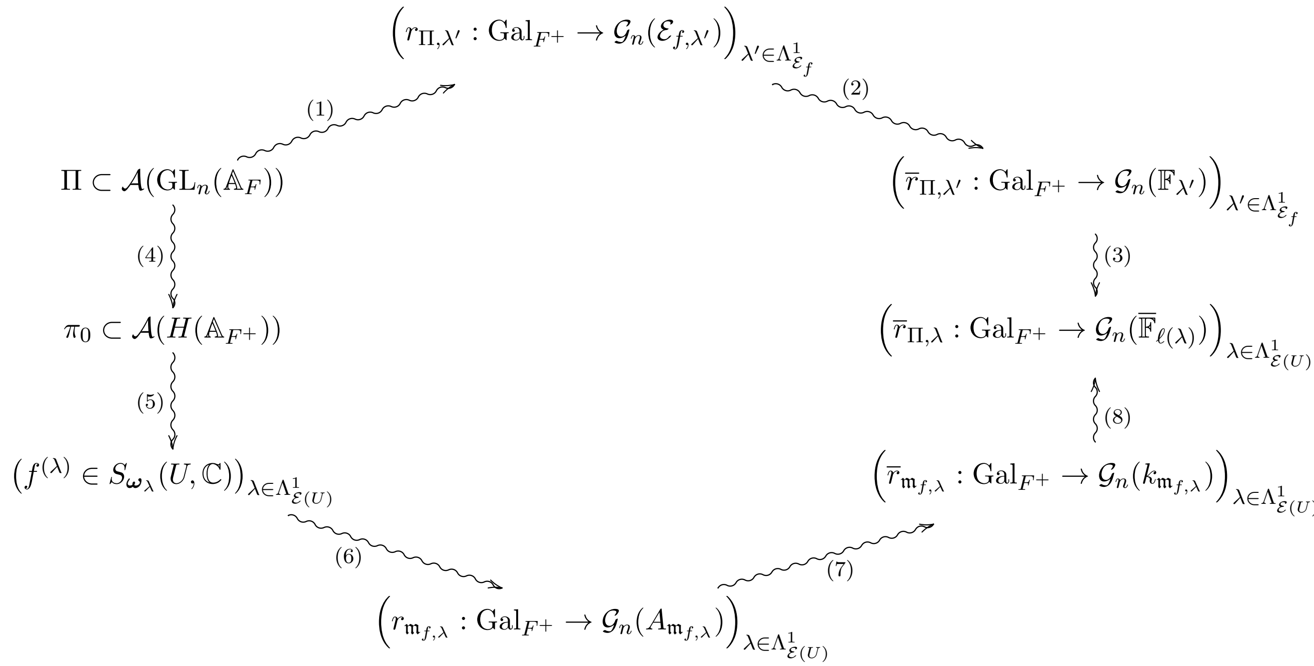
We can visualize the compatibility of this theorem with the assignment from Theorem 6.4 as follows:

- Recall that Π is a RACSDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, admitting an avatar $\pi_0 = \langle f \rangle$ via Theorem 6.28; Denote the level of f by U and the weight by $\boldsymbol{\omega}$.

- For each finite place λ of $\mathcal{E}(U)$ we fix an F-big enough field extension $\mathcal{K}_\lambda$ of $\mathcal{E}(U)_\lambda$. Our initial choice of isomorphisms $(\iota_\ell)_\ell$ between $\overline{\mathbb{Q}}_\ell$ and $\mathbb{C}$ thus provides us with isomorphisms $\iota_{\mathcal{K}_\lambda} : \overline{\mathcal{K}}_\lambda \cong \mathbb{C}$. We denote the corresponding isomorphisms between the complex and the ℓ-adic weights from part 1 of Proposition 6.31 by ${}_\lambda\iota_*^+$.

- For each place λ as above, denote by λ' the place of $\mathcal{E}_f$ lying below λ and by $\mathbb{F}_\lambda$ the residue field of $\mathcal{E}_{f,\lambda'}$.

Then the following diagram commutes:



(6.9)

where

- (1) denotes the association induced by Theorem 6.4;
- (2) and (7) denote the respective reduction processes;
- (3) and (8) are the appropriate inclusions into the algebraic closure;
- (4) denotes the association of an avatar to Π, cf. Theorem 6.28;
- (5) comes from the identification of complex automorphic forms with ℓ-adic models from Proposition 6.31; (here, $\boldsymbol{\omega}_\lambda$ is short for ${}_{\lambda}\iota_*^{+,-1}(\boldsymbol{\omega}^\vee)$.)
- (6) maps each $f^{(\lambda)}$ to $r_{\mathfrak{m}_{f,\lambda}}$ via Proposition 6.33, where $\mathfrak{m}_{f,\lambda}$ is the unique maximal ideal of ${}^{\mathcal{O}_{\mathcal{K}_\lambda}}\mathbb{T}^{T_\ell}_{\boldsymbol{\omega}_\lambda}(U)$ containing
$$\mathfrak{p}_{f,\lambda} := \ker\Big(\varphi_{f^{(\lambda)}} : {}^{\mathcal{O}_{\mathcal{K}_\lambda}}\mathbb{T}^{T_\ell}_{\boldsymbol{\omega}_\lambda}(U) \to \mathcal{K}_\lambda\Big).$$
- $A_{\mathfrak{m}_{f,\lambda}}$ denotes ${}^{\mathcal{O}_{\mathcal{K}}}\mathbb{T}^{\mathcal{T}_\ell}_{\boldsymbol{\omega}_\lambda}(U)_{\mathfrak{m}_{f,\lambda}}$ and $k_{\mathfrak{m}_{f,\lambda}}$ denotes ${}^{\mathcal{O}_{\mathcal{K}}}\mathbb{T}^{\mathcal{T}_\ell}_{\boldsymbol{\omega}_\lambda}(U)/\mathfrak{m}_{f,\lambda}$.

6.4 Isomorphism theorems

6.4.1 A (conjectural) minimal $R = T$ theorem

For this section, we keep the following list, which is in part a repetition of the notation and assumptions made up to here:

R.1) ℓ denotes a rational prime fulfilling $\ell > \max(2, n)$;

R.2) F denotes a CM field of the form $F = F^+E$ for a totally real field F^+ and an imaginary quadratic field[14] E in which ℓ splits.

R.3) Assumption 6.26 is fulfilled: $F|F^+$ is unramified at all finite places and, if n is even, then also $\frac{n}{2}[F^+ : \mathbb{Q}]$ is even.

R.4) We fix a finite non-empty set $\Sigma_{\mathtt{ram}} \subset (\mathrm{Pl}_{F^+}^{fin} - \Omega_\ell^{F^+})$ such that

   i) each $\nu \in \Sigma_{\mathtt{ram}}$ splits in $F|F^+$;

   ii) if n is even, then
$$\frac{n}{2}[F^+ : \mathbb{Q}] + \#\Sigma_{\mathtt{ram}} = 0 \mod 2;$$

   (Of course, assuming condition R.3, this simply amounts to $\#\Sigma_{\mathtt{ram}}$ being even.)

R.5) We fix a finite non-empty set $\Sigma_{\mathtt{aux}} \subset (\mathrm{Pl}_{F^+}^{fin} - \Omega_\ell^{F^+})$ of primes which split in $F|F^+$, which is disjoint from $\Sigma_{\mathtt{ram}}$ and such that
$$\nu \in \Sigma_{\mathtt{aux}} \Rightarrow [F(\zeta_{\ell(\nu)}) : F] > n.$$

R.6) We consider the sets $\mathcal{T} = \Sigma_{\mathtt{aux}} \sqcup \Sigma_{\mathtt{ram}}$, $\mathcal{T}_\ell = \mathcal{T} \sqcup \Omega_\ell^{F^+}$ and lifts $\widetilde{\mathcal{T}}_{(\ell)} \subset \mathrm{Pl}_F$ of the same cardinality as $\mathcal{T}_{(\ell)}$ such that $\widetilde{\mathcal{T}}_{(\ell)} \sqcup \widetilde{\mathcal{T}}^c_{(\ell)}$ contains precisely the places above $\mathcal{T}_{(\ell)}$.

R.7) We fix a weight $\boldsymbol{\omega}$ and the level subgroup $U = U_{(\Sigma_{\mathtt{ram}}, \Sigma_{\mathtt{aux}})}$ as in Section 6.2.

R.8) We fix a number field $\mathcal{E}$ containing the Hecke eigenvalues of all the (finitely many) eigenspaces of weight $\boldsymbol{\omega}$ and level U, i.e. $\mathcal{E} \supset \mathcal{E}(U)$.

R.9) We fix a prime $\lambda \in \Omega_\ell^{\mathcal{E}}$ and a finite extension $\mathcal{K}_\lambda$ of $\mathcal{E}_\lambda$ which is F-big enough. We denote by $\boldsymbol{\omega}_\lambda$ the ℓ-adic weigth corresponding to $\boldsymbol{\omega}$ via part 1. of Proposition 6.31 (with respect to our choice of $\mathcal{K}_\lambda$).

Let $\mathfrak{m}$ be a non-Eisenstein maximal ideal of ${}^{\mathcal{O}_{\mathcal{K}_\lambda}}\mathbb{T}^{\mathcal{T}_\ell}_{\boldsymbol{\omega}_\lambda}(U)$ in the sense of [CHT08], Definition 3.4.3, and set $k_\lambda = {}^{\mathcal{O}_{\mathcal{K}_\lambda}}\mathbb{T}^{\mathcal{T}_\ell}_{\boldsymbol{\omega}_\lambda}(U)/\mathfrak{m}$. This implies that the associated residual Galois representations $\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible and that we have an extension $\overline{r}_{\mathfrak{m}}$ to $\mathcal{G}_n(k_\lambda)$ as in Proposition 6.33. We will assume

R.10) The image $X_\ell \coloneqq \overline{\rho}_{\mathfrak{m}}(\mathrm{Gal}_{F(\zeta_\ell)})$ is *adequate* in the sense of Thorne [Tho15, Definition 2.20]:

- $H^1(X_\ell, k_\lambda) = 0$ and $H^1(X_\ell, \mathfrak{gl}_n^0) = 0$;
- For any simple $k_\lambda[X_\ell]$-submodule $W \subset \mathfrak{gl}_n$, there exists a semi-simple element $\sigma \in X_\ell$ with eigenvalue $\alpha \in k_\lambda$ such that $\mathrm{tr}\, e_{\sigma,\alpha} W \neq 0$. (Here, $e_{\sigma,\alpha} \in \mathfrak{gl}_n$ denotes the unique idempotent in $k_\lambda[\sigma]$ with image equal to the α-eigenspace of σ.)

R.11) For each $\nu \in \Sigma_{\mathtt{aux}}$, $\overline{\rho}_{\mathfrak{m}}$ is unramified at ν and
$$H^0(F_\nu, \mathrm{ad}(\overline{\rho}_{\mathfrak{m}})(1)) = 0.$$

[14] We remark that this assumption does not introduce a loss of generality as the existence of such an E can be guaranteed by arguments as in [Tay08, Theorem 5.2]; see also [Gue11, proof of Theorem 4.1].

R.12) For each $\nu \in \Sigma_{\mathtt{ram}}$, $\overline{\rho}_{\mathfrak{m}}$ is unipotently ramified at ν (cf. Definition 4.25; recall in particular that this includes the possibility that $\overline{\rho}_{\mathfrak{m}}$ is unramified at ν).

Remark 6.34. We remark that we will later consider a compatible system $\mathcal{R} = (\overline{\rho}_\lambda)_{\lambda\in\mathrm{Pl}_E^{\mathrm{fin}}}$, where for almost all λ the representation $\overline{\rho}_\lambda$ will fulfill conditions R.1 - R.12, presuming Assumption 6.6.

Remark 6.35. Observe that condition R.11 implies

$$H^0(\mathrm{Gal}_{F_\nu}, k_\lambda(1)) = 0$$

for $\ell \not| n$: For such an ℓ one has $\mathrm{ad}(\overline{\rho}_{\mathfrak{m}}) \cong \mathrm{ad}^0(\overline{\rho}_{\mathfrak{m}}) \oplus k_\lambda$ and hence $\mathrm{ad}(\overline{\rho}_{\mathfrak{m}})(1) \cong \mathrm{ad}^0(\overline{\rho}_{\mathfrak{m}})(1) \oplus k_\lambda(1)$. Therefore

$$0 = H^0(F_\nu, \mathrm{ad}(\overline{\rho}_{\mathfrak{m}})(1)) = H^0(F_\nu, \mathrm{ad}(\overline{\rho}_{\mathfrak{m}})^0(1)) \oplus H^0(F_\nu, k_\lambda(1)).$$

Now, recall from [CHT08, Chapter 3.5] that $m \circ \overline{r}_{\mathfrak{m}} : \mathrm{Gal}_{F^+} \to \mathcal{G}_n(k_\lambda) \to \mathrm{GL}_1(k_\lambda)$ equals $\overline{\epsilon}_\ell^{1-n}\delta_{F|F^+}^{\mu_{\mathfrak{m}}}$ for a suitable element $\mu_{\mathfrak{m}} \in \mathbb{Z}/2\mathbb{Z}$ and where $\delta_{F|F^+}$ denotes the non-trivial character of $\mathrm{Gal}(F|F^+)$. We consider the global deformation problem

$$\mathcal{S}^{\mathtt{min,crys}} = (F|F^+, \mathcal{T}_\ell, \widetilde{\mathcal{T}}_\ell, \mathcal{O}_{\mathcal{K}}, \overline{r}_{\mathfrak{m}}, \epsilon_\ell^{1-n}\delta_{F|F^+}^{\mu_{\mathfrak{m}}}, \{\mathcal{D}_\nu\})$$

which parametrizes deformations of $\overline{r}_{\mathfrak{m}}$ to coefficient $\mathcal{O}_{\mathcal{K}}$-algebras which are unramified outside $\mathcal{T}_\ell$, of determinant $\overline{r}_{\mathfrak{m}}, \epsilon_\ell^{1-n}\delta_{F|F^+}^{\mu_{\mathfrak{m}}}$ and which fulfill locally the condition $\mathcal{D}_\nu$. Here, $\mathcal{D}_\nu$ parametrizes

- arbitrary lifts, if $\nu \in \Sigma_{\mathtt{aux}}$;
- crystalline lifts in the sense of Section 4.3, if $\nu \in \Omega_\ell^{F^+}$;
- minimally ramified lifts in the sense of Section 4.4, if $\nu \in \Sigma_{\mathtt{ram}}$.

The associated deformation functor is representable by an object we call $R^{\mathtt{min,crys}}(\overline{r}_{\mathfrak{m}})$ (or, closer to our notation from Section 2, $R_{\mathcal{T}_\ell}^{\epsilon_\ell^{1-n}\delta_{F|F^+}^{\mu_{\mathfrak{m}}},\{\mathcal{D}_\nu\}}(\overline{r}_{\mathfrak{m}})$).

Remark 6.36. We remark that we can equivalently consider $\overline{r}_{\mathfrak{m}}$ as a representation of $\mathrm{Gal}_{F,\mathcal{T}_\ell}$ and waive the constraint that our deformations must be unramified outside $\mathcal{T}_\ell$, as we did already during Definiton 6.23. This is mainly a matter of taste, but the convention we take from now on (that $\overline{r}_{\mathfrak{m}}$ is a representation of Gal_F) has the advantage that it fits more nicely with the concept of residual representations occurring as entries of a compatible systems.

Let $\widetilde{\Sigma}_{\mathtt{ram}} = \{\tilde{\nu} \mid \nu \in \Sigma_{\mathtt{ram}}\}$ denote the set of fixed lifts of the places in $\Sigma_{\mathtt{ram}}$ to F.

Conjecture 6.37. Assume the notation and all assumptions from the list R.1-R.12. Then there exists a tuple $\underline{\sigma} = (\sigma_\nu)_{\nu\in\widetilde{\Sigma}_{\mathtt{ram}}}$ as in (6.7) such that there is an isomorphism

$$R^{\mathtt{min,crys}}(\overline{r}_{\mathfrak{m}}) \xrightarrow{\cong} {}^{\mathcal{O}_{\mathcal{K}_\lambda}}_{\underline{\sigma}}\mathbb{T}^{\mathcal{T}_\ell}_{\boldsymbol{\omega}_\lambda}(U)_{\mathfrak{n}}$$

and $\mu_{\mathfrak{m}} \equiv n \mod 2$. (Here, $\mathfrak{n}$ denotes the image of $\mathfrak{m}$ under the projection ${}_{\underline{\sigma}}\theta$ from (6.8).)

Remark 6.38. We remark that this conjecture becomes more convincing in light of the fixed-type deformation condition at the end of Section 4.4: For each $\nu \in \widetilde{\Sigma}_{\mathtt{ram}}$ there exists an inertial type τ_ν, associated to $\overline{\rho}_{\mathfrak{m},\nu}$ in the same manner as we did in preparation for Theorem 4.30. To each τ_ν one can associate a certain representation $\sigma_\nu = \sigma(\tau_\nu)$ of $K = \mathrm{GL}_n(\mathcal{O}_{F_\nu})$ (which is then the K-type of the

$\mathrm{GL}_n(F_\nu)$-representation associated to an extension of τ_ν to Gal_{F_ν}). (For more details on the construction of the K-type $\sigma(\tau)$, see [Sho15, Section 4.6], [BC09, Section 6.5.2],[SZ99] and our Remark 6.39 below.) Now for the tuple $\underline{\sigma} := (\sigma_\nu)_{\nu\in\widetilde{\Sigma}_{\mathtt{ram}}}$ we conjecture that ${}^{\mathcal{O}_{K_\lambda}}_{\underline{\sigma}}\mathbb{T}_a^{\mathcal{T}_\ell}(U)_{\mathfrak{n}}$ is isomorphic to a deformation ring parametrizing lifts of $\overline{r}_{\mathfrak{m}}$ as in $\mathcal{S}^{\mathtt{min,crys}}$ above, but with the requirement that the associated GL_n-valued representation ρ_ν at ν is a lift of type τ_ν of $\overline{\rho}_{\mathfrak{m},\nu}$ (instead of being minimally ramified). (This conjecture is plausible in light of [Sho15, Theorem 2.16] together with a suitable local-global compatibility.) Our wording of Conjecture 6.37 in terms of minimally ramified deformations is then justified by Theorem 4.30 together with condition R.12.

Remark 6.39. Recall the following:

- Consider the finite general linear group $\mathfrak{G} = \mathrm{GL}_n(\ell(\nu))$ and its standard Borel subgroup $\mathfrak{B} \subset \mathfrak{G}$. Then the irreducible constituents of the (complex) representation $\mathrm{ind}_{\mathfrak{B}}^{\mathfrak{G}}(1)$ are called the unipotent representations of $\mathfrak{G}$. These representations can (canonically) be parametrized by the irreducible representations of the Weyl group $\mathcal{W}(\mathfrak{G}) \cong S_n$, see e.g. [Pra14, Corollary 4.4]. The irreducible representations of S_n in turn can be parametrized by partitions of n in terms of Specht modules, cf. [JK81]. In other words, we get a canonical bijection

  $$h : \mathcal{Y}_n \xrightarrow{\cong} \mathrm{Rep}(\mathfrak{G})^{\mathtt{uni}},$$

  where $\mathrm{Rep}(\mathfrak{G})^{\mathtt{uni}}$ denotes the set of all unipotent representations of $\mathfrak{G}$ up to isomorphism. The map h can be explicitly described in terms of induction from certain Levi subgroups (see [Sho15, Definition 4.34]) and sends $(1,\dots,1)$ to the trivial representation and (n) to the Steinberg representation.

- Under the unipotent ramification assumption, the set of inertial types $\mathcal{I}^{\mathtt{uni}}$ is in bijection with the set $\mathcal{Y}_n$ of partitions of n via the map ∇, cf. Section 4.4.1.

Then

$$\mathrm{ind}_I^K(1) \cong \mathrm{infl}_{\mathfrak{G}}^K \mathrm{ind}_{\mathfrak{B}}^{\mathfrak{G}}(1) \cong \bigoplus_{\pi\in\mathrm{Rep}(\mathfrak{G})^{\mathtt{uni}}} m_\pi \,\mathrm{infl}_{\mathfrak{G}}^K(\pi),$$

where $I \subset K$ denotes the Iwahori subgroup, $\mathrm{infl}_{\mathfrak{G}}^K$ denotes the inflation along the pro-$\ell(\nu)$ radical of K and the $m_\pi \geq 1$ are suitable multiplicities. Analogously to [BC09, Remark 6.5.2 iii)] one can thus check that the assignment $\tau \mapsto \sigma(\tau)$ is described in terms of partitions as

$$\tau \mapsto \sigma(\tau) = \mathrm{infl}_{\mathfrak{G}}^K\big(h \circ \nabla(\tau)\big).$$

Observe that the special case $n = 2$ of Remark 6.39 is precisely [BC09, Remark 6.5.2 iii)] and [Sho15, Example 2.17].

6.4.2 A $T = O$-theorem

Retain the notation from Section 6.2.

Proposition 6.40. *1. Let $K|\mathcal{E}(U)$ be a field extension. Then*

$$\mathcal{O}_K \otimes_{\mathcal{O}_{\mathcal{E}(U)}} {}^{\mathcal{O}_{\mathcal{E}(U)}}\boldsymbol{T}_{\boldsymbol{\omega}}^{\mathcal{T}}(U) \cong {}^{\mathcal{O}_K}\boldsymbol{T}_{\boldsymbol{\omega}}^{\mathcal{T}}(U) \quad \textit{and} \quad K \otimes_{\mathcal{O}_{\mathcal{E}(U)}} {}^{\mathcal{O}_{\mathcal{E}(U)}}\boldsymbol{T}_{\boldsymbol{\omega}}^{\mathcal{T}}(U) \cong {}^{K}\boldsymbol{T}_{\boldsymbol{\omega}}^{\mathcal{T}}(U).$$

2. *There exists a constant C depending on ${}^{\mathcal{O}_{\mathcal{E}(U)}}\boldsymbol{T}^{\mathcal{T}}_{\boldsymbol{\omega}}(U)$ such that the following holds for all places λ of $\mathcal{E}(U)$ which fulfill $\ell := \ell(\lambda) > C$: Let $\mathcal{K}$ be an F-big enough field extension of $\mathcal{E}(U)_\lambda$, then*

$$\mathcal{O}_{\mathcal{K}} \otimes_{\mathcal{O}_{\mathcal{E}(U)}} {}^{\mathcal{O}_{\mathcal{E}(U)}}\boldsymbol{T}^{\mathcal{T}}_{\boldsymbol{\omega}}(U) \cong {}^{\mathcal{O}_{\mathcal{K}}}\mathbb{T}^{\mathcal{T}_\ell}_{\boldsymbol{\omega}_\lambda}(U).$$

Proof. Concerning part 1., we will only prove the K-case (the other case being analogous). First recall from Section 6.2 that $\mathcal{S}_{\boldsymbol{\omega}}(U)$ admits a basis $(f_1, \dots, f_r)$ consisting of eigenforms for ${}^{\mathcal{O}_{\mathcal{E}(U)}}\mathbf{T}^{\mathcal{T}}_{\boldsymbol{\omega}}(U)$. As all eigenvalues are contained in $\mathcal{O}_{\mathcal{E}(U)}$, we can consequently embed

$$ {}^{\mathcal{O}_{\mathcal{E}(U)}}\mathbf{T}^{\mathcal{T}}_{\boldsymbol{\omega}}(U) \hookrightarrow \mathcal{O}_{\mathcal{E}(U)} \times \ldots \times \mathcal{O}_{\mathcal{E}(U)} \qquad (r \text{ factors}) \tag{6.10}$$

as $\mathcal{O}_{\mathcal{E}(U)}$-algebras. K is a torsion-free $\mathcal{O}_{\mathcal{E}(U)}$-module (hence flat, as $\mathcal{O}_{\mathcal{E}(U)}$ is a Dedekind ring), so this gives an injection

$$K \otimes_{\mathcal{O}_{\mathcal{E}(U)}} {}^{\mathcal{O}_{\mathcal{E}(U)}}\mathbf{T}^{\mathcal{T}}_{\boldsymbol{\omega}}(U) \hookrightarrow K \otimes_{\mathcal{O}_{\mathcal{E}(U)}} (\mathcal{O}_{\mathcal{E}(U)} \times \ldots \times \mathcal{O}_{\mathcal{E}(U)}) \cong K \times \ldots \times K.$$

The image of this map clearly lies in ${}^{K}\mathbf{T}^{\mathcal{T}}_{\boldsymbol{\omega}}(U)$ and contains all the operators $T^{(j)}_{F_w}$, hence it equals ${}^{K}\mathbf{T}^{\mathcal{T}}_{\boldsymbol{\omega}}(U)$. Concerning part 2., we conclude from the inclusion (6.10) that ${}^{\mathcal{O}_{\mathcal{E}(U)}}\mathbf{T}^{\mathcal{T}}_{\boldsymbol{\omega}}(U)$ is finitely generated as a $\mathbb{Z}$-module, hence as a $\mathbb{Z}$-algebra. It follows that there exists a Sturm-like bound C such that ${}^{\mathcal{O}_{\mathcal{E}(U)}}\mathbf{T}^{\mathcal{T}}_{\boldsymbol{\omega}}(U)$ is already generated by those $T^{(j)}_{F_w}$ for which $\ell(w) \leq C$. Hence, using part 1. and the compatibility from Proposition 6.31, we see that

$$\mathcal{O}_{\mathcal{K}} \otimes_{\mathcal{O}_{\mathcal{E}(U)}} {}^{\mathcal{O}_{\mathcal{E}(U)}}\mathbf{T}^{\mathcal{T}}_{\boldsymbol{\omega}}(U) = \mathcal{O}_{\mathcal{K}} \otimes_{\mathcal{O}_{\mathcal{E}(U)}} {}^{\mathcal{O}_{\mathcal{E}(U)}}\mathbf{T}^{\mathcal{T}_\ell}_{\boldsymbol{\omega}}(U) \cong {}^{\mathcal{O}_{\mathcal{K}}}\mathbb{T}^{\mathcal{T}_\ell}_{\boldsymbol{\omega}_\lambda}(U).$$ ☐

Now let $E \supset \mathcal{E}(U)$ be a number field with ring of integers $\mathcal{O}_E$. We get the following corollary:

Corollary 6.41. *${}^{\mathcal{O}_E}\boldsymbol{T}^{\mathcal{T}}_{\boldsymbol{\omega}}(U)$ is finite and torsion-free as an $\mathcal{O}_E$-module and $E \otimes_{\mathcal{O}_E} {}^{\mathcal{O}_E}\boldsymbol{T}^{\mathcal{T}}_{\boldsymbol{\omega}}(U)$ is semisimple (so, in particular, we have a decomposition*

$$E \otimes_{\mathcal{O}_E} {}^{\mathcal{O}_E}\boldsymbol{T}^{\mathcal{T}}_{\boldsymbol{\omega}}(U) \cong k_1 \times \ldots \times k_m \tag{6.11}$$

as a finite product of fields.)

Note that the k_i are in fact isomorphic to E (as we supposed $E \supset \mathcal{E}(U)$).

Proof. That ${}^{\mathcal{O}_E}\mathbf{T}^{\mathcal{T}}_{\boldsymbol{\omega}}(U)$ is finite and torsion-free follows directly from the proof of Proposition 6.40. So let us show that $E \otimes_{\mathcal{O}_E} {}^{\mathcal{O}_E}\mathbf{T}^{\mathcal{T}}_{\boldsymbol{\omega}}(U)$ is semisimple, i.e. that its Jacobson radical is trivial. As $E \otimes_{\mathcal{O}_E} {}^{\mathcal{O}_E}\mathbf{T}^{\mathcal{T}}_{\boldsymbol{\omega}}(U)$ is commutative and finitely generated over E, the Jacobson radical equals the nilradical, so we have to prove that $E \otimes_{\mathcal{O}_E} {}^{\mathcal{O}_E}\mathbf{T}^{\mathcal{T}}_{\boldsymbol{\omega}}(U)$ is reduced. But by the above it is clear that $E \otimes_{\mathcal{O}_E} {}^{\mathcal{O}_E}\mathbf{T}^{\mathcal{T}}_{\boldsymbol{\omega}}(U)$ does not contain nilpotent elements. The decomposition as a product of fields follows from the Artin-Wedderburn theorem. ☐

For the following, let $\mathbf{T}$ be an $\mathcal{O}_E$-algebra subject to the following conditions:

- $\mathbf{T}$ is finite and torsion-free over $\mathcal{O}_E$;
- $E \otimes_{\mathcal{O}_E} \mathbf{T}$ is semisimple (so, in particular, we have a decomposition

$$E \otimes_{\mathcal{O}_E} \mathbf{T} \cong k_1 \times \ldots \times k_m \tag{6.12}$$

  as a finite product of fields.)

By Corollary 6.41, $\mathbf{T} = {}^{\mathcal{O}_E}\mathbf{T}^{\mathcal{T}}_{\boldsymbol{\omega}}(U)$ fulfills these conditions, and this is the choice for $\mathbf{T}$ we are interested in. However, we choose to use this more general characterization in order to simplify the notation in the following proof and to emphasize that we only use these two formal properties of $\mathbf{T}$.

Theorem 6.42. *There exists a constant N (depending on $\boldsymbol{T}$) such that, for all places λ of E fulfilling $\ell(\lambda) > N$, we have a decomposition*

$$\mathcal{O}_{E_\lambda} \otimes_{\mathcal{O}_E} \boldsymbol{T} \cong \prod_{i=1}^{m} \prod_{j=1}^{n_i} \mathcal{O}_{\lambda,i,j}, \tag{6.13}$$

of $\mathcal{O}_{E_\lambda} \otimes_{\mathcal{O}_E} \boldsymbol{T}$ as a product of finitely many complete discrete valuation rings over $\mathbb{Z}_{\ell(\lambda)}$.

Proof. By the semisimplicity of $E \otimes_{\mathcal{O}_E} \mathbf{T}$, there exist orthogonal idempotents $e_1, \dots, e_r \in E \otimes_{\mathcal{O}_E} \mathbf{T}$ which fulfill $\sum_i e_i = 1$ and

$$e_i.\big(E \otimes_{\mathcal{O}_E} \mathbf{T}\big) \cong k_i.$$

Therefore, we can find a constant $N \in \mathbb{N}$ such that $e_1, \dots, e_r \in \mathbf{T}\left[\frac{1}{N}\right]$ and

$$\mathbf{T}\left[\frac{1}{N}\right] = \bigoplus_{i=1}^{r} e_i.\big(\mathbf{T}\left[\frac{1}{N}\right]\big). \tag{6.14}$$

Thus we see that for each λ with $\ell(\lambda) \not| N$ tensoring with $\mathcal{O}_{E_\lambda}$ over $\mathcal{O}_E\left[\frac{1}{N}\right]$ yields an isomorphism

$$\mathcal{O}_{E_\lambda} \otimes_{\mathcal{O}_E} \mathbf{T} = \bigoplus_{i=1}^{r} e_i.\big(\mathcal{O}_{E_\lambda} \otimes_{\mathcal{O}_E} \mathbf{T}\big).$$

Now consider the embeddings

$$\iota_i : e_i.\big(\mathbf{T}\left[\frac{1}{N}\right]\big) \hookrightarrow \mathcal{O}_{k_i}\left[\frac{1}{N}\right]$$

for $i = 1, \dots, r$. As $\mathbf{T}$ is finite and torsion-free over $\mathcal{O}_E$, it is finite and free as a $\mathbb{Z}$-module. Hence $\mathbf{T}\left[\frac{1}{N}\right]$ is finite and free as a $\mathbb{Z}\left[\frac{1}{N}\right]$-module. Moreover

$$\mathbb{Q}.e_i.\big(\mathbf{T}\left[\frac{1}{N}\right]\big) = k_i,$$

so we see that both $e_i.\big(\mathbf{T}\left[\frac{1}{N}\right]\big)$ and $\mathcal{O}_{k_i}\left[\frac{1}{N}\right]$ are $\mathbb{Z}\left[\frac{1}{N}\right]$-orders in k_i. In particular, they are both free $\mathbb{Z}\left[\frac{1}{N}\right]$-modules of the same rank (say, t_i). So let us write $\mathcal{O}_{k_i}\left[\frac{1}{N}\right] = \mathbb{Z}\left[\frac{1}{N}\right]^{t_i}$ and consider $J_i \coloneqq \mathrm{im}(\iota_i)$ as a submodule of $\mathbb{Z}\left[\frac{1}{N}\right]^{t_i}$.

Now, by the elementary divisor theorem for finitely generated $\mathbb{Z}\left[\frac{1}{N}\right]$-modules, we see that

$$\begin{aligned} \mathrm{im}(\iota_i) &\cong \bigoplus_{j=1}^{t_i} d_j \mathbb{Z}\left[\frac{1}{N}\right], \\ \mathrm{cok}(\iota_i) &\cong \bigoplus_{j=1}^{t_i} \mathbb{Z}\left[\frac{1}{N}\right] / d_j \mathbb{Z}\left[\frac{1}{N}\right], \end{aligned}$$

where all $d_j \neq 0$ (as ι_i is injective and $\operatorname{rk} e_i.\big(\mathbf{T}\left[\frac{1}{N}\right]\big) = \operatorname{rk} \mathcal{O}_{k_i}\left[\frac{1}{N}\right]$) and where $d_1|\dots|d_{t_i}$. After possibly multiplying by suitable elements of $\mathbb{Z}\left[\frac{1}{N}\right]^\times$, we can additionally assume that $d_j \in \mathbb{Z}$ and $(d_j, N) = 1$ for all j. Thus we get

$$\operatorname{cok}(\iota_i) \cong \bigoplus_{j=1}^{t_i} \mathbb{Z}\left[\frac{1}{N}\right]/d_j\mathbb{Z}\left[\frac{1}{N}\right] \cong \bigoplus_{j=1}^{t_i}\Big(\mathbb{Z}/d_j\mathbb{Z}\Big)\left[\frac{1}{N}\right] \overset{(d_j,N)=1}{\cong} \bigoplus_{j=1}^{t_i} \mathbb{Z}/d_j\mathbb{Z}.$$

(Alternatively, we can use the general fact that finitely generated torsion $\mathbb{Z}\left[\frac{1}{N}\right]$-modules are finite.) The same argument applied to $\oplus_i \iota_i$ instead of ι yields

$$c := \#\operatorname{cok}\Big(\bigoplus_i \iota_i\Big) < \infty.$$

Thus, after replacing N by cN we see that the ι_i become isomorphisms and hence (6.14) implies

$$\mathcal{O}_{E_\lambda} \otimes_{\mathcal{O}_E} \mathbf{T} \cong \bigoplus_{i=1}^{r} \mathcal{O}_{E_\lambda} \otimes_{\mathcal{O}_E} \mathcal{O}_{k_i}\left[\frac{1}{N}\right] = \bigoplus_{i=1}^{r} \mathcal{O}_{E_\lambda} \otimes_{\mathcal{O}_E} \mathcal{O}_{k_i}$$

for all λ with $\ell(\lambda) > N$.

Now we use the following general fact: If $K_2|K_1$ is an extension of number fields with rings of integers $\mathcal{O}_{K_2}, \mathcal{O}_{K_1}$ and if $\mathfrak{p}$ is a place of K_1, then

$$\mathcal{O}_{K_{1,\mathfrak{p}}} \otimes_{\mathcal{O}_{K_1}} \mathcal{O}_{K_2} \cong \prod_{\mathfrak{P}|\mathfrak{p}} \mathcal{O}_{K_{2,\mathfrak{P}}}.$$

(This follows from [Ser79, Ch. 2, §3, Proposition 4].) This implies the decomposition in (6.13). Moreover (after enlarging N if necessary) we can assume that the extensions $k_i|\mathbb{Q}$ are unramified at all primes not dividing N. It follows (e.g. from [Ser79, Ch. 2, §3, Theorem 1 (ii)]) that all the rings occurring on the right hand side of (6.13) are unramified extensions of $\mathbb{Z}_{\ell(\lambda)}$, i.e. are complete discrete valuation rings. □

Corollary 6.43. *Write $\mathcal{O}_\lambda = \mathcal{O}_{E_\lambda}$ for a place λ of E. There exists a constant C' such that*

$${}^{\mathcal{O}_\lambda}\mathbb{T}^{\mathcal{T}_\ell}_{\boldsymbol{\omega}_\lambda}(U)_{\mathfrak{m}} \cong \mathcal{O}_\lambda$$

holds for any place λ of E and for any maximal ideal $\mathfrak{m} \subset {}^{\mathcal{O}_\lambda}\mathbb{T}^{\mathcal{T}_\ell}_{\boldsymbol{\omega}_\lambda}(U)$, as long as $\ell(\lambda) > C'$.

Proof. Using Proposition 6.40 and Theorem 6.42 (with $\mathbf{T} = {}^{\mathcal{O}_E}\mathbf{T}^{\mathcal{T}}_{\boldsymbol{\omega}}(U)$) and observing that all the k_i in (6.11) are isomorphic to E in this case, we see that for each λ with $\ell(\lambda) > C' := \max(C, N)$ we get an isomorphism

$${}^{\mathcal{O}_\lambda}\mathbb{T}^{\mathcal{T}_\ell}_{\boldsymbol{\omega}_\lambda}(U) \cong \mathcal{O}_\lambda^m.$$

The claim follows. □

6.4.3 An $R = O$-theorem and independence from the auxiliary primes

Definition 6.44. Let $f \in \mathcal{S}_{\boldsymbol{\omega}}(U)$ be an eigenform (for some level subgroup of the form $U = U_{\Sigma_{\mathtt{ram}},\Sigma_{\mathtt{aux}}}$) and let ℓ be a rational prime. We say that a place $\nu \notin \Sigma_{\mathtt{ram}} \cup \Omega_\ell^{F^+} \cup \Omega_\infty^{F^+}$ of F^+ is (f,ℓ)-*auxiliary* if it fulfills the following two conditions:

- ν splits in $F|F^+$ (as, say, $\tilde{\nu}\tilde{\nu}^c$);
- For each place λ of $\mathcal{E}(U)$ which fulfills $\ell(\lambda) = \ell$, the residual representation $\overline{\rho}_{f,\lambda}$ associated to f is unramified at ν and its zeroth cohomology vanishes:

$$H^0(\mathrm{Gal}_{F_{\tilde{\nu}}}, \mathrm{ad}(\overline{\rho}_{f,\lambda})(1)) = 0. \tag{6.15}$$

The significance of the (f,ℓ)-auxiliary condition is reflected by:

Proposition 6.45. *Let ν be (f,ℓ)-auxiliary and $\lambda \in \mathrm{Pl}_{\mathcal{E}(U)}$ a place above ℓ, then the canonical surjection*

$$h : R^\square(\overline{\rho}_{f,\lambda}|\,\mathrm{Gal}_{F_{\tilde{\nu}}}) \twoheadrightarrow R^{\square,nr}(\overline{\rho}_{f,\lambda}|\,\mathrm{Gal}_{F_{\tilde{\nu}}})$$

is an isomorphism. In other words: Any lift of $\overline{\rho}_{f,\lambda}|\,\mathrm{Gal}_{F_{\tilde{\nu}}}$ is automatically unramified.

Proof. This is seen as follows: First remark that $\mathrm{Gal}_{F_{\tilde{\nu}}}^{nr} := \mathrm{Gal}(F_{\tilde{\nu}}^{nr}|F_{\tilde{\nu}}) \cong \hat{\mathbb{Z}}$ and the corresponding framed deformation ring $R^\square(\overline{\rho}_{f,\lambda}|\,\mathrm{Gal}_{F_{\tilde{\nu}}}^{nr}) = R^{\square,\mathrm{nr}}(\overline{\rho}_{f,\lambda}|\,\mathrm{Gal}_{F_{\tilde{\nu}}})$ is formally smooth of dimension $d^{\mathrm{nr}} = n^2$ (cf. Remark 4.18 and Lemma 4.23). By Assumption (6.15), also the unrestricted deformation ring is formally smooth of dimension $d = Z^1(\mathrm{Gal}_{F_{\tilde{\nu}}}, \mathrm{ad}(\overline{\rho}_{f,\lambda}))$. Therefore, we are good as soon as we can show that $d = d^{\mathrm{nr}}$ (cf. Lemma 2.18). For this, we remark that the coboundaries

$$B^1 = \mathrm{ad}(\overline{\rho}_{f,\lambda})/\,\mathrm{ad}(\overline{\rho}_{f,\lambda})^{\mathrm{Gal}_{F_{\tilde{\nu}}}^{nr}} = \mathrm{ad}(\overline{\rho}_{f,\lambda})/\,\mathrm{ad}(\overline{\rho}_{f,\lambda})^{\mathrm{Gal}_{F_{\tilde{\nu}}}}$$

are the same in the unramified and in the unrestricted situation (because $\overline{\rho}_{f,\lambda}$ was assumed to be unramified at ν). Thus we have the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B^1 & \longrightarrow & Z^1(\hat{\mathbb{Z}}, \mathrm{ad}(\overline{\rho}_{f,\lambda})) & \longrightarrow & H^1(\hat{\mathbb{Z}}, \mathrm{ad}(\overline{\rho}_{f,\lambda})) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B^1 & \longrightarrow & Z^1(\mathrm{Gal}_{F_{\tilde{\nu}}}, \mathrm{ad}(\overline{\rho}_{f,\lambda})) & \longrightarrow & H^1(\mathrm{Gal}_{F_{\tilde{\nu}}}, \mathrm{ad}(\overline{\rho}_{f,\lambda})) & \longrightarrow & 0
\end{array}$$

where the vertical maps are the inflation maps along $\mathrm{Gal}_{F_{\tilde{\nu}}} \to \mathrm{Gal}_{F_{\tilde{\nu}}}/\,\mathrm{Gal}_{F_{\tilde{\nu}}^{nr}} \cong \mathrm{Gal}_{F_{\tilde{\nu}}}^{nr}$. But, by the local Euler-Poincare formula, we know that

- $\dim H^1(\mathrm{Gal}_{F_{\tilde{\nu}}}^{nr}, \mathrm{ad}(\overline{\rho}_{f,\lambda})) = \dim H^0(\mathrm{Gal}_{F_{\tilde{\nu}}^{nr}}, \mathrm{ad}(\overline{\rho}_{f,\lambda})) = \dim H^0(\mathrm{Gal}_{F_{\tilde{\nu}}}, \mathrm{ad}(\overline{\rho}_{f,\lambda}))$;
- $\dim H^1(\mathrm{Gal}_{F_{\tilde{\nu}}}, \mathrm{ad}(\overline{\rho}_{f,\lambda})) = \dim H^0(\mathrm{Gal}_{F_{\tilde{\nu}}}, \mathrm{ad}(\overline{\rho}_{f,\lambda})) + \dim H^2(\mathrm{Gal}_{F_{\tilde{\nu}}}, \mathrm{ad}(\overline{\rho}_{f,\lambda}))$
$\overset{\text{by (6.15)}}{=} \dim H^0(\mathrm{Gal}_{F_{\tilde{\nu}}}, \mathrm{ad}(\overline{\rho}_{f,\lambda}))$.

Thus, the outer vertical maps are both isomorphisms, hence the middle map must be an isomorphism as well and we have $d = d^{\mathrm{nr}}$. □

Now, let $f \in \mathcal{S}_{\boldsymbol{\lambda}}(U_0)$ for $U_0 = U_{\Sigma_{\mathtt{ram}},\emptyset}$. For a finite collection Φ of (f,ℓ)-auxiliary places, denote $U_\Phi = U_{\Sigma_{\mathtt{ram}},\Phi}$. For any place λ of $\mathcal{E}(U_\Phi)_\lambda$ with $\ell(\lambda) = \ell$ and any extension $\widetilde{E}|\mathcal{E}(U_\Phi)_\lambda$ we have an embedding

$$^{\mathcal{O}_{\widetilde{E}}}\mathbb{T}^{\mathcal{T}_\ell}_{\boldsymbol{\omega}_\lambda}(U_0) \hookrightarrow {}^{\mathcal{O}_{\widetilde{E}}}\mathbb{T}^{\mathcal{T}_\ell}_{\boldsymbol{\omega}_\lambda}(U_\Phi).$$

Let $\mathfrak{m}_{f,0}$ (resp. $\mathfrak{m}_{f,\Phi}$) be the maximal ideal of ${}^{\mathcal{O}_{\widetilde{E}}}\mathbb{T}^{\mathcal{T}_\ell}_{\boldsymbol{\omega}_\lambda}(U_0)$ (resp. of ${}^{\mathcal{O}_{\widetilde{E}}}\mathbb{T}^{\mathcal{T}_\ell}_{\boldsymbol{\omega}_\lambda}(U_\Phi)$) containing the kernel of φ_f.

Proposition 6.46.

$$^{\mathcal{O}_{\widetilde{E}}}\mathbb{T}^{\mathcal{T}_\ell}_{\boldsymbol{\omega}_\lambda}(U_0)_{\mathfrak{m}_{f,0}} \cong {}^{\mathcal{O}_{\widetilde{E}}}\mathbb{T}^{\mathcal{T}_\ell}_{\boldsymbol{\omega}_\lambda}(U_\Phi)_{\mathfrak{m}_{f,\Phi}}.$$

Proof. We claim that the canonical injection

$$^{\mathcal{O}_{\widetilde{E}}}\mathbb{T}^{\mathcal{T}_\ell}_{\boldsymbol{\omega}_\lambda}(U_0)_{\mathfrak{m}_{f,0}} \hookrightarrow {}^{\mathcal{O}_{\widetilde{E}}}\mathbb{T}^{\mathcal{T}_\ell}_{\boldsymbol{\omega}_\lambda}(U_\Phi)_{\mathfrak{m}_{f,\Phi}}$$

is surjective: Any counter-example to this claim would lead to the existence of an eigenform

$$g \in \mathcal{S}_{\boldsymbol{\omega}}(U_\Phi) - \mathcal{S}_{\boldsymbol{\omega}}(U_0)$$

such that $\mathfrak{m}_{f,\Phi} = \mathfrak{m}_{g,\Phi}$. But then (cf. Proposition 6.33), the ℓ-adic Galois representation $\rho_{g,\ell}$ lifts $\overline{\rho}_{f,\ell}$. Then Proposition 6.45 tells us that $\rho_{g,\ell}$ is unramified at each $\nu \in \Phi$. Using the local-global compatibility of the Langlands correspondence, this implies that the local parts Π_ν must be unramified for $\nu \in \Phi$, where Π denotes the cuspidal automorphic representation generated by g, i.e. we get that $g \in \mathcal{S}_{\boldsymbol{\omega}}(U_0)$. This yields a contradiction as desired. □

Corollary 6.47. *Assume that $\ell > C'$, where C' is the constant from Corollary 6.43 for $U = U_0$. Then*

$$^{\mathcal{O}_{\widetilde{E}}}\mathbb{T}^{\mathcal{T}_\ell}_{\boldsymbol{\omega}_\lambda}(U_\Phi)_{\mathfrak{m}_{f,\Phi}} \cong \mathcal{O}_{\widetilde{E}}.$$

Let us subsume our observations so far (translated to the $\mathcal{G}_n$-valued family using Proposition 6.22):

Theorem 6.48. *Let $U_0 = U_{\Sigma_{\mathtt{ram}},\emptyset}$ be a congruence subgroup and $f \in \mathcal{S}_{\boldsymbol{\omega}}(U_0)$ be an automorphic form such that $\Pi = \langle f\rangle$ is a RACSDC automorphic representation which is unramified outside $\Sigma_{\mathtt{ram}}$ and unipotently ramified at the places in $\Sigma_{\mathtt{ram}}$. Let us assume Conjecture 6.37 and fix for each place λ of $\mathcal{E}(U_0)$ the following data:*

- *An (f,ℓ)-auxiliary place $\nu_{\lambda,1}$ such that $[F(\zeta_{\ell(\nu_{\lambda,1})}) : F] > n$;*
- *a finite field extension $\mathcal{K}_\lambda$ of $\mathcal{E}(U_0)_\lambda$ which is F-big enough.*

Write $U_\lambda = U_{\Sigma_{\mathtt{ram}},\{\nu_{\lambda,1}\}}$ and denote by $\mathfrak{m}_{f,\lambda} \subset {}^{\mathcal{O}_{\mathcal{K}_\lambda}}\mathbb{T}^{\mathcal{T}_\ell}_{\boldsymbol{\omega}_\lambda}(U_\lambda)$ the maximal ideal which contains the kernel of φ_f. Then there exists a constant K, depending on f, U_0 and $\boldsymbol{\omega}$, such that

$$\ell(\lambda) > K \;\Rightarrow\; R^{\mathtt{min,crys}}(\overline{r}_{\mathfrak{m}_{f,\lambda}}) \cong \mathcal{O}_{\mathcal{K}_\lambda},$$

where $R^{\mathtt{min,crys}}(\overline{r}_{\mathfrak{m}_{f,\lambda}})$ is the universal deformation ring of $\overline{r}_{\mathfrak{m}_{f,\lambda}}$ corresponding to the deformation condition

$$\mathcal{S}^{\mathtt{min,crys}}_\lambda := (F|F^+, \mathcal{T}_\ell, \widetilde{\mathcal{T}}_\ell, \mathcal{O}_{\mathcal{K}_\lambda}, \overline{r}_{\mathfrak{m}}, \epsilon^{1-n}\delta^{\mu_{\mathfrak{m}}}_{F|F^+}, \{\mathcal{D}_\nu\})$$

considered in Section 6.4.1 (with $\mathcal{T} = \Sigma_{\mathtt{ram}} \cup \{\nu_{\lambda,1}\}$).

Proof. This is a combination of Conjecture 6.37, Corollary 6.47 and Observation 6.32. □

6.4.4 Congruences between automorphic forms and minimal ramification

Consider two eigenforms $f_1, f_2 \in \mathcal{S}_{\boldsymbol{\omega}}(U)$, where U is a level subgroup of the form $U_{\Sigma_{\mathtt{ram}},\Sigma_{\mathtt{aux}}}$ (and, as before, we write $\mathcal{T} = \Sigma_{\mathtt{ram}} \sqcup \Sigma_{\mathtt{aux}}$).

Definition 6.49. Let λ be a place of $\mathcal{E}(U)$ and set $\ell = \ell(\lambda)$. For $i = 1, 2$ consider the maps

$$\psi_{f_i} : {}^{\mathbb{Z}}\mathbb{T}^{\mathcal{T}_\ell}_{\boldsymbol{\omega}_\lambda}(U) \longrightarrow \overline{\mathcal{O}}_{\mathcal{E}(U)_\lambda} \longrightarrow \overline{\mathbb{F}}_\ell$$

assigning to a Hecke operator T the mod-λ reduction of the eigenvalue of T acting on the ℓ-adic model $\tilde{f}_i$ of f_i. We say that f_1 and f_2 are *congruent modulo* λ (in symbols: $f_1 \equiv_\lambda f_2$) if $\psi_{f_1} = \psi_{f_2}$.

Remark 6.50. It seems most natural that one uses the same embedding $\mathcal{O}_{\mathcal{E}(U)_\lambda} \hookrightarrow \overline{\mathcal{O}}_{\mathcal{E}(U)_\lambda}$ in the definition of ψ_{f_1} and ψ_{f_2}. If one allows different embeddings, then $\psi_{f_1} = \psi^\sigma_{f_2}$ for some $\sigma \in \mathrm{Gal}_{\mathbb{Q}}$. But then $\psi_{f_1} = \psi_{f_2^\sigma}$ where f_2^σ is a form conjugate to f_2. If f_2^σ was then equal to f_1, this would simply describe a trivial congruence of f with itself, which is not interesting. If $f_2^\sigma \neq f_1$, then f_2^σ is congruent to f_1 modulo λ in the above sense, i.e. there is an interesting congruence in the sense of Definition 6.49.

We prove the following lemma under the condition $\ell \gg 0$ (or, more precisely, $\ell > C$, where C is a Sturm-like bound depending only on ${}^{\mathcal{O}_{\mathcal{E}(U)}}\mathbf{T}^{\emptyset}_{\boldsymbol{\omega}}(U)$):

Lemma 6.51. *The existence of a congruence $f_1 \equiv_\lambda f_2$ with $\ell = \ell(\lambda) \gg 0$ implies that there exists a maximal ideal $\mathfrak{M}$ in ${}^{\mathcal{O}_{\mathcal{E}(U)}}\mathbf{T}^{\mathcal{T}}_{\boldsymbol{\omega}}(U)$ which contains ℓ and both $\mathfrak{p}_1 = \ker \varphi'_{f_1}$ and $\mathfrak{p}_2 = \ker \varphi'_{f_2}$, where*

$$\varphi'_{f_i} : {}^{\mathcal{O}_{\mathcal{E}(U)}}\mathbf{T}^{T}_{\boldsymbol{\omega}}(U) \longrightarrow \overline{\mathcal{O}}_{\mathcal{E}(U)}$$

is defined by sending a Hecke operator T to the eigenvalue of T acting on f_i

Proof. First, recall that ${}^{\mathcal{O}_{\mathcal{E}(U)}}\mathbf{T}^{\mathcal{T}}_{\boldsymbol{\omega}}(U)$ is finitely generated as a $\mathbb{Z}$-module. Hence there exists a constant C such that ${}^{\mathcal{O}_{\mathcal{E}(U)}}\mathbf{T}^{\mathcal{T}}_{\boldsymbol{\omega}}(U)$ is already generated by the Hecke operators $T^{(j)}_{F_w}$ with $j \leq C$. In particular, we get isomorphisms ${}^{\mathcal{O}_{\mathcal{E}(U)}}\mathbf{T}^{\mathcal{T}_\ell}_{\boldsymbol{\omega}}(U) \cong {}^{\mathcal{O}_{\mathcal{E}(U)}}\mathbf{T}^{\mathcal{T}}_{\boldsymbol{\omega}}(U)$ for all primes $\ell > C$. (Recall that we defined $\mathcal{T}_\ell = \mathcal{T} \cup \Omega^F_\ell$.) The claim now follows from the commutative diagram

$$\begin{array}{ccccc}
{}^{\mathcal{O}_{\mathcal{E}(U)_\lambda}}\mathbb{T}^{\mathcal{T}_\ell}_{\boldsymbol{\omega}_\lambda}(U) & \longrightarrow & \overline{\mathcal{O}}_{\mathcal{E}(U)_\lambda} & \longrightarrow & \overline{\mathbb{F}}_\ell \; , \\
\uparrow & & \uparrow & & \\
{}^{\mathcal{O}_{\mathcal{E}(U)}}\mathbf{T}^{\mathcal{T}_\ell}_{\boldsymbol{\omega}}(U) & \xrightarrow{\varphi'_{f_i}} & \overline{\mathcal{O}}_{\mathcal{E}(U)} & & \\
\| & & & \eta_{f_i} & \\
{}^{\mathcal{O}_{\mathcal{E}(U)}}\mathbf{T}^{\mathcal{T}}_{\boldsymbol{\omega}}(U) & & & &
\end{array}$$

where the concatenation of the two upper horizontal maps is the obvious continuation of ψ_{f_i} to ${}^{\mathcal{O}_{\mathcal{E}(U)_\lambda}}\mathbb{T}^{\mathcal{T}_\ell}_{\boldsymbol{\omega}_\lambda}(U)$. The assumption $f_1 \equiv_\lambda f_2$ implies $\eta_{f_1} = \eta_{f_2}$. So we see that the congruence condition implies that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are both contained in the maximal ideal $\ker \eta_{f_1} = \ker \eta_{f_2}$. $\square$

Lemma 6.52. *Let A be an algebra that is finite flat over $\mathbb{Z}$ and let $\mathfrak{p}_1, \mathfrak{p}_2$ be two distinct minimal primes of A. Then there are only finitely many maximal ideals of A that contain both $\mathfrak{p}_1$ and $\mathfrak{p}_2$.*

Proof. By the going up and down theorems for $A/\mathbb{Z}$ (using finiteness and flatness), one can easily check that the ring A has dimension 1, and that the minimal primes of A are in bijection to the maximal ideals of the Artinian ring $A \otimes_{\mathbb{Z}} \mathbb{Q}$. We need to show that $B := A/(\mathfrak{p}_1 + \mathfrak{p}_2)$ contains only finitely many maximal ideals. It suffices to show that B is finite. Since B is finitely generated over $\mathbb{Z}$, we need to show that $B \otimes_{\mathbb{Z}} \mathbb{Q}$ is zero. However, $A \otimes_{\mathbb{Z}} \mathbb{Q}$ is an algebra that is finite over $\mathbb{Q}$, and hence a finite product of local Artinian rings. In particular, the sum of any two distinct prime ideals of $A \otimes_{\mathbb{Z}} \mathbb{Q}$ is all of $A \otimes_{\mathbb{Z}} \mathbb{Q}$, i.e. such ideals are relatively prime. It follows that $B \otimes_{\mathbb{Z}} \mathbb{Q}$ is zero, as was to be shown. □

Corollary 6.53. *Presume Assumption 6.6 and Conjecture 6.37. Then Assumption 4.45 holds for the compatible system* $\mathcal{R}_\Pi = (\rho_{f,\lambda})_{\lambda \in \Lambda^1_{\mathcal{E}(U)}}$ *attached to* $\Pi = \langle f \rangle$.

Proof. By Conjecture 6.37, we know that for almost all $\lambda \in \Lambda^1_{\mathcal{E}(U)}$ we can find an automorphic form $g^{(\lambda)}$ such that $\rho_{g^{(\lambda)},\lambda}$ is a minimal lift of $\overline{\rho}_{f,\lambda}$. Denote the finite failure set by $X' \subset \mathrm{Pl}^{\mathrm{fin}}_{\mathcal{E}(U)}$. We enlarge X' to

$$X := X' \cup \{\lambda \mid \ell(\lambda) \leq C\},$$

where C is the constant from (the proof of) Lemma 6.51. Then, any place $\lambda \notin X$ at which $\rho_{f,\lambda}$ is not a minimal lift of $\overline{\rho}_{f,\lambda}$ gives rise to a non-trivial congruence in $\mathcal{S}_{\boldsymbol{\omega}}(U)$, i.e. to a triple

$$(f, g^{(\lambda)}, \lambda) \in \mathcal{S}_{\boldsymbol{\omega}}(U) \times \mathcal{S}_{\boldsymbol{\omega}}(U) \times \mathrm{Pl}^{\mathrm{fin}}_{\mathcal{E}(U)} \text{ fulfilling } f \neq g^{(\lambda)} \text{ and } f \equiv_\lambda g^{(\lambda)}. \tag{6.16}$$

By Lemma 6.51, this implies the existence of a maximal ideal ${}^{\mathcal{O}_{\mathcal{E}(U)}}\mathbf{T}^{\mathcal{T}}_{\boldsymbol{\omega}}(U)$ containing ℓ, $\mathfrak{p}_1 = \ker \varphi'_{f_1}$ and $\mathfrak{p}_2 = \ker \varphi'_{f_2}$, as long as ℓ is not contained in the finite failure set $\overline{X} = \{\ell(\lambda) | \lambda \in X\} \subset \mathrm{Pl}^{\mathrm{fin}}_{\mathbb{Q}}$. As there are only finitely many eigenspaces in $\mathcal{S}_{\boldsymbol{\omega}}(U)$, Lemma 6.52 with $A = {}^{\mathcal{O}_{\mathcal{E}(U)}}\mathbf{T}^{\mathcal{T}}_{\boldsymbol{\omega}}(U)$ implies that there are only finitely many triples as in (6.16). (Here, we tacitly identify triples of the form (f, g, λ) and $(z.f, z'.g, \lambda)$ with $z, z' \in \mathbb{C}^\times$, as they represent the same congruence condition.) The claim follows. □

6.5 Unobstructedness of the minimal deformation rings

Finally, we are in a position to formulate and prove our main result. For this, recall the partial compatible system $\mathcal{R}_\Pi = (\overline{\rho}_\lambda)_{\lambda \in \Lambda^1_{\mathcal{E}}}$ associated to Π by (6.3) and the $\mathcal{G}_n$-valued family $(\overline{r}_\lambda)_{\lambda \in \Lambda^1_{\mathcal{E}}}$. If T is a set of places of F, we denote by $\overline{T}$ the set of places of F^+ below T. First we make the following technical assumption, which will be revoked later on:

Assumption 6.54. Each place $\nu \in \overline{S}_{\Pi,\infty} = \overline{S}_\Pi \sqcup \Omega^{F^+}_\infty$ splits in $F|F^+$ as, say, $\tilde{\nu}\tilde{\nu}^c$. (For $\Omega^{F^+}_\infty$, this splitting is automatic, so we only put a constraint on S_Π here.)

For each $\lambda \in \Lambda^1_{\mathcal{E}}$, we consider the global deformation problem $\mathcal{D}_\lambda = \{D_{\lambda,\nu}\}_{\nu \in \overline{S}_{\Pi,\infty}}$ for $\overline{r}_\lambda$, where $D_{\lambda,\nu}$ parametrizes

- arbitrary lifts of $\overline{\rho}_{\lambda,\tilde{\nu}}$, if $\nu \in \Omega^{F^+}_\infty$;
- minimally ramified lifts of $\overline{\rho}_{\lambda,\tilde{\nu}}$ in the sense of Section 4.4, if $\nu \in \overline{S}_\Pi$.

Write χ for the character $\epsilon^{1-n}\delta^{n (\mathrm{mod}\, 2)}_{F|F^+}$ of Gal_{F^+}. We now compile the necessary assumptions for Theorem 6.56:

Assumption 6.55.

1. **(Irreducibility):** Assumption 6.6 holds: The set of places $\Lambda_{\mathcal{E}_\Pi}$ where our residual compatible system is absolutely irreducible has Dirichlet density 1;
2. **(Availability of a minimal R=T-theorem):** Conjecture 6.37 holds;
3. **(No consecutive weights):** The sets of Hodge-Tate weights HT_τ of the system $\mathcal{R}_\Pi$ fulfill (for all embeddings τ) the condition from Theorem 4.7: If two numbers a, b occur in HT_τ, then either $a = b$ or $|a - b| \geq 2$;
4. **(Disjoint q-orbits):** For $\nu \in S_\Pi$, let (r_ν, N_ν) be the Weil-Deligne representation associated to Π_ν via the local Langlands correspondence. Write

$$r_\nu(\mathrm{Frob}_\nu) \sim \begin{pmatrix} \mathcal{H}^\nu_{l^\nu_1}(\alpha^\nu_1) & & & \\ & \mathcal{H}^\nu_{l^\nu_2}(\alpha^\nu_2) & & \\ & & \ddots & \\ & & & \mathcal{H}^\nu_{l^\nu_{k^\nu}}(\alpha^\nu_{k^\nu}) \end{pmatrix} \text{ with } \mathcal{H}^\nu_m(\alpha) = \begin{pmatrix} \alpha & & & \\ & \alpha q_\nu & & \\ & & \ddots & \\ & & & \alpha q_\nu^{m-1} \end{pmatrix}.$$

Then, for all $\nu \in S_\Pi$ and for all $0 \leq i \neq j \leq k^\nu$, the q-orbits

$$q_\nu^{\mathbb{Z}} \alpha_i^\nu = \{q_\nu^a . \alpha_i^\nu \mid a \in \mathbb{Z}\} \quad \text{and} \quad q_\nu^{\mathbb{Z}} \alpha_j^\nu = \{q_\nu^a . \alpha_j^\nu \mid a \in \mathbb{Z}\}$$

are disjoint. (This is Assumption 4.37).

(Observe that the first two parts can be understood as general conjectures, while the last two parts put a constraint on our choice of Π. Observe also that the first part implies that $\Lambda^1_{\mathcal{E}_\Pi}$ has Dirichlet density 1 by Lemma 6.7.) For the next theorem, we set $\overline{S}_{\Pi,\ell} = \overline{S}_\Pi \cup \Omega_\ell^{F^+} \cup \Omega_\infty^{F^+}$.

Theorem 6.56. *Presuming Assumptions 6.54 and 6.55, there exists a subset $\Lambda^0_{\mathcal{E}_\Pi} \subset \Lambda^1_{\mathcal{E}_\Pi}$ of Dirichlet density 1 such that the functor $D^{\Box_{\overline{S}_{\Pi,\ell}},\chi,\mathcal{D}_\lambda}_{\overline{S}_{\Pi,\ell},W(k_\lambda)}(\overline{r}_\lambda)$ is globally unobstructed if $\lambda \in \Lambda^0_{\mathcal{E}_\Pi}$.*

As a first step towards the proof, consider the following assumption and the following alternative version of Theorem 6.56:

Assumption 6.57 (**Unipotent ramification**). For all $\nu \in S_\Pi$, the Weil-Deligne representation (r_ν, N_ν) associated to Π at ν has trivial restriction to the inertia subgroup. In particular, for any choice $\lambda \in \Lambda^1_{\mathcal{E}_\Pi}, \nu \in S_\Pi$ with $\ell(\lambda) \neq \ell(\nu)$, the representation $\overline{\rho}_\lambda | \mathrm{Gal}_{F_\nu}$ is unipotently ramified.

Theorem 6.58. *Presume (in addition to Assumptions 6.54 and 6.55) that Assumptions 6.26 and 6.57 hold. Then there exists a subset $\Lambda^0_{\mathcal{E}_\Pi} \subset \Lambda^1_{\mathcal{E}_\Pi}$ of Dirichlet density 1 such that for all $\lambda \in \Lambda^0_{\mathcal{E}_\Pi}$ the following holds:*

- $D^{\Box_{\overline{S}_{\Pi,\ell}},\chi,\mathcal{D}_\lambda}_{\overline{S}_{\Pi,\ell},W(k_\lambda)}(\overline{r}_\lambda)$ *has vanishing dual Selmer group;*
- *for all $\nu \in \Omega_\ell^{F^+}$ the local deformation ring $R^{\Box,\chi_\nu}_{W(k_\lambda)}(\overline{r}_{\lambda,\nu})$ is relatively smooth.*

Before we come to the proof, let us introduce a new notation:

Definition 6.59. Let Λ be the valuation ring of a finite extension of $\mathrm{Quot}(W(k_\lambda))$ and let L^+ be a finite, totally real extension of F^+. Fix two finite sets of places $\Sigma \subset S \subset \mathrm{Pl}_{F^+}$, a residual $\mathcal{G}_n(k_\lambda)$-valued representation $\overline{r}$ of Gal_{F^+} and a global deformation condition $\mathcal{D} = (\mathcal{D}_\nu)_{\nu\in\Sigma}$, where each $\mathcal{D}_\nu$ is either the minimally ramified, the FL-crystalline or the unconditioned local deformation condition. We denote by

$$^{L^+}R^{[\square_\Sigma],(\chi),\mathcal{D}}_{S,\Lambda}(\overline{r}) := R^{[\square_{\Sigma'}],(\chi|\,\mathrm{Gal}_{L^+}),\mathcal{D}'}_{S',\Lambda}(\overline{r}|\,\mathrm{Gal}_{L^+})$$

the deformation ring of $\overline{\rho}|\,\mathrm{Gal}_{L^+}$, where S', Σ' denote the sets of places of L^+ above S, Σ, and where $\mathcal{D}' := (D'_{\nu'})_{\nu'\in\Sigma'}$ with $D'_{\nu'}$ parametrizing arbitrary (resp. minimally ramified, resp. FL-crystalline) lifts of $\overline{r}|\,\mathrm{Gal}_{L^+_{\nu'}}$ if D_ν parametrizes arbitrary (resp. minimally ramified, resp. FL-crystalline) lifts (for ν the place of F^+ below ν'). We use an analogous notational convention for the deformation functor ("D instead of R") and for the associated GL_n-valued representation ("$\overline{\rho}$ instead of $\overline{r}$ and $L = F.L^+$ instead of L^+").

Proof of Theorem 6.56 assuming Theorem 6.58. The key observation is that we can always attain the situation of Theorem 6.58 by a finite solvable base change by using Lemma 6.15 and Corollary 6.17: There exists a totally real field F_1^+ which is a finite extension of F^+ such that Assumptions 6.26, 6.54 and 6.57 are fulfilled for the compatible system associated to the base change Π_{F_1} of Π to $F_1 := F_1^+.F$. Now we use the unobstructedness framework from Section 3.2 applied to the following functors:

- $D^{\square_{\overline{S}_{\Pi,\ell}},\chi,\mathcal{D}_\lambda}_{\overline{S}_{\Pi,\ell},W(k_\lambda)}(\overline{r}_\lambda)$;
- $D^{\square_{\overline{S}_{\Pi,\ell}},\chi,\mathcal{D}^\emptyset_\lambda}_{\overline{S}_{\Pi,\ell},W(k_\lambda)}(\overline{r}_\lambda) := D^{\square_{\overline{S}_{\Pi,\ell}},\chi}_{\overline{S}_{\Pi,\ell},W(k_\lambda)}(\overline{r}_\lambda)$, i.e. $\mathcal{D}^\emptyset_\lambda$ denotes the unconditioned deformation condition for $\overline{r}_\lambda$;
- $D^{\square_{\overline{S}'_{\Pi,\ell}},\chi|\,\mathrm{Gal}_{F_1^+},\mathcal{D}_\lambda(F_1)}_{\overline{S}'_{\Pi,\ell},W(k_\lambda)}(\overline{r}_\lambda|\,\mathrm{Gal}_{F_1^+}) = {}^{F_1^+}D^{\square_{\overline{S}_{\Pi,\ell}},\chi,\mathcal{D}_\lambda}_{\overline{S}_{\Pi,\ell},W(k_\lambda)}(\overline{r}_\lambda)$, where S'_Π denotes the places of F_1 above S_Π and where $\mathcal{D}_\lambda(F_1) = (D'_{\lambda,\nu})_{\nu\in\overline{S}'_{\Pi,\infty}}$ denotes the deformation condition defined analogously to $\mathcal{D}_\lambda$, i.e. parametrizing deformations of $\overline{r}_\lambda|\,\mathrm{Gal}_{F_1^+}$ which are minimally ramified at $\overline{S}'_\Pi$ and unconditioned at the infinite places;
- $D^{\square_{\overline{S}'_{\Pi,\ell}},\chi|\,\mathrm{Gal}_{F_1^+},\mathcal{D}^\emptyset_\lambda(F_1)}_{\overline{S}'_{\Pi,\ell},W(k_\lambda)}(\overline{r}_\lambda|\,\mathrm{Gal}_{F_1^+}) = {}^{F_1^+}D^{\square_{\overline{S}_{\Pi,\ell}},\chi,\mathcal{D}^\emptyset_\lambda}_{\overline{S}_{\Pi,\ell},W(k_\lambda)}(\overline{r}_\lambda)$ i.e. $\mathcal{D}^\emptyset_\lambda(F_1)$ is the unconditioned deformation condition for $\overline{r}_\lambda|\,\mathrm{Gal}_{F_1^+}$.

Now by Corollary 6.53, Assumption 4.45 holds. Let $X \subset \mathrm{Pl}^{\mathrm{fin}}_{\mathcal{E}_\Pi}$ be the finite failure set from Assumption 4.45, then by our local $R = R^{\mathtt{min}}$-result Corollary 4.47 we have isomorphisms

$$D^{\square_{\overline{S}_{\Pi,\ell}},\chi,\mathcal{D}_\lambda}_{\overline{S}_{\Pi,\ell},W(k_\lambda)}(\overline{r}_\lambda) \cong D^{\square_{\overline{S}_{\Pi,\ell}},\chi,\mathcal{D}^\emptyset_\lambda}_{\overline{S}_{\Pi,\ell},W(k_\lambda)}(\overline{r}_\lambda), \tag{6.17}$$

$${}^{F_1^+}D^{\square_{\overline{S}_{\Pi,\ell}},\chi,\mathcal{D}_\lambda(F_1)}_{\overline{S}_{\Pi,\ell},W(k_\lambda)}(\overline{r}_\lambda) \cong {}^{F_1^+}D^{\square_{\overline{S}_{\Pi,\ell}},\chi,\mathcal{D}^\emptyset_\lambda(F_1)}_{\overline{S}_{\Pi,\ell},W(k_\lambda)}(\overline{r}_\lambda), \tag{6.18}$$

for all $\lambda \in \Lambda^1_{\mathcal{E}_\Pi} - X$. It is clear from the definition that $\mathcal{D}^\emptyset_\lambda$ is a pre-dual-$\mathcal{D}^\emptyset_\lambda(F_1)$-condition, so it follows that for $\lambda \in \Lambda^1_{\mathcal{E}_\Pi} - X$ also $\mathcal{D}_\lambda$ is a pre-dual-$\mathcal{D}_\lambda(F_1)$-condition.

Therefore, after eliminating the finitely many places which are not coprime to $[F_1^+ : F^+]$ or which are contained in the failure set X, we can use the potential unobstructedness framework from Section 3.2:

$D^{\square_{\overline{S}_{\Pi,\ell}},\chi,\mathcal{D}_\lambda}_{\overline{S}_{\Pi,\ell},W(k_\lambda)}(\overline{r}_\lambda)$ has vanishing dual Selmer group if ${}^{F_1}D^{\square_{\overline{S}_{\Pi,\ell}},\chi,\mathcal{D}_\lambda(F_1)}_{\overline{S}_{\Pi,\ell},W(k_\lambda)}(\overline{r}_\lambda)$ has vanishing dual Selmer group. This implies that there is no loss of generality if we presume Assumptions 6.26 and 6.57 (in addition to Assumption 6.55) when proving the "has vanishing dual Selmer group"-part of Theorem 6.56.

The local parts of the "globally unobstructed" notion (cf. Definition 3.7), i.e. the relative smoothness of the local deformation rings

$$R^{\square,\chi_\nu,D_{\lambda,\nu}}_{W(k_\lambda)}(\overline{r}_{\lambda,\tilde{\nu}}) = \begin{cases} R^{\square}_{W(k_\lambda)}(\overline{\rho}_{\lambda,\nu}) & \text{if } \nu \in \Omega^{F^+}_\infty, \\ R^{\square,\mathtt{min}}_{W(k_\lambda)}(\overline{\rho}_{\lambda,\nu}) & \text{if } \nu \in \overline{S}_\Pi, \end{cases} \tag{6.19}$$

and

$$R^{\square,\chi_\nu,D_{\lambda,\nu}}_{W(k_\lambda)}(\overline{r}_{\lambda,\tilde{\nu}}) = R^{\square,\chi_\nu}_{W(k_\lambda)}(\overline{r}_{\lambda,\tilde{\nu}}) \quad \text{if } \nu \in \Omega^{F^+}_\ell,$$

follow (for $\Omega^{F^+}_\infty$) from Proposition 2.70, (for $\overline{S}_\Pi$) from Lemma 4.23 and (for $\Omega^{F^+}_\ell$) from the second bullet point in the statement of Theorem 6.58. This finishes the proof.

We remark that, for $\Omega^{F^+}_\ell$, we cannot simply cite Lemma 4.11 on the level of F^+ because we don't know if all $\nu \in \Omega^{F^+}_\ell$ are split in the extension $F|F^+$: If ν is not split, then we cannot work with $\overline{\rho}_{\lambda,\nu}$ instead of $\overline{r}_{\lambda,\tilde{\nu}}$ as we cannot apply Proposition 6.22. This is also the reason why we included the second bullet point in the statement of Theorem 6.58. We also remark that the `min`-condition in the second entry of 6.19 is redundant, i.e. we have

$$R^{\square,\chi_\nu}_{W(k_\lambda)}(\overline{r}_{\lambda,\tilde{\nu}}) = R^{\square,\mathtt{min}}_{W(k_\lambda)}(\overline{\rho}_{\lambda,\nu}) = R^{\square}_{W(k_\lambda)}(\overline{\rho}_{\lambda,\nu}) \tag{6.20}$$

for $\nu \in \overline{S}_\Pi$, as long as $\ell \gg 0$. □

We give another (stronger) version of Theorem 6.56:

Theorem 6.60. *Presuming Assumption 6.55, there exists a subset $\Lambda^0_{\mathcal{E}_\Pi} \subset \Lambda^1_{\mathcal{E}_\Pi}$ of Dirichlet density* 1 *such that the functor $D^{\square_{\overline{S}_{\Pi,\ell}},\chi}_{\overline{S}_{\Pi,\ell},W(k_\lambda)}(\overline{r}_\lambda)$ is globally unobstructed if $\lambda \in \Lambda^0_{\mathcal{E}_\Pi}$.*

Remark that we do not impose Assumption 6.54 here. Remark, moreover, that the deformation condition $\mathcal{D}_\lambda$ does not show up in the claim. This has two reasons: Firstly, $\mathcal{D}_\lambda$ is already dispensable in Theorem 6.56 (i.e. $\mathcal{D}_\lambda$ coincides with the unconditioned deformation condition) for $\ell \gg 0$ by the local $R = R^{\mathtt{min}}$ result Corollary 4.47. Secondly, condition $\mathcal{D}_\lambda$ cannot be imposed on the functor $D^{\square_{\overline{S}_{\Pi,\ell}},\chi}_{\overline{S}_{\Pi,\ell},W(k_\lambda)}(\overline{r}_\lambda)$ in Theorem 6.60 because there might be places in $\overline{S}_\Pi$ which are not split in $F|F^+$, and we have no notion of minimally ramified deformations valued in other groups than GL_n. Before we come to a proof, consider the following adapted version of Theorem 6.58:

Theorem 6.61. *Presume (in addition to Assumption and 6.55) that Assumptions 6.26 and 6.57 hold. Then there exists a subset $\Lambda^0_{\mathcal{E}_\Pi} \subset \Lambda^1_{\mathcal{E}_\Pi}$ of Dirichlet density* 1 *such that for all $\lambda \in \Lambda^0_{\mathcal{E}_\Pi}$ the following holds:*

- $D^{\square_{\overline{S}_{\Pi,\ell}},\chi}_{\overline{S}_{\Pi,\ell},W(k_\lambda)}(\overline{r}_\lambda)$ *has vanishing dual Selmer group;*
- *for all $\nu \in \Omega^{F^+}_\ell$ the local deformation ring $R^{\square,\chi_\nu}_{W(k_\lambda)}(\overline{r}_{\lambda,\nu})$ is relatively smooth.*

- *for all* $\nu \in \overline{S}_\Pi$ *the local deformation ring* $R^{\square,\chi_\nu}_{W(k_\lambda)}(\overline{r}_{\lambda,\nu})$ *is relatively smooth.*

Proof of Theorem 6.60 assuming Theorem 6.61. The “Proof of Theorem 6.56 assuming Theorem 6.58” carries over verbatim, except for the local condition that $R^{\square,\chi_\nu}_{W(k_\lambda)}(\overline{r}_{\lambda,\tilde{\nu}})$ shall be relatively smooth for $\nu \in \overline{S}_\Pi$ (because we don’t have an isomorphism as in (6.20), as we allow ν to stay inert in $F|F^+$). But this condition is precisely the one added as the third bullet point of Theorem 6.61. □

The remainder of this section will be devoted to the proof of Theorem 6.58 (and Theorem 6.61). Before we come to this proof, let us first record a potential version of Theorem 6.56:

Corollary 6.62. *Let* F, E *be number fields and*

$$\mathcal{R} = \Big(\rho_\lambda : \mathrm{Gal}_F \to \mathrm{GL}_n(E_\lambda)\Big)_{\lambda\in \mathrm{Pl}_E^{fin}}$$

a compatible system which is potentially automorphic: There exists a finite solvable extension $L|F$ *such that the base change* ${}^L\mathcal{R} = (\rho_\lambda|\,\mathrm{Gal}_L)_{\lambda\in\mathrm{Pl}_E^{fin}}$ *is of the form* ${}^L\mathcal{R} = \mathcal{R}_\Pi$ *for a RAESDC automorphic form* Π *of* $\mathrm{GL}_n(\mathbb{A}_L)$*. Assume that* Π *fulfills the conditions of Assumption 6.55 and that the ramification set* S_Π *contains only places which are split in* $L|L^+$*. Let* $(r_\lambda)_{\lambda\in\mathrm{Pl}_E^{fin}-T}$ *denote the* $\mathcal{G}_n$*-valued family associated to* $\mathcal{R}$*, where* $T \subset \mathrm{Pl}_E^{fin}$ *is the failure set where* ρ_λ *is not absolutely irreducible. Let* $D^{\square_{\overline{S}_{\Pi,\ell}},\chi}_{\overline{S}_{\Pi,\ell},W(k_\lambda)}(\overline{r}_\lambda)$ *denote the functor parametrizing fixed-determinant* $\overline{S}_\ell$*-framed deformations of* $\overline{r}_\lambda$ *which are unramified outside* $\overline{S}_\ell$*. Then there exists a subset* $\Lambda_E \subset \mathrm{Pl}_E^{fin}$ *of Dirichlet density* 1 *such that* $R^{\square_{\overline{S}_\ell},\chi}_{\overline{S}_\ell,W(k_\lambda)}(\overline{r}_\lambda)$ *is globally unobstructed if* $\lambda \in \Lambda_E - T$*.*

Proof. This follows immediately from Theorem 6.56 applied to ${}^L\mathcal{R}$ and the potential unobstructedness descent from L to F applied verbatim as in the proof of Theorem 6.56 assuming Theorem 6.58. □

We remark that the assumption in Corollary 6.62 on the splitting of the places in S_Π can be avoided by referring to Theorem 6.62 instead of Theorem 6.56.

6.5.1 Auxiliary primes in extensions

Recall, that $\mathcal{R}_\Pi = (\rho_\lambda)_{\lambda\in\Lambda^1_\mathcal{E}}$ defines a pure (of some weight w) and strictly compatible $\mathcal{E}$-rational system of Gal_F-representations for any finite extension $\mathcal{E}$ of $\mathcal{E}_\Pi$. Let $S = S_\Pi$. We want to check that we have a sufficient supply of auxiliary primes as demanded by condition R.11 in Section 6.4.1: We say that a prime $\nu \in \mathrm{Pl}_{F^+}^{fin} - \Omega^{F^+}_{\ell(\lambda)}$ is λ-*auxiliary*, if

- ν splits in $F|F^+$ as $\tilde{\nu}\tilde{\nu}^c$,
- $\overline{\rho}_\lambda$ is unramified at $\tilde{\nu}$,
- $H^0(F_{\tilde{\nu}}, \mathrm{ad}(\overline{\rho}_\lambda)(1))$ or, equivalently, $\mathrm{Hom}_{F_{\tilde{\nu}}}(\overline{\rho}_\lambda, \overline{\rho}_\lambda(1))$ vanishes.

We say that λ *admits auxiliary primes* if there exist (at least) two places in $\mathrm{Pl}_{F^+}^{fin} - \Omega^{F^+}_{\ell(\lambda)}$ which are λ-auxiliary. As a first step, we have the following:

Theorem 6.63. *1. The set*

$$\Lambda_{\mathcal{E}}(\mathit{aux}, F^+) := \{\lambda \in \mathrm{Pl}_{\mathcal{E}}^{\mathit{fin}} \mid \lambda \text{ admits auxiliary primes } \} \subset \mathrm{Pl}_{\mathcal{E}}^{\mathit{fin}}$$

is cofinite;

2. The set

$$\Lambda_{\mathcal{E}}^1(\mathit{aux}, F^+) := \{\lambda \in \Lambda_{\mathcal{E}}^1 \mid \lambda' \text{ admits auxiliary primes for all } \lambda' \in \Omega_{\ell(\lambda)}^{\mathcal{E}} \} \subset \Lambda_{\mathcal{E}}^1$$

has Dirichlet density 1.

Proof. It is clear that 1. implies 2., so we only prove 1. Let ν_1 be a place of F^+ away from S which splits in $F|F^+$ as $\tilde{\nu}_1\tilde{\nu}_1^c$ and for which we want to determine those λ for which ν_1 is λ-auxiliary. By the pureness of the system $\mathcal{R}$ we know that the eigenvalues of $\overline{\rho}_\lambda(\mathrm{Frob}_{\tilde{\nu}_1})$ are q_{ν_1}-Weil numbers of some weight w, i.e. algebraic numbers fulfilling condition (3.4). Denote the set of these eigenvalues by X. Then the set of eigenvalues of $\overline{\rho}_\lambda(1)(\mathrm{Frob}_{\tilde{\nu}})$ is given by $X' = \{x.q_{\nu_1} \mid x \in X\}$. Hence, if ν_1 is not λ-auxiliary, the condition $\mathrm{Hom}_{F_{\tilde{\nu}_1}}(\overline{\rho}_\lambda, \overline{\rho}_\lambda(1)) \neq 0$ implies

$$x \equiv x'.q_{\nu_1} \qquad \mathrm{mod}\, \ell(\lambda) \tag{6.21}$$

for (at least one) suitable choice of elements $x, x' \in X$. Clearly, a congruence as in (6.21) can hold only for finitely many λ. Let Y_1 denote the (cofinite) complement of those λ in $\mathrm{Pl}_{\mathcal{E}}^{\mathrm{fin}}$.

The same procedure with respect to another place ν_2 of F^+ away from S which splits in $F|F^+$ leads to a set Y_2. Therefore, we see that

$$Y_1 \cap Y_2 \subset \Lambda_{\mathcal{E}}(\mathtt{aux}, F^+) \subset \mathrm{Pl}_{\mathcal{E}}^{\mathrm{fin}}$$

and the claim follows as $Y_1 \cap Y_2$ is cofinite in $\mathrm{Pl}_{\mathcal{E}}^{\mathrm{fin}}$. □

Now for a finite, totally real extension $L^+|F^+$ we write $\Lambda_{\mathcal{E}}^1(\mathtt{aux}, L^+)$ for the set of those $\lambda \in \Lambda_{\mathcal{E}}^1$ which admit auxiliary primes with respect to the compatible system associated to the base change $\mathcal{R}_L$, i.e. for those $\lambda \in \Lambda_{\mathcal{E}}^1$ for which there exist two places ν_1, ν_2 in $\mathrm{Pl}_{L^+}^{\mathit{fin}} - \Omega_{\ell(\lambda)}^{L^+}$ such that

- ν_i splits in $L|L^+$ (with $L := F.L^+$) as $\tilde{\nu}_i\tilde{\nu}_i^c$,
- $\overline{\rho}_\lambda | I_{L_{\tilde{\nu}_i}}$ is trivial,
- $H^0(L_{\tilde{\nu}_i}, \mathrm{ad}(\overline{\rho}_\lambda)(1)) = 0$,

where $i = 1, 2$. Then we have

Theorem 6.64. *1. The set*

$$\Lambda_{\mathcal{E}}(\mathit{aux}, L^+|F^+) := \bigcap_{M^+} \Lambda_{\mathcal{E}}(\mathit{aux}, M^+) \subset \mathrm{Pl}^{\mathit{fin}}\, \Lambda_{\mathcal{E}}$$

is cofinite, where M^+ runs through all intermediate extension fields of $L^+|F^+$.

2. *The set*

$$\Lambda^1_{\mathcal{E}}(\boldsymbol{aux}, L^+|F^+) := \bigcap_{M^+} \Lambda^1_{\mathcal{E}}(\boldsymbol{aux}, M^+) \subset \Lambda^1_{\mathcal{E}}$$

has Dirichlet density 1*, where* M^+ *runs through all intermediate extension fields of* $L^+|F^+$*.*

Proof. We easily see that the proof of Theorem 6.63 carries over: Let ν_1, ν_2 be places of L^+ away from S (Attention: *not* "away from the the ramification set of $\mathcal{R}_L$", which is a possibly weaker condition) which are completely split in $L|L^+$ and $L^+|F^+$. (There are infinitely many such places we can choose from, by applying Chebotarev's density theorem to the Galois closure of L.)

Let Y_i denote the sets of places as in the proof of Theorem 6.63, applied to L^+ instead of F^+. For an intermediate field M^+ we let $\nu_i^{M^+}$ denote the place of M^+ below ν_i. It is obvious that for any such M^+ and for any $\lambda \in Y_i$ we have

- $\nu_i^{M^+}$ splits in $M|M^+$ (with $M := F.M^+$) as $\tilde{\nu}_i^{M^+}\tilde{\nu}_i^{M^+,c}$;
- $\overline{\rho}_\lambda|I_{M_{\tilde{\nu}_i^{M^+}}}$ is trivial;
- $H^0(M_{\tilde{\nu}_i^{M^+}}, \mathrm{ad}(\overline{\rho}_\lambda)(1)) = 0$.

The claim now follows as in the proof of Theorem 6.63. □

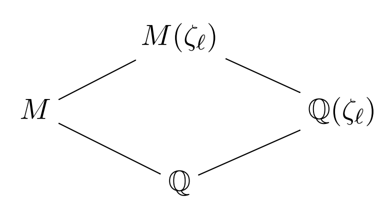
We can even get a stronger version of this: Denote

$$Z(L^+|F^+) := \bigcap_{M^+} Z(M^+) \text{ with } Z(M^+) = \{\nu \in \mathrm{Pl}^{\mathrm{fin}}_{L^+} \mid \forall w \in \mathrm{Pl}_{M^+} \text{ below } \nu : [M(\zeta_{\ell(\nu)}) : M] > n\}$$

and write $\Lambda^1_{\mathcal{E}}(\mathtt{aux}, L^+|F^+)^\diamond$ for the set of those $\lambda \in \Lambda^1_{\mathcal{E}}(\mathtt{aux}, L^+|F^+)$ for which the following holds: Any $\lambda' \in \Omega^{\mathcal{E}}_{\ell(\lambda)}$ admits two auxiliary primes which lie in $Z(L^+|F^+)$. We make the following easy observation:

Proposition 6.65. $Z(L^+|F^+)$ *is cofinite in* Pl_{L^+}*.*

Proof. As there are only finitely many intermediate extensions, it suffices to show that each $Z(M^+)$ has a finite complement in Pl_{L^+}. For this, consider the diagram



for a rational prime ℓ. We are done if we can show that $[M(\zeta_\ell) : M] > n$ holds for almost all ℓ. Denote by d the degree $[M : \mathbb{Q}]$ and recall $[\mathbb{Q}(\zeta_\ell) : \mathbb{Q}] = \ell - 1$. It follows that for all $\ell > dn$ we have $[M(\zeta_\ell) : M] > n$. □

Theorem 6.66. *The set*

$$\Lambda^1_{\mathcal{E}}(\boldsymbol{aux}, L^+|F^+)^\diamond \subset \Lambda^1_{\mathcal{E}}$$

has Dirichlet density 1*.*

Proof. The proof of Theorem 6.64 carries over, if we replace the sentence

> Let ν_1, ν_2 be places of L^+ away from S [..]

by

> Let ν_1, ν_2 be places of L^+ away from $S \cup \left(\mathrm{Pl}^{\mathrm{fin}}_{L^+} - Z(L^+|F^+)\right)$ [..].

□

6.5.2 Proof of Theorem 6.58

Let L^+ be a totally real, finite extension of F^+. We say that L^+ is *pre-admissible*, if the following conditions are met:

P.1) $L \coloneqq F.L^+$ is unramified over L^+ at every finite place;

P.2) The extension $L^+|F^+$ is Galois and solvable.

These conditions are designed to capture the following:

Observation: If L^+ is pre-admissible, then there exists a unitary group H over L^+ (as considered in Section 6.2) and a unitary avatar $\langle f \rangle = \pi_L$ of $H(\mathbb{A}_{L^+})$ of the base change Π_L of Π to L. (Here, $f \in \mathcal{S}_{\boldsymbol{\omega}}(U_0)$ denotes a suitable automorphic form on H of level $U_0 \coloneqq U_{(\Delta,\emptyset)}$ and suitable weight $\boldsymbol{\omega}$ which generates Π_L, where $\Delta \coloneqq \{\nu \in \mathrm{Pl}_{L^+} | \nu \text{ lies above } S_\Pi\}$.)

Definition 6.67. We say that a prime $\lambda \in \Lambda_1^{\mathcal{E}}$ is L^+-*procurable* if the following two conditions are fulfilled:

1. The restriction of $\overline{\rho}_\lambda$ to Gal_L remains absolutely irreducible;
2. There exists an L-big enough extension field $\mathcal{K}_\lambda$ of $\mathcal{E}_\lambda$ such that there is an isomorphism
$$ {}^{L^+}R^{\lambda,\mathtt{crys}}_{\mathcal{O}_{\mathcal{K}_\lambda}} \cong \mathcal{O}_{\mathcal{K}_\lambda}, \tag{6.22}$$
where ${}^{L^+}R^{\lambda,\mathtt{crys}}_{\mathcal{O}_{\mathcal{K}_\lambda}} \coloneqq {}^{L^+}R^{\chi,\mathcal{D}_\lambda(\mathtt{crys})}_{\overline{S}'_{\Pi,\ell},\mathcal{O}_{\mathcal{K}_\lambda}}(\overline{r}_\lambda)$ denotes the universal deformation ring parametrizing crystalline (above ℓ), minimally ramified (at $\overline{S}_\Pi$) deformations of $\overline{r}_\lambda|\,\mathrm{Gal}_{L^+}$ to coefficient $\mathcal{O}_{\mathcal{K}_\lambda}$-algebras which are unramified outside the set $\overline{S}'_{\Pi,\ell} \subset \mathrm{Pl}_{L^+}$ of places which lie above $\overline{S}_{\Pi,\ell} \subset \mathrm{Pl}_{F^+}$ and with fixed determinant χ.

We remark that the first condition is rather harmless: As we presume Assumption 6.6 (also for the restricted system $\mathcal{R}_\Pi|\,\mathrm{Gal}_L$), this can only fail for finitely many $\lambda \in \Lambda^1_{\mathcal{E}}$. We furthermore remark that in the second condition we have to consider the residual representation with values in the residue field $k_{\mathcal{O}_{\mathcal{K}_\lambda}}$ instead of k_λ: If $\iota : k_\lambda \hookrightarrow k_{\mathcal{O}_{\mathcal{K}_\lambda}}$ denotes the inclusion induced by the embedding $\mathcal{E}_\lambda \hookrightarrow \mathcal{K}_\lambda$, we are in fact considering
$$ {}^{L^+}R^{\chi,\mathcal{D}_\lambda(\mathtt{crys})}_{\overline{S}'_{\Pi,\ell},\mathcal{O}_{\mathcal{K}_\lambda}}(\mathcal{G}_n(\iota) \circ \overline{r}_\lambda). $$

In order to keep the notation simple, we will continue to abbreviate $\overline{r}_\lambda$ for $\mathcal{G}_n(\iota)\circ\overline{r}_\lambda$. (This is also justified by Definition 2.20.)

For a pre-admissible L^+, we define the following set:

$$\mathrm{Proc}(L^+) = \{\, \lambda \in \Lambda^1_{\mathcal{E}} \mid \lambda \text{ is } L^+\text{-procurable}\,\}$$

Theorem 6.68. *There exists a nested sequence $F^+ = L_0^+ \subset L_1^+ \subset \ldots$ of pre-admissible extensions of F^+ such that*

$$\lim_{i\to\infty} \delta\Big(\bigcup_{j=1}^{i} \mathrm{Proc}(L_j^+) \Big) = 1,$$

where $\delta(Y)$ denotes the density of those rational primes q for which each $\lambda \in \mathrm{Pl}_{\mathcal{E}}$ above q fulfills $\lambda \in Y$.

Proof. Let us first introduce another new notation: Let L^+ be pre-admissible, then we say that $\lambda \in \Lambda^1_{\mathcal{E}}$ is *L^+-$\star$-procurable*, if the following list of conditions is met (with $\ell = \ell(\lambda)$):

S.1) ℓ is not divisibe by any element of S_Π;

S.2) ℓ is unramified in the extension $L|\mathbb{Q}$;

S.3) All places of L above $S_{\Pi,\ell}$ are split over L^+;

S.4) The base change Π_L of Π to L is cuspidal;

S.5) There exists a place $\lambda' \in \mathrm{Pl}_{\mathcal{E}(U_0)}$ above λ such that $\lambda' \in \Lambda^1_{\mathcal{E}(U_0)}(\mathtt{aux}, L^+|F^+)^\diamond$;

S.6) If ν is a place of L above S_Π, then Π_L admits a non-trivial fixed vector for the Iwahori subgroup $\mathrm{Iw}(\nu) \subset \mathrm{GL}_n(L_\nu)$.

The set of all L^+-$\star$-procurable λ is denoted by $\mathrm{Proc}^\star(L^+)$. (Observe that condition S.4 does not depend on λ, but we intentionally include it in the list. So, if Π_L fails to be cuspidal, we have $\mathrm{Proc}^\star(L^+) = \emptyset$.)

Claim 1: $\mathrm{Proc}^\star(L^+) - \mathrm{Proc}(L^+)$ is finite.

Proof of Claim 1. We continue to denote by Π_L the base change of Π to L. By condition S.4, this is again a RACSDC representation. By the pre-conditions (P.1 and P.2), there exists a unitary group H and an avatar π_L of H over L.

Now, for $\lambda \in \mathrm{Proc}^\star(L^+)$ we pick an L-big enough field extension $\mathcal{K}_\lambda$ of $\mathcal{E}(U_0)_{\lambda'}$ and auxiliary places $\nu_1, \nu_2 \in \mathrm{Pl}_{L^+}$ as provided by condition S.5. Recall the set $\Delta = \{\nu \in \mathrm{Pl}_{L^+} \,|\nu \text{ lies above } S_\Pi\,\}$ and take $\Sigma_{\mathtt{aux}} := \{\nu_1\}$ and

$$\Sigma_{\mathtt{ram}} := \begin{cases} \Delta \sqcup \{\nu_2\} & \text{if } n \text{ is even and } \#\Delta \text{ is odd,} \\ \Delta & \text{otherwise,} \end{cases} \tag{6.23}$$

and denote $\mathcal{T} := \Sigma_{\mathtt{aux}} \cup \Sigma_{\mathtt{ram}} \subset \mathrm{Pl}_{L^+}$, $U := U_{(\Sigma_{\mathtt{ram}},\Sigma_{\mathtt{ram}})}$. The first case in (6.23) is designed to ensure that condition R.4 of Section 6.4.1 is fulfilled. Adding an auxiliary place to the set where we allow our lifts to ramify is harmless (i.e. does not change the deformation problem) as shown in Proposition 6.45. Note that we also have $\mathcal{E}(U_0) = \mathcal{E}(U)$. We consider now the complex Hecke algebra ${}^{\mathcal{O}_{\mathcal{E}(U)}}\mathbf{T}^{\mathcal{T}}_{\boldsymbol{\omega}}(U)$ and the ℓ-adic model $\mathbb{T} := {}^{\mathcal{O}_{\mathcal{K}_\lambda}}\mathbb{T}^{\mathcal{T}_\ell}_{\boldsymbol{\omega}_\lambda}(U)$.

Recall that we wrote $\pi_L = \langle f \rangle$ for the unitary avatar of the base change of Π to L and for a suitable choice $f \in \mathcal{S}_{\boldsymbol{\omega}}(U_0)$. We see that $\overline{\rho}_\lambda|\,\mathrm{Gal}_L$ (understood as a representation with values in $\mathrm{GL}_n(k_{\mathcal{O}_{\mathcal{K}_\lambda}})$)

equals the reduction of the representation attached to the maximal ideal $\mathfrak{m} := \ker(\varphi_{f^{(\lambda)}}) \subset \mathbb{T}$ by Proposition 6.33, where $f^{(\lambda)}$ is the ℓ-adic model of f.

We can now check the preconditions for Conjecture 6.37 for this choice of $L^+, \ell, \Sigma_{\mathtt{aux}}, \Sigma_{\mathtt{ram}}, \boldsymbol{\omega}, U, \mathcal{E}(U), \mathcal{K}_\lambda$ and $\mathfrak{m}$:

- If $\ell > \max(2, n)$, conditions R.1-R.6, R.8-R.9 and R.11-R.12 are either fulfilled by our choices above or are mere notational remarks which cannot fail;
- By condition S.6, Π_L admits Iwahori fix vectors for all $\nu \in \Sigma_{\mathtt{ram}}$, as demanded by the choice of the level subgroup for Conjecture 6.37, i.e. the subgroup U in condition R.7 is the right one;
- Adequateness condition R.10: Presuming Assumption 6.6, this cannot fail if $\ell \geq 2(n+1)$, see [GHTT12].

Thus, the desired isomorphism (6.22) follows from Theorem 6.48 as long as ℓ is bigger than the constant K from there. We subsume:

$$\mathrm{Proc}^\star(L^+) - \mathrm{Proc}(L^+) \subset \{\lambda \mid \ell(\lambda) \leq \max(2, n, 2(n+1), K)\}.$$

End of proof of Claim 1. ♣

Therefore, it suffices to show that there exists a nested sequence $F^+ = L_0^+ \subset L_1^+ \subset \ldots$ of pre-admissible extensions of F^+ such that

$$\lim_{i\to\infty} \delta\Big(\bigcup_{j=1}^{i} \mathrm{Proc}^\star(L_j^+) \Big) = 1.$$

For the construction of the extensions, define the set

$$\Omega_F := \{\, d \in \mathbb{N} \mid \sqrt{d} \notin F,\ \text{base change } \Pi \rightsquigarrow \Pi_{F(\sqrt{d})} \text{ remains cuspidal } \}.$$

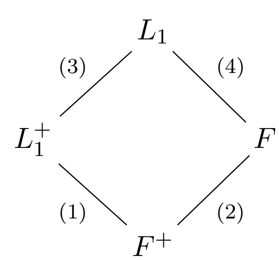
By Lemma 6.12, Ω_F is not empty, so we choose a $d_1 \in \Omega_F$ and take $L_1^+ = L^+(\sqrt{d_1})$.

Claim 2: L_1^+ is pre-admissible.

Proof of Claim 2. Condition P.2 is automatically fulfilled because $[L_1^+ : F^+] = 2$. Considering condition P.1, we have to check that $L_1 | L_1^+$ is unramified everywhere. For this, we observe that we have an identity of the discriminants

$$\Delta_{L_1|F^+} = \Delta_{L_1^+|F^+} \Delta_{F|F^+} = \Delta_{L_1^+|F^+}.$$

(This follows e.g. from [Jan96, Exercise 3 on p. 51].) Consider the following diagram:



(6.24)

Assuming there is a prime w of L_1^+ that ramifies in (3), the prime v of F^+ which lies below w must ramify in the extension $L_1 | F^+$. But then v divides $\Delta_{L_1^+|F^+} = \Delta_{L_1|F^+}$, i.e. v ramifies in (1). This would imply that v has ramification index 4 in the extension $L_1|F^+$. But in (2), v is unramified by the prerequisites, so it can at most ramify in (4), yielding a ramification index of 2 in $L_1|F^+$. This contradicts the assumption that w ramifies in (3).

End of proof of Claim 2. ♣

Claim 3: $\delta\big(\mathrm{Proc}^{\star}(F_1^+)\big) \geq \frac{1}{2}$.

Proof of Claim 3. We check which λ fail the list S.1-S.6:

- Concerning S.1 and S.2, we have to exclude the finitely many λ for which $\ell(\lambda)$ is not coprime to S or ramifies in $L_1^+|\mathbb{Q}$;
- Condition S.4 is universally fulfilled by our choice of L_1^+;
- Condition S.5 excludes a set of places λ of Dirichlet density 0, cf. Theorem 6.66.
- Concerning condition S.6, we remark that by local-global compatibility (cf. [CH13, Theorem 1.4] and the references therein) Π_L admits an $Iw(\nu)$-fixed vector if $\rho|\,\mathrm{Gal}_L$ has unipotent ramification at ν [Wed08, (4.3.6) Proposition]. Thus condition S.6 follows immediately from our Assumption 6.57.
- We first explain why the "S-part" of condition S.3 is not destroyed, i.e. why each place w of L_1^+ above S splits in $L_1|L_1^+$. For this, let v be the prime of F^+ below w and consider again diagram (6.24). We know that v splits in (2). If v stays inert in (1), then necessarily w must split in (3) because v is split in $L_1|F^+$. If v splits as $w.w'$ in (1), we can use [Neu99, Exercise 3 on p. 52] to see that v splits completely in $L_1|F^+$. This again implies that w splits in (3). (Remark that, by the same reasoning, we see that for $\lambda \in \mathrm{Proc}^{\star}(L_0^+)$ we have that any prime of L_1^+ above $\ell(\lambda)$ splits in $L_1|L_1^+$. Loosely speaking, we don't loose $\star$-procuration when base changing from F to L_1. Also remark that we used an analogous argument before, cf. the proof of Corollary 6.17.)
- It remains to count those ℓ which fulfill the condition that all primes of L_1^+ above ℓ are split in the extension $L_1|L_1^+$. By Lemma 6.10, their density is at least $\frac{1}{2}$.

End of proof of Claim 3. ♣

For the next tower step we take $F_2^+ := F_1^+(\sqrt{d_2})$ for a $d_2 \in \Omega_{F_1^+}$. It is checked as before that $\Omega_{F_1^+} \neq \emptyset$ and that F_2^+ is pre-admissible. Writing $F_2^+ = F^+(\sqrt{d_1}, \sqrt{d_2})$ we see that the extension $F_2^+|F^+$ is Galois.
Claim 4: $\delta\big(\mathrm{Proc}^{\star}(F_2^+)\big) \geq \frac{3}{4}$.

Proof of Claim 4. This follows as in the proof of Claim 3, the main points being:

- Remark that by Theorem 6.66, we can assume that the auxiliary primes chosen at the F_1^+-level are exactly the primes lying below the auxiliary primes chosen at the F_2^+-level. In other words, the (density-0) set of rational primes removed to guarantee condition S.5 during the proof of Claim 3 is the same as the one during the proof of Claim 4;
- Analogously as in the proof of Claim 3 we use that we don't loose $\star$-procuration when base changing from L^1 to L^2;
- The quantity $\frac{3}{4}$ follows again from Lemma 6.10.

End of proof of Claim 4. ♣

Iterating this construction of quadratic extensions we get a nested sequence of pre-admissible fields F_j^+ such that

$$\delta\Big(\bigcup_{j=1}^{i}\mathrm{Proc}^\star(F_j^+)\Big) \geq \delta(\mathrm{Proc}^\star(F_i^+) \geq 1 - \frac{1}{2^i} \quad \underset{i\to\infty}{\longrightarrow} \quad 1.$$

Together with Claim 1, this concludes the proof of Theorem 6.68. □

We now give a slight variation of Definition 6.67 and Theorem 6.68:

Definition 6.69. With regard to a pre-admissible extension L^+ of F^+, we say that a prime $\lambda \in \mathrm{Pl}_{\mathcal{E}}^{fin}$ is L^+-$\mathfrak{D}$-*procurable* if the restriction of $\overline{\rho}_\lambda$ to Gal_L (with $L = F.L^+$) remains absolutely irreducible and if there is an isomorphism

$$ {}^{L^+}R^{\square,\lambda,\mathtt{crys}}_{W(k_\lambda)} \cong W(k_\lambda)[[x_1,\dots,x_u]], \tag{6.25}$$

where ${}^{L^+}R^{\square,\lambda,\mathtt{crys}}_{W(k_\lambda)} := {}^{L^+}R^{\square,\chi,\mathcal{D}_\lambda(\mathtt{crys})}_{\overline{S}_{\Pi,\ell},W(k_\lambda)}(\overline{r}_\lambda)$ and $u = \dim(\mathfrak{g}_n^{\mathtt{der}}) = n^2$.

The set of all L^+-$\mathfrak{D}$-procurable λ is denoted by $\mathrm{Proc}^{\mathfrak{D}}(L^+)$.

Theorem 6.70. *There exists a nested sequence $F^+ = L_0^+ \subset L_1^+ \subset \dots$ of pre-admissible extensions of F^+ such that*

$$\lim_{i\to\infty}\delta\Big(\bigcup_{j=1}^{i}\mathrm{Proc}^{\mathfrak{D}}(L_j^+)\Big) = 1.$$

Proof. For $i \in \mathbb{N}$ denote

$$\Delta_i = \bigcup_{j=1}^{i}\mathrm{Proc}(L_j^+).$$

Also fix for each $\lambda \in \Delta_i$ a $j \leq i$ such that $\lambda \in \mathrm{Proc}(L_j^+)$. Denote the corresponding field extension from the proof of Theorem 6.68 by $L_{(\lambda)} = L_{(\lambda)}^+.F$. By Theorem 6.68, for such a $\lambda \in \Delta_i$ we have

$$ {}^{L^+_{(\lambda)}}R^{\lambda,\mathtt{crys}}_{\mathcal{O}_{\mathcal{K}_\lambda}} \cong \mathcal{O}_{\mathcal{K}_\lambda}$$

for a suitable extension $\mathcal{O}_{\mathcal{K}_\lambda}$ of $W(k_\lambda)$. Proposition 2.62 then yields

$$ {}^{L^+_{(\lambda)}}R^{\square,\lambda,\mathtt{crys}}_{\mathcal{O}_{\mathcal{K}_\lambda}} \cong \mathcal{O}_{\mathcal{K}_\lambda}[[x_1,\dots,x_u]].$$

Now we can use Corollary 2.35 to deduce the isomorphism (6.25). □

Corollary 6.71. *There exists a subset $\Lambda_{\mathcal{E}}^2 \subset \Lambda_{\mathcal{E}}^1$ of Dirichlet density 1 such that for each $\lambda \in \Lambda_{\mathcal{E}}^2$ there exists a finite, totally real extension $L_{(\lambda)}^+$ of F and an isomorphism*

$$ {}^{L^+_{(\lambda)}}R^{\square_{\overline{S}_{\Pi,\ell}},\chi,\mathcal{D}_\lambda(\mathtt{crys})}_{\overline{S}_{\Pi,\ell},W(k_\lambda)}(\overline{r}_\lambda) \cong W(k_\lambda)[[x_1,\dots,x_{w(\lambda)}]]$$

with $w(\lambda) = n^2.\#\overline{S}'_{\Pi,\ell} - 1$ and where $\overline{S}'_{\Pi,\ell}$ denotes the places of $L_{(\lambda)}^+$ above S_Π.

Proof. This follows from another application of Proposition 2.62. □

Next, we will apply the framework of Section 3.1 to the attained λ. We remind the reader that $\mathcal{D}_\lambda$ denotes the deformation condition parametrizing lifts of $\overline{r}_\lambda$ which are minimally ramified at $\overline{S}_\Pi$.

Theorem 6.72. *There exists a cofinite subset $\Lambda^3_{\mathcal{E}} \subset \Lambda^2_{\mathcal{E}}$ such that the following holds: Let $\lambda \in \Lambda^3_{\mathcal{E}}$ and $L^+_{(\lambda)}$ the corresponding extension from Corollary 6.71. Then the deformation functor*

$$^{L^+_{(\lambda)}}D^{\square_{\overline{S}_{\Pi,\ell}},\chi,\mathcal{D}_\lambda}_{\overline{S}_{\Pi,\ell},W(k_\lambda)}(\overline{r}_\lambda) = D^{\square_{\overline{S}'_{\Pi,\ell}},\chi|\operatorname{Gal}_{L^+_{(\lambda)}},\mathcal{D}'_\lambda}_{\overline{S}'_{\Pi,\ell},W(k_\lambda)}(\overline{r}_\lambda|\operatorname{Gal}_{L^+_{(\lambda)}})$$

has vanishing dual Selmer group (i.e. $H^1_{\mathcal{L}^\perp_\lambda}(\operatorname{Gal}_{L^+_{(\lambda)}}, \mathfrak{g}^{\mathtt{der},\vee}) = 0$, where $\mathcal{L}_\lambda$ is the system of local conditions corresponding to the deformation condition $\mathcal{D}_\lambda$).

Proof. When applying the framework, we take

- $\mathtt{sm}$ as the condition parametrizing all deformations;
- $\mathtt{crys}$ as the condition parametrizing all crystalline deformations (see Section 4.3);
- $\mathtt{min}$ as the condition parametrizing all minimally ramified lifts (see Section 4.4);
- $\chi = \epsilon_\ell^{1-n}\delta_{F|F^+}^{n(\mathrm{mod}\,2)}$.

We first check the following list of conditions (and we abbreviate $L^+ = L^+_{(\lambda)}$ as we check this for a fixed $\lambda \in \Lambda^2_{\mathcal{E}}$):

1. **(Representability)**: The $S'_{\ell.\infty}$-framed deformation functor

   $$^{L^+}D^{\square_{S'_\ell},\chi}_{S'_\ell,W(k_\lambda)}(\overline{r}_\lambda)$$

   is representable (by an object $^{L^+}R^{\square_{S'_\ell},\chi}_{S'_\ell,W(k_\lambda)}(\overline{r}_\lambda)$).

   Answer: This follows from our Proposition 2.61.

2. **($\mathbf{sm}/k$)**: As we took for $\mathtt{sm}$ the unrestricted deformation condition, we have to check that for each $\nu \in \Omega_\ell$ the functor

   $$^{L^+_\nu}D^{\square,\chi_\nu}_{W(k_\lambda)}(\overline{r}_{\lambda,\nu}) = D^{\square,\chi_\nu}_{W(k_\lambda)}(\overline{r}_\lambda|\operatorname{Gal}_{L^+_\nu})$$

   is representable and that the representing object $R^{\chi,\square,\mathtt{sm}}_\nu$ is formally smooth of relative dimension

   $$d^{\square,\mathtt{sm}}_\nu = \dim(\mathfrak{g}^{\mathtt{der}}_n)([L_{\tilde{\nu}} : \mathbb{Q}_\ell] + 1) = n^2([L_{\tilde{\nu}} : \mathbb{Q}_\ell] + 1) = n^2([L^+_\nu : \mathbb{Q}_\ell] + 1).$$

   (This also amounts to the vanishing of the error terms δ_ν in Theorem 3.12.2.)

   Answer: Representability follows from the first part of Theorem 2.22. For the remaining claim, we first refer to Proposition 6.22 in order to get an isomorphism

   $$^{L^+_\nu}R^{\square,\chi_\nu}_{W(k_\lambda)}(\overline{r}_{\lambda,\nu}) \cong {}^{L_{\tilde{\nu}}}R^{\square}_{W(k_\lambda)}(\overline{\rho}_{\lambda,\tilde{\nu}}).$$

   Now everything follows from Lemma 4.11.

3. **(crys)**: For each $\nu \in \Omega_\ell$, the subfunctor
$$^{L_\nu^+}D^{\square,\chi_\nu,\mathtt{crys}}_{W(k_\lambda)}(\overline{r}_{\lambda,\nu}) \hookrightarrow {}^{L_\nu^+}D^{\square,\chi_\nu}_{W(k_\lambda)}(\overline{r}_{\lambda,\nu})$$
is relatively representable and the representing object is formally smooth of relative dimension
$$d_\nu^{\square,\mathtt{crys}} = \dim(\mathfrak{g}_n^{\mathtt{der}}) + (\dim(\mathfrak{g}_n^{\mathtt{der}}) - \dim(\mathfrak{b}_n^{\mathtt{der}}))[L_\nu^+ : \mathbb{Q}_\ell],$$
where $\mathfrak{b}_n$ denotes the Lie algebra of a Borel subgroup of $\mathcal{G}_n$.
<u>Answer</u>: By definition,
$$^{L_\nu^+}R^{\square,\chi_\nu,\mathtt{crys}}_{W(k_\lambda)}(\overline{r}_{\lambda,\nu}) \cong {}^{L_{\tilde\nu}}R^{\square,\mathtt{crys}}_{W(k_\lambda)}(\overline{\rho}_{\lambda,\tilde\nu}).$$
Thus, the condition is fulfilled by Lemma 4.14.

4. **(min)**: For each $\nu \in S$, the subfunctor
$$^{L_\nu^+}D^{\square,\chi_\nu,\mathtt{min}}_{W(k_\lambda)}(\overline{r}_{\lambda,\nu}) \hookrightarrow {}^{L_\nu^+}D^{\square,\chi_\nu}_{W(k_\lambda)}(\overline{r}_{\lambda,\nu})$$
is relatively representable and the representing object is formally smooth of relative dimension
$$d_\nu^{\square,\mathtt{min}} = \dim(\mathfrak{g}_n^{\mathtt{der}}).$$
<u>Answer</u>: Again, by definition,
$$^{L_\nu^+}R^{\square,\chi_\nu,\mathtt{min}}_{W(k_\lambda)}(\overline{r}_{\lambda,\nu}) \cong {}^{L_{\tilde\nu}}R^{\square,\mathtt{min}}_{W(k_\lambda)}(\overline{\rho}_{\lambda,\tilde\nu}).$$
Thus, the condition is fulfilled by Lemma 4.23.

5. **(∞)**: For each $\nu \in \Omega_\infty$, the local deformation ring $^{L_\nu^+}R^{\square,\chi_\nu}_{W(k_\lambda)}(\overline{r}_{\lambda,\nu})$ is formally smooth of relative dimension $d_\nu^\square = \dim(\mathfrak{b}_n^{\mathtt{der}})$.
<u>Answer</u>: We get from Proposition 2.70 that $^{L_\nu^+}R^{\square,\chi_\nu}_{W(k_\lambda)}(\overline{r}_{\lambda,\nu})$ is formally smooth of relative dimension $\dim\big((\mathfrak{g}_n^{\mathtt{der}})^{c_\nu=-1}\big) = \dim(\mathfrak{gl}_n^{c_\nu=-1})$, where c_ν is the non-trivial element of the decomposition group at ν. By construction (see Lemma 2.1.4 and Proposition 3.4.4 of [CHT08]), the image of $\overline{r}_\lambda(c_\nu)$ is not contained in $\mathrm{GL}_n \times \mathrm{GL}_1$. Moreover,
$$\mathscr{m} \circ \overline{r}_\lambda(c_\nu) = \overline{\epsilon}_\ell^{1-n}(c_\nu)\delta^{\mu_\mathfrak{m}}(c_\nu) = \begin{cases} (-1).(-1)^{\mu_\mathfrak{m}} & \text{if } n \text{ is even;} \\ (-1)^{\mu_\mathfrak{m}} & \text{if } n \text{ is odd.} \end{cases}$$
Here, ϵ_ℓ denotes the cyclotomic character (which sends c_ν to -1), δ denotes the non-trivial character of $\mathrm{Gal}(F|F^+)$ and $\mu_\mathfrak{m}$ is a suitable element of $\mathbb{Z}/2\mathbb{Z}$. It follows from our $R = T$-theorem (Conjecture 6.37) that $\mu_\mathfrak{m} \equiv n \pmod 2$, so we have $\mathscr{m} \circ \overline{r}(c_\nu) = -1$, independent from the parity of n. Using [CHT08, Lemma 2.1.3], this implies $\dim(\mathfrak{gl}_n^{c_\nu=-1}) = \frac{n(n+1)}{2} = \dim(\mathfrak{b}_n^{\mathtt{der}})$.

6. **(Presentability)**: Consider the ring
$$^{L^+}R^{\mathtt{loc}} := \widehat{\bigotimes}_{\nu \in \overline{S}'_{\Pi,\ell}} {}^{L^+}\widetilde{R}_\nu \qquad \text{with } {}^{L^+}\widetilde{R}_\nu = \begin{cases} ^{L_\nu^+}D^{\square,\chi_\nu,\mathtt{min}}_{W(k_\lambda)}(\overline{r}_{\lambda,\nu}) & \text{if } \nu \in S; \\ ^{L_\nu^+}D^{\square,\chi_\nu}_{W(k_\lambda)}(\overline{r}_{\lambda,\nu}) & \text{if } \nu \in \Omega_\ell \sqcup \Omega_\infty. \end{cases} \tag{6.26}$$
Then there exists a presentation
$$^{L^+}R^{\square_{\overline{S}_{\Pi,\ell}},\chi}_{\overline{S}_{\Pi,\ell},W(k_\lambda)}(\overline{r}_\lambda) \cong {}^{L^+}R^{\mathtt{loc}}[[x_1,\ldots,x_a]]_{/(f_1,\ldots,f_b)}$$

with $a-b=(\#\overline{S}'_{\Pi,\ell}-1).\dim(\mathfrak{g}_n^{\mathtt{ab}})$.

Answer: This is the content of Corollary 2.68, but we have to check Assumption 2.63. As $\mathfrak{g}^{\mathtt{der}}=\mathfrak{gl}_n$ (Proposition 6.19), this condition holds by Corollary 2.73 for almost all λ.

7. **(R=T)**: The ring ${}^{L^+_{(\lambda)}}R^{\square_{\overline{S}_{\Pi,\ell}},\chi,\mathcal{D}_\lambda(\mathtt{min,crys})}_{\overline{S}_{\Pi,\ell},W(k_\lambda)}(\overline{r}_\lambda)$ is formally smooth of relative dimension

$$r_0=\dim(\mathfrak{g}).\#\overline{S}'_{\Pi,\ell}-\dim(\mathfrak{g}^{\mathtt{ab}}).$$

Answer: This follows from Corollary 6.71.

We see that the requirements of Theorem 3.12.2 are met. So we can conclude the proof if we verify the three requirements of Corollary 3.16:

8. ℓ must be big enough so that $\mathfrak{g}_n=\mathfrak{g}_n^{\mathtt{der}}\oplus\mathfrak{g}_n^{\mathtt{ab}}$.

   Answer: This can be achieved by excluding finitely many λ.

9. $H^0(\mathrm{Gal}_{L^+},\mathfrak{g}_n^{\mathtt{der},\vee})=0$.

   Answer: This can be proved analogously as in Section 2.6. First, remark that

$$H^0(\mathrm{Gal}_{L^+},\mathfrak{g}_n^{\mathtt{der},\vee})\subset H^0(\mathrm{Gal}_L,\mathfrak{g}_n^{\mathtt{der},\vee})\cong H^0(\mathrm{Gal}_L,\mathfrak{gl}_n^\vee),$$

   for which we have to recall

   - $\overline{r}_\lambda|\mathrm{Gal}_L$ equals $\overline{\rho}_\lambda$ (via the embedding $\mathrm{GL}_n\subset\mathcal{G}_n$) by construction, see Lemma 6.21;
   - $\mathfrak{g}_n^{\mathtt{der}}\cong\mathfrak{gl}_n$, see Proposition 6.19;
   - the adjoint representation of Gal_L on $\mathfrak{g}_n^{\mathtt{der}}$ (via $\overline{r}_\lambda$) corresponds to the adjoint representation of Gal_L on $\mathfrak{gl}_n$ (via $\overline{\rho}_\lambda$) with respect to the identifications from the above two bullet points (see [CHT08, Section 2.1]).

   Thus we are good if we can show that $H^0(\mathrm{Gal}_L,\mathfrak{gl}_n^\vee)$ vanishes for almost all λ, which follows from Corollary 2.73.

10. For $\nu\in\overline{S}'_\Pi$, $\dim(L_{\lambda,\nu})=h^0(\mathrm{Gal}_{L^+_\nu},\mathfrak{g}_n^{\mathtt{der}})$.

    Answer: As ν is split, we can use Proposition 6.22 to get

$$h^0(\mathrm{Gal}_{L^+_\nu},\mathfrak{g}_n^{\mathtt{der}})=h^0(\mathrm{Gal}_{L_{\tilde{\nu}}},\mathfrak{gl}_n),$$

    where the action on $\mathfrak{gl}_n$ is via $\overline{\rho}_{\lambda,\tilde{\nu}}$. The claim now follows from [CHT08, Corollary 2.4.21].

The finitely many exclusions as required in parts 6., 8. and 9. are now the places we exclude from $\Lambda^2_{\mathcal{E}}$ to get $\Lambda^3_{\mathcal{E}}$. □

Now we can finally complete the desired proof:

Proof of Theorem 6.58. First remark that Theorem 6.72 is not far from the "has vanishing dual Selmer group"-part of Theorem 6.58, the main difference is that we have introduced field extensions $L^+_{(\lambda)}|F^+$ which we can eliminate with the potential unobstructedness methods of Section 3.2. As each index $[L^+_{(\lambda)} : F^+]$ is a power of 2 (and k_λ has odd characteristic for all $\lambda \in \Lambda^3_{\mathcal{E}}$), the "of degree coprime to ℓ"-part of Lemma 3.21 is universally fulfilled. For each extension $L^+_{(\lambda)}|F^+$ we can now argue exactly as in the proof of Theorem 6.56 assuming Theorem 6.58, but we have to take care of the (finite) failure sets $X = X(L^+_{(\lambda)})$ for which the local $R = R^{\mathtt{min}}$-result Corollary 4.47 fails at the $L^+_{(\lambda)}$-level. For this, recall that the $L^+_{(\lambda)}$ show up in the tower $F^+ = L^+_0 \subset L^+_1 \subset \dots$ and that we have

$$\lim_{i\to\infty} \delta\Big\{\lambda \;\Big|\; {}^{L^+_{(\lambda)}}D^{\square_{S'_\ell},\chi,\mathcal{D}_\lambda(\mathtt{min})}_{S'_\ell,W(k_\lambda)}(\overline{r}_\lambda) \text{ has vanishing dual Selmer group}, L^+_{(\lambda)} \subset L^+_i \Big\} = 1,$$

by Corollary 6.71 and Corollary 6.72. Hence,

$$\lim_{i\to\infty} \delta\Big\{\lambda \;\Big|\; {}^{L^+_{(\lambda)}}D^{\square_{\overline{S}_{\Pi,\ell}},\chi,\mathcal{D}_\lambda}_{\overline{S}_{\Pi,\ell},W(k_\lambda)}(\overline{r}_\lambda) \text{ has vanishing dual Selmer group}, L^+_{(\lambda)} \subset L^+_i, \lambda \notin X(L^+_{(\lambda)}) \Big\} = 1.$$

The first bullet point of Theorem 6.58 follows.

It remains to show that the local deformation ring $R^{\square,\chi_\nu}_{W(k_\lambda)}(\overline{r}_{\lambda,\nu})$ is relatively smooth for $\nu \in \Omega^{F^+}_\ell$. By Corollary 4.7 we know that

$$\lim_{i\to\infty} \delta\Big\{\lambda \;\Big|\; D^{\square,\chi_\nu}_{W(k_\lambda)}(\overline{r}_\lambda|\operatorname{Gal}_{L^+_{(\lambda),\nu'}}) \text{ is unobstructed for all } \nu' \in \Omega^{L^+_{(\lambda)}}_{\ell(\lambda)}, L^+_{(\lambda)} \subset L^+_i \Big\}$$

$$= \lim_{i\to\infty} \delta\Big\{\lambda \;\Big|\; \text{any } \nu' \in \Omega^{L^+_{(\lambda)}}_{\ell(\lambda)} \text{ is split in the extension } L_{(\lambda)}|L^+_{(\lambda)}, L^+_{(\lambda)} \subset L^+_i \Big\} = 1.$$

Using Lemma 3.18.2 and Proposition 3.2, the second bullet point of Theorem 6.58 follows. □

Proof of Theorem 6.61. The claim of Theorem 6.61 follows by performing a base change towards a finite solvable extension F' of F (which needs to be a CM field) such that the restricted system $\mathcal{R}_\Pi|\operatorname{Gal}_{F'^+}$ fulfills the preconditions for Theorem 6.58, i.e. such that all places above $\overline{S}_\Pi$ are split in the extension $F'|F'^+$, where F'^+ is the maximal totally real subfield of F'. This is possible by Corollary 6.17. Using Lemma 3.18 and Proposition 3.2, the first two bullet points of Theorem 6.61 then follow directly from Theorem 6.58 (for all λ such that $\ell(\lambda)$ does not divide $[F' : F]$). It remains to show the third bullet point in Theorem 6.61: From Lemma 4.23 it follows that for all $\nu \in \mathrm{Pl}_{F'^+}$ above $\overline{S}_\Pi$ the local deformation functor $D^{\square,\chi_\nu}_{W(k_\lambda)}(\overline{r}_{\lambda,\nu}|\operatorname{Gal}_{F'_\nu})$ is unobstructed. Using Lemma 3.18.2 and Proposition 3.2, the third bullet point follows. □

We repeat that this also completes the proof of Theorem 6.56 (using the "Proof of Theorem 6.56 assuming Theorem 6.58" following the statement of Theorem 6.58 at the beginning of Section 6.5). Likewise, this also completes the proof of Theorem 6.60.

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